2023 Spring Probability & Statistics for EECS March 5, 2023

Homework 04

Professor: Ziyu Shao & Dingzhu Wen *Due: 23:59 on March 12, 2023*

1. Consider the original Monty Hall problem, except that Monty enjoys opening door 2 more than he enjoys opening door 3, and if he has a choice between opening these two doors, he opens door 2 with probability p, where $\frac{1}{2} \leq p \leq 1$.

   To recap: there are three doors, behind one of which there is a car (which you want), and behind the others there are goats (which you don't want). Initially, all possibilities are equally likely for where the car is. You choose a door, which for concreteness we assume is door 1. Monty (knows which door has the car) then opens a door to reveal a goat, and offers you the option of switching.

   (a) Find the unconditional probability that the strategy of always switching succeeds (unconditional in the sense that we do not condition on which of doors 2 or 3 Monty opens).

   (b) Find the probability that the strategy of always switching succeeds, given that Monty opens door 2 (assume we always choose door 1 first).

   (c) Find the probability that the strategy of always switching succeeds, given that Monty opens door 3 (assume we always choose door 1 first).

2. (a) Is there a discrete distribution with support $\{1, 2, 3, \dots\}$, such that the value of the PMF at n is proportional to $1/n$?

   (b) Is there a discrete distribution with support $\{1, 2, 3, \dots\}$, such that the value of the PMF at n is proportional to $1/n^2$?

3. Let X be a random day of the week, coded so that Monday is 1, Tuesday is 2, *etc.* (so X takes values $1, 2, \dots, 7$ with equal probabilities). Let Y be the next day after X. Do X and Y have the same distribution? What is $P(X < Y)$?

4. There are two coins, one with probability p_1 of Heads and the other with probability p_2 of Heads. One of the coins is randomly chosen (with equal probabilities for the two coins). It is then flipped $n \geq 2$ times. Let X be the number of times it lands Heads.

   (a) Find the PMF of X.

   (b) What is the distribution of X if $p_1 = p_2$?

   (c) Give an intuitive explanation of why X is not Binomial for $p_1 \neq p_2$.

5. For x and y binary digits (0 or 1), let $x \bigoplus y$ be 0 if $x = y$ and 1 if $x \neq y$ (this operation is called exclusive or (often abbreviated to XOR), or addition mod 2).

   (a) Let $X \sim \text{Bern}(p)$ and $Y \sim \text{Bern}(1/2)$, independently. What is the distribution of $X \bigoplus Y$?

   (b) With notation as in sub-problem (a), is $X \bigoplus Y$ independent of X? Is $X \bigoplus Y$ independent of Y? Be sure to consider both the case $p = 1/2$ and the case $p \neq 1/2$.

   (c) Let $X_1, \ldots, X_n$ be i.i.d. (*i.e.*, independent and identically distributed) Bern(1/2) R.V.s. For each nonempty subset J of $\{1, 2, \ldots, n\}$, let

   $$Y_J = \bigoplus_{j \in J} X_j.$$

   Show that $Y_J \sim \text{Bern}(1/2)$ and that these $2^n - 1$ R.V.s are pairwise independent, but not independent.