1 Basic Homology Algebra

1.1 Basic definition

Definition 1.1.1

Consider a direct sum

$$C_{.} := C_* := \oplus_{n\in\mathbb{Z}} C_n$$

of R-modules[a]. Often we call C_* a ***graded-module*** with its n^{th} ***graded component*** C_n. Members of C_n are also called ***homogeneous elements*** of C_* of degree n.

1. A **R**-***module homomorphism*** $f : C_* \to C'_*$ is called a graded homomorphism if there exists d such that $f(C_r) \subseteq C'_{r+d}$ for all r. We then call d the degree of f. We shall denote $f|_{C_r}$ by f_r, and often we may simply write f itself for f_r provided that there is no confusion.
2. By a ***chain complex*** (C_*, ∂) of R -modules, we mean a graded R-module C_*, together with an endomorphism $\partial := \partial_* : C_* \to C_*$ of degree -1 with the property $\partial \circ \partial = 0$. The endomorphism ∂ is called the ***differential*** or the ***boundry map*** of the chain complex. Often we shall not mention the ∂ at all and merely say C_* is a chain complex.
3. If C and C_* are two chain complexes then by a ***chain map*** $f = f_* : C_* \to C'_*$ we mean a graded module homomorphism of degree 0 that commutes with the corresponding differentials.

[a] is also a R-module

Remark.

1. The direct sum C_* is also an R-module.
2. Observe that ∂ consists of a sequence $\{\partial_n : C_n \to C_{n-1}\}$ of R-module homomorphisms such that $\partial_n \circ \partial_{n-1} = 0$ for all n.
3. f consists of a sequence $\{f_n : C_n \to C'_n\}$ of R-module homomorphisms such that $\partial'_n \circ f_n = f_{n-1} \circ \partial_n$ for all n. Expressed with an

diagram,that is

$$\begin{tikzcd} C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[d, "f_{n-1}"] \\ C'_n \arrow[r, "\partial'_n"] & C'_{n-1} \end{tikzcd}$$

Proposition 1.1.2

There is a category of chain complexes of R-modules and chain maps. We shall denote this category by $\mathbf{Ch_R}$.

Proof. The objects are chain complexes. The morphisms are chain maps.For composition of morphisms,consider two chain maps $f : C_* \to C'_*, g : C'_* \to C''_*$. Only need to show that the following diagram commutes

$$\begin{tikzcd} C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[d, "f_{n-1}"] \\ C'_n \arrow[r, "\partial'_n"] \arrow[d, "f'_n"] & C'_{n-1} \arrow[d, "f'_{n-1}"] \\ C''_n \arrow[r, "\partial''_n"] & C''_{n-1} \end{tikzcd}$$

which is obvious. Finally , the existence of the identity map is also obvious.

□

Definition 1.1.3 ▶ Direct Sum of Chain Complexes

Definthe the direct sum of a family of chain complexes $\{(C^\alpha, \partial^\alpha)\}_{\alpha\in\Lambda}$ as the chain complexes $(C, \partial) := (\oplus_\alpha C^\alpha, \oplus_\alpha \partial^\alpha)$, where the n^{th} graded component of C is

$$C_n = (C_n^\alpha)_{\alpha\in\Lambda},$$

∂ is defined as

$$\partial\left((c^\alpha)_{\alpha\in\Lambda}\right) := (\partial^\alpha(c^\alpha))_{\alpha\in\Lambda}$$

1.2 Exact Sequence

Definition 1.2.1 ▶ Exact Sequence

1. A sequence of R-modules
$$M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$$
is said to be ***exact*** at M if $\ker\beta = \operatorname{Im}\alpha$.
2. A sequence
$$\cdots \longrightarrow M_{n-1} \longrightarrow M_n \longrightarrow M_{n+1} \longrightarrow \cdots$$
is said to be exact if it is exact at each M_n.
3. By a short exact sequence we mean an exact sequence of the form
$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

Definition 1.2.2 ▶ Exact Sequence of Chain Complexes

A sequence of chain complexes and chain maps
$$0 \longrightarrow C'_{\cdot} \xrightarrow{f_{\cdot}} C_{\cdot} \xrightarrow{g_{\cdot}} C''_{\cdot} \longrightarrow 0$$
is said to be exact if for each n the corresponding sequence of modules
$$0 \longrightarrow C'_n \xrightarrow{f_n} C_n \xrightarrow{g_n} C''_n \longrightarrow 0$$
is exact.

Lemma 1.2.3 ▶ Snake lemma

Given a commutative diagram of R-module homomorphisms: where the two horizontal sequences are exact, there exists a R-module homomorphism

$$\begin{tikzcd}
 & M' \arrow[r, "\alpha"] \arrow[d, "f'"] & M \arrow[r, "\beta"] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & N' \arrow[r, "\alpha'"] & N \arrow[r, "\beta'"] & N'' &
\end{tikzcd}$$

$\delta : \operatorname{Ker} f'' \longrightarrow \operatorname{Coker} f'$, called the *connecting homomorphism* such that the sequence

$$\operatorname{Ker} f' \xrightarrow{\bar\alpha} \operatorname{Ker} f \xrightarrow{\bar\beta} \operatorname{Ker} f'' \xrightarrow{\delta} \operatorname{Coker} f' \xrightarrow{\bar\alpha'} \operatorname{Coker} f \xrightarrow{\bar\beta'} \operatorname{Coker} f''$$

is exact. Moreover, the ***connecting homomorphism*** δ has the naturality properties, so that the above assignment of a ‘snake’ to the corresponding ‘six-term’ exact sequence of modules defines a covariant functor.

Note.
最关键的信息是 ***connetcting homomorphism*** δ. 构造的过程就是借助 β 的是 surjective, α' 是 injectinve, 来调整使得 β 和 α' 对应的箭头是可以反转的. 事实上在应用时, 这个 connetcting homomorphism 的具体构造往往比它的存在性更重要.

Proof. ***to be finished*** We need to show

1. $\bar{\alpha}, \bar{\beta}$ are well defined, where $\bar{\alpha} := \alpha|_{\ker f'}, \bar{\beta} := \beta|_{\ker f}$
2. $\operatorname{Im} \bar{\alpha} = \ker \bar{\beta}$.
3. $\bar{\alpha}', \bar{\beta}'$ are well defined, where $\bar{\alpha}'([x]) := [\alpha'(x)]$, $\bar{\beta}'([y]) := [\beta'(y)]$
4. $\operatorname{Im} \bar{\alpha}' = \ker \bar{\beta}'$
5. Construct a δ.
6. $\operatorname{Im} \bar{\beta} = \ker \delta$
7. $\operatorname{Im} \delta = \ker \bar{\alpha}'$

We prove the above points below.

1. By commutative, $f \circ \alpha(\ker f') = \alpha' \circ f'(\ker f') = 0$, which shows that $\alpha(\ker f') \subseteq \ker f$. Thus $\bar{\alpha}$ is well defined. $\bar{\beta}$'s is similar.
2. $$\operatorname{Im} \bar{\alpha} = \alpha(\ker f') \subseteq \operatorname{Im} \alpha = \ker \beta \implies \beta(\operatorname{Im} \bar{\alpha}) = 0 \implies \operatorname{Im} \bar{\alpha} \subseteq \ker \bar{\beta}$$
   The other side,
   $$\ker \bar{\beta} = \ker \beta \cap \ker f = \operatorname{Im} \alpha \cap \ker f$$
   Take $y \in \ker \bar{\beta}$, there exists $x \in M'$ such that $y = \alpha(x)$ and $f(y) = f(\alpha(x)) = 0$. Since $f \circ \alpha = \alpha' \circ f'$, $\alpha' \circ f'(x) = 0$. Since α' is injective, $x \in \ker f'$. Thus $y \in \operatorname{Im} \bar{\alpha}$.
3. For $\bar{\alpha}'$, we need to show that for $x_1, x_2 \in N'$ such that $x_1 - x_2 \in \operatorname{Im} f'$, $\alpha'(x_1) - \alpha'(x_2) \in \operatorname{Im} f$. It is true by commutative
   $$\alpha'(x_1) - \alpha'(x_2) = \alpha'(x_1 - x_2) \in \operatorname{Im}(\alpha' \circ f') = \operatorname{Im}(f \circ \alpha) \subseteq \operatorname{Im} f$$

$\bar{\beta}'$'s is similar.

4. Take $[y]_{\sim \operatorname{Im} f} \in \operatorname{Im} \bar{\alpha}'$, then there exists $[x]_{\sim \operatorname{Im} f'} \in \operatorname{Coker} f'$, such that $\alpha'\left([x]_{\sim \operatorname{Im} f'}\right) = [y]_{\sim \operatorname{Im} f}$. Then $\alpha'(x) - y \in \operatorname{Im} f$, there exists $z \in M$, such that
$$\alpha'(x) - y = f(z)$$
By commutative and exactness: $\ker \beta' = \operatorname{Im} \alpha'$,
$$\beta'(y) = \beta'(\alpha'(x) - f(z)) = 0 - \beta' \circ f(z) = -f'' \circ \beta(z) \in \operatorname{Im} f''$$
, which shows that $\bar{\beta}'\left([y]_{\sim \operatorname{Im} f}\right) = [0]_{\sim \operatorname{Im} f''}$, $\operatorname{Im} \bar{\alpha}' \subseteq \ker \bar{\beta}'$.

   Then, take $[y]_{\sim \operatorname{Im} f} \in \ker \bar{\beta}'$. Then $y \in \ker \beta' = \operatorname{Im} \alpha'$. There exists $x \in N'$ such that $y = \alpha'(x)$. Then $[y]_{\sim \operatorname{Im} f} = \alpha^{\bar{\prime}}\left([x]_{\sim \operatorname{Im} f'}\right)$, $[y]_{\sim \operatorname{Im} f} \in \operatorname{Im} \bar{\alpha}'$, $\ker \bar{\beta}' \subseteq \operatorname{Im} \bar{\alpha}'$.

5. Take $z \in \ker f''$. Since β is surjective, there exists $y \in M$ such that $\beta(y) = z$. Once we show that $f(y) \in \operatorname{Im} \alpha'$, we can define
$$\delta(z) = [(\alpha')^{-1}(f(y))]_{\sim \operatorname{Im} f'}$$
provided that this definition is well defined. By commutative
$$\beta'(f(y)) = f''(\beta(y)) = f''(z) = 0.$$
Hence $f(y) \in \ker \beta' = \operatorname{Im} \alpha'$. Finally, to show that it is well defined, we take $z_1, z_2 \in \ker f''$ such that $\beta(y_1) = \beta(y_2) = z$. We need to show that $(\alpha')^{-1}(f(y_1 - y_2)) \in \operatorname{Im} f'$. Since $\beta(y_1) - \beta(y_2) = 0$, $\beta(y_1 - y_2) = 0$, $y_1 - y_2 \in \ker \beta = \operatorname{Im} \alpha$. Suppose $\alpha(x) = y_1 - y_2$. Then
$$(\alpha')^{-1}(f(y_1 - y_2)) = (\alpha')^{-1}(f\alpha(x)) = (\alpha')^{-1}(\alpha' \circ f')(x) = f'(x) \in \operatorname{Im} f'$$
, which completes the proof.

□

Corollary 1.2.4

Consider the following commutative diagram of R-modules and R-linear maps in which the two rows are exact. If f_1 and f_3 are isomorphisms then so is f_2.

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & M_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & M_3 \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & N_1 \arrow[r, "\beta_1"] & N_2 \arrow[r, "\beta_2"] & N_3 \arrow[r] & 0
\end{tikzcd}$$

Proof. By snake lemma, we have

$$0 \xrightarrow{\alpha_1} \ker f_2 \xrightarrow{\alpha_2} 0 \xrightarrow{\delta} 0 \xrightarrow{\beta_1} \operatorname{Coker} f_2 \xrightarrow{\beta_2} 0$$

is exact. Then we have

$$\begin{aligned} 0 &= \ker \alpha_2 = \operatorname{Im} \alpha_1 = 0 \\ &= \ker f_2 \end{aligned}$$

which shows that $\ker f_2 = 0$. Similarly, $\operatorname{Coker} f_2 = 0$. The above shows that f_2 is an isomorphisim. □

Corollary 1.2.5 ▶ Four lemma

Consider the following commutative diagram of R modules and R-linear maps in which the two rows are exact. Suppose that f_1 is surjective and f_4 is injective. Then

(i) f_2 is injective $\implies$ f_3 is injective.
(ii) f_3 is surjective $\implies$ f_2 is surjective.

$$\begin{tikzcd}
M_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & M_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & M_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"] & M_4 \arrow[d, "f_4"] \\
N_1 \arrow[r, "\beta_1"] & N_2 \arrow[r, "\beta_2"] & N_3 \arrow[r, "\beta_3"] & N_4
\end{tikzcd}$$

Proof sketch.
如果 f_3 被两个单射 f_2, f_4 夹在中间, 得益于 f_1 是满射, 可以将右三列保持单射性地调整为一条 snake, 从而导出 f_3 是单射.
类似地, 如果 f_2 被两个满射 f_1, f_3 夹在中间, 得益于 f_4 是单射, 左三列可以保持满射性地调整为一条 snake, 导出 f_2 是满射.

Proof. 1. We can construct a snake

$$\begin{tikzcd} & M_2/\ker\alpha_2 \arrow[r, "\alpha_2"] \arrow[d, "\bar{f}_2"] & M_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"] & \operatorname{Im}\alpha_3 \arrow[r] \arrow[d, "f_4|_{\operatorname{Im}\alpha_3}"] & 0 \\ 0 \arrow[r] & N_2/\ker\beta_2 \arrow[r, "\beta_2"] & N_3 \arrow[r, "\beta_3"] & N_4 & \end{tikzcd}$$

By snake lemma, the following sequence is exact

$$\ker\bar{f}_2 \to \ker f_3 \to \ker f_4|_{\operatorname{Im}\alpha_3}$$

Since f_4 is injective, $f_4|_{\operatorname{Im}_{\alpha_3}}$ is as well. Furthermore, by applying the snake lemma to the following diagram

$$\begin{tikzcd} & M_1 \arrow[r, "\alpha_1"] \arrow[d, "\bar{f}_1"'] & M_2 \arrow[r, "p"] \arrow[d, "f_2"] & M_2/\ker\alpha_2 \arrow[r] \arrow[d, "\bar{f}_2"] & 0 \\ 0 \arrow[r] & N_1/\ker\beta_1 \arrow[r, "\bar{\beta}_1"'] & N_2 \arrow[r, "p"'] & N_2/\ker\beta_2 & \end{tikzcd}$$

We get

$$0 = \ker f_2 \to \ker\bar{f}_2 \xrightarrow{\delta} \operatorname{Coker}\bar{f}_1$$

is a exact suquence,where $\operatorname{Coker}\bar{f}_1 = 0$ since f_1 is surjective and $\bar{f}_1$ is as well. It shows that $\ker\bar{f}_2 = 0$. Finally, we have the following sequence is exact

$$0 = \ker\bar{f}_2 \to \ker f_3 \to \ker f_4|_{\operatorname{Im}\alpha_3} = 0$$

It follows that $\ker f_3 = 0$, f_3 is injective.

2. Another snake we can construct is

$$\begin{tikzcd} & M_1 \arrow[r, "\alpha_1"] \arrow[d, "\bar{f}_1"] & M_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & \operatorname{Im}\alpha_2 = \ker\alpha_3 \arrow[r] \arrow[d, "f_3|_{\operatorname{Im}\alpha_2}"] & 0 \\ 0 \arrow[r] & N_1/\ker\beta_1 \arrow[r, "\bar{\beta}_1"] & N_2 \arrow[r, "\beta_2"] & \operatorname{Im}\beta_2 = \ker\beta_3 & \end{tikzcd}$$

Then by snake lemma ,the following sequence is exact

$$\ker\bar{f}_1 \to \ker f_2 \to \ker f_3|_{\operatorname{Im}\alpha_2} \to \operatorname{Coker}\bar{f}_1 \to \operatorname{Coker} f_2 \to \operatorname{Coker} f_3|_{\operatorname{Im}\alpha_2}$$

Furthermore, by applying the snake lemma to the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker}\alpha_3 \arrow[r, "\iota_M"] \arrow[d, "f_3|_{\operatorname{Im}\alpha_2}"] & M_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"] & \operatorname{Im}\alpha_3 \arrow[r] \arrow[d, "f_4|_{\operatorname{Im}\alpha_3}"] & 0 \\
0 \arrow[r] & \operatorname{Ker}\beta_3 \arrow[r, "\iota_N"] & N_3 \arrow[r, "\beta_3"] & \operatorname{Im}\beta_3 \arrow[r] & 0
\end{tikzcd}$$

We get

$$0 = \ker f_4|_{\operatorname{Im}\alpha_3} \xrightarrow{\delta} \operatorname{Coker}\left(f_3|_{\operatorname{Im}\alpha_2}\right) \to \operatorname{Coker} f_3 = 0$$

which shows that $\operatorname{Coker}\left(f_3|_{\operatorname{Im}\alpha_2}\right)$. Finally, we have the following sequence exact

$$0 = \operatorname{Coker}\bar{f}_1 \to \operatorname{Coker} f_2 \to \operatorname{Coker} f_3|_{\operatorname{Im}\alpha_2} = 0$$

Thus $\operatorname{Coker} f_2 = 0$, f_2 is surjective.

□

Corollary 1.2.6 ▸ Five lemma

In the following diagram of R-modules, the two rows are given to be exact. If f_1, f_2, f_4 and f_5 are isomorphisms then f_3 is also an isomorphism.

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d, "f_1"] & M_2 \arrow[r] \arrow[d, "f_2"] & M_3 \arrow[r] \arrow[d, "f_3"] & M_4 \arrow[r] \arrow[d, "f_4"] & M_5 \arrow[d, "f_5"] \\
M_1' \arrow[r] & M_2' \arrow[r] & M_3' \arrow[r] & M_4' \arrow[r] & M_5'
\end{tikzcd}$$

Proof. By applying the Four lemma to the left four columns, we get f_3 is injective. And by applying the Four lemma to the right four columns , we get f_3 is surjective .

□

1.3 Homology

Definition 1.3.1

Given a chain complex C_*, define the ***homology group*** of C_* to be the graded R-module

$$H_*(C_*) := \bigoplus_{n\in\mathbb{Z}} H_n(C_*)$$

by taking

$$H_n(C_*) := \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}, \quad \forall n \in \mathbb{Z}.$$

Proposition 1.3.2

If $f : C_* \to C'_*$ is a chain map then f induces a graded homomorphism

$$H_*(f) : H_*(C_*) \to H_*(C'_*)$$

In addition, this has the natruality property,viz.,

1. $H_*(\mathrm{Id}) = \mathrm{Id}$
2. If g is another chain map such that $f \circ g$ is defined, then

$$H_*(f \circ g) = H_*(f) \circ H_*(g)$$

Thus, H_* is a ***covariant functor*** from the category of chain complexe to the category of graded modules.

Theorem 1.3.3

The homology of a direct sum of chain complexes is isomorphic to the direct sum of the homology of chain complexes.

Proof.

$$H_n(\oplus_\alpha C_*^\alpha) = \ker(\oplus_\alpha \partial_n^\alpha) / \operatorname{Im}(\oplus_\alpha \partial_{n+1}^\alpha)$$

$$\bigoplus_\alpha H_n(C_*^\alpha) = \bigoplus_\alpha \left(\ker \partial_n^\alpha / \operatorname{Im} \partial_{n+1}^\alpha\right)$$

There are obvious isomorphics :

$$\ker(\oplus_\alpha \partial_n^\alpha) \simeq \oplus_\alpha \ker \partial_n^\alpha, \quad \operatorname{Im}(\oplus_\alpha \partial_{n+1}^\alpha) \simeq \oplus_\alpha \operatorname{Im} \partial_{n+1}^\alpha$$

A standard result in algebra is that if K_α is a submodule of M_α, then

$$\frac{\bigoplus_\alpha M_\alpha}{\bigoplus_\alpha K_\alpha} \cong \bigoplus_\alpha \frac{M_\alpha}{K_\alpha}$$

. Hence , we have

$$\oplus_\alpha \ker \partial_n^\alpha / \oplus_\alpha \operatorname{Im} \partial_{n+1}^\alpha \simeq \bigoplus_\alpha \left(\ker \partial_n^\alpha / \operatorname{Im}_{n+1}^\alpha\right)$$

□

Theorem 1.3.4

Given a short exact sequence of chain complexes

$$0 \to C'_* \xrightarrow{\alpha} C_* \xrightarrow{\beta} C''_* \to 0$$

there is a functorial long exact sequence of homology groups

$$\to H_n(C'_*) \xrightarrow{H_n(\alpha)} H_n(C_*) \xrightarrow{H_n(\beta)} H_n(C''_*) \xrightarrow{\delta_n} H_{n-1}(C'_*) \xrightarrow{H_{n-1}(\alpha)} H_{n-1}(C_*) \to$$

Proof sketch.
Consider the diagram

$$\begin{tikzcd}
& C'_n/\mathrm{Im}\partial'_{n+1} \arrow[r, "\bar{\alpha}_n"] \arrow[d, "\partial'_n"] & C_n/\mathrm{Im}\partial_{n+1} \arrow[r, "\bar{\beta}_n"] \arrow[d, "\partial_n"] & C''_n/\mathrm{Im}\partial''_{n+1} \arrow[r] \arrow[d, "\partial''_{n+1}"] & 0 \\
0 \arrow[r] & \mathrm{Ker}\partial'_{n-1} \arrow[r, "\alpha'_{n-1}"] & \mathrm{Ker}\partial_{n-1} \arrow[r, "\beta'_{n-1}"] & \mathrm{Ker}\partial''_{n-1} &
\end{tikzcd}$$