Homework 7 (due Saturday, December 14, 2024)

1. `(2 points)` **Random Walk on $\mathbb{Z}/n\mathbb{Z}$.**

   In Lecture 17, we discussed the result of a random walk (1/2 chance to go left and 1/2 chance to go right) on a clock dial.

   > *Let $n \geq 2$ be a positive integer. For the random walk starting at $0 \in \mathbb{Z}/n\mathbb{Z}$, the chance that number k, $1 \leq k \leq n-1$, is the last number to be visited is $\dfrac{1}{n-1}$.*

   Here we investigate the "*second-to-last number*" version of this problem.

   (a) Write a program that takes $n, n \geq 3$, as input and return k where k, $1 \leq k \leq n-1$, is the *second-to-last number* to be visited on a random walk that starts at $0 \in \mathbb{Z}/n\mathbb{Z}$.

   (b) Make a (correct) conjecture about the probability that k, $1 \leq k \leq n-1$ is the *second-to-last* number to be visited.

   (c) (`extra credit +2 points`) Prove your conjecture!

2. `(1 point)` Let $n \geq 2$ be a positive integer. Let $\mathfrak{w} \sim \mathrm{Unif}(S_n)$ be a uniformly random permutation in S_n. We denote the X-ray of $\mathfrak{w}$ by $\mathfrak{X}(\mathfrak{w})$. Evaluate the probability

$$\mathbb{P}\{(\mathfrak{X}(\mathfrak{w}))_n = n-1\}\,.$$

3. `(2 points)` Consider the following variant of the coupon collector problem. Suppose that each cereal box is independent of other boxes. Each box contains one of the seven different types of toy, where the probability that the i^{th} type is contained in a box is $i/28$, for $i = 1, 2, \ldots, 7$.

   Let X be the number of cereal boxes one has to buy until all seven types of toy are collected. Write a program to guess the answer to the following question: "What is the integer closest to the expected value of X?"

4. Suppose that there are exactly 693 students in a certain classroom. Each of them independently samples a value from the Expo(1) distribution.

   a) `(1 point)` What is the probability that at least one student in the classroom obtains a value smaller than 0.001?

      Round your answer to the closest integer percentage. ($x\,\%$, where x is an integer.)

   b) `(extra credit + 1 point)` What is the probability that *every student* in the classroom obtains a value smaller than 0.001?

      For part b), you can simply give an approximation.

      Just for fun (optional), consider the table on this Wikipedia page:

link to the table.

Suppose that this class meets every day including on the weekends. How often on average does this occur?