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1 Introduction

1.1 Arithmetic in [0,infty]

1.2 Metric Space

Lemma 1.2.1

X is a metric space, $x \in X$, B_1, B_2 are twos balls in X. If $x \in B_1 \cap B_2$, then x is the center of an open ball $B \subseteq B_1 \cap B_2$.

Proof. We may assume that B_1, B_2 have different center. Let $y_1 \neq y_2$,

$$B_1 = \{x : d(x, y_1) < r_1\}, \quad B_2 = \{x : d(x, y_2) < r_2\}$$

Take $x \in B_1 \cap B_2$, we claim that there exists $r_0 > 0$ such that $B := \{y : d(x, y) < r_0\} \subseteq B_1 \cap B_2$. Otherwise , for all $r > 0$, there exists a point y_r such that the following two hold at the same time

1. $d(x, y_r) < r$
2. $d(y_r, y_1) \geq r_1$ or $d(y_r, y_2) \geq r_2$.

Thus

$$d(x, y_1) \geq d(y_r, y_1) - d(x, y_r) > r_1 - r, \quad \text{or} \quad d(x, y_2) > r_2 - r$$

The above shows that

$$r > \min\{r_1 - d(x, y_1), r_2 - d(x, y_2)\}, \quad \forall r > 0$$

which is a contradiction. □

1.3 Limits of Set Sequences

Definition 1.3.1 ▸ Limit Inferior and Limit Superior

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of sets.

1. By the ***Limit Inferior of the sequence***, we mean

$$\liminf_{n\to\infty} A_n = \bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty} A_n = \{x : \exists N_0 \in \mathbb{N}, \forall n > N_0, x \in A_n\}$$

2. By the ***Limit Superior of the sequence***,we mean

$$\limsup_{n\to\infty} A_n = \bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty} A_n = \{x : \forall N \in \mathbb{N}, \exists n \geq N, x \in A_n\}$$

Note.

1. Limit Inferior 包含那些“最终稳定下来”的元素，即从某个点之后就永远属于序列中所有后续集合的元素。
2. Limit Superior 包含那些“反复出现”的元素，即在无限多个集合中出现的元素。

Proposition 1.3.2

For any sequence of sets $\{A_n\}$, it always holds that:

$$\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$$

Proof sketch.
直觉上是显然的，因为“最终稳定下来”的元素一定会“反复出现”．

Proof. It is obvious by the $\{x : x \in P\}$ form representation of the Limit Inferior and Superior. □

Definition 1.3.3 ▸ Limit of a Set Sequence

If the limit inferior and limit superior of a set sequence $\{A_n\}_{n=1}^{\infty}$ are equal, i.e., $\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$, then we say the ***limit*** of the sequence exists, and it is defined as:

$$\lim_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$$

Proposition 1.3.4

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of sets.

1. If $\{A_n\}$ is an increasing sequence, then the limit of $\{A_n\}$ exists and

$$\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n$$

2. If $\{A_n\}$ is a decreasing sequence, then the limit of $\{A_n\}$ exists and

$$\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$$

Proof. 1. If $\{A_n\}$ is increasing , then

$$\bigcap_{n=N}^{\infty} A_n = A_N$$

and

$$\bigcup_{n=N}^{\infty} A_n = \bigcup_{n=1}^{\infty} A_n, \forall N \in \mathbb{N}.$$

Thus

$$\liminf_{n\to\infty} A_n = \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} A_n = \bigcup_{N=1}^{\infty} A_N$$

and

$$\limsup_{n\to\infty} A_n = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} A_n = \bigcap_{N=1}^{\infty} \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} A_n$$

which are the same.

2. If $\{A_n\}$ is decreasing, then

$$\bigcap_{n=N}^{\infty} A_n = \bigcap_{n=1}^{\infty} A_n$$

and

$$\bigcup_{n=N}^{\infty} A_n = A_N$$

Thus

$$\liminf_{n\to\infty} A_n = \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} A_n = \bigcup_{N=1}^{\infty} \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} A_n$$

and

$$\limsup_{n\to\infty} A_n = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} A_n = \bigcap_{N=1}^{\infty} A_N$$

which are the same.

□

Proposition 1.3.5

For a sequence of sets $\{A_n\}_{n=1}^{\infty}$ and their corresponding sequence of indicator functions $\{\chi_{A_n}(x)\}_{n=1}^{\infty}$:

- $\chi_{\liminf_{n\to\infty} A_n}(x) = \liminf_{n\to\infty} \chi_{A_n}(x)$
- $\chi_{\limsup_{n\to\infty} A_n}(x) = \limsup_{n\to\infty} \chi_{A_n}(x)$
- $\lim_{n\to\infty} A_n$ exists, then $\chi_{\lim_{n\to\infty} A_n}(x) = \lim_{n\to\infty} \chi_{A_n}(x)$

Proof sketch.

- 若 x 在 limit inferior 里面, 则 x 是“最终稳定”的, $\chi_{A_n}(x)$ 是关于 n “最终”恒为 1 的.
- 若 x 在 limit superior 里面, 则 x 是“反复出现”的, 即相当于 N 多大, 总会出现之后的某个 n 使得 $\chi_{A_n}(x) = 1$.
- 当极限存在时, 函数列 $\left\{\chi_{A_n}(x)\right\}_{n=1}^{\infty}$ 的 limit inferior 和 supperior 根据上两条相等, 等于极限集合的 χ.

2 Abstract Integration

2.1 The Concept of Measurbility

Definition 2.1.1

1. A collection $\mathfrak{M}$ of subsets of a set X is said to be a *σ-algebra* in X if $\mathfrak{M}$ has the following properties:
   (a) $X \in \mathfrak{M}$.
   (b) If $A \in \mathfrak{M}$, then $A^c \in \mathfrak{M}$.
   (c) If $A = \bigcup_{n=1}^{\infty} A_n$ and $A_n \in \mathfrak{M}$ for $n = 1, 2, 3, \cdots$, then $A \in \mathfrak{M}$.
2. If $\mathfrak{M}$ is a σ-algebra in X, then X is called a ***measurable*** provided that $f^{-1}(V)$ is measurable set in X for every open set V in Y.
3. If X is a measurable space, Y is a topological space, and f is a mapping of X into Y, then f is said to be ***measurable*** provided that $f^{-1}(V)$ is a measurable set in X for every open set V in Y.

Note.
对比 topological space 的定义, σ-algebra 的定义是对称的, 即可测集的补集仍是可测的, 而对于拓扑空间, 开集的补集不是开集, 而是被定义为了闭集这样的对象.

Lemma 2.1.2

1. Let X be a measurable space , Y, Z be topological spaces. If $f : X \to Y$ is measurable , and if $g : Y \to Z$ is continuous, then $g \circ f : X \to Z$ is measurable.
2. Let u, v be real measurable functions on a measurable space X, let Φ be a continuous mapping of the plane into a topological space Y, and define
$$h(x) = \Phi(u(x), v(x))$$
for $x \in X$. Then $h : X \to Y$ is measurable.

Corollary 2.1.3

Let X be a measurable space. The following propositions hold

1. If $f = u + iv$, where u and v are real measurable functions on X, then f is a complex measurable function on X.
2. If $f = u + iv$ is a complex measurable function on X, then u, v, and $|f|$ are real measurable functions on X.
3. If f and g are complex measurable functions on X, then so are $f + g$ and fg.
4. If E is a measurable set in X and if
$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$
then χ_E is a measurable function.
5. If f is a complex measurable function on X , there is a complex measurable function α on X such that $|\alpha| = 1$ and $f = \alpha |f|$.

Theorem 2.1.4

If $\mathscr{F}$ is any collection of subsets of X , there exists a smallest σ-algebra $\mathfrak{M}^*$ such that $\mathscr{F} \subseteq \mathfrak{M}^*$.

Definition 2.1.5 ▶ Borel

Let X be a topological space.

1. By ***Borel σ-algebra*** , we mean the smallest σ-algebra $\mathscr{B}$ in X such that every open set in X belongs to $\mathscr{B}$. The members of $\mathscr{B}$ are called the ***Borel sets*** of X.
2. All countable unions of closed sets and all countable intersections of open sets are Borel sets, which we called ***F_σ's and G_δ's***, respectively.
3. By a ***Borel function***, we mean a measurable function on the measurable space $(X, \mathscr{B})$.

Remark.

1. The letters F and G were used for colsed and opensets,respectively, and σ refers to union, δ to intersection.
2. A continuous function is Borel-measurable, since the preimage of any open set is open and therefore a Borel set.

Theorem 2.1.6

Suppose $\mathfrak{M}$ is a σ-algbera in X ,and Y is a topological space. Let f map X into Y.

1. If Ω is the collection of all sets $E \subseteq Y$ such that $f^{-1}(E) \in \mathfrak{M}$, then Ω is a σ- algebra in Y.
2. If f is measurable and E is a Borel set in Y , then $f^{-1}(E) \subseteq \mathfrak{M}$.
3. If $Y = [-\infty, \infty]$ and $f^{-1}((\alpha, \infty]) \subseteq \mathfrak{M}$ for every real α, then f is measurable .
4. If f is measurable, if Z is topological space, if $g : Y \to Z$ is a Borel mapping, and if $h = g \circ f$, then $h : X \to Z$ is measurable.

Theorem 2.1.7

If $f_n : X \to [-\infty, \infty]$ is measurable,for $\neq 1, 2, 3, \cdots$, and

$$g = \sup_{n \geq 1} f_n, \quad h = \lim_{n \to \infty} \sup f_n,$$

then g and h are measurable.

Corollary 2.1.8

1. The limit of every pointwise convergent sequence of complex measurable functions is measurable.
2. If f and g are measurable(with range in $[-\infty, \infty]$), then so are $\max\{f, g\}$ and $\min\{f, g\}$. In particular , this is true of the functions

$$f^+ = \max\{f, 0\}, \quad f^- = -\min\{f, 0\}.$$

   which are called the ***positive part*** and ***negative part*** of f , respectively.

Remark.

1. There are the standrd representation

$$|f| = f^+ + f^-, \quad f = f^+ - f^-$$

2. And easy but (may) useful observation is : If $f = g - h, g \geq 0, h \geq 0$, then $f^+ \leq g$ and $f^- \leq h$.

2.2 Simple Functions

Definition 2.2.1

A complex function s on a measurable space X whose range consists of only finitely many points will be called a ***simple function***. Among these are the nonnegative simple functions, whose range is a finite subset of $[0, \infty)$.
Specifically, if $\alpha_1, \cdots, \alpha_n$ are distinct values of a simple function s, and if we set $A_i = \{x : s(x) = \alpha_i\}$, then clearly

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}.$$

Where χ_{A_i} is the characteristic function of A_i.

Remark.

- Here, we explicity exclude ∞ from the values of a simple function.
- It is clear that s is mearuable if and only if each of the sets A_i is measurable.

Theorem 2.2.2

Let $f : X \to [0, \infty]$ be measurable space. There exists simple measurable functions s_n on X such that

1. $0 \le s_1 \le s_2 \le \cdots \le f$.
2. $s_n(x) \to f(x)$ as $n \to \infty$, for every $x \in X$.

Proof sketch.
We construst a sequence of Borel simple functions $\{\varphi_n(x)\}$ to act as an ***identify***. 当 n 增大的同时，我们同时让 $\varphi_n(x)$ 的单位逼近精度和单位逼近范围随着 n 提升. 并且在舍弃误差时，总是向下取整，使得该单位逼近是自下而上的.

Proof. For every $x \in [0, \infty]$, and for every $n \in \mathbb{N}$, there exists a unique integer $k_n(x)$, such that

$$k_n(x) 2^{-n} \le x < (k_n(x) + 1) 2^{-n}$$

For every $n \in \mathbb{N}$, we define

$$\varphi_n(x) = \begin{cases} k_n(x)\,2^{-n}, & 0 \le x \le n \\ n, & x \ge n \end{cases}$$

Each $\varphi_n(x)$ is then a Borel simple function. It is not hard to show that $0 \le \varphi_1 \le \varphi_2 \cdots \le \cdots \le \mathrm{Id}$. For each n, we define

$$s_n(x) := (\varphi_n \circ f)(x)$$

Then $\{s_n\}$ is a suquence of simple measurable functions such that $0 \le s_1 \le s_2 \le \cdots \le f$. Since $\lim_{n\to\infty} \varphi_n(x) = x$, then $\lim_{n\to\infty} s_n(x) = f(x)$. □

2.3 Measure

Definition 2.3.1 ▸ Measure and Measure Space

(a) A ***positive measure*** is a function μ, defined on a σ-algebra $\mathfrak{M}$, whose range is in $[0,\infty]$ and which is ***countably additive***. This means that if $\{A_i\}$ is a disjoint countable collection of members of $\mathfrak{M}$, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

To avoid trivialities, we shall also assume that $\mu(A) < \infty$ for at least one $A \in \mathfrak{M}$.

(b) A ***measure space*** is a measurable space which has a positive measure defined on the σ-algebra of its measurable sets.

(c) A ***complex measure*** is a complex-valued countably additive function defined on a σ-algebra.

Theorem 2.3.2

Let μ be a positive measure on a σ-algebra $\mathfrak{M}$. Then

1. $\mu(\varnothing) = 0$.
2. $\mu(A_1 \cup \cdots \cup A_n) = \mu(A_1) + \cdots + \mu(A_n)$ if $A_1, \ldots, A_n$ are pairwise disjoint members of $\mathfrak{M}$.
3. $A \subset B$ implies $\mu(A) \le \mu(B)$ if $A \in \mathfrak{M}, B \in \mathfrak{M}$.

4. $\mu(A_n) \to \mu(A)$ as $n \to \infty$ if $A = \bigcup_{n=1}^{\infty} A_n, A_n \in \mathfrak{M}$, and
$$A_1 \subset A_2 \subset A_3 \subset \dots$$
5. $\mu(A_n) \to \mu(A)$ as $n \to \infty$ if $A = \bigcap_{n=1}^{\infty} A_n, A_n \in \mathfrak{M}$,
$$A_1 \supset A_2 \supset A_3 \supset \dots,$$
and $\mu(A_1)$ is finite.

Proof. 1. Take $A \in \mathfrak{M}$ such that $\mu(A) < \infty$. [1] And let $A_2 = A_3 = \cdots = \varnothing$, then $\mu(\varnothing) > 0$ leads to a contradition to the countably additive.

2. Take $A_{n+1} = A_{n+2} = \cdots = \varnothing$.
3. Note that $B = (B \setminus A) \cup A$, then by additivity
$$\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$$
4. Let $A_0 = \varnothing$, and let $B_n = A_n \setminus A_{n-1}$ for all $n \in \mathbb{N}$. Then $B_1, \cdots, B_n$ are pairwise disjoint members of $\mathfrak{M}$ such that $A = \bigcup_{n=1}^{\infty} B_n$. We have
$$\mu(A) = \sum_{n=1}^{\infty} \mu(B_n) = \sum_{n=1}^{\infty} \mu(A_n \setminus A_{n-1})$$
If one of the $\mu(A_n)$ is ∞, then $\lim_{n\to\infty} \mu(A_n)$ and $\mu(A)$ both are ∞ . Otherwise, we have
$$\mu(A_n \setminus A_{n-1}) = \mu(A_n) - \mu(A_{n-1}), \quad \forall n \mathbb{N}$$
Thus
$$\mu(A) = \sum_{n=1}^{\infty} \mu(A_n \setminus A_{n-1}) = \lim_{n\to\infty} \mu(A_n)$$
5. Let $B_n = A_n \setminus A_{n+1}$ for all $n \in \mathbb{N}$. $B_1, \cdots, B_n$ are pariwise disjoint members of $\mathfrak{M}$ with finite measure, such that
$$A_1 \setminus A = A_1 \setminus \left(\bigcap_{n=1}^{\infty} A_n \right) = \bigcup_{n=1}^{\infty} (A_1 \setminus A_n) = \bigcup_{n=1}^{\infty} \left(\bigcup_{k=1}^{n} B_k \right) = \bigcup_{n=1}^{\infty} B_n$$

[1] That is what we supposed at the definition of measure.

. We have

$$\mu\left(A_1 \setminus A\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right),$$

where the RHS is $\mu\left(A_1\right) - \mu\left(A\right)$, and the LHS is

$$\sum_{n=1}^{\infty}\left(\mu\left(A_n\right) - \mu\left(A_{n+1}\right)\right) = \mu\left(A_1\right) - \lim_{n\to\infty}\mu\left(A_{n+1}\right)$$

Since $\mu\left(A_1\right) < \infty$, we have

$$\mu\left(A\right) = \lim_{n\to\infty}\mu\left(A_n\right)$$

□

Example 2.3.3

1. For any $E \subset X$, where X is any set, define $\mu(E) = \infty$ if E is an infinite set, and let $\mu(E)$ be the number of points in E if E is finite. This μ is called the ***counting measure*** on X.
2. Fix $x_0 \in X$, define $\mu(E) = 1$ if $x_0 \in E$ and $\mu(E) = 0$ if $x_0 \notin E$, for any $E \subset X$. This μ may be called the ***unit mass concentrated at*** x_0.
3. Let μ be the counting measure on the set $\{1, 2, 3, \dots\}$, let $A_n = \{n, n+1, n+2, \dots\}$. Then $\bigcap A_n = \varnothing$ but $\mu(A_n) = \infty$ for $n = 1, 2, 3, \dots$. This shows that the hypothesis

$$\mu(A_1) < \infty$$

is not superfluous in Theorem 2.3.2(5).

2.4 Integration of Positive Funtions

Definition 2.4.1 ▸ Integration of Positive Funtions

1. If $s : X \to [0, \infty)$ is a measurable simple function, of the form

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i},$$

where $\alpha_1, \dots, \alpha_n$ are the distinct values of s, and if $E \in \mathfrak{M}$, we define

$$\int_E s\, d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E).$$

The convention $0 \cdot \infty = 0$ is used here; it may happen that $\alpha_i = 0$ for some i and that $\mu(A_i \cap E) = \infty$.

2. If $f : X \to [0, \infty]$ is measurable, and $E \in \mathfrak{M}$, we define

$$\int_E f\, d\mu = \sup \int_E s\, d\mu,$$

the supremum being taken over all simple measurable functions s such that $0 \le s \le f$. The left member above is called the ***Lebesgue integral*** of f over E, with respect to the measure μ. It is a number in $[0, \infty]$.

Remark.
We apparently have two definitions for $\int_E f\, d\mu$ if f is simple, they are the same.

Corollary 2.4.2

(a) If $0 \le f \le g$, then $\int_E f\, d\mu \le \int_E g\, d\mu$.
(b) If $A \subset B$ and $f \ge 0$, then $\int_A f\, d\mu \le \int_B f\, d\mu$.
(c) If $f \ge 0$ and c is a constant, $0 \le c < \infty$, then

$$\int_E cf\, d\mu = c \int_E f\, d\mu.$$

(d) If $f(x) = 0$ for all $x \in E$, then $\int_E f\, d\mu = 0$, even if $\mu(E) = \infty$.
(e) If $\mu(E) = 0$, then $\int_E f\, d\mu = 0$, even if $f(x) = \infty$ for every $x \in E$.
(f) If $f \ge 0$, then $\int_E f\, d\mu = \int_X \chi_E f\, d\mu$.

Remark.
We can also regard the $\int_E f\, d\mu$ as the restricted function $f|_E$ integrates on the induced measure space E.

Proposition 2.4.3

Let s and t be nonnegative measurable simple functions on X. For $E \in \mathfrak{M}$, define

$$\varphi(E) = \int_E s\,d\mu.$$

Then φ is a measure on $\mathfrak{M}$. Also

$$\int_X (s+t)\,d\mu = \int_X s\,d\mu + \int_X t\,d\mu.$$

Theorem 2.4.4 ▸ Lebesgue's Monotone Convergence Theorem

Let $\{f_n\}$ be a sequence of measurable functions on X, and suppose that
(a) $0 \le f_1(x) \le f_2(x) \le \cdots \le \infty$ for every $x \in X$,
(b) $f_n(x) \to f(x)$ as $n \to \infty$, for every $x \in X$.
Then f is measurable, and

$$\int_X f_n\,d\mu \to \int_X f\,d\mu \quad \text{as } n \to \infty.$$

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