Probability & Statistics for EECS: Homework #010

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Problem 1

(1) As for X discrete, Y discrete, we have that

$$P(X=x)=\sum_{y}P(X=x,Y=y)=\sum_{y}P(X=x|Y=y)P(Y=y)$$

(2) As for X continuous, Y discrete, we have that

$$P(X=x)=\sum_{-\infty}^{\infty}P(X=x|Y=y)f_Y(y)dy$$

Then we have that,

$$\lim_{\varepsilon\to 0}P(X\in(x-\varepsilon,x+\varepsilon))=\lim_{\varepsilon\to 0}\sum_{y}P(X\in(x-\varepsilon,x+\varepsilon)|Y=y)P(Y=y)$$

So we have that

$$f_X(x)=\sum_{y}f_X(x|Y=y)P(Y=y)$$

(3) As for X discrete, Y continuous, as we have that

$$P(X=x|Y=y)=\frac{f_Y(y|X=x)P(X=x)}{f_Y(y)}$$

Then we have

$$P(X=x|Y=y)f_Y(y)=f_Y(y|X=x)P(X=x)$$

Then we integrate both sides with respect to y, we have that

$$\int_{-\infty}^{\infty}P(X=x|Y=y)f_Y(y)dy=\int_{-\infty}^{\infty}f_Y(y|X=x)P(X=x)dy$$

Then we have that

$$f_X(x)=\int_{-\infty}^{\infty}f_Y(y|X=x)P(X=x)dy=P(X=x)$$

So we get that

$$P(X=x)=\int_{-\infty}^{\infty}P(X=x|Y=y)f_Y(y)dy$$

(4) As for X continuous, Y continuous, we have

$$f_X(x)=\int_{-\infty}^{\infty}f_{X,Y}(x,y)dy=\int_{-\infty}^{\infty}f_{Y|X}(y|x)f_X(x)dy=\int_{-\infty}^{\infty}f_{Y|X}(y|x)dy$$

So we get that

$$f_X(x)=\int_{-\infty}^{\infty}f_{Y|X}(y|x)dy$$

Problem 2

1. First, we let U be that the arrival time of the next Blissville company bus, then $U \sim Unif(0, 15)$ as the bus comes every 15 minutes, then we let $X \sim Expo(\frac{1}{15})$ is that be the arrival time of the next Blotchville company bus, then we have that

$$P(X < U) = \int_0^{15} P(X < U|U = u)\frac{1}{15}du$$

So we have that

$$P(X < U) = \frac{1}{15}\int_0^{15}(1 - e^{-\frac{u}{15}})du = \frac{1}{e}$$

2. As for the wait time, that is the wait until the first bus comes. Denote that the wait time is W, then we have $W = min(X, U)$ Then as for the CDF, we firstly calculate the $P(W > t)$, then we have that

$$P(W > t) = P(X > t, U > t) = P(X > t)P(U > t) = e^{-\frac{t}{15}}(1 - \frac{t}{15})$$

So we get that the CDF of the waiting time is that

$$P(W \leq t) = 1 - P(W > t) = 1 - e^{-\frac{t}{15}}(1 - \frac{t}{15}),$$

where $t \in (0, 10)$ and CDF is 0 for $t \leq 0$ and 1 for $t \geq 10$.

Problem 3

(a) Firstly we denote that p is the probability that an egg hatch, and $q = 1 - p$. As we have that X is the number which hatch, and that Y is the number which do not hatch, we have that $N = X + Y$, then we have that N, X, Y are dependent, as N is the sum of two variables. Then we have that

$$\begin{aligned}
P(N = n, X = x, Y = y) &= P(X = x, Y = y) \\
&= \sum_{n=0}^{\infty} P(X = x, Y = y | N = n) P(N = n) \\
&= P(X = x, Y = y | N = x + y) P(N = x + y) \\
&= P(X = x | N = x + y) P(N = x + y) \\
&= \binom{x+y}{x} p^x q^y \frac{e^{-\lambda} \lambda^{x+y}}{(x+y)!} \\
&= \frac{e^{-\lambda p} (\lambda p)^x}{x!} \frac{e^{-\lambda p} (\lambda p)^y}{y!}
\end{aligned}$$

Where n, x, y are nonnegative integers and $n = x + y$. N, X, Y are not independent, but as X and Y are independent, we also get that $X \sim Pois(\lambda p)$, $Y \sim Pois(\lambda q)$

(b) As for the joint PMF of N, X, as from (a) we have that $X \sim Pois(\lambda p)$ and that $Y \sim Pois(\lambda q)$ Then we have

$$P(N = n, X = x) = P(X = x, Y = n - x) = \frac{e^{-\lambda p} \lambda p^x}{x!} \frac{e^{-\lambda q} \lambda q^{n-x}}{(n-x)!},$$

where $n \geq x$ and that X and N are dependent as $N \geq X$

(c) As for joint PMF of X, Y, from (a) we have that

$$P(X = x, Y = y) = \frac{e^{-\lambda p} (\lambda p)^x}{x!} \frac{e^{-\lambda p} (\lambda p)^y}{y!}$$

where x and y are nonnegative integers

(d) As for the relationship of X and N, from (a) we have that $X \sim Pois(\lambda p)$ and $Y \sim Pois(\lambda q)$, then we have that

$$Cov(N, X) = Cov(X + Y, X) = Cov(X, X) + Cov(Y, X)$$

As X and Y are independent, then

$$Cov(N, X) = Var(X) = \lambda p$$

We then have

$$Corr(N, X) = \frac{Cov(X, Y)}{\sqrt{Var(X) Var(Y)}} = \frac{\lambda p}{\sqrt{\lambda \lambda p}} = \sqrt{p}$$

Problem 4

Denote that the two measurements are X, Y, as they are 2 independent standard Normal random variables, so X, Y i.i.d. $\sim N(0,1)$ and denote that $M = max(X, Y)$, $L = min(X, Y)$
So $max(x, y) + min(x, y) = x + y$, and $max(x, y) - min(x, y) = |x - y|$, we then have that

$$E(M) + E(L) = E(M + L) = E(X + Y) = E(X) + E(Y) = 0$$

Problem 5