Probability & Statistics for EECS: Homework #09

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Problem 1

(a) As for the case X, Y are discrete, we have that use the definition of conditioning probability

$$P(Y=y|X=x)=\frac{P(X=x,Y=Y)}{P(X=x)}$$

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So that we have that

$$P(Y=y|X=x)=\frac{P(X=x,Y=Y)}{P(X=x)}=\frac{P(X=x|Y=y)P(Y=y)}{P(X=x)}$$

(b) As for the case X, Y are continuous, we have that use the definition of conditioning probability

$$f_{Y|X}(y|x)=\frac{f_{Y,X}(y,x)}{f_X(x)}$$

$$f_{X|Y}(x|y)=\frac{f_{X,Y}(x,y)}{f_Y(y)}$$

So that we have that

$$f_{Y|X}(y|x)=\frac{f_{Y,X}(y,x)}{f_X(x)}=\frac{f_{X|Y}(x|y)f_Y(y)}{f_X(x)}$$

(c) As for the case X is discrete, Y is continuous, we have that use the definition of conditioning probability Denote that ε is a small number, then we have that

$$P[Y\in(y-\varepsilon,y+\varepsilon)|X=x]=\frac{P[X=x|Y\in(y-\varepsilon,y+\varepsilon)]P[Y\in(y-\varepsilon,y+\varepsilon)]}{P(X=x)}$$

$$\begin{aligned}f_Y(y|X=x)&=\lim_{\varepsilon\to 0}\frac{P[Y\in(y-\varepsilon,y+\varepsilon)|X=x]}{2\varepsilon}\\&=\lim_{\varepsilon\to 0}\frac{P[X=x|Y\in(y-\varepsilon,y+\varepsilon)]\frac{P[Y\in(y-\varepsilon,y+\varepsilon)]}{2\varepsilon}}{P(X=x)}\\&=\frac{P(X=x|Y=y)f_Y(y)}{P(X=x)}\end{aligned}$$

So that we have that

$$f_Y(y|X=x)=\frac{P(X=x|Y=y)f_Y(y)}{P(X=x)}$$

(d) As for the case X is continuous, Y is discrete, we have that use the definition of conditioning probability Denote that ε is a small number, then we have that

$$\begin{aligned}\lim_{\varepsilon\to 0}P(Y=y|X\in(x-\varepsilon,x+\varepsilon))&=lim_{\varepsilon\to 0}\frac{P[X\in(x-\varepsilon,x+\varepsilon)|Y=y]P(Y=y)}{P[X\in(x-\varepsilon,x+\varepsilon)]}\\&=lim_{\varepsilon\to 0}\frac{\frac{P[X\in(x-\varepsilon,x+\varepsilon)|Y=y]}{2\varepsilon}P(Y=y)}{\frac{P[X\in(x-\varepsilon,x+\varepsilon)]}{2\varepsilon}}\\&=\frac{f_X(x|Y=y)P(Y=y)}{f_X(x)}\end{aligned}$$

So we get that $P(Y=y|X=x)=\dfrac{f_X(x|Y=y)P(Y=y)}{f_X(x)}$

Problem 2

(a) As for the joint PMF of X, Y, N, we have that the PMF is $P(X = x, Y = y, N = n)$, then as we have that $N = X + Y$, then only when $x + y = n$, will the PMF be non-zero. So we have that

$$P(X = x, Y = y, N = n) = P(X = x, Y = y) = (1 - p)^x * p * (1 - p)^y * p = (1 - p)^{x+y} p^2$$

, as we have that $x + y = n$, so we get that

$$P(X = x, Y = y, N = n) = (1 - p)^n p^2$$

(b) As for the joint PMF os X, N, we have that the PMF is $P(X = x, N = n)$, as only when $n = x + y$ will the PMF be non-zero, so we have that

$$P(X = x, N = n) = P(X = x, Y = n - x) = (1 - p)^x p(1 - p)^{n-x} p = (1 - p)^n p^2$$

(c) As for the conditional PMF of X given $N = n$, we have that the PMF is

$$P(X = x|N = n) = \frac{P(X = x, N = n)}{P(N = n)}.$$

The numerator is the joint PMF of X and N, which is $P(X = x, N = n) = (1 - p)^n p^2$, and the denominator is PMF of N, which is $P(N = n) = \sum_{x=0}^{n} (1 - p)^n p^2 = (n + 1)(1 - p)^n p^2$, so we have that

$$P(X = x|N = n) = \frac{P(X = x, N = n)}{P(N = n)} = \frac{(1 - p)^n p^2}{(n + 1)(1 - p)^n p^2} = \frac{1}{n + 1}.$$

where x = 0, 1, 2, ..., n.
Description: The conditional PMF of X given $N = n$ is a uniform distribution, which is $P(X = x|N = n) = \frac{1}{n+1}$. The event $P(X = x)$ is a Geom distribution, while the event $N = n$ is actuallu a negative binomial distribution, which denote the fail times before the second success. So the conditional PMF of X given $N = n$ is $\frac{1}{n+1}$, which denote that the first success between the first and the second success is uniformly distributed.

Problem 3

(a) To verify that the conditional distribution of X given X > c is the same as the distribution of c + X, firstly we can find the corresponding CDF of X given X > c, which is $P(X \leq x|X > c) = \dfrac{P(c < X \leq x)}{P(X > c)} = \dfrac{F(x) - F(c)}{1 - F(c)}$. As $X \sim Expo(\lambda)$, so we have $F(x) = 1 - e^{-\lambda x}$. So, the $P(X \leq x|X > c) = \dfrac{e^{-\lambda c} - e^{-\lambda x}}{e^{-\lambda c}} = 1 - e^{-\lambda(x-c)}$.
As for the CDF of c + X, we have that $P(c + X \leq x) = P(X \leq x - c) = 1 - e^{-\lambda(x-c)}$. So we have that $P(X \leq x|X > c) = P(c+X \leq x)$. So that the conditional CDF of X given X > c is the same as the c + X.

(b) As for the CDF of X given X < c, we have that for x < c, $P(X \leq x|X < c) = \dfrac{P(X \leq x, X < c)}{P(X < c)} = \dfrac{P(X \leq x)}{P(X < c)} = \dfrac{1 - e^{-\lambda x}}{1 - e^{-\lambda c}}$ As for the PDF, we have that $f(x|X < c) = (P(X \leq x|X < c))' = (\dfrac{P(X \leq x)}{P(X < c)})' = (\dfrac{1 - e^{-\lambda x}}{1 - e^{-\lambda c}})' = \dfrac{\lambda e^{-\lambda x}}{1 - e^{-\lambda c}}$ for x < c, as for $x \geq c$, PDF is zero.

Problem 4

As we have that U_1, U_2, U_3 be i.i.d. Unif(0, 1), and let $L = min(U_1, U_2, U_3)$, $M = max(U_1, U_2, U_3)$

(a) 1. As for the marginal CDF of M is $F_M(m) = P(M \leq m) = P(U_1 \leq m, U_2 \leq m, U_3 \leq m) = P(U_1 \leq m)P(U_2 \leq m)P(U_3 \leq m) = m^3$. For $m \in [0, 1]$

2. As for the marginal PDF of M, we have that $f_M(m) = (F_M(m))' = (m^3)' = 3m^2$. For $m \in [0, 1]$

3. As for the joint CDF of M and L, firstly we consider the event $L > l, M \leq m$, which is easy to calculate, that is $P(L > l, M \leq m) = (m-l)^3$, we have that $P(L \leq l, M \leq m) = P(M \leq m) - P(L > l, M \leq m) = m^3 - (m-l)^3$ for $m \geq l$ and that $m, l \in [0, 1]$.

4. As for the joint PDF of M and L, we have that $f(l, m) = \dfrac{\partial^2 P(L \leq l, M \leq m)}{\partial l \partial m} = 6(m - l)$ for $m, l \in [0, 1]$ and that $m \geq l$.

(b) As for the conditional PDF of M given L, firstly we have that $P(L > l) = P(U_1 \geq l, U_2 \geq l, U_3 \geq l) = (1-l)^3$. Then we have that $P(L \leq l) = 1 - P(L > l) = 1 - (1-l)^3$, then we get that $f_L(l) = 3(1-l)^2$, where $l \in [0, 1]$. So we get $f_{M|L}(m|l) = \dfrac{f(l, m)}{f_L(l)} = \dfrac{6(m-l)}{3(1-l)^2} = \dfrac{2(m-l)}{(1-l)^2}$, where $m, l \in [0, 1]$ and that $m \geq l$.

Problem 5

Firstly, we denote that $q = 1 - p$

(a) As we have that $L = min(X, Y), M = max(X, Y)$, then we have that
for $l < m$, we have $P(L = l, M = m) = P(X = l, Y = m) + P(X = m, Y = l) = q^l pq^m p + q^m pq^l p = 2p^2 q^{l+m}$
for $l = m$, we have $P(L = l, M = m) = P(X = l, Y = l) = q^l pq^l p = p^2 q^{2l}$
for $l > m$, we have $P(L = l, M = m) = P(X = l, Y = m) + P(X = m, Y = l) = 0$
So we get that the joint PMF of L and M is

$$P(L = l, M = m) = \begin{cases} 2p^2 q^{l+m} & l < m \\ p^2 q^{2l} & l = m \\ 0 & l > m \end{cases}$$

However, as we have that the joint PMF of X and Y depend on l and m's size, it is clearly that the joint can't be get by multiplying the function of l and m, so L and M are no independent.

(b) As for the marginal distribution of L

(1) By using the joint PMF, we have that

$$\begin{aligned} P(L = l) &= \sum_{m=l}^{\infty} P(L = l, M = m) \\ &= p^2 q^{2l} + \sum_{m=l+1}^{\infty} 2p^2 q^{l+m} \\ &= p^2 q^{2l} + 2p^2 q^l \sum_{m=l+1}^{\infty} q^m \\ &= p^2 q^{2l} + 2p^2 q^l q^{l+1} \left(\frac{1 - q^{\infty}}{1 - q} \right) \\ &= p^2 q^{2l} + 2p^2 q^l q^{l+1} \frac{1}{p} \\ &= p^2 q^{2l} + 2pq^{2l+1} \end{aligned}$$

(2) Also we we can use story, that is, consider 2 independent Bernuoilli independent sequence, at the time n, at least one of the 2 sequences has a success, then, we have that the probability of the event is $1 - (1 - p)^2$, so we have that $P(L = l) = (1 - (1 - q^2))^l (1 - q^2) = q^{2l}(1 - q^2)$, as we have that $(p + q)^2 = 1$, so $1 - q^2 = p^2 + 2pq$, so the $P(L = l) = q^{2l}(p^2 + 2pq) = p^2 q^{2l} + 2pq^{2l+1}$.

So, both using joint PMF and using story, can we find the marginal distribution of L.

(c) As we have that $L = min(X, Y), M = max(X, Y)$, then we have that $L + M = X + Y$, then $EL + EM = E(L + M) = E(X + Y) = E(X) + E(Y)$, as X and Y i.i.d. $Geom(p)$, we have that $E(X) = E(Y) =$

$2\frac{q}{p} = \frac{2q}{p}$ Then, we find the $E(L)$, as $P(L = l) = p^2q^{2l} + 2pq^{2l+1}$, so

$$\begin{aligned}
E(L) &= \sum_{l=0}^{\infty} lP(L = l) \\
&= (p^2 + 2pq)\sum_{l=1}^{\infty} lq^{2l} \\
q^2E(L) &= (p^2 + 2pq)\sum_{l=1}^{\infty} lq^{2l+2} \\
E(L) - q^2E(l) &= (p^2 + 2pq)\sum_{l=1}^{\infty} lq^{2l+2} \\
&= (1 - q^2)\frac{q^2}{1 - q^2} \\
E(L) &= \frac{q^2}{1 - q^2}
\end{aligned}$$

So we get that $EM = \frac{2q}{p} - EL = \frac{2q}{p} - \frac{q^2}{1 - q^2} = \frac{(1 - p)(3 - p)}{p(2 - p)}$

(d) As for the joint PMF of L and $M - L$, we have that for $k \geq 0$

$$P(L = l, M - L = k) = P(L = l, M = k + l),$$

when $k > 0$, we have

$$\begin{aligned}
P(L = l, M - L = k) &= P(X = l, Y = k + l) + P(X = k + l, Y = l) \\
&= q^l pq^{k+l}p + q^{k+l}pq^l p \\
&= 2p^2q^{2l+k}.
\end{aligned}$$

When $k = 0$, we have that $P(L = l, M - L = k) = p^2q^{2l+k}$.
As for the $P(L = l)$, we have $P(L = l) = (1 - q^2)q^{2l}$
As for $P(M - L = k)$ For $k > 0$, we have that

$$P(M - L = k) = \sum_{l=0}^{\infty} P(L = l, M = l + k)P(L = l) = \frac{2p^2q^k}{1 - q^2},$$

where $P(M - L = k)P(L = l) = P(L = l, M - L = k)$, L and $M - L$ are independent and for $k = 0$, we have that $P(M - L = k) = P(M = L) = \sum_{l=0}^{\infty} P(M = L = l) = \frac{p^2q^k}{1 - q^2}$.
So we have for k = 0, $P(L = l, M - l = k) = P(M - L = k)P(L = l)$.
In conclusion, L and M - L are independent.