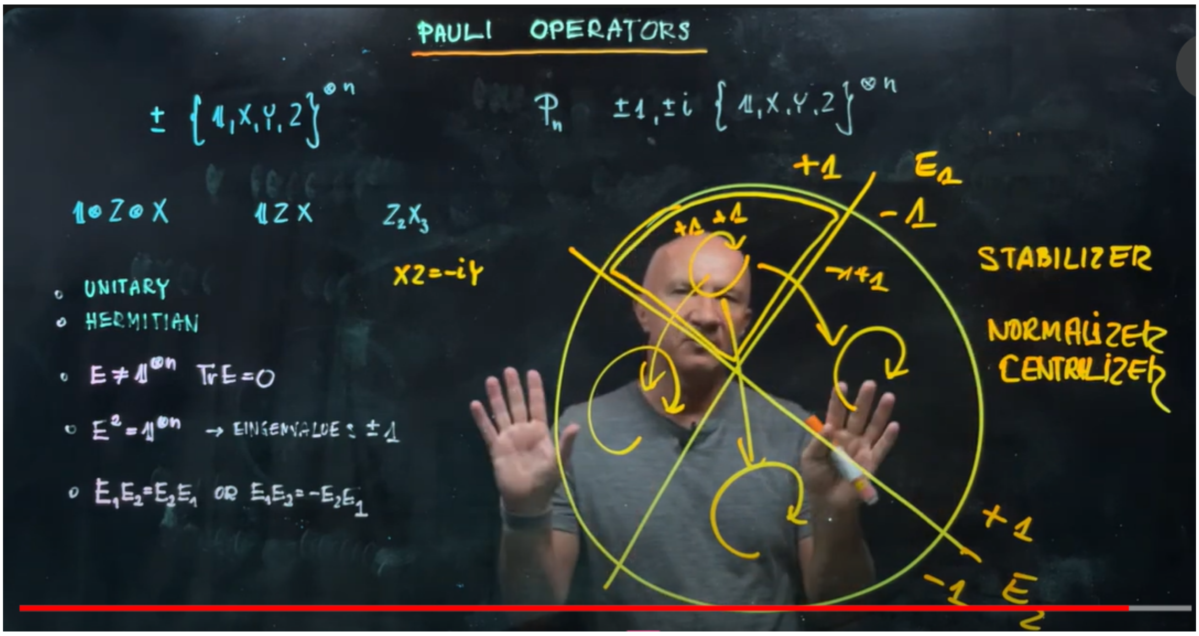
Error Correction: from classic to quantum & stabilizer

> In this lecture, we learn some basic knowledge of QEC, which lay the ground for later toric code and topological code.

Pauli Operators

- Consists of X, Y, Z, I, which is $\{X, Y, Z, I\}^{\otimes n}$.
- Unitary.
- Hermitian.
- $E \neq I^{\otimes n}$, $TrE = 0$.
- $E^2 = I^{\otimes n}$, which means the eigen values of E is +1 and -1 .
- $E1E2 = E2E1 \ or \ E1E2 = -E2E1$, which mean either commute or anti-commute.



> 如图所绘，我们可以用两个Pauli operator将整个Hilbert Space按照特征子空间分为四个部分，而 +1 +1 的特征空间一般用于code space，剩下的子空间为 error space.
>
> 当一个Pauli operator作用于一个state $|\psi\rangle$后，可能仍然处于code space，也可能进入error space，从而被检测出错误.

Stabilizer Formalism

Stabilizer Codes are defined by specifying two sets of operators:

- (Stabilizer) Generator.
- Encoded logical operators.

The stabilizer group

Let $\{|\psi_j\rangle\}$ be the code-word basis states, for three qubits repetition code, it's $|000\rangle$ and $|111\rangle$.

Stabilizer group: the set of Pauli operators which leave all codeword basis states $|\psi_j\rangle$ invariant.

$$P_k|\psi_j\rangle = |\psi_j\rangle \quad \forall P_k \in \mathscr{P}$$

Stabilizer operator (or stabilizer element): member of the stabilizer group

Claim: The set of stabilizer operators must commute.

Stabilizer Generators

Any group G can be specified by a set of generators $\{g_j\}_{j=1}^{m}$.

Theorem: For an Abelian group of self-inverse operators, any element $g \in G$ can be written as $g = \prod_J g_j^{\alpha_j}$ where $\alpha_j \in \{0,1\}$.

$$|G| = 2^m$$

- k: logical qubits.
- n: physical qubits.
- m: stabilizer generators.
- $m = n - k$

Example:

Three qubit repetition code

$$|0\rangle_L = |000\rangle \qquad |1\rangle_L = |111\rangle$$

$$n = 3 \quad k = 1 \qquad \Longrightarrow \quad m = 3 - 1 = 2$$

Order of the stabilizer group $= 2^m = 4$

$$ZZI, ZIZ, IZZ, III$$

Error detection in the stabilizer formalism

- We can detect errors on stabilizer codes **by measuring the stabilizer operators**.
- **m measurements suffice**(Just measure m generators will suffice).
- **Syndrome**: outcome of the measurement of a given stabilizer generator.

Encoded logical operators in the stabilizer formalism

For the three qubit repetition code, we had

$$|0\rangle_L = |000\rangle \qquad |1\rangle_L = |111\rangle \qquad |\psi\rangle = \alpha|0\rangle + \beta|1\rangle \to |\psi\rangle_L = \alpha|000\rangle + \beta|111\rangle$$

$$\bar{X} = XXX \qquad \bar{Z} = ZII$$

> Instead of ZII, we can also use IZI or IIZ.

- Equivalent set of logical operators

  Let S be the stabilizer group.

  $|\psi\rangle$ be the state in the codespace.

  L be a logical operator.

  $$\begin{aligned} &S_j|\psi\rangle = |\psi\rangle,\ S_j \in S \\ \implies &LS_j|\psi\rangle = L|\psi\rangle,\ S_j \in S \end{aligned}$$

  > Given a logical operator L, there exists a family of $|s| = 2^m$ operators$\{LS_j\}$ that act equivalently on the codespace.

- Centralizer or Normalizer

  The centralizer of an element z of a group G is the set of elements G which commute with z,

  $$C_G(z) = \{x \in G,\ xz = zx\}.$$

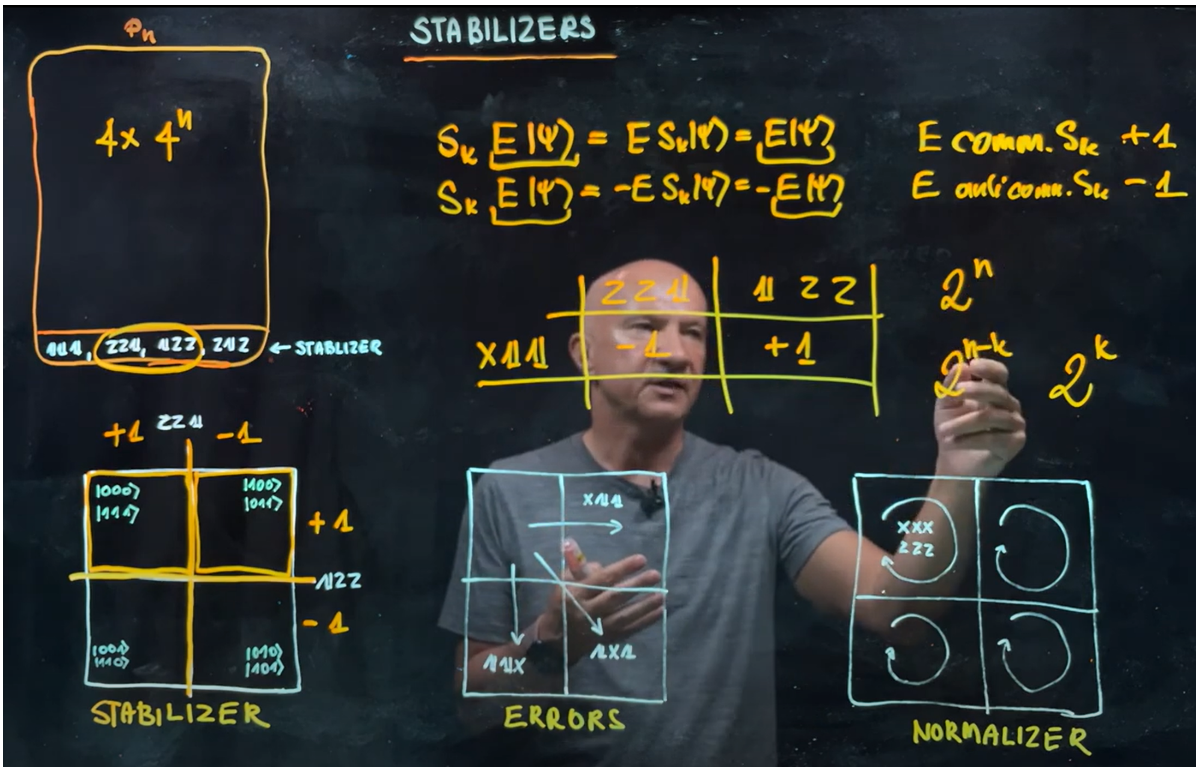
  Likewise, the centralizer of a subgroup H of group G is the set of elements of G which commute with every element of H,

  $$C_G(H) = \{x \in G,\ \forall h \in H,\ xh = hx\}.$$

  > **A logical Pauli operator must belong to the centralizer of the stabilizer group.**

- Distance

  The minimal weight of any operator in the centralizer of the code.



> 以三位重复码为例:
>
> 在这张图中,我们选取ZZI和IZZ为Stabilizer Generator,而Stabilizer Group则由$III\ ZZI\ IZZ\ ZIZ$组成,刚好为2^2. 两个generator将Hilbert Space 划分为了四个子空间,以在两个算子上的测量值区分.
>
> +1 +1子空间为code-space,剩余三个均为error-space. errors对应的算子会将code-space中的state map到error-space中去,而normalizer(centralizer)为子空间上的不变算子,所以logical operator必须是normalizer,因为其必须将code-space上的向量映射回code-space.
>
> 需要注意的是,本图中可能存在的错误有三种,即从code-space分别到三个error-space,但是每种错误可能对应多个算子(根据上面的Equivalent set of logical operators就可以构造出多个). 在一般情况下这个事实也成立,而我们一般选择weight最小的operator作为错误进行处理,因为weight最小代表其概率最大.