Homework 3 (due Monday, October 21, 2024)

Remark. Please also turn in your code and worksheet for any program that you use for this homework.

1. `(2 points)` **Compound probability distribution.** The chain of distribution is another way to gain a sequence that satisfies Benford's law. Here, we ask you to simulate the chain of uniform distributions. For example,

$$U(0, U(0, U(0, U(0, 371))))$$

   is the uniform distribution chaining 4 times. How many times do you need to chain the uniform distribution to get the sequence for which the first digits bear a resemblance to Benford's law?

2. `(1.5 points)` Pick any real life data (i.e. numbers of population in each country, the prices of the food in Bangkok, etc) and verify whether the leading digits of your data satisfy Benford's Law.

3. `(2 points)` Use the program to guess the recurrence of the following sequences.

   (a) a_n is the *square* of n^{th} Fibonacci number where Fibo(0)=0 and Fibo(1)=1.

   (b) $a_n = \sum_{i=1}^{n} i^2, \quad n = 0, 1, 2, \ldots.$

   (c) $a_n = \left\lfloor \left(\frac{n}{2}\right)^3 \right\rfloor, \quad n = 0, 1, 2, \ldots.$

4. `(2 points)` Recall that palindromic or self-inverse compositions are those that read the same from left to right as from right to left. Guess the C-finite relation of the number of palindromic compositions of n.

   If possible, try to prove the C-finite relation that you found rigorously as well.

5. Define the sequence $\{a_n\}_{n=0}^{\infty}$ inductively by

$$(a_0, a_1, a_2, a_3, a_4, a_5) := (0, -14, -121, -548, -1915, -5774),$$

   and for $n \in \mathbb{Z}_{\geq 6}$,

$$a_n := 14a_{n-1} - 81a_{n-2} + 248a_{n-3} - 424a_{n-4} + 384a_{n-5} - 144a_{n-6}.$$

   (a) `(0.5 point)` Find a formula for a_n.

   (b) `(0.5 point)` The above recurrence formula is a linear recurrence of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_d a_{n-d},$$

   with $d = 6$. Is there a linear recurrence for the same sequence $\{a_n\}_{n=0}^{\infty}$ for a smaller value of d? If not, explain why. If so, find one such formula.

(c) `(0.5 point)` Many initial terms of the sequence are negative integers. Are there infinitely many negative terms in the sequence, or are there only a finite number of them? If there are infinite, explain why. If there are finite, find the largest index n for which $a_n < 0$.

(d) **(extra credit)** `(+0.5 extra point)` Determine absolute constants C_1, C_2 such that

$$\log(a_n) = C_1 \cdot n + C_2 \cdot \log(n) + O(1),$$

for $n \in \mathbb{Z}_{\geq 1}$. Here, log denotes the natural logarithm.

6. **(extra credit)** `(+1 extra point)` **Monty Hall with Two Dice**

Time and again, Patrick has been told that when he goes to the Monty Hall game show, he *must* choose to switch the door. "Always switch the door," they said. "Your probability of winning a car would be doubled," they said. Patrick, after hearing these things said to him a myriad of times, makes a mental note to himself that he would switch the door if he ever gets onto the Monty Hall game show.

One day, Patrick somehow finds himself participating in the Monty Hall game show. Patrick, internally very ready to switch the door, met Monty for the first time. The host tells Patrick, "We operate things a little differently now," and proceeds to hand him a fair die, to Patrick's confusion.

In Monty's hand is another (independent) fair die. "I'm going to roll *my* die, and you're going to roll *yours*," said Monty to Patrick, who is patiently listening, probably still fixing to switch the door. "Now there are still three doors, labeled D_1, D_2, D_3 (D stands for *Door*). There are still 1 prized car and 2 non-prized goats. I roll my die, but I'm not going to show the result of the roll to you yet. If it turns up $1, 2, 3$, or 4, I put the car behind D_1. If it's 5, I put the car behind D_2. If it's 6, I put the car behind D_3." says the host. Patrick, half listening, stares mindlessly at the die in his hand, wondering what he's going to do with it.

"Next, you roll yours," explains Monty, "and if it turns up $1, 2, 3$, or 4, you pick D_1." Patrick looks at Monty's face, wondering if the host knows what the verb *pick* means, but he's not going to protest about his lack of freedom to choose. "If it's 5, you pick D_2. If it's 6, you pick D_3. Sounds good?" asks Monty. Patrick nods.

Monty, noticing the sadness in Patrick's eyes, continues "Well, then the rest is the same. I open one door you don't pick with a goat behind it. Let the revealed goat run around, yada yada." Patrick interjects, "and can I choose whether or not to **switch the door**?"

Monty beams, and continues mysteriously, "Yes, of course, you can, but..." There is a second of pause before Monty resumes, "but you have to decide *now*, not after you see the outcome of your die roll. Tell me now, please, are you going to switch the door?"

(a) If Patrick has to decide now, should he decide to switch the door, or should he decide to not switch the door?

"Hey! That's quite unfair, no?" yelled Patrick, "In your original version, I would be able to decide *after* I pick my door!" It was very nice weather that afternoon, and Monty just had a lovely cup of tea. Somehow Monty is particularly generous at the moment. "Okay, you can decide after you pick your door," says Monty.

(b) Describe the best strategy for Patrick. (A *strategy* here is of the form "For each $i = 1, 2, 3$, if Patrick picks D_i, then he should or should not switch (possibly depending on i).") If he follows the strategy, what's his overall probability of winning a car if he can decide to switch or not after he picks the door?