MATH 570, Homework 4

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Solutions

Problem 1. Prove that if a coproduct exists in a category, then it is unique up to isomorphism. That is, prove that if $(S', (\iota'_\alpha))$ and $(S'', (\iota''_\alpha))$ are both coproducts of the family of objects $(X_\alpha)_{\alpha \in A}$ then S' and S'' are isomorphic.

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Proof. Let $(S', (\iota'_\alpha))$ and $(S'', (\iota''_\alpha))$ be coproducts for the family of objects $(X_\alpha)_{\alpha \in A}$. Then we are guaranteed unique morphisms $f' \colon S' \to S''$ and $f'' \colon S'' \to S'$ which satisfy $\iota''_\alpha \circ f' = \iota'_\alpha$ and $\iota'_\alpha \circ f'' = \iota''_\alpha$. Considering the diagram on page 214 of our text, we have that we can let $W = S'$ and $S'' = S$ and then note that $f_\alpha = \iota'_\alpha$ and the diagram commutes with $f'' \circ f'$ or $\mathrm{Id}_{S'}$ in place of f'. So then $f'' \circ f' = \mathrm{Id}_{S'}$. An analogous argument shows that $f' \circ f'' = \mathrm{Id}_{S''}$. And so we have that S' and S'' are isomorphic by uniqueness of f' and f''. $\square$

Problem 2. Let $(X_\alpha)_{\alpha\in A}$ be a family of topological spaces, and equip $\coprod_{\alpha\in A} X_\alpha$ with the disjoint union topology. Prove that $\coprod_{\alpha\in A} X_\alpha$ is the coproduct of $(X_\alpha)_{\alpha\in A}$ in the category of topological spaces as follows.

(a) Define the maps $\iota_\alpha \colon X_\alpha \to \coprod_{\alpha\in A} X_\alpha$.

(b) Prove that $(\coprod_{\alpha\in A} X_\alpha, (\iota_\alpha))$ satisfies the necessary universal property.

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Proof (Part (a)). We have that $\iota_\alpha \colon X_\alpha \to \coprod_{\alpha\in A} X_\alpha$ by $\iota_\alpha(X) = (X, \alpha)$. □

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Proof (Part (b)). Suppose that W is a space with morphism $f_\alpha \colon X_\alpha \to W$ with $x \mapsto w$ and define $f \colon \coprod_{\alpha\in A} X_\alpha \to W$ by $(x, \alpha) \mapsto f_\alpha(X)$. These satisfy the universal property. □

Problem 3. Let $X = (\mathbb{R} \times \{0\}) \cup (\mathbb{R} \times \{1\}) \subseteq \mathbb{R}^2$ (equipped with the standard topology). Define an equivalence relation on X by declaring $(x,0) \sim (x,1)$ if $x \neq 0$. The quotient space $X/\sim$ is called the *line with two origins.*

(a) Show that $X/\sim$ is not Hausdorff (and hence not a manifold).

(b) Show that $X/\sim$ is locally Euclidean.

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Proof (Part (a)). Consider the points $p_1 = (0,0)$ and $p_2 = (0,1)$ which are distinct in the quotient space. Then consider arbitrary neighborhoods $N_{\epsilon_1}(p_1) = ((-\epsilon_1, \epsilon_1), 0)$ and $N_{\epsilon_2}(p_1) = ((-\epsilon_2, \epsilon_2), 1)$ which are indeed open sets as their preimage in X are open. But note that no matter the choice of ϵ_1 and ϵ_2 we have that $N_{\epsilon_1}(p_1) \cap N_{\epsilon_2}(p_2) \neq \emptyset$ since $(x,0) \sim (x,1)$ for $x \neq 0$. □

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Proof (Part (b)). If $p = (x,0) \sim (x,1)$ for $x \neq 0$, then a neighborhood of p is of the form $N_\epsilon(p) = ((x-\epsilon, x+\epsilon), 0)$ for $0 < \epsilon < |x|$. Clearly this set is open as the preimage of $N_\epsilon(p)$ is open in X. Thus a homeomorphism for these points $f\colon X/\sim \to \mathbb{R}$ is given by $f(x,0) = x$. Which is clearly a homeomorphism. We then define $f(0,0) = 0$ and $f(0,1) = 0$ which again are clearly homeomorphisms. Just take $N_\epsilon(0,0)$ which in which we have $f(N_\epsilon(0,0)) = N_\epsilon(0) \subseteq \mathbb{R}$. A similar argument is used for $(0,1)$. □

Problem 4. Let X and Y be topological spaces, and let $f\colon X \to Y$. Suppose that $X = \cup_{\alpha\in A} U_\alpha$, with U_α open in X for all α, and that $f|_{U_\alpha}\colon U_\alpha \to Y$ is continuous for all α. Prove that $f\colon X \to Y$ is continuous.

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Proof. Consider an arbitrary open set $V \subseteq Y$. Then we have that $(f|_{U_\alpha})^{-1}(V) = U_\alpha \cap f^{-1}(V)$ which is open in X. So consider now $\cup_{\alpha\in A}(f|_{U_\alpha})^{-1}(V) = \cup_{\alpha\in A}(U_\alpha \cap f^{-1}(V)) = f^{-1}(V) \cap (\cup_{\alpha\in A}(U_\alpha)) = f^{-1}(V) \cap X = f^{-1}(V)$ which is open since the arbitrary union of open sets is open. Thus f is continuous. $\square$

Problem 5. Make a Möbius band out of a strip of paper, and then cut it along its central circle. Now, draw a picture to show that identifying diametrically opposite points on one of the boundary circles of a cylinder creates a Möbius band. That is, draw a picture to show that if you take a cylinder $S^1 \times [0,1] = \{(x,y,z) | x^2 + y^2 = 1 \text{ and } 0 \leq z \leq 1\}$ and identify each $(x,y,1)$ with $(-x,-y,1)$, then you get a Möbius band.

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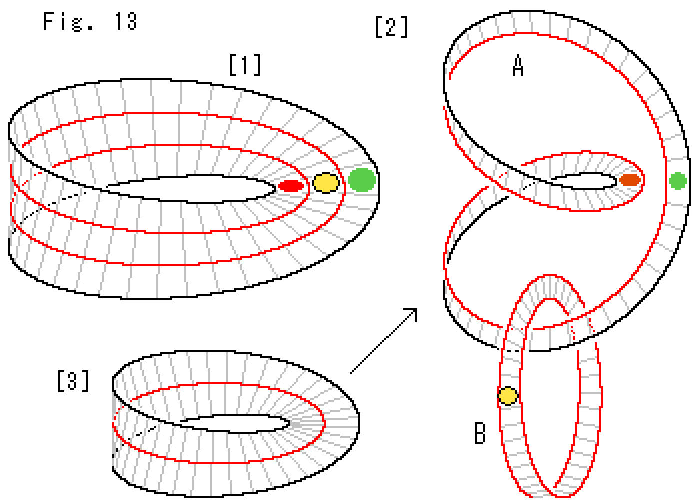

Proof. In this picture we can see that we can glue the antipodal edges of the cylinder A shown in [2] (disregard the extra M$\acute{}$obius band cut out) and achieve the band shown in [3]. I remember doing this problem with you last year and drawing is hard, so hopefully this suffices. Also I think it's cool you can cut the twice twisted cylinder in half again and you get two disjoint pieces which in some sense shows that it is topologically a cylinder! □