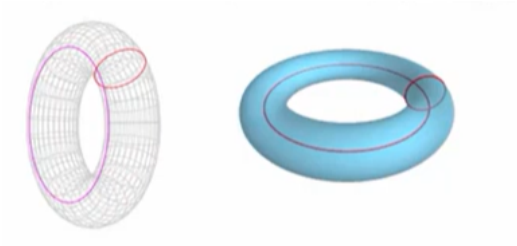
Toric Code

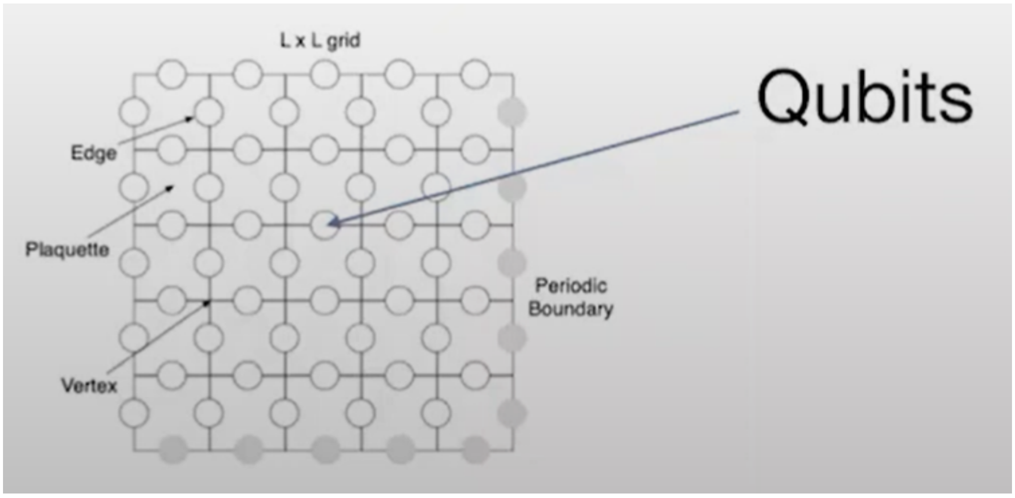
Introduction

Toric code is the simplest example of a topological code.



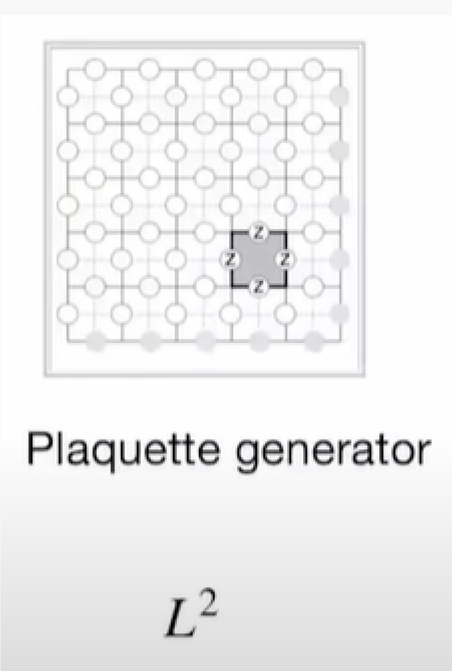
> These two loops are called none-trivial loop, because they can not be deformed into a point or to each other.

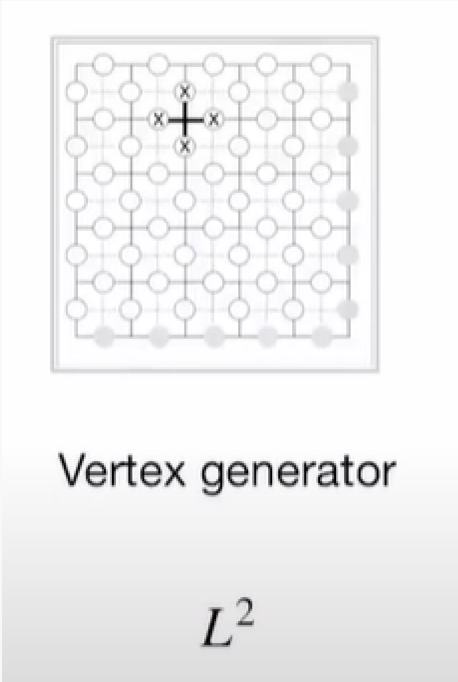
Physical Qubits



- of edges $L^2 + L^2 = 2L^2$.
- Each edge correspond to a physical qubit.
- of physical qubits $2L^2$.

Stabilizer Generator

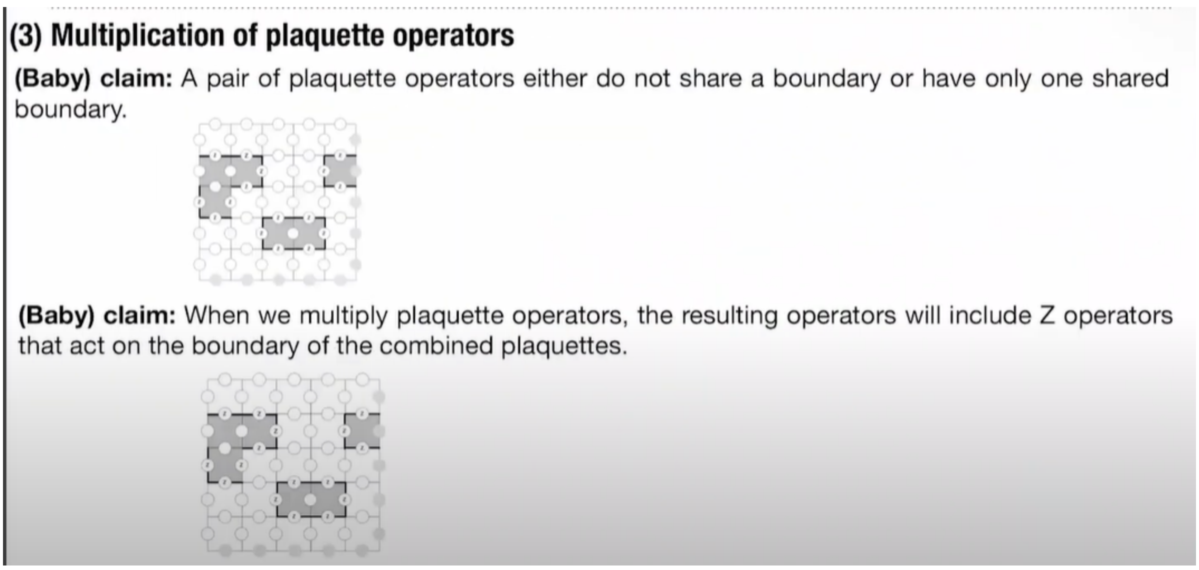




- Plaquette Operator = $\bigotimes_{S\in P} Z_s$.
- Vertex Operator = $\bigotimes_{S\in V} X_v$.
- Plaquette and Vertex operators commute.

> If two operator dose not overlap, then they are surely commute.
>
> If they overlap, the overlapping qubits will be exactly two, and since $XZ = ZX$, they are commute.

Multiplication of Plaquette Operators

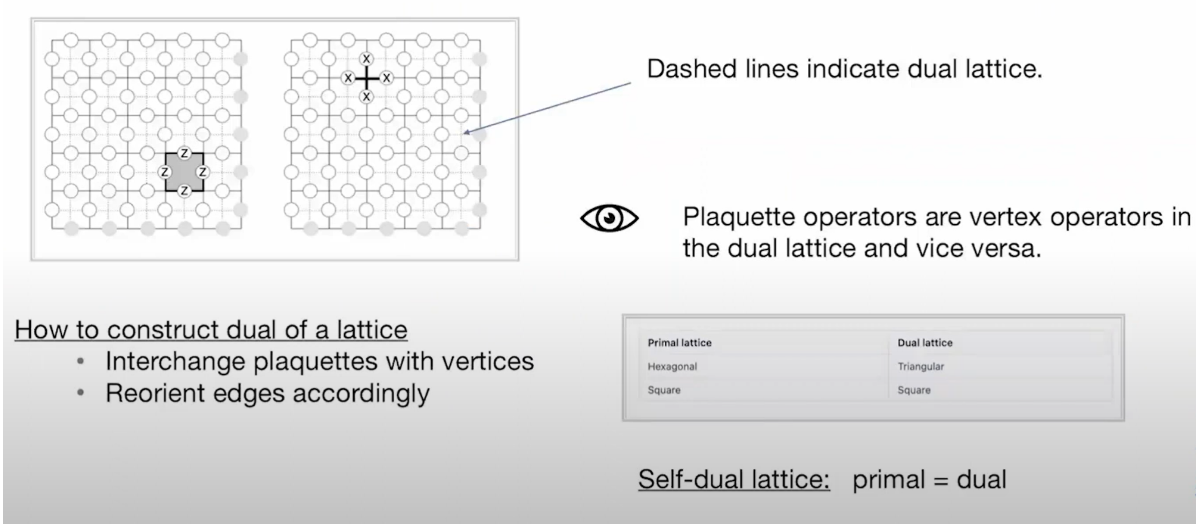


> 几个块状算子的张量积即为他们边界上的一圈Z的张量积，因为被share 的边界两个Z抵消，$ZZ = I$.

Claim: Independent Generators $L^2 - 1$.

> Since $\prod_{\alpha} P_\alpha = I$.

Dual Lattice



> 这里感觉和线代学的对偶空间有点联系，但是没有看得特别明白，但大致意思应该是说分析plaquette算子，相当于在对偶空间分析vertex算子，所以前面说的那些plaquette的性质应当对于vertex也成立.

Encoded Qubits

The encoded qubits of toric code is **2**.

- We have total $2L^2$ physical qubits, which means $n = 2L^2$.
- We have total $2(L^2-1)$ generators, which means $m = 2(L^2-1)$.
- Since $k = n - m$, we encoded totally 2 qubits.

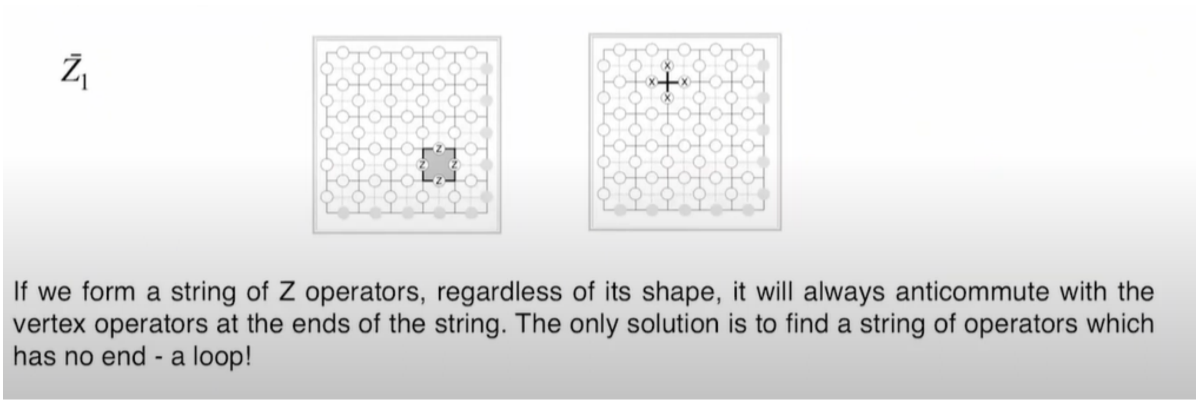
Encoded Logical Operators

Recall that the requirements for the encoded logical operators:

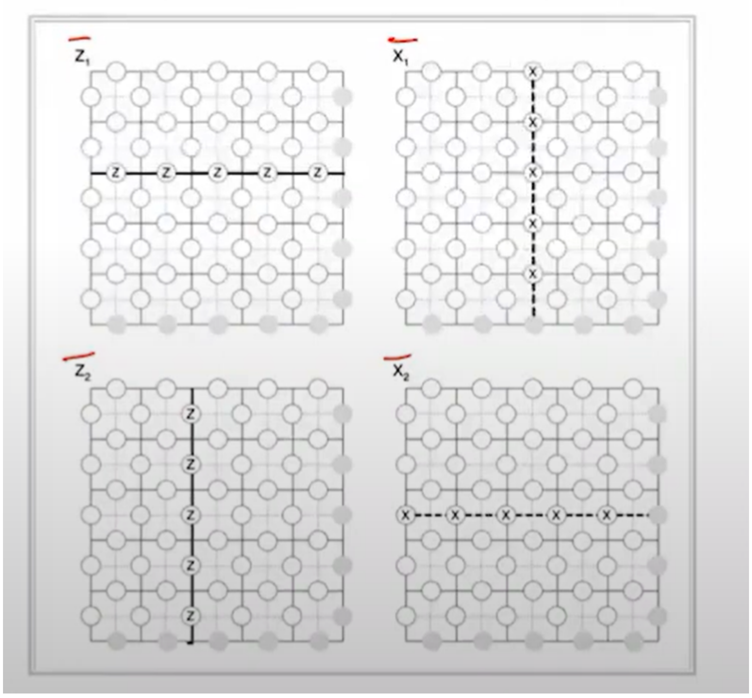
- Must commute with all elements of the stabilizer group.
- Must not be an element of the stabilizer group.
- Must satisfy the communication and anti-communication relation of the Pauli operators they encode.

Now let's try to construct a Z operator:

> 首先我们考虑能否使一个封闭的二维图形？不行，因为封闭的二维图形必定由一些Plaquette组成，所以其属于Stabilizer Group.
>
> 所以下面我们考虑string，即一维的线性结构.

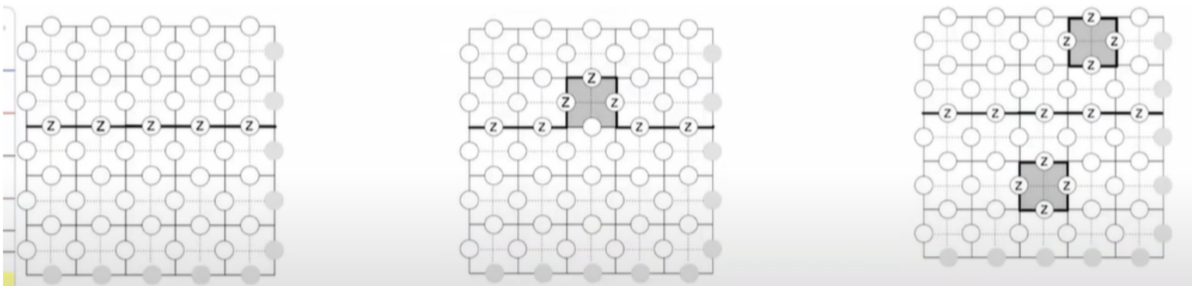


> $XZ = -ZX$



> 这就是我们最终构造出来的Z和X，其实他们也不一定要是一条笔直的线，可以有一些zigzag，但只要保证是loop即可.
>
> 另外，水平和竖直的环恰好对应了本篇笔记开头的那两个none-trivial的环.

Equivalent of Logical Operators under Stabilizer Multiplication.



> 正如Lec01所讲的那样，一个算子乘上一些列的stabilizer 都与原来等效.

Code Distance

- Minimum weight of any logical operator in the code.
- 可以看到我们的logical operator最短应当为水平和竖直方向笔直的直线(环)，因为zigzag只会增加weight.
- Thus, $d = L$.
- $[[n = 2L^2,\ k = 2,\ d = L]]$.

Error Correction via Toric Code

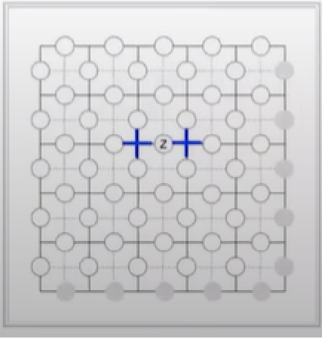
Recall that:

- We can detect errors on stabilizer codes by measuring the stabilizer generators.
- Syndrome: outcome of the measurement of a given stabilizer generator.

- When error E happens, the stabilizer generators that don't commute with E will output -1.

Example: Z error on a single qubit

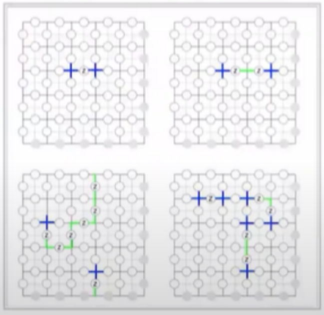
Which stabilizer generators anticommute with it?



Vertex operators immediately adjacent to it

(Baby) claim: Given any string of errors on the primal lattice, the only stabilizer generators have their outcome -1 are the vertices at the two ends of the string.

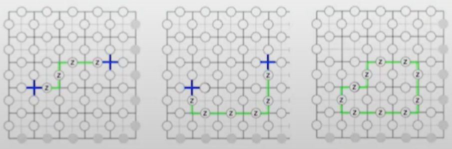
Proof:



The ends of a string can be considered its "bounday"

- **Main task in error correction:** identification of the error operator to apply given the syndrome
- For exa: apply the inverse of the error operator
- For self-inverse Pauli errors, apply the same operator

aby) claim: If $E'E = S$, where S is a stabilizer, then E' will correct E.

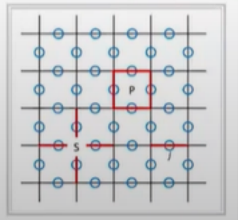


> 像图三这样，如果有两次犯错他们的路径构成了一个环，他们形成了一个stabilizer，相当于负负得正，自我修复了.

The Toric Code Hamiltonian

$$A_s = \prod_{j \in s} X_j \qquad B_p = \prod_{j \in p} Z_j$$

$$H_{tc} = -\sum_s A_s - \sum_p B_p$$



> There are actually four kinds of ground states for toric code, which make sense because $2^k = 2^2 = 4$, the size of the logical qubits space is 4.