Harmonic Maps and Gradient Flow

Math 546 Project

Colin Roberts

July 13, 2020

The Motivating Questions

Question

What is a physically meaningful energy functional?

Question

What are the functions that minimize this energy?

Question

Is there a method or algorithm to search for an optimizer by starting at any point in our space?

Applications

Real World Applications

- Building 3D (printing) structures.
- Machine learning.
- Smoothing of data.

Mathematics Applications

- Framework for optimization problems.
- Existence of minimal mappings.
- Solution to the Poincaré conjecture.

The Gyroid

Applications

- Existence of minimal surfaces follows from existence of solutions to heat equation.
- Ricci flow with surgery- Perelmen's solution to the Poincare conjecture

The Motivating Problems from Geometry and Physics

Geometry

- Geodesics
- Minimal Submanifolds
- Gradient Flow

Physics

- Free Particles
- Elastic Materials
- Heat Flow

Geodesics

Geodesics γ have a few equivalent interpretations.

- γ is the shortest path between two points on M.
- γ is the least curved path between two points on M.
- γ is a trajectory of a free particle on M.
- γ minimizes the Dirichlet energy.

Geodesics

Geodesics γ have a few equivalent interpretations.

- γ is the shortest path between two points on M.
- γ is the least curved path between two points on M.
- γ is a trajectory of a free particle on M.
- γ minimizes the Dirichlet energy.

Minimal Surfaces

Fix a closed curve Γ and a surface Σ with $\partial\Sigma = \Gamma$. The following are equivalent definitions of a *minimal surface*.

- Σ minimizes the area functional.
- Σ has zero mean curvature.
- Σ is a critical point of the mean curvature flow.
- Σ minimizes the Dirichlet energy.

Minimal Surfaces

Fix a closed curve Γ on M and a surface Σ with $\partial\Sigma = \Gamma$. The following are equivalent definitions of a *minimal surface*.

- Σ minimizes the area functional.
- Σ has zero mean curvature.
- Σ is a critical point of the mean curvature flow.
- Σ minimizes the Dirichlet energy.

The Working Example

- Consider an n-dimensional membrane N that undergoes stretching.
- This stretching costs energy and is unfavorable.
- We wish to find the membrane N that minimizes this energy cost.

One Dimension

For a 1-dimensional membrane given by $y = u(x)$ the increase in length from stretching is approximately

$$\sqrt{(\Delta x)^2 + (\Delta y)^2} - \Delta x = \left(\sqrt{1 + \left(\frac{\Delta y}{\Delta x} \right)^2} - 1 \right) \Delta x$$

$$\implies = \frac{1}{2} \left\| \frac{du}{dx} \right\|^2 dx$$

The Dirichlet Energy

Definition

The *Dirichlet energy* measures stretching of the whole n-dimensional membrane. It is defined as the functional

$$\mathcal{E}\colon H^1(\Omega) \to \mathbb{R}$$

defined by

$$\mathcal{E}[u] = \int_\Omega \frac{1}{2}\|\nabla u\|^2 = \int_\Omega \frac{1}{2}\langle \nabla u, \nabla u\rangle$$

Harmonic Maps

Definition

A *harmonic map* is a map that stationary point the Dirichlet energy functional

Remark

We can extend the Dirichlet energy to maps between Riemannian manifolds.

Stationary Points

Question

What is a stationary point of a functional?

Stationary Points

Question

What is a stationary point of a functional?

Answer

It is an analogous definition to stationary points for functions.

Stationary Points of Functions

Definition

Consider a function $f\colon \Omega \to \mathbb{R}$, then a *stationary point* $(p_1, \ldots, p_n) \in \Omega$ satisfies

$$\nabla_{\mathbf{e}_i} f(p_1, \ldots, p_n) = 0 \quad \forall \mathbf{e}_i \in \{\mathbf{e}_1, \ldots, \mathbf{e}_n\},$$

where $\nabla_{\mathbf{e}_i}$ is the directional derivative in direction $\mathbf{e}_i$.

Stationary Points of Functionals

Definition

Consider a functional $\mathcal{E}\colon H^1(\Omega) \to \mathbb{R}$, then a *stationary point* $u \in H^1(\Omega)$ satisfies

$$\delta_v \mathcal{E}[u] = 0 \quad \forall v \in H_0^1(\Omega),$$

where δ_v is a variation in "direction" v.

The Laplace Equation Example

Consider a variation in direction v of the Dirichlet energy functional

$$\delta_v \mathcal{E}[u] := \frac{d}{d\epsilon} \mathcal{E}[u + \epsilon v]\Big|_{\epsilon=0} = 0$$

$$\int_\Omega \nabla u \cdot \nabla v = 0,$$

which is the weak form of the Laplace equation,

$$-\Delta u = 0.$$

Remark

$\implies$ Solutions to the Laplace equation are harmonic maps.

Finding Harmonic Maps

Question

Is there any easier way to find harmonic maps?

Finding Harmonic Maps

Question

Is there any easier way to find harmonic maps?

Answer

Yes. We can flow to the stationary points of functionals via the gradient flow.

Gradient Flow

In finite dimensions, (negative) gradient flow is given by following the path of steepest descent that limits to a stationary point.

Finite Dimensional Gradient Flow

Definition

The *gradient flow* is a curve $\boldsymbol{\gamma}$ such that

$$\frac{\partial \gamma}{\partial t} = -\nabla f.$$

Equivalently, the gradient flow in direction $\mathbf{e}_i$ is given by

$$\langle \mathbf{e}_i, \dot{\boldsymbol{\gamma}} \rangle = -\nabla_{\mathbf{e}_i} f \quad \forall \mathbf{e}_i \in \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}.$$

Computational Method

This is feasible to compute given any starting position by

Gradient Flow in H^1

Definition

The *gradient flow* in $H^1(\Omega)$ in direction v is

$$\left\langle v, \frac{\partial u}{\partial t} \right\rangle = -\delta_v \mathcal{E}[u]$$

and will give us a weak form of a PDE.

The Heat Equation Example

We can then take the gradient flow in direction v by

$$\left\langle v, \frac{\partial u}{\partial t} \right\rangle = -\delta_v \mathcal{E}[u]$$

$$\downarrow$$

$$\int_\Omega v \frac{\partial u}{\partial t} = -\int_\Omega \nabla u \cdot \nabla v$$

Which is the weak form of the source free Heat equation

$$\frac{\partial u}{\partial t} - \Delta u = 0.$$

Answering the Question

Question

What is a stationary point for the functional $\mathcal{E}$?

Answering the Question

Question

What is a stationary point for the functional $\mathcal{E}$?

Answer

The limit as $t \to \infty$ of our gradient flow.

Computational Method

- The process for computing the gradient flow is similar to that in $\mathbb{R}^n$.
- Here and we use a finite element basis and compute the directional variation along this basis.

Geodesics in $\mathbb{R}^n$

Take a curve $\boldsymbol{\gamma}\colon [0,1] \to \mathbb{R}^n$. Then the Dirichlet energy is

$$\mathcal{E}[\boldsymbol{\gamma}] = \int_{[0,1]} \frac{1}{2}\|\dot{\boldsymbol{\gamma}}\|^2 = \int_{[0,1]} \frac{1}{2}\langle \dot{\boldsymbol{\gamma}}, \dot{\boldsymbol{\gamma}} \rangle.$$

Note that the quantity

$$\frac{1}{2}\langle \dot{\boldsymbol{\gamma}}, \dot{\boldsymbol{\gamma}} \rangle$$

is the kinetic energy of a free particle.

Geodesics in $\mathbb{R}^n$

1. Fix the endpoints to $\boldsymbol{\gamma}(s)$, $\boldsymbol{\gamma}(0) = p$ and $\boldsymbol{\gamma}(1) = q$.
2. Define
$$\mathbf{u}\colon [0,1] \times [0,\infty) \to \mathbb{R}^n$$
with
$$\mathbf{u}(s,0) = \boldsymbol{\gamma}(s), \quad \mathbf{u}(0,t) = p, \quad \mathbf{u}(1,t) = q.$$
3. Define $\mathbf{v}(s,t)$ and force
$$\mathbf{v}(0,t) = \mathbf{0}, \quad \mathbf{v}(1,t) = \mathbf{0}.$$

Geodesics in $\mathbb{R}^n$

4. Then the gradient flow is

$$\begin{aligned}\int_{[0,1]} \mathbf{v} \cdot \frac{\partial \mathbf{u}}{\partial t} ds &= -\delta_{\mathbf{v}} \mathcal{E}[\mathbf{u}] \\ &= -\int_{[0,1]} \frac{\partial \mathbf{u}}{\partial s} \cdot \frac{\partial \mathbf{v}}{\partial s} ds \\ &= \int_{[0,1]} \frac{\partial^2 \mathbf{u}}{\partial s^2} \cdot \mathbf{v} ds - \underbrace{\int_{\{0,1\}} \frac{\partial \mathbf{u}}{\partial s} \cdot \mathbf{v} ds}_{\text{equals } 0}.\end{aligned}$$

Geodesics in $\mathbb{R}^n$

5. With sufficient smoothness we get

$$\frac{\partial \mathbf{u}}{\partial t} - \frac{\partial^2 \mathbf{u}}{\partial s^2} = \mathbf{0}.$$

6. If we take the limit $t \to \infty$, then

$$-\frac{\partial^2 \mathbf{u}}{\partial s^2} = \mathbf{0}.$$

7. So we have the components

$$u_i = a_i s + b_i$$

and $\mathbf{u}$ is a straight line.

Geodesics on M

On a smooth Riemannian manifold M, the Dirichlet energy for a curve $\boldsymbol{\gamma}\colon [0,1] \to M$ is

$$\mathcal{E}[\boldsymbol{\gamma}] = \int_{[0,1]} \frac{1}{2} \langle \dot{\boldsymbol{\gamma}}, \dot{\boldsymbol{\gamma}} \rangle_g = \int_{[0,1]} \frac{1}{2} g_{ij} \dot{\gamma}^i \dot{\gamma}^j \, dt,$$

where $\langle \cdot, \cdot \rangle_g$ is the position dependent inner product (Riemannian metric).

Advantage: Can handle constraints such as curves confined to a sphere.

Minimal Surfaces in $\mathbb{R}^3$

1. Fix a closed curve $\Gamma\colon S^1 \to \mathbb{R}^3$.
2. Let Σ be a surface defined by a function $\mathbf{u}\colon D \times [0, \infty) \to \mathbb{R}^3$.
3. We require $\text{Graph}(\mathbf{u}|_{\partial D}(\mathbf{x}, t)) = \Gamma$.
4. Then do the gradient flow with the Dirichlet energy

$$\int_D v \frac{\partial u}{\partial t} d\mathbf{x} = \int_D \nabla v \cdot \nabla u d\mathbf{x}$$

On a Manifold

The Dirichlet energy for a map $u\colon M \to N$ (M compact) is

$$\mathcal{E}[u] = \frac{1}{2}\int_M \|du\|^2 d\mathrm{Vol}_M = \frac{1}{2}\int_M g^{ij}h_{\alpha\beta}\frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\beta}{\partial x^j} d\mathrm{Vol}_M.$$

A stationary point of this energy is a *weakly harmonic map*.

- The matrix $\frac{\partial u^\alpha}{\partial x^i}$ is the Jacobian of the transformation.
- The Jacobian describes the stretching of space.
- Geodesics, minimal surfaces, and minimal submanifolds are harmonic.

From Ω to $\mathbb{R}^n$

Let $\mathbf{u}\colon \Omega \to \mathbb{R}^n$ so we can realize $\mathbf{u}$ as a vector field on Ω. Then $g^{ij} = \delta^{ij}$ and $h_{\alpha\beta} = \delta_{\alpha\beta}$. We then get

$$\mathcal{E}[u] = \frac{1}{2}\int_\Omega \|\mathrm{Jac}(\mathbf{u})\|_F^2 d\mathbf{x}.$$

with the Frobenius norm

$$\|A\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2}.$$

Things to add/change

- Non-unique minimal surfaces with tennis ball
- Analogies with stretchable membranes. Especially when talking about minimal submanifolds. Like stretching over a sphere and stuff.
- Geodesics as rubber bands.
- The flow to a geodesic as pulling a rope tight.
- Organize the motivation/conclusion type stuff more
- Maybe just define the notion of a harmonic map
- `https://en.wikipedia.org/wiki/Harmonic_map`
  uhh... should the ϕ^{-1} be a pullback instead of a preimage?

The Benefit of the Results

What's the point of what we'll do here?

- Provide a method to search for optimizers from arbitrary initial conditions.
- Create a framework to prove theorems such as existence and uniqueness.