Introduction to topology

Basic Topology

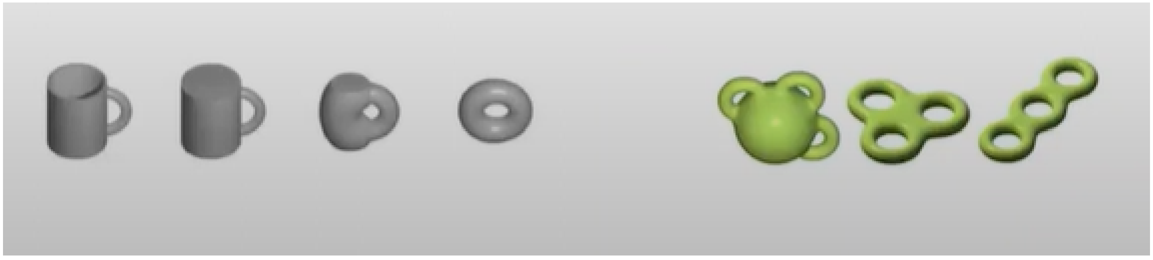
Topological Equivalence

> When we say two objects are "topologically equivalent", we mean equivalent up to **homeomorphism**.

Homeomorphism: a continuous function between two objects or topological spaces which

1. is continuous.
2. is a bijection (one to one and onto, hence invertible).
3. has a continuous inverse function.

Examples: continuous stretching, bending ···

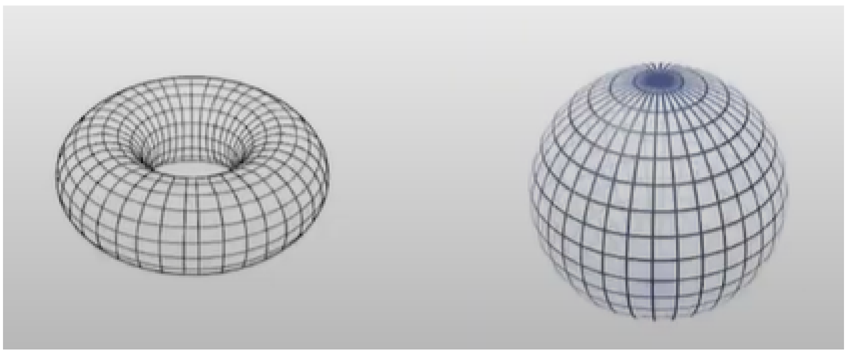


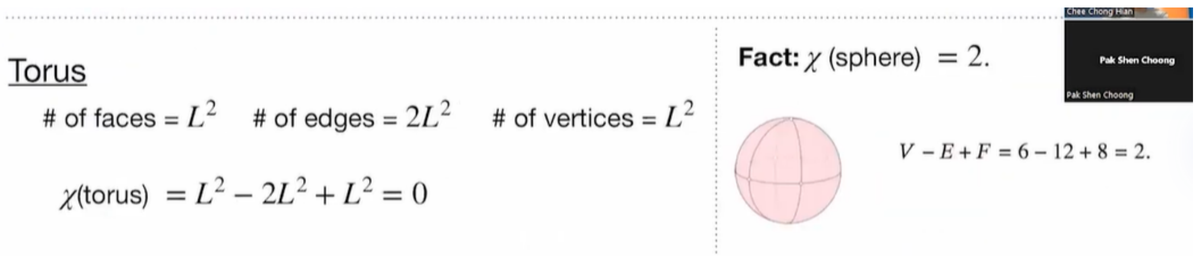
Topological Invariants

Properties or numeric quantities that remain unchanged under homeomorphism, such as Euler Characteristic.

$$\chi = V - E + F$$

> The Euler Characteristic can be generalized to parametrise general spaces via the process of "cellulation", which is division of surface into polygonal cells, such as triangulation.
>
> 比如上节课学的toric code，就是将一个曲面cellulation 成了很多方形.





> Since $\chi(torus) \neq \chi(sphere)$, the sphere and torus are topologically inequivalent.

Genus: The formal term used to count the number of handles in a surface.

> 可以理解为洞的个数，sphere没有，torus有一个.

Theorem:

The Euler Characteristic of any compact orientable surface of genus g is $2-2g$.

$$\chi = 2 - 2g$$

> **Orientable:** A triangulatable surface is orientable if we can assign an orientation to every cell in the triangulations, such that the orientations of edges of adjacent cells agree. E.g. A sphere is orientable, a mobius strip is not. We do not need to consider orientations in the study of qubit topological codes (but they are important in the definition of topological codes for qudits with $d > 2$.)

Theorem:

All compact surfaces with genus g are homeomorphic to a sphere with g handles.

Application

> For toric code, the number of qubits encoded is $2-\chi$.

$$physical\ qubits = E$$
$$generators = (F-1)+(V-1)$$
$$logical\ qubits = physical\ qubits - generators = 2+E-F-V = 2-\chi$$

> 其实这个公式不止对toric code成立，为toric code时，$\chi = 0$，所以逻辑比特为2，如果换用其他拓扑图形，使得χ减小，我们就可以获得更多的逻辑比特.

Generalizing to genus g g-torus:

1. Cellulate a general torus with a square lattice.
2. Define stabilizer generators.

$$\chi(g-torus) = 2-2g$$
$$qubits\ encoded = 2g$$

Homology

<u>Homology:</u> the mathematical field which abstracts and generalises the notion of a "boundary

Our focus:
1. Concrete aspects of homology relevant to qubit topological codes
2. Homology on the group $\mathbb{Z}_2$, the simplest form of homology
3. Square lattice cellulations

Recall toric code
1. Square lattice with periodic boundary conditions
2. This lattice is an example of a cellulation of the torus.
3. Consists of:
   A. 0-dimensional objects: the vertices
   B. 1-dimensional objects: the edges
   C. 2-dimensional objects: the plaquettes

RECALL

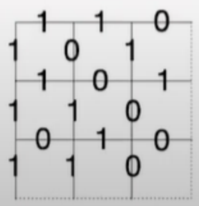
n-cell: an n-dimensional object in a cellulation.

- 0-cell: vertices.
- 1-cell: edges.
- 2-cell: plaquettes.

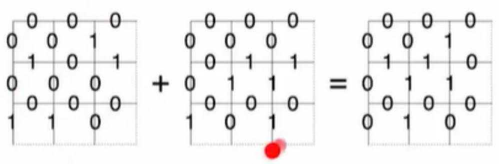
Chains

1-chains

1-chain: an assignment of elements of $\mathbb{Z}_2$ to every 1-dimensional object on the lattice



> 可以理解为选取了边集中的一个子集(标记为1的部分).

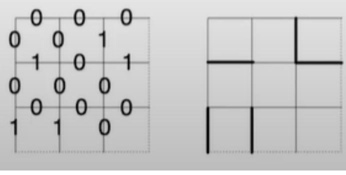


The set of 1-chains form a group.
1. Group composition: edge-wise (bit-wise) addition modulo 2
2. Associativity: follows from associativity of addition
3. Identity operator: all-zeros 1-chain
4. Inverse: (for $\mathbb{Z}_2$), every 1-chain is self-inverse

Abelian group

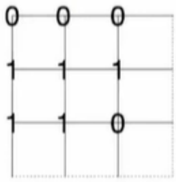
$|G| = 2^E$

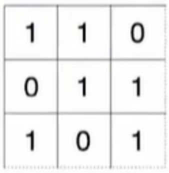
E: # of edges



Similarly, we can define 0-chains, 2-chains and n-chains.

0-chains, 2-chains and n-chains





0-chain: assign an element of $\mathbb{Z}_2$ to every vertex
2-chain: assign 0 or 1 to every plaquette
n-chain: assign an element of $\mathbb{Z}_2$ to every n-cell

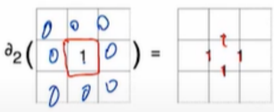
Notation

C_n : group of n-chains
Lower case Roman letters to label chains
1-chain $c \in C_1$

The Boundary Map

n-boundary map ∂_n: a group-structure preserving map from the set of n-chains c to the (n-1)-chains d.

The 2-boundary map

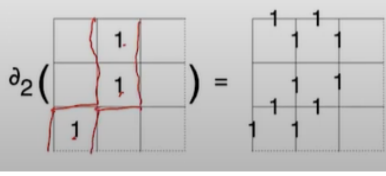


Let $c = \sum_j a_j c_j$

2-chain generators

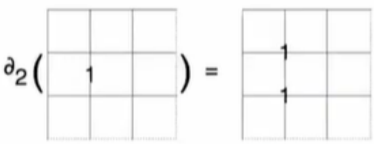
bit valued coefficients

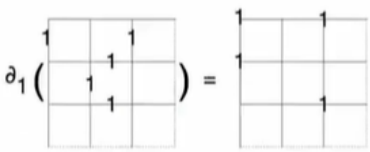
$$\partial_2(c) = \partial\left(\sum_j a_j c_j\right) = \sum_j a_j \partial_2 c_j$$



The 1-boundary map

maps 1-chains to 0-chains





The 0-boundary map

What is the boundary map for 0-cells?
There are no -1 dim objects.
Nonetheless, it is useful to define ∂_0 .

0_n : null n-chain
For any 0-chain c, $\partial_0(c) = 0_{n-1}$

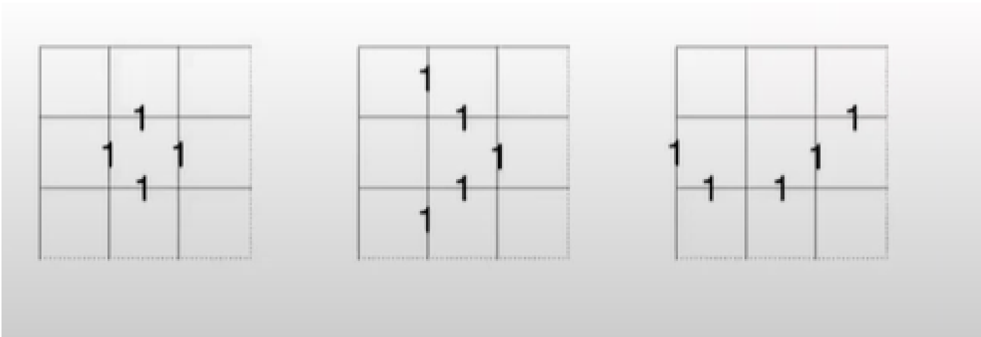
Cycles

Central concept in homology.

n-cycle: an n-cycle is an n-chain c satisfying:

$$\partial_n c = 0_{n-1}$$

A cycle is a chain with null boundary, i.e. "without a boundary".



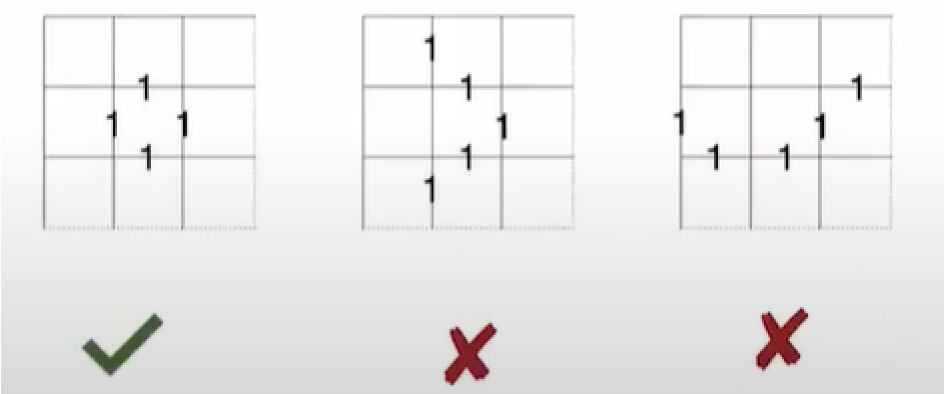
Boundary

n-boundary chains: any n-chain which is the boundary of (n+1)-chain is called n-boundary chain or n-boundary.

Notations

$$\begin{aligned} C_n: &\quad \text{The group of n-chains.} \\ Z_n: &\quad \text{The group of n-cycles.} \\ B_n: &\quad \text{The group of n-boundaries.} \end{aligned}$$

- **Every n-boundary is an n-cycle, but not every n-cycle is a n-boundary.**



> 如上图，左侧的trivial cycle恰好为一个boundary，而右侧的两个none-trivial cycle不是boundary.

- $\partial_{n-1}\partial_n c_n = 0_{n-2}$

> Since the boundary of a boundary is null.

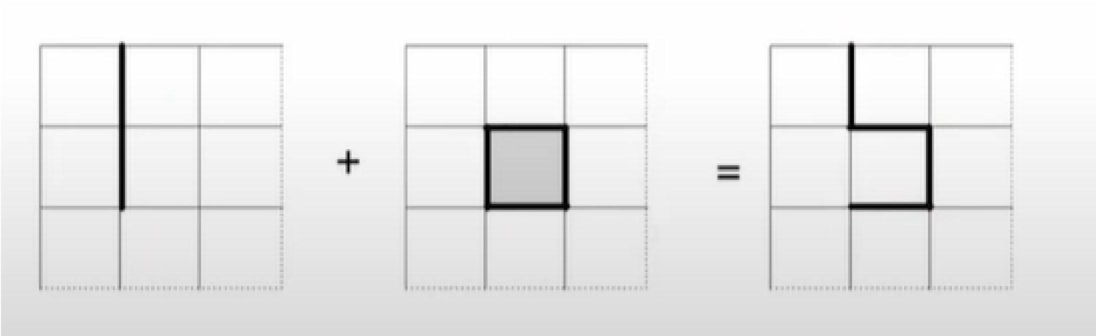
- $B_n \subseteq Z_n \subseteq \mathbb{C}_n$.

> Homology studies the properties and interplay of these groups.

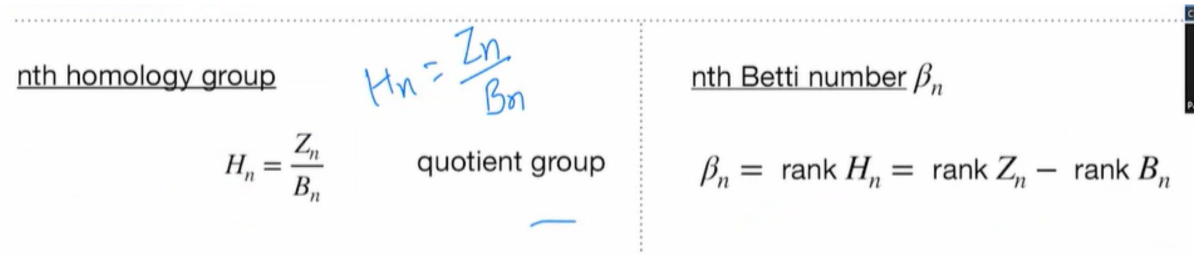
Homology Equivalence

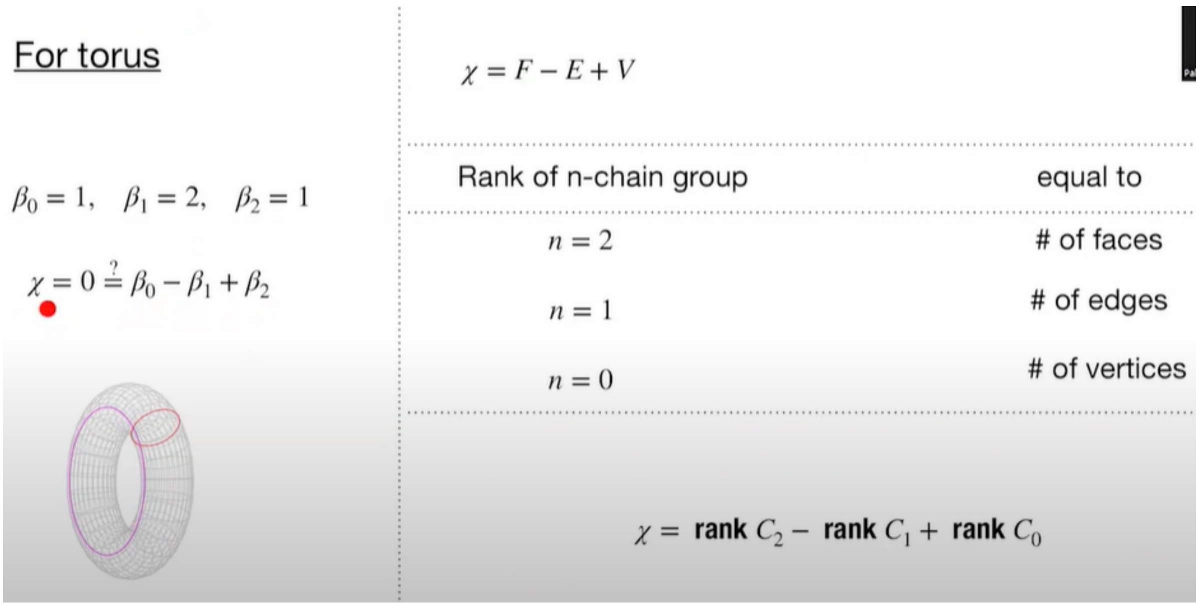
Two n-chains c and d are homologically equivalent if they are equal up to composition with an n-boundary b,

$$c = d + b,\ where\ b \in B_n$$



> 我们可以使用这一性质,将n-chains划分为不同的等价类.





> 说实话这里我还没太懂这几个群的rank怎么计算.

Claim: $rank(C_n) = rank(Z_n) + rank(B_{n-1})$.

> 对于边界算子, kernel为Z_N, range 为B_{n-1}.

Claim: $\chi = \sum_k (-1)^k \beta_k$.

> β is the betti number.

Cohomology

In cohomology, we develop duals for the key aspects of homology

1. Chains
2. Boundaries
3. Cycles

Co-chains

Object	Dual
Ket	Bra
Chain	Co-chain

Set of bras: set of linear functionals $f_\phi()$ on the vector space of kets, i.e linear maps from to scalars

$$f_\phi(|\psi\rangle) = \langle\phi|\psi\rangle \in \mathbb{C}$$

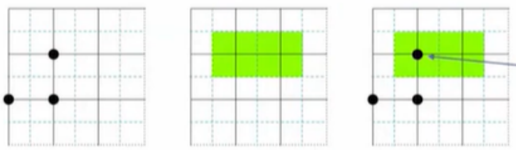
Set of n co-chains: the set of linear functionals which map each n-chain to the group $\mathbb{Z}_2$.

$$\langle\tilde{p}, c\rangle \in \mathbb{Z}_2$$

$c \in C_n$ and $\tilde{p}$ is a n co-chain (an element of the n co-chain group).

> co-chain 和 chain 做内积得到数.

0-Cochains



c $\quad$ $\tilde{p}$ $\quad$ $\langle\tilde{p}, c\rangle = 1$

1 vertex inside the dual plaquette

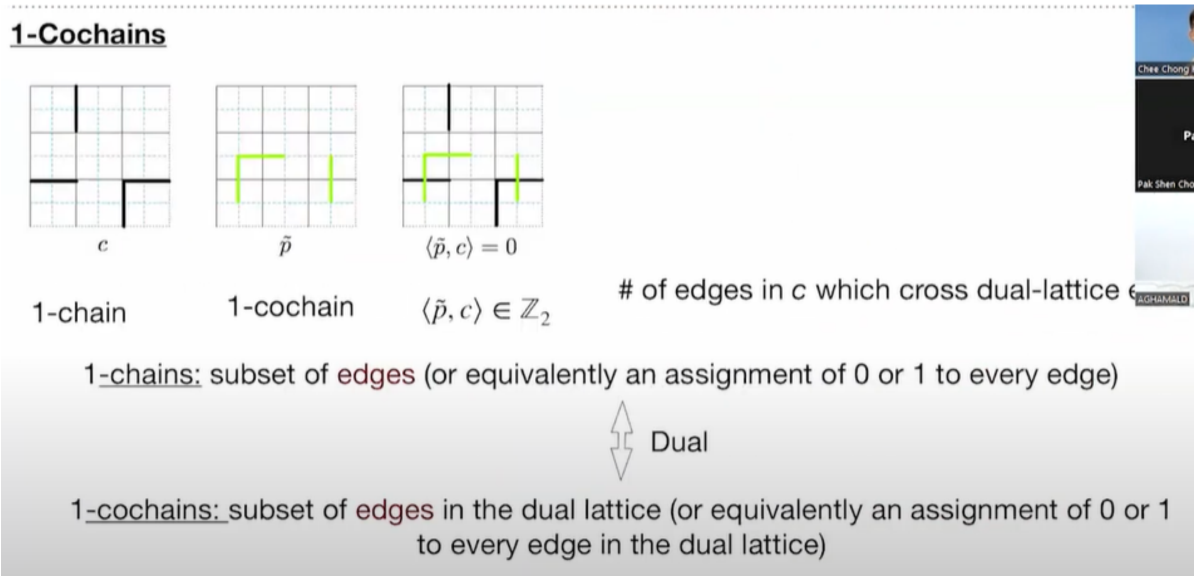
0-chain $\quad$ 0-cochain $\quad$ $\langle\tilde{p}, c\rangle \in \mathbb{Z}_2$ $\quad$ # of vertices inside the dual plaquette mo

0-chains: subset of vertices (or equivalently an assignment of 0 or 1 to every vertex)

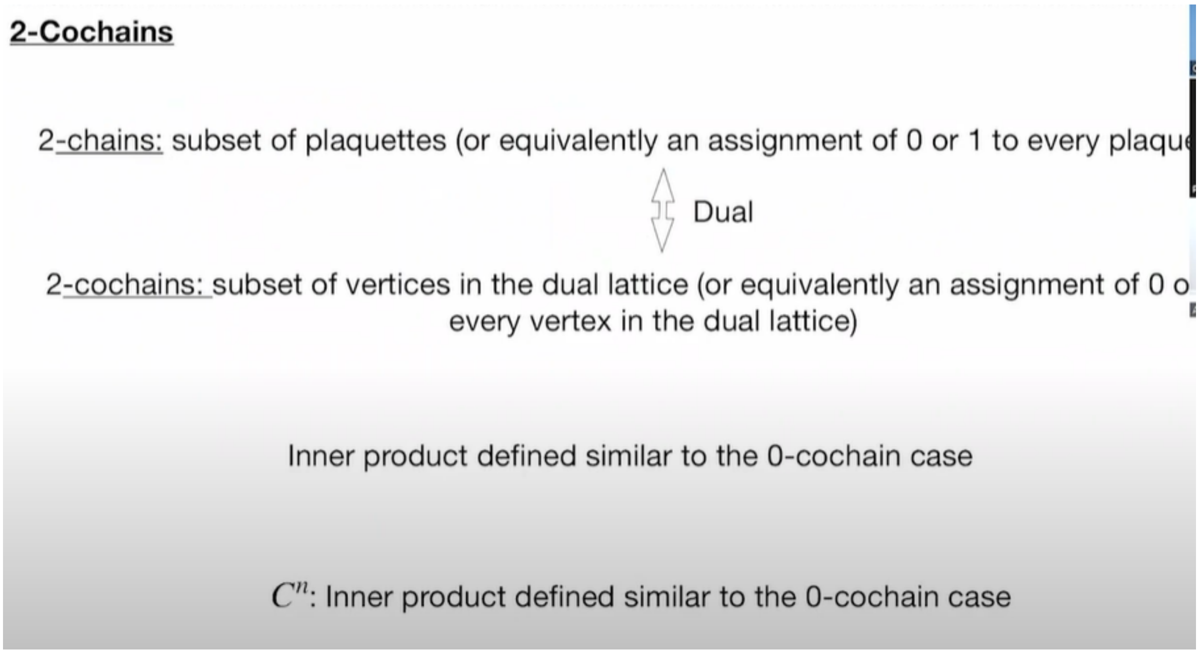
$\Updownarrow$ Dual

0-cochains: subset of plaquettes in the dual lattice (or equivalently an assignment of 0 or 1 to every plaquette in the dual lattice)

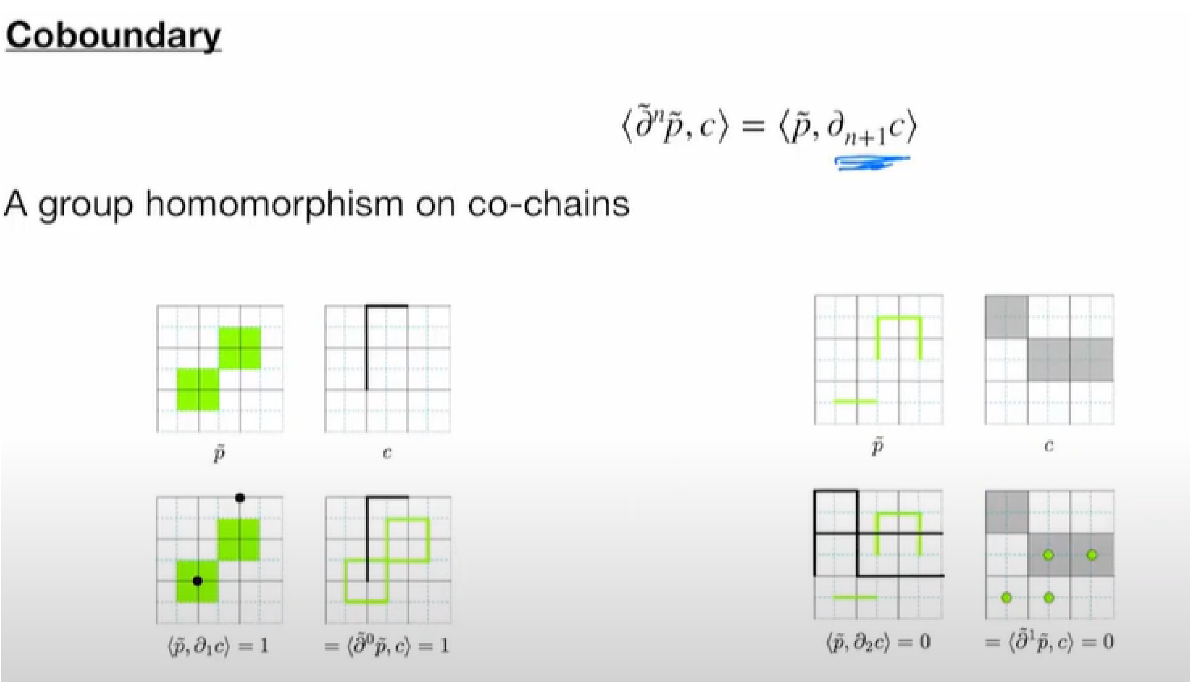
> 注意dual space中lattice的中心为原来space的一个vertex.



> 注意算内积的时候都是modulo 2.



> 最后一行写错了，应该想说的是C_n表示n-chain，而C^n表示n-cochian.



Cohomology Groups

$$
\begin{aligned}
\widetilde{\partial}^n : &\quad \text{n-coboundary map.} \\
Z^n : &\quad \text{the group of cocycles(kernel).} \\
B^n : &\quad \text{the image of coboundaries(iamge/range).} \\
H^n = \frac{Z^n}{B^n} &\quad \text{nth cohomology group.} \\
Betti\ numbers\ \beta^n : &\quad \text{rank of the nth cohomology group.}
\end{aligned}
$$