2023 Spring Probability & Statistics for EECS March 26, 2023

Homework 07

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1. Let
$$F(x) = \frac{2}{\pi}\sin^{-1}(\sqrt{x}), \text{ for } 0 < x < 1,$$
and let $F(x) = 0$ for $x \leq 0$ and $F(x) = 1$ for $x \geq 1$.
   (a) Check that F is a valid CDF, and find the corresponding PDF f.
   (b) Explain how it is possible for f to be a valid PDF even though $f(x)$ goes to ∞ as x approaches 0 and as x approaches 1.

2. Let F be a CDF which is continuous and strictly increasing. Let μ be the mean of the distribution. The quantile function, F^{-1}, has many applications in statistics and econometrics. Show that the area under the curve of the quantile function from 0 to 1 is μ.

3. Let $U_1, \ldots, U_n$ be i.i.d. Unif$(0,1)$, and $X = \max(U_1, ..., U_n)$. What is the PDF of X? What is $E(X)$?

4. A stick of length 1 is broken at a uniformly random point, yielding two pieces. Let X and Y be the lengths of the shorter and longer pieces, respectively, and let $R = X/Y$ be the ratio of the lengths X and Y.
   (a) Find the CDF and PDF of R.
   (b) Find the expected value of R (if it exists).
   (c) Find the expected value of $1/R$ (if it exists).

5. The Exponential is the analog of the Geometric in continuous time. This problem explores the connection between Exponential and Geometric in more detail, asking what happens to a Geometric in a limit where the Bernoulli trials are performed faster and faster but with smaller and smaller success probabilities.

   Suppose that Bernoulli trials are being performed in continuous time; rather than only thinking about first trial, second trial, etc., imagine that the trials take place at points on a timeline. Assume that the trials are at regularly spaced times 0, Δt, $2\Delta t, \ldots$, where Δt is a small positive number. Let the probability of success of each trial be $\lambda\Delta t$, where λ is a positive constant. Let G be the number of failures before the first success (in discrete time), and T be the time of the first success (in continuous time).

(a) Find a simple equation relating G to T.

(b) Find the CDF of T.

(c) Show that as $\Delta t \to 0$, the CDF of T converges to the Expo(λ) CDF, evaluating all the CDFs at a fixed $t \geq 0$.

6. Let $Z \sim \mathcal{N}(0, 1)$, and c be a nonnegative constant. Find $E(\max(Z - c, 0))$, in terms of the standard Normal CDF Φ and PDF φ.