2023 Spring Probability & Statistics for EECS April 30, 2023

Homework 12

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1. Laplace's law of succession says that if $X_1, X_2, \ldots, X_{n+1}$ are conditionally independent $\text{Bern}(p)$ r.v.s given p, but p is given a $\text{Unif}(0,1)$ prior to reflect ignorance about its value, then

$$P\left(X_{n+1}=1 \mid X_1+\cdots+X_n=k\right)=\frac{k+1}{n+2}$$

As an example, Laplace discussed the problem of predicting whether the sun will rise tomorrow, given that the sun did rise every time for all n days of recorded history; the above formula then gives $(n+1)/(n+2)$ as the probability of the sun rising tomorrow.

(a) Find the posterior distribution of p given $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$, and show that it only depends on the sum of the x_j (so we only need the one-dimensional quantity $x_1+x_2+\cdots+x_n$ to obtain the posterior distribution, rather than needing all n data points).

(b) Prove Laplace's law of succession, using a form of LOTP to find

$$P\left(X_{n+1}=1 \mid X_1+\cdots+X_n=k\right)$$

by conditioning on p.

(c) Reinterpret the Laplace's law of succession from the perspective of Beta-Binomial Conjugacy.

2. (a) Let $p \sim \text{Beta}(a,b)$, where a and b are positive real numbers. Find $E(p^2(1-p)^2)$, fully simplified (Γ should not appear in your final answer).

Two teams, A and B, have an upcoming match. They will play five games and the winner will be declared to be the team that wins the majority of games. Given p, the outcomes of games are independent, with probability p of team A winning and $(1-p)$ of team B winning. But you don't know p, so you decide to model it as an r.v., with $p \sim \text{Unif}(0,1)$ a priori (before you have observed any data).

To learn more about p, you look through the historical records of previous games between these two teams, and find that the previous outcomes were, in chronological order, $AAABBAABAB$. (Assume that the true value of p has not been changing over time and will be the same for the match, though your *beliefs* about p may change over time.)

(b) Does your posterior distribution for p, given the historical record of games between A and B, depend on the specific order of outcomes or only on the fact that A won exactly 6 of the 10 games on record? Explain.

(c) Find the posterior distribution for p, given the historical data.

The posterior distribution for p from (c) becomes your new prior distribution, and the match is about to begin!

(d) Conditional on p, is the indicator of A winning the first game of the match positively correlated with, uncorrelated with, or negatively correlated with the indicator of A winning the second game of the match? What about if we only condition on the historical data?

(e) Given the historical data, what is the expected value for the probability that the match is not yet decided when going into the fifth game (viewing this probability as an r.v. rather than a number, to reflect our uncertainty about it)?

3. Let $U_1, \ldots, U_n$ be i.i.d. Unif$(0,1)$. Let $U_{(j)}$ be the corresponding jth order statistic, where $1 \leq j \leq n$.

   (a) Find the joint PDF of $U_{(1)}, \ldots, U_{(n)}$.

   (b) Find the joint PDF of $U_{(j)}$ and $U_{(k)}$, where $1 \leq j < k \leq n$.

   (c) Let $X \sim \text{Bin}(n, p)$ and $B \sim \text{Beta}(j, n-j+1)$, where n is a positive integer and j is a positive integer with $j \leq n$. Show using a story about order statistics that

   $$P(X \geq j) = P(B \leq p).$$

   This shows that the CDF of the continuous r.v. B is closely related to the CDF of the discrete r.v. X, and is another connection between the Beta and Binomial.

   (d) Show that

   $$\int_0^x \frac{n!}{(j-1)!(n-j)!} t^{j-1}(1-t)^{n-j} dt = \sum_{k=j}^{n} \binom{n}{k} x^k (1-x)^{n-k},$$

   without using calculus, for all $x \in [0,1]$ and j, n positive integers with $j \leq n$.

4. If $X \sim \text{Pois}(\lambda)$, $Z \sim \text{Gamma}(k+1, 1)$, where k is a nonnegative integer. Show the Poisson-Gamma Duality holds:

   $$P(X \leq k) = P(Z > \lambda).$$

   Hint: Two possible methods, where one is based on the identity in 3(d), the other is based on the model of Poisson process.

5. Programming Assignment:

(a) Use the Acceptance-Rejection Method to obtain the samples from distribution Beta$(2,4)$. You need to plot the pictures of both histogram and the theoretical PDF.

(b) Use the Acceptance-Rejection Method to obtain the samples from the standard Normal distribution $\mathcal{N}(0,1)$. You are required to show the correctness of your algorithm in theory.

(c) Both the Acceptance-Rejection Method and Box-Mulller Method can obtain the samples from the standard Normal distribution $\mathcal{N}(0,1)$. Discuss the pros and cons of such two methods.

(d) Use the importance sampling method to evaluate the probability of rare event $c = P(Y > 8)$, where $Y \sim N(0,1)$.