ColoState Spring 2018 MATH 617 Assignment 5

Due Fri. 04/06/2018

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(20 points) *Problem 1.* Let $f(x)$ be a real-valued continuous function defined on $\mathbb{R}$. Prove that the inverse image $f^{-1}(B)$ of a Borel subset B is also a Borel subset.

(20 points) *Problem 2.* Prove that if g is absolutely continuous on $[a, b]$, then g has bounded variation.

(20 points) *Problem 3.* Let f be a real-valued increasing function on $[0, 1]$ and $\int_0^1 f' d\lambda = f(1) - f(0)$. Prove that f is absolutely continuous.

(20 points) *Problem 4.* Suppose that $F\colon [a, b] \to \mathbb{R}$ is increasing. Prove that there exist unique real-valued functions $G(x)$, $H(x)$ on $[a, b]$ satisfying the following

(i) $F(x) = G(x) + H(x)$, $\forall x \in [a, b]$;

(ii) $G(x)$ is absolutely continuous and $G(a) = F(a)$;

(iii) $H'(x) = 0$ a.e. on $[a, b]$.

(20 points) *Problem 5.* Suppose $F_n(x)$, $n \in \mathbb{N}$, is a sequence of absolutely continuous increasing functions on $[a, b]$ such that

(1) $F_n(a) = 0$, $\forall n \in \mathbb{N}$;

(2) $F_n'(x)$ is a decreasing sequence for almost every $x \in [a, b]$.

Prove that

(i) $F_n(x)$ is a decreasing sequence for all $x \in [a, b]$;

(ii) If $\lim\limits_{n\to\infty} F_n(x) = F(x)$ on $[a, b]$, then $F(x)$ is absolutely continuous.

Problem 1. Let $f(x)$ be a real-valued continuous function defined on $\mathbb{R}$. Prove that the inverse image $f^{-1}(B)$ of a Borel subset B is also a Borel subset.

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Proof. Define the set

$$\mathcal{A} = \{B \in \mathcal{B}_{\mathbb{R}} \, : \, f^{-1}(B) \in \mathcal{B}_{\mathbb{R}}\},$$

where $\mathcal{B}_{\mathbb{R}}$ denotes the Borel sets on $\mathbb{R}$. By definition $\mathcal{A} \subseteq \mathcal{B}_{\mathbb{R}}$. So, we need to show that $\mathcal{A} \supseteq \mathcal{B}_{\mathbb{R}}$. To see this, we show that $\mathcal{A}$ is a σ-algebra on subsets of $\mathbb{R}$ generated by open intervals and since $\mathcal{B}_{\mathbb{R}}$ is the smallest σ-algebra on $\mathbb{R}$ which is generated by open intervals, we will have that $\mathcal{A} \supseteq \mathcal{B}_{\mathbb{R}}$. First, we have that continuity of f implies that f^{-1} pulls back open sets in $\mathcal{B}_{\mathbb{R}}$ to open sets. Hence, we will have that $\mathcal{A}$ is generated by open sets.

(i) Note that $f^{-1}(\mathbb{R}) = \mathbb{R}$ since $f(x)$ is a continuous defined on all of $\mathbb{R}$ and so $\mathbb{R} \in \mathcal{A}$. Then we also have that $f^{-1}(\emptyset) = \emptyset$ and so $\emptyset \in \mathcal{A}$.

(ii) Let $A \in \mathcal{A}$. This means $A = f^{-1}(A')$ for $A' \in \mathcal{B}_{\mathbb{R}}$. Then we have that $A'^c \in \mathcal{B}_{\mathbb{R}}$

$$A^c = \mathbb{R} \setminus A = \mathbb{R} \setminus f^{-1}(A') = f^{-1}(\mathbb{R} \setminus A') = f^{-1}(A'^c).$$

So we have $A^c \in \mathcal{A}$.

(iii) Let $A_i \in \mathcal{A}$ for all $i \in \mathbb{N}$ which means that $A_i = f^{-1}(A_i')$ for $A_i' \in \mathcal{B}_{\mathbb{R}}$. Then, consider

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} f^{-1}(A_i') = f^{-1}\left(\bigcup_{i=1}^{\infty} A_i'\right).$$

Since $\bigcup_{i=1}^{\infty} A_i' \in \mathcal{B}_{\mathbb{R}}$, we have that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

These three items above show that $\mathcal{A}$ is a σ-algebra on $\mathbb{R}$ and thus $\mathcal{A} \supseteq \mathcal{B}_{\mathbb{R}}$ and so we have that $\mathcal{A} = \mathcal{B}_{\mathbb{R}}$, which proves the original statement. $\square$

Problem 2. Prove that if g is absolutely continuous on $[a, b]$, then g has bounded variation.

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Proof. Fix $\epsilon = 1$ and we have that there exists a $\delta > 0$ such that

$$\sum_{i=1}^{n} |g(b_i) - g(a_i)| < 1,$$

when (a_i, b_i), $i = 1, \ldots, n$, are mutually disjoint subintervals of $[a, b]$ satisfying

$$\sum_{i=1}^{n} (b_i - a_i) < \delta$$

by the absolutely continuity of g. This implies that if we take any subinterval $[c, d]$ of $[a, b]$ with $(d - c) < \delta$ that we have $V_c^d(g) < 1$ by absolute continuity. So we fix any partition of $[a, b]$ satsifying $\|P\| < \delta$ which yields

$$V_a^b(g) = \sum_{i=1}^{k} V_{x_{i-1}}^{x_i}(g) < \sum_{i=1}^{k} 1 = k.$$

Hence, g has bounded variation. $\square$

Problem 3. Let f be a real-valued increasing function on $[0,1]$ and $\int_0^1 f' d\lambda = f(1) - f(0)$. Prove that f is absolutely continuous.

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Proof. We wish to show that $f(x) = \int_0^x f' d\lambda + f(0)$. First, if $x = 0$, then

$$f(0) = \int_0^0 f' d\lambda + f(0) = f(0).$$

And for $x = 1$, we have

$$f(1) = \int_0^1 f' d\lambda + f(0) = f(1) - f(0) + f(0) = f(1).$$

Now, suppose for a contradiction that for any $x \in (0,1)$ that we have $f(x) \neq \int_0^x f'(x) d\lambda - f(0)$. Since f is an increasing function, we have that

$$\int_0^x f' d\lambda + f(0) < f(x).$$

However, this implies

$$\int_0^1 f' d\lambda = \int_0^x f' d\lambda + \int_x^1 f' d\lambda < f(1) - f(0)$$

which contradicts our original statement. Hence, we have that

$$f(x) = \int_0^x f' d\lambda + f(0)$$

and since $f(x)$ is defined via an integral, we have that f is absolutely continuous. $\square$

Problem 4. Suppose that $F\colon [a,b] \to \mathbb{R}$ is increasing. Prove that there exist unique real-valued functions $G(x)$, $H(x)$ on $[a,b]$ satisfying the following

(i) $F(x) = G(x) + H(x)$, $\forall x \in [a,b]$;

(ii) $G(x)$ is absolutely continuous and $G(a) = F(a)$;

(iii) $H'(x) = 0$ a.e. on $[a,b]$.

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Proof. Since F is increasing on $[a,b]$ we have that $F' \in L_1[a,b]$. So, we define

$$G(x) = \int_a^x F'(x)d\lambda + F(a)$$

and note that $G(a) = F(a)$ and that $G(x)$ is absolutely continuous by definition. Now, we have that

$$\begin{aligned} H(x) &= F(x) - G(x) \\ &= F(x) - \int_a^x F'(x)d\lambda - F(a) \\ \implies H'(x) &= F'(x) - F'(x) = 0 \qquad \text{almost everywhere since } F'(x) \text{ exists almost everywhere.} \end{aligned}$$

Note that $G(a) = F(a)$ implies that $H(a) = 0$ and since $H'(x) = 0$ almost everywhere, we have that $H(x) = 0$ for all $x \in [a,b]$.

Now, to see that G and H are unique, suppose $\exists g, h$ satisfying the same criteria above but $g(x) \neq G(x)$, $g(x) \neq H(x)$, $h(x) \neq G(x)$, and $h(x) \neq H(x)$. Then

$$G(x) = g(x) + h(x).$$

Evaluating at $x = a$ implies that $h(a) = 0$ and so $G(a) = g(a) = F(a)$. Again, $h'(x) = 0$ almost everywhere and so $h(x) = 0 = H(x)$ and thus we have

$$G(x) = g(x).$$

So G and H are unique. $\square$

Problem 5. Suppose $F_n(x)$, $n \in \mathbb{N}$, is a sequence of absolutely continuous increasing functions on $[a, b]$ such that

(1) $F_n(a) = 0$, $\forall n \in \mathbb{N}$;

(2) $F_n'(x)$ is a decreasing sequence for almost every $x \in [a, b]$.

Prove that

(i) $F_n(x)$ is a decreasing sequence for all $x \in [a, b]$;

(ii) If $\lim\limits_{n\to\infty} F_n(x) = F(x)$ on $[a, b]$, then $F(x)$ is absolutely continuous.

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Proof.

(i) We have

$$\begin{aligned} F_n'(x) &\geq F_{n+1}'(x) \\ \implies \int_a^x F_n'(x)d\lambda &\geq \int_a^x F_{n+1}'(x)d\lambda \\ \implies F_n(x) - F_n(a) &\geq F_{n+1}(x) - F_{n+1}(a) \\ \implies F_n(x) &\geq F_{n+1}(x) \qquad \text{since } F_n(a) = 0 \ \forall n. \end{aligned}$$

(ii) Suppose we have $\lim_{n\to\infty} F_n(x) = F(x)$. First, since each F_n is absolutely continuous, we have that for each n and any collection of m mutually disjoint intervals $[a_i, b_i]$ that if $\sum_{i=1}^m (b_i - a_i) < \delta$ then

$$\sum_{i=1}^m |F_n(b_i) - F_n(a_i)| < \frac{\epsilon}{3}$$

Then note that $F_n(x)$ is a monotone sequence of absolutely continuous (and hence, continuous) functions and $[a, b]$ is compact which means that F_n converges uniformly to F. By uniform convergence, we have for some N that for $n \geq N$

$$|F(x) - F_n(x)| < \frac{\epsilon}{3m}.$$

Now, using these same arbitrary intervals above and for $n \geq N$ we have

$$\begin{aligned} \sum_{i=1}^m |F(b_i) - F(a_i)| &= \sum_{i=1}^m |F(b_i) - F_n(b_i) + F_n(a_i) - F(a_i) - F_n(a_i) + F_n(b_i)| \\ &\leq \sum_{i=1}^m (|F(b_i) - F_n(b_i)| + |F(a_i) - F_n(a_i)| + |F_n(b_i) - F_n(a_i)|) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + m\frac{\epsilon}{3m} \\ &= \epsilon. \end{aligned}$$

Hence, F is absolutely continuous. $\square$