目录

第 1 章　分离变量

Example 1.1

$$u_{tt}=a^2u_{xx},\ u\left(x,0\right)=\sin\frac{3\pi x}{l},\ u_t\left(x,0\right)=x\left(l-x\right),\ u\left(0,t\right)=u\left(l,t\right)=0$$

Solution 令 $u=X\left(x\right)T\left(t\right)$, 则

$$X''+\lambda X=0,\quad T''+a^2\lambda T=0$$

$$X\left(0\right)T\left(t\right)=X\left(l\right)T\left(t\right)=0$$

故

$$X\left(0\right)=X\left(l\right)=0$$

若方程有界, 则 $\lambda>0$, 设 $\lambda=k^2$, 则

$$X\left(x\right)=C_1\cos kx+C_2\sin kx$$

则 $C_1=0$, 取 $C_2=1$, 则

$$\sin kl=0\implies k=\frac{n\pi}{l},n=1,2,\cdots$$

令

$$X_n\left(x\right)=\sin\frac{n\pi}{l}x$$

$$T_n\left(t\right)=A_n\cos\frac{n\pi a}{l}t+B_n\sin\frac{n\pi a}{l}t$$

则

$$u=\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi a}{l}t+B_n\sin\frac{n\pi a}{l}t\right)\sin\frac{n\pi}{l}x$$

带入初值, 得到

$$\sin\frac{3\pi x}{l}=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi}{l}x$$

恰好是它的正弦展开, 于是

$$A_n=\frac{2}{l}\int_0^l\sin\frac{3\pi x}{l}\sin\frac{n\pi}{l}x\,\mathrm{d}x=\begin{cases}0 & ,n\neq 3\\ 2, & n=3\end{cases}$$

此外

$$x\left(l-x\right)=\sum_{n=1}^{\infty}\frac{n\pi a}{l}B_n\sin\frac{n\pi a}{l}x$$

于是

$$\frac{n\pi a}{l}B_n=\frac{2}{l}\int_0^l x\left(l-x\right)\sin\frac{n\pi}{l}x\,\mathrm{d}x$$

$$\begin{aligned}
\frac{2}{l}\int_0^l x\left(l-x\right)\sin\frac{n\pi}{l}x\,\mathrm{d}x &= -\frac{l}{n\pi}\frac{2}{l}\int_0^l x\left(l-x\right)\,\mathrm{d}\cos\frac{n\pi}{l}x\\
&=\frac{2}{n\pi}\int_0^l\cos\frac{n\pi x}{l}\left(l-2x\right)\,\mathrm{d}x\\
&=\frac{2}{n\pi}\int_0^l\left(-2x\right)\cos\frac{n\pi x}{l}\,\mathrm{d}x\\
&=\frac{-4l}{n^2\pi^2}\int_0^l x\,\mathrm{d}\sin\frac{n\pi x}{l}\\
&=\frac{4l}{n^2\pi^2}\int_0^l\sin\frac{n\pi x}{l}\,\mathrm{d}x\\
&=\frac{4l}{n^2\pi^2}\frac{l}{n\pi}\left(-\cos\frac{n\pi x}{l}\right)|_{x=0}^{x=l}\\
&=\frac{4l^2}{n^{3\pi^3}}\left(1-\left(-1\right)^n\right)
\end{aligned}$$

于是

$$B_n=\frac{l}{n\pi}\frac{4l^2}{n^3\pi^3}\left[1-\left(-1\right)^n\right]=\frac{4l^3}{n^4\pi^4a}\left[1-\left(-1\right)^n\right]$$

于是

$$u\left(x,t\right)=\sum_{n=1}^{\infty}\frac{4l^3}{n^4\pi^4a}\left[1-\left(-1\right)^n\right]\sin\frac{n\pi a}{l}t\sin\frac{n\pi}{l}x+\left(2\cos\frac{2\pi a}{l}t+\frac{4l^3}{2^4\pi^4a}\cdot 2\right)\sin\frac{3\pi}{l}x$$