Appendix I
Proofs and Parameter Settings

A. Proof of Theorem 2

Proof: According to (4) and (7), we have $\mathbf{X}_t = \sqrt{\alpha_t}\mathbf{X}_{t-1} + \sqrt{\beta_t}\mathbf{\Omega} = \sqrt{\alpha_t}\mathbf{X}_{t-1} + \sqrt{\eta(1-\alpha_t)}\mathbf{\Omega}$. Let $\psi \sim \mathcal{N}(0, \eta\mathbf{I})$, it observed that $\mathbf{X}_t = \sqrt{\alpha_t}\mathbf{X}_{t-1} + \sqrt{1-\alpha_t}\psi$. Then, recursively applying from $\{\mathbf{X}_T, \mathbf{X}_{T-1}, \ldots, \mathbf{X}_1\}$, we can derive the expression for $\mathbf{X}_T$ as

$$\begin{cases} \mathbf{X}_T = \sqrt{\alpha_T}\mathbf{X}_{T-1} + \sqrt{(1-\alpha_T)}\psi \\ \mathbf{X}_{T-1} = \sqrt{\alpha_{T-1}}\mathbf{X}_{T-2} + \sqrt{(1-\alpha_{T-1})}\psi \\ \vdots \\ \mathbf{X}_1 = \sqrt{\alpha_1}\mathbf{X}_0 + \sqrt{(1-\alpha_1)}\psi \end{cases}$$

$$\Rightarrow \mathbf{X}_T = \sqrt{\overline{\alpha}_T}\mathbf{X}_0 + \sqrt{(1-\overline{\alpha}_T)}\psi.$$

Let $\sigma = \sqrt{\eta\left(1-\overline{\alpha}_T\right)}$, we have $\sqrt{1-\overline{\alpha}_T}\psi \sim \mathcal{N}\left(0, \sigma^2\mathbf{I}\right)$. For given (ϵ, δ), according to line 12 in Algorithm 1, we can always find an appropriate $\omega \ll 1$, such that the value of $1-\overline{\alpha}_T$ is not small, while $\eta \gg 1$ satisfies $\sigma \geq \frac{\sqrt{2\log(1.25/\delta)}\Delta f}{\epsilon}$. Finally, based on the Definition 3, we conclude that the output $\mathbf{X}_T$ of Algorithm 1 satisfies (ϵ, δ)-DP.

B. Proof of Theorem 3

Proof: For a given input p in range $(0,1)$, the query function is $f(\mathbf{R}) = \sqrt{\overline{\alpha}_T}count(\mathbf{R}) = \sqrt{\overline{\alpha}_T}\mathbf{X}_0$, we have $\Delta f = \sqrt{\overline{\alpha}_T} = p < 1$. Let

$$\sqrt{\eta(1-\overline{\alpha}_T)} = \sigma = \frac{\sqrt{2\log(1.25/\delta)}}{\epsilon},$$

it holds $\sigma > \frac{\sqrt{2\log(1.25/\delta)}\Delta f}{\epsilon}$. Thus, $\mathbf{X}_T = \sqrt{\overline{\alpha}_T}\mathbf{X}_0 + \sqrt{\eta(1-\overline{\alpha}_T)}\mathbf{\Omega}$ is satisfies (ϵ,δ)-DP. Then, we deduce $\eta = \frac{\sigma^2}{1-\overline{\alpha}_{Tm}} = \frac{\sigma^2}{1-p^2}$, and $\omega = \frac{1}{\eta}$ as per (7). Thus, line 4 in Algorithm 2 is proven.

Next, we assume that Tm exists and has an upper bound Tup. Given β_0 and β_T such that $0 < \beta_0 < \beta_T < 1$, and the growth ratio k is ensured to be greater than 0. We have

$$\overline{\alpha}_T = \prod_{i=1}^{Tm}\left[1-\omega\left(ki+\beta_0\right)\right] < \prod_{i=1}^{Tm}\left(1-\omega\beta_0\right) \leq \left(1-\omega\beta_0\right)^{Tup}.$$

Moreover, $\left(1-\omega\beta_0\right)^{Tup} \leq p^2 = \overline{\alpha}_{Tm}$ always has a solution. Finally, by searching for Tm within a specific range $\{1,2,\ldots,Tup\}$, it can be found that the smallest Tm ensures that lines 5 to 8 of Algorithm 2 hold true.

From the above, Theorem 3 is proven.

C. Proof of Theorem 4

Proof: Based on the Bayes rule, we define the probability of $\mathbf{X}_{t-1}$ given $\mathbf{X}_t$ as

$$p\left(\mathbf{X}_{t-1} \,|\mathbf{X}_t\right) = \frac{p\left(\mathbf{X}_t \,|\mathbf{X}_{t-1}\right)\cdot p\left(\mathbf{X}_{t-1}\right)}{p\left(\mathbf{X}_t\right)}.$$

While giving $\mathbf{X}_0$ as a known condition , the probability does not change, i.e.,

$$p\left(\mathbf{X}_{t-1} \,|\mathbf{X}_t, \mathbf{X}_0\right) = \frac{p\left(\mathbf{X}_t \,|\mathbf{X}_{t-1}, \mathbf{X}_0\right)\cdot p\left(\mathbf{X}_{t-1} \,|\mathbf{X}_0\right)}{p\left(\mathbf{X}_t \,|\mathbf{X}_0\right)},$$

where, $\begin{cases} p\left(\mathbf{X}_t \,|\mathbf{X}_{t-1}, \mathbf{X}_0\right) = \sqrt{\alpha_t}\mathbf{X}_{t-1} + \sqrt{(1-\alpha_t)}\psi \\ p\left(\mathbf{X}_{t-1} \,|\mathbf{X}_0\right) = \sqrt{\overline{\alpha}_{t-1}}\mathbf{X}_0 + \sqrt{(1-\overline{\alpha}_{t-1})}\psi \\ p\left(\mathbf{X}_t \,|\mathbf{X}_0\right) = \sqrt{\overline{\alpha}_t}\mathbf{X}_0 + \sqrt{(1-\overline{\alpha}_t)}\psi \end{cases}$, it further follows as

$$\begin{cases} p\left(\mathbf{X}_t \,|\mathbf{X}_{t-1}, \mathbf{X}_0\right) \sim \mathcal{N}\left(\sqrt{\alpha_t}\mathbf{X}_{t-1}, \left(1-\alpha_t\right)\eta\mathbf{I}\right) \\ p\left(\mathbf{X}_{t-1} \,|\mathbf{X}_0\right) \sim \mathcal{N}\left(\sqrt{\overline{\alpha}_{t-1}}\mathbf{X}_0, \left(1-\overline{\alpha}_{t-1}\right)\eta\mathbf{I}\right) \\ p\left(\mathbf{X}_t \,|\mathbf{X}_0\right) \sim \mathcal{N}\left(\sqrt{\overline{\alpha}_t}\mathbf{X}_0, \left(1-\overline{\alpha}_t\right)\eta\mathbf{I}\right) \end{cases}.$$

Meanwhile, the PDF of the Gaussian distribution is defined as

$$f\left(x\right) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(\frac{-|x-\mu|^2}{2\sigma^2}\right).$$

According to the properties of the reparameterization, we get PDF of $p\left(\mathbf{X}_{t-1} \,|\mathbf{X}_t, \mathbf{X}_0\right)$ by taking $p\left(\mathbf{X}_t \,|\mathbf{X}_{t-1}, \mathbf{X}_0\right)$, $p\left(\mathbf{X}_{t-1} \,|\mathbf{X}_0\right)$, and $p\left(\mathbf{X}_t \,|\mathbf{X}_0\right)$ into $f(\cdot)$. The result is calculated as

$$f\left(\mathbf{X}_{t-1}\right) = \theta\exp\left(\chi\right),$$

where $\theta = \frac{\sqrt{1-\overline{\alpha}_t}}{\sqrt{2\pi}\sqrt{\eta(1-\alpha_t)(1-\overline{\alpha}_{t-1})}} = \frac{1}{\sqrt{2\pi}\sigma}$ and $\chi = -\frac{1}{2\eta}\left[\frac{(\mathbf{X}_t-\sqrt{\alpha_t}\mathbf{X}_{t-1})^2}{1-\alpha_t} + \frac{\left(\mathbf{X}_{t-1}-\sqrt{\overline{\alpha}_{t-1}}\mathbf{X}_0\right)^2}{1-\overline{\alpha}_{t-1}} - \frac{\left(\mathbf{X}_t-\sqrt{\overline{\alpha}_t}\mathbf{X}_0\right)^2}{1-\overline{\alpha}_t}\right]$. By further merging χ with similar items, we have $\chi = -\frac{1}{2\eta}\left[\theta_1\mathbf{X}_{t-1}^2 - \theta_2\mathbf{X}_{t-1} + \mathbf{c}\right]$, where $\theta_1 = \frac{\alpha_t}{1-\alpha_t} + \frac{1}{1-\overline{\alpha}_t}$, $\theta_2 = \frac{2\sqrt{\overline{\alpha}_t}}{1-\alpha_t}\mathbf{X}_t + \frac{2\sqrt{\overline{\alpha}_{t-1}}}{1-\overline{\alpha}_{t-1}}\mathbf{X}_0$, and $\mathbf{c}$ is a constant term that excludes $\mathbf{X}_{t-1}$. Thus, we conduct the parameters of $f\left(\mathbf{X}_{t-1}\right)$, which are the mean μ and the deviation σ^2, as follows:

$$\begin{cases} \frac{1}{\sigma^2} = \frac{1}{\eta}\frac{1-\overline{\alpha}_t}{(1-\alpha_t)(1-\overline{\alpha}_{t-1})} \\ \frac{2\mu}{\sigma^2} = \frac{1}{\eta}\left(\frac{2\sqrt{\overline{\alpha}_t}}{1-\alpha_t}\mathbf{X}_t + \frac{2\sqrt{\overline{\alpha}_{t-1}}}{1-\overline{\alpha}_{t-1}}\mathbf{X}_0\right) \end{cases}$$

$$\Rightarrow \begin{cases} \mu = \frac{\sqrt{\alpha_t}(1-\overline{\alpha}_{t-1})}{1-\overline{\alpha}_t}\mathbf{X}_t + \frac{\sqrt{\overline{\alpha}_{t-1}}(1-\alpha_t)}{1-\overline{\alpha}_t}\mathbf{X}_0 \\ \sigma^2 = \frac{\eta(1-\alpha_t)(1-\overline{\alpha}_{t-1})}{\alpha_t(1-\overline{\alpha}_{t-1})+(1-\alpha_t)} = \frac{\beta_t(1-\overline{\alpha}_{t-1})}{1-\overline{\alpha}_t} \end{cases}.$$

It can be observed that both μ and σ^2 are independent of the parameter η. Thus, the reverse process is not affected by the reconstruction relationship between α_t and β_t, as shown in (7). Finally, we solve $\mathbf{X}_0$ in reverse based on $\mathbf{X}_t$, i.e., $\mathbf{X}_0 = \frac{1}{\sqrt{\overline{\alpha}_t}}\left(\mathbf{X}_t - \sqrt{1-\overline{\alpha}_t}\psi\right)$. Finally, we get $\mu = \frac{1}{\sqrt{\alpha_t}}\left(\mathbf{X}_t - \frac{\beta_t}{\sqrt{1-\overline{\alpha}_t}}\mathbf{\Omega}\right)$. Then the parameters for $f\left(\mathbf{X}_{t-1}\right)$ are

$$\begin{cases} \mu = \frac{1}{\sqrt{\alpha_t}}\left(\mathbf{X}_t - \frac{\beta_t}{\sqrt{1-\overline{\alpha}_t}}\mathbf{\Omega}\right) \\ \sigma^2 = \frac{\beta_t(1-\overline{\alpha}_{t-1})}{1-\overline{\alpha}_t} \end{cases},$$

which are the same as the backward process of approach [24]. Thus, the Theorem 4 is proven.

D. Proof of Theorem 5

Proof: According to (15), let $\alpha = \frac{tc(i)}{tc(i)+tc(j)}$ and $\beta = 1-\alpha = \frac{tc(j)}{tc(i)+tc(j)}$, then, plugging them into (13) to get

$$\begin{cases} D_\alpha\left(P_i \,\|P_j\right) = \frac{1}{\alpha-1}\log\left(\sum_{r\in Scale} p_{ir}^{\alpha}\cdot p_{jr}^{1-\alpha}\right) \\ \frac{\alpha}{1-\alpha}D_\beta\left(P_j \,\|P_i\right) = -\frac{\alpha}{\beta}\cdot\frac{1}{\alpha}\log\left(\sum_{r\in Scale} p_{jr}^{\beta}\cdot p_{ir}^{1-\beta}\right) \\ \qquad = \frac{1}{\alpha-1}\log\left(\sum_{r\in Scale} p_{jr}^{1-\alpha}\cdot p_{ir}^{\alpha}\right) \end{cases}.$$

After the rearrangement, the equation $D_\alpha\left(P_i \,\|P_j\right) = \frac{\alpha}{1-\alpha}D_{1-\alpha}\left(P_j \,\|P_i\right)$ is proven.

TABLE I: Parameters of Tm, ω, and η

p	Params.	ϵ=0.1	ϵ=0.2	ϵ=0.3	ϵ=0.4	ϵ=0.5
0.9	ω (10^{-4})	1.10	4.10	9.10	16.20	25.20
	η (10^3)	9.091	2.439	1.099	0.617	0.367
	Tm (10^3)	3.831	1.027	0.463	0.260	0.167
0.8	ω (10^{-4})	2.00	7.70	17.20	30.60	47.70
	η (10^3)	5.000	1.299	0.581	0.329	0.210
	Tm (10^3)	4.462	1.159	0.518	0.291	0.187

E. Parameter settings

To achieve a target compression rate for matrix $\mathbf{C}$, we can configure the parameters of (7) and determine the minimum Tm to ensure sufficient noise injection while conserving computational resources. The parameter values calculated by Algorithm 2 are detailed in Table I above.