Problem 4. Let $f\colon X \to Y$ be a continuous injective function with X compact and Y Hausdorff. Prove that X and $f(X)$ are homeomorphic.

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Proof. Since f is injective we have that f is a bijection between X and $f(X)$. Thus, we just need to show that f^{-1} is continuous. So, let $U \in X$ be open and then consider $f(U) \in Y$. Since f is injective, any $x \in U$ is mapped to a unique point $f(x) = y \in f(X)$. Since Y Hausdorff, we have that there exist disjoint neighborhoods about y and any $p \in f(X) \setminus f(U)$. Note that X compact and f continuous implies that $f(X)$ is also compact. Let $N_p(y)$ be an open set containing y such that it is disjoint from the open set $V_p \in f(X) \setminus f(U)$ containing the point p. Note then that this collection $\{V_p\}_{p \in f(X) \setminus f(U)}$ with the subspace topology (meaning each $V_p = O_p \cap f(X) \setminus f(U)$ for some O_p open in X) is a cover of $f(X) \setminus f(U)$ and has a finite subcover given by the collection $\{V_{p_i}\}_{i=1,\ldots,n}$ since $f(X)$ is compact. Finally we have that $\cap_{i=1}^{n} N_{p_i}$ is an open set containing y in $f(U)$ disjoint from $f(X) \setminus f(U)$ and this implies that $f(U)$ is open in $f(X)$. Thus, X and $f(X)$ are homeomorphic. $\square$