Probability & Statistics for EECS: Homework #12

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Problem 1

(a) As for the posterior distribution, we have that is

$$p|X_1 = x_1, X_2 = x_2, ..., X_n = x_n$$

As the prior is Beta(1, 1), then we have that by using the Beta Binomial conjugacy, we get that the posterior with $X_1 = x_1$ is that

$$Beta(1 + x_1, 1 + (1 - x_1))$$

The same, with additional given $X_2 = x_2$, we get that the posterior is that

$$Beta(1 + x_1 + x_2, 1 + (1 - x_1) + (1 - x_2))$$

So, given that $X_1 = x_1, X_2 = x_2, ..., X_n = x_n$ we get that the posterior is that

$$Beta(1 + \sum_{j=1}^{n} x_j, 1 + n - \sum_{j=1}^{n} x_j)$$

So we get that

$$p|X_1 = x_1, X_2 = x_2, ..., X_n = x_n \sim Beta(1 + \sum_{j=1}^{n} x_j, 1 + n - \sum_{j=1}^{n} x_j)$$

From where we see that the parameters of the distribution is only consisted of $\sum_{j=1}^{n} x_j$ so we get that the distribution only depends on the sum of x_j.

(b) Proof of Laplace's law of succession, denote the sum of X_j is that

$$S_n = \sum_{j=1}^{n} X_j,$$

use the LOTP, denote that $f(p|S_n = k)$ is the posterior PDF, we get that

$$P(X_{n+1} = 1|S_n = k) = \int_0^1 P(X_{n+1} = 1|p, S_n = k)f(p|S_n = k)dp$$

then we have that

$$P(X_{n+1} = 1|S_n = k) = \frac{\Gamma(n+2)}{\Gamma(k+1)\Gamma(n-k+1)} \int_0^1 pp^k(1-p)^{n-k}dp$$

$$= \frac{\Gamma(n+2)}{\Gamma(k+1)\Gamma(n-k+1)} \frac{\Gamma(k+2)\Gamma(n-k+1)}{\Gamma(n+3)}$$

$$= \frac{k+1}{n+2}.$$

(c) From the perspective of Beta-Binomial conjugacy, we have that as the Laplace's Law of Succession states that if we observe an event N times, of which k times are successful, then the probability of success in the next event is $\frac{k+1}{N+2}$. This can be presented as Beta distribution. If we assume a prior probability of beta(a,b), then after observing k successes and (N-k) failures, the posterior probability is beta(a+k,b+(N-k)). The Beta distribution has conjugacy with the binomial distribution, which means that we can use the Beta distribution to represent the posterior distribution of the binomial distribution. Therefore, Laplace's Law of Succession can be reinterpreted as a Bayesian inference process on the binomial distribution using the Beta-Binomial conjugacy relationship.

Problem 2

(a) By using LOTUS, as we have that $p \sim Beta(a, b)$ we have that $E(p^2(1-p)^2)$ can be written as

$$E(p^2(1-p)^2) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 p^2(1-p)^2 p^{a-1}(1-p)^{b-1} dp$$
$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 p^{a+1}(1-p)^{b+1} dp$$
$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+2)\Gamma(b+2)}{\Gamma(a+b+4)}$$

With the property of Gamma function, we have that

$$\Gamma(a+2) = (a+1)\Gamma(a+1) = (a+1)a\Gamma(a),$$

and that

$$\Gamma(a+b+4) = (a+b+3)\Gamma(a+b+3) = (a+b+3)(a+b+2)(a+b+1)(a+b)\Gamma(a+b).$$

So we get that
$$E(p^2(1-p)^2) = \frac{(a+1)a(b+1)b}{(a+b+3)(a+b+2)(a+b+1)(a+b)}$$

(b) The posterior distribution only depends on the fact that A won exactly 6 of the 10 games on record. As for a sequence event, we need to update the distribution, firstly, we get Beta(a, b), where a = 1, b = 1, then if a A wins occurs, the a += 1, if a B wins occurs, the b += 1. From the update process, we get that the order of outcomes do not effect the final parameters of Beta distribution, so we get that it is the fact that A won exactly 6 of 10 games on record.

(c) As the input event is that AAABBAABAB, where A wins 6 times, and B wins 4 times. So we get that the final posterior distribution for p given historical data is

$$p|historical data \sim Beta(6+1, 4+1)$$

which is

$$p|historical data \sim Beta(7, 5)$$

(d) As for the indicator of A, we have that they are conditionally independent given p, however, without given p, they are not independent. Thus, given p, the indicators are uncorrelated, without given p, the indicators are postively correlated as if A wins at the first match, the p will increased, as the A is more believed to win.

(e) As for the probability that the match is not yet decided when going into the fifth game There must be 2 wins for A and 2 wins for B, as in (c), we get the posterior distribution is Beta(7, 5), so we get that the probability is

$$\binom{4}{2} p^2(1-p)^2 = 6\frac{(7+1)7(5+1)5}{(7+5+3)(7+5+2)(7+5+1)(7+5)} = \frac{4}{13}.$$

Problem 3

Problem 4

Problem 5