Knot Invariants: the Jones Polynomial & Kauffman Bracket

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1 Introduction

The first thing one notices when looking into knot theory is it's not immediately obvious how the field links to what many would consider to be more traditional areas of Mathematics. Fundamental questions about knots arise quite naturally after a very small amount of exposition, but faced with even the most simply posed problems it is not easy to apply the standard approaches to proof shared by so many other fields, meaning that other more novel techniques are required.

After a brief motivation on the history of knot theory and some definitions, this essay will focus on an approach to investigate one such problem:

> *"Given two knots how can we tell if they are fundamentally different or in fact the same knot?"*

This is often referred to as the recognition problem. To prove two knots are "the same" -a notion we will soon define properly- it suffices to list a sequence of manipulations of the string to turn one into the other; to prove two knots are different is a much less trivial exercise. Consider the knots below, are they different?

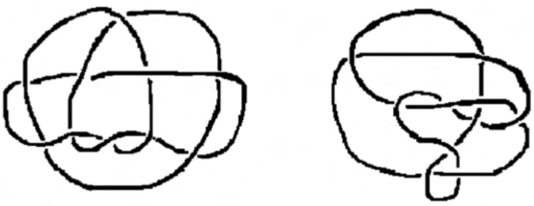


Figure 1: The Perko Pair

Known as the Perko Pair these knots were listed in all knot tables as distinct for 75 years, until an American lawyer named Ken Perko proved them to be the same knot despite looking rather different from one another. The Perko Pair is a perfect demonstration of the challenge the recognition problem poses.

There are endless possibilities for manipulating a string in three dimensions, so to tackle the recognition problem knot theorists use properties of knots which remain constant from any perspective; so called knot *invariants*. Here we will focus on one invariant in particular the: Jones Polynomial.

In the 1980s, Vaughan Jones observed in his work on von Neumann algebras a relation which was very similar to one which occurs in Knot Theory. In studying this he invented the first new polynomial invariant for knots since 1929, transforming the study of knot invariants and low-dimensional topology. The Jones polynomial merits investigation since it demonstrates an unexpected link between Knot Theory and fields such as Statistical Mechanics. So startling was this discovery, that Jones was awarded the Fields Medal for it in 1990 and the mysterious connection between these fields remains an active area of research. [1]

2 Motivation

2.1 Knot Tabulation & the Need for Invariants

Knots have had symbolic meaning and practical uses which are recorded as far back as Ancient Roman and Celtic art. It is perhaps surprising then to learn that knot theory as a mathematical field of research is relatively young. Knot Theory as studied today was kick-started in 1867 by the prominent scientist Lord Kelvin, who whilst pondering the structure of the atom formed an elegant, though sadly incorrect, theory which motivated years of investigation into the structure of knots [2].

Kelvin conjectured that atoms are simply knots in the invisible ether (a perfect fluid which the Victorians believed permeated all of space), whose topological structures account for their properties. The "vortex model" was debated for nearly 20 years and had some important proponents in the scientific community, such as James Maxwell [3, p. 22].

Immediately convinced, PG Tait (a friend and scientific peer of Kelvin) set out to create a kind of "periodic table" of unique knots, convinced that it would provide deep insight into the behaviour of atoms. As we know this did not prove to be the case, but PG Tait's pioneering work continued and in 1876 he published the first ever table of knots, grouped by the their minimum number of crossings points. Soon after, American mathematician Charles Little managed to extend Tait's tables from 7 crossings to 10 crossings; though both lacked the topological methods to guarantee they had not omitted or repeated any knots.

At the dawn of the 20^{th} century, with the vortex model of the atom now firmly out of mind, mathematicians working on knot theory began to realise there was not much further insight to be gained from continuing to enumerate increasingly larger knots and began to focus their research on creating methods to prove the distinctness of knots. In the late 1920s, James Alexander worked out a way to assign an invariant polynomial to a knot and this proved particularly effective. By the 1970s, Little and Tait were proved to be almost unerring: Little's 10 crossing table was found to contain only one erroneous duplication: the Perko Pair.

3 Definitions

3.1 Defining & Constructing Knots

Anybody without velcro on their shoes knows what a knot is, but mathematical knots differ in form slightly to a practical knot in that they are always a closed loop. This captures the entanglement of the string, preventing untying (Figure 2).

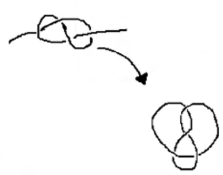


Figure 2: Practice vs Theory

Informally, visualise an elastic band being cut in one place, knotted in some way and then having its ends glued back together. For our discussion of knot invariants this intuition will largely suffice, but for a more complete understanding we will look at two more formal definitions.

Definition 3.1.1. *A **Knot** is often defined as a simple, closed, differentiable curve embedded in* $\mathbb{R}^3$

Note that differentiability is absolutely necessary to avoid "wild knottings". A function which is only continuous could define a "wild knot". For example a loop which is knotted infinitely many times with the size of these tangles converging to a single point, a notion far removed from our rubber band intuition. [4, p. 11]

There is an alternative knot definition which is often more suitable for proving important foundational theorems of knot theory, which support some of the more high-level results we will use in this essay.

Definition 3.1.2. *Alternatively a **Knot** is a closed polygonal curve in* $\mathbb{R}^3$ *i.e. the union of the ordered set of distinct points* $(p_1, p_2, ..., p_n)$ *and the edges joining the pairs* $(p_1, p_2), (p_2, p_3), ..., (p_n, p_1)$ *(provided also that the edges only intersect at their end points).*

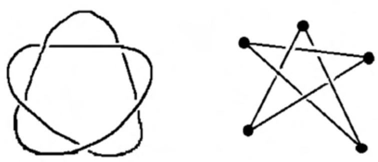


Figure 3: The same knot constructed via both definitions

Definition 3.1.3. *A **Link** is simply a union of multiple knots in* $\mathbb{R}^3$*. Links are called **splittable** when their component knots can be separated without "cutting" the string i.e. without breaking the rules of knot equivalence*

For example the union of two unknots is splittable whereas figure 5 is not. For aesthetic reasons the diagrams in this essay will be smooth curves but since a smooth curve can be arbitrarily well approximated by a polygonal curve these definitions are equally valid. [4]. All the knots we will discuss live in $\mathbb{R}^3$, but for obvious reasons of practicality knots are best studied in the plane, on flat paper:

Definition 3.1.4. *A **projection** of a link, K, is its image under the function* $\pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ *s.t.* $\pi((x, y, z)) \mapsto (x, y) \quad \forall p \in K$

Different knots can have the same projection since under a projection information is lost at crosssing points where two distinct points in $\mathbb{R}^3$ are mapped to the same point in the plane. To be make well-defined **knot diagrams** we must use projections which are injective except at a finite number of points (i.e. the crossing points) and then remove all ambiguity by leaving gaps in the line at these points to indicate which one strand goes beneath the other (figure 4). As explored in the introduction every knot has a set of equivalent knot diagrams, many of which will be very different depending on the way in which we choose to take our projection.

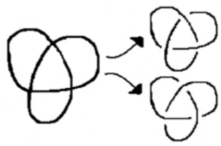


Figure 4: Creating a diagram from a projection

3.2 Common Knot Terminology

Definition 3.2.1. *We can assign an* ***Orientation*** *to a knot or link diagram by choosing a direction of travel around each loop comprising the diagram. e.g. figure 5.*

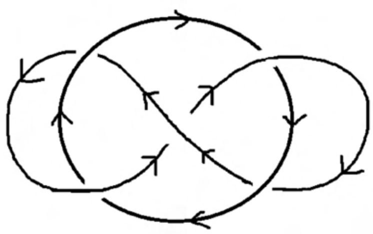
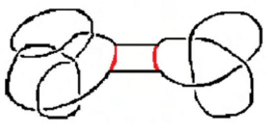


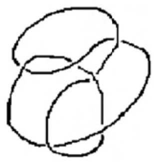
Figure 5: An oriented diagram of an "unsplittable" link

Definition 3.2.2. *A* ***Knot Composition*** *is a way to define a new knot by cutting the knots in two places and "splicing" them together as in Figure 6a. This is done in such a way so as not to add any more crossings to the knot.*

Definition 3.2.3. *Any knot which cannot be created via composition of two other knots is called a* ***Prime Knot****.*



(a) Knot Composition



(b) 6_3

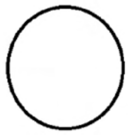
Figure 6

Practicality requires knot tables only to include prime knots and not to duplicate knots which are equivalent up to reflection. Note well that this does not mean that all knots are equivalent to their mirror image! For example the two knots in figure 4 are **chiral**

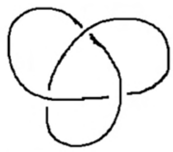
Definition 3.2.4. *A **chiral knot** is a knot which is not equivalent to its mirror image.*

The word chiral is apt since it is derived from the Greek word for "hand"; the left and right human hand display chirality since one cannot be superimposed onto the other without a reflection. This is why it makes sense to talk about some knots as "left-handed" and "right-handed".

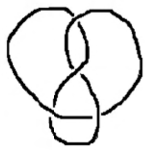
Most knots are unnamed, and referred to by a their crossing number with an identifying subscript index. For example, Livingstone [4] and most other knot tables list Figure 6b as 6_3. Only the simplest and most common knots have a name; see in figure 7 the three simplest knots which we will be seeing quite often.



(a) The Unknot or Trivial Knot



(b) 3_1: The Trefoil Knot



(c) 4_1: The Figure of Eight Knot

Figure 7

3.3 Knot Equivalence

Before we can meaningfully discuss a knot invariant it will be necessary to pin down exactly when two knots are "equivalent". To do this we will examine the link between equivalence of knots and basic manipulations of their diagrams.

3.3.1 Planar Isotopies

Definition 3.3.1. *A **Planar Isotopy** is a function which deforms the projection plane in a way such that the string "does not pass through itself". Essentially it preserves the topological properties of the knot.*

We would like knots sharing essentially the same form to be equivalent to each other, so we will consider two knots equivalent if they are related by a Planar Isotopy:

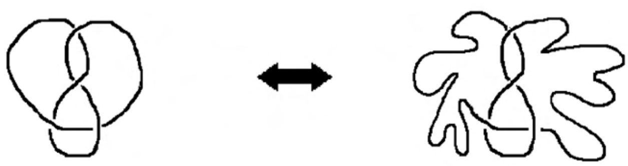


Figure 8: Two Knots Equivalent under a planar isotopy

3.3.2 The Reidemeister Moves

During the 1920s knot theorist Kurt Reidemeister described three simple moves we can use to change the relation between arcs of string on a knot diagram:

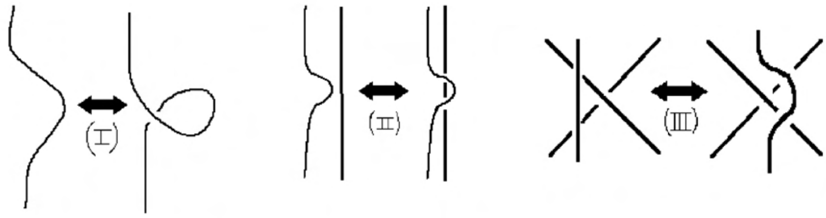


Figure 9: The Reidemeister Moves: R1, R2 & R3

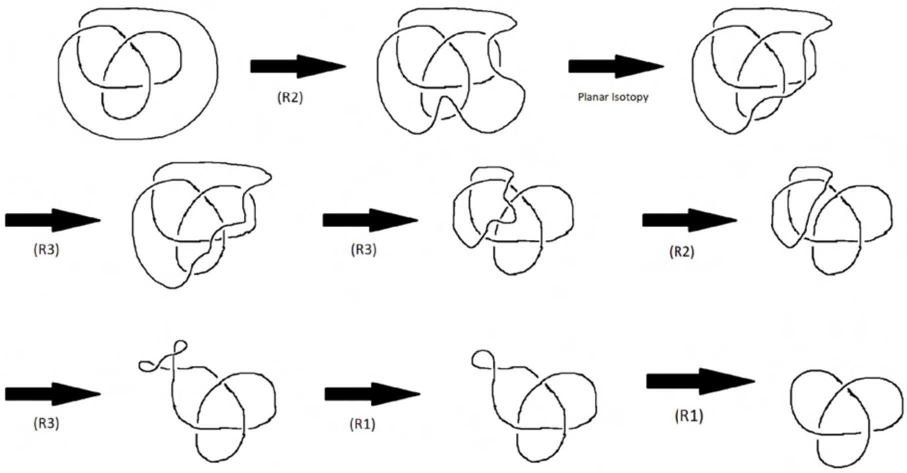
The first allows us to create or untangle a loop in a section of a knot. The second allows us to push one section of string in a knot under (or over) a different section of string, creating two new crossing points. The third and final Reidemeister move allows us to move a section of string to the other side of a crossing point whenever the string is above (or below) *both* of the two strands which form the junction. He went on to prove a theorem which is fundamental to our study of invariants:

Theorem 3.1. *Two link diagrams represent equivalent links if and only if there exists a finite sequence of Reidemeister Moves and Planar Isotopies relating them.*

The proof of this is quite involved, with many cases each requiring the use of foundational topological tools and definitions which will not be necessary for a discussion of the Jones Polynomial. As such it would be counter-productive to reproduce it here but a complete proof is offered in Murasugi's "Knot Theory and its applications" [5, p. 50]

However this result is very intuitive after trying a few examples such as the one below. To develop some intuition on why the Reidemeister moves are sufficient it is helpful to think about cases in which knot diagrams appear to be ill defined. In any such case we can use a perturbation (a small rotation or change of the plane of projection) to create a well defined diagram to which we may apply a Reidemeister move. [4, p. 31]

Example 3.3.1. *(From an exercise given by Colin Adams [6])*
Here we use the moves (R1), (R2) & (R3) to demonstrate the equivalence of two projections of a trefoil knot



It is important to realise that in most cases attempting to find an explicit sequence of Reidemeister moves is a poor approach to proving the equivalence of two knots. Even when the knots are obviously equivalent, as above, we can require a very large number of moves. Tackling diagrams such as the Perko Pair (featured in the introduction) is a very daunting task since there is no obvious starting point. Moreover, applying the Reidemeister moves can never assure us that two knots are distinct since we cannot be sure to have exhausted every possibility.

So for a systematic approach to tabulating knots, this method will not suffice, however Theorem 3.1 yields an extremely powerful corollary which we will use to derive a better approach:

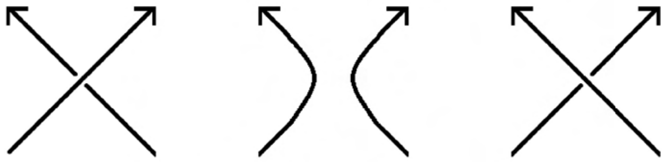
Corollary 3.3.1. *To prove that any property of a knot, K, is invariant (i.e. it holds for any knot equivalent to K) it will suffice to show that this property is invariant under the three Reidemeister moves.*

4 The Jones Polynomial

Now that we have the tools to define a knot invariant we will discuss the Jones Polynomial from two perspectives: an overview of the original approach taken by Jones followed by an in-depth look at a significantly simpler approach derived later by Louis Kauffman.

At the heart of the Jones Polynomial lies an equation called the Skein Relation, which relates the Jones Polynomial of any three oriented links which are identical except at one neighbourhood around a crossing point, where they differ as in figure 10

The skein relation is a powerful tool since it allows us to calculate the Jones Polynomial of a knot diagram by "splicing" or "smoothing" its crossings, i.e. considering the polynomials of simpler diagrams where a L_+ or L_- crossings



(a) L_+ (b) L_0 (c) L_-

Figure 10: The differing sections of Links related by the skein relation

becomes an L_0 neighbourhood. Thus if we assign a value to the unknot and clearly define a suitable skein relation we can calculate the Jones polynomial of any knot recursively, simply by continuing to splice crossings.

4.1 The Original Approach

Definition 4.1.1. *A **braid** of n strings, is a set of n intertwined strings between two parallel "bars" such that the strings begin at a fixed set of n points on the "top bar" and end at a set of points at the same positions on a "bottom bar".*

Note that a given string needn't begin and end in the same position on each bar, and we do not allow the strings to change direction vertically i.e. the path of each string could be traced out by an object falling under gravity and horizontal forces alone.

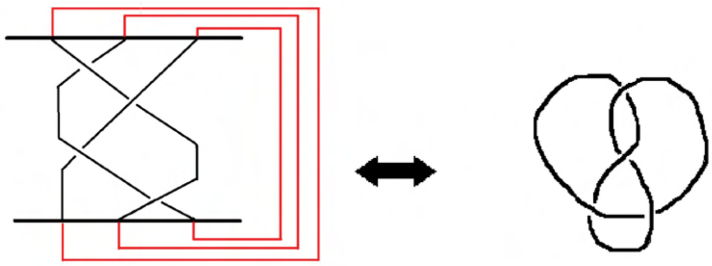


Figure 11: A Braid whose closure is the figure of 8

Theorem 4.1. ***(James Alexander, 1923)***
For any knot (or link) there exists a braid which can be closed in the obvious way shown in figure 11 to form it.

Braids are a useful concept because the set of n-string braids, $\mathbb{B}_n$, forms a group with respect to the binary operation, $\otimes$, of direct product ("glueing" the "bottom bar" of one braid to the "top bar" of another). $\mathbb{B}_n$ is generated by the set $\{\sigma_1, \sigma_2, \cdots, \sigma_{n-1}\}$ where σ_i denotes the n-string braid where the only crossing is made by the i^{th} string passing over the $(i+1)^{th}$ string. In this way, due to the theorem above, the study of knots is reduced to a study of the algebra of $(\mathbb{B}_n,\otimes)$. This is the approach Jones took to constructing his invariant. [7]

By describing the Reidemeister moves as mappings on the braid group, Jones was able to construct a function, $V : \mathbb{B}_n \longrightarrow \mathbb{Z}[t]$, which is not only an invariant but also has the following desirable properties [7, p. 125] which allow us to calculate it:

1. Normalisation; The Jones polynomial of the unknot is equal to 1
2. Skein Relation; Jones recognised his function to obey the following relation (where L_0, L_+, L_- defined as in figure 10)

$$t^{-1}V(L_+) - tV(L_-) = (t^{1/2} - t^{-1/2})V(L_0)$$

Example 4.1.1. ***Jones polynomial of the "Unlink"***

V("the unlink")= $V(\bigcirc\bigcirc)$ *so by the skein relation above:*

$V(\bigcirc\bigcirc) = (t^{1/2} - t^{-1/2})^{-1}(t^{-1}V(8) - tV(8))$

$= (t^{1/2} - t^{-1/2})^{-1}(t^{-1}V(\bigcirc) - tV(\bigcirc))$ *by invariance of V*

$= (t^{1/2} - t^{-1/2})^{-1}(t^{-1} - t)$ *by normalisation property*

$= -t^{1/2} - t^{-1/2}$

In fact this example can be generalised. As a consequence of the skein relation we have another useful property:

Lemma 4.1.1. *The Disjoint Union Rule;*

$$V(\bigcirc \cup K) = (-t^{1/2} - t^{-1/2})V(K)$$

Since Jones' paper on his polynomial won him a Fields Medal it is perhaps unsurprising that the original proof that this construction is invariant, which involves Von Neumann and Hecke Algebras, is beyond the scope of this essay. However, we will be examining the Kauffman version of this proof in detail.

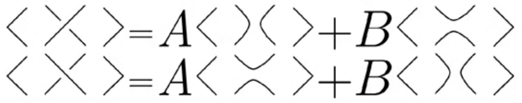
4.2 Kauffman's Bracket Approach

Fortunately we are able to derive and calculate the Jones Polynomial of knot without a knowledge of the underlying braid algebra. This is thanks to an equivalent yet simpler approach by the mathematician Louis Kauffman. This version avoids use of the braid group altogether and is calculated only from properties of the knot diagram; it is a **combinatorial** approach. The following derivation of Kauffman's polynomial invariant is adapted from Colin Adams [6].

To define the Jones polynomial Kauffman first defines another polynomial:

Definition 4.2.1. ***The Bracket polynomial*** *is a polynomial of three variables (A,B and C) defined on the diagram of a knot or link, K. It is denoted* $< K >$ *and has the following defining properties:*

1. *Normalisation Property;* $\langle unknot \rangle = \langle \bigcirc \rangle = 1$
2. *The Skein Relation;*



or equivalently

where  *represent the bracket polynomials of any 4 knot or link diagrams which are identical except for one neighbourhood where they differ as shown within the brackets.*

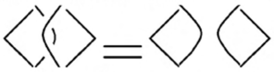
3. *The Disjoint Union Property;*

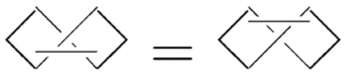
$$\langle K \cup \bigcirc \rangle = C \langle K \rangle$$

For the disjoint union of the diagrams of any link K with the unknot i.e. a link diagram where K and the unknot have been "split".

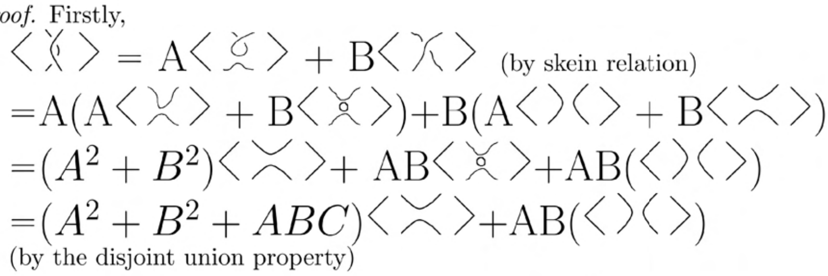
Lemma 4.2.1. *For $B := A^{-1}$ and $C := -A^2 - A^{-2}$, the Bracket Polynomial in A is invariant under type II and type III Reidemeister moves.*

We must show that for these choices of B and C the following holds:

Type II; 

Type III; 

Proof. Firstly,

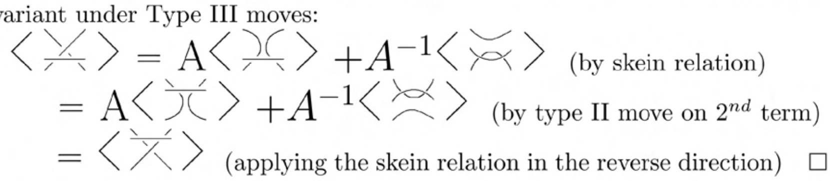


Comparing coefficients we see that for invariance under type II we require

$$AB = 1 \qquad A^2 + B^2 + ABC = 0$$

Solving the system for B and C we get the values of B and C stated in the Lemma.

Now we check that for this value of B and C the Bracket polynomial is invariant under Type III moves:



Since this is such a useful result we will include these choices of B and C to be part of the definition of the bracket polynomial.

Lemma 4.2.2. *Unfortunately the bracket polynomial **is not** invariant under type 1 moves.*

Proof. $\langle \ \rangle = A\langle \ \rangle + A^{-1}\langle \ \rangle$

$= A(-A^2 - A^{-2})\langle - \rangle + A^{-1}\langle - \rangle$

$= (-A^3)\langle - \rangle$

whereas $\langle \ \rangle = (-A^{-3})\langle - \rangle$

(a very easy exercise analogous to what was just proven)

Since these knots are related by two type I moves yet have different bracket polynomials the result is proven. □

Kauffman seeks to fix this problem by orienting the knots whose polynomial we would like, and using a property closely related to orientation to adjust his polynomial by a coefficient. This is a plausible idea to try as the difference between an L_+ crossing and an L_- crossing is really only a matter of orientation.

Definition 4.2.2. *The **writhe** of a diagram is calculated by orienting it in any direction and the counting the number of L_+ and $L-$ crossings as defined in figure 10. Then $\omega(K) = (\#L_+ crossings) - (\#L_- crossings)$.*

Example 4.2.1. *For example summing the crossings on the diagram below gives a total writhe of +1*

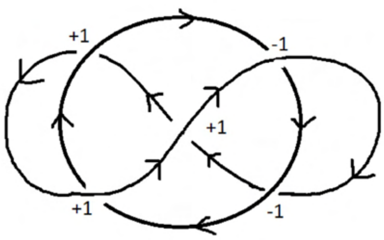


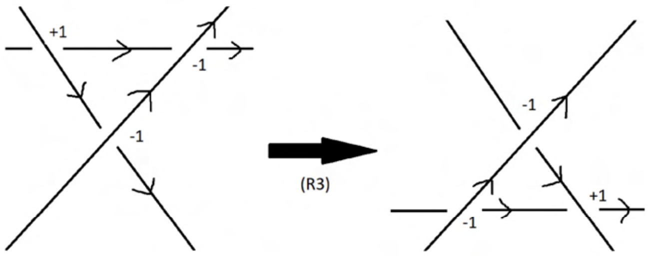
Figure 12: Calculating the writhe of a link

Lemma 4.2.3. *Like the Bracket polynomial, the writhe of a knot diagram is invariant under type II and type III Reidemeister moves.*

Proof. Sketch:

It is easily observed that regardless of choice of orientation, a type II move always involves creating or removing two crossings: one of type L_+ and another of type L_-. These crossings have a zero sum effect on the writhe hence writhe is invariant under type II moves.

For type III moves a case analysis of the finite number of combinations of orientations for three strands (one example case is included below) will confirm that writhe is invariant under type III moves.



□

Theorem 4.2. *For an oriented knot or link, K, the* ***Kauffman Invariant,***

$$X(K) := (-A^3)^{-\omega(K)} < K >$$

is a knot invariant.

Proof. By lemmas 4.2.3 and 4.2.1 $\omega(K)$ and $< K >$ are unaffected by type II and III Reidemeister moves. Thus by its construction X(K) is also invariant type II and III moves.

For type I invariance we consider two knots, K and $\widetilde{K}$, which differ in one neighbourhood as shown in figure 13.

(a) K (b) $\widetilde{K}$

Figure 13

Clearly the two are separated by a type I move and $\omega(\widetilde{K}) = \omega(K) + 1$ since by adding this half twist to K we introduce a new L_+ type crossing. Hence,

$X(\widetilde{K}) = (-A^3)^{-\omega(\widetilde{K})} \langle\widetilde{K}\rangle$
$= (-A^3)^{-\omega(\widetilde{K})}(-A^3) \langle K\rangle$ (see lemma 4.2.2)
$= (-A^3)^{-\omega(\widetilde{K})+1} \langle K\rangle$
$= (-A^3)^{-(\omega(K)+1)+1} \langle K\rangle$
$= (-A^3)^{-\omega(K)} \langle K\rangle = X(K)$

So we have demonstrated this Reidemeister move does not affect the Kauffman invariant of a knot. Analogous reasoning demonstrates that the other version of the type I move (a half twist in the opposite direction) is similarly inert.

Thus Kauffman's invariant is invariant under all three Reidemeister moves and thus by corollary 3.3.1 it is a knot invariant. □

Is this the Jones Polynomial? So the Kauffman Invariant derived using the Bracket Polynomial is indeed an invariant, but how can we be sure that we have derived the Jones Polynomial and not a completely new polynomial?

Theorem 4.3. *The Skein Relation used by Kauffman as a defining property of his invariant is equivalent to the Original Skein Relation Jones recognised his polynomial to obey (presented in section 4.1). i.e. for $A = t^{-1/4}$*

$$= A \quad + A^{-1} \quad \Leftrightarrow t^{-1}X(L_+) - tX(L_-) = (t^{1/2} - t^{-1/2})X(L_0)$$

Proof. Without loss of generality take $\omega(L_0) = \lambda$, then from definition of Kauffman Invariant we have:

$$X(L_0) = (-A^3)^{-\lambda}$$
$$X(L_+) = (-A^3)^{-(\lambda+1)} \quad = (-A^3)^{-\lambda}(-A^3)[A \quad + A^{-1} \quad]$$
$$X(L_-) = (-A^3)^{-(\lambda-1)} \quad = (-A^3)^{-\lambda}(-A^3)^{-1}[A\langle \quad \rangle + A^{-1}\langle \quad \rangle]$$

$$t^{-1}X(L_+) - tX(L_-) = A^4X(L_+) - A^{-4}X(L_-)$$
$$= (-A^3)^{-\lambda}[(-A^2 + A^{-2}) \quad - \langle \quad \rangle + \langle \quad \rangle]$$
$$= (-A^2 + A^{-2})(-A^3)^{-\lambda}$$
$$= (A^{-2} - A^2)X(L_0) = (t^{1/2} - t^{-1/2})X(L_0)$$

□

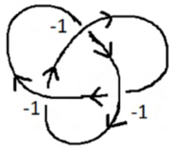
Corollary 4.2.1. *The Kauffman Invariant and Jones Polynomial are the same invariant i.e. for any link K, $V_K(t) = X_K(t^{-1/4})$*

This is because, since the two invariants are also normalised in the same way, the fact that the skein relations are identical means that the algorithm we would apply to calculate them is identical so we will always obtain the same polynomial.

Example 4.2.2. ***Calculate the Jones Invariant of a left-handed trefoil***

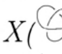
$$\langle \quad \rangle = A\langle \quad \rangle + A^{-1}\langle \quad \rangle$$
$$= A(A\langle \quad \rangle + A^{-1}\langle \quad \rangle) + A^{-1}\langle \quad \rangle$$
$$= A(A(-A^2 - A^{-2})\langle \quad \rangle + A^{-1}\langle \quad \rangle) + A^{-1}\langle \quad \rangle$$
$$= (-A^4)\langle \quad \rangle + A^{-1}\langle \quad \rangle$$
$$= (-A^4)\langle \quad \rangle + A^{-1}(A\langle \quad \rangle + A^{-1}\langle \quad \rangle)$$
$$= (1 - A^4)\langle \quad \rangle + A^{-2}\langle \quad \rangle$$
$$= (1 - A^4)(A\langle \quad \rangle + A^{-1}\langle \quad \rangle) + A^{-2}\langle \quad \rangle$$
$$= (1 - A^4)(A(-A^2 - A^{-2}) + A^{-1}) + A^{-2}\langle \quad \rangle$$
$$= (-A^3)(1 - A^4) + A^{-2}(A\langle \quad \rangle + A^{-1}\langle \quad \rangle)$$
$$= (-A^3)(1 - A^4) + A^{-2}(A + A^{-1}(-A^2 - A^{-2}))$$

$=A^7 - A^3 - A^{-5}$



gives writhe of knot; $\omega($$)= -3$

hence by previous theorem we calculate the Jones Polynomial:

$X($$)= (-A^3)^{-(-3)}.\langle$  $\rangle$

$= (-A^9)(A^7 - A^3 - A^{-5})$

$= -A^{16} - A^{12} + A^4$

4.3 Strengths and Limitations

In summary, thanks to Kaufmann's work we have a simple combinatorial algorithm which yields the Jones polynomial, an invariant which demonstrates a remarkable ability to distinguish between prime knots. *Every known prime knot composed of 9 or fewer crossing points has a distinct Jones Polynomial* [6, p. 55], effectively meaning we can use the techniques developed in this chapter to uniquely identify any of these knots. Moreover, the Jones Polynomial succeeds where many other knot invariants fail in that it can distinguish between some (not all) *chiral knots* e.g. the left and right handed trefoils. It's predecessor,the Alexander Polynomial, was unable to do this. In addition there is an important open question in knot theory about whether there exists non-trivial knots with a Jones Polynomial equal to 1. Many conjecture that no such knots exist and if this could be proven it would be a very useful property indeed.

Unfortunately, as the example calculation of a trefoil demonstrates, the Jones Polynomial is limited in that calculations even using the simplified Kauffman method are very time and labour intensive. In order to calculate the Jones polynomial of the trefoil, we had to apply the skein relation to each of it's three crossings. Breaking down the first crossing created two diagrams both of which spawned another two diagrams and so on.

Whilst the use of the Skein relation to break down complex knots is a neat trick it creates a problem: *each time we have to evaluate a crossing the number of diagrams we are handling doubles.* If we program a computer with an algorithm to calculate the Kauffman Invariant of a knot of n crossings, after the last application of the skein relation the computer would be handling 2^n knot diagrams. In other words the Jones Polynomial requires **exponential time** to compute. Anything more than 6 crossings ($2^6 = 64$ diagrams) would be extremely tedious to attempt by hand and as crossing number increased we would reach the limits of what is possible with a conventional computer very quickly.

Undoubtedly the Jones Polynomial is a fascinating tool, invaluable to elementary knot classification. Some see in it scientific applications: from helping to unravel complicated knots of bacterial DNA to exploiting its link with statistical mechanics in order to model lattices of magnetically charged particles. [6, p. 205] Unfortunately, however, those who hope to unlock the full potential of the Jones Polynomial may have to wait until the arrival of quantum computers.

5 References

References

[1] Vaughan Jones Personal Webpage. https://royalsociety.org/people/vaughan-jones-11708/.

[2] Daniel S. Silver. Knot theory's odd origins. *American Scientist*, vol. 94(2):158–165, 2006.

[3] Peter Cromwell. *Knots and Links*. Cambridge University Press, 2004.

[4] Charles Livingstone. *Knot Theory*. The Mathematical Association of America, 1993.

[5] Kunio Murasugi. *Knot Theory & its applications*. Birkhauser, 2008.

[6] Colin C. Adams. *The Knot Book*. American Mathematical Society, 1994.

[7] Weiping Li. *Lecure Notes on Knot Invariants*. World Scientific Publishing, 2016.