Probability & Statistics for EECS: Homework #11

Due on Apr 30, 2023 at 23:59

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Problem 1

(a) $(X, Y, X+Y)$ is a MVN, as for $(X, Y, X+Y)$, we have that assume that a, b, c are parameters that $a, b, c \in R$, then we have that according to the definition of MVN, we let

$$M = aX + bY + c(X+Y) = (a+c)X + (b+c)Y.$$

Then we have that as X, Y be i.i.d. $N(0,1)$, then with the linear property of Normal distribution, we have that the M is a linear combination of 2 independent normal distribution, so for any $a, b, c \in R$, the linear combination of the X, Y has a Normal distribution. So we have that the linear combination of $X, Y, X+Y$ also has a Normal distribution.
So we have that $(X, Y, X+Y)$ is a multivariate normal distribution.

(b) $(X, Y, SX+SY)$ is not a MVN, as for $(X, Y, SX+SY)$, we could prove it is not a MVN by showing that one linear combination is not continuous. As for all parameters are 1, we have $M = X+Y+SX+SY$, then as for $P(M=0)$, as Normal distribution is a continuous distribution, then $P(M=0)=0$, then as for $M = X+Y+SX+SY$, we have that there 2 cases

  (a) $S=-1$, then as S is a random sign with equal probabilities, then the probability of $S=-1$ is $\frac{1}{2}$, then we have that under this case, $P(M=0)=\frac{1}{2}$
  (b) $X+Y=0$, then as X, Y i.i.d. $N(0,1)$, we have that the probability is 0 due to continuous variable.

Then we have that $P(M=0) = P(S=-1) + P(X+Y=0) - P(S=-1, X+Y=0) = \dfrac{1}{2} + 0 - 0 = \dfrac{1}{2}$, due to that S is independent with X, Y. So we have that this is contradictory, so we have that $(X, Y, SX+SY)$ is not a MVN.

(c) (SX, SY) is a MVN, we firstly denote that $M = aX + bY$, where $a, b \in R$, then we use the linear property of Normal distribution, we have that the M is also a Normal distribution. Then as we have $O = aSX + aSY = SM$, we then need to prove that SM is a Normal distribution. As for $o \in R$, we have by LOTP

$$P(O \le o) = P(SM \le o) = P(SM \le o|S=1)P(S=1) + P(SM \le o|S=-1)P(S=-1)$$

Then as S has equal probability of 1 and -1, we have

$$P(O \le o) = P(M \le o|S=1)\frac{1}{2} + P(-M \le o|S=-1)\frac{1}{2}$$

As S and X, Y are independent and that M is a Normal distribution, we have that

$$P(O \le o) = P(M \le o)\frac{1}{2} + P(-M \le o)\frac{1}{2} = P(M \le o)\frac{1}{2} + P(M \le o)\frac{1}{2} = P(M \le o)$$

So we have that $O = aSX + bSY = SM$ is also a Normal distribution, where $a, b \in R$. So we have that (SX, SY) is a MVN.

Problem 2

(1) Firstly we prove T and W are independent using property of MVN, we have that as X, Y i.i.d. $N(0,1)$, then as for $T = X - Y$ and $W = X - Y$, we use the property of Normal distribution, we have that as both T and W are linear combinations of Normal distribution, then T, W are also Normal distribution. We denote $H = aT + bW$, we have that

$$H = aT + bW = a(X+Y) + b(X-Y) = (a+b)X + (a-b)Y,$$

as $a, b \in R$, then $(a+b), (a-b)$ also $\in R$. So we have that H also has a Normal distribution. So that the (T, W) is a MVN. Then we attempt to use the Theorem, that if (X, Y) is Bivariate Normal and $Corr(X,Y) = 0$, then X and Y are independent. From the above proof, we already have that (T, W) is a MVN, then we also have that they are bivariate Normal. Then as for $Corr(T, W)$, we have that as for the $Cov(T, W)$, we have that

$$Cov(T, W) = Cov(X+Y, X-Y) = Cov(X, X) - Cov(X, Y) + Cov(Y, X) - Cov(Y, Y)$$

As X, Y are independent, we have that

$$Cov(T, W) = Var(X) - Var(Y)$$

as X, Y i.i.d. $N(0,1)$, we have that $Var(X) - Var(Y) = 0$, so we get that $Cov(T, W) = 0$ Then we have that

$$Corr(T, W) = \frac{Cov(T, W)}{\sqrt{Var(T)Var(W)}} = 0$$

So with the theorem, we get that T, W are independent.

(2) Then we prove T and W are independent using change of variables. We again need to prove that $Cov(T, W) = 0$, we make tranformation first, we have that the relation between X, Y and T, W: denote that the joint PMF of X, Y is $f_{X,Y}(x, y)$, the joint PMF of T, W is $f_{T,W}(t, w)$, then we have that

$$f_{T,W}(t, w) = f_{X,Y}(x, y)|\frac{\partial(x, y)}{\partial(t, w)}| = f_{X,Y}(x, y)| - \frac{1}{4} - \frac{1}{4}| = f_{X,Y}(x, y)| - \frac{1}{2}| = \frac{1}{2} f_{X,Y}(x, y)$$

So we get that as $X = \dfrac{T+W}{2}$ and that $Y = \dfrac{T-W}{2}$, we get

$$f_{T,W}(t, w) = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{t+w}{2})^2} \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{t-w}{2})^2} = \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4}t^2} \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4}w^2}$$

from where we see that T, W are also Normal distribution $N(0, 2)$, so that T, W are independent.

Problem 3

Firstly we find the relationship between R, θ and X, Y, we get that

$$X = (cos\theta)R$$

$$Y = (sin\theta)R$$

Then we make tranformation between R, θ and X, Y, we denote that the joint PDF of X, Y are $f_{X,Y}(x, y)$ the joint PDF of R, θ are $f_{R,\theta}(r, \theta)$, so we get that

$$f_{R,\theta}(r, \theta) = f_{X,Y}(x, y)|J|$$

where the $|J|$ is

$$\left|\frac{\partial(x, y)}{\partial(r, \theta)}\right| = \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix} = r$$

So we get that as X, Y i.i.d. $N(0, 1)$, we have that

$$f_{R,\theta}(r, \theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} r = \frac{1}{2\pi} r e^{-\frac{1}{2}r^2}$$

Then we can divide this into 2 parts, we have that $f_R(r) = re^{-\frac{1}{2}r^2}$, $f_\theta(\theta) = \frac{1}{2\pi}$, where we have that as $\theta \in (0, 2\pi)$ and that $r \in (0, \infty)$

$$\int_0^{2\pi} f_\theta(\theta) d\theta = \int_0^{2\pi} \frac{1}{2\pi} d\theta = 1.$$

$$\int_0^{\infty} f_R(r) = \int_0^{\infty} re^{-\frac{1}{2}r^2} dr = \int_0^{\infty} -e^{-\frac{r^2}{2}} d(-\frac{r^2}{2}) = 1$$

where $f_\theta(\theta)$ is a valid PDF, and the $f_R(r)$ is also a valid PDF. So we get that R, θ are independent.

Problem 4

1. As for the marginal PDF of T and W, we again apply change of variable to this. As we have that $T = X + Y$ and that $W = \dfrac{X}{Y}$, then we firstly find the distribution of T and W. We have that

$$f_{T,W}(t,w) = f_{X,Y}(x,y)\,|J| = f_{X,Y}(x,y)\left|\frac{\partial(x,y)}{\partial(t,w)}\right|$$

Where as $X = \dfrac{WT}{X+1}$, $Y = \dfrac{T}{W+1}$

$$\left|\frac{\partial(x,y)}{\partial(t,w)}\right| = \begin{vmatrix} \dfrac{w}{w+1} & \dfrac{t}{(w+1)^2} \\ \dfrac{1}{(w+1)} & \dfrac{-t}{(w+1)^2} \end{vmatrix} = \frac{t}{(w+1)^2},$$

as $t \in (0, +\infty)$. Then as X, Y i.i.d. $Expo(\lambda)$, we have that

$$f_{T,W}(t,w) = \lambda e^{-\lambda x}\lambda e^{-\lambda y}\frac{t}{(w+1)^2} = \frac{\lambda^2 t e^{-\lambda t}}{(w+1)^2}$$

Then we divide it into 2 parts, where $f_T(t) = \lambda^2 t e^{-\lambda t}$, $f_W(w) = \dfrac{1}{(w+1)^2}$, we then have that as $w \in (0, +\infty)$ and $t \in (0, \infty)$

$$\int_0^\infty \frac{1}{(w+1)^2}dw = -\frac{1}{\infty+1} + \frac{1}{1} = 1$$

$$\int_0^\infty \lambda^2 t e^{-\lambda t}dt = -\lambda\int_0^\infty td(e^{-\lambda t}) = 1$$

both of them is a valid PDF, so we get that the joint PDF of T, W is

$$f_{T,W}(t,w) = \frac{\lambda^2 t e^{-\lambda t}}{(w+1)^2}$$

and the marginal PDF of T, W are

$$f_T(t) = \lambda^2 t e^{-\lambda t}$$

$$f_W(w) = \frac{1}{(w+1)^2}$$

where $t \in (0, \infty)$, $w \in (0, \infty)$

2. As we have that X, Y, Z i.i.d. $Unif(0,1)$ and that $W = X + Y + Z$, so we have that $W \in [0,3]$. Then as for $M = X + Y$, we have that $M \in [0,2]$

   (a) $m \in [0,1]$, we have that $f_M(m) = \int_0^m f_X(x)f_Y(m-x)dx = \int_0^m dx = m$.

   (b) $m \in (1,2]$, we have that $f_M(m) = \int_{m-1}^1 f_X(x)f_Y(m-x)dx = 2 - m$.

   Then as for the $W = X + Y + Z$, we have that $W = M + Z$, then there are 3 cases

   (a) $w \in [0,1]$, $f_W(w) = \int_0^w tdt = \dfrac{1}{2}w^2$

   (b) $w \in (1,2]$, $f_W(w) = \int_{w-1}^1 tdt + \int_1^w (2-t)dt = -w^2 + 3w - \dfrac{3}{2}$

   (c) $w \in (2,3]$, $f_W(w) = \int_{w-1}^2 (2-t)dt = \dfrac{1}{2}w^2 - 3w + \dfrac{9}{2}$

Problem 4 continued on next page...

So finally we get that

$$w \in [0,1], f_W(w) = \frac{1}{2}w^2$$

$$w \in (1,2], f_W(w) = -w^2 + 3w - \frac{3}{2}$$

$$w \in (2,3], f_W(w) = \frac{1}{2}w^2 - 3w + \frac{9}{2}$$

for other w, $f_W(w) = 0$

3. To show M has the same distribution as $X + \frac{1}{2}Y$, we use 2 methods

   (a) Property of the Exponential, we have that as $M = max(X,Y)$, we denote that $L = min(X,Y)$, we then have that with the property of expoential distribution, $L \sim Expo(2\lambda)$, then as for $\frac{1}{2}Y$, we denote that $\frac{1}{2}Y = N$, then $P(N \leq n) = P(\frac{1}{2}Y \leq n) = P(Y \leq 2n)$, so we get that $\frac{1}{2}Y \sim Expo(2\lambda)$. So $L = \frac{1}{2}Y$, as $X + Y = M + L$, we get that $M = X + Y - L = X + Y - \frac{1}{2}Y = X + \frac{1}{2}Y$. So we get that M has the same distribution as $X + \frac{1}{2}Y$.

   (b) Convolution, we have that firstly as for the

   $$F_M(m) = P(M \leq m) = P(max(X,Y) \leq m) = P(X \leq m, Y \leq m)$$

   As we have that X, Y are independent, we get

   $$F_M(m) = P(X \leq m)P(Y \leq m) = (1 - e^{-\lambda m})^2.$$

   Then we have

   $$f_M(m) = F_M(m)' = 2\lambda e^{-\lambda m} - 2\lambda e^{-2\lambda m}$$

   Then as for $X + \frac{1}{2}Y$, we have that denote that $X + \frac{1}{2}Y = P$, and $\frac{1}{2}Y = Q$, then we get $P = X + Q$

   $$f_P(p) = \int_0^p = f_X(x) f_Q(p-q) dp = \int_0^t \lambda e^{-\lambda x} 2\lambda e^{-2\lambda(p-x)} dx = 2\lambda^2 e^{-2\lambda t} \int_0^t e^{\lambda x} dx$$

   We get that

   $$f_P(p) = 2\lambda e^{-\lambda p} - 2\lambda e^{-2\lambda p}$$

   which has the same form with $f_M(m)$. So we get that M and $X + \frac{1}{2}Y$ has the same distribution.

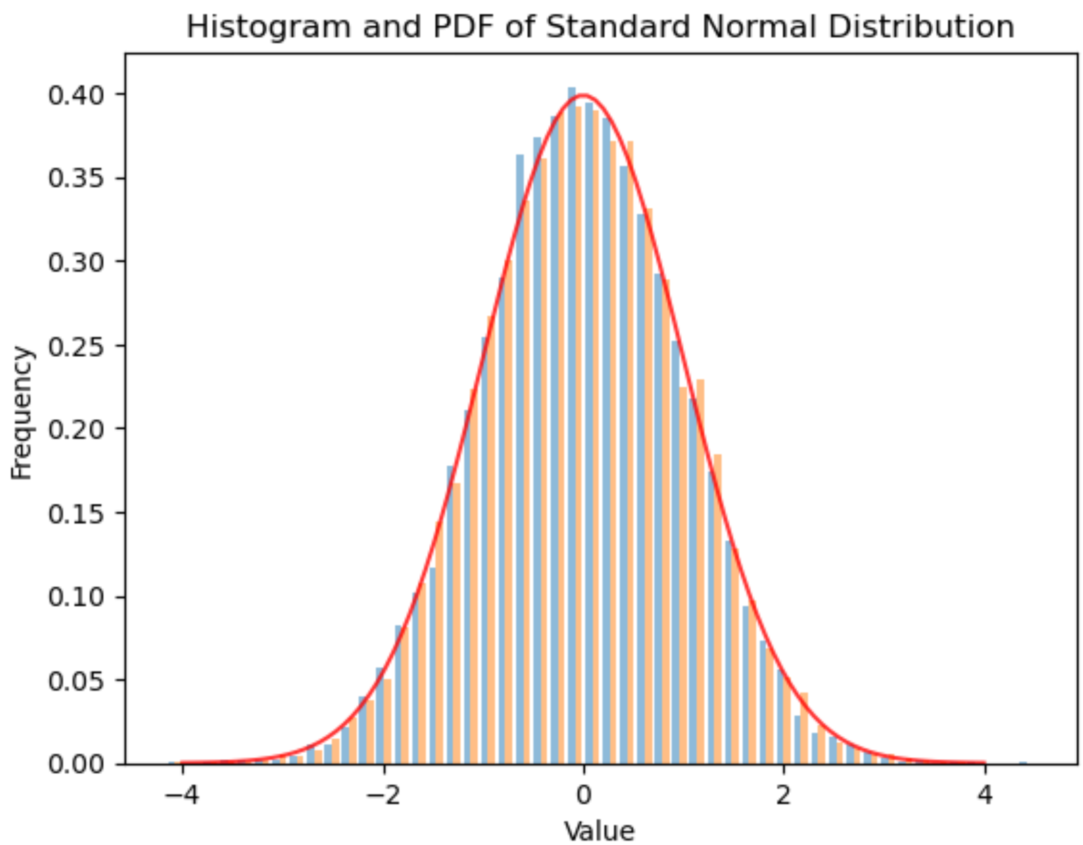
Problem 5

HW11-sol-p05

May 1, 2023

1 P5 question(a)

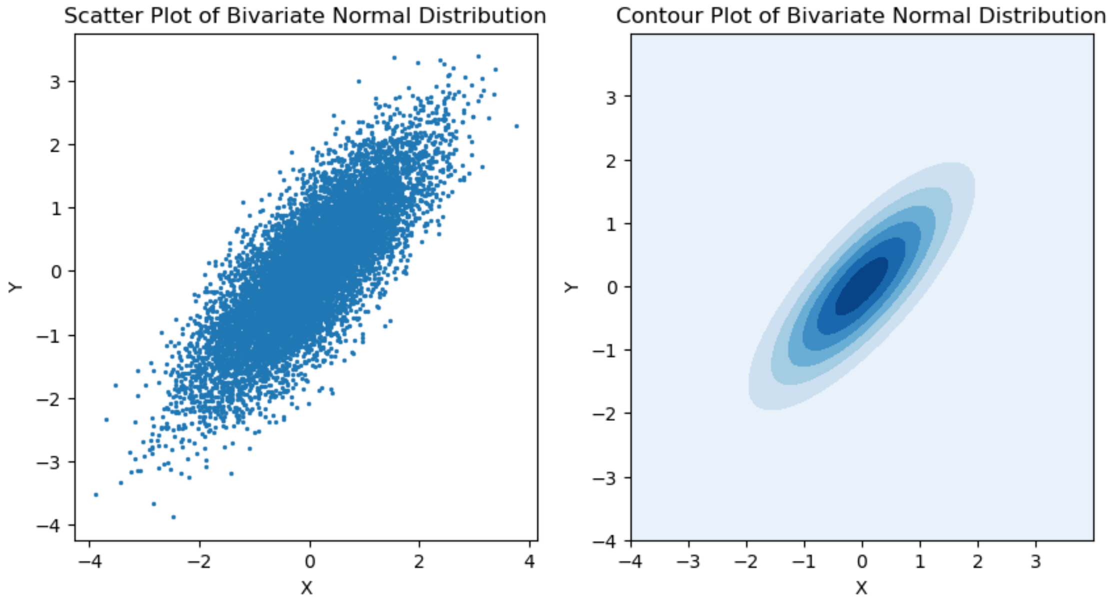
[ ]:
```
import numpy as np
import matplotlib.pyplot as plt

# Box-Muller
def box_muller(num_samples):
    # generate random variable u1 and u2
    u1 = np.random.rand(num_samples)
    u2 = np.random.rand(num_samples)

    # normal distribution
    x = np.sqrt(-2*np.log(u1)) * np.cos(2*np.pi*u2)
    y = np.sqrt(-2*np.log(u1)) * np.sin(2*np.pi*u2)

    return x, y

# 10000 samples
samples = box_muller(10000)

fig, ax = plt.subplots()

# histogram
ax.hist(samples, bins=50, density=True, alpha=0.5)

# PDF
x = np.linspace(-4, 4, 100)
pdf = (1/np.sqrt(2*np.pi)) * np.exp(-(x**2)/2)
ax.plot(x, pdf, 'r', alpha=0.8)

ax.set_title('Histogram and PDF of Standard Normal Distribution')
ax.set_xlabel('Value')
ax.set_ylabel('Frequency')

plt.show()
```



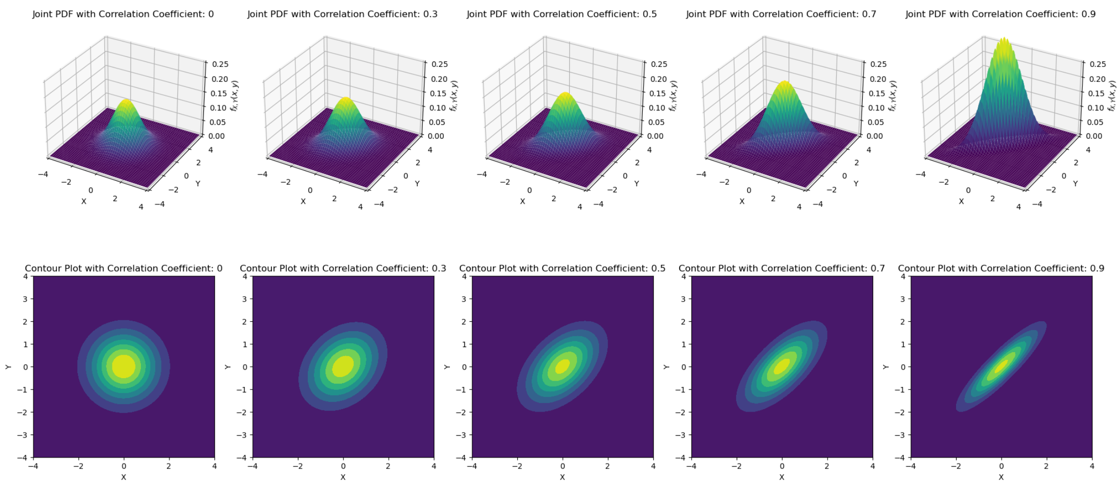
2 P5 question(b)

[ ]:
```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import multivariate_normal

# box-muller
def box_muller(num_samples):
    # u1 and u2
    u1 = np.random.rand(num_samples)
    u2 = np.random.rand(num_samples)

    # normal distribution
    x = np.sqrt(-2*np.log(u1)) * np.cos(2*np.pi*u2)
    y = np.sqrt(-2*np.log(u1)) * np.sin(2*np.pi*u2)

    return x, y

def bivariate_normal(num_samples, rho):
```

```python
    x, y = box_muller(num_samples)

    # transformation
    x_new = x
    y_new = rho*x + np.sqrt(1-rho**2)*y

    return x_new, y_new

# sample with 10000, 0.8
samples = bivariate_normal(10000, 0.8)

fig, axs = plt.subplots(nrows=1, ncols=2, figsize=(10, 5))

axs[0].scatter(samples[0], samples[1], s=2)

x, y = np.mgrid[-4:4:.01, -4:4:.01]
pos = np.dstack((x, y))
cov = np.array([[1, 0.8], [0.8, 1]])
rv = multivariate_normal([0, 0], cov, 10000)
axs[1].contourf(x, y, rv.pdf(pos), cmap='Blues')

axs[0].set_title('Scatter Plot of Bivariate Normal Distribution')
axs[0].set_xlabel('X')
axs[0].set_ylabel('Y')
axs[1].set_title('Contour Plot of Bivariate Normal Distribution')
axs[1].set_xlabel('X')
axs[1].set_ylabel('Y')

plt.show()
```



[ ]:
```python
import numpy as np
from scipy.stats import multivariate_normal
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

mean = np.array([0, 0])
x, y = np.meshgrid(np.linspace(-4, 4, 100), np.linspace(-4, 4, 100))
x_y_grid = np.stack((x, y), axis=-1)
labels = [0, 0.3, 0.5, 0.7, 0.9]

fig = plt.figure(figsize=(20, 10))

i = 1
for rho in labels:
    cov = np.array([[1, rho], [rho, 1]])

    rv = multivariate_normal(mean, cov)

    # pdf
    pdf = rv.pdf(x_y_grid)

    ax = fig.add_subplot(2, 5, i, projection='3d')
    ax.plot_surface(x, y, pdf, cmap='viridis')
    ax.set_xlabel('X')
    ax.set_ylabel('Y')
    ax.set_zlabel(r"$f_{X, Y}(x, y)$")
    ax.set_title(f'Joint PDF with Correlation Coefficient: {rho}')
    ax.set_xlim(-4, 4)
    ax.set_ylim(-4, 4)
    ax.set_zlim(0.00, 0.25)

    ax = fig.add_subplot(2, 5, i+5)
    # ax.axis('equal')
    ax.set_aspect('equal')
    c = ax.contourf(x, y, pdf, cmap='viridis')
    ax.set_xlabel('X')
    ax.set_ylabel('Y')
    ax.set_title(f'Contour Plot with Correlation Coefficient: {rho}')

    i += 1

plt.tight_layout()
plt.show()
```


Joint PDF with Correlation Coefficient: 0
Joint PDF with Correlation Coefficient: 0.3
Joint PDF with Correlation Coefficient: 0.5
Joint PDF with Correlation Coefficient: 0.7
Joint PDF with Correlation Coefficient: 0.9
Contour Plot with Correlation Coefficient: 0
Contour Plot with Correlation Coefficient: 0.3
Contour Plot with Correlation Coefficient: 0.5
Contour Plot with Correlation Coefficient: 0.7
Contour Plot with Correlation Coefficient: 0.9
X
Y
$f_{X,Y}(x,y)$