

Basic Homology Algebra

1.1 Basic definition

Definition 1.1.1

Consider a direct sum

$$C_{\bullet} := C_* := \bigoplus_{n \in \mathbb{Z}} C_n$$

of R -modules^a. Often we call C_* a **graded-module** with its n^{th} **graded component** C_n . Members of C_n are also called **homogeneous elements** of C_* of degree n .

1. A **R -module homomorphism** $f : C_* \rightarrow C'_*$ is called a graded homomorphism if there exists d such that $f(C_r) \subseteq C'_{r+d}$ for all r . We then call d the degree of f . We shall denote $f|_{C_r}$ by f_r , and often we may simply write f itself for f_r provided that there is no confusion.
2. By a **chain complex** (C_*, ∂) of R -modules, we mean a graded R -module C_* , together with an endomorphism $\partial := \partial_* : C_* \rightarrow C_*$ of degree -1 with the property $\partial \circ \partial = 0$. The endomorphism ∂ is called the **differential** or the **boundry map** of the chain complex. Often we shall not mention the ∂ at all and merely say C_* is a chain complex.
3. If C and C_* are two chain complexes then by a **chain map** $f = f_* : C_* \rightarrow C'_*$ we mean a graded module homomorphism of degree 0 that commutes with the corresponding differentials.

^ais also a R -module

Remark.

1. The direct sum C_* is also an R -module.
2. Observe that ∂ consists of a sequence $\{\partial_n : C_n \rightarrow C_{n-1}\}$ of R -module homomorphisms such that $\partial_n \circ \partial_{n-1} = 0$ for all n .
3. f consists of a sequence $\{f_n : C_n \rightarrow C'_n\}$ of R -module homomorphisms such that $\partial'_n \circ f_n = f_{n-1} \circ \partial_n$ for all n . Expressed with an

diagram, that is

$$\begin{array}{ccc} C_n & \xrightarrow{\partial_n} & C_{n-1} \\ \downarrow f_n & & \downarrow f_{n-1} \\ C'_n & \xrightarrow{\partial'_n} & C'_{n-1} \end{array}$$

Proposition 1.1.2

There is a category of chain complexes of R -modules and chain maps. We shall denote this category by \mathbf{Ch}_R .

Proof. The objects are chain complexes. The morphisms are chain maps. For composition of morphisms, consider two chain maps $f : C_* \rightarrow C'_*$, $g : C'_* \rightarrow C''_*$. Only need to show that the following diagram commutes

$$\begin{array}{ccc} C_n & \xrightarrow{\partial_n} & C_{n-1} \\ \downarrow f_n & & \downarrow f_{n-1} \\ C'_n & \xrightarrow{\partial'_n} & C'_{n-1} \\ \downarrow f'_n & & \downarrow f'_{n-1} \\ C''_n & \xrightarrow{\partial''_n} & C''_{n-1} \end{array}$$

which is obvious. Finally, the existence of the identity map is also obvious. □

Definition 1.1.3 ▶ Direct Sum of Chain Complexes

Define the direct sum of a family of chain complexes $\{(C^\alpha, \partial^\alpha)\}_{\alpha \in \Lambda}$ as the chain complexes $(C, \partial) := (\bigoplus_\alpha C^\alpha, \bigoplus_\alpha \partial^\alpha)$, where the n^{th} graded component of C is

$$C_n = (C_n^\alpha)_{\alpha \in \Lambda},$$

∂ is defined as

$$\partial((c^\alpha)_{\alpha \in \Lambda}) := (\partial^\alpha(c^\alpha))_{\alpha \in \Lambda}$$

1.2 Exact Sequence

Definition 1.2.1 ► Exact Sequence

1. A sequence of R -modules

$$M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$$

is said to be **exact** at M if $\ker \beta = \operatorname{Im} \alpha$.

2. A sequence

$$\cdots \longrightarrow M_{n-1} \longrightarrow M_n \longrightarrow M_{n+1} \longrightarrow \cdots$$

is said to be exact if it is exact at each M_n .

3. By a short exact sequence we mean an exact sequence of the form

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

Definition 1.2.2 ► Exact Sequence of Chain Complexes

A sequence of chain complexes and chain maps

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} C'' \longrightarrow 0$$

is said to be exact if for each n the corresponding sequence of modules

$$0 \longrightarrow C'_n \xrightarrow{f_n} C_n \xrightarrow{g_n} C''_n \longrightarrow 0$$

is exact.

Lemma 1.2.3 ► Snake lemma

Given a commutative diagram of R -module homomorphisms: where the two horizontal sequences are exact, there exists a R -module homomorphism

$$\begin{array}{ccccccc} M' & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M'' & \longrightarrow & 0 \\ \downarrow f' & & \downarrow f & & \downarrow f'' & & \\ 0 & \longrightarrow & N' & \xrightarrow{\alpha'} & N & \xrightarrow{\beta'} & N'' \end{array}$$

$\delta : \operatorname{Ker} f'' \longrightarrow \operatorname{Coker} f'$, called the *connecting homomorphism* such that the sequence

$$\operatorname{Ker} f' \xrightarrow{\bar{\alpha}} \operatorname{Ker} f \xrightarrow{\bar{\beta}} \operatorname{Ker} f'' \xrightarrow{\delta} \operatorname{Coker} f' \xrightarrow{\bar{\alpha}'} \operatorname{Coker} f \xrightarrow{\bar{\beta}'} \operatorname{Coker} f''$$

is exact. Moreover, the **connecting homomorphism** δ has the naturality properties, so that the above assignment of a ‘snake’ to the corresponding ‘six-term’ exact sequence of modules defines a covariant functor.

Note.

最关键的信息是 **connecting homomorphism** δ . 构造的过程就是借助 β 的是 surjective, α' 是 injective, 来调整使得 β 和 α' 对应的箭头是可以反转的. 事实上在应用时, 这个 connecting homomorphism 的具体构造往往比它的存在性更重要.

Proof. to be finished We need to show

1. $\bar{\alpha}, \bar{\beta}$ are well defined, where $\bar{\alpha} := \alpha|_{\ker f'}, \bar{\beta} := \beta|_{\ker f}$
2. $\text{Im } \bar{\alpha} = \ker \bar{\beta}$.
3. $\bar{\alpha}', \bar{\beta}'$ are well defined, where $\bar{\alpha}'([x]) := [\alpha'(x)], \bar{\beta}'([y]) := [\beta'(y)]$
4. $\text{Im } \bar{\alpha}' = \ker \bar{\beta}'$
5. Construct a δ .
6. $\text{Im } \bar{\beta} = \ker \delta$
7. $\text{Im } \delta = \ker \bar{\alpha}'$

We prove the above points below.

1. By commutative, $f \circ \alpha(\ker f') = \alpha' \circ f'(\ker f') = 0$, which shows that $\alpha(\ker f') \subseteq \ker f$. Thus $\bar{\alpha}$ is well defined. $\bar{\beta}$'s is similar.
- 2.

$$\text{Im } \bar{\alpha} = \alpha(\ker f') \subseteq \text{Im } \alpha = \ker \beta \implies \beta(\text{Im } \bar{\alpha}) = 0 \implies \text{Im } \bar{\alpha} \subseteq \ker \bar{\beta}$$

The other side,

$$\ker \bar{\beta} = \ker \beta \cap \ker f = \text{Im } \alpha \cap \ker f$$

Take $y \in \ker \bar{\beta}$, there exists $x \in M'$ such that $y = \alpha(x)$ and $f(y) = f(\alpha(x)) = 0$. Since $f \circ \alpha = \alpha' \circ f'$, $\alpha' \circ f'(x) = 0$. Since α' is injective, $x \in \ker f'$. Thus $y \in \text{Im } \bar{\alpha}$.

3. For $\bar{\alpha}'$, we need to show that for $x_1, x_2 \in N'$ such that $x_1 - x_2 \in \text{Im } f'$, $\alpha'(x_1) - \alpha'(x_2) \in \text{Im } f$. It is true by commutative

$$\alpha'(x_1) - \alpha'(x_2) = \alpha'(x_1 - x_2) \in \text{Im } (\alpha' \circ f') = \text{Im } (f \circ \alpha) \subseteq \text{Im } f$$

$\bar{\beta}'$'s is similar.

4. Take $[y]_{\sim \text{Im } f} \in \text{Im } \bar{\alpha}'$, then there exists $[x]_{\sim \text{Im } f'} \in \text{Coker } f'$, such that $\alpha'([x]_{\sim \text{Im } f'}) = [y]_{\sim \text{Im } f}$. Then $\alpha'(x) - y \in \text{Im } f$, there exists $z \in M$, such that

$$\alpha'(x) - y = f(z)$$

By commutative and exactness: $\ker \beta' = \text{Im } \alpha'$,

$$\beta'(y) = \beta'(\alpha'(x) - f(z)) = 0 - \beta' \circ f(z) = -f'' \circ \beta(z) \in \text{Im } f''$$

, which shows that $\bar{\beta}'([y]_{\sim \text{Im } f}) = [0]_{\sim \text{Im } f''}$, $\text{Im } \bar{\alpha}' \subseteq \ker \bar{\beta}'$.

Then, take $[y]_{\sim \text{Im } f} \in \ker \bar{\beta}'$. Then $y \in \ker \beta' = \text{Im } \alpha'$. There exists $x \in N'$ such that $y = \alpha'(x)$. Then $[y]_{\sim \text{Im } f} = \alpha'([x]_{\sim \text{Im } f'})$, $[y]_{\sim \text{Im } f} \in \text{Im } \bar{\alpha}'$, $\ker \bar{\beta}' \subseteq \text{Im } \bar{\alpha}'$.

5. Take $z \in \ker f''$. Since β is surjective, there exists $y \in M$ such that $\beta(y) = z$. Once we show that $f(y) \in \text{Im } \alpha'$, we can define

$$\delta(z) = [(\alpha')^{-1}(f(y))]_{\sim \text{Im } f'}$$

provided that this definition is well defined. By commutative

$$\beta'(f(y)) = f''(\beta(y)) = f''(z) = 0.$$

Hence $f(y) \in \ker \beta' = \text{Im } \alpha'$. Finally, to show that it is well defined, we take $z_1, z_2 \in \ker f''$ such that $\beta(y_1) = \beta(y_2) = z$. We need to show that $(\alpha')^{-1}(f(y_1 - y_2)) \in \text{Im } f'$. Since $\beta(y_1) - \beta(y_2) = 0$, $\beta(y_1 - y_2) = 0$, $y_1 - y_2 \in \ker \beta = \text{Im } \alpha$. Suppose $\alpha(x) = y_1 - y_2$. Then

$$(\alpha')^{-1}(f(y_1 - y_2)) = (\alpha')^{-1}(f\alpha(x)) = (\alpha')^{-1}(\alpha' \circ f')(x) = f'(x) \in \text{Im } f'$$

, which completes the proof. □

Corollary 1.2.4

Consider the following commutative diagram of R -modules and R -linear maps in which the two rows are exact. If f_1 and f_3 are isomorphisms then so is f_2 .

$$\begin{array}{ccccccc}
0 & \longrightarrow & M_1 & \xrightarrow{\alpha_1} & M_2 & \xrightarrow{\alpha_2} & M_3 \longrightarrow 0 \\
& & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 \\
0 & \longrightarrow & N_1 & \xrightarrow{\beta_1} & N_2 & \xrightarrow{\beta_2} & N_3 \longrightarrow 0
\end{array}$$

Proof. By snake lemma, we have

$$0 \xrightarrow{\alpha_1} \ker f_2 \xrightarrow{\alpha_2} 0 \xrightarrow{\delta} 0 \xrightarrow{\beta_1} \operatorname{Coker} f_2 \xrightarrow{\beta_2} 0$$

is exact. Then we have

$$\begin{aligned}
0 &= \ker \alpha_2 = \operatorname{Im} \alpha_1 = 0 \\
&= \ker f_2
\end{aligned}$$

which shows that $\ker f_2 = 0$. Similarly, $\operatorname{Coker} f_2 = 0$. The above shows that f_2 is an isomorphism. \square

Corollary 1.2.5 ▶ Four lemma

Consider the following commutative diagram of R modules and R -linear maps in which the two rows are exact. Suppose that f_1 is surjective and f_4 is injective. Then

- (i) f_2 is injective $\implies f_3$ is injective.
- (ii) f_3 is surjective $\implies f_2$ is surjective.

$$\begin{array}{ccccccc}
M_1 & \xrightarrow{\alpha_1} & M_2 & \xrightarrow{\alpha_2} & M_3 & \xrightarrow{\alpha_3} & M_4 \\
\downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 \\
N_1 & \xrightarrow{\beta_1} & N_2 & \xrightarrow{\beta_2} & N_3 & \xrightarrow{\beta_3} & N_4
\end{array}$$

Proof sketch.

如果 f_3 被两个单射 f_2, f_4 夹在中间, 得益于 f_1 是满射, 可以将右三列保持单射性地调整为一条 snake, 从而导出 f_3 是单射.

类似地, 如果 f_2 被两个满射 f_1, f_3 夹在中间, 得益于 f_4 是单射, 左三列可以保持满射性地调整为一条 snake, 导出 f_2 是满射.

Proof. 1. We can construct a snake

$$\begin{array}{ccccccc}
 M_2/\ker \alpha_2 & \xrightarrow{\alpha_2} & M_3 & \xrightarrow{\alpha_3} & \operatorname{Im} \alpha_3 & \longrightarrow & 0 \\
 \downarrow \bar{f}_2 & & \downarrow f_3 & & \downarrow f_4|_{\operatorname{Im} \alpha_3} & & \\
 0 \longrightarrow & N_2/\ker \beta_2 & \xrightarrow{\beta_2} & N_3 & \xrightarrow{\beta_3} & N_4 &
 \end{array}$$

By snake lemma, the following sequence is exact

$$\ker \bar{f}_2 \rightarrow \ker f_3 \rightarrow \ker f_4|_{\operatorname{Im} \alpha_3}$$

Since f_4 is injective, $f_4|_{\operatorname{Im} \alpha_3}$ is as well. Furthermore, by applying the snake lemma to the following diagram

$$\begin{array}{ccccccc}
 M_1 & \xrightarrow{\alpha_1} & M_2 & \xrightarrow{p} & M_2/\ker \alpha_2 & \longrightarrow & 0 \\
 \bar{f}_1 \downarrow & & \downarrow f_2 & & \downarrow \bar{f}_2 & & \\
 0 \longrightarrow & N_1/\ker \beta_1 & \xrightarrow{\bar{\beta}_1} & N_2 & \xrightarrow{p} & N_2/\ker \beta_2 &
 \end{array}$$

We get

$$0 = \ker f_2 \rightarrow \ker \bar{f}_2 \xrightarrow{\delta} \operatorname{Coker} \bar{f}_1$$

is a exact suquence, where $\operatorname{Coker} \bar{f}_1 = 0$ since f_1 is surjective and \bar{f}_1 is as well. It shows that $\ker \bar{f}_2 = 0$. Finally, we have the following sequence is exact

$$0 = \ker \bar{f}_2 \rightarrow \ker f_3 \rightarrow \ker f_4|_{\operatorname{Im} \alpha_3} = 0$$

It follows that $\ker f_3 = 0$, f_3 is injective.

2. Another snake we can construct is

$$\begin{array}{ccccccc}
 M_1 & \xrightarrow{\alpha_1} & M_2 & \xrightarrow{\alpha_2} & \operatorname{Im} \alpha_2 = \ker \alpha_3 & \longrightarrow & 0 \\
 \downarrow \bar{f}_1 & & \downarrow f_2 & & \downarrow f_3|_{\operatorname{Im} \alpha_2} & & \\
 0 \longrightarrow & N_1/\ker \beta_1 & \xrightarrow{\bar{\beta}_1} & N_2 & \xrightarrow{\beta_2} & \operatorname{Im} \beta_2 = \ker \beta_3 &
 \end{array}$$

Then by snake lemma ,the following sequence is exact

$$\ker \bar{f}_1 \rightarrow \ker f_2 \rightarrow \ker f_3|_{\operatorname{Im} \alpha_2} \rightarrow \operatorname{Coker} \bar{f}_1 \rightarrow \operatorname{Coker} f_2 \rightarrow \operatorname{Coker} f_3|_{\operatorname{Im} \alpha_2}$$

Furthermore, by applying the snake lemma to the following diagram

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \text{Ker } \alpha_3 & \xrightarrow{\iota_M} & M_3 & \xrightarrow{\alpha_3} & \text{Im } \alpha_3 & \longrightarrow & 0 \\
 & & \downarrow f_3|_{\text{Im } \alpha_2} & & \downarrow f_3 & & \downarrow f_4|_{\text{Im } \alpha_3} & & \\
 0 & \longrightarrow & \text{Ker } \beta_3 & \xrightarrow{\iota_N} & N_3 & \xrightarrow{\beta_3} & \text{Im } \beta_3 & \longrightarrow & 0
 \end{array}$$

We get

$$0 = \ker f_4|_{\text{Im } \alpha_3} \xrightarrow{\delta} \text{Coker}(f_3|_{\text{Im } \alpha_2}) \rightarrow \text{Coker } f_3 = 0$$

which shows that $\text{Coker}(f_3|_{\text{Im } \alpha_2}) = 0$. Finally, we have the following sequence exact

$$0 = \text{Coker } \bar{f}_1 \rightarrow \text{Coker } f_2 \rightarrow \text{Coker } f_3|_{\text{Im } \alpha_2} = 0$$

Thus $\text{Coker } f_2 = 0$, f_2 is surjective.

□

Corollary 1.2.6 ▶ Five lemma

In the following diagram of R -modules, the two rows are given to be exact. If f_1, f_2, f_4 and f_5 are isomorphisms then f_3 is also an isomorphism.

$$\begin{array}{ccccccccc}
 M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 & \longrightarrow & M_5 \\
 \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\
 M'_1 & \longrightarrow & M'_2 & \longrightarrow & M'_3 & \longrightarrow & M'_4 & \longrightarrow & M'_5
 \end{array}$$

Proof. By applying the Four lemma to the left four columns, we get f_3 is injective. And by applying the Four lemma to the right four columns, we get f_3 is surjective.

□

1.3 Homology

Definition 1.3.1

Given a chain complex C_* , define the **homology group** of C_* to be the graded R -module

$$H_*(C_*) := \bigoplus_{n \in \mathbb{Z}} H_n(C_*)$$

by taking

$$H_n(C_*) := \text{Ker } \partial_n / \text{Im } \partial_{n+1}, \quad \forall n \in \mathbb{Z}.$$

Proposition 1.3.2

If $f : C_* \rightarrow C'_*$ is a chain map then f induces a graded homomorphism

$$H_*(f) : H_*(C_*) \rightarrow H_*(C'_*)$$

In addition, this has the naturality property, viz.,

1. $H_*(\text{Id}) = \text{Id}$
2. If g is another chain map such that $f \circ g$ is defined, then

$$H_*(f \circ g) = H_*(f) \circ H_*(g)$$

Thus, H_* is a **covariant functor** from the category of chain complexes to the category of graded modules.

Theorem 1.3.3

The homology of a direct sum of chain complexes is isomorphic to the direct sum of the homology of chain complexes.

Proof.

$$\begin{aligned} H_n(\bigoplus_{\alpha} C_*^{\alpha}) &= \text{ker}(\bigoplus_{\alpha} \partial_n^{\alpha}) / \text{Im}(\bigoplus_{\alpha} \partial_{n+1}^{\alpha}) \\ \bigoplus_{\alpha} H_n(C_*^{\alpha}) &= \bigoplus_{\alpha} (\text{ker } \partial_n^{\alpha} / \text{Im } \partial_{n+1}^{\alpha}) \end{aligned}$$

There are obvious isomorphisms :

$$\text{ker}(\bigoplus_{\alpha} \partial_n^{\alpha}) \simeq \bigoplus_{\alpha} \text{ker } \partial_n^{\alpha}, \quad \text{Im}(\bigoplus_{\alpha} \partial_{n+1}^{\alpha}) \simeq \bigoplus_{\alpha} \text{Im } \partial_{n+1}^{\alpha}$$

A standard result in algebra is that if K_α is a submodule of M_α , then

$$\frac{\bigoplus_\alpha M_\alpha}{\bigoplus_\alpha K_\alpha} \cong \bigoplus_\alpha \frac{M_\alpha}{K_\alpha}$$

. Hence , we have

$$\bigoplus_\alpha \ker \partial_n^\alpha / \bigoplus_\alpha \operatorname{Im} \partial_{n+1}^\alpha \simeq \bigoplus_\alpha (\ker \partial_n^\alpha / \operatorname{Im} \partial_{n+1}^\alpha)$$

□

Theorem 1.3.4

Given a short exact sequence of chain complexes

$$0 \rightarrow C'_* \xrightarrow{\alpha} C_* \xrightarrow{\beta} C''_* \rightarrow 0$$

there is a functorial long exact sequence of homology groups

$$\rightarrow H_n(C'_*) \xrightarrow{H_n(\alpha)} H_n(C_*) \xrightarrow{H_n(\beta)} H_n(C''_*) \xrightarrow{\delta_n} H_{n-1}(C'_*) \xrightarrow{H_{n-1}(\alpha)} H_{n-1}(C_*) \rightarrow$$

Proof sketch.

Consider the diagram

$$\begin{array}{ccccccc} C'_n / \operatorname{Im} \partial'_{n+1} & \xrightarrow{\bar{\alpha}_n} & C_n / \operatorname{Im} \partial_{n+1} & \xrightarrow{\bar{\beta}_n} & C''_n / \operatorname{Im} \partial''_{n+1} & \longrightarrow & 0 \\ \downarrow \partial'_n & & \downarrow \partial_n & & \downarrow \partial''_{n+1} & & \\ 0 \longrightarrow & \operatorname{Ker} \partial'_{n-1} & \xrightarrow{\alpha'_{n-1}} & \operatorname{Ker} \partial_{n-1} & \xrightarrow{\beta'_{n-1}} & \operatorname{Ker} \partial''_{n-1} & \end{array}$$