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A SIMPLE METHOD FOR CONSTRUCTING ORTHOGONAL ARRAYS BY THE KRONECKER SUM*

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Abstract In this article, we propose a new general approach to constructing asymmetrical orthogonal arrays, namely the Kronecker sum. It is interesting since a lot of new mixed-level orthogonal arrays can be obtained by this method.

Key words Difference matrices, Kronecker sum, mixed-level orthogonal arrays, permutation matrices, projection matrices.

1 Introduction

An $n \times m$ matrix A, having k_i columns with p_i levels, $i = 1, 2, \cdots, t$, t is an integer, $m = \sum_{i=1}^{t} k_i$, $p_i \neq p_j$, for $i \neq j$, is called an orthogonal array (OA) of strength d and size n if each $n \times d$ submatrix of A contains all possible $1 \times d$ row vectors with the same frequency. Unless stated otherwise, we consider an orthogonal array of strength 2, using the notation $L_n(p_1^{k_1} p_2^{k_1} \cdots p_t^{k_t})$ for such an array. An orthogonal array is said to have mixed level (or asymmetrical) if $t \geq 2$.

An essential concept for the construction of asymmetrical orthogonal arrays is that of difference matrices. Bose and Bush[1] were the first to use it with this objective. Using the notation for additive (or Abelian) groups, a difference matrix (or difference scheme) with level p is an $\lambda p \times m$ matrix with the entries from a finite additive group G of order p such that the vector differences of any two columns of the array, say $d_i - d_j$ if $i \neq j$, contains every element of G exactly λ times. We denote such an array by $D(\lambda p, m; p)$, although this notation suppresses the relevance of group G. In most of our examples G corresponds to the additive group associated with a Galois field $GF(p)$. The difference matrix $D(\lambda p, m; p)$ is called a generalized Hadamard matrix if $\lambda p = m$. In particular, $D(\lambda 2, \lambda 2; 2)$ is the usual Hadamard matrix.

If a $D(\lambda p, m; p)$ exists, it can always be constructed so that one of its rows and one of its columns have only the zero element of G. Deleting this column from $D(\lambda p, m; p)$, we obtain a

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difference matrix, denoted by $D^0(\lambda p, m-1; p)$, called an atom of difference matrix $D(\lambda p, m; p)$ or an atomic difference matrix. Without loss of generality, the $D(\lambda p, m; p)$ can be written as

$$D(\lambda p, m; p) = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} = (0 \quad D^0(\lambda p, m-1; p)).$$

The property is important for the following discussions.

For two matrices $A = (a_{ij})_{n\times m}$ and $B = (b_{ij})_{s\times t}$ both with the entries from group G, define their Kronecker sum[2] to be

$$A \oplus B = (a_{ij} \oplus B)_{1\le i\le n, 1\le j\le m},$$

where each submatrix $a_{ij} \oplus B$ of $A \oplus B$ stands for the matrix obtained by adding a_{ij} to each entry of B. Shrikhande[2] showed that $A\oplus B$ is a difference matrix if both A and B are difference matrices. And Zhang[3] showed that A is a difference matrix if both $A \oplus B$ and B are difference matrices.

Today orthogonal arrays have many important applications in statistics, and they play important roles in coding theory and cryptography. The current emphasis is in the area of asymmetrical factorial design, or asymmetrical orthogonal arrays. Many new construction methods on the asymmetrical orthogonal arrays have been proposed, but most of these methods are only or mainly from the construction of symmetrical orthogonal arrays. Refer to the book [4] by Hedayat et al. for further references.

A new theory or procedure of constructing asymmetrical orthogonal arrays by using the orthogonal decompositions of projection matrices has been given by Zhang, Lu and Pang[5], Zhang, Pang, and Wang[6], and Zhang, Duan, Lu and Zheng[7]. The first who used it with this objective was Zhang[8−13]. The idea comes from the theory of multilateral matrices – a mathematical technique to solve the problems of system with complexity. In general, the procedure of constructing asymmetrical orthogonal arrays in our theory has been partitioned mainly into five parties: orthogonal-array addition, subtraction, multiplication, division and replacement. The technique, namely generalized Kronecker product, has also been proposed for the construction of asymmetrical orthogonal arrays by Zhang[3] in the theory of multilateral matrices. In this article the technique will be further explained and extended to construct some new asymmetrical (or mixed-level) orthogonal arrays by using the orthogonal decompositions of projection matrices.

Section 2 contains the basic concepts and main theorems. Some new mixed level OAs are constructed in Section 3.

2 Basic Concepts and Main Theorems

In our procedure, an important idea is to find the relationship among difference matrices, projection matrices, and permutation matrices. The following notations are used.

Let 1_r be the $r\times 1$ vector of 1's, 0_r the $r\times 1$ vector of 0's, I_r the identity matrix of order r and $J_{r,s}$ the $r \times s$ matrix of 1's, also $J_r = J_{r,r}$. Of course, the two matrices $P_r = (\frac{1}{r})1_r 1_r^{\mathrm{T}} = (\frac{1}{r})J_r$ and $\tau_r = I_r - P_r$ are projection matrices.

Define

$$(r) = (0, \cdots, r-1)^{\mathrm{T}}_{1\times r}, \qquad e_i(r) = (0\cdots 0 \overset{i}{1} 0 \cdots 0)^{\mathrm{T}}_{1\times r},$$

where $e_i(r)$ is the base vector of R^r (r-dim vector space) for any i. We can construct two permutation matrices as follows:

$$N_r = e_1(r)e_2^{\mathrm{T}}(r) + \cdots + e_{r-1}(r)e_r^{\mathrm{T}}(r) + e_r(r)e_1^{\mathrm{T}}(r)$$

and

$$K(p,q)=\sum_{i=1}^{p}\sum_{j=1}^{q}e_i(p)e_j^{\mathrm{T}}(q)\otimes e_j(q)e_i^{\mathrm{T}}(p), \tag{1}$$

where $\otimes$ is the usual Kronecker product in the theory of matrices. The permutation matrices N_r and $K(p,q)$ have the following properties:

$$N_r(r)=1\oplus(r), \qquad \bmod r, \qquad \text{and } K(p,\lambda p)((\lambda p)\oplus(p))=(p)\oplus(\lambda p).$$

Let $D=(d_{ij})_{\lambda p\times m}$ be a matrix over an additive group G of order p. Then for any given $d_{ij}\in G$ there exists a permutation matrix $\sigma(d_{ij})$ such that

$$\sigma(d_{ij})(p)=d_{ij}\oplus(p),$$

where the vector (p) is with elements from G. Define $H(\lambda p,m;p)=(\sigma(d_{ij}))_{\lambda p^2\times mp}$, where each entry or submatrix $\sigma(d_{ij})$ of $H(\lambda p,m;p)$ is a $p\times p$ permutation matrix. And Zhang[3] has proved that the matrix $D=(d_{ij})_{\lambda p\times m}$ over some group G is a difference matrix $D(\lambda p,m;p)$ if and only if

$$H^{\mathrm{T}}(\lambda p,m;p)H(\lambda p,m;p)=\lambda p(I_m\otimes\tau_p+J_m\otimes P_p).$$

On the other hand, the permutation matrices $\sigma(d_{ij})$ are often obtained by the permutation matrices N_r and $K(p,q)$. Furthermore, by the permutation matrices $\sigma(d_{ij})$ and $K(\lambda p,p)$, the Kronecker sum[2] of difference matrices can be written as

$$\begin{aligned}(p)\oplus D(\lambda p,m;p)&=K(p,\lambda p)D(\lambda p,m;p)\oplus(p)\\&=K(p,\lambda p)(\sigma(d_{ij})(p))_{\lambda p^2\times m}\\&=K(p,\lambda p)(S_1(0_{\lambda p}\oplus(p)),\cdots,S_m(0_{\lambda p}\oplus(p)))\\&=[Q_1((p)\oplus 0_{\lambda p}),\cdots,Q_m((p)\oplus 0_{\lambda p})],\end{aligned}$$

where

$$Q_j=K(p,\lambda p)S_jK(p,\lambda p)^{\mathrm{T}}, \qquad S_j=\ \mathrm{diag}(\sigma(d_{1j}),\cdots,\sigma(d_{rj})), \qquad (r=\lambda p), \tag{2}$$

are permutation matrices for any $j=1,2,\cdots,m$ and where $0_{\lambda p}\oplus(p)=1_{\lambda p}\otimes(p)$ holds for the additive group associated with Galois Field GF(p). Therefore, both the projection matrices P_r and τ_r and the permutation matrices $N_r, K(p,q), Q_j$ and S_j (defined in (1) and (2)) are often used to construct the asymmetrical orthogonal arrays in our procedure.

Definition 1 Let A be an orthogonal array of strength 1, i.e.,

$$A=(a_1,a_2,\cdots,a_m)=(T_1(0_{r_1}\oplus(p_1)),T_2(0_{r_2}\oplus(p_2)),\cdots,T_m(0_{r_m}\oplus(p_m))),$$

where $r_ip_i=n$, and T_i is a permutation matrix for any $i=1,2,\cdots,m$. The following projection matrix:

$$A_j=T_j(P_{r_j}\otimes\tau_{p_j})T_j^{\mathrm{T}} \tag{3}$$

is called the matrix image (MI) of the jth column a_j of A, denoted by $m(a_j)=A_j$ for $j=1,2,\cdots,m$. In general, the MI of a subarray of A is defined as the sum of the MIs of all its columns. In particular, we denote the MI of A by $m(A)$.

In Definition 1, for a given column $a_j=T_j(0_{r_j}\oplus(p_j))$, the matrices A_j defined in equation (3) are unique though the permutation matrix T_j introduced here is not unique.

If a design is an orthogonal array, then the MIs of its columns have some interesting properties that can be used to construct orthogonal arrays. For example, by the definition, we have $m(0_r) = P_r$ and $m((r)) = \tau_r$.

Theorem 1 *For any permutation matrix T and any orthogonal array L with strength at least one, we have*

$$m(T(L \oplus 0_r)) = T(m(L) \otimes P_r)T^{\mathrm{T}} \quad \textit{and} \quad m(T(0_r \oplus L)) = T(P_r \otimes m(L))T^{\mathrm{T}}.$$

Theorem 2 *Let the array A be an orthogonal array of strength 1, i.e.,*

$$A = (a_1, a_2, \cdots, a_m) = (T_1(0_{r_1} \oplus (p_1)), T_2(0_{r_2} \oplus (p_2)), \cdots, T_m(0_{r_m} \oplus (p_m))),$$

where $r_i p_i = n$, T_i is a permutation matrix, for $i = 1, 2, \cdots, m$.

The following statements are equivalent:

(1) *A is an orthogonal array of strength 2.*

(2) *The MI of A is a projection matrix.*

(3) *The MIs of any two columns of A are orthogonal, i.e, $m(a_i)m(a_j) = 0$ $(i \neq j)$.*

(4) *The projection matrix τ_n can be decomposed as*

$$\tau_n = m(a_1) + m(a_2) + \cdots + m(a_m) + \triangle,$$

where $rk(\triangle) = n - 1 - \sum_{j=1}^{m}(p_j - 1)$ is the rank of the matrix $\triangle$.

Definition 2 An orthogonal array A is said to be saturated if $\sum_{j=1}^{m}(p_j - 1) = n - 1$ (or, equivalently, $m(A) = \tau_n$).

Corollary 1 *Let (L, H) and K be orthogonal arrays of run size n. Then (K, H) is an orthogonal array if $m(K) \leq m(L)$, where $m(K) \leq m(L)$ means that the difference $m(L) - m(K)$ is nonnegative definite.*

Corollary 2 *Suppose L and H are orthogonal arrays. Then $K = (L, H)$ is also an orthogonal array if $m(L)$ and $m(H)$ are orthogonal, i.e., $m(L)m(H) = 0$. In this case $m(K) = m(L) + m(H)$.*

These theorems and corollaries can be found in [3,12,13].

By Corollaries 1 and 2, to construct an orthogonal array L_n of run size n, we should decompose the projection matrix τ_n into $C_1 + C_2 + \cdots + C_k$ such that $C_i C_j = 0$ for $i \neq j$ and find orthogonal arrays H_j such that $m(H_j) \leq C_j$ for $j = 1, 2, \cdots, k$, because the array $L_n = (H_1, H_2, \cdots, H_k)$ is an orthogonal array of run size n. The method of constructing orthogonal arrays by using the orthogonal decompositions of projection matrices is also called orthogonal-array addition[5].

The following theorem is a main result in our procedure of generalized Kronecker product for constructing the asymmetrical orthogonal arrays.

Theorem 3 *Let $D^0(\lambda p, m-1; p)$ be an atom of difference matrix $D(\lambda p, m; p)$. Then $D^0(\lambda p, m-1; p) \oplus (p)$ is an orthogonal array whose MI (defined in (3)) is less than or equal to $\tau_{\lambda p} \otimes \tau_p$.*

Proof From Theorems 1 and 2 and the construction of [1], we have that

$$\begin{aligned} & m[(0_r \oplus (p), D^0(r, m-1; p) \oplus (p)] \\ &= P_r \otimes \tau_p + m(D^0(r, m-1; p) \oplus (p)) \\ &= P_r \otimes \tau_p + S_1(P_r \otimes \tau_p)S_1^{\mathrm{T}} + \cdots + S_m(P_r \otimes \tau_p)S_m^{\mathrm{T}} \end{aligned}$$

is a projection matrix where the atomic difference matrix $D^0(r, m-1; p) = (d_{ij})_{r \times (m-1)}$, $S_j = \mathrm{diag}(\sigma(d_{1j}), \sigma(d_{2j}), \cdots, \sigma(d_{rj}))$, $r = \lambda p$, (defined in (2)) and the MIs of any two columns of $(0_r \oplus (p), D^0(r, m-1; p) \oplus (p))$ are orthogonal, i.e.,

$$(P_r \otimes \tau_p) S_j (P_r \otimes \tau_p) S_j^{\mathrm{T}} = 0 \quad \text{and} \quad S_i (P_r \otimes \tau_p) S_i^{\mathrm{T}} S_j (P_r \otimes \tau_p) S_j^{\mathrm{T}} = 0, \quad i \neq j.$$

Thus we now only need to prove that

$$S_j (P_r \otimes \tau_p) S_j^{\mathrm{T}} \leq I_r \otimes \tau_p,$$

since $m(D^0(r, m-1; p) \oplus (p)) \leq I_r \otimes \tau_p - P_r \otimes \tau_p = \tau_r \otimes \tau_p$.

In fact, by the matrix properties $P_r \otimes \tau_p \leq I_r \otimes \tau_p$, we have

$$\begin{aligned} S_j (P_r \otimes \tau_p) S_j^{\mathrm{T}} &\leq S_j (I_r \otimes \tau_p) S_j^{\mathrm{T}} \\ &= \mathrm{diag}(\sigma(d_{1j}) \tau_p \sigma(d_{1j})^{\mathrm{T}}, \cdots, \sigma(d_{rj}) \tau_p \sigma(d_{rj})^{\mathrm{T}}) \\ &= \mathrm{diag}(\tau_p, \tau_p, \cdots, \tau_p) \\ &= I_r \otimes \tau_p. \end{aligned}$$

This completes the proof. ∎

Theorem 4 (Three-factor method) *Let $D^0(q, m-1; p)$ be an atom of difference matrix $D(q, m; p)$. Yet let $n = prq$ and L_{rq} be orthogonal arrays of run sizes rq. If there exist orthogonal arrays $L_{pr} = L_{pr}(r^1 p^{m_1}) = [0_p \oplus (r), L_{pr}^{(-)}(p^{m_1})]$ and $L_{pr} = L_{pr}(r^1 \cdots) = [0_p \oplus (r), L_{pr}^{(-)}]$, then*

$$L_{prq} = [L_{pr}^{(-)} \oplus 0_q, 0_p \oplus L_{rq}, L_{pr}^{(-)}(p^{m_1}) \oplus D^0(q, m-1; p)]$$

is an orthogonal array.

Proof The proof follows from Theorem 3 and the orthogonal decomposition of τ_{prq}:

$$\tau_{prq} = [\tau_p \otimes I_r] \otimes P_q + P_p \otimes \tau_{rq} + [\tau_p \otimes I_r] \otimes \tau_q.$$

In fact, let $L_{pr}^{(-)}(p^{m_1}) = [S_1(0_\lambda \oplus (p)), \cdots, S_{m_1}(0_\lambda \oplus (p))]$, then

$$m(L_{pr}^{(-)}(p^{m_1})) = m(L_{pr}) - P_p \otimes \tau_r \leq \tau_p \otimes I_r.$$

By Theorem 3, we have

$$\begin{aligned} & m(L_{pr}^{(-)}(p^{m_1}) \oplus D^0(q, m-1; p)) \\ = & \sum_{j=1}^{m_1} [S_j \otimes I_q][P_\lambda \otimes m((p) \oplus D^0(q, m-1; p))][S_j \otimes I_q]^{\mathrm{T}} \\ \leq & \sum_{j=1}^{m_1} [S_j \otimes I_q][P_\lambda \otimes \tau_p \otimes \tau_q][S_j \otimes I_q]^{\mathrm{T}} \\ = & \sum_{j=1}^{m_1} [S_j (P_\lambda \otimes \tau_p) S_j^{\mathrm{T}}] \otimes \tau_q \\ \leq & \tau_p \otimes I_r \otimes \tau_q. \end{aligned}$$

On the other hand, by Theorem 1, we have

$$m(L_{pr}^{(-)} \oplus 0_q) = m(L_{pr}^{(-)}) \otimes P_q \leq \tau_p \otimes I_r \otimes P_q,$$

and

$$m(0_p \oplus L_{rq}) = P_p \otimes m(L_{rq}) \leq P_p \otimes \tau_{rq}.$$

By orthogonal array addition, we obtain an orthogonal array

$$\left[L_{pr}^{(-)} \oplus 0_q, 0_p \oplus L_{rq}, L_{pr}^{(-)}(p^{m_1}) \oplus D^0(q, m-1; p)\right]. \qquad \blacksquare$$

3 Constructions of OAs with Run Size 72

3.1 Construction of OA $L_{72}(\cdots 4^1)$

Since $72 = 18 \times 2 \times 2$, by the three-factor method, it follows that

$$\left[L_{36}^{(-)} \oplus 0_2, 0_{18} \oplus (4), L_{36}^{(-)}(2^{34}) \oplus (2)\right]$$

is an orthogonal array for any orthogonal arrays $L_{36}^{(-)}$ and $L_{36}^{(-)}(2^{34})$ if $L_{36} = [0_{18} \oplus (2), L_{36}^{(-)}]$ and $L_{36}(2^{35}) = [0_{18} \oplus (2), L_{36}^{(-)}(2^{34})]$ are orthogonal arrays.

The orthogonal array $L_{36}(2^{35}) = [0_{18} \oplus (2), L_{36}^{(-)}(2^{34})]$ exists in Table 2.

There are at least 19 new orthogonal arrays of run size 72 for this family which are included in Table 1.

Table 1 Orthogonal arrays $L_{72}(\cdots)$ obtained in Sections 3.1, 3.2 and 3.3

No.	Krown OAs $L_{36}(\cdots)$	Obtained OAs $L_{72}(\cdots 4^1)$	Obtained OAs $L_{72}(\cdots 6^x)$	Obtained OAs $L_{72}(\cdots 12^1)$
1	$L_{36}(2^{35})$	$L_{72}(2^{68}4^1)$	–	–
2	$L_{36}(2^{28}3^1)$	$L_{72}(2^{61}3^14^1)$(new)	$L_{72}(2^{56}6^1)$ (new)	–
3	$L_{36}(2^{20}3^2)$	$L_{72}(2^{53}3^24^1)$(new)	$L_{72}(2^{48}3^16^1)$	–
4	$L_{36}(2^{18}3^16^1)$	$L_{72}(2^{51}3^16^14^1)$(new)	$L_{72}(2^{46}6^2)$	$L_{72}(2^{36}3^112^1)$
5	$L_{36}(2^{16}3^4)$	$L_{72}(2^{49}3^44^1)$(new)	$L_{72}(2^{44}3^36^1)$	–
6	$L_{36}(2^{16}9^1)$	$L_{72}(2^{49}9^14^1)$(new)	–	–
7	$L_{36}(2^{13}6^2)$	$L_{72}(2^{46}6^24^1)$(new)	–	$L_{72}(2^{31}6^112^1)$ (new)
8	$L_{36}(2^{11}3^{12})$	$L_{72}(2^{44}3^{12}4^1)$	$L_{72}(2^{39}3^{11}6^1)$	–
9	$L_{36}(2^{11}3^26^1)$	$L_{72}(2^{44}3^26^14^1)$(new)	$L_{72}(2^{39}3^16^2)$	$L_{72}(2^{29}3^212^1)$
10	$L_{36}(2^{10}3^86^1)$	$L_{72}(2^{43}3^86^14^1)$(new)	$L_{72}(2^{38}3^76^2)$ (new)	$L_{72}(2^{28}3^812^1)$
11	$L_{36}(2^{10}3^16^2)$	$L_{72}(2^{43}3^16^24^1)$(new)	$L_{72}(2^{38}6^3)$ (new)	$L_{72}(2^{28}3^16^112^1)$ (new)
12	$L_{36}(2^93^46^2)$	$L_{72}(2^{42}3^46^24^1)$(new)	$L_{72}(2^{37}3^36^3)$	$L_{72}(2^{27}3^46^112^1)$ (new)
13	$L_{36}(2^86^3)$	$L_{72}(2^{41}6^34^1)$(new)	–	$L_{72}(2^{26}6^212^1)$ (new)
14	$L_{36}(2^43^{13})$	$L_{72}(2^{37}3^{13}4^1)$	$L_{72}(2^{32}3^{12}6^1)$	–
15	$L_{36}(2^43^16^3)$	$L_{72}(2^{37}3^16^34^1)$(new)	$L_{72}(2^{32}6^4)$ (new)	$L_{72}(2^{22}3^16^212^1)$ (new)
16	$L_{36}(2^33^96^1)$	$L_{72}(2^{36}3^96^14^1)$(new)	$L_{72}(2^{31}3^86^2)$	$L_{72}(2^{21}3^912^1)$
17	$L_{36}(2^33^26^3)$	$L_{72}(2^{36}3^26^34^1)$(new)	$L_{72}(2^{31}3^16^4)$ (new)	$L_{72}(2^{21}3^26^212^1)$ (new)
18	$L_{36}(2^23^{12}6^1)$	$L_{72}(2^{35}3^{12}6^14^1)$(new)	$L_{72}(2^{30}3^{11}6^2)$(new)	$L_{72}(2^{20}3^{12}12^1)$
19	$L_{36}(2^23^56^2)$	$L_{72}(2^{35}3^56^24^1)$(new)	$L_{72}(2^{30}3^46^3)$(new)	$L_{72}(2^{20}3^56^112^1)$ (new)
20	$L_{36}(2^218^1)$	$L_{72}(2^{35}18^14^1)$(new)	–	–
21	$L_{36}(2^13^86^2)$	$L_{72}(2^{34}3^86^24^1)$(new)	$L_{72}(2^{29}3^76^3)$(new)	$L_{72}(2^{19}3^86^112^1)$ (new)
22	$L_{36}(2^13^36^3)$	$L_{72}(2^{34}3^36^34^1)$(new)	$L_{72}(2^{29}3^26^4)$(new)	$L_{72}(2^{19}3^36^212^1)$ (new)
23	$L_{36}(3^{13}4^1)$	–	$L_{72}(2^{28}3^{12}6^14^1)$	–
24	$L_{36}(3^{12}12^1)$	–	$L_{72}(2^{28}3^{11}6^112^1)$(new)	–
25	$L_{36}(3^76^3)$	–	$L_{72}(2^{28}3^66^4)$(new)	$L_{72}(2^{18}3^76^212^1)$ (new)
26	$L_{36}(4^19^1)$	–	–	–
⋮	⋮	⋮	⋮	⋮

Table 2 Orthogonal arrays $L_{36}(\cdots)$ used in Sections 3.1, 3.2, and 3.3

No.	$B_1 - B_8$	$B_9 - B_{17}$	$B_{18} - B_{26}$	$B_{27} - B_{35}$	CF
1	0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0	0 0
2	1 0 0 1 1 1 1 1	0 0 0 0 0 0 0 0 0	1 0 1 1 0 0 1 1 1	0 1 0 0 1 1 1 1 1	1 1
3	0 1 1 1 0 0 1 1	1 0 0 0 1 0 1 0 1	1 1 1 0 1 0 0 1 0	1 0 0 0 0 1 1 1 0	2 2
4	1 1 0 0 1 0 0 1	0 1 1 1 1 0 0 0 1	1 1 1 0 0 1 1 0 0	0 0 1 0 0 1 0 1 1	0 3
5	0 0 0 1 1 0 1 0	1 1 1 1 1 1 1 1 1	1 0 1 1 0 0 0 0 0	0 1 0 0 1 1 0 0 0	1 4
6	1 1 1 0 1 0 1 0	1 0 0 0 0 1 1 1 0	1 0 0 0 0 1 1 1 0	0 1 1 1 0 1 0 1 0	2 5
7	0 0 1 0 1 1 0 1	0 0 0 1 0 1 1 0 1	0 1 1 1 0 1 0 1 0	1 1 1 0 1 0 0 1 0	0 0
8	1 1 0 1 0 1 0 0	0 0 0 1 0 1 1 0 1	1 0 1 0 1 1 0 0 1	1 1 0 1 0 1 0 0 1	1 1
9	0 1 1 1 0 0 1 1	0 0 1 0 0 1 0 1 1	1 1 0 1 0 1 0 0 1	0 1 1 0 0 0 1 0 1	2 2
10	1 1 0 0 1 0 0 1	1 0 0 1 1 1 0 1 0	0 1 1 1 1 0 0 0 1	0 1 0 1 0 0 1 1 0	0 3
11	0 1 1 1 1 1 0 0	1 1 1 0 1 0 0 1 0	0 0 1 0 0 1 0 1 1	1 1 0 0 0 0 0 1 1	1 4
12	1 1 1 0 1 0 1 0	0 1 1 0 0 0 1 0 1	0 1 1 0 0 0 1 0 1	1 1 0 1 1 0 1 0 0	2 5
13	0 0 0 0 0 0 0 0	0 1 0 0 1 1 1 1 1	0 1 0 0 1 1 1 1 1	0 1 0 0 1 1 1 1 1	0 0
14	1 0 0 1 1 1 1 1	0 1 0 0 1 1 1 1 1	0 0 0 0 0 0 0 0 0	1 0 1 1 0 0 1 1 1	1 1
15	0 1 1 1 0 0 1 1	0 1 0 1 0 0 1 1 0	0 0 1 1 1 1 1 0 0	0 0 0 1 1 0 0 1 1	2 2
16	1 1 0 0 1 0 0 1	1 1 1 0 0 1 1 0 0	1 0 0 1 1 1 0 1 0	1 0 0 0 1 0 1 0 1	0 3
17	0 0 0 1 1 0 1 0	1 0 1 1 0 0 0 0 0	0 1 0 0 1 1 0 0 0	1 1 1 1 1 1 1 1 1	1 4
18	1 1 1 0 1 0 1 0	0 0 0 1 1 0 0 1 1	0 0 0 1 1 0 0 1 1	1 0 1 0 1 1 0 0 1	2 5
19	0 0 1 0 1 1 0 1	0 0 1 0 1 0 1 1 0	1 1 0 1 1 0 1 0 0	1 1 0 1 0 1 0 0 1	0 0
20	1 1 0 1 0 1 0 0	0 0 1 0 1 0 1 1 0	0 1 1 1 0 1 0 1 0	0 0 1 1 1 1 1 0 0	1 1
21	0 1 0 0 0 1 1 1	0 1 1 1 0 1 0 1 0	1 1 0 0 0 0 0 1 1	1 0 0 1 1 1 0 1 0	2 2
22	1 0 1 1 0 0 0 1	0 1 1 1 0 1 0 1 0	0 0 1 0 1 0 1 1 0	1 1 1 0 0 1 1 0 0	0 3
23	0 1 1 1 1 1 0 0	1 1 0 1 0 1 0 0 1	0 1 0 1 0 0 1 1 0	0 0 0 1 0 1 1 0 1	1 4
24	1 0 1 0 0 1 1 0	0 1 1 1 1 0 0 0 1	1 0 0 1 1 1 0 1 0	0 1 0 1 0 0 1 1 0	2 5
25	0 0 0 0 0 0 0 0	1 0 1 1 0 0 1 1 1	1 0 1 1 0 0 1 1 1	1 0 1 1 0 0 1 1 1	0 0
26	1 0 0 1 1 1 1 1	1 0 1 1 0 0 1 1 1	0 1 0 0 1 1 1 1 1	0 0 0 0 0 0 0 0 0	1 1
27	0 1 0 0 0 1 1 1	1 1 0 1 1 0 1 0 0	0 0 0 1 0 1 1 0 1	1 1 1 0 0 1 1 0 0	2 2
28	1 0 1 1 0 0 0 1	1 1 0 1 1 0 1 0 0	1 1 0 0 0 0 0 1 1	0 1 1 1 1 0 0 0 1	0 3
29	0 0 0 1 1 0 1 0	0 1 0 0 1 1 0 0 0	1 1 1 1 1 1 1 1 1	1 0 1 1 0 0 0 0 0	1 4
30	1 0 1 0 0 1 1 0	1 0 0 1 1 1 0 1 0	1 1 1 0 0 1 1 0 0	1 0 0 0 1 0 1 0 1	2 5
31	0 0 1 0 1 1 0 1	1 1 0 0 0 0 0 1 1	1 0 1 0 1 1 0 0 1	0 0 1 1 1 1 1 0 0	0 0
32	1 1 0 1 0 1 0 0	1 1 0 0 0 0 0 1 1	1 1 0 1 1 0 1 0 0	1 1 1 0 1 0 0 1 0	1 1
33	0 1 0 0 0 1 1 1	1 0 1 0 1 1 0 0 1	0 0 1 0 1 0 1 1 0	0 1 1 1 1 0 0 0 1	2 2
34	1 0 1 1 0 0 0 1	1 0 1 0 1 1 0 0 1	0 0 0 1 0 1 1 0 1	1 0 0 1 1 1 0 1 0	0 3
35	0 1 1 1 1 1 0 0	0 0 1 1 1 1 1 0 0	1 0 0 0 1 0 1 0 1	0 0 1 0 1 0 1 1 0	1 4
36	1 0 1 0 0 1 1 0	1 1 1 0 0 1 1 0 0	0 1 1 1 1 0 0 0 1	0 0 1 0 0 1 0 1 1	2 5

$$L_{36}(2^{35}) = (B_1 - B_{35}) = [0_{18} \oplus (2), L_{36}^{(-)}(2^{34})]$$
$$L_{36}(3^1 2^{28}) = (CB_1 B_9 - B_{35}) = [0_{12} \oplus (3), L_{36}^{(-)}(2^{28})]$$
$$L_{36}(6^1 2^{18}) = (FB_{18} - B_{35}) = [0_6 \oplus (6), L_{36}^{(-)}(2^{18})]$$

3.2 Construction of OA $L_{72}(\cdots 6^x)$

Since $72 = 12 \times 3 \times 2$, by the three-factor method, it follows that

$$[L_{36}^{(-)} \oplus 0_2, 0_{12} \oplus (6), L_{36}^{(-)}(2^{28}) \oplus (2)]$$

is an orthogonal array for any orthogonal arrays $L_{36}^{(-)}$ and $L_{36}^{(-)}(2^{28})$ if $L_{36} = [0_{12} \oplus (3), L_{36}^{(-)}]$ and $L_{36}(3^1 2^{28}) = [0_{12} \oplus (3), L_{36}^{(-)}(2^{28})]$ are orthogonal arrays.

The orthogonal array $L_{36}(3^1 2^{28}) = [0_{12} \oplus (3), L_{36}^{(-)}(2^{28})]$ exists in Table 2.

There are at least 11 new orthogonal arrays of run size 72 for this family which are included in Table 1.

3.3 Construction of OA $L_{72}(\cdots 12^1)$

Since $72 = 6 \times 6 \times 2$, by the three-factor method, it follows that

$$[L_{36}^{(-)} \oplus 0_2, 0_{12} \oplus (12), L_{36}^{(-)}(2^{18}) \oplus (2)]$$

is an orthogonal array for any orthogonal arrays $L_{36}^{(-)}$ and $L_{36}^{(-)}(2^{18})$ if $L_{36} = [0_{12} \oplus (6), L_{36}^{(-)}]$ and $L_{36}(6^1 2^{18}) = [0_{12} \oplus (6), L_{36}^{(-)}(2^{18})]$ are orthogonal arrays.

The orthogonal array $L_{36}(6^1 2^{18}) = [0_{12} \oplus (6), L_{36}^{(-)}(2^{18})]$ exists in Table 2.

There are at least 10 new orthogonal arrays of run size 72 for this family which are included in Table 1.

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