2023 Spring Probability & Statistics for EECS	February 26, 2023

Homework 03

Professor: Ziyu Shao & Dingzhu Wen — *Due: 23:59 on March 05, 2023*

1. A system composed of 5 homogeneous devices is shown in the following figure. It is said to be functional when there exists at least one end-to-end path that devices on such path are all functional. For such a system, if each device, which is independent of all other devices, functions with probability p, then what is the probability that the system functions? Such a probability is also called the system reliability.

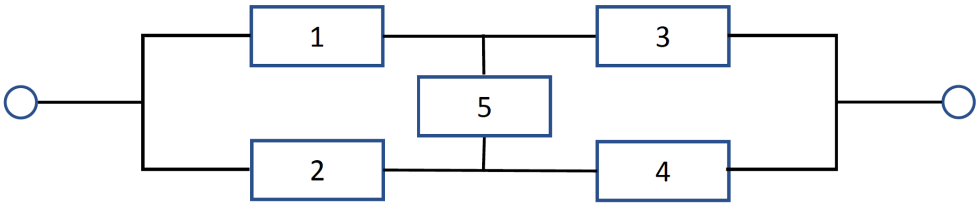

Figure 1: An illustration of the system composed of 5 homogeneous devices.

2. (a) Suppose that in the population of gamers, being good at Genshin Impact is independent of being good at Apex (with respect to some measure of "good"). A certain school club has a simple admission procedure: admit an applicant if and only if the applicant is good at Genshin Impact or is good at Apex.

   Intuitive explain why it makes sense that among gamers that the club admits, being good at Apex is negatively associated with being good at Genshin Impact, *i.e.*, conditioning on being good at Apex decreases the chance of being good at Genshin Impact.

   (b) (Berkson's paradox) Show that if A and B are independent and $C = A \cup B$, then A and B are conditionally dependent given C (as long as $P(A \cap B) > 0$ and $P(A \cup B) < 1$), with

$$P(A \mid B \cap C) < P(A \mid C).$$

3. A fair die is rolled repeatedly, and a running total is kept (which is, at each time, the total of all the rolls up until that time). Let p_n be the probability that the running total is ever exactly n (assume the die will always be rolled enough times so that the running total will eventually exceed n, but it may or may not ever equal n).

(a) Write down a recursive equation for p_n. Your equation should be true for all positive integers n, so give a definition of p_0 and p_k for $k < 0$ so that the recursive equation is true for small values of n.

(b) Find p_7.

(c) Give an intuitive explanation for the fact that $p_n \to 1/3.5 = 2/7$ as $n \to \infty$.

4. There are n types of toys, which you are collecting one by one. Each time you buy a toy, it is randomly determined which type it has, with equal probabilities. Let $p_{i,j}$ be the probability that just after you have bought your ith toy, you have exactly j toy types in your collection, for $i \geq 1$ and $0 \leq j \leq n$.

   (a) Find a recursive equation expressing $p_{i,j}$ in terms of $p_{i-1,j}$ and $p_{i-1,j-1}$, for $i \geq 2$ and $1 \leq j \leq n$.

   (b) Describe how the recursion from (a) can be used to calculate $p_{i,j}$.

5. Link is an immortal drunk man who wanders around randomly on the integers. He starts at the origin, and at each step he moves 1 unit to the right or 1 unit to the left, with probabilities p and $q = 1 - p$ respectively, independently of all his previous steps. Let S_n be his position after n steps.

   (a) Let p_k be the probability that the drunk ever reaches the value k, for all $k \geq 0$. Write down a recursive equation for p_k (you do not need to solve it for this part).

   (b) Find p_k, fully simplified; be sure to consider all 3 cases: $p < 1/2, p = 1/2$, and $p > 1/2$.