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# Fundamental Group

# Covering Space

## 2.1 Basic Definitions

### Definition 2.1.1 ► Covering Space

Let  $p : \bar{X} \rightarrow X$  be a surjective map.

- We say an open subset  $V$  of  $X$  is **evenly covered** by  $p$ , if  $p^{-1}(V)$  is a disjoint union of open subsets of  $\bar{X}$ :

$$p^{-1}(V) = \coprod_i U_i$$

where, each  $U_i$  is mapped homeomorphically onto  $V$  by  $p$ .

- In this case, we shall refer to each  $U_i$  as a **sheet** for  $p$  over  $V$ .
- If the space  $X$  can be covered by open subsets which are evenly covered by  $p$ , then we say that  $p$  is a **covering projection**; the space  $\bar{X}$  is called a covering space of  $X$ .

### Theorem 2.1.2

Let  $p : \bar{X} \rightarrow X$  be a continuous map, where  $X$  is locally path connected.

1. The map  $p$  is a covering space iff for each component  $\bar{C}$  of  $\bar{X}$ , the restriction map  $p : p^{-1}(C) \rightarrow C$  is a covering projection.
2. If  $p$  is a covering projection then for each component  $\bar{C}$  of  $\bar{X}$ , the map  $p : \bar{C} \rightarrow p(\bar{C})$  is a covering projection and  $p(\bar{C})$  is a component of  $X$ .

#### Remark.

Thus, we always assume that a covering space is locally path connected and connected. For the following content, unless otherwise specified, we will all adopt this assumption.

**Proposition 2.1.3**

Let  $p : \bar{X} \rightarrow X$  be a covering projection. If  $X$  is Hausdorff then  $\bar{X}$  is Hausdorff. Further, if  $p$  is finite-to-one, then show that if  $\bar{X}$  is Hausdorff then  $X$  is Hausdorff.

*Note.*

I haven't finish the proof of the second part yet. Actually, if a covering projection is proper it is also closed. Once we have this conclusion we can then finish the proof easily.

*Proof.*

1. Take two different points  $\bar{x}, \bar{y} \in \bar{X}$ . Denote  $x := p(\bar{x}), y := p(\bar{y})$ .
  - (a) If  $x \neq y$ : Since  $X$  is Hausdorff, there exists two open sets  $U, V$  such that  $x \in U, y \in V, U \cap V = \emptyset$ . We have  $p^{-1}(U), p^{-1}(V)$  are two open subsets of  $\bar{X}$  with  $p^{-1}(U) \cap p^{-1}(V) = \emptyset$ . Since  $\bar{x} \in p^{-1}(U), \bar{y} \in p^{-1}(V)$ , it shows that  $\bar{X}$  is Hausdorff.
  - (b) If  $x = y$ : Let  $W$  be a open neighbourhood of  $x = y$ . We write  $p^{-1}(W) = \coprod_i W_i$ . Since  $\bar{x}, \bar{y} \in \coprod_i W_i$  and any two  $W_i$  are disjoint, there are two different  $W_{i_1}, W_{i_2}$  such that  $\bar{x} \in W_{i_1}, \bar{y} \in W_{i_2}$ .

The above shows that  $\bar{X}$  is Hausdorff.

2. Take two different points  $x, y \in X$ . Then  $p^{-1}(x), p^{-1}(y)$  are two finite sets of  $\bar{X}$  with empty intersection. We write

$$p^{-1}(x) = \{\bar{x}_i\}_{i=1}^n, \quad p^{-1}(y) = \{\bar{y}_i\}_{i=1}^n$$

For each  $i = 1, \dots, n$ , let  $U_i \ni \bar{x}_i, V_i \ni \bar{y}_i, U_i \cap V_i = \emptyset$ . Then  $p(\bigcup_i U_i) \cap p(\bigcup_i V_i) = \emptyset, x \in p(\bigcup_i U_i), y \in p(\bigcup_i V_i)$ .

□

# Basic Homology Algebra

## 3.1 Basic definition

### Definition 3.1.1

Consider a direct sum

$$C_{\bullet} := C_* := \bigoplus_{n \in \mathbb{Z}} C_n$$

of  $R$ -modules<sup>a</sup>. Often we call  $C_*$  a **graded-module** with its  $n^{\text{th}}$  **graded component**  $C_n$ . Members of  $C_n$  are also called **homogeneous elements** of  $C_*$  of degree  $n$ .

1. A  **$R$ -module homomorphism**  $f : C_* \rightarrow C'_*$  is called a **graded homomorphism** if there exists  $d$  such that  $f(C_r) \subseteq C'_{r+d}$  for all  $r$ . We then call  $d$  the **degree** of  $f$ . We shall denote  $f|_{C_r}$  by  $f_r$ , and often we may simply write  $f$  itself for  $f_r$  provided that there is no confusion.
2. By a **chain complex**  $(C_*, \partial)$  of  $R$ -modules, we mean a graded  $R$ -module  $C_*$ , together with an endomorphism  $\partial := \partial_* : C_* \rightarrow C_*$  of degree  $-1$  with the property  $\partial \circ \partial = 0$ . The endomorphism  $\partial$  is called the **differential** or the **boundary map** of the chain complex. Often we shall not mention the  $\partial$  at all and merely say  $C_*$  is a chain complex.
3. If  $C$  and  $C_*$  are two chain complexes then by a **chain map**  $f = f_* : C_* \rightarrow C'_*$  we mean a graded module homomorphism of degree 0 that commutes with the corresponding differentials.

---

<sup>a</sup>is also a  $R$ -module

### Remark.

1. The direct sum  $C_*$  is also an  $R$ -module.
2. Observe that  $\partial$  consists of a sequence  $\{\partial_n : C_n \rightarrow C_{n-1}\}$  of  $R$ -module homomorphisms such that  $\partial_n \circ \partial_{n-1} = 0$  for all  $n$ .
3.  $f$  consists of a sequence  $\{f_n : C_n \rightarrow C'_n\}$  of  $R$ -module homomorphisms such that  $\partial'_n \circ f_n = f_{n-1} \circ \partial_n$  for all  $n$ . Expressed with an

diagram, that is

$$\begin{array}{ccc} C_n & \xrightarrow{\partial_n} & C_{n-1} \\ \downarrow f_n & & \downarrow f_{n-1} \\ C'_n & \xrightarrow{\partial'_n} & C'_{n-1} \end{array}$$

### Proposition 3.1.2

There is a category of chain complexes of  $R$ -modules and chain maps. We shall denote this category by  $\mathbf{Ch}_R$ .

*Proof.*

The objects are chain complexes. The morphisms are chain maps. For composition of morphisms, consider two chain maps  $f : C_* \rightarrow C'_*$ ,  $g : C'_* \rightarrow C''_*$ . Only need to show that the following diagram commutes

$$\begin{array}{ccc} C_n & \xrightarrow{\partial_n} & C_{n-1} \\ \downarrow f_n & & \downarrow f_{n-1} \\ C'_n & \xrightarrow{\partial'_n} & C'_{n-1} \\ \downarrow f'_n & & \downarrow f'_{n-1} \\ C''_n & \xrightarrow{\partial''_n} & C''_{n-1} \end{array}$$

which is obvious. Finally, the existence of the identity map is also obvious.  $\square$

### Definition 3.1.3 ▶ Direct Sum of Chain Complexes

Define the direct sum of a family of chain complexes  $\{(C^\alpha, \partial^\alpha)\}_{\alpha \in \Lambda}$  as the chain complexes  $(C, \partial) := (\bigoplus_{\alpha} C^\alpha, \bigoplus_{\alpha} \partial^\alpha)$ , where the  $n^{\text{th}}$  graded component of  $C$  is

$$C_n = (C_n^\alpha)_{\alpha \in \Lambda},$$

$\partial$  is defined as

$$\partial((c^\alpha)_{\alpha \in \Lambda}) := (\partial^\alpha(c^\alpha))_{\alpha \in \Lambda}$$

## 3.2 Exact Sequence

### Definition 3.2.1 ▶ Exact Sequence

1. A sequence of  $R$ -modules

$$M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$$

is said to be **exact** at  $M$  if  $\ker \beta = \text{Im } \alpha$ .

2. A sequence

$$\cdots \longrightarrow M_{n-1} \longrightarrow M_n \longrightarrow M_{n+1} \longrightarrow \cdots$$

is said to be exact if it is exact at each  $M_n$ .

3. By a short exact sequence we mean an exact sequence of the form

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

### Definition 3.2.2 ▶ Exact Sequence of Chain Complexes

A sequence of chain complexes and chain maps

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} C'' \longrightarrow 0$$

is said to be exact if for each  $n$  the corresponding sequence of modules

$$0 \longrightarrow C'_n \xrightarrow{f_n} C_n \xrightarrow{g_n} C''_n \longrightarrow 0$$

is exact.

### Lemma 3.2.3 ▶ Snake lemma

Given a commutative diagram of  $R$ -module homomorphisms: where the two horizontal sequences are exact, there exists a  $R$ -module homomorphism

$$\begin{array}{ccccccc} M' & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M'' & \longrightarrow & 0 \\ \downarrow f' & & \downarrow f & & \downarrow f'' & & \\ 0 & \longrightarrow & N' & \xrightarrow{\alpha'} & N & \xrightarrow{\beta'} & N'' \end{array}$$

$\delta : \text{Ker } f'' \longrightarrow \text{Coker } f'$ , called the *connecting homomorphism* such that the sequence

$$\text{Ker } f' \xrightarrow{\bar{\alpha}} \text{Ker } f \xrightarrow{\bar{\beta}} \text{Ker } f'' \xrightarrow{\delta} \text{Coker } f' \xrightarrow{\bar{\alpha}'} \text{Coker } f \xrightarrow{\bar{\beta}'} \text{Coker } f''$$

is exact. Moreover, the **connecting homomorphism**  $\delta$  has the naturality properties, so that the above assignment of a ‘snake’ to the corresponding ‘six-term’ exact sequence of modules defines a covariant functor.

*Note.*

最关键的信息是 **connecting homomorphism**  $\delta$ . 构造的过程就是借助  $\beta$  的是 surjective,  $\alpha'$  是 injective, 来调整使得  $\beta$  和  $\alpha'$  对应的箭头是可以反转的. 事实上在应用时, 这个 connecting homomorphism 的具体构造往往比它的存在性更重要.

*Proof.*

**to be finished** We need to show

1.  $\bar{\alpha}, \bar{\beta}$  are well defined, where  $\bar{\alpha} := \alpha|_{\ker f'}, \bar{\beta} := \beta|_{\ker f}$
2.  $\text{Im } \bar{\alpha} = \ker \bar{\beta}$ .
3.  $\bar{\alpha}', \bar{\beta}'$  are well defined, where  $\bar{\alpha}'([x]) := [\alpha'(x)], \bar{\beta}'([y]) := [\beta'(y)]$
4.  $\text{Im } \bar{\alpha}' = \ker \bar{\beta}'$
5. Construct a  $\delta$ .
6.  $\text{Im } \bar{\beta} = \ker \delta$
7.  $\text{Im } \delta = \ker \bar{\alpha}'$

We prove the above points below.

1. By commutative,  $f \circ \alpha(\ker f') = \alpha' \circ f'(\ker f') = 0$ , which shows that  $\alpha(\ker f') \subseteq \ker f$ . Thus  $\bar{\alpha}$  is well defined.  $\bar{\beta}$ 's is similar.
- 2.

$$\text{Im } \bar{\alpha} = \alpha(\ker f') \subseteq \text{Im } \alpha = \ker \beta \implies \beta(\text{Im } \bar{\alpha}) = 0 \implies \text{Im } \bar{\alpha} \subseteq \ker \bar{\beta}$$

The other side,

$$\ker \bar{\beta} = \ker \beta \cap \ker f = \text{Im } \alpha \cap \ker f$$

Take  $y \in \ker \bar{\beta}$ , there exists  $x \in M'$  such that  $y = \alpha(x)$  and  $f(y) = f(\alpha(x)) = 0$ . Since  $f \circ \alpha = \alpha' \circ f'$ ,  $\alpha' \circ f'(x) = 0$ . Since  $\alpha'$  is injective,  $x \in \ker f'$ . Thus  $y \in \text{Im } \bar{\alpha}$ .

3. For  $\bar{\alpha}'$ , we need to show that for  $x_1, x_2 \in N'$  such that  $x_1 - x_2 \in \text{Im } f'$ ,



$\alpha' (x_1) - \alpha' (x_2) \in \text{Im } f$  . It is true by commutative

$$\alpha' (x_1) - \alpha' (x_2) = \alpha' (x_1 - x_2) \in \text{Im } (\alpha' \circ f') = \text{Im } (f \circ \alpha) \subseteq \text{Im } f$$

$\bar{\beta}'$ 's is similar.

4. Take  $[y]_{\sim \text{Im } f} \in \text{Im } \bar{\alpha}'$ , then there exists  $[x]_{\sim \text{Im } f'} \in \text{Coker } f'$ , such that  $\alpha' ([x]_{\sim \text{Im } f'}) = [y]_{\sim \text{Im } f}$ . Then  $\alpha' (x) - y \in \text{Im } f$ , there exists  $z \in M$ , such that

$$\alpha' (x) - y = f (z)$$

By commutative and exactness:  $\ker \beta' = \text{Im } \alpha'$ ,

$$\beta' (y) = \beta' (\alpha' (x) - f (z)) = 0 - \beta' \circ f (z) = -f'' \circ \beta (z) \in \text{Im } f''$$

, which shows that  $\bar{\beta}' ([y]_{\sim \text{Im } f}) = [0]_{\sim \text{Im } f''}$ ,  $\text{Im } \bar{\alpha}' \subseteq \ker \bar{\beta}'$ .

Then, take  $[y]_{\sim \text{Im } f} \in \ker \bar{\beta}'$ . Then  $y \in \ker \beta' = \text{Im } \alpha'$ . There exists  $x \in N'$  such that  $y = \alpha' (x)$ . Then  $[y]_{\sim \text{Im } f} = \alpha' ([x]_{\sim \text{Im } f'})$ ,  $[y]_{\sim \text{Im } f} \in \text{Im } \bar{\alpha}'$ ,  $\ker \bar{\beta}' \subseteq \text{Im } \bar{\alpha}'$ .

5. Take  $z \in \ker f''$ . Since  $\beta$  is surjective, there exists  $y \in M$  such that  $\beta (y) = z$ . Once we show that  $f (y) \in \text{Im } \alpha'$ , we can define

$$\delta (z) = [(\alpha')^{-1} (f (y))]_{\sim \text{Im } f'}$$

provided that this definition is well defined. By commutative

$$\beta' (f (y)) = f'' (\beta (y)) = f'' (z) = 0.$$

Hence  $f (y) \in \ker \beta' = \text{Im } \alpha'$ . Finally, to show that it is well defined, we take  $z_1, z_2 \in \ker f''$  such that  $\beta (y_1) = \beta (y_2) = z$ . We need to show that  $(\alpha')^{-1} (f (y_1 - y_2)) \in \text{Im } f'$ . Since  $\beta (y_1) - \beta (y_2) = 0$ ,  $\beta (y_1 - y_2) = 0$ ,  $y_1 - y_2 \in \ker \beta = \text{Im } \alpha$ . Suppose  $\alpha (x) = y_1 - y_2$ . Then

$$(\alpha')^{-1} (f (y_1 - y_2)) = (\alpha')^{-1} (f \alpha (x)) = (\alpha')^{-1} (\alpha' \circ f') (x) = f' (x) \in \text{Im } f'$$

, which completes the proof.

□

**Corollary 3.2.4**

Consider the following commutative diagram of  $R$ -modules and  $R$ -linear maps in which the two rows are exact. If  $f_1$  and  $f_3$  are isomorphisms then so is  $f_2$ .

$$\begin{array}{ccccccc}
 0 & \longrightarrow & M_1 & \xrightarrow{\alpha_1} & M_2 & \xrightarrow{\alpha_2} & M_3 \longrightarrow 0 \\
 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 \\
 0 & \longrightarrow & N_1 & \xrightarrow{\beta_1} & N_2 & \xrightarrow{\beta_2} & N_3 \longrightarrow 0
 \end{array}$$

*Proof.*

By snake lemma, we have

$$0 \xrightarrow{\alpha_1} \ker f_2 \xrightarrow{\alpha_2} 0 \xrightarrow{\delta} 0 \xrightarrow{\beta_1} \text{Coker } f_2 \xrightarrow{\beta_2} 0$$

is exact. Then we have

$$\begin{aligned}
 0 &= \ker \alpha_2 = \text{Im } \alpha_1 = 0 \\
 &= \ker f_2
 \end{aligned}$$

which shows that  $\ker f_2 = 0$ . Similarly,  $\text{Coker } f_2 = 0$ . The above shows that  $f_2$  is an isomorphism.  $\square$

**Corollary 3.2.5 ▶ Four lemma**

Consider the following commutative diagram of  $R$  modules and  $R$ -linear maps in which the two rows are exact. Suppose that  $f_1$  is surjective and  $f_4$  is injective. Then

- (i)  $f_2$  is injective  $\implies f_3$  is injective.
- (ii)  $f_3$  is surjective  $\implies f_2$  is surjective.

$$\begin{array}{ccccccc}
 M_1 & \xrightarrow{\alpha_1} & M_2 & \xrightarrow{\alpha_2} & M_3 & \xrightarrow{\alpha_3} & M_4 \\
 \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 \\
 N_1 & \xrightarrow{\beta_1} & N_2 & \xrightarrow{\beta_2} & N_3 & \xrightarrow{\beta_3} & N_4
 \end{array}$$

*Proof sketch.*

如果  $f_3$  被两个单射  $f_2, f_4$  夹在中间, 得益于  $f_1$  是满射, 可以将右三列保持单射性地调整为一条 snake, 从而导出  $f_3$  是单射.

类似地, 如果  $f_2$  被两个满射  $f_1, f_3$  夹在中间, 得益于  $f_4$  是单射, 左三列可以保持满射性地调整为一条 snake, 导出  $f_2$  是满射.

*Proof.*

1. We can construct a snake

$$\begin{array}{ccccccc}
 M_2 / \ker \alpha_2 & \xrightarrow{\alpha_2} & M_3 & \xrightarrow{\alpha_3} & \operatorname{Im} \alpha_3 & \longrightarrow & 0 \\
 \downarrow \bar{f}_2 & & \downarrow f_3 & & \downarrow f_4|_{\operatorname{Im} \alpha_3} & & \\
 0 \longrightarrow & N_2 / \ker \beta_2 & \xrightarrow{\beta_2} & N_3 & \xrightarrow{\beta_3} & N_4 & 
 \end{array}$$

By snake lemma, the following sequence is exact

$$\ker \bar{f}_2 \rightarrow \ker f_3 \rightarrow \ker f_4|_{\operatorname{Im} \alpha_3}$$

Since  $f_4$  is injective,  $f_4|_{\operatorname{Im} \alpha_3}$  is as well. Furthermore, by applying the snake lemma to the following diagram

$$\begin{array}{ccccccc}
 M_1 & \xrightarrow{\alpha_1} & M_2 & \xrightarrow{p} & M_2 / \ker \alpha_2 & \longrightarrow & 0 \\
 \bar{f}_1 \downarrow & & \downarrow f_2 & & \downarrow \bar{f}_2 & & \\
 0 \longrightarrow & N_1 / \ker \beta_1 & \xrightarrow{\bar{\beta}_1} & N_2 & \xrightarrow{p} & N_2 / \ker \beta_2 & 
 \end{array}$$

We get

$$0 = \ker f_2 \rightarrow \ker \bar{f}_2 \xrightarrow{\delta} \operatorname{Coker} \bar{f}_1$$

is a exact suquence, where  $\operatorname{Coker} \bar{f}_1 = 0$  since  $f_1$  is surjective and  $\bar{f}_1$  is as well. It shows that  $\ker \bar{f}_2 = 0$ . Finally, we have the following sequence is exact

$$0 = \ker \bar{f}_2 \rightarrow \ker f_3 \rightarrow \ker f_4|_{\operatorname{Im} \alpha_3} = 0$$

It follows that  $\ker f_3 = 0$ ,  $f_3$  is injective.

2. Another snake we can construct is

$$\begin{array}{ccccccc}
 M_1 & \xrightarrow{\alpha_1} & M_2 & \xrightarrow{\alpha_2} & \operatorname{Im} \alpha_2 = \ker \alpha_3 & \longrightarrow & 0 \\
 \downarrow \bar{f}_1 & & \downarrow f_2 & & \downarrow f_3|_{\operatorname{Im} \alpha_2} & & \\
 0 \longrightarrow & N_1 / \ker \beta_1 & \xrightarrow{\bar{\beta}_1} & N_2 & \xrightarrow{\beta_2} & \operatorname{Im} \beta_2 = \ker \beta_3 & 
 \end{array}$$

Then by snake lemma ,the following sequence is exact

$$\ker \bar{f}_1 \rightarrow \ker f_2 \rightarrow \ker f_3|_{\text{Im } \alpha_2} \rightarrow \text{Coker } \bar{f}_1 \rightarrow \text{Coker } f_2 \rightarrow \text{Coker } f_3|_{\text{Im } \alpha_2}$$

Furthermore, by applying the snake lemma to the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Ker } \alpha_3 & \xrightarrow{\iota_M} & M_3 & \xrightarrow{\alpha_3} & \text{Im } \alpha_3 & \longrightarrow & 0 \\ & & \downarrow f_3|_{\text{Im } \alpha_2} & & \downarrow f_3 & & \downarrow f_4|_{\text{Im } \alpha_3} & & \\ 0 & \longrightarrow & \text{Ker } \beta_3 & \xrightarrow{\iota_N} & N_3 & \xrightarrow{\beta_3} & \text{Im } \beta_3 & \longrightarrow & 0 \end{array}$$

We get

$$0 = \ker f_4|_{\text{Im } \alpha_3} \xrightarrow{\delta} \text{Coker } (f_3|_{\text{Im } \alpha_2}) \rightarrow \text{Coker } f_3 = 0$$

which shows that  $\text{Coker } (f_3|_{\text{Im } \alpha_2}) = 0$ . Finally, we have the following sequence exact

$$0 = \text{Coker } \bar{f}_1 \rightarrow \text{Coker } f_2 \rightarrow \text{Coker } f_3|_{\text{Im } \alpha_2} = 0$$

Thus  $\text{Coker } f_2 = 0$ ,  $f_2$  is surjective.

□

### Corollary 3.2.6 ► Five lemma

In the following diagram of  $R$ -modules, the two rows are given to be exact. If  $f_1, f_2, f_4$  and  $f_5$  are isomorphisms then  $f_3$  is also an isomorphism.

$$\begin{array}{ccccccccc} M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 & \longrightarrow & M_5 \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\ M'_1 & \longrightarrow & M'_2 & \longrightarrow & M'_3 & \longrightarrow & M'_4 & \longrightarrow & M'_5 \end{array}$$

*Proof.*

By applying the Four lemma to the left four columns, we get  $f_3$  is injective. And by applying the Four lemma to the right four columns , we get  $f_3$  is surjective .

□

### 3.3 Homology

#### Definition 3.3.1

Given a chain complex  $C_*$ , define the **homology group** of  $C_*$  to be the graded  $R$ -module

$$H_*(C_*) := \bigoplus_{n \in \mathbb{Z}} H_n(C_*)$$

by taking

$$H_n(C_*) := \text{Ker } \partial_n / \text{Im } \partial_{n+1}, \quad \forall n \in \mathbb{Z}.$$

#### Proposition 3.3.2

If  $f : C_* \rightarrow C'_*$  is a chain map then  $f$  induces a graded homomorphism

$$H_*(f) : H_*(C_*) \rightarrow H_*(C'_*)$$

In addition, this has the naturality property, viz.,

1.  $H_*(\text{Id}) = \text{Id}$
2. If  $g$  is another chain map such that  $f \circ g$  is defined, then

$$H_*(f \circ g) = H_*(f) \circ H_*(g)$$

Thus,  $H_*$  is a **covariant functor** from the category of chain complexes to the category of graded modules.

#### Theorem 3.3.3

The homology of a direct sum of chain complexes is isomorphic to the direct sum of the homology of chain complexes.

*Proof.*

$$H_n(\bigoplus_{\alpha} C_*^{\alpha}) = \text{ker}(\bigoplus_{\alpha} \partial_n^{\alpha}) / \text{Im}(\bigoplus_{\alpha} \partial_{n+1}^{\alpha})$$

$$\bigoplus_{\alpha} H_n(C_*^{\alpha}) = \bigoplus_{\alpha} (\text{ker } \partial_n^{\alpha} / \text{Im } \partial_{n+1}^{\alpha})$$

There are obvious isomorphisms :

$$\text{ker}(\bigoplus_{\alpha} \partial_n^{\alpha}) \simeq \bigoplus_{\alpha} \text{ker } \partial_n^{\alpha}, \quad \text{Im}(\bigoplus_{\alpha} \partial_{n+1}^{\alpha}) \simeq \bigoplus_{\alpha} \text{Im } \partial_{n+1}^{\alpha}$$

A standard result in algebra is that if  $K_\alpha$  is a submodule of  $M_\alpha$ , then

$$\frac{\bigoplus_\alpha M_\alpha}{\bigoplus_\alpha K_\alpha} \cong \bigoplus_\alpha \frac{M_\alpha}{K_\alpha}$$

. Hence , we have

$$\bigoplus_\alpha \ker \partial_n^\alpha / \bigoplus_\alpha \operatorname{Im} \partial_{n+1}^\alpha \simeq \bigoplus_\alpha (\ker \partial_n^\alpha / \operatorname{Im} \partial_{n+1}^\alpha)$$

□

### Theorem 3.3.4

Given a short exact sequence of chain complexes

$$0 \rightarrow C'_* \xrightarrow{\alpha} C_* \xrightarrow{\beta} C''_* \rightarrow 0$$

there is a functorial long exact sequence of homology groups

$$\rightarrow H_n(C'_*) \xrightarrow{H_n(\alpha)} H_n(C_*) \xrightarrow{H_n(\beta)} H_n(C''_*) \xrightarrow{\delta_n} H_{n-1}(C'_*) \xrightarrow{H_{n-1}(\alpha)} H_{n-1}(C_*) \rightarrow$$

*Proof sketch.*

Consider the diagram

$$\begin{array}{ccccccc} C'_n / \operatorname{Im} \partial'_{n+1} & \xrightarrow{\bar{\alpha}_n} & C_n / \operatorname{Im} \partial_{n+1} & \xrightarrow{\bar{\beta}_n} & C''_n / \operatorname{Im} \partial''_{n+1} & \longrightarrow & 0 \\ \downarrow \partial'_n & & \downarrow \partial_n & & \downarrow \partial''_{n+1} & & \\ 0 \longrightarrow & \operatorname{Ker} \partial'_{n-1} & \xrightarrow{\alpha'_{n-1}} & \operatorname{Ker} \partial_{n-1} & \xrightarrow{\beta'_{n-1}} & \operatorname{Ker} \partial''_{n-1} & \end{array}$$

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