

Riemannian Geometry

for Dummies

Colin Roberts

Colorado
State
University

Section 1

Introduction

Riemannian geometry is the study of a *smooth manifold* M along with a *Riemannian metric* g .

The point of Riemannian geometry is to generalize the differentiable and metric structure of \mathbb{R}^n .

We think of living on the manifold. We refer to this as *intrinsic*.

We generalize to spaces that have interesting topology and geometry.

This will require us to rethink some notions we found “easy”
in \mathbb{R}^n .

But we will gain a very general framework for working with differentiable objects.

Section 2

Motivation

Why study this in the first place?

Example: Partial differential equations (PDEs) on spaces that are not flat.

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- General relativity

Example: Optimization in interesting spaces.

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- Curved spacetime

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- EIT
- Polymer growth
- Electrodynamics

Section 3

Preliminaries

Subsection 1

Smooth Manifolds

Our To-Do List:

- Start with a topological space M

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- Look at open sets U that cover M

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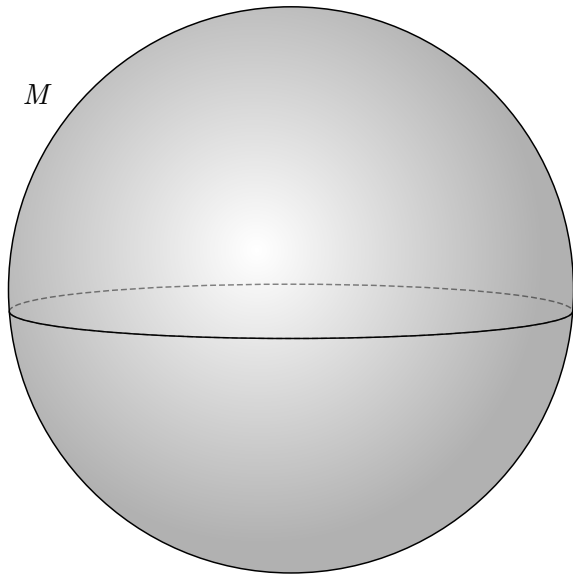
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- Look at open sets U that cover M
- Construct local coordinates φ
- Show coordinate transition functions are smooth

Working example: 2-sphere

$$S^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$



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$$\varphi_\alpha\colon \mathcal{O}_\alpha \rightarrow U_\alpha \subset M \qquad \varphi_\beta\colon \mathcal{O}_\beta \rightarrow U_\beta \subset M.$$

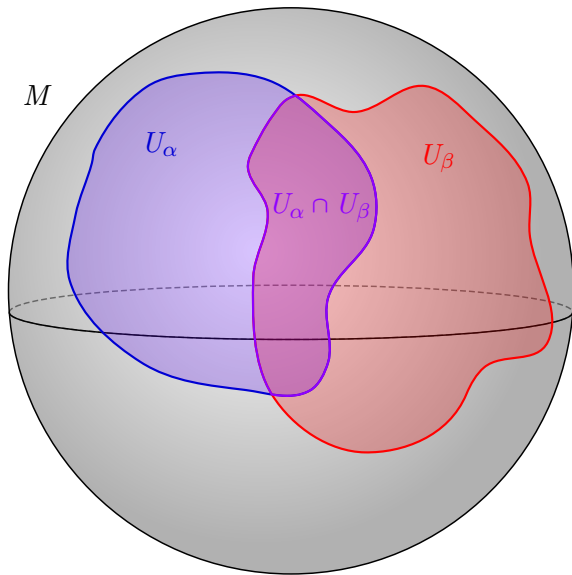
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$$\varphi_\alpha: \mathcal{O}_\alpha \rightarrow U_\alpha \subset M \quad \varphi_\beta: \mathcal{O}_\beta \rightarrow U_\beta \subset M.$$

These are our *local coordinates*.



Our local coordinates must work together on overlaps

$$U_\alpha \cap U_\beta.$$

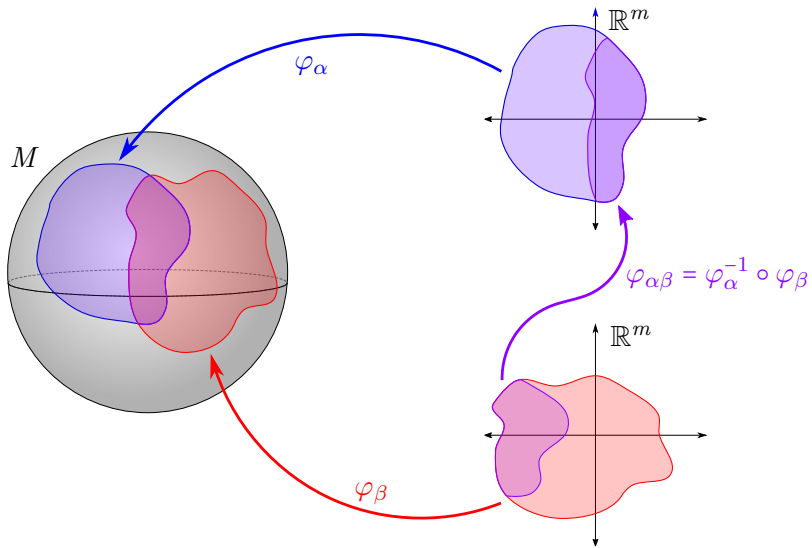
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We check the *transition function*

$$\phi_{\alpha\beta} = \varphi_\alpha^{-1} \circ \varphi_\beta$$

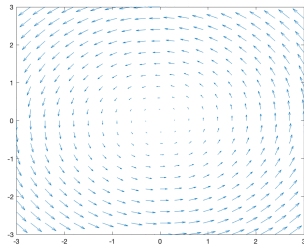
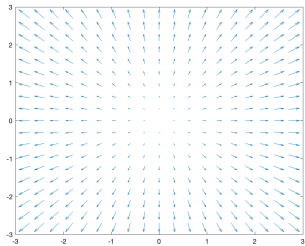
is smooth and invertible as a function on \mathbb{R}^m .



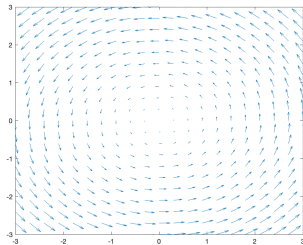
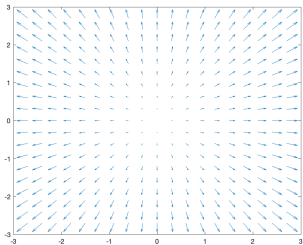
Subsection 2

Vector Fields

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Intrinsic vector fields on manifolds carry geometric information.

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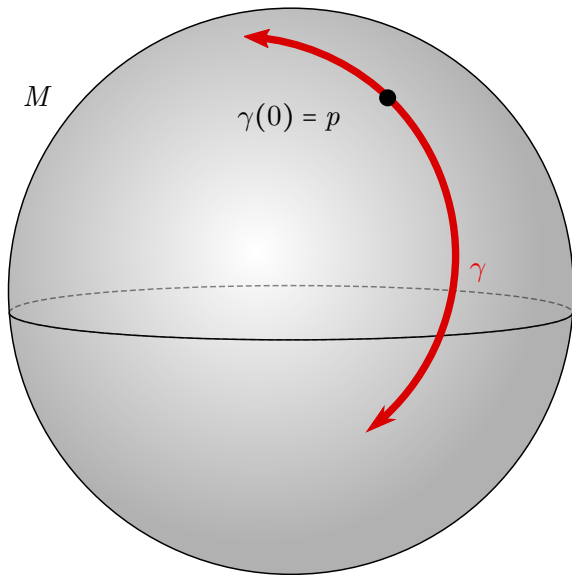
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- Glue together tangent spaces to form the *tangent bundle* TM
- Properly define vector fields X as *sections* of the tangent bundle

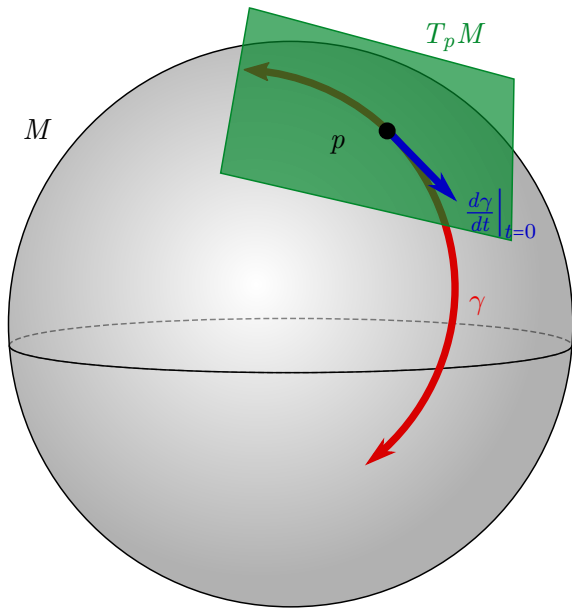
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- All possible tangent vectors form the tangent space $T_p M$.



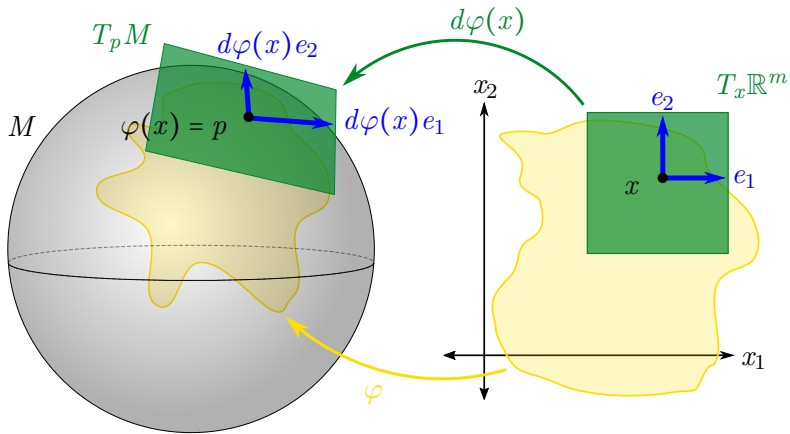


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- If $\varphi(x) = p$, then $d\varphi(x): T_x\mathbb{R}^m \rightarrow T_pM$



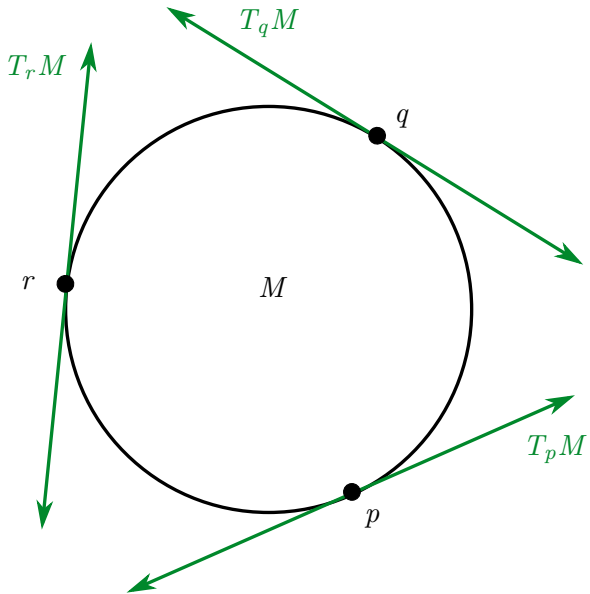
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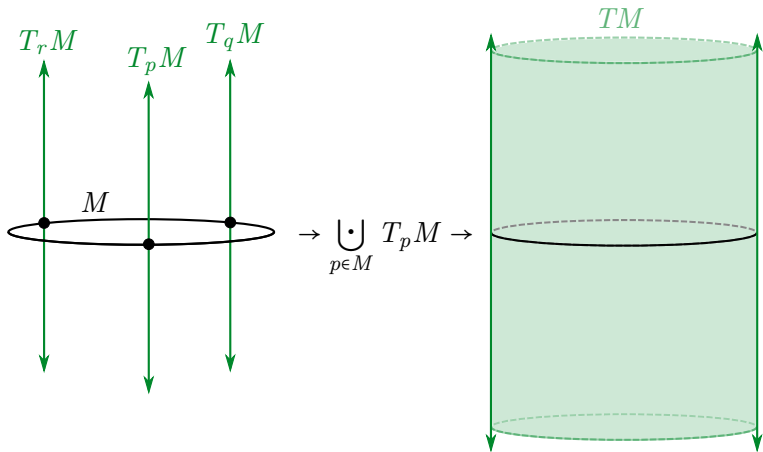
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- This allows us to see how tangent vectors move around the whole manifold.

We briefly drop a dimension to the 1-sphere

$$S^1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$





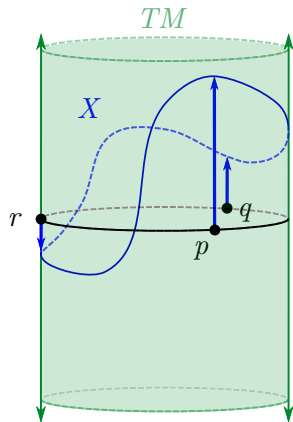
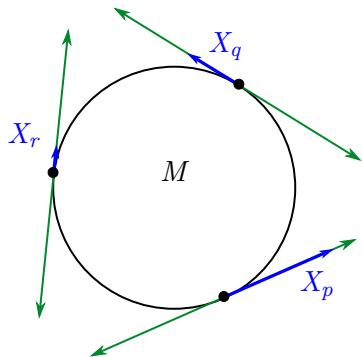
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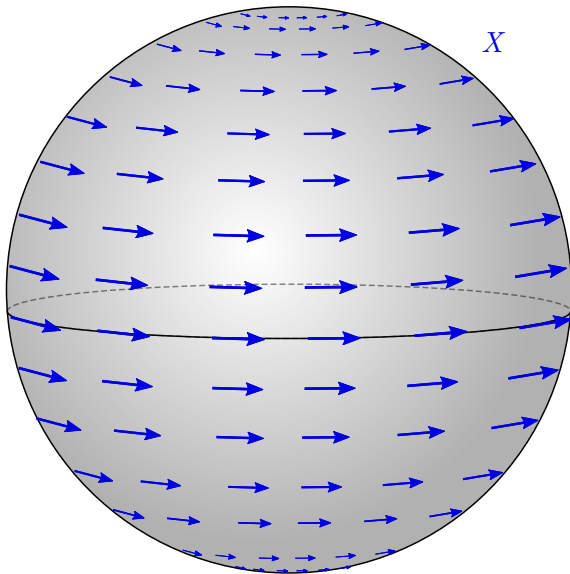
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- X is a *section* if $\pi \circ X = \text{Id}_M$ (vertical line test)



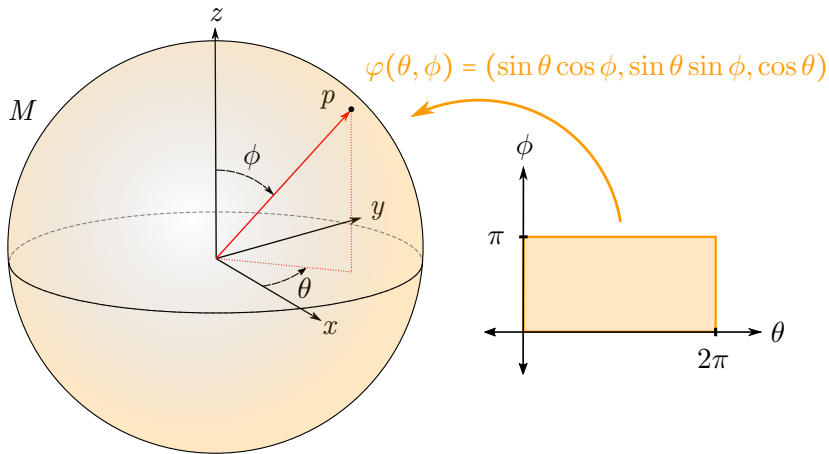
Back to the 2-sphere.

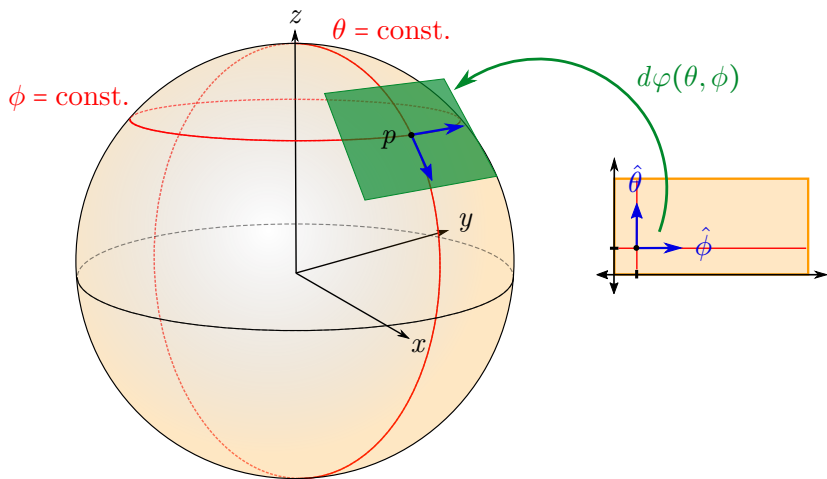


Subsection 3

Specific Coordinates

We should work with specific coordinates on S^2 .

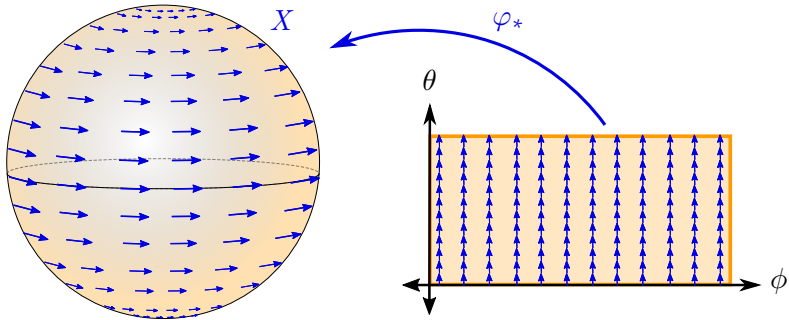




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- This bundle map $\varphi_*: T\mathbb{R}^m \rightarrow TM$ is the *pushforward*



Section 4

Riemannian Geometry

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- Have the inner product vary smoothly as we vary the point p ;
- Define this as our Riemannian metric tensor field g ;
- Extract geometrical and analytical qualities of the underlying manifold M .

Subsection 1

Riemannian Metric

We use the differential and dot product to form a matrix at each point

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This matrix is the *Riemannian metric*.

Riemannian metric provides an inner product for tangent vectors on M . Thus, we know

- how lengths are distorted;
- how volume is distorted.

This allows us to integrate or differentiate in our coordinates but think of it as intrinsic to the manifold.

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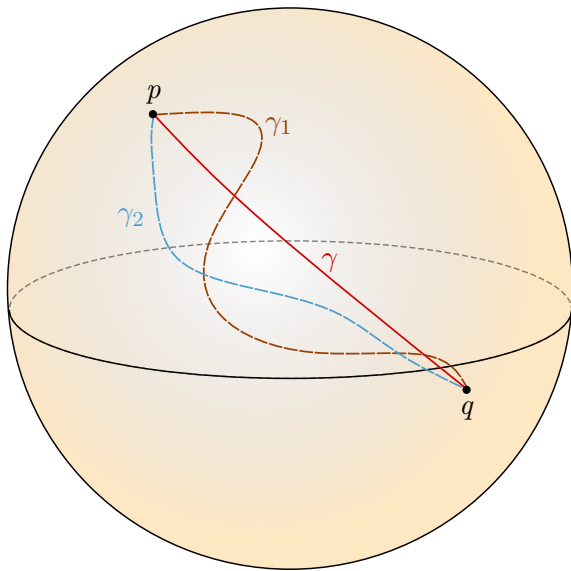
We need to solve

$$\inf_{\gamma} \ell(\gamma) := \int_0^1 \sqrt{g(\dot{\gamma}, \dot{\gamma})} dt$$

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- $g(\dot{\gamma}, \dot{\gamma})$ is the speed on M
- We put $g(\dot{\gamma}, \dot{\gamma})$ to mean $\sum_{i,j=1}^m g_{ij} \dot{\gamma}_i \dot{\gamma}_j$.



Solving this optimization problem yields the *geodesic equation*

$$\ddot{x}^l + \dot{x}^j \dot{x}^k \Gamma_{jk}^l = 0$$

where Γ_{jk}^l are the *Christoffel symbols* which are formed by derivatives of the metric.

This defines an intrinsic derivative ∇ called the
Levi-Civita connection

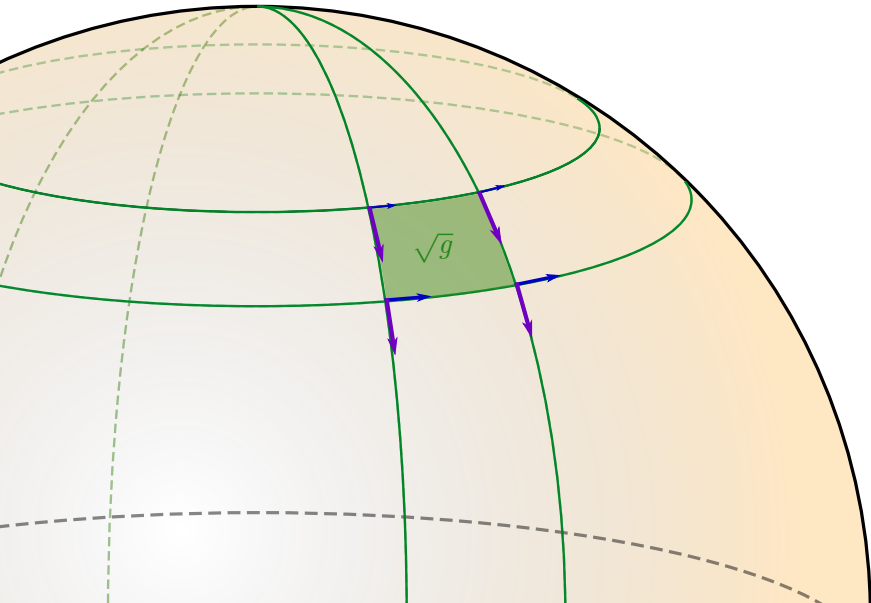
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- The determinant gives us area information.
- Then $\sqrt{|\det(g(x))|}$ gives us the volume on M

In spherical coordinates, $\sqrt{|\det(g)|} = \sin \varphi$ which gives us the integrand

$$\sin \varphi d\varphi d\theta.$$



Section 5

Conclusions

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- No measurement depends on the choice of coordinates

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- Hence, we can define lengths and volumes
- Thus, we can integrate

- g induces a derivative ∇

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- g provides an intrinsic length function on M

This is just the beginning!