Probability & Statistics for EECS: Homework #03

Due on Mar 5, 2023 at 23:59

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Problem 1

Consider such functional situations, differed by the number of function ways:

1. 2 devices functions: then there are: 1->3; 2->4; The probability is $2p^2(1-p)^3$
2. 3 devices functions: then there are: 132/4/5; 241/3/5; 154; 253; The probability is $8p^3(1-p)^2$
3. 4 devices functions: then there are: 1345, 2345, 1234, 1253, 1245, the probability is $5p^4(1-p)$.
4. 5 devices functions: then there is: 12345, the probability is p^5.

Add there together, we get the probability that this system functions is $p^5 + 5p^4(1-p) + 8p^3(1-p)^2 + 2p^2(1-p)^3 = 2p^5 - 5p^4 + 2p^3 + 2p^2$.

Problem 2

1. Intuitive explatioon:
   By simply all the people, being good at Genshin Impact is independent of being good at Apex, however, by consider the people that are admitted to the club, as they are admitted when being good at Genshin Impact, then being good at Apex is dependent on being good at Genshin Impact or Apex, then now assume a admitted people who is not good at Genshin Impact, then the people must be good at Apex as the people is admitted. So with the additional condition (among the admitted players), being good at Apex is dependent on being good at Genshin Impact.

2. Firstly consider the inequation's left side $P(A|B \cap C)$, as A and B are independent and $C = A \cup B$, then we have that $B \cap C = B$, so $P(A|B \cap C) = P(A|B) = P(A)$. Then consider the right sider of the inequation, $P(A|C) = \dfrac{P(A \cap C)}{P(C)} = \dfrac{P(A)}{P(C)}$, as $0 < P(C) < 1$, then we have $P(A|C) = \dfrac{P(A)}{P(C)} > P(A)$, so we get that $P(A|B \cap C) < P(A|C)$.

Problem 3

1. To get p_n recursively, we can consider the first roll (as the running total is ever exactly n), then there is 6 types of the first roll: (1, 2, 3, 4, 5, 6). Then we can get the $p_n = \frac{1}{6}p_{n-1} + \frac{1}{6}p_{n-2} + \frac{1}{6}p_{n-3} + \frac{1}{6}p_{n-4} + \frac{1}{6}p_{n-5} + \frac{1}{6}p_{n-6}$. Then we also need to define the base case, $p_0 = 1$ as we can when 0 time roll, we always get 0 in total, $p_k = 0 (k < 0)$, as the total is negative makes no sense.

2. As we get the recursively equation of p_n, then $p_7 = \frac{1}{6}p_6 + \frac{1}{6}p_5 + \frac{1}{6}p_4 + \frac{1}{6}p_3 + \frac{1}{6}p_2 + \frac{1}{6}p_1$. Then we need to get the value of p_1 to p_6.

   (a) $p_1 = \frac{1}{6}p_0 + \frac{1}{6}p_{-1} + \frac{1}{6}p_{-2} + \frac{1}{6}p_{-3} + \frac{1}{6}p_{-4} + \frac{1}{6}p_{-5} = \frac{1}{6}$

   (b) $p_2 = \frac{1}{6}p_1 + \frac{1}{6}p_0 + \frac{1}{6}p_{-1} + \frac{1}{6}p_{-2} + \frac{1}{6}p_{-3} + \frac{1}{6}p_{-4} = \frac{1}{6}(\frac{1}{6} + 1)$

   (c) $p_3 = \frac{1}{6}p_2 + \frac{1}{6}p_1 + \frac{1}{6}p_0 + \frac{1}{6}p_{-1} + \frac{1}{6}p_{-2} + \frac{1}{6}p_{-3} = \frac{1}{6}(\frac{1}{6} + 1)^2$

   (d) $p_4 = \frac{1}{6}p_3 + \frac{1}{6}p_2 + \frac{1}{6}p_1 + \frac{1}{6}p_0 + \frac{1}{6}p_{-1} + \frac{1}{6}p_{-2} = \frac{1}{6}(\frac{1}{6} + 1)^3$

   (e) $p_5 = \frac{1}{6}p_4 + \frac{1}{6}p_3 + \frac{1}{6}p_2 + \frac{1}{6}p_1 + \frac{1}{6}p_0 + \frac{1}{6}p_{-1} = \frac{1}{6}(\frac{1}{6} + 1)^4$

   (f) $p_6 = \frac{1}{6}p_5 + \frac{1}{6}p_4 + \frac{1}{6}p_3 + \frac{1}{6}p_2 + \frac{1}{6}p_1 + \frac{1}{6}p_0 = \frac{1}{6}(\frac{1}{6} + 1)^5$

   Add them together, we will get that $p_7 = \frac{1}{6}(\frac{1}{6} + \frac{1}{6}(\frac{1}{6}+1) + ... + \frac{1}{6}(\frac{1}{6}+1)^5) = \frac{1}{36}\frac{1-(\frac{7}{6})^6}{1-\frac{7}{6}} = \frac{1}{6}((\frac{7}{6})^6 - 1)$

3. Intuitive explanation:
   As there is 6 types of each roll (1, 2, 3, 4, 5, 6), then each time we add $\frac{1+2+3+4+5+6}{6} = \frac{21}{6} = \frac{7}{2}$. Then as for a n, every 2 rolls, we get that 7 added on the total number, then consider on the total number, as in every 7 numbers, there are 2 rolls that get the number. Then we get the probability of $p_n = 2/7$ when $n \to \infty$.

Problem 4

1. $p_{i,j}$ can be get when $p_{i-1,j}$ continue select a toy other than the j types of toys, the probability is $\frac{n-(j-1)}{n}$, as for $p_{i-1,j}$, $p_{i,j}$ can be get by continue select one of the j types of toys, the probability is $\frac{j}{n}$. So the recursive equation is $p_{i,j} = \frac{n-j+1}{n} p_{i-1,j-1} + \frac{j}{n} p_{i-1,j}$.
2. We can calculate by firstly calculate $p_{i,1}, p_{i-1,1}, ..., p_{1,1}$, as when $j = 0$, the $p_{i,j} = 0$, then $p_{i,1} = 0 + \frac{j}{n} p_{i-1,j}$, as $p_{1,1} = 1$, then we can calculate all the $p_{i,1}, p_{i-1,1}, ..., p_{1,1}$. Then we calculate $p_{i,2}, p_{i-1,2}, ..., p_{2,2}$, as $p_{i,2} = \frac{n-1}{n} p_{i-1,1} + \frac{j}{n} p_{i-1,2}$, then we can recursively calculate the $p_{i,2}$. By continuing doing so, we will get the value of $p_{i,j}$.

Problem 5

1. As p_k is the probability that drunk ever reaches the value k, the as for p_k, there are 2 situations to reach the S_k, that is

   (a) From S_{k-1} moves one unit to the right, then the probability is $p * p_{k-1}$

   (b) From S_{k+1} moves one unit to the left, then the probability is $q * p_{k+1}$

   So, the p_k can be represented as $p_k = p * p_{k-1} + q * p_{k+1}$ for $k \geq 1$, note that $p_0 = 1$.

2. Firstly we give such definition: as the drunk man will reaches the negative positions, then we can define events C_j that $j \in R$, which means that the drunk man reaches k before ever reaching -j. Then we will get that $A_j \subseteq A_{j+1}$ as if Link manage to reach -j-1, then he need to reach -j firstly. Then, by using C_j, we will get that $\cup_{j=1}^{\infty} C_j$ be Link ever reaches k. Then we can present the p_k by using $\cup_{j=1}^{\infty} C_j$. We can firstly get the C_j by using the equation.

$$
\begin{aligned}
p_k &= p * p_{k-1} + q * p_{k+1} \\
\frac{p_{k+1} - p_k}{p_k - p_{k-1}} &= \frac{q}{p} \\
\frac{p_{k+1} - p_k}{p_k - p_{k-1}} &= \frac{q}{p}
\end{aligned}
$$

   Then we define that, $D_k = p_{k+1} - p_k$. Then we will get that: $\frac{D_k}{D_{i-1}} = \frac{q}{p}$. So we will get that $\frac{D_{N-1}}{D_{N-2}} = \frac{q}{p}$, ... , $\frac{D_2}{D_1} = \frac{q}{p}$. So $D_{N-1} = D_1(\frac{q}{p})^{N-2}$. Then we make sum of α, $\alpha_1 + \alpha_2 + ... + \alpha_{N-1} = P_N - P_0 = 1$. Then we get $p_i = \alpha_0 + \alpha_1 + ... + \alpha_{i-1} = \alpha_1 \frac{1 - (\frac{q}{p})^i}{1 - \frac{q}{p}}$. Then we need to elimnate the α_1, as the sum of α_i is 1, then the $p_i = \frac{1 - (\frac{q}{p})^i}{1 - (\frac{q}{p})^N}$. Note that when p $= \frac{1}{2}$, we will get that:

$$
\lim_{\frac{q}{p} \to 1} \frac{1 - (\frac{q}{p})^i}{1 - (\frac{q}{p})^N} = \frac{i}{N}.\text{Using L'hospital therom.}
$$

   Then consider the $\cup_{j=1}^{\infty} C_j$, that is let $j \to \infty$. Consider 3 cases:

   (a) p $= \frac{1}{2}$, then we will get that $P(A_j) = \frac{j}{j+k} \to 1$ when $j \to \infty$

   (b) p $> \frac{1}{2}$, then we will get that $P(A_j) = \frac{1 - (\frac{q}{p})^j}{1 - (\frac{q}{p})^{j+k}} \to 1$ when $j \to \infty$

   (c) p $< \frac{1}{2}$, then we will get that $P(A_j) = \frac{1 - (\frac{q}{p})^j}{1 - (\frac{q}{p})^{j+k}} \to (\frac{p}{q})^k$ when $j \to \infty$