Probability & Statistics for EECS: Homework #05

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Problem 1

1. As the treasure has equally probability to be in the realm from 1 to 9. Then we have that:

   (a) If treasure in realm 1, then we need to ask 1 questions.

   (b) If treasure in realm 2, then we need to ask 2 questions.

   (c) ...

   (d) If treasure in realm 8, then we need to ask 8 questions.

   (e) Note that if treasure in realm 9, then we need to ask 8 questions.

   Then we denote that event X_i is that treasure in realm i. Then we have that:

   $$P(X_i) = \frac{1}{9}.$$

   Then we denote that Y is the number of questions we need to ask. Then we have that:

   $$P(Y = k) = P(X_k) = \frac{1}{9}$$

   Then we have that:

   $$E(X_i) = 1 * \frac{1}{9} + 2 * \frac{1}{9} + ... + 8 * \frac{2}{9} = \frac{44}{9}.$$

2. By using the bisection method, we could calculate the expected questions for each case that the treasure is in a specific realm.

   (a) If treasure in realm 1, then we need to ask 4 questions.

   (b) If treasure in realm 2, then we need to ask 4 questions.

   (c) If treasure in realm 3, then we need to ask 3 questions.

   (d) If treasure in realm 4, then we need to ask 3 questions.

   (e) If treasure in realm 5, then we need to ask 3 questions.

   (f) If treasure in realm 6, then we need to ask 3 questions.

   (g) If treasure in realm 7, then we need to ask 3 questions.

   (h) If treasure in realm 8, then we need to ask 3 questions.

   (i) If treasure in realm 9, then we need to ask 3 questions.

   Then we have that: the expected questions is $\frac{1}{9} * 4 + ... + \frac{1}{9} * 3 = \frac{29}{9}$.

Problem 2

Problem 3

1. Denote the times that they are simulataneously successful is A. As we looking for the cases that they are simultaneously successful, where the probability that simulataneously successful is $p_1 * p_2 = p_1p_2$, then this is the geometric distribution, we have

$$A - 1 \sim Geom(p_1p_2)$$

and

$$A \sim Fs(p_1p_2)$$

Then we denote that $p = p_1p_2$ and that $q = 1 - p$, we have :

$$E(A) = \sum_{k=0}^{\infty} kP(X = k) = \sum_{k=0}^{\infty} kq^{k-1}p = \frac{p}{q}\sum_{k=0}^{\infty} kq^k.$$

Firstly we denote the $RES = \sum_{k=0}^{n} kq^k$, as $q * RES = q\sum_{k=0}^{n} kq^k = \sum_{k=0}^{\infty} kq^{k+1}$, then make subtraction and get $RES - q * RES = -nq^{n+1} + \sum_{i=1}^{n} q^n$, so we get that $RES = \dfrac{q(1 - q^n - nq^n + nq^{n+1})}{(1 - q)^2}$.
Take it into $E(A)$, we have $E(A) = \dfrac{1}{p}\lim_{n\to\infty}(1 - q^n - nq^n + nq^{n+1}) = \dfrac{1}{p}$.
Then we get $p = p_1p_2$ and that $q = 1 - p$ into $E(A)$, we have that

$$E(A) = \frac{1}{p_1p_2}.$$

2. Denote the times that at least one has a success is B, the event's probability is $1 - (1 - p_1)(1 - p_2) = p_1 + p_2 - p_1p_2$, then the same as this problem's question 1, we denote that $p = p_1 + p_2 - p_1p_2, q = 1 - p$, we have

$$E(B) = \frac{1}{p} = \frac{1}{p_1 + p_2 - p_1p_2}.$$

3. Firstly we denote that $p = p_1 = p_2$, and $q = 1-p$, then we find the probability that they simultaneously success, denote that C_1, C_2 are the first time they success, then we have the probability $P(C_1 = C_2) = \sum_{n=1}^{\infty} P(C_1 = n, C_2 = n) = \sum_{n=1}^{\infty} p^2q^{2(n-1)} = \dfrac{p^2}{1 - q^2} = \dfrac{p}{2 - p}$.
Then, as we have that $P(C_1 = C_2) + P(C_1 < C_2) + P(C_1 > C_2) = P(C_1 = C_2) + 2P(C_1 < C_2) = 1$, we get the

$$P(C_1 < C_2) = \frac{1}{2}(1 - \frac{p}{2 - p}) = \frac{1 - p}{2 - p}.$$

Problem 4

1. According to the problem, we have that $X \sim Geom(p), Y \sim Geom(q)$. We have that $P(X = k) = (1-p)^k p, P(Y = k) = (1-q)^k q$. Then $P(X = Y) = \sum_{k=0}^{\infty} P(X = k, Y = k) = \sum_{k=0}^{\infty} (1-p)^k p(1-q)^k q = pq \sum_{k=0}^{\infty} ((1-p)(1-q))^k = pq \dfrac{1}{1-(1-p)(1-q)} = \dfrac{pq}{p+q-pq}$.So, we have
$$P(X = Y) = \frac{pq}{(1-p)(1-q)}.$$

2. To solve $E[max(X, Y)]$, we firstly calculate the $P(max(X, Y) = k)$.
$$\begin{aligned} P[max(X,Y) = k] &= P(X = k, Y \leq k) + P(Y = k, X < k) \\ &= P(X = k)P(Y \leq k) + P(Y = k)P(X < k) \\ &= (1-p)^k p \sum_{n=0}^{k} (1-q)^n q + (1-q)^k q \sum_{n=0}^{k-1} (1-p)^n p \\ &= (1-p)^k p[1-(1-q)^{k+1}] + (1-q)^k q[1-(1-p)^k] \end{aligned}$$
We have that
$$\begin{aligned} E[max(X,Y)] &= \sum_{k=0}^{\infty} kP[max(X,Y) = k] \\ &= \sum_{k=0}^{\infty} k\{(1-p)^k p[1-(1-q)^k] + (1-q)^k q[1-(1-p)^{k-1}]\} \\ &= \sum_{k=0}^{\infty} k(1-p) \end{aligned}$$

3. According to the question 2 of this problem, we get that
$$\begin{aligned} P[max(X,Y) = k] &= P(X = k, Y \geq k) + P(Y = k, X > k) \\ &= P(X = k)P(Y \geq k) + P(Y = k)P(X > k) \\ &= (1-p)^k p \sum_{n=k}^{\infty} (1-q)^n q + (1-q)^k q \sum_{n=k+1}^{\infty} (1-p)^n p \\ &= (1-p)^k p(1-q)^k + (1-q)^k q(1-p)^{k+1} \\ &= [p + (1-p)q](1-p)^k (1-q)^k \\ &= (p+q-pq)[(1-p)(1-q)]^k \end{aligned}$$

4. We calculate the $E[X|X \leq Y]$
$$\begin{aligned} E[X|X \leq Y] &= \sum_{i=0}^{\infty} iP(X = i|X \leq Y) \\ &= \sum_{i=0}^{\infty} i\frac{P(X = i, X \leq Y)}{P(X \leq Y)} \\ &= \sum_{i=0}^{\infty} i\frac{P(X = i)P(Y \geq i)}{P(X \leq Y)} \\ &= \sum_{i=0}^{\infty} i\frac{(1-p)^i p \sum_{j=i}^{\infty} (1-q)^j q}{P(X \leq Y)} \\ &= \sum_{i=0}^{\infty} i\frac{(1-p)^i p \sum_{j=i}^{\infty} (1-q)^j q}{P(X \leq Y)} \end{aligned}$$

Problem 4 continued on next page...

Problem 5