

# Abstract Integration

## 1.1 The Concept of Measurability

### Definition 1.1.1

1. A collection  $\mathfrak{M}$  of subsets of a set  $X$  is said to be a  $\sigma$ -**algebra** in  $X$  if  $\mathfrak{M}$  has the following properties:
  - (a)  $X \in \mathfrak{M}$ .
  - (b) If  $A \in \mathfrak{M}$ , then  $A^c \in \mathfrak{M}$ .
  - (c) If  $A = \bigcup_{n=1}^{\infty} A_n$  and  $A_n \in \mathfrak{M}$  for  $n = 1, 2, 3, \dots$ , then  $A \in \mathfrak{M}$ .
2. If  $\mathfrak{M}$  is a  $\sigma$ -algebra in  $X$ , then  $X$  is called a **measurable** provided that  $f^{-1}(V)$  is measurable set in  $X$  for every open set  $V$  in  $Y$ .
3. If  $X$  is a measurable space,  $Y$  is a topological space, and  $f$  is a mapping of  $X$  into  $Y$ , then  $f$  is said to be **measurable** provided that  $f^{-1}(V)$  is a measurable set in  $X$  for every open set  $V$  in  $Y$ .

*Note.*

对比 topological space 的定义,  $\sigma$ -algebra 的定义是对称的, 即可测集的补集仍是可测的, 而对于拓扑空间, 开集的补集不是开集, 而是被定义为了闭集这样的对象.

### Lemma 1.1.2

1. Let  $X$  be a measurable space,  $Y, Z$  be topological spaces. If  $f : X \rightarrow Y$  is measurable, and if  $g : Y \rightarrow Z$  is continuous, then  $g \circ f : X \rightarrow Z$  is measurable.
2. Let  $u, v$  be real measurable functions on a measurable space  $X$ , let  $\Phi$  be a continuous mapping of the plane into a topological space  $Y$ , and define

$$h(x) = \Phi(u(x), v(x))$$

for  $x \in X$ . Then  $h : X \rightarrow Y$  is measurable.

**Corollary 1.1.3**

Let  $X$  be a measurable space. The following propositions hold

1. If  $f = u + iv$ , where  $u$  and  $v$  are real measurable functions on  $X$ , then  $f$  is a complex measurable function on  $X$ .
2. If  $f = u + iv$  is a complex measurable function on  $X$ , then  $u$ ,  $v$ , and  $|f|$  are real measurable functions on  $X$ .
3. If  $f$  and  $g$  are complex measurable functions on  $X$ , then so are  $f + g$  and  $fg$ .
4. If  $E$  is a measurable set in  $X$  and if

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

then  $\chi_E$  is a measurable function.

5. If  $f$  is a complex measurable function on  $X$ , there is a complex measurable function  $\alpha$  on  $X$  such that  $|\alpha| = 1$  and  $f = \alpha |f|$ .

**Theorem 1.1.4**

If  $\mathcal{F}$  is any collection of subsets of  $X$ , there exists a smallest  $\sigma$ -algebra  $\mathfrak{M}^*$  such that  $\mathcal{F} \subseteq \mathfrak{M}^*$ .

**Definition 1.1.5 ▶ Borel**

Let  $X$  be a topological space.

1. By **Borel  $\sigma$ -algebra**, we mean the smallest  $\sigma$ -algebra  $\mathcal{B}$  in  $X$  such that every open set in  $X$  belongs to  $\mathcal{B}$ . The members of  $\mathcal{B}$  are called the **Borel sets** of  $X$ .
2. All countable unions of closed sets and all countable intersections of open sets are Borel sets, which we called  $F_\sigma$ 's **and**  $G_\delta$ 's, respectively.
3. By a **Borel function**, we mean a measurable function on the measurable space  $(X, \mathcal{B})$ .

*Remark.*

1. The letters  $F$  and  $G$  were used for closed and open sets, respectively, and  $\sigma$  refers to union,  $\delta$  to intersection.
2. A continuous function is Borel-measurable, since the preimage of any open set is open and therefore a Borel set.

**Theorem 1.1.6**

Suppose  $\mathfrak{M}$  is a  $\sigma$ -algebra in  $X$ , and  $Y$  is a topological space. Let  $f$  map  $X$  into  $Y$ .

1. If  $\Omega$  is the collection of all sets  $E \subseteq Y$  such that  $f^{-1}(E) \in \mathfrak{M}$ , then  $\Omega$  is a  $\sigma$ -algebra in  $Y$ .
2. If  $f$  is measurable and  $E$  is a Borel set in  $Y$ , then  $f^{-1}(E) \in \mathfrak{M}$ .
3. If  $Y = [-\infty, \infty]$  and  $f^{-1}((\alpha, \infty]) \in \mathfrak{M}$  for every real  $\alpha$ , then  $f$  is measurable.
4. If  $f$  is measurable, if  $Z$  is topological space, if  $g : Y \rightarrow Z$  is a Borel mapping, and if  $h = g \circ f$ , then  $h : X \rightarrow Z$  is measurable.

**Theorem 1.1.7**

If  $f_n : X \rightarrow [-\infty, \infty]$  is measurable, for  $n = 1, 2, 3, \dots$ , and

$$g = \sup_{n \geq 1} f_n, \quad h = \lim_{n \rightarrow \infty} \sup f_n,$$

then  $g$  and  $h$  are measurable.

**Corollary 1.1.8**

1. The limit of every pointwise convergent sequence of complex measurable functions is measurable.
2. If  $f$  and  $g$  are measurable (with range in  $[-\infty, \infty]$ ), then so are  $\max\{f, g\}$  and  $\min\{f, g\}$ . In particular, this is true of the functions

$$f^+ = \max\{f, 0\}, \quad f^- = -\min\{f, 0\}.$$

which are called the **positive part** and **negative part** of  $f$ , respectively.

*Remark.*

1. There are the standard representation

$$|f| = f^+ + f^-, \quad f = f^+ - f^-$$

2. An easy but (may) useful observation is: If  $f = g - h$ ,  $g \geq 0$ ,  $h \geq 0$ , then  $f^+ \leq g$  and  $f^- \leq h$ .

## 1.2 Simple Functions

### Definition 1.2.1

A complex function  $s$  on a measurable space  $X$  whose range consists of only finitely many points will be called a **simple function**. Among these are the nonnegative simple functions, whose range is a finite subset of  $[0, \infty)$ .

Specifically, if  $\alpha_1, \dots, \alpha_n$  are distinct values of a simple function  $s$ , and if we set  $A_i = \{x : s(x) = \alpha_i\}$ , then clearly

$$s = \sum_{i=1}^n \alpha_i \chi_{A_i}.$$

Where  $\chi_{A_i}$  is the characteristic function of  $A_i$ .

*Remark.*

- Here, we explicitly exclude  $\infty$  from the values of a simple function.
- It is clear that  $s$  is measurable if and only if each of the sets  $A_i$  is measurable.

### Theorem 1.2.2

Let  $f : X \rightarrow [0, \infty]$  be measurable space. There exists simple measurable functions  $s_n$  on  $X$  such that

1.  $0 \leq s_1 \leq s_2 \leq \dots \leq f$ .
2.  $s_n(x) \rightarrow f(x)$  as  $n \rightarrow \infty$ , for every  $x \in X$ .

*Proof sketch.*

We construct a sequence of Borel simple functions  $\{\varphi_n(x)\}$  to act as an **identity**. 当  $n$  增大的同时, 我们同时让  $\varphi_n(x)$  的单位逼近精度和单位逼近范围随着  $n$  提升. 并且在舍弃误差时, 总是向下取整, 使得该单位逼近是自下而上的.

*Proof.* For every  $x \in [0, \infty]$ , and for every  $n \in \mathbb{N}$ , there exists a unique integer  $k_n(x)$ , such that

$$k_n(x) 2^{-n} \leq x < (k_n(x) + 1) 2^{-n}$$

For every  $n \in \mathbb{N}$ , we define

$$\varphi_n(x) = \begin{cases} k_n(x) 2^{-n}, & 0 \leq x \leq n \\ n, & x \geq n \end{cases}$$

Each  $\varphi_n(x)$  is then a Borel simple function. It is not hard to show that  $0 \leq \varphi_1 \leq \varphi_2 \leq \dots \leq \text{Id}$ . For each  $n$ , we define

$$s_n(x) := (\varphi_n \circ f)(x)$$

Then  $\{s_n\}$  is a suquence of simple measurable functions such that  $0 \leq s_1 \leq s_2 \leq \dots \leq f$ . Since  $\lim_{n \rightarrow \infty} \varphi_n(x) = x$ , then  $\lim_{n \rightarrow \infty} s_n(x) = f(x)$ .  $\square$

## 1.3 Measure

### Definition 1.3.1 ► Measure and Measure Space

- (a) A **positive measure** is a function  $\mu$ , defined on a  $\sigma$ -algebra  $\mathfrak{M}$ , whose range is in  $[0, \infty]$  and which is **countably additive**. This means that if  $\{A_i\}$  is a disjoint countable collection of members of  $\mathfrak{M}$ , then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

To avoid trivialities, we shall also assume that  $\mu(A) < \infty$  for at least one  $A \in \mathfrak{M}$ .

- (b) A **measure space** is a measurable space which has a positive measure defined on the  $\sigma$ -algebra of its measurable sets.
- (c) A **complex measure** is a complex-valued countably additive function defined on a  $\sigma$ -algebra.

### Theorem 1.3.2

Let  $\mu$  be a positive measure on a  $\sigma$ -algebra  $\mathfrak{M}$ . Then

1.  $\mu(\emptyset) = 0$ .
2.  $\mu(A_1 \cup \dots \cup A_n) = \mu(A_1) + \dots + \mu(A_n)$  if  $A_1, \dots, A_n$  are pairwise disjoint members of  $\mathfrak{M}$ .
3.  $A \subset B$  implies  $\mu(A) \leq \mu(B)$  if  $A \in \mathfrak{M}, B \in \mathfrak{M}$ .

4.  $\mu(A_n) \rightarrow \mu(A)$  as  $n \rightarrow \infty$  if  $A = \bigcup_{n=1}^{\infty} A_n, A_n \in \mathfrak{M}$ , and

$$A_1 \subset A_2 \subset A_3 \subset \dots$$

5.  $\mu(A_n) \rightarrow \mu(A)$  as  $n \rightarrow \infty$  if  $A = \bigcap_{n=1}^{\infty} A_n, A_n \in \mathfrak{M}$ ,

$$A_1 \supset A_2 \supset A_3 \supset \dots,$$

and  $\mu(A_1)$  is finite.

*Proof.* 1. Take  $A \in \mathfrak{M}$  such that  $\mu(A) < \infty$ .<sup>1</sup> And let  $A_2 = A_3 = \dots = \emptyset$ , then  $\mu(\emptyset) > 0$  leads to a contradiction to the countably additive.

2. Take  $A_{n+1} = A_{n+2} = \dots = \emptyset$ .

3. Note that  $B = (B \setminus A) \cup A$ , then by additivity

$$\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$$

4. Let  $A_0 = \emptyset$ , and let  $B_n = A_n \setminus A_{n-1}$  for all  $n \in \mathbb{N}$ . Then  $B_1, \dots, B_n$  are pairwise disjoint members of  $\mathfrak{M}$  such that  $A = \bigcup_{n=1}^{\infty} B_n$ . We have

$$\mu(A) = \sum_{n=1}^{\infty} \mu(B_n) = \sum_{n=1}^{\infty} \mu(A_n \setminus A_{n-1})$$

If one of the  $\mu(A_n)$  is  $\infty$ , then  $\lim_{n \rightarrow \infty} \mu(A_n)$  and  $\mu(A)$  both are  $\infty$ . Otherwise, we have

$$\mu(A_n \setminus A_{n-1}) = \mu(A_n) - \mu(A_{n-1}), \quad \forall n \in \mathbb{N}$$

Thus

$$\mu(A) = \sum_{n=1}^{\infty} \mu(A_n \setminus A_{n-1}) = \lim_{n \rightarrow \infty} \mu(A_n)$$

5. Let  $B_n = A_n \setminus A_{n+1}$  for all  $n \in \mathbb{N}$ .  $B_1, \dots, B_n$  are pairwise disjoint members of  $\mathfrak{M}$  with finite measure, such that

$$A_1 \setminus A = A_1 \setminus \left( \bigcap_{n=1}^{\infty} A_n \right) = \bigcup_{n=1}^{\infty} (A_1 \setminus A_n) = \bigcup_{n=1}^{\infty} \left( \bigcup_{k=1}^n B_k \right) = \bigcup_{n=1}^{\infty} B_n$$

<sup>1</sup>That is what we supposed at the definition of measure.

. We have

$$\mu(A_1 \setminus A) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right),$$

where the RHS is  $\mu(A_1) - \mu(A)$ , and the LHS is

$$\sum_{n=1}^{\infty} (\mu(A_n) - \mu(A_{n+1})) = \mu(A_1) - \lim_{n \rightarrow \infty} \mu(A_{n+1})$$

Since  $\mu(A_1) < \infty$ , we have

$$\mu(A) = \lim_{n \rightarrow \infty} \mu(A_n)$$

□

### Example 1.3.3

1. For any  $E \subset X$ , where  $X$  is any set, define  $\mu(E) = \infty$  if  $E$  is an infinite set, and let  $\mu(E)$  be the number of points in  $E$  if  $E$  is finite. This  $\mu$  is called the **counting measure** on  $X$ .
2. Fix  $x_0 \in X$ , define  $\mu(E) = 1$  if  $x_0 \in E$  and  $\mu(E) = 0$  if  $x_0 \notin E$ , for any  $E \subset X$ . This  $\mu$  may be called the **unit mass concentrated at  $x_0$** .
3. Let  $\mu$  be the counting measure on the set  $\{1, 2, 3, \dots\}$ , let  $A_n = \{n, n+1, n+2, \dots\}$ . Then  $\bigcap A_n = \emptyset$  but  $\mu(A_n) = \infty$  for  $n = 1, 2, 3, \dots$ . This shows that the hypothesis

$$\mu(A_1) < \infty$$

is not superfluous in Theorem 1.3.2(5).