Probability & Statistics for EECS: Homework #02

Due on Feb 26, 2023 at 23:59

Name: **Penghao Wang**
Student ID: 2021533138

Problem 1

1. As there the definiton of the bootstrap is a sequence formed from the a_j's sampling with replacement, then as for $(a_1, ..., a_n)$, the number of possible bootstrap samples is $n * n * ... * n = n^n$.

2. As order does not matter, then we need to consider the choose times of each numbers, note that the sum of total choose times of all the numberes is n. Then this problem can be considered as partition of n into k parts and the number of ways to partition n into k parts is $\binom{n+k-1}{k-1}$. Here we consider that $k = n$, so the total number is $\binom{2n-1}{n-1}$.

3. (a) Consider such 2 conditions: $b_1 = (a_1, a_2, ..., a_n)$, $b_2 = (a_1, a_1, ..., a_1)$. As order does not matter, then there are $n!$ cases of b_1, but as for b_1, there is only 1 cases. So we find the 2 samples and shows that not all unordered bootstrap samples are equally likely.

   (b) As defined below, we can find that the probability p_1 of b_1 is $\dfrac{n!}{n^n}$, and the probability p_2 of b_2 is $\dfrac{1}{n^n}$. Then we can calculate $p_1/p_2 = n!$.

   (c) As for the case b_1, there is only one case. But as for the case b_2, there is n cases as there is totally n numbers. Then the ratio is $\dfrac{\frac{n!}{n^n}}{\frac{n}{n^n}} = \dfrac{n!}{n}$, so the ratio is $(n-1)!$.

Problem 2

As for all the cases, there is totally 108^n cases. Then we define that event A_i is that n boxes do not include 1 type coupons, event C is that choose all 108 types of coupons. Then we then use the inclusion-exclusion formula, we have $P(C^c) = P(\cup_{i=1}^{108} A_i) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + ... + (-1)^{109} P(A_1 \cap ... \cap A_n) = \frac{\binom{108}{1} 107^n}{108^n} - \frac{\binom{108}{2} 106^n}{108^n} + ... + \frac{\binom{108}{107} 1^n}{108^n} - \frac{0^n}{108^n} = \frac{\sum_{i=1}^{107} \binom{108}{i} (-1)^{i+1}(108-i)^n}{108^n}$. Then we have $P(C) = 1 - \frac{\sum_{i=1}^{107} \binom{108}{i} (-1)^{i+1}(108-i)^n}{108^n}$. As shown in the following figure, we can conclude that the when such probability is no less than 95%, the minimum number of n is 823.

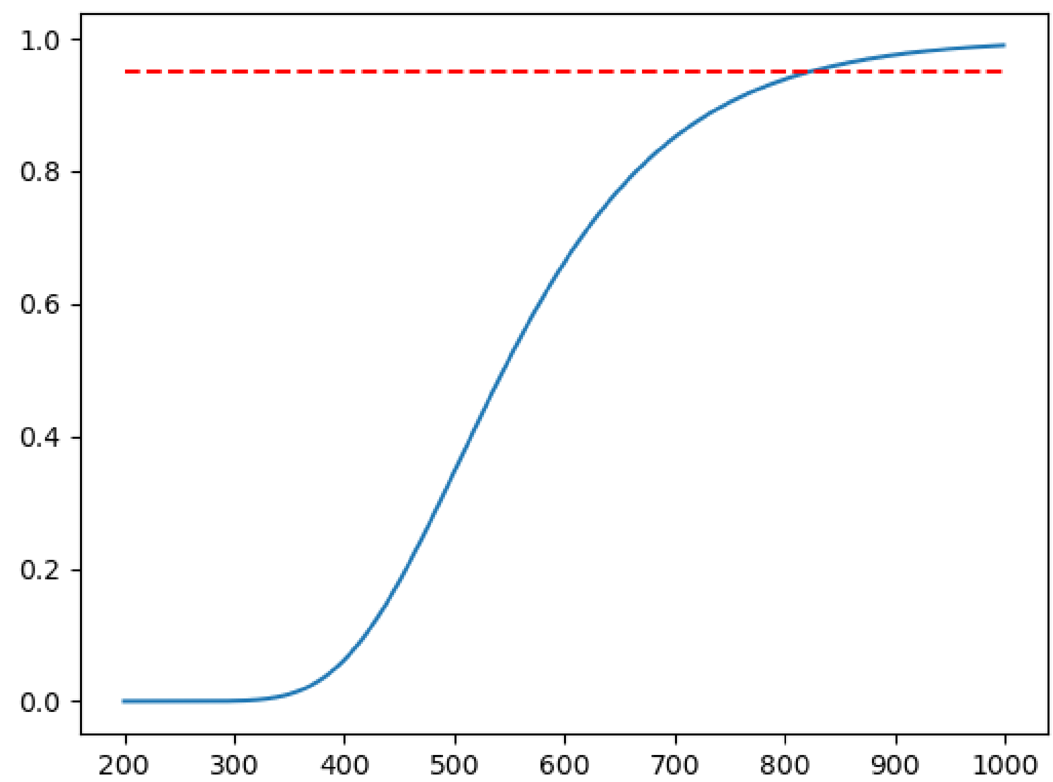

Figure 1: The probability of choose all 108 types of coupons, n range from 200 to 1000

Problem 3

Firstly choose four objects from the 100 garage kits, as there is 5 defectives, then the probability of accepted is choose the four kits from the 95 kits, that is $P = \dfrac{\dbinom{95}{4}}{\dbinom{100}{4}} = \dfrac{636709}{784245}$

Problem 4

Define event A is first time draw a gold coin, define event B is at the second draw, draw a gold coin, define C_0 as the event of take the box a, C_1 as the box b, C_2 as the box c. Then the probability $P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B \cap A)}{P(A|C_0) + P(A|C_1) + P(A|C_2)} = \frac{\frac{1}{3}}{\frac{1}{3} + 0 + \frac{1}{3} * \frac{1}{2}} = \frac{2}{3}$

Problem 5

1. As when Mirana play bold, Mirana won't draw, she wins with probability p_w, then loses with probability $(1 - p_w)$, then there are 2 cases:

   (a) Win both in games 1 and 2, then the probability is $p_w * p_w = p_w^2$.

   (b) Win a game and lose a game, then win in the third game, then the probability is $2 * p_w * (1 - p_w) * p_w = 2p_w^2(1 - p_w)$.

   Then the total probability is $p_w^2 + 2p_w^2(1 - p_w) = p_w^2(3 - 2p_w)$.

2. As when Mirana play timid, Mirana won't win, she draws with probability p_d, then loses with probability $(1 - p_d)$, then there is only 1 case:
   Draw both in games 1 and 2, then at games 3, Mirana play bold, and wins. The probability is $p_d * p_d * p_w = p_d^2 p_w$.

3. As when Mirana play timid, Mirana won't win, she draws with probability p_d, then loses with probability $(1 - p_d)$, and firstly Mirana plays bold then there are 3 case:

   (a) Win in game 1, draw in game 2. The probability is $p_w * p_d = p_w p_d$.

   (b) Win in game 1. lose in game 2, then win in game 3. Then the probability is $p_w * (1 - p_d) * p_w = p_w^2 - p_w^2 p_d$.

   (c) Lose in game 1, win in game 2, then win in game 3. Then the probability is $(1 - p_w) * p_w * p_w = p_w^2 - p_w^3$.

   Add the three cases together, we will get that Mirana wins the match with probability $(p_w p_d) + (p_w^2 - p_w^2 p_d) + (p_w^2 - p_w^3) = p_w p_d + 2p_w^2 - p_w^2 p_d - p_w^3$.

4. With the strategy in (c) above, we will get that, Mirana wins the match with probability $p_w p_d + 2p_w^2 - p_w^2 p_d - p_w^3$. Then we analyse the probability: $p_w p_d + 2p_w^2 - p_w^2 p_d - p_w^3$. To make the probability big than 1/2, we can calculate the probability by give an example of p_w, p_d. Take $p_w = \frac{9}{20}$, $p_d = \frac{19}{20}$, we will get that the win probability is $\frac{549}{1000}$, where have a better than a 50-50 chance to win the match. Intuitively, we can make partial derivative for p_d, then we will get that $p_w - p_w^2$, as $0 < p_w < \frac{1}{2}$, then we will get that the derivative bigger than 0, so the probability will grow bigger as p_d grow bigger. As we have get a example that make the probability bigger than $\frac{1}{2}$, so there may have the chance. Also, as Mirana plays timid when she get ahead, and bold when she get behind, which increase the probablity of Mirana wins, when ahead, she play timid to decrease the probablity to fall behind, when behind, plays bold to increase the probablity to win in order to get ahead, so the probability will may bigger than $\frac{1}{2}$.