Probability & Statistics for EECS: Homework #04

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Problem 1

1. Denote that event A is that the strategy of always switching succeeds, and denote that $B_i, i \in 1, 2, 3$ is the event that the car is behind door i. Then we have

$$\begin{aligned} P(A) &= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3) \\ &= 0 * \frac{1}{3} + 1 * \frac{1}{3} + 1 * \frac{1}{3} \\ &= \frac{2}{3} \end{aligned}$$

   So we get the unconditional probability that the strategy of always switching succeeds.

2. Denote that event C_j is that Monty opens the door j. Then we need to calculate $P(A|C_2)$, as we assume that we always choose the door 1 first, and event A is that strategy of always switching succeeds, then $P(A|C_2) = P(B_3|C_2)$ as we door 2 is opened and after switch, we win.

$$\begin{aligned} P(B_3|C_2) &= \frac{P(C_2|B_3)P(B_3)}{P(C_2)} \\ &= \frac{P(C_2|B_3)P(B_3)}{P(C_2|B_1)P(B_1) + P(C_2|B_2)P(B_2) + P(C_2|B_3)P(B_3)} \\ &= \frac{1 * \frac{1}{3}}{p * \frac{1}{3} + 0 * \frac{1}{3} + 1 * \frac{1}{3}} \\ &= \frac{1}{1+p} \end{aligned}$$

3. The same as the question (b). Then we need to calculate $P(A|C_3)$, as we assume that we always choose the door 1 first, and event A is that strategy of always switching succeeds, then $P(A|C_3) = P(B_2|C_3)$ as we door 3 is opened and after switch, we win.

$$\begin{aligned} P(B_2|C_3) &= \frac{P(C_3|B_2)P(B_3)}{P(C_3)} \\ &= \frac{P(C_3|B_2)P(B_2)}{P(C_3|B_1)P(B_1) + P(C_3|B_2)P(B_2) + P(C_3|B_3)P(B_3)} \\ &= \frac{1 * \frac{1}{3}}{(1-p) * \frac{1}{3} + 1 * \frac{1}{3} + 0 * \frac{1}{3}} \\ &= \frac{1}{2-p} \end{aligned}$$

Problem 2

1. Consider the PMF be $\frac{a}{n}$, where a is a constant, then we calculate the sum of PMF, which should equal 1. However, $\sum_{i=1}^{\infty} \frac{a}{n} = a \sum_{i=1}^{\infty} \frac{1}{n}$ is a series that do not converge, so there do not exist a constant a that makes the sum of PMF is not equal to 1. So there is not a discrete distribution that makes the value of the PMF at n is proportional to $\frac{1}{n}$.

2. Consider the PMF be $\frac{b}{n^2}$, where b is a constatn, then we calculate the sum of PMF, which should equal 1.

$$\sum_{i=1}^{n} \frac{b}{n^2} = b \sum_{i=1}^{n} \frac{1}{n^2} = b\frac{\pi^2}{6} = \frac{b\pi^2}{6} = 1.$$

$$b = \frac{6}{\pi^2}.$$

Then find the satisfied discrete distribution with PMF of $\frac{6}{\pi^2 n^2}$ such that the value of the PMF at n is proportional to $\frac{1}{n^2}$.

Problem 3

1. X and Y have the same distribution, as X is a random day of the week, so the probability of X = any of the 7 days are equal and is $\frac{1}{7}$. Then as Y is the day after X, so the probability of Y = any of the 7 days are also equal and is also $\frac{1}{7}$. So X and Y have the same distribution.

2. To calculate the $P(X < Y)$, we can calculate the $P(X \geq Y)$ first, there is only case, that is X = 7, Y = 1, the cases' probability is $\frac{1}{7}$ as there are 7 days in total. Then the probability $P(X < Y) = 1 - \frac{1}{7} = \frac{6}{7}$.

Problem 4

1. Firstly we find the range of X, that is $[0, n]$ as there are n flips in total. Define that k is the number of times it lands Heads, then we calculate the PMF of X is that:

$$P(X = k) = \frac{1}{2}\binom{n}{k}p_1^k(1-p_1)^{n-k} + \frac{1}{2}\binom{n}{k}p_2^k(1-p_2)^{n-k}, \text{where k} \in [0, n].$$

2. If $p_1 = p_2$, then

$$P(X = k) = \binom{n}{k}p_1^k(1-p_1)^{n-k}.$$

   Which is the binomial distribution.

3. As the 2 coins has different probability of landing Heads, then we need to consider the case of choose coin 1 and coin 2, as each time, the coin is fixed and not choose from the 2 coins, so each flip is not independent. So the distribution of X is not binomial distribution.

Problem 5

1. Consider conditioning on X, then we have that

$$\begin{aligned}P(X \oplus Y = 1|X = 0) &= P(Y = 1)P(X = 0)\\ &= \frac{1-p}{2}.\end{aligned}$$

$$\begin{aligned}P(X \oplus Y = 1|X = 1) &= P(Y = 0)P(X = 1)\\ &= \frac{p}{2}.\end{aligned}$$

Then we have that $P(X \oplus Y = 0) = \frac{1}{2} * p + \frac{1}{2} * (1-p) = \frac{1}{2}$, then we have $P(X \oplus Y = 1) = \frac{1}{2} * p + \frac{1}{2} * (1-p) = \frac{1}{2}$. So the distribution of $X \oplus Y$ is Bern($\frac{1}{2}$).

2. (a) X: From question (a) we can see that $X \oplus Y$ is independent with X as conditioning on X do not have connection with X.

   (b) Y:

   i. If p = $\frac{1}{2}$, then we conditioning on Y, we get that with $m \in 0, 1, n \in 0, 1$

$$\begin{aligned}P(X \oplus Y = m, Y = n) &= \frac{1}{2}P(X \oplus n = m|m = 0) + \frac{1}{2}P(X \oplus n = m|m = 1)\\ &= \frac{1}{2}P(X \oplus n = 0) + \frac{1}{2}P(X \oplus n = 1)\\ &= \frac{1}{2} * \frac{1}{2} * \frac{1}{2} + \frac{1}{2} * \frac{1}{2} * \frac{1}{2}\\ &= \frac{1}{4}\end{aligned}$$

$$P(X \oplus Y = m)P(Y = n) = \frac{1}{2} * \frac{1}{2} = \frac{1}{4}$$
$$P(X \oplus Y = m, Y = n) = P(X \oplus Y = m)P(Y = n)$$

   then the distribution of $X \oplus Y$ is independent with Y.

   ii. If p $\neq \frac{1}{2}$, then we conditioning on Y, we get that, for example:

$$P(X \oplus Y = 1, Y = 1) = \frac{1}{2}P(X \oplus 1 = 1) = \frac{1}{2}P(X = 0) = \frac{1-p}{2}$$

$$P(X \oplus Y = 1)P(Y = 1) = \frac{1}{2} * \frac{1}{2} = \frac{1}{4} \neq \frac{1-p}{2}$$

   then the distribution of $X \oplus Y$ is not independent with Y.

3. (a) Proof of $Y_j \sim Bern(\frac{1}{2})$:

   When $j = 1$, then $Y_j = X_1$, then we have $P(Y_j = 1) = P(X_1 = 1) = \frac{1}{2}$, so we get $Y_j \sim Bern(1/2)$, with $j = 1$
   When $j = 2$, then $Y_j = X_1 \oplus X_2$, then we have $P(Y_j = 1) = P(X_1 \oplus X_2 = 1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, so $Y_j \sim Bern(1/2)$.

Now assume that $j = k$, $Y_j \sim Bern(1/2)$, then we consider when $j' = k+1$, then $j' = k+1$. Then $Y_j' = Y_j \oplus X_{k+1}$. Then $P(Y_j' = 1) = P(Y_j \oplus X_{k+1} = 1) = \frac{1}{2}P(Y_j \oplus X_{k+1} = 1|X_{k+1} = 0) + \frac{1}{2}P(Y_j \oplus X_{k+1} = 1|X_{k+1} = 1) = \frac{1}{2} * \frac{1}{2} + \frac{1}{2} * \frac{1}{2} = \frac{1}{2}$.
The same for $P(Y_j = 0)$, so with mathematical induction we have $Y_J \sim Bern(1/2)$.

(b) Pairwise independent:
Consider 2 subsets J, K of {1, 2, ..., n}, then we have that $J = A \oplus B, K = A \oplus C$, $J \cup K$ can parted into $J \cap K$, $J \cap K^c$, $J^c \cap K$.
Assume that $J \cap K$ is nonempty, then for $y \in 0,1, z \in 0,1$, we have that

$$\begin{aligned}
P(Y_j = y, Y_k = z) &= \frac{1}{2}P(A \oplus B = y, A \oplus C = z|A = 1) + \frac{1}{2}P(A \oplus B = y, A \oplus C = z|A = 0) \\
&= \frac{1}{2}P(1 \oplus B = y)P(1 \oplus C = z) + \frac{1}{2}P(0 \oplus B = y)P(0 \oplus C = z) \\
&= \frac{1}{8} + \frac{1}{8} \\
&= \frac{1}{4} \\
&= P(Y_J = y)P(Y_K = z)
\end{aligned}$$

As A, B, C are independent, A, B, C, Y_J, $Y_K \sim Bern(1/2)$, we have Y_J and Y_K are independent.

(c) Not independent:
Give an example that set A and set B are 2 subsets of {1, 2, ..., n}, and satisfy that $A \cap B = \emptyset$, then $A \cup B$ is also a subset of {1, 2, ..., n}. Then $P(Y_A = 0, Y_B = 0, Y_{A\cup B} = 0) = 0$, while $P(Y_A = 0)P(Y_B = 0)P(Y_{A\cap B} = 0) = \frac{1}{2} * \frac{1}{2} * \frac{1}{2} = \frac{1}{8}$, so not independent.