Quiz 1 Practice

The following are some practice problems for the first quiz.[1] The problems are supposed to be solved without help from calculators. We provide solutions in this document after the problems.

Note! This practice is supposed to be *much longer* than the actual quiz. Do not be afraid.

Problems

Problem 1. The dataset

$$2, 4, 5, 10, 14$$

contains 5 data points. What is the sample variance of the dataset?

Problem 2. Let X be a real-valued discrete random variable with support

$$\text{supp}(X) = \{3^1, 3^2, 3^3, 3^4, \dots\}.$$

Suppose that there exists a real number λ such that for every positive integer k,

$$\mathbb{P}\{X = 3^k\} = \frac{\lambda}{5^k}.$$

(a) Compute λ.

(b) Compute $\mathbb{E}(X)$.

Problem 3. Suppose that Y is a real-valued random variable with mean $\mathbb{E}(Y) = 2$ and variance $\text{Var}(Y) = 5$. What is $\mathbb{E}((Y-1)^2)$?

[1] The first quiz of ICMA 393/487, Trimester 1 Academic Year 2024-2025, Mahidol University International College. **Instructors:** Thotsaporn Thanatipanonda and Pakawut Jiradilok.
(Quiz 1 Practice Version: Oct 9, 2024.)

Problem 4. Let Z be a real-valued random variable such that $\mathbb{E}(Z) = 4$, $\mathbb{E}(Z^2) = 25$, and $\mathbb{E}(Z^3) = 190$. Let $X_1, X_2, \ldots$ be a sequence of independently and identically distributed random variables which follow the same probability distribution as Z.

(a) Compute

$$\lim_{n\to\infty} \mathbb{P}\left\{-0.6 \le \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}} - 3\sqrt{n} \le +1.2\right\}.$$

(b) Compute

$$\lim_{n\to\infty} \mathbb{P}\left\{-1.2 \le \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}} - 4\sqrt{n} \le +1.8\right\}.$$

(c) Compute

$$\lim_{n\to\infty} \mathbb{P}\left\{-1.8 \le \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}} - 5\sqrt{n} \le 0\right\}.$$

Give your answers in percentage (e.g. something like 25%). Round each answer to the nearest integer (in the form `integer` %).

The following table of approximated values for

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,\mathrm{d}t$$

might be useful:

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6
$\Phi(x) \approx$	0.5000	0.5398	0.5793	0.6179	0.6554	0.6915	0.7257
x	0.7	0.8	0.9	1.0	1.1	1.2	1.3
$\Phi(x) \approx$	0.7580	0.7881	0.8159	0.8413	0.8643	0.8849	0.9032
x	1.4	1.5	1.6	1.7	1.8	1.9	2.0
$\Phi(x) \approx$	0.9192	0.9332	0.9452	0.9554	0.9641	0.9713	0.9772

Problem 5. Recall the following Taylor series expansions for the sine and the tangent functions (at $x = 0$):

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots,$$
$$\tan(x) = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots.$$

In class, we saw that

$$n\sin\left(\frac{\pi}{n}\right) \qquad \text{and} \qquad n\tan\left(\frac{\pi}{n}\right)$$

are approximations for π.

(a) In the regime $n \to \infty$, which absolute constants $\lambda_1, \lambda_2 \in \mathbb{R}$ make

$$A(n) := \lambda_1 \cdot n\sin\left(\frac{\pi}{n}\right) + \lambda_2 \cdot n\tan\left(\frac{\pi}{n}\right)$$

the *best* approximation for π (among all other values of λ_1 and λ_2)?

(b) Let $A(n)$ be as above, with the best absolute constants λ_1 and λ_2. Consider using the number

$$A(10^{15})$$

as an approximation for π. Estimate the number of correct digits in $A(10^{15})$ (counting the leftmost 3 as the first correct digit, and the next 1 as the second correct digit, and so on). Provide a reasoning behind your estimation.

The following approximations might be useful:

$$\pi \approx 3.14159265358979,$$
$$\pi^5 \approx 306.01968478528145.$$

Problem 6. Suppose you have three cats, Mochi, Nibbles, and Whiskers. You are sitting in your office with all three cats in another room. You know that if you *meow*, cats may or may not come. More precisely, independently per meow, Mochi comes to you with probability 50%, Nibbles does so with probability 60%, and Whiskers does so with probability 80%. Whether each cat decides to come or not is independent of others' decisions.

Now, if more than one cats arrive to your office, it is a certainty (somehow) that all cats that arrive will simply stare at you in unanimous silence. If Mochi comes alone, she has a 90% probability of meowing back at you, independent of previous events. If Nibbles comes alone, she has a 50% probability of meowing back, independent of previous events. But if Whiskers comes alone, he has only a probability of 15% of meowing back.

At the moment, you are in your office alone. If you meow now, what is the overall probability that some cats arrive and meow back at you?

Problem 7. In yet another variant of the Monty Hall game show, there are 12 doors. Monty first rolls a fair die to get a number $N \in \{1, 2, \ldots, 6\}$, and hides this number from the participant. Monty then selects uniformly at random N doors and puts a car behind each of these doors. As usual, he puts a goat behind each of the $12 - N$ other doors.

The participant uniformly at random, independently from Monty's random variables, points at one of the 12 doors. Monty then rolls another independent fair die to get a new number $M \in \{1, 2, \ldots, 6\}$, and continues to open M doors, not selected by the participant, which have goats behind them.

The participant then switches to a different unopened door, uniformly at random, independent from previous actions, and opens the door to claim the object behind.

What's the overall probability that the participant gets a car? Give your answer in percentage (e.g. something like 25%). Round the answer to the nearest integer (in the form `integer` %).

The following calculation might be useful:

$$\frac{7}{144} \cdot \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \right) \approx 0.041107.$$

Problem 8. Let X and Y be independent random variables following $\text{Unif}\{1, 2, 3, 4\}$. Let $Z := |X - Y|$. The variance $\text{Var}(Z)$ is a rational number. Let's write it as the reduced fraction

$$\text{Var}(Z) = \frac{m}{n},$$

where m and n are relatively prime[2] (coprime) positive integers. Evaluate $m + n$.

[2]Two positive integers m and n are said to be **relatively prime** (or, synonymously, **coprime**) if $\gcd(m, n) = 1$.

Problem 9. Let A and B be two events. Suppose that

$$\mathbb{P}\{A|B\} = \frac{2}{5}, \qquad \mathbb{P}\{B|A\} = \frac{2}{3}, \qquad \text{and} \qquad \mathbb{P}\{A\} + \mathbb{P}\{B\} = \frac{4}{5}.$$

(a) What's the probability that at least one of event A or event B happens? Give your answer in percentage (e.g. something like 25%). Round the answer to the nearest integer (in the form `integer` %).

(b) Are events A and B independent? (Recall that two events A and B are *independent* if and only if $\mathbb{P}\{A \cap B\} = \mathbb{P}\{A\} \cdot \mathbb{P}\{B\}$.)

Problem 10. Toss a fair coin three times independently. Conditional on the event that the outcome has at least 2 `HEADS`, what is the probability that the first result is `HEAD`?

Problem 11. Suppose we have two independent fair dice. Toss them both simultaneously (independently), and observe the sum of the two resulting numbers. If the sum is at most 9, we stop and record the sum. Otherwise, do not record, and throw the dice again. Repeat until we obtain a sum that is at most 9, and record. Let the recorded number be S.

What's the probability that $S \leq 5$?

Problem 12. Let X and Y be independent and identically distributed $\mathrm{Unif}\{1, 2, 3, 4, 5, 6\}$. Compute

$$\mathbb{P}\{X + 2Y \leq 8 \,|\, X - Y \geq 2\}.$$

Problem 13. Let X_1 and X_2 be independent and identically distributed $\mathrm{Unif}\{3, 6, 9\}$. Define the random variable

$$Y := \begin{cases} 5 & \text{if } X_2 \leq 7, \\ 8 & \text{if } X_2 > 7. \end{cases}$$

Compute the following.

(a) $\mathbb{E}(X_1 + Y)$.

(b) $\mathbb{E}(X_2 + Y)$.

(c) $\mathbb{E}(X_1 \cdot Y)$.

(d) $\mathbb{E}(X_2 \cdot Y)$.

(e) $\mathrm{Var}(X_1 + Y)$.

(f) $\mathrm{Var}(X_2 + Y)$.

(g) $\mathrm{Var}(X_1 \cdot Y)$.

(h) $\mathrm{Var}(X_2 \cdot Y)$.

Problem 14. Let $G \sim \mathrm{Geom}(1/4)$. Let $U \sim \mathrm{Unif}\{1, 2, \ldots, G\}$. What is the probability that $U = 1$?

Problem 15. Let $X \sim \mathrm{Unif}[4]$. Let $Y \sim \mathrm{Unif}[X]$. Let $Z \sim \mathrm{Unif}[Y]$. What is the probability that $Z = 2$? Give your answer in percentage (e.g. something like 25%). Round the answer to the nearest integer (in the form `integer` %).

Recall that the notation $[n]$ denotes $\{1, 2, \ldots, n\}$, and so $\mathrm{Unif}[n]$ denotes the discrete uniform distribution on the set $\{1, 2, \ldots, n\}$.

Solutions

Solution to Problem 1. *Answer:* $\boxed{24}$.

We first calculate the sample mean:

$$\overline{x} = \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{2 + 4 + 5 + 10 + 14}{5} = 7.$$

To calculate the sample variance, we use the formula

$$s^2 = \frac{(a_1 - \overline{x})^2 + (a_2 - \overline{x})^2 + \cdots + (a_n - \overline{x})^2}{n-1}$$

to find

$$s^2 = \frac{(-5)^2 + (-3)^2 + (-2)^2 + 3^2 + 7^2}{4} = \frac{25 + 9 + 4 + 9 + 49}{4} = 24.$$

Therefore, the sample variance is $\boxed{24}$.

Solution to Problem 2. *Answers:* **(a)** $\boxed{4}$. **(b)** $\boxed{6}$.

(a) Recall that

$$\sum_{a \in \operatorname{supp}(X)} \mathbb{P}\{X = a\} = 1.$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{\lambda}{5^k} = 1. \tag{1}$$

Since we have

$$\sum_{k=1}^{\infty} \frac{1}{5^k} = \frac{1}{4},$$

the left-hand side of (1) equals $\frac{\lambda}{4}$, and thus $\lambda = \boxed{4}$.

(b) The expectation is

$$\mathbb{E}(X) = \sum_{a \in \operatorname{supp}(X)} \mathbb{P}\{X = a\} \cdot a = \sum_{k=1}^{\infty} \frac{4}{5^k} \cdot 3^k = 4 \sum_{k=1}^{\infty} \left(\frac{3}{5}\right)^k = 4 \cdot \frac{3}{2} = \boxed{6}.$$

Solution to Problem 3. *Answer:* $\boxed{6}$.

Recall that $\text{Var}(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2$. Therefore,

$$\mathbb{E}(Y^2) = \text{Var}(Y) + \mathbb{E}(Y)^2 = 5 + 2^2 = 9.$$

By linearity of expectation, we have

$$\mathbb{E}((Y-1)^2) = \mathbb{E}(Y^2 - 2Y + 1) = \mathbb{E}(Y^2) - 2\mathbb{E}(Y) + 1 = 9 - 2 \cdot 2 + 1 = \boxed{6}.$$

Solution to Problem 4. *Answers:* **(a)** $\boxed{0\ \%}$. **(b)** $\boxed{38\ \%}$. **(c)** $\boxed{0\ \%}$.

(a) & (c) Since $\mathbb{E}(X_1) = 4$ (not 3 or 5), the answers to parts (a) and (c) are $\boxed{0\ \%}$ by the strong law of large numbers.

(b) Note that

$$\sqrt{\text{Var}(X_1)} = \sqrt{\mathbb{E}(X_1^2) - \mathbb{E}(X_1)^2} = \sqrt{25 - 4^2} = 3.$$

Therefore, by the central limit theorem, we have

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}\left\{-1.2 \le \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}} - 4\sqrt{n} \le +1.8\right\} \\
&= \lim_{n\to\infty} \mathbb{P}\left\{-0.4 \le \frac{X_1 + X_2 + \cdots + X_n - 4 \cdot n}{3 \cdot \sqrt{n}} \le +0.6\right\} \\
&= \Phi(0.6) - \Phi(-0.4) = \Phi(0.6) + \Phi(0.4) - 1 \approx 0.3812,
\end{aligned}$$

where we have used the fact that $\forall x \in \mathbb{R}$, $\Phi(-x) = 1 - \Phi(x)$.

Round the answer to the nearest integer to obtain $\boxed{38\ \%}$.

Solution to Problem 5. *Answers:* **(a)** $\boxed{(\lambda_1, \lambda_2) = (2/3, 1/3)}$. **(b)** $\boxed{59}$.

(a) From the Taylor expansions, we find that if

$$A(n) = \lambda_1 \cdot n \sin\left(\frac{\pi}{n}\right) + \lambda_2 \cdot n \tan\left(\frac{\pi}{n}\right),$$

then

$$A(n) = (\lambda_1 + \lambda_2) \cdot \pi + (-\lambda_1 + 2\lambda_2) \cdot \frac{\pi^3}{6} \cdot \frac{1}{n^2} + (\lambda_1 + 16\lambda_2) \cdot \frac{\pi^5}{120} \cdot \frac{1}{n^4} + O\left(\frac{1}{n^6}\right).$$

In the regime where $n \to \infty$, for $A(n)$ to be a good approximation for π, we should make $A(n) = \pi + O(n^{-4})$. This happens if and only if

$$\lambda_1 + \lambda_2 = 1 \qquad \text{and } -\lambda_1 + 2\lambda_2 = 0,$$

which is if and only if $\lambda_1 = 2/3$ and $\lambda_2 = 1/3$.

(b) For these values of λ_1 and λ_2, the formula above gives

$$A(n) = \pi + \frac{\pi^5}{20} \cdot \frac{1}{n^4} + O\left(\frac{1}{n^6}\right),$$

so $A(10^{15})$ would be approximately $\frac{\pi^5}{20} \cdot 10^{-60}$ more than π. From the given calculation, we know that

$$\frac{\pi^5}{20} \approx 15.30.$$

Hence,

$$\frac{\pi^5}{20} \cdot 10^{-60} \approx 1.53 \times 10^{-59},$$

from which we can guess that the first incorrect digit of this approximation would happen at the 59$^{\text{th}}$ digit after the decimal point. Totally, we can guess that there would be 58 digits correct after the decimal point, and the first digit (3) to the left of the decimal point correct. Hence, we expect approximately 59 correct digits. (Note that the actual number of correct digits is indeed 59.)

Solution to Problem 6. *Answer:* $\boxed{9\ \%}$.

Let B_1 denote the event that Mochi comes alone. Let B_2 denote the event that Nibbles comes alone. Let B_3 denote the event that Whiskers comes alone. Let B_4 denote the event that either no cats come or at least two cats come. Let A be the event that a cat meows back. We can compute:

$$\begin{aligned}
\mathbb{P}\{B_1\} &= (0.5)(0.4)(0.2) = 0.04,\\
\mathbb{P}\{B_2\} &= (0.5)(0.6)(0.2) = 0.06,\\
\mathbb{P}\{B_3\} &= (0.5)(0.4)(0.8) = 0.16,\\
\mathbb{P}\{B_4\} &= 1 - \mathbb{P}\{B_1\} - \mathbb{P}\{B_2\} - \mathbb{P}\{B_3\} = 0.74.
\end{aligned}$$

Now, we have

$$\begin{aligned}
\mathbb{P}\{A|B_1\} &= 0.9,\\
\mathbb{P}\{A|B_2\} &= 0.5,\\
\mathbb{P}\{A|B_3\} &= 0.15,\\
\mathbb{P}\{A|B_4\} &= 0.
\end{aligned}$$

Therefore,

$$\mathbb{P}\{A\} = \sum_{i=1}^{4} \mathbb{P}\{B_i\}\mathbb{P}\{A|B_i\} = 0.09.$$

Solution to Problem 7. *Answer:* $\boxed{45\ \%}$.

We define the following events. For each $m, n \in [6]$, let $B_{0,m,n}$ denote the event that the participant points at a goat door, and $M = m$, and $N = n$. For each $m, n \in [6]$, let $B_{1,m,n}$ denote the event that the participant points at a car door, and $M = m$, and $N = n$. Let A denote the event that the participant wins a car after switching.

First, let's compute $\mathbb{P}\{B_{0,m,n}\}$ and $\mathbb{P}\{B_{1,m,n}\}$. Since M and N are independent random variables following $\mathrm{Unif}\{1, 2, \ldots, 6\}$, the probability that $M = m$ and $N = n$ is $1/36$. Now, given that there are $N = n$ cars, the probability that the participant selects a goat door is $(12 - n)/12$. In total,

$$\mathbb{P}\{B_{0,m,n}\} = \frac{1}{36} \cdot \frac{12 - n}{12}.$$

Similarly,

$$\mathbb{P}\{B_{1,m,n}\} = \frac{1}{36} \cdot \frac{n}{12}.$$

Now note that

$$\mathbb{P}\{A|B_{0,m,n}\} = \frac{n}{11 - m},$$

and

$$\mathbb{P}\{A|B_{1,m,n}\} = \frac{n-1}{11 - m}.$$

Combining the quantities above, we find

$$\begin{aligned}
\mathbb{P}\{A\} &= \sum_{m=1}^{6}\sum_{n=1}^{6}\left(\frac{1}{36}\cdot\frac{n}{12}\cdot\frac{n-1}{11-m}+\frac{1}{36}\cdot\frac{12-n}{12}\cdot\frac{n}{11-m}\right)\\
&= \frac{1}{432}\sum_{m=1}^{6}\sum_{n=1}^{6}\frac{11n}{11-m}\\
&= \frac{77}{144}\sum_{m=1}^{6}\frac{1}{11-m}\\
&= 11\cdot\frac{7}{144}\left(\frac{1}{5}+\frac{1}{6}+\cdots+\frac{1}{10}\right),
\end{aligned}$$

which, by the given approximation, is approximately $11\cdot 0.041107 \approx \boxed{45\ \%}$.

Solution to Problem 8. *Answer:* $\boxed{31}$.

Note that for each $x \in \{1,2,3,4\}$ and for each $y \in \{1,2,3,4\}$, we have

$$\mathbb{P}\{X = x, Y = y\} = \frac{1}{16}.$$

By writing out the sixteen cases, we find that

$$Z = \begin{cases} 0, & \text{with probability } 4/16,\\ 1, & \text{with probability } 6/16,\\ 2, & \text{with probability } 4/16,\\ 3, & \text{with probability } 2/16.\end{cases}$$

Thus,

$$\mathbb{E}(Z) = \frac{4}{16}\cdot 0+\frac{6}{16}\cdot 1+\frac{4}{16}\cdot 2+\frac{2}{16}\cdot 3 = \frac{5}{4},$$

and

$$\mathbb{E}(Z^2) = \frac{4}{16}\cdot 0+\frac{6}{16}\cdot 1+\frac{4}{16}\cdot 4+\frac{2}{16}\cdot 9 = \frac{5}{2}.$$

This implies $\mathrm{Var}(Z) = \mathbb{E}(Z^2) - \mathbb{E}(Z)^2 = \frac{5}{2} - \left(\frac{5}{4}\right)^2 = \frac{15}{16}$, and so $m = 15$ and $n = 16$. Hence, $m + n = \boxed{31}$.

Solution to Problem 9. *Answers:* **(a)** $\boxed{60\ \%}$. **(b)** $\boxed{\text{No}}$.

(a) We have

$$\frac{\mathbb{P}\{A \cap B\}}{\mathbb{P}\{B\}} = \frac{2}{5} \qquad \text{and} \qquad \frac{\mathbb{P}\{A \cap B\}}{\mathbb{P}\{A\}} = \frac{2}{3}.$$

Therefore,

$$\frac{\mathbb{P}\{A\} + \mathbb{P}\{B\}}{\mathbb{P}\{A \cap B\}} = \frac{3}{2} + \frac{5}{2} = 4.$$

Using $\mathbb{P}\{A\} + \mathbb{P}\{B\} = \frac{4}{5}$, we deduce that $\mathbb{P}\{A \cap B\} = \frac{1}{5}$. Thus,

$$\mathbb{P}\{A \cup B\} = \mathbb{P}\{A\} + \mathbb{P}\{B\} - \mathbb{P}\{A \cap B\} = \frac{4}{5} - \frac{1}{5} = \frac{3}{5} = \boxed{60\ \%}.$$

(b) From

$$\frac{\mathbb{P}\{A \cap B\}}{\mathbb{P}\{B\}} = \frac{2}{5} \qquad \text{and} \qquad \frac{\mathbb{P}\{A \cap B\}}{\mathbb{P}\{A\}} = \frac{2}{3},$$

knowing that $\mathbb{P}\{A \cap B\} = \frac{1}{5}$ now, we find $\mathbb{P}\{A\} = 3/10$ and $\mathbb{P}\{B\} = 1/2$. Thus,

$$\mathbb{P}\{A\} \cdot \mathbb{P}\{B\} = \frac{3}{20} \neq \frac{1}{5} = \mathbb{P}\{A \cap B\},$$

whence A and B are *not* independent.

Solution to Problem 10. *Answer:* $\boxed{3/4}$.

Let A denote the event that the first result is `HEAD`. Let B denote the event that there are at least 2 `HEADS` in the three tosses. Note that

$$A \cap B = \{\texttt{HHH}, \texttt{HHT}, \texttt{HTH}\},$$

and

$$B = \{\texttt{HHH}, \texttt{HHT}, \texttt{HTH}, \texttt{THH}\}.$$

Therefore,

$$\mathbb{P}\{A|B\} = \frac{\mathbb{P}\{A \cap B\}}{\mathbb{P}\{B\}} = \frac{3/8}{4/8} = \frac{3}{4}.$$

Solution to Problem 11. *Answer:* $\boxed{1/3}$.

For each roll, each of the 36 outcomes (e.g. the first die shows 3 and the second shows 5) is equally likely. The desired probability is

$$\mathbb{P}\{S \leq 5 \,|\, S \leq 9\} = \frac{\mathbb{P}\{S \leq 5 \ \text{ and } \ S \leq 9\}}{\mathbb{P}\{S \leq 9\}} = \frac{\mathbb{P}\{S \leq 5\}}{\mathbb{P}\{S \leq 9\}}.$$

There are 10 outcomes for which $S \leq 5$ and there are 30 outcomes for which $S \leq 9$. Therefore, the answer is

$$\frac{10}{30} = \frac{1}{3}.$$

Solution to Problem 12. *Answer:* $\boxed{1/2}$.

We proceed in a similar manner to the previous problem. The event $X - Y \geq 2$ is the same as the event that (X, Y) is one of $(3, 1)$, $(4, 1)$, $(4, 2)$, $(5, 1)$, $(5, 2)$, $(5, 3)$, $(6, 1)$, $(6, 2)$, $(6, 3)$, $(6, 4)$. The event

$$\{X + 2Y \leq 8 \text{ and } X - Y \geq 2\}$$

is the same as the event that (X, Y) is one of $(3, 1)$, $(4, 1)$, $(4, 2)$, $(5, 1)$, $(6, 1)$. Therefore,

$$\mathbb{P}\{X + 2Y \leq 8 \,|\, X - Y \geq 2\} = \frac{5}{10} = \frac{1}{2}.$$

Solution to Problem 13. *Answers:* $\boxed{12, 12, 36, 39, 8, 14, 300, 582}$.

First, we have

$$\mathbb{E}(X_1) = \mathbb{E}(X_2) = \frac{3 + 6 + 9}{3} = 6.$$

Second, note that Y is 5 with probability $2/3$ and is 8 with probability $1/3$. Therefore,

$$\mathbb{E}(Y) = \frac{2}{3} \cdot 5 + \frac{1}{3} \cdot 8 = 6.$$

By linearity of expectation, we have

$$\mathbb{E}(X_1 + Y) = \mathbb{E}(X_2 + Y) = 6 + 6 = 12.$$

Note that X_1 and Y are independent, so

$$\mathbb{E}(X_1 \cdot Y) = \mathbb{E}(X_1) \cdot \mathbb{E}(Y) = 6 \cdot 6 = 36.$$

We cannot do the same thing with $X_2 \cdot Y$, since X_2 and Y are *not* independent. Alternatively, we note that $X_2 \cdot Y$ is either 15, 30, or 72, equally likely. Thus,

$$\mathbb{E}(X_2 \cdot Y) = \frac{15 + 30 + 72}{3} = 39.$$

Now, note that

$$\mathbb{E}(X_1^2) = \mathbb{E}(X_2^2) = \frac{3^2 + 6^2 + 9^2}{3} = 42,$$

and

$$\mathbb{E}(Y^2) = \frac{2}{3} \cdot 25 + \frac{1}{3} \cdot 64 = 38.$$

Hence, by linearity of expectation,

$$\mathbb{E}((X_1+Y)^2) = \mathbb{E}(X_1^2) + 2\mathbb{E}(X_1Y) + \mathbb{E}(Y^2) = 42 + 2\cdot 36 + 38 = 152.$$

Thus,

$$\mathrm{Var}(X_1+Y) = \mathbb{E}((X_1+Y)^2) - \mathbb{E}(X_1+Y)^2 = 152 - 144 = 8.$$

(Alternatively, we can say, by independence of X_1 and Y, that $\mathrm{Var}(X_1+Y) = \mathrm{Var}(X_1) + \mathrm{Var}(Y) = (42-36) + (38-36) = 6 + 2 = 8$.)

Next,

$$\mathrm{Var}(X_2+Y) = \mathbb{E}((X_2+Y)^2) - \mathbb{E}(X_2+Y)^2 = 42 + 2\cdot 39 + 38 - 144 = 14.$$

Now,

$$\mathrm{Var}(X_1\cdot Y) = \mathbb{E}(X_1^2Y^2) - \mathbb{E}(X_1Y)^2 = 42\cdot 38 - 36^2 = 300,$$

and

$$\mathrm{Var}(X_2\cdot Y) = \mathbb{E}(X_2^2Y^2) - \mathbb{E}(X_2Y)^2 = 2103 - 1221 = 582.$$

Note that even though X_1 and X_2 are distributed the same, $\mathbb{E}(X_1\cdot Y)$ and $\mathbb{E}(X_2\cdot Y)$ are *not* the same. This is due to how X_1 is independent of Y, while X_2 is not.

Solution to Problem 14. *Answer:* $\boxed{\frac{2}{3}\log(2)}$.

Conditioning on the value G takes, we find

$$\begin{aligned}
\mathbb{P}\{U=1\} &= \sum_{g=1}^{\infty} \mathbb{P}\{G=g\}\cdot\mathbb{P}\{U=1\,|\,G=g\} \\
&= \sum_{g=1}^{\infty} \left(\frac{1}{4}\right)\cdot\left(\frac{3}{4}\right)^{g-1}\cdot\frac{1}{g} \\
&= \frac{1}{3}\sum_{g=1}^{\infty}\frac{(3/4)^g}{g} \\
&= -\frac{1}{3}\log\left(1-\frac{3}{4}\right) = \frac{2}{3}\log(2).
\end{aligned}$$

Solution to Problem 15. *Answer:* $\boxed{20\%}$.

First, we compute the distribution of Y. Note that

$$\begin{aligned}
\mathbb{P}\{Y=1\} &= \sum_{x=1}^{4} \mathbb{P}\{X=x\}\cdot\mathbb{P}\{Y=1\,|\,X=x\} \\
&= \sum_{x=1}^{4}\frac{1}{4}\cdot\frac{1}{x} = \frac{1}{4}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right) = \frac{25}{48}.
\end{aligned}$$

Similarly, we find $\mathbb{P}\{Y = 2\} = \frac{13}{48}$, $\mathbb{P}\{Y = 3\} = \frac{7}{48}$, and $\mathbb{P}\{Y = 4\} = \frac{3}{48}$. Now observe that

- $\mathbb{P}\{Z = 2 \,|\, Y = 1\} = 0$,
- $\mathbb{P}\{Z = 2 \,|\, Y = 2\} = 1/2$,
- $\mathbb{P}\{Z = 2 \,|\, Y = 3\} = 1/3$,
- $\mathbb{P}\{Z = 2 \,|\, Y = 4\} = 1/4$.

Hence,

$$\mathbb{P}\{Z = 2\} = \frac{25}{48} \cdot 0 + \frac{13}{48} \cdot \frac{1}{2} + \frac{7}{48} \cdot \frac{1}{3} + \frac{3}{48} \cdot \frac{1}{4} = \frac{115}{576} \approx 19.965\%.$$