Quiz 2 Practice

The following are some practice problems for the second quiz. You can use computer program to help with calculation but no access to the internet.

Rule: Quiz 2 is on Monday November 18 from 4-5pm (the regular lecture starts from 5.10-5.50pm). The cheat sheet of one A4 double-sided is allowed (just like in quiz 1).

Format: 10% of total grade.
One programming problem 5%, Two non-programming problems (like in quiz 1) 5%

Sample programming problems

1. (Simulation) Define a bias coin to have a probability of getting head =0.75 and probability of getting tail =0.25. Then let a random variable X to be the number of heads after tossing this bias coin twice.

   a) Generate 10,000 random values of X. Then, use the data to find the mean (expected value), standard deviation. Also plot the distribution of X.

   b) Let the sample mean $\bar{X} = \dfrac{X_1 + X_2 + ... + X_{25}}{25}$. Generate 10,000 random values of $\bar{X}$. Then, use the data to find the mean (expected value), standard deviation. Also plot the distribution of $\bar{X}$.

2. (Benford's law) Investigate the distribution of the leading digits from the 200-by-200 multiplication table. Compare the distribution with the Benford's law.

   For example, 4-by-4 multiplication table:

m/n	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16

   The list of leading digits is $[1, 2, 3, 4, 2, 4, 6, 8, 3, 6, 9, 1, 4, 8, 1, 1]$.

3. (Alarm clock) Recall that you can generate $X = Exp(\lambda)$, the exponential random variable of mean λ by
$$x = -\frac{1}{\lambda}\log(1-U),$$
   where U is a uniform $(0, 1)$.

   Let $X_1 \sim Exp(1), X_2 \sim Exp(2), X_3 \sim Exp(3)$.

a) Simulate 10,000 values of X_1. Plot the distribution and compare it with the probability density function of $Exp(1)$ i.e. $f(x) = e^{-x}, x \geq 0$.

b) Use the simulation to approximate the probability $\mathbb{P}(X_1 = \min\{X_1, X_2, X_3\})$.

c) Use the simulation to approximate the probability $\mathbb{P}(X_1 = \max\{X_1, X_2, X_3\})$.

Sample non-programming problems

1. (Exponential distribution) Show the memoryless property of exponential function: Given $X \sim Exp(\lambda)$,
$$\mathbb{P}\{X \geq a \,|\, X \geq b\} = \mathbb{P}\{X \geq a - b\}.$$

2. Let X_1 and X_2 be independent and identically distributed Unif$\{3, 6, 9\}$. Define the random variable
$$Y := \begin{cases} 5 & \text{if } X_2 \leq 7, \\ 8 & \text{if } X_2 > 7. \end{cases}$$
Compute the following.
   (a) $Cov(X_1, X_2)$ and $corr(X_1, X_2)$.
   (b) $Cov(X_1, Y)$ and $corr(X_1, Y)$.
   (c) $Cov(X_2, Y)$ and $corr(X_2, Y)$.

3. (Descent) We define the indicator variable of $w \in S_n$ at position i, where $i = 1, 2, ..., n-1$ as
$$I_i = \begin{cases} 1, & \text{if } w_i > w_{i+1} \\ 0, & \text{otherwise.} \end{cases}$$
Show that $Cov(I_i, I_j) = \begin{cases} 0, & \text{if } |i-j| \geq 2 \\ \dfrac{1}{4}, & \text{if } i = j \\ -\dfrac{1}{12}, & \text{if } |i-j| = 1. \end{cases}$

4. (Toilet problem) Calculate the probability of getting the best toilet when there are 4 toilets and the learning phase is of length 1. Show your work.