Probability & Statistics for EECS: Homework #05

Due on Mar 19, 2023 at 23:59

Name: **Penghao Wang**
Student ID: 2021533138

Problem 1

1. As the treasure has equally probability to be in the realm from 1 to 9. Then we have that:

   (a) If treasure in realm 1, then we need to ask 1 questions.

   (b) If treasure in realm 2, then we need to ask 2 questions.

   (c) ...

   (d) If treasure in realm 8, then we need to ask 8 questions.

   (e) Note that if treasure in realm 9, then we need to ask 8 questions.

   Then we denote that event X_i is that treasure in realm i. Then we have that:

   $$P(X_i) = \frac{1}{9}.$$

   Then we denote that Y is the number of questions we need to ask. Then we have that:

   $$P(Y = k) = P(X_k) = \frac{1}{9}$$

   Then we have that:

   $$E(X_i) = 1 * \frac{1}{9} + 2 * \frac{1}{9} + ... + 8 * \frac{2}{9} = \frac{44}{9}.$$

2. By using the bisection method, we could calculate the expected questions for each case that the treasure is in a specific realm.

   (a) If treasure in realm 1, then we need to ask 4 questions.

   (b) If treasure in realm 2, then we need to ask 4 questions.

   (c) If treasure in realm 3, then we need to ask 3 questions.

   (d) If treasure in realm 4, then we need to ask 3 questions.

   (e) If treasure in realm 5, then we need to ask 3 questions.

   (f) If treasure in realm 6, then we need to ask 3 questions.

   (g) If treasure in realm 7, then we need to ask 3 questions.

   (h) If treasure in realm 8, then we need to ask 3 questions.

   (i) If treasure in realm 9, then we need to ask 3 questions.

   Then we have that: the expected questions is $\frac{1}{9} * 4 + ... + \frac{1}{9} * 3 = \frac{29}{9}$.

Problem 2

Use the model Coupon Collector in Lecture, we firstly denote that X_k is the number of days needed to see the Youthuber eating steaks with k types deneness at least once.
Firstly we have $E[X_1] = 1$, then $X_2 - X_1 \sim Geom(\frac{n-1}{n})$, then we have $X_k - X_{k-1} \sim Geom(\frac{n-k}{n})$, then donote that $W_k = X_k - X_{k-1}$
Then we have

$$\begin{aligned} E[X_n] &= E[X_1 + X_2 - X_1 + ... + X_n - X_{n-1}] \\ &= E[W_1 + W_2 + ... + W_n] \\ &= E[W_1] + E[W_2] + E[W_n] \\ &= 1 + \frac{n}{n-1} + ... + \frac{n}{1} \\ &= n \sum_{i=1}^{n} \frac{1}{i} \end{aligned}$$

In this problem, k = 5, then we have $E[X_5] = 5 * (\frac{1}{1} + \frac{1}{2} + ... + \frac{1}{5}) = \frac{137}{12}$.

Problem 3

1. Denote the times that they are simulataneously successful is A. As we looking for the cases that they are simultaneously successful, where the probability that simulataneously successful is $p_1 * p_2 = p_1p_2$, then this is the geometric distribution, we have

$$A - 1 \sim Geom(p_1p_2)$$

and

$$A \sim Fs(p_1p_2)$$

Then we denote that $p = p_1p_2$ and that $q = 1 - p$, we have :

$$E(A) = \sum_{k=0}^{\infty} kP(X = k) = \sum_{k=0}^{\infty} kq^{k-1}p = \frac{p}{q}\sum_{k=0}^{\infty} kq^k.$$

Firstly we denote the $RES = \sum_{k=0}^{n} kq^k$, as $q * RES = q\sum_{k=0}^{n} kq^k = \sum_{k=0}^{\infty} kq^{k+1}$, then make subtraction and get $RES - q * RES = -nq^{n+1} + \sum_{i=1}^{n} q^n$, so we get that $RES = \dfrac{q(1 - q^n - nq^n + nq^{n+1})}{(1-q)^2}$.
Take it into $E(A)$, we have $E(A) = \dfrac{1}{p}\lim_{n\to\infty}(1 - q^n - nq^n + nq^{n+1}) = \dfrac{1}{p}$.
Then we get $p = p_1p_2$ and that $q = 1 - p$ into $E(A)$, we have that

$$E(A) = \frac{1}{p_1p_2}.$$

2. Denote the times that at least one has a success is B, the event's probability is $1 - (1 - p_1)(1 - p_2) = p_1 + p_2 - p_1p_2$, then the same as this problem's question 1, we denote that $p = p_1 + p_2 - p_1p_2, q = 1 - p$, we have

$$E(B) = \frac{1}{p} = \frac{1}{p_1 + p_2 - p_1p_2}.$$

3. Firstly we denote that $p = p_1 = p_2$, and $q = 1-p$, then we find the probability that they simultaneously success, denote that C_1, C_2 are the first time they success, then we have the probability $P(C_1 = C_2) = \sum_{n=1}^{\infty} P(C_1 = n, C_2 = n) = \sum_{n=1}^{\infty} p^2q^{2(n-1)} = \dfrac{p^2}{1-q^2} = \dfrac{p}{2-p}$.
Then, as we have that $P(C_1 = C_2) + P(C_1 < C_2) + P(C_1 > C_2) = P(C_1 = C_2) + 2P(C_1 < C_2) = 1$, we get the

$$P(C_1 < C_2) = \frac{1}{2}(1 - \frac{p}{2-p}) = \frac{1-p}{2-p}.$$

Problem 4

Firstly we denote the $RES = \sum_{k=0}^{n} kq^k$, as $q * RES = q\sum_{k=0}^{n} kq^k = \sum_{k=0}^{\infty} kq^{k+1}$, then make subtraction and get $RES - q * RES = -nq^{n+1} + \sum_{i=1}^{n} q^n$, so we get that $RES = \dfrac{q(1-q^n-nq^n+nq^{n+1})}{(1-q)^2}$. Then the $\lim_{n\to\infty}\sum_{k=0}^{n} kq^k = \dfrac{q}{(1-q)^2}$. So we get that:

$$\sum_{k=0}^{\infty} kq^k = \frac{q}{(1-q)^2}$$

1. According to the problem, we have that $X \sim Geom(p), Y \sim Geom(q)$. We have that $P(X=k) = (1-p)^k p, P(Y=k) = (1-q)^k q$. Then $P(X=Y) = \sum_{k=0}^{\infty} P(X=k, Y=k) = \sum_{k=0}^{\infty}(1-p)^k p(1-q)^k q = pq\sum_{k=0}^{\infty}((1-p)(1-q))^k = pq\dfrac{1}{1-(1-p)(1-q)} = \dfrac{pq}{p+q-pq}$.So, we have
$$P(X=Y) = \frac{pq}{p+q-pq}.$$

2. To solve $E[max(X,Y)]$, we firstly calculate the $P(max(X,Y)=k)$.
$$\begin{aligned} P[max(X,Y)=k] &= P(X=k, Y\le k) + P(Y=k, X<k) \\ &= P(X=k)P(Y\le k) + P(Y=k)P(X<k) \\ &= (1-p)^k p\sum_{n=0}^{k}(1-q)^n q + (1-q)^k q\sum_{n=0}^{k-1}(1-p)^n p \\ &= (1-p)^k p[1-(1-q)^{k+1}] + (1-q)^k q[1-(1-p)^k] \end{aligned}$$
We have that
$$\begin{aligned} E[max(X,Y)] &= \sum_{k=0}^{\infty} kP[max(X,Y)=k] \\ &= \sum_{k=0}^{\infty} k\{(1-p)^k p[1-(1-q)^{k+1}] + (1-q)^k q[1-(1-p)^k]\} \\ &= \sum_{k=0}^{\infty} k(1-p)^k p - k(1-p)^k p(1-q)^{k+1} + k(1-q)^k q - k(1-q)^k q(1-p)^k \\ &= \frac{1-p}{p} - p(1-q)\frac{(1-p)(1-q)}{(p+q-pq)^2} + \frac{1-q}{q} - q\frac{(1-p)(1-q)}{(p+q-pq)^2} \\ &= \frac{1-p}{p} + \frac{1-q}{q} - \frac{(1-p)(1-q)}{p+q-pq} \\ &= -1 - \frac{1}{p+q-pq} + \frac{1}{p} + \frac{1}{q}. \end{aligned}$$

3. According to the question 2 of this problem, we get that
$$\begin{aligned} P[min(X,Y)=k] &= P(X=k, Y\ge k) + P(Y=k, X>k) \\ &= P(X=k)P(Y\ge k) + P(Y=k)P(X>k) \\ &= (1-p)^k p\sum_{n=k}^{\infty}(1-q)^n q + (1-q)^k q\sum_{n=k+1}^{\infty}(1-p)^n p \\ &= (1-p)^k p(1-q)^k + (1-q)^k q(1-p)^{k+1} \\ &= [p+(1-p)q](1-p)^k(1-q)^k \\ &= (p+q-pq)[(1-p)(1-q)]^k \end{aligned}$$

Problem 4 continued on next page...

4. Firstly we calculate the probability $P(X=i|X\leq Y)$, we have

$$P(X=i|X\leq Y)=\frac{P(X=i,Y\geq i)}{P(X\leq Y)}=\frac{(1-p)^i p(1-q)^i}{\sum_{j=0}^{\infty}P(X=j)P(Y\geq j)}=\frac{(1-p)^i p(1-q)^i}{\sum_{j=0}^{\infty}(1-q)^j(1-p)^j p}$$

We get that

$$\begin{aligned}P(X=i|X\leq Y)&=(1-p)^n(1-q)^n(p+q-pq)\\E[X|X\leq Y]&=\sum_{k=0}^{\infty}kP(X=k|X\leq Y)\\&=\sum_{k=0}^{\infty}k(p+q-pq)(1-p)^k(1-q)^k\\&=(p+q-pq)\frac{(1-p)(1-q)}{(p+q-pq)^2}\\&=\frac{(1-p)(1-q)}{p+q-pq}\end{aligned}$$

So, the $E[X|X\leq Y]=\dfrac{(1-p)(1-q)}{p+q-pq}$

Problem 5

1. Denote that A_j is the rank of the jth dish you try, A is the sum of the ranks, X_i is the ith dish's rank. We have $A = A_1 + ... + A_k + (m-k)X$, as X is the rank of the best dish that you find in the exploration phase.

$$
\begin{aligned}
E(A_i) &= E(Y_i|X = X_1)P(X = X_1) + ... + E(Y_i|X = X_k)P(X = X_k) \\
&= \frac{\sum_{j=1}^{X_1-1} j}{k(X_1-1)} + ... + \frac{\sum_{j=1}^{X_k-1} j}{k(X_k-1)} \\
&= \frac{X_1}{2k} + \frac{X_2}{2k} + ... + \frac{X_k}{2k} \\
&= \frac{E(X)}{2}
\end{aligned}
$$

   We then have

$$E(A) = E(A_1) + E(A_2) + ... + E(A_k) + E(X) + (m-k)E(X) = \frac{E(X)(k+1)}{2} + (m-k)E(X)$$

   So we get

$$E(A) = (m - \frac{k}{2} + \frac{1}{2})E(X)$$

2. To find the PMF of X, which is $P(X = i)$, we need to choose a dish with rank i and other $k-1$ dishes with rank $\leq i$ from 1 to $j-1$, we have

$$P(X = i) = \frac{\binom{i-1}{k-1}}{\binom{n}{k}}.$$

3. As we have get the PMF of X, then we get that

$$E(X) = \sum_{i=k}^{n} iP(X = i) = \sum_{i=k}^{n} i\frac{\binom{i-1}{k-1}}{\binom{n}{k}} = \frac{k}{\binom{n}{k}}\sum_{i=k}^{n}\binom{i}{k} = k\frac{\binom{n+1}{k+1}}{\binom{n}{k}} = \frac{k(n+1)}{k+1}$$

4. To make the $E[X]$ maxmimze, we have

$$E[X] = (m - \frac{k}{2} + \frac{1}{2})\frac{k(n+1)}{k+1}$$

   To make this maxmimize, we need to get the differential to k, then we have

$$E[X]' = -\frac{1}{2}\frac{k(n+1)}{k+1} + (m - \frac{k}{2} + \frac{1}{2})\frac{n+1}{(k+1)^2} = 0$$

   Then we get

$$k = \frac{-2 \pm \sqrt{4 - 4(-2m-1)}}{2} = \sqrt{2(m+1)} \pm 1$$

   As $k \in [0, m]$, we get finally k is

$$k = \sqrt{2(m+1)} - 1.$$