Homework 5 (due Friday, November 22, 2024)

1. (`2 points`) By modifying the program `SimToilet(n,k)`, we can obtain the whole distribution for the rank of the chosen toilet. In particular, let R be a random variable representing the "rank" (in terms of the score) of the toilet we end up choosing. For example, $R = 1$ means the best toilet is chosen, $R = 2$ means the second best toilet is chosen, and $R = n$ means the worst toilet is chosen. Output the complete $P(R = r)$, the probability that we ended up choosing the toilet of rank r (by setting the learning phase of length k), for $r = 1, \ldots, n$ as a vector of length n. Make sure they sum to 1. Name the program `ToiletDistribution(n,k)`.

Program
Name of function: `ToiletDistribution(n,k)`
Input: Total number of toilets, n;
Input: The size of learning phase, k
Output: The probability distribution of the toilet rank
Example: Input: `ToiletDistribution(100,37);`
Example: Output: `[0.373, 0.143, 0.077, ..., 0.003, 0.002]`

2. (`2 points`) Use command `ToiletDistribution(n,k)` to find the expected number for the rank of the chosen toilet (mean rank). What is the optimal k if we use the expected number as a criterion to choose the optimal learning size? Discuss.

3. (`2 points`) Recall that the Stirling numbers of the second kind $S(n, k)$ are given by: (i) for every positive integer n, we have $S(n, 1) = S(n, n) = 1$, and (ii) for every pair of positive integers n and k with $1 < k < n$, we have

$$S(n, k) = S(n - 1, k - 1) + k \cdot S(n - 1, k). \qquad (\heartsuit)$$

In class, we considered the operator $\mathcal{T}$ given by $\mathcal{T}(Q(x)) := x \cdot \frac{\mathrm{d}}{\mathrm{d}x} Q(x)$, and we stated the proposition which says that for every positive integer k, we have

$$\mathcal{T}^k(Q(x)) = \sum_{i=1}^{k} S(k, i) x^i \cdot \frac{\mathrm{d}^i}{\mathrm{d}x^i} Q(x).$$

(a) Prove this proposition.

Suggested Approach: You might do a mathematical induction on k in the proposition, using the recurrence $(\heartsuit)$.

(b) Use the proposition to compute

$$F(x) := \mathcal{T}^7(\sin(x) + \cos(x)),$$

and evaluate

$$F\left(\frac{\pi}{2}\right).$$

4. (`1 point`) Let $\mathfrak{w} \sim \mathrm{Unif}(S_{20})$ be a uniformly random permutation in S_{20}. Let $X := \mathrm{des}(\mathfrak{w})$ be the number of descents in $\mathfrak{w}$. Evaluate $\mathbb{E}[(X+2)^5]$.

5. (`2 points`) Consider the following (potentially biased) version of Gambler's Ruin. The gambler starts with 100 gold coins. In each step, there is a probability p of getting 1 more gold coin, and a probability $1-p$ of losing 1 gold coin. The gambler stops when the number of gold coins reaches either 80 (where the gambler "loses") or 120 (where the gambler "wins").

(a) If $p = 50\%$, what's the probability that the gambler wins?

(b) If $p = 40\%$, what's the probability that the gambler wins?

6. (`extra credit +1 point`) Find an algebraic proof for the "toilet identity":

$$\frac{(n-k)!}{(n-1)!} \sum_{a=k}^{n-1} \frac{(a-1)!}{(a-k)!} \cdot \frac{1}{n-a} = \sum_{a=k}^{n-1} \frac{1}{a},$$

for positive integers k and n such that $n - 1 \geq k$.