Probability & Statistics for EECS: Homework #04

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Problem 1

1. Denote that event A is that the strategy of always switching succeeds, and denote that $B_i, i \in 1, 2, 3$ is the event that the car is behind door i. Then we have

$$\begin{aligned} P(A) &= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3) \\ &= 0 * \frac{1}{3} + 1 * \frac{1}{3} + 1 * \frac{1}{3} \\ &= \frac{2}{3} \end{aligned}$$

So we get the unconditional probability that the strategy of always switching succeeds.

2. Denote that event C_j is that Monty opens the door j. Then we need to calculate $P(A|C_2)$, as we assume that we always choose the door 1 first, and event A is that strategy of always switching succeeds, then $P(A|C_2) = P(B_3|C_2)$ as we door 2 is opened and after switch, we win.

$$\begin{aligned} P(B_3|C_2) &= \frac{P(C_2|B_3)P(B_3)}{P(C_2)} \\ &= \frac{P(C_2|B_3)P(B_3)}{P(C_2|B_1)P(B_1) + P(C_2|B_2)P(B_2) + P(C_2|B_3)P(B_3)} \\ &= \frac{1 * \frac{1}{3}}{p * \frac{1}{3} + 0 * \frac{1}{3} + 1 * \frac{1}{3}} \\ &= \frac{1}{1 + p} \end{aligned}$$

3. The same as the question (b). Then we need to calculate $P(A|C_3)$, as we assume that we always choose the door 1 first, and event A is that strategy of always switching succeeds, then $P(A|C_3) = P(B_2|C_3)$ as we door 3 is opened and after switch, we win.

$$\begin{aligned} P(B_2|C_3) &= \frac{P(C_3|B_2)P(B_3)}{P(C_3)} \\ &= \frac{P(C_3|B_2)P(B_2)}{P(C_3|B_1)P(B_1) + P(C_3|B_2)P(B_2) + P(C_3|B_3)P(B_3)} \\ &= \frac{1 * \frac{1}{3}}{(1 - p) * \frac{1}{3} + 1 * \frac{1}{3} + 0 * \frac{1}{3}} \\ &= \frac{1}{2 - p} \end{aligned}$$

Problem 2

1. Consider the PMF be $\frac{a}{n}$, where a is a constant, then we calculate the sum of PMF, which should equal 1. However, $\sum_{i=1}^{\infty} \frac{a}{n} = a \sum_{i=1}^{\infty} \frac{1}{n}$ is a series that do not converge, so there do not exist a constant a that makes the sum of PMF is not equal to 1. So there is not a discrete distribution that makes the value of the PMF at n is proportional to $\frac{1}{n}$.

2. Consider the PMF be $\frac{b}{n^2}$, where b is a constatn, then we calculate the sum of PMF, which should equal 1.

$$\sum_{i=1}^{n} \frac{b}{n^2} = b \sum_{i=1}^{n} \frac{1}{n^2} = b\frac{\pi^2}{6} = \frac{b\pi^2}{6} = 1.$$

$$b = \frac{\pi^2}{6}.$$

   Then find the satisfied discrete distribution with PMF of $\frac{\pi^2}{6n^2}$ such that the value of the PMF at n is proportional to $\frac{1}{n^2}$.

Problem 3

1. X and Y have the same distribution, as X is a random day of the week, so the probability of X = any of the 7 days are equal and is $\frac{1}{7}$. Then as Y is the day after X, so the probability of Y = any of the 7 days are also equal and is also $\frac{1}{7}$. So X and Y have the same distribution.

2. To calculate the $P(X < Y)$, we can calculate the $P(X \geq Y)$ first, there is only case, that is X = 7, Y = 1, the cases' probability is $\frac{1}{7}$ as there are 7 days in total. Then the probability $P(X < Y) = 1 - \frac{1}{7} = \frac{6}{7}$.

Problem 4

1. Firstly we find the range of X, that is $[0, n]$ as there are n flips in total. Define that k is the number of times it lands Heads, then we calculate the PMF of X is that:

$$P(X = k) = \frac{1}{2}\binom{n}{k}p_1^k(1-p_1)^{n-k} + \frac{1}{2}\binom{n}{k}p_2^k(1-p_2)^{n-k}, \text{where k} \in [0, n].$$

2. If $p_1 = p_2$, then

$$P(X = k) = \binom{n}{k}p_1^k(1-p_1)^{n-k}.$$

   Which is the binomial distribution.

3. As the 2 coins has different probability of landing Heads, then we need to consider the case of choose coin 1 and coin 2, as each time, the coin is fixed and not choose from the 2 coins, so each flip is not independent. So the distribution of X is not binomial distribution.

Problem 5

1. Consider conditioning on X, then we have that

$$
\begin{aligned}
P(X \oplus Y = 1 | X = 0) &= P(Y = 1)P(X = 0) \\
&= \frac{1-p}{2}.
\end{aligned}
$$

$$
\begin{aligned}
P(X \oplus Y = 1 | X = 1) &= P(Y = 0)P(X = 1) \\
&= \frac{p}{2}.
\end{aligned}
$$

Then we have that $P(X \oplus Y = 0) = \frac{1}{2} * p + \frac{1}{2} * (1-p) = \frac{1}{2}$, then we have $P(X \oplus Y = 1) = \frac{1}{2} * p + \frac{1}{2} * (1-p) = \frac{1}{2}$. So the distribution of $X \oplus Y$ is Bern($\frac{1}{2}$).

2. (a) X: From question (a) we can see that $X \oplus Y$ is independent with X as conditioning on X do not have connection with X.

   (b) Y:

      i. If p $= \frac{1}{2}$, then we conditioning on Y, which is the same as conditioning on X. So still independent on Y.

      ii. If p $\neq \frac{1}{2}$, then we conditioning on Y, we get that

$$
P(X \oplus Y = 1 | Y = k) = P(X \oplus k = 1) = p(1-k) + (1-p)k
$$

      where $k \in 0, 1$, then the distribution of $X \oplus Y$ is not independent with Y.

3. as there is