Geometric Algebra and Calculus on Riemannian Manifolds

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1 Eigenfields of Laplace-Beltrami Operator

Consider the clifford-laplace-beltrami operator Δ_g and the eigen equation

$$\Delta_q F = \lambda F.$$

1.1 Spinors on 2-dimensional Manifold

If we let $F = f_0 + f_2$ be an even clifford field on a 2-dimensional manifold, we are interested in the spectral equation

$$\Delta_a F = \lambda F$$

which is then

$$\Delta_g f_0 + \Delta_g f_2 = \lambda f_0 + \lambda f_2,$$

hence we are simultaneously diagonalizing the laplace operator. For functions f_0 , we have

$$\delta df_0 = \lambda f_0$$

and for bivectors we simultaneously have

$$(d\delta + \delta d)f_2 = \lambda f_2.$$

Working in coordinates, we locally have F(x,y) (use the definition of laplace beltrami in coordinates?)

2 Clifford Bundles

What are the eigenspinors of laplace beltrami on a manifold? Can one show that with some Q the Clifford sections on M form a Banach algebra?

https://en.wikipedia.org/wiki/Clifford_algebra Look at "basis and dimension" to show that we will work with orthonormal frames. Let (M, Q) be an m-dimensional smooth manifold with a quadratic form Q on the tangent bundle. By polarisation, this induces

$$g(u,v) = \frac{1}{2}(Q(u+v) - Q(u) - Q(v)).$$

Thus any quadratic manifold is a Riemannian manifold (M, g). Then we can consider an induced Clifford bundle $C\ell(TM, g)$ on M with the quadratic form Q(v) = g(v, v). Other sources make the choice of Q(v) = -g(v, v) in order to mimic the complex and quaternionic fields.

Specifically, we define $C\ell(T_pM,g_p)$ to be the Clifford algebra on the tangent space T_pM and let

$$C\ell(TM,g) := \dot{\bigcup}_{p \in M} C\ell(T_pM,g_p)$$

be the Clifford bundle.

We can then define the space of Clifford sections by noting we have a natural projection

$$\pi \colon C\ell(TM,g) \to M$$

that maps a Clifford element to the point at which it is based and putting

$$\Gamma C\ell(TM, g) := \{ \sigma \colon M \to C\ell(TM, g) \mid \pi \circ \sigma = \mathrm{Id}_M \}.$$

How do we define smoothness here? Maybe just consider a section of the tensor algebra. Then we can show the smoothness and algebra properties from a quotient map from there.

Definition 2.1. Define a k-blade field as a smooth section of

$$\odot^k(TM)$$

Proposition 2.1. The space $\Gamma C\ell(TM,g)$ forms an algebra.

Proposition 2.2. There exists a norm on $\Gamma C\ell(TM,g)$ via involution (multiplication by the pseudoscalar).

Proposition 2.3. With this norm, $\Gamma C\ell(TM,g)$ is a Banach algebra.

Proposition 2.4. With (blah) we have that $\Gamma C\ell(TM, g)$ is a Banach *-algebra.

We need that the algebra of clifford sections is a Banach *-algebra. https://en.wikipedia.org/wiki/Gelfand%E2%80%93Naimark_theorem

https://en.wikipedia.org/wiki/Banach_algebra#Spectral_theory This ties up the knot for the relationship of different notions of spectra.

Can we define an inner product on the clifford algebra in a natural way? https://math.stackexchange.com/questions/2606319/is-the-natural-norm-on-the-exterior-algebra-submult That is, multiply the components of rank n in a meaningful way like in the link.

Relationship to frame bundles and flags?

Two versions of a Clifford norm and mention of $C\ell(p,q,r)$ https://math.stackexchange.com/questions/1128844/about-the-definition-of-norm-in-clifford-algebra

Also this https://mathoverflow.net/questions/176140/norms-on-clifford-algebra-c-norm

I feel like a good Ph.D. project would be to fucking make a single source that has information on clifford algebras on manifolds with all the relations to function algebras and such.

This is cool https://www.jstor.org/stable/pdf/1970397.pdf

Clifford homology/cohomology/elliptic complexes with dirac operator (derivative)

How do we define this dirac operator in a coordinate free way? Will it be able to give us $\operatorname{Im} \subseteq \operatorname{Ker}$? Maybe there is some way to say "yes, up to non-harmonic functions" as D^2 is a k-blade laplacian.

This definition is essentially defined throughout [1]

Definition 2.2. Define

$$D_L = \sum_{i=1}^n e_i \frac{\partial}{\partial x^i}$$

and

$$D_R = \sum_{i=1}^n \frac{\partial}{\partial x^i} e_i$$

Definition 2.3. Define the exterior derivative $D \wedge$ by

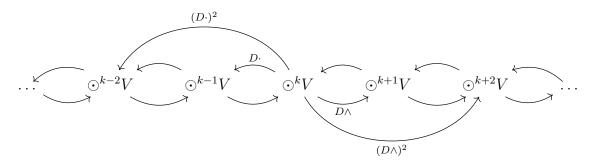
$$\frac{1}{2}\left(D_L - D_R\right)$$

and the interior derivative D by

$$\frac{1}{2}\left(D_L+D_R\right).$$

And again, in [1], it's shown that

$$(D\cdot)^2 = 0 = (D\wedge)^2.$$



probabilistic (norm one) clifford sections?

left/right dirac operators

in lectures on clifford algebras - clifford analysis lecture, there is a generalization of cauchy integral formula

3 C^* -Algebras

References

[1] Chris Doran. Geometric Algebra for Physicists. Cambridge University Press, 2003.