**2023 Spring Probability & Statistics for EECS** May 6, 2023

# Homework 13

*Professor: Ziyu Shao & Dingzhu Wen* *Due: 23:59 on May 14, 2023*

1. Let $X_1, X_2, \dots$ be i.i.d. Expo(1).

   (a) Let $N = \min\{n : X_n \geq 1\}$ be the index of the first $X_j$ to exceed 1. Find the distribution of $N-1$ (give the name and parameters), and hence find $E(N)$.

   (b) Let $M = \min\{n : X_1 + X_2 + \dots + X_n \geq 10\}$ be the number of $X_j$'s we observe until their sum exceeds 10 for the first time. Find the distribution of $M-1$ (give the name and parameters), and hence find $E(M)$.

   (c) Let $\bar{X}_n = (X_1 + \dots + X_n)/n$. Find the exact distribution of $\bar{X}_n$ (give the name and parameters), as well as the approximate distribution of $\bar{X}_n$ for $n$ large (give the name and parameters).

2. Let the random variables $X_1, X_2, \dots, X_n$ be independent with $E(X_i) = \mu$, $a \leq X_i \leq b$ for each $i = 1, \dots, n$, where $a, b$ are constants. Then for any $\epsilon \geq 0$, show the Hoeffding Bound holds:
$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq \epsilon\right) \leq 2\exp\left(-\frac{2n\epsilon^2}{(b-a)^2}\right).$$
   **Hint**: Hoeffding Lemma + Chernoff Inequality.

3. Given a random variable $X$ with expectation $\mu$ and variance $\sigma^2$. For any $a \geq 0$, show the following inequality holds:
$$P\left(X - \mu \geq a\right) \leq \frac{\sigma^2}{\sigma^2 + a^2}.$$

4. We observe a collection $X = (X_1, \dots, X_n)$ of random variables, with an unknown common mean whose value we wish to infer. We assume that given the value of the common mean, the $X_i$ are normal and independent, with known variances $\sigma_1^2, \dots, \sigma_n^2$. We model the common mean as a random variable $\Theta$, with a given normal prior (known mean $x_0$ and known variance $\sigma_0^2$). Find the posterior PDF of $\Theta$.

5. (a) We wish to estimate the parameter for an exponential distribution, denoted by $\theta$, based on the observations of $n$ independent random variables $X_1, \dots, X_n$, where $X_i \sim \text{Expo}(\theta)$. Find the MLE of $\theta$.

   (b) We wish to estimate the mean $\mu$ and variance $\nu$ of a normal distribution using $n$ independent observations $X_1, \dots, X_n$, where $X_i \sim \mathcal{N}(\mu, \nu)$. Find the MLE of the parameter vector $\theta = (\mu, \nu)$.