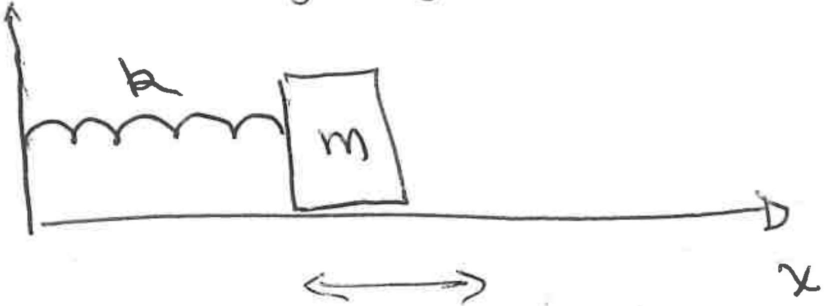
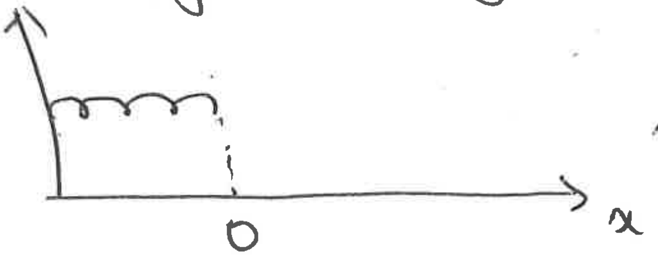
Lecture 1 : The harmonic oscillator

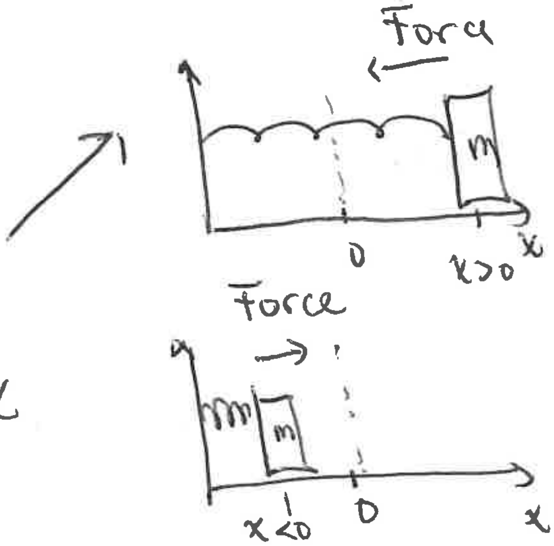
Mass-spring system



Motion $x(t)$. Newton's 2nd Law $ma = F$

Hooke's Law for springs





$x = 0$ Equilibrium, Force $\propto x$

definition : $a(t)$ acceleration

$$a = \frac{d^2x}{dt^2} \quad ; \quad v(t) = \frac{dx}{dt} \quad \text{velocity}$$

$$\Rightarrow \boxed{m\frac{d^2x}{dt^2} = -kx}$$

Restoring force

$x > 0 \Rightarrow -kx < 0$ $\xleftarrow{\text{Force}}$

$x < 0 \Rightarrow -kx > 0$ $\xrightarrow{\text{Force}}$

MATH : Equation in the box represents a 2nd order ODE or a system of first order ODE's

$$\frac{dx}{dt} = v \qquad \frac{dv}{dt} = -\frac{k}{m}x$$

$$\frac{d^2x}{dt^2} + \frac{k}{m} x = 0$$

2^{nd} order linear ordinary differential equations

General solution

$x(t) = A_1 \cos(\Omega t) + A_2 \sin(\Omega t)$ $\Omega = \sqrt{\frac{k}{m}}$

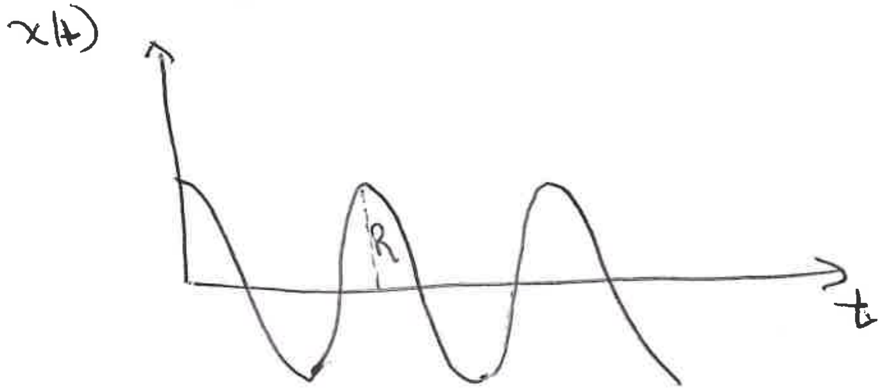
frequency

units

$w = \frac{1}{time}$ (Hertz),

Oscillatory behavior $x(t) = x(t + \frac{2\pi}{\Omega})$ $T = \frac{2\pi}{\Omega}$ (seconds)

period



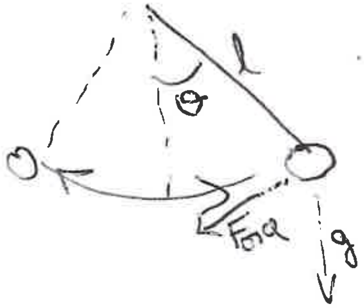
Alternative solution form $x(t) = R\cos(\Omega t + \phi)$

amplitude

phase

Similar model

θ(t) angle wrt vertical axis



$$l m\frac{d^2\theta}{dt^2} = -mg \sin\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l} \sin\theta$$

θ in radians

$\frac{d\theta}{dt}$ $\frac{rad}{sec}$ angular velocity

Taylor series $\sin(\theta) = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} + \cdots$

Linear approximation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0 \qquad \theta(t) = \alpha\cos(\Omega t) + \beta\sin(\Omega t)$$

$$\Omega = \sqrt{\frac{g}{l}}, \quad T = 2\pi\sqrt{\frac{l}{g}}$$

weakly nonlinear oscillator

$$\frac{d^2\theta}{dt^2} + \Omega_L^2\theta = \epsilon\theta^3 \qquad 0 < \epsilon \ll 1 \qquad \text{Ex } \epsilon = 0.01$$

A natural simplification is to ignore nonliner Term $\epsilon\theta^{③}$

$$\theta_0(t) \approx \alpha\cos(\Omega_L t) + \beta\sin(\Omega_L t) \qquad \Omega_L \to \Omega_{NL}$$

Assume $\beta = 0$ Error $\epsilon\theta_0^3 = \frac{\epsilon\alpha^3}{4}\cos(3\Omega t) + \frac{3\epsilon}{4}\alpha^3\cos(\Omega t)$

Correction

$$\theta_{NL}(t) \approx \alpha\cos(\Omega_{Lt}t) + \overbrace{D\cos(3\Omega_{NL}t)}^{\text{Generation of 3rd harmonic}}$$

$$\Omega_{NL}^2 = \Omega_L^2 - \beta\frac{\alpha^2}{4} \rightarrow \text{frequency Shif}$$

$$D = \frac{3\beta\alpha^3}{4(\Omega_L^2 - 9\Omega_{NL}^2)}$$

Forced pendulum

$$\frac{d^2x}{dt^2} + \Omega^2 x = C\sin(\omega t) \qquad C, \omega \text{ known (given) constants}$$

Particular solution $x(t) = r\sin(\omega t)$ $\quad \frac{d^2x}{dt^2} = -r\omega^2\sin(\omega t)$

"plug in" $\quad r(\Omega^2 - \omega^2)\sin\omega t = C\sin(\omega t)$

$$r = \frac{C}{\Omega^2 - \omega^2}$$

General solution
$$x(t) = \underbrace{A\cos(\Omega t) + B\sin(\Omega t)}_{\substack{\text{Homog.} \\ \text{(natural mode)}}} + \underbrace{\frac{C}{\Omega^2-\omega^2}\sin(\omega t)}_{\substack{\text{response} \\ \text{to forcing} \\ \text{(Part. sol)}}}$$

O.K. as long as $\omega \neq \Omega$

What happens at $\omega = \Omega$: <u>RESONANCE</u>

$$\frac{d^2x}{dt^2} + x = 0.1\,x^3 \qquad x(0)=2\,, \quad \frac{dx}{dt}(0)=0$$

Linear model
$\Omega_L = 1$

$$x(t) = x_p(t) + x_2 \qquad |x_1| << |x_0|$$

$$x_p(t) = A\cos(\Omega_{NL}t)$$

Plug in

$$-\Omega_{NL}^2 A\cos(\Omega_{NL}t) + A\cos(\Omega_{NL}t) = 0.1\left(A\cos(\Omega_{NL}t)\right)^3$$
$$+\frac{d^2x_2}{dt^2} + x_2 \qquad = 0.1\,A^3\cos^3(\Omega_{NL}t)$$

Trigonometric identity ⟶

$$\cos^3(\Omega_{NL}t) = \cos(\Omega_{NL}t)\cos^2(\Omega_{NL}t) = \frac{1}{2}\cos(\Omega_{NL}t)\left[1+\cos(2\Omega_{NL}t)\right]$$

$$= \frac{1}{2}\cos(\Omega_{NL}t) + \frac{1}{2}\cos(\Omega_{NL}t)\cos(2\Omega_{NL}t) = \frac{1}{2}\cos\Omega_{NL}t$$

$$+\frac{1}{4}\Big[\cos(\underbrace{(2\Omega_{NL}+\Omega_{NL})}_{\text{sum}}t) + \cos(\underbrace{(2\Omega_{NL}-\Omega_{NL})}_{\text{difference}}t)\Big]$$

$$= \frac{3}{4}\cos(\Omega_{NL}t) + \frac{1}{4}\cos(\Omega_{NL}t)$$

$$(1-\Omega_{NL}^2)A\cos(\Omega_{NL}t) = 0.1\,A^3\left[\frac{3}{4}\cos(\Omega_{NL}t) + \boxed{\frac{1}{4}\cos(3\Omega_{NL}t)}\right]$$
$$\boxed{+\frac{d^2x_2}{dt^2} + x_2}$$

$$(1-\Omega_{NL}^2)A = \frac{0.3}{4}A^3 \qquad \boxed{\Omega_{NL}^2 = 1 - 0.075\,A^2}$$

Frequency shift of natural frequency
$\Omega_L = 1$

Balaning remaining term

$$\frac{d^2 x_1}{dt^2} + x_1 = 0.025 A^3 \cos(3\Omega_{NL} t)$$

$$x_1(t) = C \cos(3\Omega_{NL} t)$$

$$(1 - 9\Omega_{NL}^2)\, C = 0.025 A^3$$

$$\boxed{C = -\frac{0.025}{8 + .675 A^2} A^3}$$

SUMMARY

Imploved approximation

$$x(t) \approx x_1(t) + x_2(t)$$

$$= A \cos(\Omega_{NL} t) + C \cos(\underset{\uparrow\ 3^{rd}\ \text{harmonic}}{3\Omega_{NL} t})$$

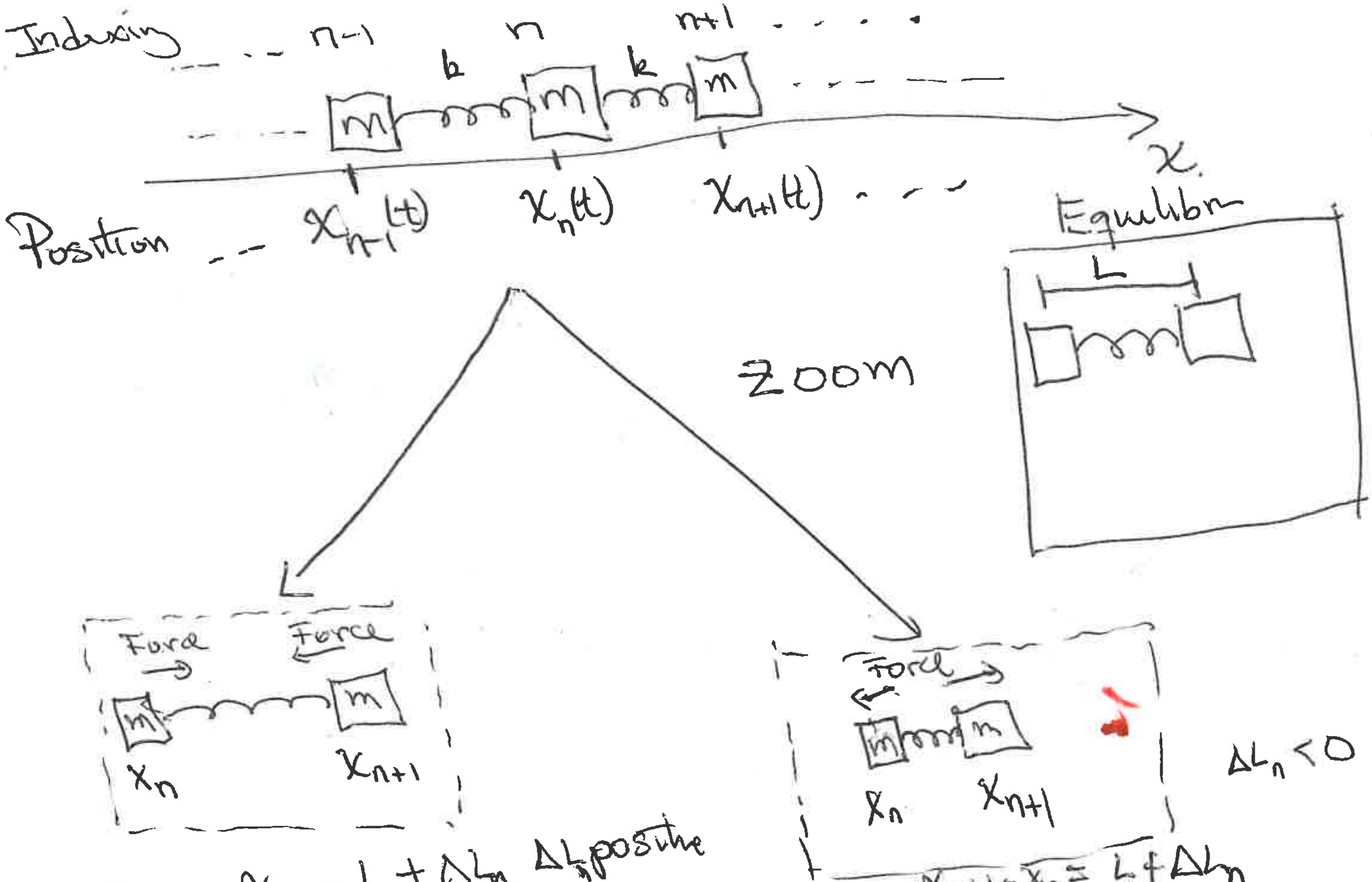
$$\Omega_{NL}^2 = 1 - 0.075 A^2 \quad \text{freq shift}$$

$$C = -\frac{0.025 A^3}{8 + 0.675 A^2} \quad \text{small}$$

If we keep improving the solution

$$x(t) \approx x_1(t) + x_2(t) + \text{higher } (5^{th}, 7^{th}, 9^{th}, \ldots) \text{ harmonics}$$

Systems of coupled oscillators



$x_{n+1} - x_n = L + \Delta L_n$ ΔL_n positive

Hooke's Law Force $\propto$ Relative displacement

$$= k(x_{n+1} - x_n) - kL$$

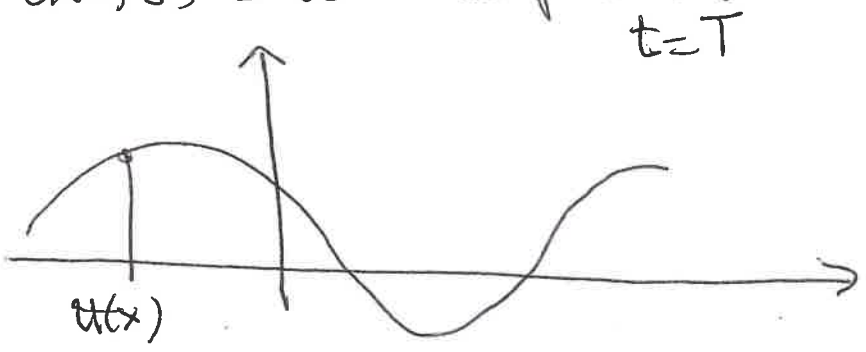
Newton's 2nd Law for nth mass

$$m\frac{d^2x_n}{dt^2} = k(x_{n+1} - x_n) - kL - \left[k(x_n - x_{n-1}) - kL\right]$$

$$= k\left[x_{n+1} - 2x_n + x_{n-1}\right]$$

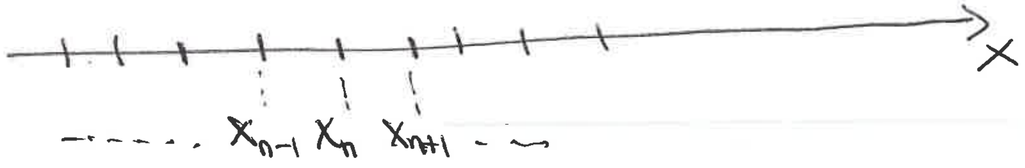
FROM A CONTINUUM to a disrete model

$u(x,t)$ = wave amplitude



Wave equation $$\underbrace{\frac{\partial^2 u}{\partial t^2}}_{\text{acceleration}} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

1. Discretize in space



$u(x_n,t) \equiv u_n(t)$ $\qquad x_{n+1}-x_n=h$

11. Use Taylor series

$$u(x_n+h,t) = u_{n+1} \approx u(x_n,t) + \frac{\partial u(x_n,t)}{\partial x} h + \frac{1}{2}\frac{\partial^2 u(x_n)}{\partial x^2} h^2$$

$$u(x_n-h,t) = u_{n-1} \approx u(x_n,t) - \frac{\partial u(x_n,t)}{\partial x} h + \frac{1}{2}\frac{\partial^2 u(x_n)}{\partial x^2} h^2$$

$$\Rightarrow u_{n+1}+u_{n-1} = 2u_n + h^2\frac{\partial^2 u_n}{\partial x^2} \quad \text{or} \quad \frac{\partial^2 u_n}{\partial x^2} = \frac{u_{n+1}-2u_n+u_{n-1}}{h^2}$$

then at each site x_n

$$\frac{\partial^2 u(x_n,t)}{\partial t^2} - c^2 \frac{\partial^2 u(x_n,t)}{\partial t^2} = 0$$

$$\downarrow$$

$$\frac{d^2 u_n}{dt^2} - \frac{c^2}{h^2}\left(u_{n+1} - 2u_n + u_{n-1}\right) = 0$$

or $$\frac{d^2 u_n}{dt^2} = \frac{c^2}{h^2}\left[\left(u_{n+1} - u_n\right) - \left(u_n - u_{n-1}\right)\right]$$

From ODEs To Map
(Buildup for numerical simulations)

$$M \frac{d^2 x_n(t)}{dt^2} = \frac{k}{h^2}\left(x_{n+1} - 2x_n + x_{n+1}\right)$$

Approximate the <u>time</u> evolution for discrete Times $t_0, t_1, \ldots t_n \ldots$ $t_{n+1} - t_n = \delta$

Taylor series (same Math! just replace $x \to t$)

$$\cancel{d^2} \frac{d^2 x_n(t_m)}{dt^2} \approx \frac{x_n(t_{m+1}) - 2x_n(t_m) + x_n(t_{m-1})}{\delta^2}$$

Notation $x_n(t_m) = x_n^{(m)}$

$$M\left(x_n^{(m+1)} - 2x_n^{(m)} + x_n^{(m-1)}\right) = k\frac{\delta^2}{h^2}\left[x_{n+1}^{(m)} - 2x_n^{(m)} + x_{n-1}^{(m)}\right]$$

or

$$x_n^{(m+1)} = \frac{\bar{k}}{M}\left[x_{n+1}^{(m)} - 2\left(1 - \frac{M}{\bar{k}}\right)x_n^{(m)} + x_{n-1}^{(m)}\right]$$

$$\bar{k} = k\frac{\delta^2}{h^2}$$