

Probability & Statistics for EECS: Homework #05

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Problem 1

1. As the treasure has equally probability to be in the realm from 1 to 9. Then we have that:

- (a) If treasure in realm 1, then we need to ask 1 questions.
- (b) If treasure in realm 2, then we need to ask 2 questions.
- (c) ...
- (d) If treasure in realm 8, then we need to ask 8 questions.
- (e) Note that if treasure in realm 9, then we need to ask 8 questions.

Then we denote that event X_i is that treasure in realm i . Then we have that:

$$P(X_i) = \frac{1}{9}.$$

Then we denote that Y is the number of questions we need to ask. Then we have that:

$$P(Y = k) = P(X_k) = \frac{1}{9}$$

Then we have that:

$$E(X_i) = 1 * \frac{1}{9} + 2 * \frac{1}{9} + \dots + 8 * \frac{1}{9} = \frac{44}{9}.$$

2. By using the bisection method, we could calculate the expected questions for each case that the treasure is in a specific realm.

- (a) If treasure in realm 1, then we need to ask 4 questions.
- (b) If treasure in realm 2, then we need to ask 4 questions.
- (c) If treasure in realm 3, then we need to ask 3 questions.
- (d) If treasure in realm 4, then we need to ask 3 questions.
- (e) If treasure in realm 5, then we need to ask 3 questions.
- (f) If treasure in realm 6, then we need to ask 3 questions.
- (g) If treasure in realm 7, then we need to ask 3 questions.
- (h) If treasure in realm 8, then we need to ask 3 questions.
- (i) If treasure in realm 9, then we need to ask 3 questions.

Then we have that: the expected questions is $\frac{1}{9} * 4 + \dots + \frac{1}{9} * 3 = \frac{29}{9}$.

Problem 2

Use the model Coupon Collector in Lecture, we firstly denote that X_k is the number of days needed to see the Youthuber eating steaks with k types deneness at least once.

Firstly we have $E[X_1] = 1$, then $X_2 - X_1 \sim \text{Geom}(\frac{n-1}{n})$, then we have $X_k - X_{k-1} \sim \text{Geom}(\frac{n-k}{n})$, then denote that $W_k = X_k - X_{k-1}$

Then we have

$$\begin{aligned} E[X_n] &= E[X_1 + X_2 - X_1 + \dots + X_n - X_{n-1}] \\ &= E[W_1 + W_2 + \dots + W_n] \\ &= E[W_1] + E[W_2] + \dots + E[W_n] \\ &= 1 + \frac{n}{n-1} + \dots + \frac{n}{1} \\ &= n \sum_{i=1}^n \frac{1}{i} \end{aligned}$$

In this problem, $k = 5$, then we have $E[X_5] = 5 * (\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{5}) = \frac{137}{12}$.

Problem 3

1. Denote the times that they are simultaneously successful is A . As we looking for the cases that they are simultaneously successful, where the probability that simultaneously successful is $p_1 * p_2 = p_1 p_2$, then this is the geometric distribution, we have

$$A - 1 \sim \text{Geom}(p_1 p_2)$$

and

$$A \sim \text{Fs}(p_1 p_2)$$

Then we denote that $p = p_1 p_2$ and that $q = 1 - p$, we have :

$$E(A) = \sum_{k=0}^{\infty} k P(X = k) = \sum_{k=0}^{\infty} k q^{k-1} p = \frac{p}{q} \sum_{k=0}^{\infty} k q^k.$$

Firstly we denote the $RES = \sum_{k=0}^{\infty} k q^k$, as $q * RES = q \sum_{k=0}^{\infty} k q^k = \sum_{k=0}^{\infty} k q^{k+1}$, then make subtraction and get $RES - q * RES = -nq^{n+1} + \sum_{i=1}^n q^n$, so we get that $RES = \frac{q(1 - q^n - nq^n + nq^{n+1})}{(1 - q)^2}$.

Take it into $E(A)$, we have $E(A) = \frac{1}{p} \lim_{n \rightarrow \infty} (1 - q^n - nq^n + nq^{n+1}) = \frac{1}{p}$.

Then we get $p = p_1 p_2$ and that $q = 1 - p$ into $E(A)$, we have that

$$E(A) = \frac{1}{p_1 p_2}.$$

2. Denote the times that at least one has a success is B , the event's probability is $1 - (1 - p_1)(1 - p_2) = p_1 + p_2 - p_1 p_2$, then the same as this problem's question 1, we denote that $p = p_1 + p_2 - p_1 p_2$, $q = 1 - p$, we have

$$E(B) = \frac{1}{p} = \frac{1}{p_1 + p_2 - p_1 p_2}.$$

3. Firstly we denote that $p = p_1 = p_2$, and $q = 1 - p$, then we find the probability that they simultaneously success, denote that C_1, C_2 are the first time they success, then we have the probability $P(C_1 = C_2) = \sum_{n=1}^{\infty} P(C_1 = n, C_2 = n) = \sum_{n=1}^{\infty} p^2 q^{2(n-1)} = \frac{p^2}{1 - q^2} = \frac{p}{2 - p}$.

Then, as we have that $P(C_1 = C_2) + P(C_1 < C_2) + P(C_1 > C_2) = P(C_1 = C_2) + 2P(C_1 < C_2) = 1$, we get the

$$P(C_1 < C_2) = \frac{1}{2} \left(1 - \frac{p}{2 - p} \right) = \frac{1 - p}{2 - p}.$$

Problem 4

Firstly we denote the $RES = \sum_{k=0}^n kq^k$, as $q * RES = q \sum_{k=0}^n kq^k = \sum_{k=0}^{\infty} kq^{k+1}$, then make subtraction and get $RES - q * RES = -nq^{n+1} + \sum_{i=1}^n q^n$, so we get that $RES = \frac{q(1 - q^n - nq^n + nq^{n+1})}{(1 - q)^2}$. Then the $\lim_{n \rightarrow \infty} \sum_{k=0}^n kq^k = \frac{q}{(1 - q)^2}$. So we get that:

$$\sum_{k=0}^{\infty} kq^k = \frac{q}{(1 - q)^2}$$

1. According to the problem, we have that $X \sim \text{Geom}(p), Y \sim \text{Geom}(q)$. We have that $P(X = k) = (1 - p)^k p, P(Y = k) = (1 - q)^k q$. Then $P(X = Y) = \sum_{k=0}^{\infty} P(X = k, Y = k) = \sum_{k=0}^{\infty} (1 - p)^k p (1 - q)^k q = pq \sum_{k=0}^{\infty} ((1 - p)(1 - q))^k = pq \frac{1}{1 - (1 - p)(1 - q)} = \frac{pq}{p + q - pq}$. So, we have

$$P(X = Y) = \frac{pq}{p + q - pq}.$$

2. To solve $E[\max(X, Y)]$, we firstly calculate the $P(\max(X, Y) = k)$.

$$\begin{aligned} P[\max(X, Y) = k] &= P(X = k, Y \leq k) + P(Y = k, X < k) \\ &= P(X = k)P(Y \leq k) + P(Y = k)P(X < k) \\ &= (1 - p)^k p \sum_{n=0}^k (1 - q)^n q + (1 - q)^k q \sum_{n=0}^{k-1} (1 - p)^n p \\ &= (1 - p)^k p [1 - (1 - q)^{k+1}] + (1 - q)^k q [1 - (1 - p)^k] \end{aligned}$$

We have that

$$\begin{aligned} E[\max(X, Y)] &= \sum_{k=0}^{\infty} k P[\max(X, Y) = k] \\ &= \sum_{k=0}^{\infty} k \{ (1 - p)^k p [1 - (1 - q)^{k+1}] + (1 - q)^k q [1 - (1 - p)^k] \} \\ &= \sum_{k=0}^{\infty} k (1 - p)^k p - k (1 - p)^k p (1 - q)^{k+1} + k (1 - q)^k q - k (1 - q)^k q (1 - p)^k \\ &= \frac{1 - p}{p} - p(1 - q) \frac{(1 - p)(1 - q)}{(p + q - pq)^2} + \frac{1 - q}{q} - q \frac{(1 - p)(1 - q)}{(p + q - pq)^2} \\ &= \frac{1 - p}{p} + \frac{1 - q}{q} - \frac{(1 - p)(1 - q)}{p + q - pq} \\ &= -1 - \frac{1}{p + q - pq} + \frac{1}{p} + \frac{1}{q}. \end{aligned}$$

3. According to the question 2 of this problem, we get that

$$\begin{aligned} P[\min(X, Y) = k] &= P(X = k, Y \geq k) + P(Y = k, X > k) \\ &= P(X = k)P(Y \geq k) + P(Y = k)P(X > k) \\ &= (1 - p)^k p \sum_{n=k}^{\infty} (1 - q)^n q + (1 - q)^k q \sum_{n=k+1}^{\infty} (1 - p)^n p \\ &= (1 - p)^k p (1 - q)^k + (1 - q)^k q (1 - p)^{k+1} \\ &= [p + (1 - p)q](1 - p)^k (1 - q)^k \\ &= (p + q - pq)[(1 - p)(1 - q)]^k \end{aligned}$$

4. Firstly we calculate the probability $P(X = i|X \leq Y)$, we have

$$P(X = i|X \leq Y) = \frac{P(X = i, Y \geq i)}{P(X \leq Y)} = \frac{(1-p)^i p (1-q)^i}{\sum_{j=0}^{\infty} P(X = j) P(Y \geq j)} = \frac{(1-p)^i p (1-q)^i}{\sum_{j=0}^{\infty} (1-q)^j (1-p)^j p}$$

We get that

$$P(X = i|X \leq Y) = (1-p)^n (1-q)^n (p+q-pq)$$

$$\begin{aligned} E[X|X \leq Y] &= \sum_{k=0}^{\infty} k P(X = k|X \leq Y) \\ &= \sum_{k=0}^{\infty} k (p+q-pq) (1-p)^k (1-q)^k \\ &= (p+q-pq) \frac{(1-p)(1-q)}{(p+q-pq)^2} \\ &= \frac{(1-p)(1-q)}{p+q-pq} \end{aligned}$$

So, the $E[X|X \leq Y] = \frac{(1-p)(1-q)}{p+q-pq}$

Problem 5

1. Denote that A_j is the rank of the j th dish you try, A is the sum of the ranks, X_i is the i th dish's rank. We have $A = A_1 + \dots + A_k + (m - k)X$, as X is the rank of the best dish that you find in the exploration phase. Suppose that the i 'th dish with highest rank in the first k dishes, then we have,

$$A = \sum_{1 \leq i \leq k, i \neq i'} A_i + A_{i'} + (m - k)X = \sum_{1 \leq i \leq k, i \neq i'} A_i + (m - k + 1)X$$

We then have

$$E(A) = E(A_1) + E(A_2) + \dots + E(A_k) + E(X) + (m - k)E(X) = \frac{E(X)(k + 1)}{2} + (m - k)E(X)$$

Then as we have

$$E(A_i | 1 \leq A_i \leq X) = \frac{1}{X - 1}(1 + 2 + \dots + X - 1) = \frac{X}{2}.$$

Then we get

$$E(A_i) = E(E(A_i | 1 \leq A_i < X)) = \frac{E(X)}{2}.$$

Finally we make sum of each A_i , we get

$$E(A) = \sum_{1 \leq i \leq k, i \neq i'} \frac{E(X)}{2} + (m - k + 1)E(X) = (m - \frac{k}{2} + \frac{1}{2})E(X)$$

So we get

$$E(A) = (m - \frac{k}{2} + \frac{1}{2})E(X)$$

2. To find the PMF of X , which is $P(X = i)$, we need to choose a dish with rank i and other $k - 1$ dishes with rank $\leq i$ from 1 to $j - 1$, we have

$$P(X = i) = \frac{\binom{i - 1}{k - 1}}{\binom{n}{k}}.$$

3. As we have get the PMF of X , then we get that

$$E(X) = \sum_{i=k}^n i P(X = i) = \sum_{i=k}^n i \frac{\binom{i - 1}{k - 1}}{\binom{n}{k}} = \frac{k}{\binom{n}{k}} \sum_{i=k}^n \binom{i}{k} = k \frac{\binom{n + 1}{k + 1}}{\binom{n}{k}} = \frac{k(n + 1)}{k + 1}$$

4. To make the $E[X]$ maximize, we have

$$E[X] = (m - \frac{k}{2} + \frac{1}{2}) \frac{k(n + 1)}{k + 1}$$

To make this maximize, we need to get the differential to k , then we have

$$E[X]' = -\frac{1}{2} \frac{k(n + 1)}{k + 1} + (m - \frac{k}{2} + \frac{1}{2}) \frac{n + 1}{(k + 1)^2} = 0$$

Then we get

$$k = \frac{-2 \pm \sqrt{4 - 4(-2m - 1)}}{2} = \sqrt{2(m + 1)} \pm 1$$

As $k \in [0, m]$, we get finally k is

$$k = \sqrt{2(m + 1)} - 1.$$