Basic Homology Algebra

1.1 Basic definition

Definition 1.1.1

Consider a direct sum

$$C_{\cdot} := C_* := \oplus_{n \in \mathbb{Z}} C_n$$

of R-modules[a]. Often we call C_* a **graded-module** with its nth **graded component** C_n. Members of C_n are also called **homogeneous elements** of C_* of degree n.

1. A **R-module homomorphism** $f : C_* \to C'_*$ is called a graded homomorphism if there exists d such that $f(C_r) \subseteq C'_{r+d}$ for all r. We then call d the degree of f. We shall denote $f|_{C_r}$ by f_r, and often we may simply write f itself for f_r provided that there is no confusion.
2. By a **chain complex** (C_*, ∂) of R-modules, we mean a graded R-module C_*, together with an endomorphism $\partial := \partial_* : C_* \to C_*$ of degree -1 with the property $\partial \circ \partial = 0$. The endomorphism ∂ is called the **differential** or the **boundry map** of the chain complex. Often we shall not mention the ∂ at all and merely say C_* is a chain complex.
3. If C and C_* are two chain complexes then by a **chain map** $f = f_* : C_* \to C'_*$ we mean a graded module homomorphism of degree 0 that commutes with the corresponding differentials.

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[a]is also a R-module

Remark.

1. The direct sum C_* is also an R-module.
2. Observe that ∂ consists of a sequence $\{\partial_n : C_n \to C_{n-1}\}$ of R-module homomorphisms such that $\partial_n \circ \partial_{n-1} = 0$ for all n.
3. f consists of a sequence $\{f_n : C_n \to C'_n\}$ of R-module homomorphisms such that $\partial'_n \circ f_n = f_{n-1} \circ \partial_n$ for all n. Expressed with an

diagram,that is

$$C_n \xrightarrow{\partial_n} C_{n-1}$$
$$\downarrow{f_n} \qquad\qquad \downarrow{f_{n-1}}$$
$$C'_n \xrightarrow{\partial'_n} C'_{n-1}$$

Proposition 1.1.2

There is a category of chain complexes of R-modules and chain maps. We shall denote this category by $\mathbf{Ch_R}$.

Proof. The objects are chain complexes. The morphisms are chain maps.For composition of morphisms,consider two chain maps $f : C_* \to C'_*, g : C'_* \to C''_*$. Only need to show that the following diagram commutes

$$C_n \xrightarrow{\partial_n} C_{n-1}$$
$$\downarrow{f_n} \qquad\qquad \downarrow{f_{n-1}}$$
$$C'_n \xrightarrow{\partial'_n} C'_{n-1}$$
$$\downarrow{f'_n} \qquad\qquad \downarrow{f'_{n-1}}$$
$$C''_n \xrightarrow{\partial''_n} C''_{n-1}$$

which is obvious. Finally , the existence of the identity map is also obvious.

□

Definition 1.1.3 ▶ Direct Sum of Chain Complexes

Definthe the direct sum of a family of chain complexes $\{(C^\alpha, \partial^\alpha)\}_{\alpha\in\Lambda}$ as the chain complexes $(C, \partial) := (\oplus_\alpha C^\alpha, \oplus_\alpha \partial^\alpha)$, where the nth graded component of C is

$$C_n = (C^\alpha_n)_{\alpha\in\Lambda},$$

∂ is defined as

$$\partial\left((c^\alpha)_{\alpha\in\Lambda}\right) := (\partial^\alpha(c^\alpha))_{\alpha\in\Lambda}$$

1.2   Exact Sequence

Definition 1.2.1 ▶ Exact Sequence

1. A sequence of R-modules
$$M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$$
is said to be **exact** at M if $\ker \beta = \operatorname{Im} \alpha$.
2. A sequence
$$\cdots \longrightarrow M_{n-1} \longrightarrow M_n \longrightarrow M_{n+1} \longrightarrow \cdots$$
is said to be exact if it is exact at each M_n.
3. By a short exact sequence we mean an exact sequence of the form
$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

Definition 1.2.2 ▶ Exact Sequence of Chain Complexes

A sequence of chain complexes and chain maps
$$0 \longrightarrow C'. \xrightarrow{f.} C. \xrightarrow{g.} C''. \longrightarrow 0$$
is said to be exact if for each n the corresponding sequence of modules
$$0 \longrightarrow C'_n \xrightarrow{f_n} C_n \xrightarrow{g_n} C''_n \longrightarrow 0$$
is exact.

Lemma 1.2.3 ▶ Snake lemma

Given a commutative diagram of R-module homomorphisms: where the two horizontal sequences are exact, there exists a R-module homomorphism

$$
\begin{array}{ccccccc}
M' & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M'' & \longrightarrow & 0 \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\
0 & \longrightarrow & N' & \xrightarrow{\alpha'} & N & \xrightarrow{\beta'} & N''
\end{array}
$$

$\delta : \operatorname{Ker} f'' \longrightarrow \operatorname{Coker} f'$, called the *connecting homomorphism* such that the sequence

$$\operatorname{Ker} f' \xrightarrow{\bar{\alpha}} \operatorname{Ker} f \xrightarrow{\bar{\beta}} \operatorname{Ker} f'' \xrightarrow{\delta} \operatorname{Coker} f' \xrightarrow{\bar{\alpha}'} \operatorname{Coker} f \xrightarrow{\bar{\beta}'} \operatorname{Coker} f''$$

is exact. Moreover, the ***connecting homomorphism*** δ has the naturality properties, so that the above assignment of a 'snake' to the corresponding 'six-term' exact sequence of modules defines a covariant functor.

> *Note.*
> 最关键的信息是 ***connetcting homomorphism*** δ. 构造的过程就是借助 β 的是 surjective, α' 是 injectinve, 来调整使得 β 和 α' 对应的箭头是可以反转的. 事实上在应用时, 这个 connetcting homomorphism 的具体构造往往比它的存在性更重要.

Proof. **to be finished** We need to show

1. $\bar{\alpha}, \bar{\beta}$ are well defined, where $\bar{\alpha} := \alpha|_{\ker f'}, \bar{\beta} := \beta|_{\ker f}$

2. $\operatorname{Im} \bar{\alpha} = \ker \bar{\beta}$.

3. $\bar{\alpha}', \bar{\beta}'$ are well defined, where $\bar{\alpha}'([x]) := [\alpha'(x)]$, $\bar{\beta}'([y]) := [\beta'(y)]$

4. $\operatorname{Im} \bar{\alpha}' = \ker \bar{\beta}'$

5. Construct a δ.

6. $\operatorname{Im} \bar{\beta} = \ker \delta$

7. $\operatorname{Im} \delta = \ker \bar{\alpha}'$

We prove the above points below.

1. By commutative, $f \circ \alpha(\ker f') = \alpha' \circ f'(\ker f') = 0$, which shows that $\alpha(\ker f') \subseteq \ker f$. Thus $\bar{\alpha}$ is well defined. $\bar{\beta}$'s is similar.

2.

   $$\operatorname{Im} \bar{\alpha} = \alpha(\ker f') \subseteq \operatorname{Im} \alpha = \ker \beta \implies \beta(\operatorname{Im} \bar{\alpha}) = 0 \implies \operatorname{Im} \bar{\alpha} \subseteq \ker \bar{\beta}$$

   The other side,

   $$\ker \bar{\beta} = \ker \beta \cap \ker f = \operatorname{Im} \alpha \cap \ker f$$

   Take $y \in \ker \bar{\beta}$, there exists $x \in M'$ such that $y = \alpha(x)$ and $f(y) = f(\alpha(x)) = 0$. Since $f \circ \alpha = \alpha' \circ f', \alpha' \circ f'(x) = 0$. Since α' is injective, $x \in \ker f'$. Thus $y \in \operatorname{Im} \bar{\alpha}$.

3. For $\bar{\alpha}'$, we need to show that for $x_1, x_2 \in N'$ such that $x_1 - x_2 \in \operatorname{Im} f'$, $\alpha'(x_1) - \alpha'(x_2) \in \operatorname{Im} f$. It is true by commutative

   $$\alpha'(x_1) - \alpha'(x_2) = \alpha'(x_1 - x_2) \in \operatorname{Im}(\alpha' \circ f') = \operatorname{Im}(f \circ \alpha) \subseteq \operatorname{Im} f$$

$\bar{\beta}''$'s is similar.

4. Take $[y]_{\sim \operatorname{Im} f} \in \operatorname{Im} \bar{\alpha}'$, then there exists $[x]_{\sim \operatorname{Im} f'} \in \operatorname{Coker} f'$, such that $\alpha'\left([x]_{\sim \operatorname{Im} f'}\right) = [y]_{\sim \operatorname{Im} f}$. Then $\alpha'(x) - y \in \operatorname{Im} f$, there exists $z \in M$, such that

$$\alpha'(x) - y = f(z)$$

By commutative and exactness: $\ker \beta' = \operatorname{Im} \alpha'$,

$$\beta'(y) = \beta'(\alpha'(x) - f(z)) = 0 - \beta' \circ f(z) = -f'' \circ \beta(z) \in \operatorname{Im} f''$$

, which shows that $\bar{\beta}'\left([y]_{\sim \operatorname{Im} f}\right) = [0]_{\sim \operatorname{Im} f''}$, $\operatorname{Im} \bar{\alpha}' \subseteq \ker \bar{\beta}'$.

Then, take $[y]_{\sim \operatorname{Im} f} \in \ker \bar{\beta}'$. Then $y \in \ker \beta' = \operatorname{Im} \alpha'$. There exists $x \in N'$ such that $y = \alpha'(x)$. Then $[y]_{\sim \operatorname{Im} f} = \alpha^{\bar{}}\left([x]_{\sim \operatorname{Im} f'}\right), [y]_{\sim \operatorname{Im} f} \in \operatorname{Im} \bar{\alpha}'$, $\ker \bar{\beta}' \subseteq \operatorname{Im} \bar{\alpha}'$.

5. Take $z \in \ker f''$. Since β is surjective, there exists $y \in M$ such that $\beta(y) = z$. Once we show that $f(y) \in \operatorname{Im} \alpha'$, we can define

$$\delta(z) = [(\alpha')^{-1}(f(y))]_{\sim \operatorname{Im} f'}$$

provided that this definition is well defined. By commutative

$$\beta'(f(y)) = f''(\beta(y)) = f''(z) = 0.$$

Hence $f(y) \in \ker \beta' = \operatorname{Im} \alpha'$. Finally, to show that it is well defined, we take $z_1, z_2 \in \ker f''$ such that $\beta(y_1) = \beta(y_2) = z$. We need to show that $(\alpha')^{-1}(f(y_1 - y_2)) \in \operatorname{Im} f'$. Since $\beta(y_1) - \beta(y_2) = 0$, $\beta(y_1 - y_2) = 0$, $y_1 - y_2 \in \ker \beta = \operatorname{Im} \alpha$. Suppose $\alpha(x) = y_1 - y_2$. Then

$$(\alpha')^{-1}(f(y_1 - y_2)) = (\alpha')^{-1}(f\alpha(x)) = (\alpha')^{-1}(\alpha' \circ f')(x) = f'(x) \in \operatorname{Im} f'$$

, which completes the proof.

$\square$

Corollary 1.2.4

Consider the following commutative diagram of R-modules and R-linear maps in which the two rows are exact. If f_1 and f_3 are isomorphisms then so is f_2.

$$0 \longrightarrow M_1 \xrightarrow{\alpha_1} M_2 \xrightarrow{\alpha_2} M_3 \longrightarrow 0$$
$$\downarrow{f_1} \qquad \downarrow{f_2} \qquad \downarrow{f_3}$$
$$0 \longrightarrow N_1 \xrightarrow{\beta_1} N_2 \xrightarrow{\beta_2} N_3 \longrightarrow 0$$

Proof. By snake lemma, we have

$$0 \xrightarrow{\alpha_1} \ker f_2 \xrightarrow{\alpha_2} 0 \xrightarrow{\delta} 0 \xrightarrow{\beta_1} \mathrm{Coker}\, f_2 \xrightarrow{\beta_2} 0$$

is exact. Then we have

$$0 = \ker \alpha_2 = \mathrm{Im}\, \alpha_1 = 0$$
$$= \ker f_2$$

which shows that $\ker f_2 = 0$. Similarly, $\mathrm{Coker}\, f_2 = 0$. The above shows that f_2 is an isomorphisim. $\qquad \square$

Corollary 1.2.5 ▶ Four lemma

Consider the following commutative diagram of R modules and R-linear maps in which the two rows are exact. Suppose that f_1 is surjective and f_4 is injective. Then
  (i) f_2 is injective $\implies$ f_3 is injective.
 (ii) f_3 is surjective $\implies$ f_2 is surjective.

$$M_1 \xrightarrow{\alpha_1} M_2 \xrightarrow{\alpha_2} M_3 \xrightarrow{\alpha_3} M_4$$
$$\downarrow{f_1} \qquad \downarrow{f_2} \qquad \downarrow{f_3} \qquad \downarrow{f_4}$$
$$N_1 \xrightarrow{\beta_1} N_2 \xrightarrow{\beta_2} N_3 \xrightarrow{\beta_3} N_4$$

Proof sketch.
如果 f_3 被两个单射 f_2, f_4 夹在中间, 得益于 f_1 是满射, 可以将右三列保持单射性地调整为一条 snake, 从而导出 f_3 是单射.
类似地, 如果 f_2 被两个满射 f_1, f_3 夹在中间, 得益于 f_4 是单射, 左三列可以保持满射性地调整为一条 snake, 导出 f_2 是满射.

Proof.   1. We can construct a snake

$$M_2/\ker\alpha_2 \xrightarrow{\ \alpha_2\ } M_3 \xrightarrow{\ \alpha_3\ } \operatorname{Im}\alpha_3 \longrightarrow 0$$
$$\Big\downarrow{\bar{f}_2} \qquad\qquad \Big\downarrow{f_3} \qquad\qquad \Big\downarrow{f_4|_{\operatorname{Im}\alpha_3}}$$
$$0 \longrightarrow N_2/\ker\beta_2 \xrightarrow{\ \beta_2\ } N_3 \xrightarrow{\ \beta_3\ } N_4$$

By snake lemma, the following sequence is exact

$$\ker\bar{f}_2 \to \ker f_3 \to \ker f_4|_{\operatorname{Im}\alpha_3}$$

Since f_4 is injective, $f_4|_{\operatorname{Im}\alpha_3}$ is as well. Furthermore, by applying the snake lemma to the following diagram

$$M_1 \xrightarrow{\ \alpha_1\ } M_2 \xrightarrow{\ p\ } M_2/\ker\alpha_2 \longrightarrow 0$$
$$\bar{f}_1\Big\downarrow \qquad\qquad \Big\downarrow{f_2} \qquad\qquad \Big\downarrow{\bar{f}_2}$$
$$0 \longrightarrow N_1/\ker\beta_1 \xrightarrow[\bar{\beta}_1]{} N_2 \xrightarrow[p]{} N_2/\ker\beta_2$$

We get

$$0 = \ker f_2 \to \ker\bar{f}_2 \xrightarrow{\ \delta\ } \operatorname{Coker}\bar{f}_1$$

is a exact suquence,where $\operatorname{Coker}\bar{f}_1 = 0$ since f_1 is surjective and $\bar{f}_1$ is as well. It shows that $\ker\bar{f}_2 = 0$. Finally, we have the following sequence is exact

$$0 = \ker\bar{f}_2 \to \ker f_3 \to \ker f_4|_{\operatorname{Im}\alpha_3} = 0$$

It follows that $\ker f_3 = 0$, f_3 is injective.

2. Another snake we can construct is

$$M_1 \xrightarrow{\ \alpha_1\ } M_2 \xrightarrow{\ \alpha_2\ } \operatorname{Im}\alpha_2 = \ker\alpha_3 \longrightarrow 0$$
$$\Big\downarrow{\bar{f}_1} \qquad\qquad \Big\downarrow{f_2} \qquad\qquad \Big\downarrow{f_3|_{\operatorname{Im}\alpha_2}}$$
$$0 \longrightarrow N_1/\ker\beta_1 \xrightarrow{\ \bar{\beta}_1\ } N_2 \xrightarrow{\ \beta_2\ } \operatorname{Im}\beta_2 = \ker\beta_3$$

Then by snake lemma ,the following sequence is exact

$$\ker\bar{f}_1 \to \ker f_2 \to \ker f_3|_{\operatorname{Im}\alpha_2} \to \operatorname{Coker}\bar{f}_1 \to \operatorname{Coker}f_2 \to \operatorname{Coker}f_3|_{\operatorname{Im}\alpha_2}$$

Furthermore, by applying the snake lemma to the following diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker}\alpha_3 & \xrightarrow{\iota_M} & M_3 & \xrightarrow{\alpha_3} & \operatorname{Im}\alpha_3 & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle f_3|_{\operatorname{Im}\alpha_2}} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4|_{\operatorname{Im}\alpha_3}} & & \\
0 & \longrightarrow & \operatorname{Ker}\beta_3 & \xrightarrow{\iota_N} & N_3 & \xrightarrow{\beta_3} & \operatorname{Im}\beta_3 & \longrightarrow & 0
\end{array}
$$

We get

$$
0 = \ker f_4|_{\operatorname{Im}\alpha_3} \xrightarrow{\delta} \operatorname{Coker}\left(f_3|_{\operatorname{Im}\alpha_2}\right) \to \operatorname{Coker} f_3 = 0
$$

which shows that $\operatorname{Coker}\left(f_3|_{\operatorname{Im}\alpha_2}\right)$. Finally, we have the following sequence exact

$$
0 = \operatorname{Coker}\bar{f}_1 \to \operatorname{Coker} f_2 \to \operatorname{Coker} f_3|_{\operatorname{Im}\alpha_2} = 0
$$

Thus $\operatorname{Coker} f_2 = 0$, f_2 is surjective.

$\square$

Corollary 1.2.6 ▶ Five lemma

In the following diagram of R-modules, the two rows are given to be exact. If f_1, f_2, f_4 and f_5 are isomorphisms then f_3 is also an isomorphism.

$$
\begin{array}{ccccccccc}
M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 & \longrightarrow & M_5 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
M_1' & \longrightarrow & M_2' & \longrightarrow & M_3' & \longrightarrow & M_4' & \longrightarrow & M_5'
\end{array}
$$

Proof. By applying the Four lemma to the left four columns, we get f_3 is injective. And by applying the Four lemma to the right four columns , we get f_3 is surjective .

$\square$

1.3　Homology

Definition 1.3.1

Given a chain complex C_*, define the **homology group** of C_* to be the graded R-module

$$H_*(C_*) := \bigoplus_{n \in \mathbb{Z}} H_n(C_*)$$

by taking

$$H_n(C_*) := \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}, \quad \forall n \in \mathbb{Z}.$$

Proposition 1.3.2

If $f : C_* \to C'_*$ is a chain map then f induces a graded homomorphism

$$H_*(f) : H_*(C_*) \to H_*(C'_*)$$

In addition, this has the natruality property,viz.,
  1. $H_*(\operatorname{Id}) = \operatorname{Id}$
  2. If g is another chain map such that $f \circ g$ is defined, then

$$H_*(f \circ g) = H_*(f) \circ H_*(g)$$

Thus, H_* is a **covariant functor** from the category of chain complexe to the category of graded modules.

Theorem 1.3.3

The homology of a direct sum of chain complexes is isomorphic to the direct sum of the homology of chain complexes.

Proof.

$$H_n\left(\oplus_\alpha C_*^\alpha\right) = \ker\left(\oplus_\alpha \partial_n^\alpha\right) / \operatorname{Im}\left(\oplus_\alpha \partial_{n+1}^\alpha\right)$$

$$\bigoplus_\alpha H_n\left(C_*^\alpha\right) = \bigoplus_\alpha \left(\ker \partial_n^\alpha / \operatorname{Im} \partial_{n+1}^\alpha\right)$$

There are obvious isomorphics :

$$\ker\left(\oplus_\alpha \partial_n^\alpha\right) \simeq \oplus_\alpha \ker \partial_n^\alpha, \quad \operatorname{Im}\left(\oplus_\alpha \partial_{n+1}^\alpha\right) \simeq \oplus_\alpha \operatorname{Im} \partial_{n+1}^\alpha$$

A standard result in algebra is that if K_α is a submodule of M_α, then

$$\frac{\bigoplus_\alpha M_\alpha}{\bigoplus_\alpha K_\alpha} \cong \bigoplus_\alpha \frac{M_\alpha}{K_\alpha}$$

. Hence , we have

$$\bigoplus_\alpha \ker \partial_n^\alpha / \bigoplus_\alpha \operatorname{Im} \partial_{n+1}^\alpha \simeq \bigoplus_\alpha \left(\ker \partial_n^\alpha / \operatorname{Im}_{n+1}^\alpha \right)$$

$\square$

Theorem 1.3.4

Given a short exact sequence of chain complexes

$$0 \to C'_* \xrightarrow{\alpha} C_* \xrightarrow{\beta} C''_* \to 0$$

there is a functorial long exact sequence of homology groups

$$\to H_n(C'_*) \xrightarrow{H_n(\alpha)} H_n(C_*) \xrightarrow{H_n(\beta)} H_n(C''_*) \xrightarrow{\delta_n} H_{n-1}(C'_*) \xrightarrow{H_{n-1}(\alpha)} H_{n-1}(C_*) \to$$

Proof sketch.
Consider the diagram

$$
\begin{array}{ccccccc}
C'_n/\operatorname{Im}\partial'_{n+1} & \xrightarrow{\bar{\alpha}_n} & C_n/\operatorname{Im}\partial_{n+1} & \xrightarrow{\bar{\beta}_n} & C''_n/\operatorname{Im}\partial''_{n+1} & \longrightarrow & 0 \\
\downarrow{\partial'_n} & & \downarrow{\partial_n} & & \downarrow{\partial''_{n+1}} & & \\
0 \longrightarrow \operatorname{Ker}\partial'_{n-1} & \xrightarrow{\alpha'_{n-1}} & \operatorname{Ker}\partial_{n-1} & \xrightarrow{\beta'_{n-1}} & \operatorname{Ker}\partial''_{n-1} & &
\end{array}
$$