ICMA393 Discrete Simulation: HW4

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1. Benford's Law: Chains of Exponential Distributions

I ran the simulation for $N = 100000$ iterations, for the $Expo(Expo(Expo(Expo(371))))$.

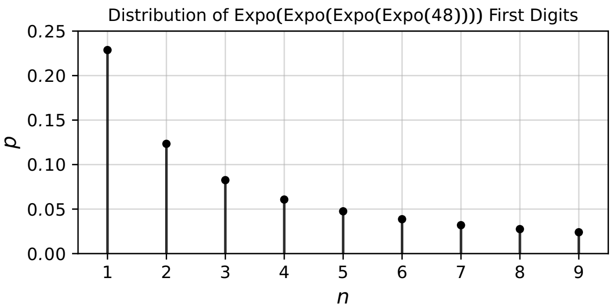

Figure 1: The chained exponential distributions seems to follow Benford's law.

2. Inter-Arrival Times Are Independent and Follow the Exponential Distribution

I ran the simulation for the given intervals for $N = 100000$ iterations.

Q	Method	λ	k	Interval	Result	Diff
a	Simulation	2	2	[0, 2)	0.14834	
	Formula	2	2	[0, 2)	0.146525111	0.001814889
b	Simulation	2	2	[2, 4)	0.14606	
	Formula	2	2	[2, 4)	0.146525111	0.000465111

Figure 2: Comparison of probability results from simulation and theoretical formula for a Poisson process over intervals $[0, 2)$ and $[2, 4)$

Notice that both simulations a and b have very similar results despite their intervals being different. This is due to the idea that both of their intervals have the same sized "window" that do not change over time (memoryless thing), the length of their intervals are the same. They effectively have the equal chances in this scenario which is quite intuitive, reminds me of roulette wheels.

Now comparing both simulations a and b to the Poisson formula we can see that the formula seems to faithfully model the probability of the scenarios. This is really nice because computing the formula is way faster than iterativley computing the simulation.

3. Poisson Random Variables

4. Exponential Random Variables

a) Prove that $E[X] = \lambda^{-1}$

$$= \int_0^\infty x \cdot \lambda e^{-\lambda x} dx$$

Integrate by parts using $\int u dv = uv - \int v du$, let $u = x, dv = \lambda e^{-\lambda x}$.

$$= [-xe^{-\lambda x}]_0^\infty - \int_0^\infty -e^{-\lambda x} dx$$
$$= [-xe^{-\lambda x}]_0^\infty - [-\frac{e^{-\lambda x}}{\lambda}]_0^\infty$$
$$= [-xe^{-\lambda x}]_0^\infty - [(-\frac{e^{-\lambda(\infty)}}{\lambda}) - (-\frac{e^{-\lambda(0)}}{\lambda})]$$

For the outside term $xe^{-\lambda x} \to 0$, When $x \to \infty$ then $xe^{-\lambda x} \to 0$, and when $x = 0$ then $xe^{-\lambda x} = 0$. Then we evaluate the rest of the equation to get λ^{-1}

$$= 0 - [(0) - (-\frac{1}{\lambda})]$$
$$= \frac{1}{\lambda} = \lambda^{-1}$$

b) Prove that $E[E^2] = \lambda^{-2}$

$$= \int_0^\infty x^2 \cdot \lambda e^{-\lambda x} dx$$

Integrate by parts using $\int u dv = uv - \int v du$, let $u = x^2, dv = \lambda e^{-\lambda x}$.

$$= [-x^2 e^{-\lambda x}]_0^\infty + \int_0^\infty 2xe^{-\lambda x} dx$$

We learned that the first term $[-x^2e^{-\lambda x}]_0^\infty$ evaluates to 0 from computing the last question.

$$= 0 + \int_0^\infty 2xe^{-\lambda x}dx$$

Then I apply integrate by parts again, let $u = 2x, dv = e^{-\lambda x} / -\lambda$.

$$= [-\frac{2x}{\lambda}e^{-\lambda x}]_0^\infty + \int_0^\infty \frac{2}{\lambda}e^{-\lambda x}dx$$
$$= 0 + \frac{2}{\lambda}\int_0^\infty e^{-\lambda x}dx$$
$$= \frac{2}{\lambda}[-\frac{e^{-\lambda x}}{\lambda}]_0^\infty = \frac{1}{\lambda}$$
$$= \frac{2}{\lambda^2}$$

5. Approximate Exponential Random Value from Tossing a Physical Coin!

a) Toss your coin 74 times!

H	T	H	T	H	H	H	H	T	T
T	T	H	H	H	T	T	H	T	T
H	T	H	H	H	H	H	T	H	H
H	H	T	H	T	H	T	H	T	H
H	H	T	T	H	H	H	H	H	H
T	H	H	T	T	H	H	T	T	H
T	H	T	H	T	T	H	T	H	H
T	T	H	H						

Figure 3: My outcomes recorded of tossing a coin 74 times.

b) Throw away baddies!

H	T	H	T	H	T	T	H	H	T
H	T	T	H	T	H	T	H	T	H
T	H	H	T	T	H	H	T	T	H
T	H	T	H	H	T				

Figure 4: 18 good pairs.

c) Binary-stringify!

$$b_1, b_2, ..., b_{18} = 000100111110101110$$

d,e) Compute u and x

$$\mathbf{u} = 0.07781410217285156$$
$$\mathbf{x} = -log(\mathbf{u}) \approx 2.553433$$

6. Spooky Buses for Spooktober

6.1) Report simulation outputs:

Outputs	Mean	Varience
Avg Inter-Arrival Time	0.9992874503839420	0.003348972072732149
Spooky Time	1.9929870517863757	1.981517643804638000

Figure 5

6.2) Theoretical Results:

Tath really helped me out with this one.

a) What are the expectation and the variance of the length of the interval that contains t = 163?

To frame the simulation by defining X_1 and X_2 as the pair of buses inter-arrival times that sandwich t=163 st X_1 and X_2 are Expo(1) and we define T as the total interval length. Then compute E[T]...

$$\begin{aligned} E[T] &= E[X_1 + X_2] \\ &= E[X_1] + E[X_2] \\ &= \frac{1}{\lambda} + \frac{1}{\lambda} \\ &= 2 \end{aligned}$$

Then similarly we compute Var[T]...

$$\begin{aligned} V[T] &= V[X_1 + X_2] \\ &= E[X_1] + E[X_2] \\ &= \frac{1}{\lambda^2} + \frac{1}{\lambda^2} \\ &= 2 \end{aligned}$$

b) What are the expectation and the variance of the length of the interval between the fifth and the sixth buses?

Suppose $T_i = X_1 + X_2 ... + X_i$ sum of buses inter arrival times, then we can compute the expected length between buses 5 and 6...

$$\begin{aligned} T_6 &= T_5 + X_6 \\ T_6 - T_5 &= X_6 \\ E[T_6 - T_5] &= E[X_6] \\ &= \frac{1}{\lambda} \\ &= 1 \end{aligned}$$

Again similarly we can compute the variance...

$$\begin{aligned} T_6 &= T_5 + X_6 \\ T_6 - T_5 &= X_6 \\ Var[T_6 - T_5] &= Var[X_6] \\ &= \frac{1}{\lambda^2} \\ &= 1 \end{aligned}$$

c) What are the expectation and the variance of the length of the interval between the last bus before the end (24:00) of October 29 and the first bus after the beginning (0:00) of October 30?

I think it would be the same as expectation and variance computed in Q6a. This is because in Q6b we showed that on average the time interval and variance between buses is 1 hour thus and drawing on the memoryless property thing, It said that last bus passed 1 hour ago at most and the bus that is on its way is 1 hour at most. Thus we say that the expected waiting time and variance is 2 hours which lines up with Q6a.

Source Code

https://github.com/AustinMaddison/discrete-simulation/tree/main/hw4/source