Probability & Statistics for EECS: Homework #01

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Problem 1

1. Using story proof, as for the left side of the equation, $\begin{Bmatrix} n+1 \\ k \end{Bmatrix}$ is defined as the number of ways to partition 1, 2,...,n+1 th students into k groups, then consider about the n+1 th student, there are 2 situations:

   (a) The $n+1$ th student is parted into a single group, then as for the remaining students and the groups, there will be $\begin{Bmatrix} n \\ k-1 \end{Bmatrix}$ ways to part 1, 2,...,n th student into k-1 groups.

   (b) The $n+1$ th student is parted into a non-empty group with other students, then as for the remaining n students and the k non-empty groups, there will be $\begin{Bmatrix} n \\ k \end{Bmatrix}$ ways to part 1, 2,...,n th student into k non-empty groups. Then we need to part this student into one of these k non-empty groups, and so there will be $k\begin{Bmatrix} n \\ k \end{Bmatrix}$ ways.

   Add these 2 situations together, we will have $\begin{Bmatrix} n \\ k-1 \end{Bmatrix} + k\begin{Bmatrix} n \\ k \end{Bmatrix}$, then as it is the same whether to consider about the n+1 th student independently, we will have $\begin{Bmatrix} n+1 \\ k \end{Bmatrix} = \begin{Bmatrix} n \\ k-1 \end{Bmatrix} + k\begin{Bmatrix} n \\ k \end{Bmatrix}$.

2. Using story proof, as $\begin{Bmatrix} n+1 \\ k \end{Bmatrix}$ is defined as the number of ways to partition 1, 2,...,n+1 th student into k groups, then consider about the n+1 th student, firstly assign it into a group, then n students and k groups are left. We need to consider the number of students that are not in the same group with the $n+1$ th student, as it is required that groups are non-empty, then the maximum number is n, the minmium number is k. Suppose the number of students that are not in the same group with the n + 1 th student is j, then $j \in [k, n]$, and for each j, we need to choose the j students that not in the same group with the n+1 th student, which has $\binom{n}{j}$ situations. As partition the j students into k groups has the number of ways of $\begin{Bmatrix} j \\ k \end{Bmatrix}$, then for each j, there is number of ways of $\binom{n}{j}\begin{Bmatrix} j \\ k \end{Bmatrix}$. As $j \in [k, n]$, then the total number of ways is $\sum_{j=k}^{n} \binom{n}{j}\begin{Bmatrix} j \\ k \end{Bmatrix}$. So we will have that

$$\sum_{j=k}^{n} \binom{n}{j}\begin{Bmatrix} j \\ k \end{Bmatrix} = \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix}.$$

Problem 2

Firstly, we need to calculate the number of all the nonrepeatwords, this can be considered as basic counting without replacement and order matters. Suppose the word length is j, then $j \in [1, 26]$, and for each j, the number of norepeated words is 26(26-1)...(26-j+1) which is the same with $\binom{26}{j} j!$, then the sum will be $\sum_{j=1}^{26} \binom{26}{j} j!$.

Secondly, we need to calculate the number of the nonrepeatwords that uses all 26 characters, that is j=26, so is $\binom{26}{26} 26! = 26!$.

Then we can calculate the probability of the nonrepeatwords that uses all 26 characters, which is $\frac{26!}{\sum_{j=1}^{26} \binom{26}{j} j!}$.

$$\begin{aligned}
\frac{26!}{\sum_{j=1}^{26} \binom{26}{j} j!} &= \frac{26!}{\binom{26}{1} 1! + \binom{26}{2} 2! + ... + \binom{26}{26} 26!} \\
&= \frac{26!}{\frac{26!}{25!} + \frac{26!}{24!} + ... + \frac{26!}{0!}} \\
&= \frac{1}{\frac{1}{25!} + \frac{1}{24!} + \frac{1}{23!} + \frac{1}{22!} + ... + \frac{1}{0!}}
\end{aligned}.$$

According to the taylor expands of e^x is $e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + ...$, by set $x = 1$ we will get $e = 1 + \frac{1}{1!} + \frac{1}{2!} + ... = \sum_{n=0}^{\infty} \frac{1}{n!}$. As this series converges quickly, then we will have the probability that it uses all 26 letters is very close to $\frac{1}{e}$.

Problem 3

1. As a valid curriculum consists of 4 lower level courses and 3 higher level courses, then we need to choose 4 low level courses from $\{L_1, L_2, ..., L_8\}$, and choose 3 higher level courses from $\{H_1, H_2, ..., H_10\}$. Firstly 4 lower level courses takes possibilities of $\binom{8}{4}$, then 3 higher level courses takes possibilities of $\binom{10}{3}$.

   By using the multiplication rule, the total possibile number of different curriculum is $\binom{8}{4}\binom{10}{3} = 8400$.

2. We can consider 3 situations of the higher level courses:

   (a) Take courses of $\{H_1, H_2, ..., H_5\}$, not take courses of $\{H_6, H_7, ..., H_{10}\}$.
   Then we need to take L_1 and choose 3 courses from $\{L_2, L_3, ..., L_8\}$, the number of ways is $\binom{7}{3}$.

   We also need to take 3 courses from $\{H_1, H_2, ..., H_5\}$, the number of ways is $\binom{5}{3}$. Using the multiplication rule, the total number of ways under this situations is $\binom{7}{3}\binom{5}{3}$.

   (b) Not take courses of $\{H_1, H_2, ..., H_5\}$, take courses of $\{H_6, H_7, ..., H_{10}\}$.
   Then we need to take L_2 and L_3 and choose 2 courses from $\{L_1, L_4, ..., L_8\}$, the number of ways is $\binom{6}{2}$. We also need to take 4 courses from $\{H_6, H_7, ..., H_{10}\}$, the number of ways is $\binom{5}{4}$. Using the multiplication rule, the total number of ways under this situations is $\binom{6}{2}\binom{5}{3}$.

   (c) Take courses of $\{H_1, H_2, ..., H_5\}$, take courses of $\{H_6, H_7, ..., H_{10}\}$.
   Then we need to take L_1 and L_2 and L_3 and choose 1 courses from $\{L_4, L_5, ..., L_8\}$, the number of ways is $\binom{5}{1}$. We also need to take 3 courses both from $\{H_1, H_2, ..., H_5\}$ and $\{H_6, H_7, ..., H_{10}\}$, we can first take 1 course in the $(H_1, H_2, ..., H_5)$ and then take 2 courses from $(H_6, H_7, ..., H_{10})$, then we swap the choice, choose 2 courses in the first, 1 course in the second. Using the multiplication rule, the total number of ways under this situations is $\binom{5}{1}\binom{5}{1}\binom{5}{2} + \binom{5}{1}\binom{5}{2}\binom{5}{1}$.

So, in total, the total possibile number of different curriculum is $\binom{7}{3}\binom{5}{3} + \binom{6}{2}\binom{5}{3} + \binom{5}{1}\binom{5}{1}\binom{5}{2} + \binom{5}{1}\binom{5}{2}\binom{5}{1} = 1000$.

Problem 4

1. Consider the opposite of the event that there is at least one birthday match, which is no ones' birthday match. To satisfy this situation, we need to choose k days, then assign these days to the k people, then as for choosen k days, the probability is $k!p_1p_2...p_k$. According to the definition of $e_k(x_1, x_2, ..., x_n)$, the probability of no ones birthday match is $k!e_k(\mathbf{p})$. Then the probability of at least one birthday match is $1 - k!e_k(\mathbf{p})$

2. Simple case:
   Consider change the days of a year, by consider there is 2 days in a year, and define the probability of birthday on the 2 days is p_i, p_j, then the probability of there is at least one birthday match is $p_i^2 + p_j^2$, as $p_i + p_j = 1$, then the probability equals $p_i^2 + (1 - p_i)^2 = 1 - 2p_i + 2p_i^2$, using the properity of quadratic function, by setting $p_i = p_j = \frac{1}{2}$, we will make the probability minmium.
   Extreme case:
   Consider re-assign the probability of birth at some days, as for a j = k, let $p_k = 1$, then the probability of at least one birthday match must be 1.
   Therefore, by consider this 2 examples, the probability of at least one birthday match will be minmium when $p_1 = p_2 = ... = p_{365} = \frac{1}{365}$. If makes one $p_j > \frac{1}{365}$, then there is bigger probability of birth at the j th day, and less probability at another day. As more probability birth at j th day, then there is also bigger probability for at least one birthday match. Therefore by minmium all the days probability to $\frac{1}{365}$, we will minmium the probability that at least one birthday match.

3. (a) Verify of the fact: As each term of $e_k(x_1, ..., x_n)$ is k elements of $x_1, x_2, ..., x_n$, consider such 4 cases: Firstly, the term contains both x_1, x_2, then the probability is $x_1x_2e_{k-2}(x_3, ..., x_n)$. Secondly, the term only contains x_1, the probability is $x_1e_{k-1}(x_3, ..., x_n)$. Thirdly, the term only contains x_2, the probability is $x_2e_{k-1}(x_3, ..., x_n)$. Fourthly, the term not contains x_1 and x_2, the probability is $e_k(x_3, ..., x_n)$. By adding these situations together, we will get the fact that $e_k(x_1, ..., x_n) = x_1x_2e_{k-2}(x_3, ..., x_n) + (x_1 + x_2)e_{k-1}(x_3, ..., x_n) + e_k(x_3, ..., x_n)$

   (b) By using the arithmetic mean-geometric mean inequality, and $r_1 = r_2 = (p_1 + p_2)/2$, we will get that $\frac{p_1 + p_2}{2} \geq \sqrt{p_1p_2}$, then $(\frac{p_1 + p_2}{2})^2 \geq p_1p_2$, as $r_1r_2 = (\frac{p_1 + p_2}{2})^2$, we will get that $r_1r_2 \geq p_1p_2$. Also, as $r_1 = r_2 = (p_1 + p_2)/2$, we will get $r_1 + r_2 = p_1 + p_2$. As we get the fact that $e_k(x_1, ..., x_n) = x_1x_2e_{k-2}(x_3, ..., x_n) + (x_1 + x_2)e_{k-1}(x_3, ..., x_n) + e_k(x_3, ..., x_n)$, we will get that
   $$\begin{aligned} P(\text{at leas one birthday match} \mid \mathbf{p}) &= 1 - k!e_k(p_1, ..., p_{365}) \\ &= 1 - k![p_1p_2e_{k-2}(p_3, ..., p_{365}) + (p_1 + p_2)e_{k-1}(p_3, ..., p_{365}) + e_k(p_3, ..., p_n)] \end{aligned}$$
   As we have that $r_3 = p_3, r_4 = p_4, ..., r_{365} = p_{365}$
   $$\begin{aligned} P(\text{at leas one birthday match} \mid \mathbf{r}) &= 1 - k!e_k(r_1, ..., r_{365}) \\ &= 1 - k![r_1r_2e_{k-2}(r_3, ..., r_{365}) + (r_1 + r_2)e_{k-1}(r_3, ..., r_{365}) + e_k(r_3, ..., r_n)] \\ &\leq 1 - k![p_1p_2e_{k-2}(p_3, ..., p_{365}) + (p_1 + p_2)e_{k-1}(p_3, ..., p_{365}) + e_k(p_3, ..., p_n)] \end{aligned}$$
   So we can get that $P(\text{at leas one birthday match} \mid \mathbf{p}) \geq P(\text{at leas one birthday match} \mid \mathbf{r})$.
   As for the properity of arithmetic mean-geometric mean bound, the inequality only equals when $x = y$, the same for this inequation, the inequation equals only when $p_1 = p_2$, as $r_1 = r_2$ this condition also means that $r_1 = r_2 = p_1 = p_2$, which is $\mathbf{p} = \mathbf{r}$. So this inequation is strict inequality if $\mathbf{p} \neq \mathbf{r}$.

   (c) By using the inequation, suppose that there exists a vector $\mathbf{p}$ that made the probability that at least one birthday match minmium, but $\mathbf{p}$ satisfy that exists m, n that m, n$\in [1, 365]$, and m $\neq$ n, $p_m \neq p_n$, consider them as p_1, p_2, we can construct a vector $\mathbf{r}$ that $r_1 = r_2 = (p_1 + p_2)/2$, according to the inequation, $P(\text{at leas one birthday match} \mid \mathbf{p}) > P(\text{at leas one birthday match} \mid \mathbf{r})$, we find that $\mathbf{p}$ don't make the probability minmium, so is contradictory. Therefore the value of $\mathbf{p}$ that minmizes the probability of at last one birthday match is given by $p_j = \dfrac{1}{365}$ for all j.

Problem 5

1. Consider choose 2 students without replacement from many students of $\{H_1, H_2, ..., H_{n+1}\}$ which has n+1 students in total, order not matters, then the number of ways is $\binom{n+1}{2}$.
   Then consider for each student: if we choose the first student, then we have n students left, and we need to choose 1 student from them, the number of ways is $\binom{n}{1}$. If we choose the second student, as the first student has been considered, then we need to choose 1 student from the left $n-1$ students, the number of ways is $\binom{n-1}{1}$. If we choose the third student, then we have $n-2$ students left, and we need to choose 1 student from them, the number of ways is $\binom{n-2}{1}$. The same for other students, but as for the $n+1$ th students, there is no other students to choose, so is 0. So add these together, we will get the total number of choices is $0+\binom{1}{1}+\binom{2}{1}+...+\binom{n}{1} = 1+2+...+n$. As mentioned before, the number of ways is $\binom{n+1}{2}$, so we get $1+2+...+n = \binom{n+1}{2}$.

2. (a) Story proof:
   Consider such a situation, there is a bag with n+1 objects, labeled 0, 1, 2, 3, ..., n, then as for j th student ($j \in [1, n]$), we make pair by choosing 3 objects from the bag with replacement whose labels must less than the j th object's label, then for each object, we will make pairs of four objects. So as for each object, the number of pairs is $1^3, 2^3, 3^3, ..., n^3$, the total number of pairs is $1^3+2^3+3^3+...+n^3$ (from label 1 to n). Then we consider it in the following situations:

      i. no same object, then we need to choose 4 labels from bag, so the number of ways is $\binom{n+1}{4}$, but except for the biggest label object, we need to consider of the other objects' order in pair, which has 6 ways of permutation. So the number of ways is $6\binom{n+1}{4}$.

      ii. 2 same objects, then we need to choose 3 labels from bag, then the number of ways is $\binom{n+1}{3}$, but except for the biggest label object, we need to consider of the other objects' order in pair, which has 3 ways of permutation, also by swap whose lable is same has 2 cases, for example: $\{1,1,2\} and \{2,2,1\}$, so this total number of ways is $6\binom{n+1}{3}$.

      iii. 3 same objects, then we need to choose 2 labels from bag, then the number of ways is $\binom{n+1}{2}$, there is only 1 permutation, so the number of ways is $\binom{n+1}{2}$.

   By adding these situations together, we will get that:

$$1^3+2^3+3^3+, ..., +n^3 = 6\binom{n+1}{4} + 6\binom{n+1}{3} + \binom{n+1}{2}.$$

   (b) Basic algebra for the equation:
   as we get $1+2+...+n = \binom{n+1}{2}$ in the first part of this problem, by making square of this equation, we will get: $(1+2+...+n)^2 = \left(\frac{(n+1)!}{2!(n-1)!}\right)^2 = \frac{(n+1)^2n^2}{4}$. We also get that

$1^3 + 2^3 + ... + n^3 = 6\binom{n+1}{4} + 6\binom{n+1}{3} + \binom{n+1}{2}$ in this problem, consider for the right side of the equation, we get that:

$$\begin{aligned}
6\binom{n+1}{4} + 6\binom{n+1}{3} + \binom{n+1}{2} &= 6\frac{(n+1)!}{4!(n-3)!} + 6\frac{(n+1)!}{3!(n-2)!} + \frac{(n+1)!}{2!(n-1)!} \\
&= \frac{(n+1)n(n-1)(n-2)}{4} + (n+1)n(n-1) + \frac{(n+1)n}{2} \\
&= (n+1)n(\frac{(n-1)(n-2)}{4} + (n-1) + \frac{1}{2}) \\
&= (n+1)n\frac{n^2 - 3n + 2 + 4n - 4 + 2}{4} \\
&= (n+1)n\frac{n^2 + n}{4} \\
&= \frac{(n+1)^2 n^2}{4}
\end{aligned}$$

So we get that $1^3 + 2^3 + ... + n^3 = (1 + 2 + ... + n)^2$.