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# Uncertainty-aware Accumulated Local Effects (UALE) for quantifying the heterogeneity of instance-level feature effects

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## Abstract

Accumulated Local Effects (ALE) is a popular approach to explainable AI that quantifies how a feature influences the decisions of a model taking into account feature correlations. However, in case of strong interactions between features, instance-level feature effects may deviate from the average curve provided by ALE. Therefore, it is crucial to quantify this deviation, namely, the uncertainty of the effect. In this work, we define Uncertainty-aware ALE (UALE) to quantify and visualize, on a single plot, both the average effect and the uncertainty. Furthermore, as in ALE, UALE’s approximation requires partitioning the feature domain into non-overlapping intervals (bin-splitting). The average effect and the uncertainty are then computed from the instances that lie in each bin. In this work, we formally prove that to achieve an unbiased approximation of the uncertainty in each bin, bin-splitting must follow specific constraints. Based on this, we propose a method for automatically finding the optimal intervals, balancing the trade-off between estimation bias and variance [check if this phrase explains well the concept.] We demonstrate through synthetic and real datasets (a) the advantages of modeling the uncertainty with UALE compared to other methods and (b) the importance of appropriate bin splitting for an accurate approximation of the average effect and uncertainty.

## 1 INTRODUCTION

Recently, Machine Learning (ML) has been adopted in critical domains, such as healthcare and finance. In these ar-

eas, we need a combination of accurate predictions along with meaningful explanations to support them. For this reason, there is an increased interest in Explainable AI (XAI) to understand the decision mechanism of complex black-box models. XAI literature distinguishes between local and global techniques (Molnar et al., 2020a). Local methods provide instance-level explanations, i.e. explain the prediction for a specific instance, whereas global methods summarize the entire model behavior.

Global methods create a global explanation by aggregating the instance-level explanations into a single interpretable outcome, usually a number or a plot. When strong interactions between features exist, the instance-level explanations may significantly deviate from the aggregated outcome (Herbinger et al., 2022), a phenomenon called *aggregation bias* (Mehrabani et al., 2021). Therefore, global methods should quantify not only the aggregated (averaged) effect, but also the level of heterogeneity of instance-level explanations. The same holds for Feature Effect (FE) (Grömping, 2020), a class of global XAI methods that isolate the impact of a single feature on the output.<sup>1</sup> For example, a feature with zero average effect may indicate no effect on the output or a highly positive effect in some cases and a highly negative effect in others. The ambiguity about the correct interpretation can be resolved, if the FE method informs about the heterogeneity of instance-level explanations, a quantity we refer to as the *uncertainty of the effect*.

There are three widely-used FE methods: *Partial Dependence Plots* (PDP)(Friedman, 2001), *Marginal Plots* (MP)(Apley and Zhu, 2020) and *Aggregated Local Effects* (ALE)(Apley and Zhu, 2020). ALE has been established as the state-of-the-art for measuring the average effect, since PDP and MP have been criticized (Grömping, 2020) to compute erroneous effects when the input features are (highly) correlated. However, ALE faces two limitations, the first concerns ALE definition and the second ALE approximation. Regarding the definition, ALE does not inform the user about the uncertainty. In contrast,

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<sup>1</sup>FE methods also isolate the combined effect of a pair of features to the output. Combinations of more than two features are uncommon, since they are difficult to estimate and visualize.

in PDP the heterogeneity of instance-level effects can be identified by exploring the Individual Conditional Expectations (ICE) (Goldstein et al., 2015). Regarding the approximation, ALE requires partitioning the feature domain into non-overlapping intervals (bin-splitting) and estimate the effect inside each interval (bin-effect) from the training set instances that fall inside. Finding an appropriate sequence of bins is of particular importance, since ALE’s interpretation is meaningful only inside each bin (Molnar, 2022). However, this crucial step has not raised the appropriate attention. With the current approaches (Apley and Zhu, 2020; Gkolemis et al.), the domain is split *blindly*, i.e. without considering the characteristics of the instance-level effects, in  $K$  equisized bins.

In this paper, we present Uncertainty-aware Accumulated Local Effects (UALE), a method that visualizes in a single plot both the average feature effect and the uncertainty, i.e. the level of heterogeneity of instance-level effects. The average effect is defined as in ALE and the uncertainty inherits ALE’s advantages, i.e. takes into account the correlations between features. We also propose an accurate UALE approximation. We formally prove that for an unbiased estimation of the uncertainty, the partitioning of the feature domain must respect specific conditions. Using that, we set-up an optimization problem where we search for a partitioning, a sequence of *variable-size* intervals, (a) with enough instances (accurate estimation) and (b) that respect the above conditions (unbiased estimation). Finally, we propose a method for automatically finding the optimal partitioning. We apply our method in multiple synthetic and real world datasets for evaluating its performance. The contributions of this paper are:

- UALE, a feature effect method that quantifies both the average effect and the uncertainty, i.e., the heterogeneity of instance level effects.
- A formal proof of the conditions the feature domain partitioning must fulfill for an unbiased approximation of the uncertainty.
- An accurate UALE approximation by automatically partitioning into variable-size intervals that respect the above conditions.

The implementation of our method and the code for reproducing all the experiments is provided in the submission and will become publicly available upon acceptance.

## 2 BACKGROUND AND RELATED WORK

We here review the existing methods for modeling the average effect and the heterogeneity of instance-level effects and we focus on ALE definition (Apley and Zhu, 2020) and approximation (Apley and Zhu, 2020; Gkolemis et al.).

**Notation.** We refer to random variables (rv) using uppercase  $X$ , to simple variables with plain lowercase  $x$  and to vectors with bold  $\mathbf{x}$ . Often, we partition the input vector  $\mathbf{x} \in \mathbb{R}^D$  to the feature of interest  $x_s \in \mathbb{R}$  and the rest of the features  $\mathbf{x}_c \in \mathbb{R}^{D-1}$ . For convenience we denote it as  $(x_s, \mathbf{x}_c)$ , but we it refers the vector  $(x_1, \dots, x_s, \dots, x_D)$ . Equivalently, we denote the corresponding rv as  $\mathbf{X} = (X_s, \mathbf{X}_c)$ . When we refer to an instance of the training set, we use  $\mathbf{x}^i = (x_s^i, \mathbf{x}_c^i)$ .

### 2.1 Feature Effect Methods

The three well-known feature effect methods are: PDP, MP and ALE. PDP formulates the FE of the  $s$ -th feature as an expectation over the marginal distribution  $\mathbf{X}_c$ , i.e.,  $f^{\text{PDP}}(x_s) = \mathbb{E}_{\mathbf{X}_c}[f(x_s, \mathbf{X}_c)]$ , whereas MP formulates it as an expectation over the conditional  $\mathbf{X}_c|X_s$ , i.e.,  $f^{\text{MP}}(x_s) = \mathbb{E}_{\mathbf{X}_c|X_s}[f(x_s, \mathbf{X}_c)]$ . ALE defines the local effect at  $x_s = z$  as  $f^s(z, \mathbf{x}_c) = \frac{\partial f}{\partial x_s}(z, \mathbf{x}_c)$ . The local explanations are then weighted by the conditional distribution  $p(\mathbf{x}_c|x_s = z)$  to compute the averaged effect  $\mu(z)$ . ALE is the accumulation of the averaged local effects:

$$f^{\text{ALE}}(x_s) = \int_{x_{s,\min}}^{x_s} \underbrace{\mathbb{E}_{\mathbf{X}_c|X_s=z}[f^s(z, \mathbf{X}_c)]}_{\mu(z)} \partial z \quad (1)$$

Eq. (1) has important advantages compared to PDP and MP in cases of correlated features. In these cases, PDP integrates over unrealistic instances due to the use of the marginal distribution  $\mathbf{X}_c$ , and MP computes aggregated effects, i.e., imputes the combined effect of sets of features to a single feature. ALE resolves both issues.

### 2.2 Heterogeneity Of Instance-Level Effects

FE methods show what is the average (global) effect on the output, if the value of a specific feature is increased/decreased. For completing the explanation about a feature, we need to also know to what extent the local effects deviate from the global explanation, i.e. to quantify the uncertainty of the average effect.

ICE and d-ICE(Goldstein et al., 2015) provide a set of curves that are plot on top-of PDP. Both methods produce one curve for each instance of the training set;  $f_i^{\text{ICE}}(x_s) = f(x_s, \mathbf{x}_c^i)$  for ICE and  $f_i^{\text{d-ICE}}(x_s) = \frac{\partial f}{\partial x_s}(x_s, \mathbf{x}_c^i)$  for d-ICE. The user then visually observes if the curves are homogeneous and to what extent they deviate from the PDP. Some methods try to automate the aforementioned visual exploration, by grouping (d-)ICE plots into clusters (Molnar et al., 2020b; Herbinger et al., 2022; Britton, 2019). Other approaches, like H-Statistic(Friedman and Popescu, 2008), Greenwell’s interaction index(Greenwell et al., 2018) or SHAP interaction values(Lundberg et al., 2018) inform indirectly about heterogeneous effects, quan-

tifying interactions between the input features. A strong interaction index is an indicator for the existence of heterogeneous effects.

The aforementioned approaches are under two limitations. They either do not quantify the uncertainty of the FE directly or they are based on PDPs, and, therefore, they are subject to the failure modes of PDPs in cases of correlated features (Baniecki et al., 2021), as we will show in-depth in Section 4.1. To the best of our knowledge, no work exist so far that quantifies the heterogeneous effects based on the formulation of ALE.

### 2.3 ALE Approximation

In ML scenarios, ALE is estimated from the instances of the training set. (Apley and Zhu, 2020) proposed dividing the feature domain in  $K$  equisized bins and estimating the local effects in each bin by evaluating the black box-function at the bin limits:

$$\hat{f}^{\text{ALE}}(x_s) = \sum_{k=1}^{k_x} \frac{1}{|\mathcal{S}_k|} \sum_{i: \mathbf{x}^i \in \mathcal{S}_k} [f(z_k, \mathbf{x}_c^i) - f(z_{k-1}, \mathbf{x}_c^i)] \quad (2)$$

We denote as  $k_x$  the index of the bin that  $x_s$  belongs to, i.e.  $k_x : z_{k_x-1} \leq x_s < z_{k_x}$  and  $\mathcal{S}_k$  is the set of training instances of the  $k$ -th bin, i.e.  $\mathcal{S}_k = \{\mathbf{x}^i : z_{k-1} \leq x_s^i < z_k\}$ . Afterwards, (Gkolemis et al.) proposed the Differential ALE (DALE) that uses the partial derivatives:

$$\hat{f}^{\text{DALE}}(x_s) = \Delta x \sum_{k=1}^{k_x} \frac{1}{|\mathcal{S}_k|} \sum_{i: \mathbf{x}^i \in \mathcal{S}_k} f^s(\mathbf{x}^i) \quad (3)$$

Their method has the advantages of remaining on-distribution when bins become wide and, most importantly, allows the recomputation of the accumulated effect with different bin-splitting with near-zero computational cost.

Both approximations ask from the user to blindly decide the bin density for splitting the feature axis into  $K$  equisized bins, an approach that is error prone. Setting  $K$  to a small value may hide fine-grain effects due to large bins while setting  $K$  to a high value leads to poor bin-effect estimations due to few samples per bin. In general, the user may receive contradictory explanations for different  $K$  without a clue for which one to trust.

## 3 Uncertainty-Aware ALE (UALE)

UALE extends ALE for quantifying the level of heterogeneous effects (uncertainty) and automates the bin-splitting. At Section 3.1 we define UALE, at Section 3.2 we reformulate it using an interval-based formulation, at Section 3.3 we provide UALE approximation and at Section 3.4 we define and solve the problem of optimal bin-splitting.

### 3.1 UALE: ALE With Uncertainty Quantification

UALE extends ALE definition of Eq. (1) with a component for quantifying the uncertainty. We denote as  $\mathcal{H}(z)$  the uncertainty of the local effects at a specific point  $x_s = z$  and we quantify it with the standard deviation of the local explanations  $\mathcal{H}(z) = \sigma(z)$ , where:

$$\sigma^2(z) = \mathbb{E}_{\mathbf{x}_c|z} [(f^s(z, X_c) - \mu(z))^2] \quad (4)$$

The uncertainty emerges from the natural characteristics of the experiment, i.e., the feature correlations existent in the data generating distribution and the implicit interactions of the black-box function. We also define the accumulated uncertainty at  $x_s$ , as the accumulation of the standard deviation of the local effects along the axis:

$$f_{\sigma}^{\text{ALE}}(x_s) = \int_{x_{s,\min}}^{x_s} \sigma(z) dz \quad (5)$$

UALE formulates the effect at a specific point  $x_s$  with a tuple that consists of the average effect and the uncertainty  $(\mu(z), \sigma(z))$  and visualizes it with a continuous curve and a confidence region  $f^{\text{UALE}}(x_s) := f_{\mu}^{\text{ALE}}(x_s) \pm f_{\sigma}^{\text{ALE}}(x_s)$ .

### 3.2 Interval-Based Formulation and Approximation

In real ML scenarios, all estimations are based on the instances of the training set. Therefore, we cannot estimate  $\mu(z), \sigma(z)$  at the granularity of a point, because the probability of observing a sample inside the region  $[x_s - h, x_s + h]$  tends to zero, when  $h \rightarrow 0$ . We define the regional-effect  $\mu(z_1, z_2)$  and the regional-uncertainty  $\mathcal{H}(z_1, z_2) = \sigma(z_1, z_2)$ , as the summary statistics that characterize the effect and the uncertainty inside the interval  $[z_1, z_2]$ :

$$\mu(z_1, z_2) = \mathbb{E}_{z \sim \mathcal{U}(z_1, z_2)} [\mu(z)] = \frac{\int_{z_1}^{z_2} \mu(z) dz}{z_2 - z_1} \quad (6)$$

$$\sigma^2(z_1, z_2) = \mathbb{E}_{z \sim \mathcal{U}(z_1, z_2)} [\sigma^2(z)] = \frac{\int_{z_1}^{z_2} \sigma^2(z) dz}{z_2 - z_1} \quad (7)$$

Intuitively, the regional-effect and the regional-uncertainty quantify the expected average effect and the expected uncertainty if we randomly draw a point  $z^*$  given a uniform distribution  $z^* \sim \mathcal{U}(z_1, z_2)$ . If we also define as  $\mathcal{Z}$  the sequence of  $K + 1$  points that partition the domain of the  $s$ -th feature into  $K$  variable-size intervals, i.e.  $\mathcal{Z} = \{z_0, \dots, z_K\}$ , we can redefine UALE using an interval-based formulation  $\tilde{f}^{\text{UALE}}(x_s) := \tilde{f}_{\mathcal{Z}, \mu}^{\text{ALE}}(x_s) \pm \tilde{f}_{\mathcal{Z}, \sigma}^{\text{ALE}}(x_s)$ , where:

$$\tilde{f}_{\mathcal{Z}, \mu}^{\text{ALE}}(x_s) = \sum_{k=1}^{k_x} \mu(z_{k-1}, z_k) (z_k - z_{k-1}) \quad (8)$$

$$\tilde{f}_{\mathcal{Z},\sigma}^{\text{ALE}}(x_s) = \sum_{k=1}^{k_x} \sigma(z_{k-1}, z_k)(z_k - z_{k-1}) \quad (9)$$

### 3.3 Interval-Based Approximation

For approximating the mean effect (Eq. (6)) and the uncertainty (Eq. (7)) we use the set  $\mathcal{S}_k$  of dataset instances with the  $s$ -th feature lying inside the interval  $\mathcal{S}_k = \{\mathbf{x}^i : z_{k-1} \leq x_s^i < z_k\}$ :

$$\hat{\mu}(z_{k-1}, z_k) = \frac{1}{|\mathcal{S}_k|} \sum_{i:\mathbf{x}^i \in \mathcal{S}_k} [f^s(\mathbf{x}^i)] \quad (10)$$

$$\hat{\sigma}^2(z_{k-1}, z_k) = \frac{1}{|\mathcal{S}_k|} \sum_{i:\mathbf{x}^i \in \mathcal{S}_k} (f^s(\mathbf{x}^i) - \hat{\mu}(z_{k-1}, z_k))^2 \quad (11)$$

Eq.(10) is an unbiased estimator of Eq. (6) under the assumption that the points are uniformly distributed inside the intervals. The same does not hold in the general case for Eq.(11). In the general case,  $\hat{\sigma}^2(z_{k-1}, z_k)$  is an unbiased estimator of the *observable* variance  $\sigma_{obs}^2(z_1, z_2) = \frac{\int_{z_1}^{z_2} \mathbb{E}_{\mathbf{x}_c | x_s = z} [(f^s(z, \mathbf{x}_c) - \mu(z_1, z_2))^2] \partial z}{z_2 - z_1}$ , which as we prove in Theorem 1 is an overestimation of  $\sigma^2(z_1, z_2)$ .

**Theorem 1.** *If we define the residual  $\rho(z)$  as the difference between the expected effect at  $x_s$  and the regional effect, i.e  $\rho(z) = \mu(z) - \mu(z_1, z_2)$ , then the regional variance  $\sigma^2(z_1, z_2)$  equals to:*

$$\sigma_{obs}^2(z_1, z_2) = \sigma^2(z_1, z_2) + \frac{\int_{z_1}^{z_2} \rho^2(z) \partial z}{z_2 - z_1} \quad (12)$$

Theorem 1 reveals an important connection between the ground-truth and the observable uncertainty inside a region. For convenience, we denote as  $\mathcal{E}^2(z_1, z_2) = \frac{\int_{z_1}^{z_2} \rho^2(z) \partial z}{z_2 - z_1}$  the error term that models the expected square residual inside the interval and as  $\mathcal{H}_{obs}(z_1, z_2) := \sigma_{obs}(z_1, z_2)$  the observable uncertainty. It holds that the observable uncertainty is an overestimation of the correct uncertainty

$$\mathcal{H}_{obs}^2(z_1, z_2) = \mathcal{H}^2(z_1, z_2) + \mathcal{E}^2(z_1, z_2) \quad (13)$$

and, therefore, the estimation is unbiased only in case  $\mathcal{E}^2(z_1, z_2) = 0$ .

### 3.4 Bin-Splitting: Finding Regions With Homogeneous Effects

In this section we formulate bin-splitting as an unsupervised clustering problem, searching for a solution that compromises two contradictory objectives. On the one hand, we want to minimize the error term  $\mathcal{E}$  for approaching an

unbiased estimation and on the other hand, we want wide-enough bins for including a fair population of samples for a robust estimation of  $\hat{\mu}(z_1, z_2), \hat{\sigma}(z_1, z_2)$ . We set-up the following optimization problem:

$$\begin{aligned} \min_{\mathcal{Z}=\{z_0, \dots, z_K\}} \quad & \mathcal{L} = \sum_{k=1}^K \tau_k \hat{\sigma}^2(z_{k-1}, z_k) \Delta z_k \\ \text{where} \quad & \Delta z_k = z_k - z_{k-1} \\ & \tau_k = 1 - \alpha \frac{|\mathcal{S}_k|}{N} \\ \text{s.t.} \quad & |\mathcal{S}_k| \geq N_{\text{NPB}} \\ & z_0 = x_{s,\min} \\ & z_K = x_{s,\max} \end{aligned} \quad (14)$$

We search for the sequence of intervals  $z_0, \dots, z_K$  that minimizes the sum of the bin costs. The cost of each bin is the approximated variance  $\hat{\sigma}_k^2$  scaled by the bin length  $\Delta z_k$  and discounted by the term  $\tau_k$ . The term  $\tau_K$  favors the selection of a bigger bin in case it has similar variance with the aggregate variance of many smaller ones. The constraint of at least  $N_{\text{PPB}}$  points per bin sets the lowest-limit for a *robust* estimation. The user can choose to what extent they favor the creation of wide bins through the discount parameter by setting the parameter  $\alpha$  and where they set the minimum for robust approximation with the parameter  $N_{\text{PPB}}$ . For providing a rough idea, in our experiments we set  $\alpha = 0.2$  which means that the discount ranges between [0%, 20%] and  $N_{\text{PPB}} = \frac{N}{20}$ . It is also important to clarify that by minimizing  $\hat{\sigma}_k^2 \approx \mathcal{H}_{obs}^2$ , we actually minimize the squared error term  $\mathcal{E}^2$ , since the term  $\mathcal{H}_{true}^2$  is independent of the bin splitting.

#### 3.4.1 Solve Bin-Splitting with Dynamic Programming

[Review all the description] For achieving a computationally-grounded solution we set a threshold  $K_{max}$  on the maximum number of bins which also discretizes the solution space. The width of the bin can take discrete values that are multiple of the minimum step  $u = \frac{x_{s,\max} - x_{s,\min}}{K_{max}}$ . For defining the solution, we use two indexes. The index  $i \in \{0, \dots, K_{max}\}$  denotes the point  $(z_i)$  and the index  $j \in \{0, \dots, K_{max}\}$  denotes the position of the  $j$ -th multiple of the minimum step, i.e.,  $x_j = x_{s,\min} + j \cdot u$ . The recursive cost function  $T(i, j)$  is the cost of setting  $z_i = x_j$ :

$$\mathcal{T}(i, j) = \min_{l \in \{0, \dots, K_{max}\}} [\mathcal{T}(i-1, l) + \mathcal{B}(x_l, x_j)] \quad (15)$$

where  $\mathcal{T}(0, j)$  equals zero if  $j = 0$  and  $\infty$  in any other case.  $\mathcal{B}(x_l, x_j)$  denotes the cost of creating a bin with limits  $[x_l, x_j]$ :

$$\mathcal{B}(x_l, x_j) = \begin{cases} \infty, & \text{if } x_j > x_l \text{ or } |\mathcal{S}_{(x_j, x_l)}| < N \\ 0, & \text{if } x_j = x_l \\ \hat{\sigma}^2(x_j, x_l), & \text{if } x_j \leq x_l \end{cases} \quad (16)$$

The optimal solution is given by solving  $\mathcal{L} = \mathcal{T}(K_{max}, K_{max})$  and keeping track of the sequence of steps.

- Discuss more aspects (e.g. Computational complexity)

## 4 SIMULATION EXAMPLES

In XAI it is common (Aas et al., 2021; Herbringer et al., 2022) to use simulation examples for comparing methods, because the data-generating distribution  $p(\mathbf{X})$  and the predictive function  $f(\cdot)$  are known, and, therefore, the ground-truth is accessible. We follow this common XAI practice and we split the evaluation in two groups of examples. The first group (Section 4.1) compares UALE definition against PDP-ICE in modeling the average effect and the uncertainty. The second group (Section 4.2) compares UALE approximation using the automatic bin-splitting against the fixed-size alternative, in estimating the average effect and the uncertainty.

### 4.1 Case 1: UALE vs PDP-ICE

In this simulation, we compare UALE against PDP-ICE. The aim of this example is to highlight that when features are correlated, PDP and ICE fail in quantifying the average effect and the heterogeneity, even when the black-box function is relatively simple.

We use the following distribution:  $p(\mathbf{x}) = p(x_3)p(x_2|x_1)p(x_1)$  where  $x_1 \sim \mathcal{U}(0, 1)$ ,  $x_2 = x_1$  and  $x_3 \sim \mathcal{N}(0, \sigma_3^2)$  and the following piecewise linear function:

$$f(\mathbf{x}) = \begin{cases} f_{lin} + \alpha f_{int} & \text{if } f_{lin} < 0.5 \\ 0.5 - f_{lin} + \alpha f_{int} & \text{if } 0.5 \leq f_{lin} < 1 \\ \alpha f_{int} & \text{otherwise} \end{cases} \quad (17)$$

where  $f_{lin}(\mathbf{x}) = a_1 x_1 + a_2 x_2$  and  $f_{int}(\mathbf{x}) = x_1 x_3$ . The linear term  $f_{lin}$  includes the two correlated features and the interaction term  $f_{int}$  the two non-correlated. We evaluate the effect computed by UALE and PDP-ICE in three cases; (a) without interaction ( $\alpha = 0$ ) and equal weights ( $a_1 = a_2$ ), (b) without interaction ( $\alpha = 0$ ) and different weights ( $a_1 \neq a_2$ ) and (c) with interaction ( $\alpha \neq 0$ ) with equal weights ( $a_1 = a_2$ ). In all cases, we provide the ground-truth average effect and uncertainty computed

with analytical derivations (proofs at the Appendix) and we compare it against the approximations provided by each method, after generating  $N = 500$  samples.

**No Interaction, Equal weights.** In this set-up,  $\alpha = 0$  (no interaction) and the  $a_1 = a_2 = 1$  (equal weights). The ground truth effect  $f_{\mu}^{GT}(x_1)$  is:  $x_1$  in  $[0, 0.25]$ ,  $-x_1$  in  $[0.25, 0.5]$  and 0 in  $[0.5, 1]$ . The uncertainty is  $f_{\sigma^2}^{GT}(x_1) = 0$  in  $[0, 1]$  because  $x_1$  does not interact with other variables. In Figure 1, we observe that PDP effect is wrong and ICE plots imply the existence of heterogeneous effects. In contrast, UALE models correctly both the average effect and the zero uncertainty and approximates it accurately finding a zero aggregate residual error partitioning of  $x_1$  domain.

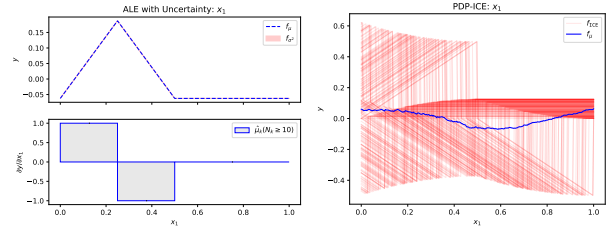


Figure 1: No interaction, Equal weights: Feature effect for  $x_1$  using UALE (Left) and PDP-ICE (Right).

**No Interaction, Non-Equal Weights.** In this set-up,  $\alpha = 0$  (no interaction),  $a_1 = 2$  and  $a_2 = 0.5$  (non-equal weights). The non-equal weights have implications at both the gradient and the interval of the piece-wise linear regions, i.e.,  $f_{\mu}^{GT}(x_1)$  is:  $2x_1$  in  $[0, 0.2]$ ,  $-2x_1$  in  $[0.2, 0.4]$  and 0 in  $[0.4, 1]$ . As before, the ground-truth uncertainty is  $f_{\sigma^2}^{GT}(x_1) = 0$  because  $x_1$  does not interact with other features. In Figure 2, we observe that PDP estimation is completely opposite to the ground-truth effect, i.e. negative in the region  $[0, 0.2)$ , positive in  $[0.2, 0.4)$ , and the ICE erroneously implies the existence of heterogeneous effects. As before, ALE quantifies correctly the ground truth effect, the zero-uncertainty and partitioning correctly the  $x_1$  domain.

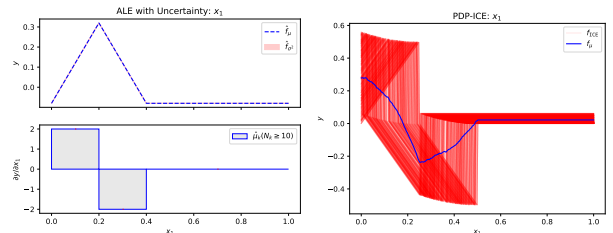


Figure 2: No interaction, Different weights: Feature effect for  $x_1$  using UALE (Left) and PDP-ICE (Right).

**Uncertainty, Equal weights.** In this set-up, we activate the interaction term, i.e.  $\alpha = 1$ , keeping the weights equal  $a_1 = a_2 = 1$ . The interaction term provokes heterogeneous

effects for features  $x_1$  and  $x_3$ , because the local effects at  $x_1$  depend on the unknown value of  $X_3$  and vice-versa. The effect of  $x_2$  has zero-uncertainty since it does not appear in any interaction term. As we observe in Figure 3, UALE correctly models the average effect and the uncertainty in all cases. On the other hand, PDP-ICE quantifies correctly the average effect and the uncertainty only in the case of feature  $x_3$ , because  $X_3$  is independent from other features. For the correlated features ( $x_1, x_2$ ) both the average effect (PDP) and the uncertainty (ICE) are wrong.

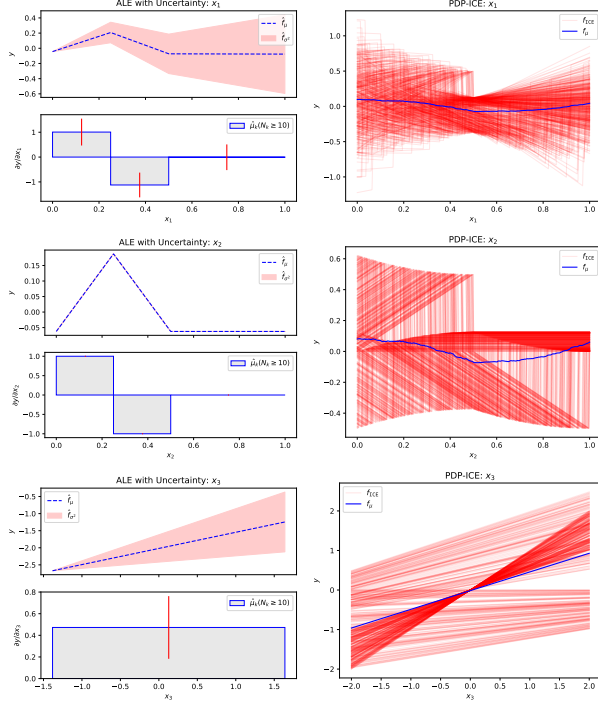


Figure 3: With interaction, equal weights: From top to bottom, feature effect for features  $\{x_1, x_2, x_3\}$  using UALE (left column) and PDP-ICE (right column).

**Discussion.** Despite the model’s simplicity, PDP-ICE fail in modeling both the average effect and the uncertainty. The error is due to the method’s formulation, i.e. the use of the marginal distribution that ignores correlations between features, and not due to poor approximation because of limited samples. In contrast, UALE models both correctly and estimates them accurately extracting the regions with constant effect. The examples above do not cover the case of an interaction term between correlated features, for example a term  $x_1x_2$ , because there is an open debate about the ground-truth effect in this case (Grömping, 2020).

## 4.2 Case 2: Bin-Splitting

In this simulation, we illustrate the advantages of automatic bin-splitting against the fixed-size alternative. For this reason, we generate a very big dataset applying dense sam-

pling ( $N = 10^6$ ) and we treat the estimation with dense fixed-size bins ( $K = 10^3$ ) as the ground-truth UALE. Afterwards, we generate less samples ( $N = 500$ ) and we compare the fixed-size estimation (for many different  $K$ ) against UALE automatic bin-splitting. In all set-ups, we sample from  $p(\mathbf{x}) = p(x_2|x_1)p(x_1)$  where  $x_1 \sim \mathcal{U}(0, 1)$  and  $x_2 \sim \mathcal{N}(x_1, \sigma_2^2 = 0.5)$ . We denote as  $\mathcal{Z}^{\text{DP}} = \{z_{k-1}^{\text{DP}}, \dots, z_K^{\text{DP}}\}$  the sequence obtained by automatic bin-splitting and with  $\mathcal{Z}^{\text{K}}$  the fixed-size splitting with  $K$  bins. The evaluation metrics we report are the average number of  $t = 30$  independent runs, using each time  $N = 500$  different samples.

**Metrics** The evaluation is done with regard to two metrics counting the mean error per bin, where the bin error is defined as the absolute difference of the approximation from the ground-truth. The first metric, Eq. (18), counts the mean bin error wrt average effect and the second, Eq. (19) the mean bin error wrt uncertainty:

$$\mathcal{L}_{\text{DP|K}}^{\mu} = \frac{1}{|\mathcal{Z}^{\text{DP|K}}|} \sum_{k \in \mathcal{Z}^{\text{DP|K}}} |\mu(z_k - z_{k-1}) - \hat{\mu}(z_k - z_{k-1})| \quad (18)$$

$$\mathcal{L}_{\text{DP|K}}^{\sigma} = \frac{1}{|\mathcal{Z}^{\text{DP|K}}|} \sum_{k \in \mathcal{Z}^{\text{DP|K}}} |\sigma(z_k - z_{k-1}) - \hat{\sigma}(z_k - z_{k-1})| \quad (19)$$

The ground truth  $\mu_k$  ( $\sigma_k$ ) is the mean value of the bin-effects (bin-uncertainties) of the bins inside the interval defined by the wide bin. For better interpretation of the uncertainty error, we also provide the mean (per bin) error term  $\mathcal{L}_{\text{DP|K}}^{\rho} = \frac{1}{|\mathcal{Z}^{\text{DP|K}}|} \sum_{k \in \mathcal{Z}^{\text{DP|K}}} (z_k - z_{k-1})$ .

**Piecewise-Linear Function.** In this set-up,  $f(\mathbf{x}) = a_1x_1 + x_1x_2$  is a piecewise-linear function with 5 different-width regions, i.e.,  $a_1$  equals to  $\{2, -2, 5, -10, 0.5\}$  in the intervals defined by the sequence  $\{0, 0.2, 0.4, 0.45, 0.5, 1\}$ . As we observe, in Figure 4, UALE’s approximation is better than all fixed-size alternatives, in both mean effect ( $\mathcal{L}^{\mu}$ ) and uncertainty ( $\mathcal{L}^{\sigma}$ ). For understanding the importance of variable-size splitting, we analyze the fixed-bin error. In the top-right of Figure 4, we observe the Law of Large Numbers (LLN);  $\mathcal{L}_K^{\mu}$  and  $K$  have positive correlation because as  $\Delta x$  becomes smaller less points contributing to each estimation. The interpretation of  $\mathcal{L}^{\sigma}$  is more complex. Given that the smallest piecewise-linear region is  $\Delta x = 0.05$  any  $K$  that is not a multiple of  $\frac{1}{\Delta x} = 20$ , adds nuisance uncertainty  $\mathcal{L}_K^{\rho} > 0$  leading to a biased uncertainty approximation. For these  $K$ , uncertainty error decreases for bigger  $K$  because the length of the bins that lie in the limit between two piecewise linear regions becomes smaller. For  $K$  that



are multiple of 20, i.e.  $K = \{20, 40, 60, 80, 100\}$ , the uncertainty approximation is unbiased  $\mathcal{L}_K^\rho \approx 0$ . Here, as in  $\mathcal{L}^\mu$ , uncertainty estimation is worse as  $K$  grows larger (LLN). As we observe in the top left of Figure 4, automatic bin-splitting separates the fine-grain bins, e.g. regions  $[0.4, 0.45]$ ,  $[0.45, 0.5]$ , and unites constant-effect regions into a single bin, e.g. region  $[0.5, 1]$ .

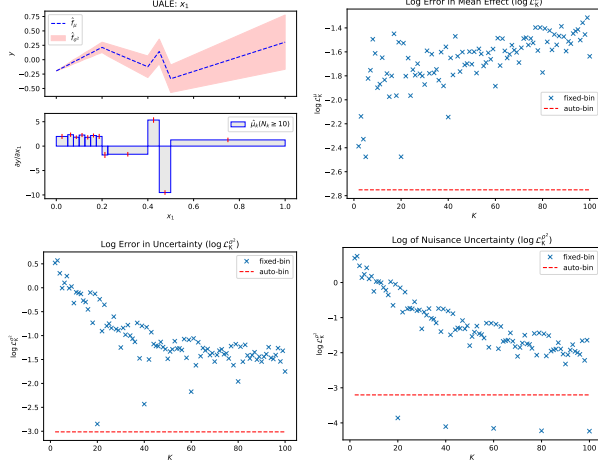


Figure 4: Figure 1

**Non-Linear Function.** In this set-up,  $f(\mathbf{x}) = 4x_1^2 + x_2^2 + x_1x_2$ , so the effect is non-linear in  $0 \leq x_1 \leq 1$ . In this set-up, wide bins increase  $\mathcal{L}^{\rho^2}$  making the uncertainty approximation more biased and narrow bins lead to a worse approximation. In Figure 5, we observe that automatic bin splitting manages to compromise the conflicting objectives, [Explain better]

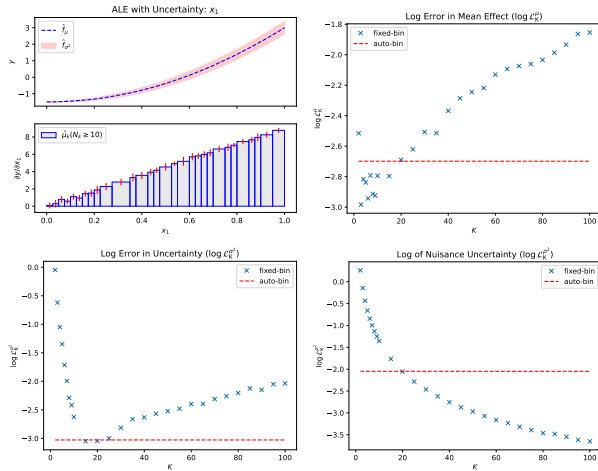


Figure 5: Figure 1

## 5 REAL-WORLD EXAMPLE

Here, we aim at demonstrating the usefulness of uncertainty quantification and the advantages of automatic bin-splitting, on the real-world California Housing dataset (Pace and Barry, 1997).

**ML setup** The California Housing is a largely-studied dataset with approximately 20000 training instances, making it appropriate for robust Monte-Carlo approximations. The dataset contains  $D = 8$  numerical features with characteristics about the building blocks of California, e.g. latitude, longitude, population of the block or median age of houses in the block. The target variable is the median value of the houses inside the block in dollars that ranges between  $[15, 500] \cdot 10^3$ , with a mean value of  $\mu_Y \approx 201 \cdot 10^3$  and a standard deviation of  $\sigma_Y \approx 110 \cdot 10^3$ .

We exclude instances with missing and outlier values. As outlier we define the feature values which are over three standard deviations away from the mean feature value. We also normalize all features to zero-mean and unit standard deviation. We split the dataset into  $N_{tr} = 15639$  training and  $N_{test} = 3910$  test examples (80/20 split) and we fit a Neural Network with 3 hidden layers of 256, 128 and 36 units respectively. After 15 epochs using the Adam optimizer with learning rate  $\eta = 0.02$ , the model achieves a normalized mean squared error (R-Squared) of 0.25 (0.75), which corresponds to a MAE of  $37 \cdot 10^3$  dollars.

Below, we illustrate the feature effect for three features: latitude  $x_2$ , population  $x_6$  and median income  $x_8$ . The particular features cover the main FE cases, e.g. positive/negative trend and linear/non-linear curve, and they are appropriate for illustration purposes. The complete results for all features, along with in-depth information about the preprocessing, training and evaluation parts are provided in the Appendix.

**Uncertainty Quantification** In real-world datasets, it is infeasible to obtain the ground truth FE for seamlessly evaluating the competitive methods. We selected the particular experiment, because, in broad terms, UALE and PDP-ICE plots agree in the estimation of the average effect and uncertainty. Therefore, we can focus on judging the quality of the information provided by the two methods.

In Figure 6, we observe, from top to bottom, the effects for the latitude, population and the median income. The effect of UALE and PDP-ICE are similar for the population and the median income. The population has a negative impact that progressively decreases: from 400 to 1500 people the house value decreases with a rate of  $-150(\pm 140)$  dollars per added person, a rate that decreases from  $-80(\pm 80)$  to  $-60(\pm 60)$  dollars per added person as we move from 1500 to 2800 people. The level of uncertainty indicates significant variance in absolute value of the rate, but in the grant

majority of instances the rate is negative. With the same inspection, we observe that the median annual income has a positive impact on the value (all numbers are thousands of dollars):  $20 \pm 15$  per 10 of added median income for incomes in  $[8, 15]$ ,  $32 \pm 20$  per 10 added income in  $[15, 60]$  and  $40 \pm 15$  per 10 added income in  $[60, 70]$ . The uncertainty indicates that there are less heterogeneous effects about the median income compared to the number of people. In both cases, we can end-up to the same conclusion by inspecting the PDP-ICE plots. For the latitude, there is a small difference in the explanations for the region  $[32, 35]$ , where UALE estimates a less negative slope with less uncertainty than PDP, while the explanations are similar for the range  $[35, 39]$ , where both methods reveal an increase in the uncertainty around the feature value 37.5.

In general, we observe that UALE complements ALE in the quantification of the heterogeneous effects, similarly to as ICE complements PDP. The automatic extraction of constant-effect and constant-uncertainty regions provided by UALE is helpful for an easier interpretation. On the other hand, ICE plots sometimes are more descriptive on locating the type of heterogeneity, whereas UALE quantifies only the level (not the type) of heterogeneity.

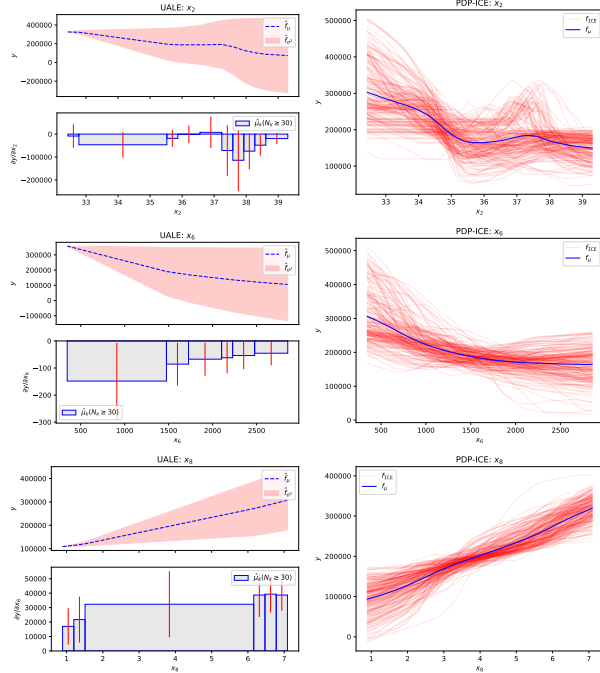


Figure 6: Figure 1

**Bin Splitting** In this part, we evaluate the robustness of the approximation using automatic bin-splitting. Following the evaluation framework of Section 4.2, we treat as ground-truth the effects computed on the full training-set  $N = 20000$  with dense fixed-size bin-splitting ( $K = 80$ ). Given the big number of samples, we make the hypothesis

that the approximation with dense binning is close to the ground truth. Afterwards, we randomly select less samples  $N = 1000$  and we compare UALE approximation with all possible fixed-size alternatives, repeating this process for 30 times for robust results. In Figure 7, we illustrate the mean values for  $\mathcal{L}^\mu, \mathcal{L}^\sigma$  of the 30 repetitions. We observe that automatic bin-splitting provides (close to the) best approximation in the three features. In the Appendix, we provide the same evaluation for all features.

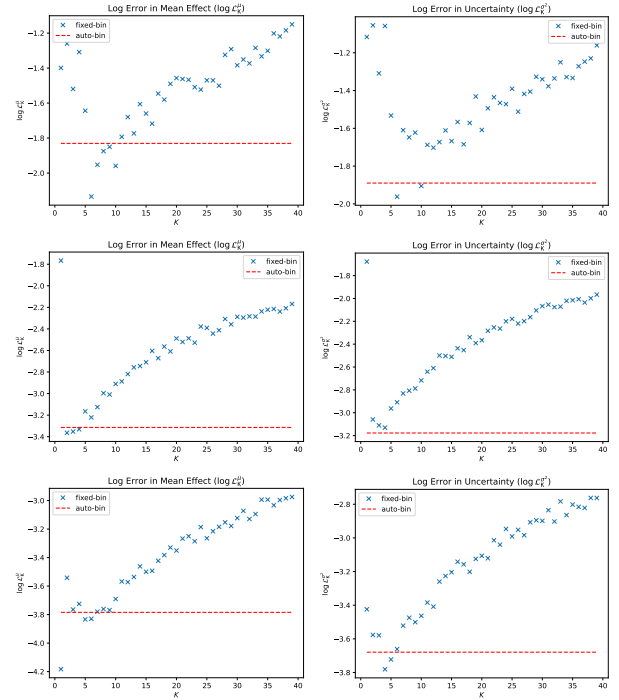


Figure 7: Figure 1

## 6 DISCUSSION

[Add Conclusion, limitations, and future work]

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