Probability & Statistics for EECS: Homework #13

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Problem 1

(a) As for the distribution of N - 1, as $X_1, X_2, ...$ i.i.d. Expo(1), we have that $X_1, X_2, ...$ are independent, and each X has probability of $\frac{1}{e}$ to exceed 1. As to the definition of Geometric distribution, which is the number of trails to get the first success, we have that N - 1 follows Geometric distribution with parameter $\frac{1}{e}$. So we get that

$$N - 1 \sim Geom(\frac{1}{e}),$$

then with the property of Geometric distribution, we have that

$$E(N) = E(N-1) + 1 = \frac{1 - 1/e}{1/e} + 1 = e - 1 + 1 = e.$$

So in conclusion, we have that the distribution is

$$N - 1 \sim Geom(\frac{1}{e})$$

and the expectation is

$$E(N) = e.$$

(b) As for the

$$min\{n : X_1 + X_2 + ... + X_n \geq 10\},$$

it is obvious that this can be considered as Poisson process as we calculate the sum of X_i and observe them until sum exceeds 10, which is the same as Poisson process is the number of arrivals until the time exceeds 10, where the arrival interval follows Expo(1). So we can consider $X_1, X_2, ..., X_n$ as the interarrival times in a Poisson process with rate 1. where the range of the time is $[0, 10)$
Then we have that

$$M - 1 \sim Pois(10)$$

and we have that with the property of Poisson distribution, the $E(M)$ is

$$E(M) = E(M - 1 + 1) = E(M - 1) + 1 = 10 + 1 = 11$$

So we have that the distribution is

$$M - 1 \sim Pois(10)$$

and the expectation is

$$E(M) = 11.$$

(c) As for the $\overline{X}_n$, we have that

$$\overline{X}_n = \frac{(X_1 + X_2 + ... + X_n)}{n} = \frac{X_1}{n} + \frac{X_2}{n} + ... + \frac{X_n}{n}$$

, As we have that

$$X_1, X_2, ...i.i.d.Expo(1),$$

we then have that

$$\frac{X_1}{n}, \frac{X_2}{n}, ... \sim Expo(n)$$

Then we have that

$$\overline{X}_n \sim Gamma(n, n)$$

As for the approximate distribution of $\overline{X}_n$ for n large, with the center limit theorem, we have that when n is large, the distribution will be approximately normal distribution with the same mean and variance as the origin distribution. As we have that the origin distribution has mean of 1 and variance of 1, we have that the approximate distribution is normal distribution

$$\overline{X}_n \sim N(1, \frac{1}{n}).$$

So we have the exact distribution is Gamma distribution

$$\overline{X}_n \sim Gamma(n, n)$$

and the approximate distribution is normal distribution

$$\overline{X}_n \sim N(1, \frac{1}{n}).$$

Problem 2

To show that the inequality

$$P(|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu| \geq \varepsilon) \leq 2exp(-\frac{2n\varepsilon^2}{(b-a)^2}).$$

holds, we use the Hoeffding Lemma + Chernoff Inequality, the Hoeffding Lemma inequality is

$$E(e^{\lambda x}) \leq e^{\frac{1}{8}\lambda^2(b-a)^2},$$

the Chernoff Inequality is

$$P(X \geq a) \leq \frac{E(e^{tX})}{e^{ta}}.$$

Proof are as follows:

Problem 3

Problem 4

Problem 5