1 Covering Space

1.1 Basic Definitions

Definition 1.1.1 ▶ Covering Space

Let $p : \bar{X} \to X$ be a surjective map.

- We say an open subset V of X is ***evenly coverd*** by p, if $p^{-1}(V)$ is a disjoint union of open subsets of $\bar{X}$:

$$p^{-1}(V) = \coprod_i U_i$$

where, each U_i is maped homeomorphically onto V by p.
- In this case, we shall refer to each U_i as a ***sheet*** for p over V.
- If the space X can be covered by open subsets which are evenly covered by p, then we say that p is a ***covering projection***; the space $\bar{X}$ is called a covering space of X.

Theorem 1.1.2

Let $p : \bar{X} \to X$ be a contininuous map, where X is locally path connected.

1. The map p is a covering space iff for each componet $\bar{C}$ of $\bar{X}$, the restriction map $p : p^{-1}(C) \to C$ is a covering projection.
2. If p is a covering projection then for each component $\bar{C}$ of $\bar{X}$, the map $p : \bar{C} \to p(\bar{C})$ is a covering projection and $p(\bar{C})$ is a component of X.

Remark.
Thus, we always assume that *a* covering space is locally path connected and connetcted. For the following content, unless otherwise specified, we will all adopt this assumption.

Proposition 1.1.3

Let $p : \bar{X} \to X$ be a covering projection.If X is Hausdorff then $\bar{X}$ is Hausdorff. Further, if p is finite-to-one, then show that if $\bar{X}$ is Hausdorff then X is Hausdorff.

Note.
I haven't finish the proof of the second part yet. Actually, if a covering projection is proper it is also closed.Once we have this conclusion we can then finish the proof easilly.

Proof. 1. Take two different points $\bar{x}, \bar{y} \in \bar{X}$. Denote $x := p(\bar{x}), y := p(\bar{y})$.

(a) If $x \neq y$: Since X is Haussdorff , there exists two open sets U, V such that $x \in U, y \in V, U \cap V = \varnothing$. We have $p^{-1}(U), p^{-1}(V)$ are two open subsets of $\bar{X}$ with $p^{-1}(U) \cap p^{-1}(V) = \varnothing$. Since $\bar{x} \in p^{-1}(U), \bar{y} \in p^{-1}(V)$, it shows that $\bar{X}$ is Haussdorff.

(b) If $x = y$: Let W be a open neighbourhood of $x = y$. We write $p^{-1}(W) = \coprod_i W_i$. Since $\bar{x}, \bar{y} \in \coprod_i W_i$ and any two W_i are disjoint , there are two different W_{i_1}, W_{i_2} such that $\bar{x} \in W_{i_1}, \bar{y} \in W_{i_2}$.

The above shows that $\bar{X}$ is Hausdorff.

2. Take two different points $x, y \in X$. Then $p^{-1}(x), p^{-1}(y)$ are two finite sets of $\bar{X}$ with empty intersection. We write

$$p^{-1}(x) = \{\bar{x}_i\}_{i=1}^n, \quad p^{-1}(y) = \{\bar{y}_i\}_{i=1}^n$$

For each $i = 1, \cdots, n$,let $U_i \ni \bar{x}_i, V_i \ni \bar{y}_i, U_i \cap V_i = \varnothing$. Then $p\left(\bigcup_i U_i\right) \cap p\left(\bigcup_i V_i\right) = \varnothing$, $x \in p\left(\bigcup_i U_i\right), y \in p\left(\bigcup_i V_i\right)$.

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