Probability & Statistics for EECS: Homework #07

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Name: **Penghao Wang**
Student ID: 2021533138

Problem 1

(a) To determine if F is a valid CDF:

(1) F is non-decreasing, as $x \in (0,1)$, then $sin^{-1}(\sqrt{x})$ is increasing, so we have that F is increasing.

(2) F is bounded, as we have known that it is non-decreasing, then the minimum value of F is $F(0) = \frac{2}{\pi} sin^{-1}(\sqrt{0}) = 0$, and the maximum value of F is $F(1) = \frac{2}{\pi} sin^{-1}(\sqrt{1}) = 1$, so F is bounded in $(0,1)$.

(3) F is continuous, as $\frac{2}{\pi}$ is constant, which is continuous, then as $sin^{-1}(\sqrt{x})$ is also continuous, we have that F is continuous.

Above all, we get that F is a valid CDF. Then we need to find the corresponding PDF f, that is $f = F'$. We have that

$$f = F' = \frac{2}{\pi} * \frac{1}{2\sqrt{x}} * \frac{1}{\sqrt{1-x}} = \frac{1}{\pi\sqrt{x(1-x)}}$$

where $x \in (0,1)$, and $f = 0$ for other x.

(b) In the question (a), we find that the PDF $f = \frac{1}{\pi\sqrt{x(1-x)}}$, then when $x \to 0$, $f \to \infty$, and when $x \to 1$, $f \to \infty$. However, f is still a valid PDF, as

(1) f is nonegative, as when $x \in (0,1)$, $f \geq 0$.

(2) f is continuous, as $\sqrt{x(1-x)}$, then f is continuous.

(3) f integrates to 1, as

$$\int_0^1 \frac{1}{\pi\sqrt{x(1-x)}} dx = \pi * arcsin(2*1-1) - \pi * arcsin(2*0-1) = 1,$$

Then we see that f is a valid PDF though $f(x)$ goes to ∞ as x approaches 0 and as x approaches to 1.

Problem 2

Using the theroem Universality of the Uniform, we have that let $U \sim Unif(0,1)$ and $X = F^{-1}(U)$, then X is an r.v. with CDF F.
Then by using LOTUS, we get that

$$\int_0^1 F^{-1}(u)du = E(F^{-1}(U)) = E(X) = \mu.$$

So we get that the area under the curve of the quantile function from 0 to 1 is μ.

Problem 3

We firstly find the CDF, that is

$$P(X \le x) = P(U_1 \le x, U_2 \le x, ..., U_n \le x)$$

As $U_1, U_2, ..., U_n$ are i.i.d. Unif(0, 1), we have for $x \in (0, 1)$,

$$P(X \le x) = (P(U_1 \le x))^n = x^n$$

for other x, $P(X \le x) = 0$.
Then the PDF is $f = nx^{n-1}$, as for $E(X)$, we have that

$$E(X) = \int_0^1 xf(x)dx = \int_0^1 nx^n dx = \frac{n}{n+1}(1^{n+1} - 0^{n+1}) = \frac{n}{n+1}$$

So, in conclusion, we get that the PDF is $f = nx^{n-1}$ for $x \in (0, 1)$ and 0 for other x, $E(x) = \dfrac{n}{n+1}$.

Problem 4

From the question, we know that $X + Y = 1$ and that $X \leq Y$.

(a) As the question said, the stick is broken at uniformly random point, then we define the break point be U, then $U \sim Unif(0,1)$. Then we have that $X = min(U, 1-U)$, and $Y = max(U, 1-U)$. Then as for CDF, that is $P(R \leq r) = P(\frac{X}{Y} \leq r) = P(X \leq Y * r) = P(X \leq \frac{r}{1+r})$. Then as the CDF of X is $P(X \leq x) = 1 - P(X > x) = 1 - P(x < U < 1-x) = 2x$. So we get that $P(R \leq r) = \frac{2r}{1+r}$ for $r \in (0,1)$ and $P(R \leq r) = 0$ for $r \leq 0$ and 1 when $r \geq 1$. Then we have that the PDF is $f = (\frac{2r}{1+r})' = \frac{2}{(1+r)^2}$ for $r \in (0,1)$ and 0 for other r.

(b) As for the $E(R)$, we get that

$$E(R) = \int_0^1 rf(r)dr = \int_0^1 \frac{2r}{(1+r)^2}dr = \int_0^1 \frac{2(1-t)}{t^2}d(1-t) = 2(\int_1^2 \frac{1}{t}dt - \int_1^2 \frac{1}{t^2}dt) = 2ln2 - 1.$$

So, $E(R) = 2ln2 - 1$.

(c) As for the $E(\frac{1}{R})$, we get that

$$E(\frac{1}{R}) = \int_0^1 \frac{1}{r}f(r)dr = \int_0^1 \frac{2}{r(1+r)^2}dr.$$

However, $\frac{2}{r(1+r)^2}$ do not converges, so $E(\frac{1}{R})$ do not exists.

Problem 5

(a) According to the problem, we have that the j th trail happens at the time $(j-1)\Delta t$, then we have that there are totally $G+1$ trails, so the $T = (G+1-1)\Delta t = G\Delta t$

(b) Firstly, we try to find the $P(T > t)$, that is $P(T > t) = P(G > \frac{t}{\Delta t})$, then as we have that $G > n$ only when the first n+1 trails all fail, so we have $P(G > n) = (1 - \lambda\Delta t)^{n+1}$, then as for noninteger x, we have that $P(G > x) = (1 - \lambda\Delta t)^{\lfloor x \rfloor + 1}$, so we get that $P(T > t) = (1 - \lambda\Delta t)^{\lfloor \frac{t}{\Delta t} \rfloor + 1}$. Then the CDF is

$$P(T \le t) = 1 - P(T \ge t) = 1 - (1 - \lambda\Delta t)^{\lfloor \frac{t}{\Delta t} \rfloor + 1}, (t \ge 0)$$

When $t < 0$, we have $P(T \le t) = 0$

(c) (1) When $t = 0$, we have that $P(T \le 0) = P(T = 0) = \lambda\Delta t$, as $\Delta t \to 0$, we have that $P(T \le 0) \to 0$.

(2) When $t > 0$, we let $\Delta = \frac{1}{n}$, and let $n \to \infty$, as we have that $nt - 1 < \lfloor nt \rfloor < nt$, we have that

$$\lim_{n\to\infty} P(T \le t) = 1 - \lim_{n\to\infty} (1 - \frac{\lambda}{n})^{nt+1} = 1 - e^{-\lambda t}.$$

So as $\Delta t \to 0$, the CDF of T converges to the $Expo(\lambda)$ CDF, evaluating all the CDFs at a fixed $t \ge 0$.

Problem 6

Use the theroem LOTUS, we get that $E(max(Z-c,0)) = \int_{-\infty}^{\infty} max(z-c,0)\phi(z)dz$. Then we have that

$$\begin{aligned} E(max(Z-c,0)) &= \int_c^{\infty} (z-c)\varphi(z)dz \\ &= \int_c^{\infty} z\varphi(z)dz - \int_c^{\infty} c\varphi(z)dz \\ &= \frac{-1}{\sqrt{2\pi}}(e^{-\frac{\infty^2}{2}} - e^{-\frac{c^2}{2}}) - c\int_c^{\infty} e^{-\frac{z^2}{2}} dz \\ &= \frac{1}{\sqrt{2\pi}} e^{\frac{-c^2}{2}} - c(1-\Phi(c)) \end{aligned}$$

So we get that $E(max(Z-c,0)) = \frac{1}{\sqrt{2\pi}} e^{\frac{-c^2}{2}} - c(1-\Phi(c))$.