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(a)假设一个反射镜左右两侧向右传播的光振幅为 1 和 x

向左传播的振幅为 $r + \frac{t(t-x)}{r}$ 和 $\frac{t-x}{r}$

由于平移对称性并考虑到相位差 $\frac{t-x}{x} = (r + \frac{t(t-x)}{r})e^{2ika}$

得到关于 x 的二次方程 $tx^2 - (1 + e^{-2ika})x + te^{-2ika} = 0$

x 的解为 $\frac{1+e^{-2ika} \pm \sqrt{1+e^{-4ika}+(2-4t^2)e^{-2ika}}}{2t} = \frac{1+e^{-2ika} \pm \sqrt{(e^{-2ika}-p)(e^{-2ika}-q)}}{2t}$

其中 $p, q = (2t^2 - 1) \pm i\sqrt{1 - (2t^2 - 1)^2}$

故高反射区间满足 $2\pi - \cos^{-1}(2t^2 - 1) \leq 2ka \leq 2\pi + \cos^{-1}(2t^2 - 1)$

即 $\frac{4\pi a}{2\pi + \cos^{-1}(2t^2 - 1)} < \lambda < \frac{4\pi a}{2\pi - \cos^{-1}(2t^2 - 1)}$

(b)

带入得

$$\begin{aligned} C^2 + X^2 &= 1 \\ C^2 E_{low} + X^2 E_{upp} &= E_1 \\ C^2 E_{upp} + X^2 E_{low} &= E_2 \\ CX(E_{low} - E_{upp}) &= g \end{aligned}$$

解得

$$E_{upp}, E_{low} = \frac{E_1 + E_2 \pm \sqrt{(E_1 - E_2)^2 + 4g^2}}{2}$$

(c)

$$\begin{aligned} k_x &= \frac{(2k+1)\pi}{a} \\ k_y &= \sqrt{k^2 - \left(\frac{(2k+1)\pi}{a}\right)^2} = k - \frac{(2k+1)^2\pi^2}{2ka^2} \end{aligned}$$

差值为 $\frac{4\pi^2}{ka^2}$ 的整数倍, 因此周期 $L = \frac{ka^2}{2\pi} = \frac{a^2}{\lambda}$

波长为 $3.0\mu\text{m}$

二

(a)假设重物下降 x, 倾斜角度 θ

竖直方向受力平衡 $2mg = \int_0^x 2a(x - \theta t)kdt$

重物所在轴力矩平衡 $mga = \int_0^x 2a(x - \theta t)ktdt$

积分得 $\frac{x}{2} \left(\frac{x}{\theta}\right)^2 - \frac{\theta}{3} \left(\frac{x}{\theta}\right)^3 = \frac{mg}{2k}, x \frac{x}{\theta} - \frac{\theta}{2} \left(\frac{x}{\theta}\right)^2 = \frac{mg}{ak}$

解得 $\frac{x}{\theta} = \frac{3a}{2}, x = \frac{4mg}{3ka^2}, a\theta = \frac{8mg}{9ka^2}$

接触面积 $3a^2$

(b)

$$\text{动能 } T = \frac{m}{2} \left((\dot{x} - a\dot{\theta})^2 + \dot{x}^2 \right) + \frac{ma^2}{6} \dot{\theta}^2 = m\dot{x}^2 - m\dot{x}a\dot{\theta} + \frac{2m}{3} a^2 \dot{\theta}^2$$

$$\text{势能 } V = \int_0^x a(x - \theta t)^2 k dt - mgx - mg(x - \theta a) = \frac{ax^3 k}{3\theta} - mg(2x - \theta a)$$

$$\text{令 } x = \frac{4mg}{3ka^2} + u, a\theta = \frac{8mg}{9ka^2} + v$$

$$T = m\dot{u}^2 - m\dot{u}\dot{v} + \frac{2m}{3} \dot{v}^2$$

$$V = \frac{3a^2 k}{2} u^2 - \frac{9a^2 k}{4} uv + \frac{9a^2 k}{8} v^2$$

带入后解得

$$\omega = \frac{24 \pm 3\sqrt{19}}{20} \sqrt{\frac{a^2 k}{m}}$$

三

$$(a) \text{ AB 杆 } x \text{ 方向投影 } a = \sqrt{1 - \left(\frac{1}{2 \sin(\frac{\pi}{5})} \right)^2} = 0.5257$$

$$\text{受力 } F_1 = \frac{F}{5b} = 0.3804F$$

$$(b) \text{ BC 杆 } x \text{ 方向投影 } b = \sqrt{1 - \left(\frac{\sin(\frac{\pi}{10})}{\sin(\frac{\pi}{5})} \right)^2} = 0.8507, yz \text{ 平面内投影 } \frac{\sin(\frac{\pi}{10})}{\sin(\frac{\pi}{5})} \text{ 刚好等于 } a$$

$$\text{受力 } F_2 = \frac{F}{10b} = 0.1176F$$

$$(c) F_3 = \frac{2aF_2 \sin(\frac{\pi}{10}) + bF_1}{2 \sin(\frac{\pi}{10})} = 0.2753F$$

四

$$(a) \frac{q \frac{qR}{x}}{4\pi\epsilon_0 \left(\frac{R^2}{x} \right)^2} = \frac{q^2 x}{4\pi\epsilon_0 R^3}$$

(b)

每一处可以近似看成平板电容器，极板间距 $d = R - r - \cos(\theta)x$

$$\text{得到电荷面密度和单位面积受力 } \sigma = \frac{\epsilon_0 U}{d}, P = \frac{\epsilon_0 U^2}{2d^2} = \frac{\epsilon_0 U^2}{2(R-r)^2} \left(1 + \frac{\cos(\theta)x}{(R-r)} \right)$$

$$\text{积分得到总静电力 } F = \int_0^\pi P \cos(\theta) 2\pi R^2 \sin(\theta) d\theta = \frac{2\epsilon_0 U^2 x \pi R^2}{3(R-r)^3} = \frac{q^2 x}{24\pi\epsilon_0 R^2 (R-r)}$$

$$\text{其中 } U = \frac{q(R-r)}{4\pi\epsilon_0 Rr}$$

五

(a)

$$\text{电场分布 } E = \frac{UrR}{(R-r)\rho^2}$$

$$\text{电子做半径 } r_0 = \frac{R+r}{2} \text{ 的圆周运动, 动能 } T = \frac{eUrR}{(R^2-r^2)}$$

(b)

$$\rho = \frac{p}{1 - e \cos(\theta)}$$
$$\frac{1}{r_0} + \frac{1}{r_1} = \frac{2}{p} = \frac{eUrR}{Tr_0^2 \cos^2(\alpha)(R-r)}$$

其中 $r_0 = \frac{R+r}{2}$ 代表狭缝位置 r_1 为落在 mcp 上的位置

$$\frac{1}{r_1} = \frac{eUrR}{Tr_0^2 \cos^2(\alpha)(R-r)} - \frac{1}{r_0} = \frac{4(R^2 - r^2)}{r_0^2 \cos^2(\alpha)(R-r)} - \frac{1}{r_0} = \frac{2 - \cos^2(\alpha)}{r_0 \cos^2(\alpha)}$$

区间为 $\frac{r_0 \cos^2(\alpha)}{2 - \cos^2(\alpha)} \sim r_0$

(c)

$$\frac{1}{r_1} = \frac{eUrR}{\left(\frac{eUrR}{(R^2 - r^2)} + \Delta E\right)r_0^2(R-r)} - \frac{1}{r_0} = \frac{2eUrR}{(eUrR + \Delta E(R^2 - r^2))r_0} - \frac{1}{r_0}$$
$$r_1 = r_0 \frac{eUrR + \Delta E(R^2 - r^2)}{eUrR - \Delta E(R^2 - r^2)}$$

六

(a)

$$\text{磁通量 } \psi = M_0 \cos(\omega t + \varphi) I_0 \cos(\omega_0 t) = \frac{M_0 I_0}{2} (\cos((\omega + \omega_0)t + \varphi) + \cos((\omega - \omega_0)t + \varphi))$$

$$\text{电动势 } E = \frac{M_0 I_0}{2} ((\omega + \omega_0) \sin((\omega + \omega_0)t + \varphi) + (\omega - \omega_0) \sin((\omega - \omega_0)t + \varphi))$$

力矩

$$\frac{EI_1}{R} \frac{dM}{d\theta}$$

$$= -\frac{M_0^2 I_0^2}{2R} ((\omega + \omega_0) \sin((\omega + \omega_0)t + \varphi) + (\omega - \omega_0) \sin((\omega - \omega_0)t + \varphi)) \sin(\omega t + \varphi) \cos(\omega_0 t)$$
$$= -\frac{M_0^2 I_0^2}{4R} ((\omega + \omega_0) \sin((\omega + \omega_0)t + \varphi) + (\omega - \omega_0) \sin((\omega - \omega_0)t + \varphi)) (\sin((\omega + \omega_0)t + \varphi) + \sin((\omega - \omega_0)t + \varphi))$$

平均值

$$-\frac{M_0^2 I_0^2 \omega}{4R}$$

(b)

$$\text{磁通量 } \psi = M_0 \cos(\omega t + \varphi) I_0 \cos(\omega_0 t) + M_0 \sin(\omega t + \varphi) I_0 \sin(\omega_0 t) = M_0 I_0 \cos((\omega - \omega_0)t + \varphi)$$

$$\text{电动势 } E = M_0 I_0 (\omega - \omega_0) \sin((\omega - \omega_0)t + \varphi)$$

力矩

$$\frac{E}{R} \left(I_1 \frac{dM_1}{d\theta} + I_2 \frac{dM_2}{d\theta} \right)$$

$$= \frac{M_0^2 I_0^2}{R} (\omega - \omega_0) \sin((\omega - \omega_0)t + \varphi) (-\cos(\omega_0 t) \sin(\omega t + \varphi) + \sin(\omega_0 t) \cos(\omega t + \varphi))$$

$$= -\frac{M_0^2 I_0^2}{R} (\omega - \omega_0) \sin^2((\omega - \omega_0)t + \varphi)$$

平均值

$$\frac{M_0^2 I_0^2 (\omega_0 - \omega)}{2R}$$

(c) 电动势 $E = \frac{M_0 I_0}{2} \left((\omega + \omega_0) \sin((\omega + \omega_0)t + \varphi) + (\omega - \omega_0) \sin((\omega - \omega_0)t + \varphi) \right)$

电流

$$I = \frac{M_0 I_0}{2} \left((\omega + \omega_0) \left(\frac{R}{R^2 + L^2 (\omega + \omega_0)^2} \sin((\omega + \omega_0)t + \varphi) - \frac{L(\omega + \omega_0)}{R^2 + L^2 (\omega + \omega_0)^2} \cos((\omega + \omega_0)t + \varphi) \right) \right. \\ \left. + (\omega - \omega_0) \left(\frac{R}{R^2 + L^2 (\omega - \omega_0)^2} \sin((\omega - \omega_0)t + \varphi) - \frac{L(\omega - \omega_0)}{R^2 + L^2 (\omega - \omega_0)^2} \cos((\omega - \omega_0)t + \varphi) \right) \right)$$

力矩

$$II_1 \frac{dM}{d\theta} = -\frac{M_0^2 I_0^2}{2} \left((\omega + \omega_0) \left(\frac{R}{R^2 + L^2 (\omega + \omega_0)^2} \sin((\omega + \omega_0)t + \varphi) - \frac{L(\omega + \omega_0)}{R^2 + L^2 (\omega + \omega_0)^2} \cos((\omega + \omega_0)t + \varphi) \right) \right. \\ \left. + (\omega - \omega_0) \left(\frac{R}{R^2 + L^2 (\omega - \omega_0)^2} \sin((\omega - \omega_0)t + \varphi) - \frac{L(\omega - \omega_0)}{R^2 + L^2 (\omega - \omega_0)^2} \cos((\omega - \omega_0)t + \varphi) \right) \right) (\sin((\omega + \omega_0)t + \varphi) \\ + \sin((\omega - \omega_0)t + \varphi))$$

平均值

$$-\frac{M_0^2 I_0^2}{4} \left(\frac{R(\omega + \omega_0)}{R^2 + L^2 (\omega + \omega_0)^2} + \frac{R(\omega - \omega_0)}{R^2 + L^2 (\omega - \omega_0)^2} \right)$$

七

考虑一个正方形框架，一个顶点上附加质量 m_1

写出动能

$$T = \frac{m_1}{2} v_2^2 + \frac{m}{3} \left(\frac{5}{2} v_1^2 + \frac{5}{2} v_2^2 + v_1 v_2 \right)$$

假设下方受到向上的大小为 F 的力

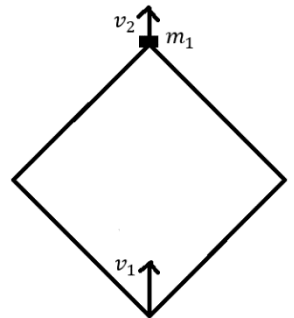
可以得到两顶点的加速度满足方程

$$\left(m_1 + \frac{5m}{3} \right) a_2 + \frac{m}{3} a_1 = 0$$

$$\frac{5m}{3} a_1 + \frac{m}{3} a_2 = F$$

解得

$$F = \frac{5m_1 + 8m}{3m_1 + 5m} m a_1$$



如果只有一个框架, $m_1 = 0$, $F = \frac{8}{5}ma_1$

如果由无穷多个框架, 由于自相似 $F = m_1 a_1$

解得 $F = \sqrt{\frac{8}{3}}ma_1$

八

(a)

假设杆上两物体向右速度为 v_1, v_2 下方物体为 $(\frac{v_1+v_2}{2}, \frac{\sqrt{3}(v_1-v_2)}{6})$

x 方向动量

$$2mv_1 + m\frac{v_1+v_2}{2} + mv_2 = 0$$

重力做功等于动能增加

$$\frac{1}{2}(2mv_1^2 + m\left(\frac{v_1+v_2}{2}\right)^2 + m\left(\frac{\sqrt{3}(v_1-v_2)}{6}\right)^2 + mv_2^2) = mgy = \frac{\sqrt{3}}{2}mgl$$

解得

$$v_1 = \sqrt{\frac{27}{74}gy} = \sqrt{\frac{27\sqrt{3}}{148}gl}$$

$$v_2 = -\sqrt{\frac{75}{74}gy} = -\sqrt{\frac{75\sqrt{3}}{148}gl}$$

下方物体

$$\vec{v}_3 = \left(-\sqrt{\frac{3}{74}gy}, \sqrt{\frac{8}{37}gy}\right) = \left(-\sqrt{\frac{3\sqrt{3}}{148}gl}, \sqrt{\frac{4\sqrt{3}}{37}gl}\right)$$

(b)

由动量守恒, 假设 $a_1 = 3a, a_2 = -5a, \vec{a}_3 = (-a, a_y)$

两侧绳上的力为 $12ma, 10ma$

Y 方向牛顿定律 $mg - 11\sqrt{3}ma = ma_y$

绳两侧物体相对速度 $v' = \sqrt{\frac{16\sqrt{3}}{37}gl}$

得到加速度关联 $2a - \frac{\sqrt{3}}{2}a_y = \frac{v'^2}{l}$

解得

$$a = \frac{69\sqrt{3}}{1369}g$$

$$a_y = \frac{908}{1369}g$$