1 Introduction

1.1 Arithmetic in [0,infty]

1.2 Metric Space

Lemma 1.2.1

X is a metric space, $x \in X$, B_1, B_2 are twos balls in X. If $x \in B_1 \cap B_2$, then x is the center of an open ball $B \subseteq B_1 \cap B_2$.

Proof. We may assume that B_1, B_2 have different center. Let $y_1 \neq y_2$,

$$B_1 = \{x : d(x, y_1) < r_1\}, \quad B_2 = \{x : d(x, y_2) < r_2\}$$

Take $x \in B_1 \cap B_2$, we claim that there exists $r_0 > 0$ such that $B := \{y : d(x, y) < r_0\} \subseteq B_1 \cap B_2$. Otherwise , for all $r > 0$, there exists a point y_r such that the following two hold at the same time

1. $d(x, y_r) < r$
2. $d(y_r, y_1) \geq r_1$ or $d(y_r, y_2) \geq r_2$.

Thus

$$d(x, y_1) \geq d(y_r, y_1) - d(x, y_r) > r_1 - r, \quad \text{or} \quad d(x, y_2) > r_2 - r$$

The above shows that

$$r > \min\{r_1 - d(x, y_1), r_2 - d(x, y_2)\}, \quad \forall r > 0$$

which is a contradiction. □

1.3 Limits of Set Sequences

Definition 1.3.1 ▶ Limit Inferior and Limit Superior

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of sets.

1. By the ***Limit Inferior of the sequence***, we mean
$$\liminf_{n\to\infty} A_n = \bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty} A_n = \{x : \exists N_0 \in \mathbb{N}, \forall n > N_0, x \in A_n\}$$
2. By the ***Limit Superior of the sequence***,we mean
$$\limsup_{n\to\infty} A_n = \bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty} A_n = \{x : \forall N \in \mathbb{N}, \exists n \geq N, x \in A_n\}$$

Note.
1. Limit Inferior 包含那些“最终稳定下来”的元素，即从某个点之后就永远属于序列中所有后续集合的元素。
2. Limit Superior 包含那些“反复出现”的元素，即在无限多个集合中出现的元素。

Proposition 1.3.2

For any sequence of sets $\{A_n\}$, it always holds that:
$$\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$$

Proof sketch.
直觉上是显然的，因为“最终稳定下来”的元素一定会“反复出现”.

Proof. It is obvious by the $\{x : x \in P\}$ form representation of the Limit Inferior and Superior. □

Definition 1.3.3 ▶ Limit of a Set Sequence

If the limit inferior and limit superior of a set sequence $\{A_n\}_{n=1}^{\infty}$ are equal, i.e., $\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$, then we say the ***limit*** of the sequence exists, and it is defined as:

$$\lim_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$$

Proposition 1.3.4

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of sets.

1. If $\{A_n\}$ is an increasing sequence, then the limit of $\{A_n\}$ exists and

$$\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n$$

2. If $\{A_n\}$ is a decreasing sequence, then the limit of $\{A_n\}$ exists and

$$\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$$

Proof. 1. If $\{A_n\}$ is increasing , then

$$\bigcap_{n=N}^{\infty} A_n = A_N$$

and

$$\bigcup_{n=N}^{\infty} A_n = \bigcup_{n=1}^{\infty} A_n, \forall N \in \mathbb{N}.$$

Thus

$$\liminf_{n\to\infty} A_n = \bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty} A_n = \bigcup_{N=1}^{\infty} A_N$$

and

$$\limsup_{n\to\infty} A_n = \bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty} A_n = \bigcap_{N=1}^{\infty}\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} A_n$$

which are the same.

2. If $\{A_n\}$ is decreasing, then

$$\bigcap_{n=N}^{\infty} A_n = \bigcap_{n=1}^{\infty} A_n$$

and

$$\bigcup_{n=N}^{\infty} A_n = A_N$$

Thus

$$\liminf_{n\to\infty} A_n = \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} A_n = \bigcup_{N=1}^{\infty} \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} A_n$$

and

$$\limsup_{n\to\infty} A_n = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} A_n = \bigcap_{N=1}^{\infty} A_N$$

which are the same.

□

Proposition 1.3.5

For a sequence of sets $\{A_n\}_{n=1}^{\infty}$ and their corresponding sequence of indicator functions $\{\chi_{A_n}(x)\}_{n=1}^{\infty}$:

- $\chi_{\liminf_{n\to\infty} A_n}(x) = \liminf_{n\to\infty} \chi_{A_n}(x)$
- $\chi_{\limsup_{n\to\infty} A_n}(x) = \limsup_{n\to\infty} \chi_{A_n}(x)$
- $\lim_{n\to\infty} A_n$ exists, then $\chi_{\lim_{n\to\infty} A_n}(x) = \lim_{n\to\infty} \chi_{A_n}(x)$

Proof sketch.

- 若 x 在 limit inferior 里面，则 x 是“最终稳定”的，$\chi_{A_n}(x)$ 是关于 n “最终”恒为 1 的.
- 若 x 在 limit superior 里面，则 x 是“反复出现”的，即相当于 N 多大，总会出现之后的某个 n 使得 $\chi_{A_n}(x) = 1$.
- 当极限存在时，函数列 $\{\chi_{A_n}(x)\}_{n=1}^{\infty}$ 的 limit inferior 和 supperior 根据上两条相等，等于极限集合的 χ.