目录

第 0 章 练习

Problem 0.1 设 M 是一个 (伪)Rimeann 流形, 令 (x^i) 是以 $p \in M$ 为中心的法坐标. 证明以下在 p 处成立:

$$R_{ijkl} = \frac{1}{2}\left(\partial_j \partial_l g_{ik} + \partial_i \partial_k g_{jl} - \partial_i \partial_l g_{jk} - \partial_j \partial_k g_{il}\right)$$

Proof

利用曲率张量的对称性, 以及法坐标的 Christoffel 在原点处退化来说明.

法坐标的 Christoffel 符号在 p 点处退化, 故在 p 点处,

$$\begin{aligned}
\nabla_{\partial_i} \nabla_{\partial_j} \partial_k &= \nabla_{\partial_i}\left(\Gamma_{jk}^m \partial_m\right) \\
&= \left(\partial_i \Gamma_{jk}^m\right) \partial_m
\end{aligned}$$

类似地

$$\nabla_{\partial_j} \nabla_{\partial_i} \partial_k = \left(\partial_j \Gamma_{ik}^m\right) \partial_m$$

于是

$$\begin{aligned}
R\left(\partial_i, \partial_j\right) \partial_k &= \nabla_{\partial_i} \nabla_{\partial_j} \partial_k - \nabla_{\partial_j} \nabla_{\partial_i} \partial_k - \left[\partial_i, \partial_j\right] \partial_k \\
&= \left(\partial_i \Gamma_{jk}^m - \partial_j \Gamma_{ik}^m\right) \partial_m
\end{aligned}$$

从而

$$\begin{aligned}
\left\langle R\left(\partial_i, \partial_j\right) \partial_k, \partial_l\right\rangle &= \left\langle \partial_i \Gamma_{jk}^m \partial_m, \partial_l\right\rangle - \left\langle \partial_j \Gamma_{ik}^m \partial_m, \partial_l\right\rangle \\
&= g_{ml} \partial_i \Gamma_{jk}^m - g_{ml} \partial_j \Gamma_{ik}^m
\end{aligned}$$

$$\begin{aligned}
\partial_j \partial_l g_{ik} &= \partial_j \partial_l \left\langle \partial_i, \partial_k\right\rangle \\
&= \partial_j \left\langle \nabla_{\partial_l} \partial_i, \partial_k\right\rangle + \partial_j \left\langle \partial_i, \nabla_{\partial_l} \partial_k\right\rangle \\
&= \partial_j \left\langle \Gamma_{li}^m \partial_m, \partial_k\right\rangle + \partial_j \left\langle \partial_i, \Gamma_{lk}^m \partial_m\right\rangle \\
&= \partial_j \left(\Gamma_{li}^m g_{mk}\right) + \partial_j \left(\Gamma_{lk}^m g_{im}\right) \\
&= g_{mk} \partial_j \Gamma_{li}^m + g_{im} \partial_j \Gamma_{lk}^m
\end{aligned}$$

类似地

$$\partial_i \partial_k g_{jl} = g_{ml} \partial_i \Gamma_{kj}^m + g_{jm} \partial_i \Gamma_{kl}^m$$

$$\partial_i \partial_l g_{jk} = g_{mk} \partial_i \Gamma_{lj}^m + g_{jm} \partial_i \Gamma_{lk}^m$$

$$\partial_j \partial_k g_{il} = g_{ml} \partial_j \Gamma_{ki}^m + g_{im} \partial_j \Gamma_{lk}^m$$

前两式减后两式除以二, 得到右式等于

$$\frac{1}{2}\left(g_{mk}\partial_j\Gamma_{li}^m + g_{ml}\partial_i\Gamma_{kj}^m - g_{mk}\partial_i\Gamma_{lj}^m - g_{ml}\partial_j\Gamma_{ki}^m\right) = \frac{1}{2}\left(-R_{ijlk} + R_{ijkl}\right) = R_{ijkl}$$

□

Problem 0.2 设 ∇ 是 (伪)Riemann 流形 (M,g) 上的 Levi-Civita 联络, ${\omega_i}^j$ 是与局部标架 (E_i) 对应的联络 1-形式矩阵. 定义一个 2-形式的矩阵 Ω_i^j, 称为 **曲率 2-形式**, 为

$$\Omega_i{}^j = \frac{1}{2}R_{kli}{}^j\varepsilon^k \wedge \varepsilon^l$$

其中 (ε^i) 是对偶余 (E_i) 的余标架. 证明以下 **Cartan 第二结构方程**成立:

$$\Omega_i^j = \mathrm{d}\omega_i{}^j - \omega_i^k \wedge \omega_k^j$$

Proof

$$\nabla_X E_i = \omega_i^j(X)\, E_j$$

$$\Gamma_{ki}^j E_j = \nabla_{E_k} E_i = \omega_i^j(E_k)\, E_j$$

故

$$\omega_i^j(E_k) = \Gamma_{ki}^j$$

$$\mathrm{d}\omega_i^j(E_k, E_l) = E_k\left(\omega_i^j(E_l)\right) - E_l\left(\omega_i^j(E_k)\right) - \omega_i^j([E_k, E_l])$$

$$\omega_i^k \wedge \omega_k^j(E_k, E_l) = \omega_i^m(E_k)\,\omega_m^j(E_l) - \omega_i^m(E_l)\,\omega_m^j(E_k)$$

$$\begin{aligned}
R_{kli} &= \nabla_{E_k}\nabla_{E_l}E_i - \nabla_{E_l}\nabla_{E_k}E_i - \nabla_{[E_k,E_l]}E_i \\
&= \nabla_{E_k}\left(\omega_i^j(E_l)\,E_j\right) - \nabla_{E_l}\left(\omega_i^j(E_k)\,E_j\right) - \omega_i^j([E_k,E_l])\,E_j \\
&= \omega_i^j(E_l)\,\nabla_{E_k}E_j + E_k\left(\omega_i^j(E_l)\right)E_j - \omega_i^j(E_k)\,\nabla_{E_l}E_j - E_l\left(\omega_i^j(E_k)\right)E_j \\
&- \omega_i^j([E_k,E_l])\,E_j \\
&= \omega_i^j(E_l)\,\omega_j^m(E_k)\,E_m + E_k\left(\omega_i^j(E_l)\right)E_j - \omega_i^j(E_k)\,\omega_j^m(E_l)\,E_m - E_l\left(\omega_i^j(E_k)\right)E_j \\
&- \omega_i^j([E_k,E_l])\,E_j \\
&= \omega_i^m(E_l)\,\omega_m^j(E_k)\,E_j + E_k\left(\omega_i^j(E_l)\right)E_j - \omega_i^m(E_k)\,\omega_m^j(E_l)\,E_j - E_l\left(\omega_i^j(E_k)\right)E_j \\
&- \omega_i^j([E_k,E_l])\,E_j
\end{aligned}$$

于是

$$\begin{aligned}
R_{kli}^j &= \omega_i^m(E_l)\,\omega_m^j(E_k) + E_k\left(\omega_i^j(E_l)\right) - \omega_i^m(E_k)\,\omega_m^j(E_l) - E_l\left(\omega_i^j(E_k)\right) \\
&- \omega_i^j([E_k,E_l])
\end{aligned}$$

$$\Omega_i^j = \frac{1}{2}R_{kli}^j\left(\varepsilon^k\otimes\varepsilon^l - \varepsilon^l\otimes\varepsilon^k\right) = \sum_{k<l} R_{kli}^j\varepsilon^k\otimes\varepsilon^l$$

于是

$$\Omega_i^j(E_k, E_l) = R_{kli}^j = \mathrm{d}\omega_i^j(E_k, E_l) - \omega_i^k \wedge \omega_k^j(E_k, E_l)$$

□