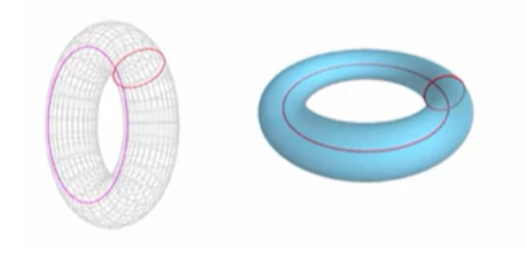


Toric Code

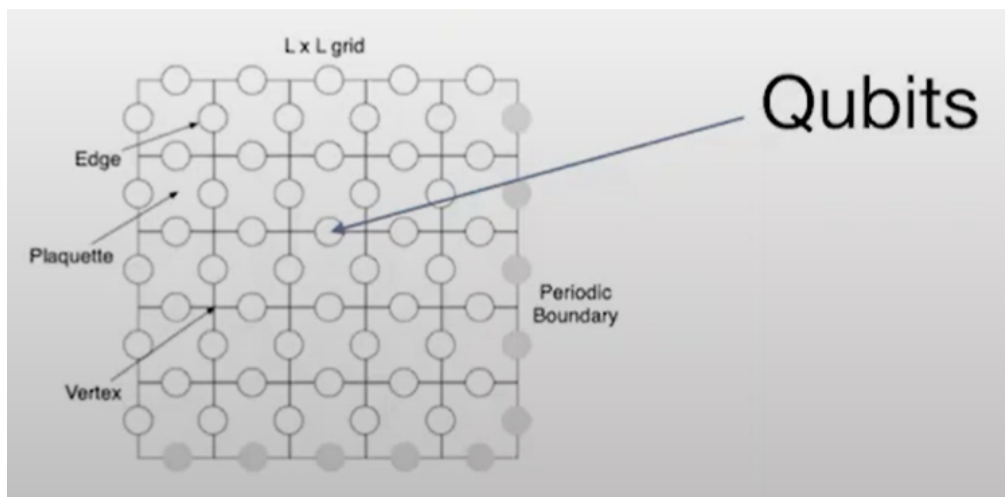
Introduction

Toric code is the simplest example of a topological code.



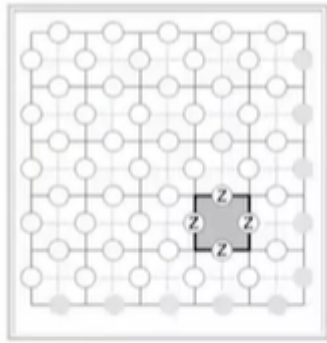
These two loops are called non-trivial loop, because they can not be deformed into a point or to each other.

Physical Qubits



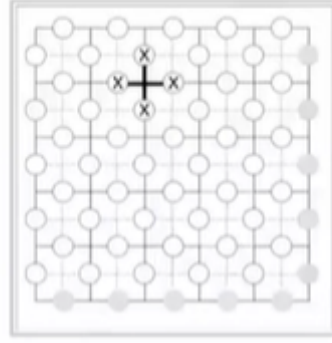
- of edges $L^2 + L^2 = 2L^2$.
- Each edge correspond to a physical qubit.
- of physical qubits $2L^2$.

Stabilizer Generator



Plaquette generator

$$L^2$$



Vertex generator

$$L^2$$

- Plaquette Operator = $\bigotimes_{s \in P} Z_s$.
- Vertex Operator = $\bigotimes_{s \in V} X_s$.
- Plaquette and Vertex operators commute.

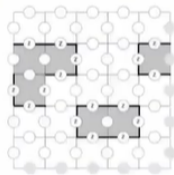
If two operators do not overlap, then they are surely commute.

If they overlap, the overlapping qubits will be exactly two, and since $XZ = ZX$, they are commute.

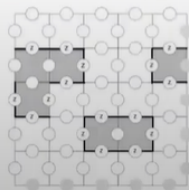
Multiplication of Plaquette Operators

(3) Multiplication of plaquette operators

(Baby) claim: A pair of plaquette operators either do not share a boundary or have only one shared boundary.



(Baby) claim: When we multiply plaquette operators, the resulting operators will include Z operators that act on the boundary of the combined plaquettes.

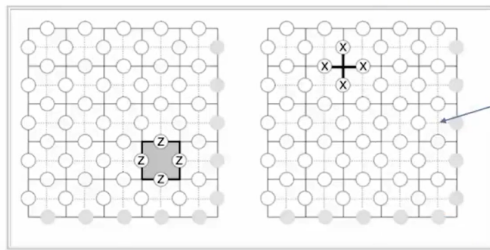


几个块状算子的张量积即为他们边界上的一圈Z的张量积, 因为被share的边界两个Z抵消, $ZZ = I$.

Claim: Independent Generators $L^2 - 1$.

Since $\prod_{\alpha} P_{\alpha} = I$.

Dual Lattice



Dashed lines indicate dual lattice.



Plaquette operators are vertex operators in the dual lattice and vice versa.

How to construct dual of a lattice

- Interchange plaquettes with vertices
- Reorient edges accordingly

Primal lattice	Dual lattice
Hexagonal	Triangular
Square	Square

Self-dual lattice: primal = dual

这里感觉和线代学的对偶空间有点联系, 但是没有看得特别明白, 但大致意思应该是说分析 plaquette 算子, 相当于在对偶空间分析 vertex 算子, 所以前面说的那些 plaquette 的性质应当对于 vertex 也成立.

Encoded Qubits

The encoded qubits of toric code is **2**.

- We have total $2L^2$ physical qubits, which means $n = 2L^2$.
- We have total $2(L^2 - 1)$ generators, which means $m = 2(L^2 - 1)$.
- Since $k = n - m$, we encoded totally 2 qubits.

Encoded Logical Operators

Recall that the requirements for the encoded logical operators:

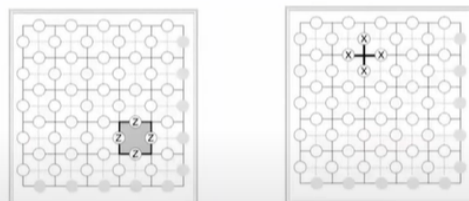
- Must commute with all elements of the stabilizer group.
- Must not be an element of the stabilizer group.
- Must satisfy the communication and anti-communication relation of the Pauli operators they encode.

Now let's try to construct a Z operator:

首先我们考虑能否使一个封闭的二维图形? 不行, 因为封闭的二维图形必定由一些 Plaquette 组成, 所以其属于 Stabilizer Group.

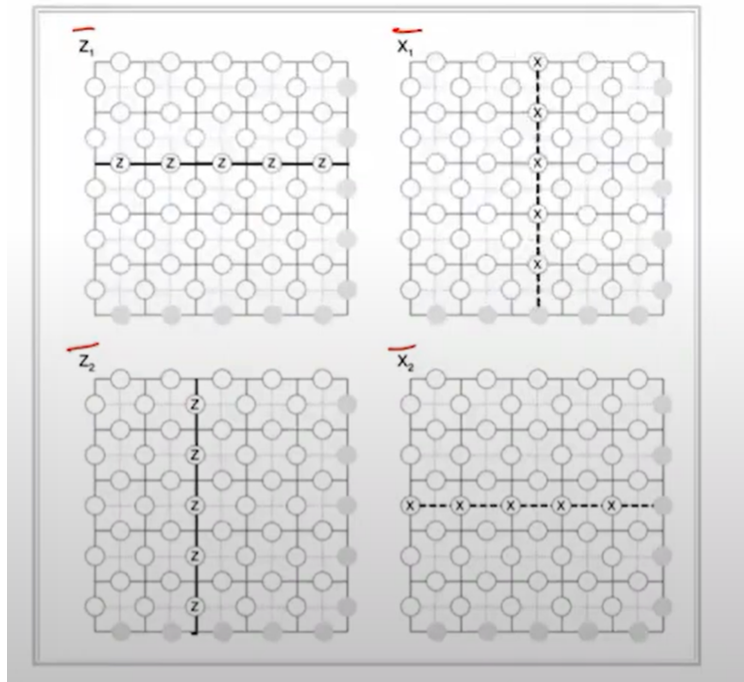
所以下面我们考虑 string, 即一维的线性结构.

\bar{Z}_1



If we form a string of Z operators, regardless of its shape, it will always anticommute with the vertex operators at the ends of the string. The only solution is to find a string of operators which has no end - a loop!

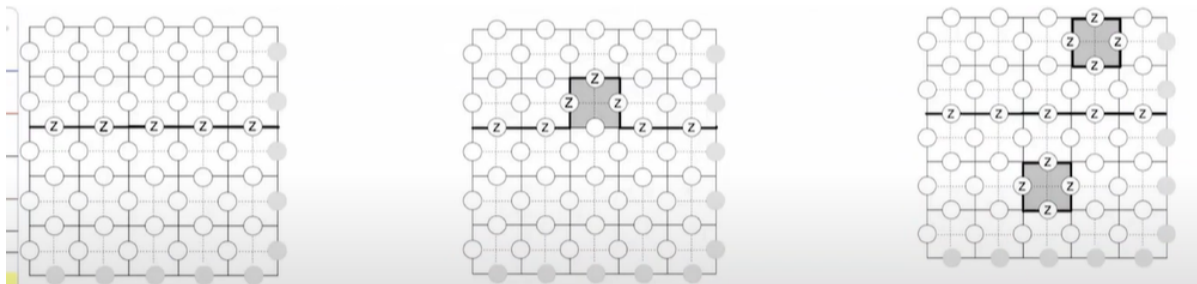
$$XZ = -ZX$$



这就是我们最终构造出来的 Z 和 X , 其实他们也不一定要是一条笔直的路, 可以有一些zigzag, 但只要保证是loop即可.

另外, 水平和竖直的环恰好对应了本篇笔记开头的那两个non-trivial的环.

Equivalent of Logical Operators under Stabilizer Multiplication.



正如Lec01所讲的那样, 一个算子乘上一些列的stabilizer都与原来等效.

Code Distance

- Minimum weight of any logical operator in the code.
- 可以看到我们的logical operator最短应当为水平和竖直方向笔直的直线(环), 因为zigzag只会增加weight.
- Thus, $d = L$.
- $[[n = 2L^2, k = 2, d = L]]$.

Error Correction via Toric Code

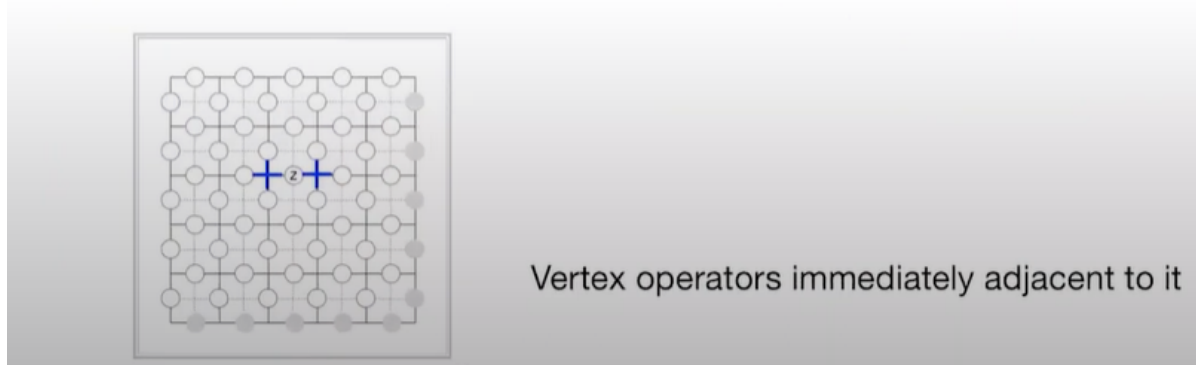
Recall that:

- We can detect errors on stabilizer codes by measuring the stabilizer generators.
- Syndrome: outcome of the measurement of a given stabilizer generator.

- When error E happens, the stabilizer generators that don't commute with E will output -1 .

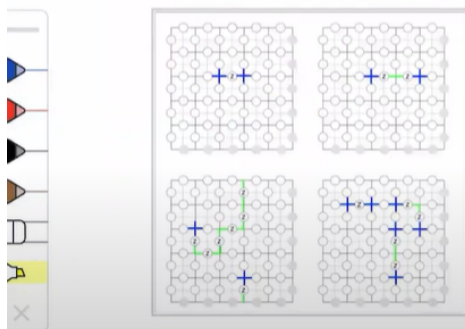
Example: Z error on a single qubit

Which stabilizer generators anticommute with it?



(Baby) claim: Given any string of errors on the primal lattice, the only stabilizer generators have their outcome -1 are the vertices at the two ends of the string.

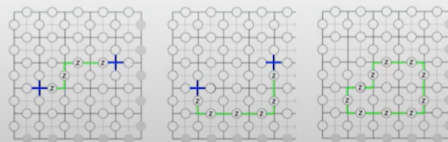
Proof:



The ends of a string can be considered its "boundary"

- **Main task in error correction:** identification of the error operator to apply given the syndrome
- For exa: apply the inverse of the error operator
- For self-inverse Pauli errors, apply the same operator

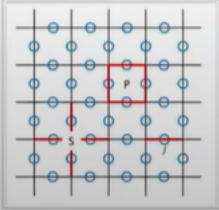
aby) claim: If $E'E = S$, where S is a stabilizer, then E' will correct E .



像图三这样, 如果有两次犯错他们的路径构成了一个环, 他们形成了一个 stabilizer, 相当于负负得正, 自我修复了.

The Toric Code Hamiltonian

$$A_s = \prod_{j \in s} X_j \quad B_p = \prod_{j \in p} Z_j$$

$$H_{tc} = - \sum_s A_s - \sum_p B_p$$


There are actually four kinds of ground states for toric code, which make sense because $2^k = 2^2 = 4$, the size of the logical qubits space is 4.