EXAM I

July 3, 2018

Name: __________________________ Instructor: __________ Time your class meets: _____

HONOR PLEDGE I have not given, received, or used any unauthorized assistance on this exam. Furthermore, I agree that I will not share any information about the questions on this exam with any other student before graded exams are returned.

Signature: _____________________ Date: ________________

- There are 6 problems, but only the highest scoring 5 problems will be counted.
- You have one hour and thirty minutes to complete this exam.
- No notes, books, or other references are allowed during this exam.
- Calculators are not allowed during the exam.
- A one-sided note sheet of $8\frac{1}{2}'' \times 11''$ or smaller is allowed during the exam.
- There are questions on the front of the page. You can use the back of each page if you need more space. If you do use any extra paper, make sure that your name is on it and that you attach it to your exam.
- You must show all work to receive credit. Answers for which no work is shown will receive no credit unless specifically stated otherwise.

Question	Score	Maximum
1		20
2		20
3		20
4		20
5		20
6		20
Total		100

1. Consider the differential equation:

$$3y' + \frac{\cos t}{y} + \frac{3}{y} = 0.$$

(a) Write the differential equation in normal form.

(b) What is the order of this differential equation? What type of differential equation is it?

(c) Find the general solution to the differential equation.

2. Consider the differential equation:

$$tx' = 4x + t^4$$

(a) Classify this differential equation: state its order, whether or not it is linear and, if it is linear, whether or not it is homogeneous.

(b) Find the general solution of this differential equation.

(c) Find a solution satisfying the initial condition $x(1) = 5$.

(d) Is your solution to (c) unique? Explain rigorously.

3. A population $P(t)$ is found to change based on the following equation:

$$P' = (P-1)(P-2)(P-3).$$

(a) What are the equilibrium point(s)?

(b) Which of the equilibrium point(s) are stable, which are unstable?'

(c) Draw a phase plane with the equilibrium point(s) labeled. Qualitatively draw possible solution curves (in the $P-t$-plane) for the following initial conditions:

$$P(0) = 0, \quad P(0) = \frac{3}{2}, \quad P(0) = \frac{5}{2}, \quad P(0) = 4.$$

4. Consider the first-order ODE:

$$(x^2 + y^2 - x)dx - ydy = 0$$

(a) Is this ODE exact? Use a calculation to justify your answer.

(b) Multiply the ODE by $\mu(x, y) = \frac{1}{x^2+y^2}$. Is this modified ODE exact? Again, use a calculation to justify your answer.

(c) Find the general solution of this differential equation.

5. You are given a circuit with a capacitor, inductor, and resistor in series. This gives the differential equation

$$\frac{d^2 I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC} I = F(t),$$

where $I(t)$ is current, $F(t)$ is an external power signal, C is capacitance, L is inductance, and R is resistance. (Note: C, L, and R are constant real numbers.)

(a) Classify this differential equation: state its order, and whether or not it is linear.

(b) Letting $C = \frac{1}{6}$, $L = 1$ and $R = 5$, what is the solution $I(t)$ to the homogenous equation (i.e., when $F(t) = 0$).

(c) Keeping the capacitance, inductance, and resistance from (a), an external power supply is attached and outputs $F(t) = \sin(t)$. What is the particular solution for this differential equation?

(d) Write down the general solution obtained from doing parts (b) and (c). What can we expect to see when we measure the current after a *very* long time is elapsed?

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6. A charged particle with mass m and charge q experiences a force $qE(t)$ where $E(t)$ is the electric field at time t. With no other forces acting, we have the following equation.:

$$mx'' - qE(t) = 0$$

If you have a switch that turns on a constant electric field at time $t = t_0$, you can write

$$E(t) = H(t - t_0),$$

where x is the position of the particle at time t and H is the Heaviside function.

(a) Given $mx'' - qH(t - t_0) = 0$ with the initial data: $x(t_0) = 0$, $x'(t_0) = 0$, find the Laplace transform of this differential equation.

(b) Going from your result in (a), what is the solution $x(t)$ to the differential equation?

(c) If you multiply your answer from (b) by $H(t - t_0)$, then this solution is valid for all times t. What is the particle doing at $t < t_0$? What about for $t > t_0$?

Table of Laplace Transforms

$f(t)$	$\mathcal{L}\{f\} = F(s)$	Domain
1	$\frac{1}{s}$	$s > 0$
t^n	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin(at)$	$\frac{a}{s^2 + a^2}$	$s > 0$
$\cos(at)$	$\frac{s}{s^2 + a^2}$	$s > 0$
$\exp(at)$	$\frac{1}{s - a}$	$s > a$
$\exp(at)\sin(bt)$	$\frac{b}{(s-a)^2 + b^2}$	$s > a$
$\exp(at)\cos(bt)$	$\frac{s-a}{(s-a)^2 + b^2}$	$s > a$
$t^n \exp(at)$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
δ_p	$\exp(-sp)$	
$H_c(t)$	$\frac{\exp(-cs)}{s}$	$s > 0$
f with period T	$\frac{F_T(s)}{1 - \exp(-Ts)}$	
$\exp(at)f(t)$	$F(s-a)$	$s > a$
$\mathrm{H}(t-c)f(t-c)$	$\exp(-cs)F(s)$	$c \geq 0$
$tf(t)$	$-F'(s)$	
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	
$f(t)$	$\int_0^\infty f(t)e^{-st}t$	
$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$	
$f * g$	$F(s) \cdot G(s)$	
y'	$sY(s) - y(0)$	
y''	$s^2Y(s) - sy(0) - y'(0)$	
$y^{(n)}$	$s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - sy^{n-2}(0) - y^{n-1}(0)$	