# MATH 570, Homework 7

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Solutions

**Problem 1.** Suppose $f, g\colon S^n \to S^n$ are continuous maps such that $f(x) \neq -g(x)$ for any $x \in S^n$. Prove that $f$ and $g$ are homotopic.

*Hint: As an easier case, what if we instead had $f, g\colon S^n \to \mathbb{R}^{n+1} \setminus \{\vec{0}\}$ with the the line segment from $f(x)$ to $g(x)$ not passing through the origin for all $x \in S^n$? Can you modify a proof of this easier case to handle the problem above where $f, g\colon S^n \to S^n$ with $f(x) \neq -g(x)$?*

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*Proof.* Consider the hint. We have, since no line segment between $f(x)$ and $g(x)$ will pass through 0, that there is a straight line homotopy between $f(x)$ and $g(x)$. In order to make this work for $f, g\colon S^n \to S^n$ instead of $f, g\colon S^n \to \mathbb{R} \setminus \{0\}$ we have the homotopy given by

$$H(x,t) = \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}.$$

Note that $H(\cdot, 0) = f$ and $H(\cdot, 1) = g$. This is a continuous homotopy since the denominator is never zero, and the numerator is addition of continuous functions. $\square$

**Problem 2.** Let $X$ be a topological space and let $g$ be a path in $X$ from $p$ to $q$. Let $\Phi_g\colon \pi_1(X,p) \to \pi_1(X,q)$ denote the group isomorphism defined in Theorem 7.13.

If $h\colon X \to Y$ is continuous, then show that the following diagram commutes:

$$\begin{tikzcd}
\pi_1(X,p) \arrow[r, "h_*"] \arrow[d, "\Phi_g"] & \pi_1(Y,h(p)) \arrow[d, "\Phi_{h\circ g}"] \\
\pi_1(X,q) \arrow[r, "h_*"] & \pi_1(Y,h(q))
\end{tikzcd}$$

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*Proof.* To show that this diagram commutes we need to show that for $[f] \in \pi_1(X,p)$ that $\phi_{h\circ g} \circ h_*[f] = \phi_g \circ h_*[f]$. First we show that $\overline{h\circ g} = h \circ \overline{g}$. We have

$$\begin{aligned}
\overline{h\circ g}(t) &= h\circ g(1-t) \\
&= h\circ \overline{g}(t).
\end{aligned}$$

Then

$$\begin{aligned}
\phi_{h\circ g} \circ h_*[f] &= \phi_{h\circ g}[h\circ f] \\
&= [\overline{(h\circ g}] \cdot (h\circ f) \cdot (h\circ g)] \\
&= [(h\circ \overline{g}) \cdot (h\circ f) \cdot (h\circ g)] \\
&= [h\circ (\overline{g} \cdot f \cdot g)] \\
&= h_*[\overline{g} \cdot f \cdot g] \\
&= h_* \circ [\overline{g}][f][g] \\
&= h_* \circ \phi_g[f].
\end{aligned}$$

Thus the diagram commutes. $\square$

**Problem 3.** Let $X$ be a path-connected topological space, and let $p, q \in X$. Show that $\pi_1(X, p)$ is abelian if and only if for any two paths $g, g'$ from $p$ to $q$ in $X$, we have $\Phi_g = \Phi_{g'}$ (as isomorphisms from $\pi_1(X, p)$ to $\pi_1(X, q)$).

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*Proof.* First, suppose that $\pi_1(X, p)$ is abelian. Denote the constant path in $\pi_1(X, p)$ by $[C_p]$. Then for $[f] \in \pi_1(X, p)$ we have

$$\begin{aligned}
\Phi_g[f] &= [\overline{g}][f][g] \\
&= [\overline{g}][g][f] \\
&= [C_p][f] \\
&= [\overline{g'}][g'][f] \\
&= [\overline{g'}][f][g'] \\
&= \Phi_{g'}[f].
\end{aligned}$$

So we have that $\Phi_g = \Phi_{g'}$.

For the converse, suppose that we have $\Phi_g = \Phi_{g'}$. Then

$$\begin{aligned}
\Phi_g[f] &= [\overline{g}][f][g] \\
&= \Phi_{C_p}[f] && \text{since the above is just homotopy equivalent to this} \\
\implies [f] &= [\overline{g}][f][g].
\end{aligned}$$

Which implies that $\pi_1(X, p)$ is abelian. $\square$

**Problem 4.** Let $F\colon C \to D$ be a (covariant) functor from category $C$ to category $D$. Prove that if $X, Y \in \mathrm{Obj}(C)$ are isomorphic objects in $C$, then $F(X), F(Y)$ are isomorphic objects in $D$.

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*Proof.* The following diagram will prove useful (at least in my visualization):

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\begin{tikzcd}
X \arrow[r, bend left, "\varphi"] \arrow[d, "F"'] & Y \arrow[l, "\varphi^{-1}"] \arrow[d, "F"] \\
F(X) \arrow[r, "F(\varphi)"] & F(Y) \arrow[l, bend left, "F(\varphi^{-1})"]
\end{tikzcd}
$$

Since $X$ and $Y$ are isomorphic in $\mathrm{Obj}(C)$ we have that $\varphi\colon X \to Y$ is an isomorphism with inverse $\varphi^{-1}$. Then we have that $F(\mathrm{Id}_{\mathrm{X}}) = F(\varphi^{-1} \circ \varphi) = F(\varphi^{-1}) \circ F(\varphi) = \mathrm{Id}_{F(X)}$. This means that $F(\varphi)$ is an isomorphism $F(\varphi)\colon F(X) \to F(Y)$ with inverse $F(\varphi^{-1})$ and we have that $F(X)$ and $F(Y)$ are isomorphic objects in $\mathrm{Obj}(D)$. $\square$

**Problem 5.** Let $X$ be a toplogical space. Prove that the following statements are equivalent:

(i) $X$ is compact.

(ii) For every collection $\{C_\alpha\}_{\alpha\in A}$ of closed subsets of $X$ with $\cap_{\alpha\in A} C_\alpha = \emptyset$, there is a finite subcollection $\{C_{\alpha_1}, \ldots, C_{\alpha_n}\}$ with $C_{\alpha_1} \cap \ldots \cap C_{\alpha_n} = \emptyset$.

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*Proof.* To show equivalence we prove that (i) implies (ii) and (ii) implies (i).

For the first implication, suppose that $X$ is compact. Then let $\{C_\alpha\}_{\alpha\in A}$ be a collection of closed subsets so that $\cap_\alpha C_\alpha = \emptyset$. Then we have that $X \setminus (\cap_\alpha C_\alpha) = \cup_\alpha X \setminus C_\alpha = X$. Thus we have that $\cup_\alpha X \setminus C_\alpha$ is an open cover of $X$ and by compactness we have that there exists a finite subcover. Put $\cup_{i=1}^n X \setminus C_{\alpha_i}$ as our finite subcover and note that $X \setminus (\cup_{i=1}^n X \setminus C_{\alpha_i}) = \cap_{i=1}^n C_{\alpha_i} = \emptyset$.

For the second implication, suppose that for every collection of closed subsets $\{C_\alpha\}_{\alpha\in A}$ satisfying $\cap_\alpha C_\alpha = \emptyset$ we have $\cap_{i=1}^n C_{\alpha_i} = \emptyset$. Note that $\cup_\alpha X \setminus C_\alpha$ is an arbitrary open cover and $X \setminus (\cap_{i=1}^n C_{\alpha_i}) = \cup_{i=1}^n X \setminus C_{\alpha_i}$ is a finite subcover. So $X$ is compact. □