MATH 517, Homework 2

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Solutions

Problem 1. A metric space is called *separable* if it contains a countable dense subset. A collection $\{V_\alpha\}$ of open subsets of a metric space X is called a *base* of X if, for all $x \in X$ and every open set $G \subseteq X$ with $x \in G$ there exists α so that $x \in V_\alpha \subseteq G$.
Prove that every separable metric space has a *countable* base.

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Proof. Let X be separable and $E = \{x_i\}_{i \in \mathbb{N}} \subseteq X$ be a countable dense subset. Then consider $B = \{N_r(x_i) | r \in \mathbb{Q} \quad \text{and} \quad x_i \in E\}$ and note that B is countable. Let $x \in G \subseteq X$ with G open. Since E is dense, $x \in E$ or x is a limit point of E. If this arbitrary $x \in E$ then B is a countable base for X since there exists a rational $r > 0$ so that $N_r(x) \subseteq G$. Note that $N_r(X) \in B$ since $x \in E$ and r is rational. Otherwise we said that x is a limit point of E. If that is the case, then every neighborhood of x contains infinitely many points of E. Since G is open, $\exists N_\delta(x) \subseteq G$ with δ rational and $N_\delta(x)$ also contains infinitely many points of E. Now choose a point $x_i \in E$ with $d(x_i, x) < \frac{delta}{4}$, then $x \in N_{\delta/3}$ and $N_{\delta/3}(x_i) \subseteq G$. Thus B is a countable base. $\square$

Problem 2. Suppose (X,d) is a metric space in which every infinite subset has a limit point. Prove that (X,d) is separable.

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Proof. Fix a $\delta > 0$ and pick an $x_1 \in X$. Then we choose $F = x_1, ..., x_j \in X$ and try to choose x_{j+1} so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, ..., j$. Suppose for a contradiction that we can create F with infinitely many points by the construction above, then for any $x \in X$ we have that $d(x, x_i) < \frac{\delta}{2}$ for some x_i. Using the triangle inequality we have that $d(x, x_i) \leq d(x_j, x_i) + d(x, x_i)$ which means that $d(x, x_j) \geq d(x_j, x_i) - d(x_i, x) \geq \frac{\delta}{2}$ for any $j \neq i$. Thus x cannot be a limit point of F since if we choose a neighborhood $N_{\delta/2}(x)$ it only contains finitely many points of F.

Now we can choose $\delta = \frac{1}{n}$ and create finite sets $F_n = \{x_i \in X | d(x_i, x_j) \geq \frac{1}{n} \forall i \neq j\}$. Then consider the union $U = \cup_{n=1}^{\infty} F_n$ which is is a countable set. Let $x \in X$ and fix $\epsilon > 0$. If $x \in U$ then U is clearly dense. However, if $x \notin U$ then we have that $\exists N \in \mathbb{N}$ so that $\frac{1}{N} < \epsilon$. Then let $y \in U$ with $y \neq x$ be a point satisfying $d(x, y) < \frac{1}{N} < \epsilon$, which must exist for some set F_n with $n \geq N$. □

Problem 3. Prove that every compact metric space K has a countable base, and that K is therefore separable.

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Proof. Let K be compact and let $B_n = \{N_{1/n}(x_1), \ldots, N_{1/n}(x_{m_n}) | n$ is a fixed natural and $x_i \in K\}$ be a finite open cover of K. Then note that $\mathscr{B} = \{B_n\}_{n\in\mathbb{N}}$ is a countable cover of K since each B_n is a finite open cover. To be clear, x_{m_n} is always some finite integer. Then let $x \in G \subseteq X$ with G open. Since $\mathscr{B}$ contains covers of K, $x \in N_{1/n}(x_i)$ for some x_{i_n} and n which satisfies $x \in N_{1/n}(x_{i_n}) \subseteq G$ since we can make $1/n$ arbitrarily small and we still have a cover of K. And thus $\mathscr{B}$ is a countable base for K. $\square$

Problem 4. Let X be a metric space in which every infinite subset has a limit point in X. Prove that X is compact.

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Proof. Let X be a metric space in which every infinite subset has a limit point. By Ex. 2.23 and 2.24 we know that X has a countable base. Thus every open cover of X has a countable subcover $\{G_n\}_{n\in\mathbb{N}}$. If no finite subcollection of $\{G_n\}_{n\in\mathbb{N}}$ covers X, then $F_n = X \setminus (G_1 \cup \ldots \cup G_n)$ is nonempty for each n but $\cap_{n\in\mathbb{N}} F_n$ is empty. Let $E = \{x_i \in F_i | i \in \mathbb{N}\}$. For a contradiction, suppose E has a limit point x, thus $\forall \epsilon > 0$, $N_\epsilon(X) \cap E$ contains infinitely many points of E. But since $\{G_n\}_{n\in\mathbb{N}}$ is a base, we have $N_\epsilon(x) \subseteq G_i$ for some i. But $G_i \cap E$ can only possibly contain $x_1, \ldots, x_i$ so we have that $N_\epsilon(x)$ contains only finitely many points. Thus x was not a limit point of E which contradicts our original statement. Thus X must be compact. □

Problem 5. Let l^1 be the set of sequences $\vec{x} = \{x_n\}$ of real numbers such that $\sum_{n=1}^{\infty} |x_n| < \infty$. Define $d(\vec{x}, \vec{y}) = \sum_{n=1}^{\infty} |x_n - y_n|$, which is a metric on l^1. Let $\vec{z} \in l^1$ be a fixed sequence of positive numbers, and prove that

$$E_{\vec{z}} := \{\vec{x} \in l^1 : |x_n| \leq z_n \text{ for all } n\}$$

is compact.

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Proof. Let $S \subseteq E_{\vec{z}}$ be infinite. Now consider the set $S_{|x|} = \{\sum_{n=1}^{\infty} |x_{\alpha_n}| : x_\alpha \in S\}$. Consider $\sup S_{|x|}$ which is some x_α. Then suppose that this sup is not a limit point of the set. Then we have that $\exists r_\alpha > 0$ so that for every $|x_\beta| \in S$ we have that $|x_\alpha| - |x_\beta| > r_\alpha$. Suppose this carries on infinitely, and thus there exists $r = \inf_\alpha(r_\alpha)$ then note that for even a countable collection $|x_{\alpha_i}|, |x_{\beta_i}|$ we have that $\sum_{i=1}^{\infty} |x_{\alpha_i}| - |x_{\beta_i} > \sum_{i=1}^{\infty} r$. But this contradicts $x_\alpha \in E_{\vec{z}}$ since every element of $E_{\vec{z}}$ are absolutely convergent series and subtracting two absolutely convergent series should be bounded. Thus we have that $S_{|x|}$ was not infinite, or we have that for some $|x_\alpha|$ that for all $epsilon > 0$ we have $N_\epsilon |x_\alpha| \cap S_{|x|}$ is infinite. Thus we have that $S_{|x|}$ must contain a limit point. I claim that this means S must also have a limit point. $\square$