MATH 317, Homework 1

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Solutions

Problem 1. Let A and B be subsets of another set U, and let $B^c = U \setminus B$.

(i) Prove that $A \setminus B = A \cap B^c$.

(ii) The *symmetric difference* (or *Boolean sum*) of A and B is defined to be $A \Delta B := A \setminus B \cup B \setminus A$. Prove that $A \Delta B = A^c \Delta B^c$.

Proof (Part (i)). Consider some $x \in A \setminus B$. Thus, $x \in \{x \in A | x \notin B\}$. Suppose that $x \notin A \cap B^c$, then $x \in B$ and $x \in A^c$. But by definition $x \in A$, so this contradicts the original statement. Thus if $x \in A \setminus B$ then $x \in A \cap B^c$ and $A \setminus B \subseteq A \cap B^c$.

Next, consider some $x \in A \cap B^c$. Thus, $x \in \{x \in A \text{ and })x \in B^c\}$. So $x \in A$ and $x \notin B$. Suppose that $x \notin A \setminus B$, then $x \in B$ or $x \in A^c$. But this is a contradiction to the original statement. Thus $x \in A \setminus B$ and $A \cap B^c \subseteq A \cap B^c$.

Since $A \setminus B \subseteq A \cap B^c$ and $A \cap B^c \subseteq A \cap B^c$, $A \setminus B = A \cap B^c$. □

Solution (Part (ii)). Instead of proving the statement itself, I found it easier to prove a lemma first, and then use that lemma to prove (ii).

Lemma (Lemma 1). — $A^c \setminus B^c = B \setminus A$

Proof (Lemma 1). Consider some $x \in B \setminus A$. Suppose for a contradiction that $x \notin A^c \setminus B^c$. By definition we have $U \supseteq A$ and $U \supseteq B$ as well as $A^c \setminus B^c = (U \setminus A) \setminus (U \setminus B)$. Since $x \in B \setminus A$ we know that $x \in U \setminus A$. Since we know that $U \setminus A \supseteq U \setminus B$ and also require that $x \in B$ we must remove any part of the set $U \setminus A$ that is *not* in B or else $x \notin B$ which contradicts our definition of x. Thus $x \in (U \setminus A) \setminus B^c = A^c \setminus B^c$ and $B \setminus A \subseteq A^c \setminus B^c$.

Next, consider some $x \in A^c \setminus B^c$. Suppose for a contradiction, $x \notin B \setminus A$. By definition, $x \in \{x \in A^c | x \notin B^c\}$. Since $x \notin B^c$, $x \in B$. But this is in fact a contradiction as we said $x \notin B \setminus A$. Thus $x \in B \setminus A$ and $A^c \setminus B^c \subseteq B \setminus A$. Since we have that, and $A^c \setminus B^c \supseteq B \setminus A$ we know that $A^c \setminus B^c = B \setminus A$. □

Now I can move on to the problem itself.

Proof (Part (ii)). Consider the two symmetric differences,

$$A \Delta B = A \setminus B \cup B \setminus A$$

and

$$A^c \Delta B^c = A^c \setminus B^c \cup B^c \setminus A^c$$

By *Lemma 1* we know that for any sets $A \subseteq U$ and $B \subseteq U$ that $A^c \setminus B^c = B \setminus A$. Thus we simply use this as a substitution to show the statement is true.

$$A \setminus B \cup B \setminus A = A^c \setminus B^c \cup B^c \setminus A^c$$

$$A \setminus B \cup B \setminus A = (B \setminus A) \cup (A \setminus B)$$
$$A \setminus B \cup B \setminus A = A \setminus B \cup B \setminus A$$

□

Problem 2. Let $a, b \in \mathbb{R}$ be such that $a < b$. Prove that for every $n \in \mathbb{N}$, if $x_1, x_2, ..., x_n \in [a, b]$, then

$$a \leq \frac{x_1 + x_2 + ... + x_n}{n} \leq b.$$

Proof. First off,

$$a \leq \frac{x_1 + x_2 + ... + x_n}{n} \leq b.$$
$$\implies na \leq x_1 + x_2 + ... + x_n \leq nb$$

Next, suppose that $x_1 + x_2 + ... + x_n \notin [na, nb]$. Since each $x_i \in [a, b]$, the minimal value for each x_i is a. If we let all $x_i = a$ then $x_1 + x_2 + ... + x_n = na \in [na, nb]$. This is a contradiction and thus $x_1 + x_2 + ... + x_n \geq na$. If we allow each x_i to take on the maximum value, b, then we find a similar contradiction. Namely, $x_1 + x_2 + ... + x_n = nb \in [na, nb]$. Since we have this contradiction as well, we know $x_1 + x_2 + ... + x_n \leq nb$. Thus $na \leq x_1 + x_2 + ... + x_n \leq nb$, which means that $a \leq \frac{x_1 + x_2 + ... + x_n}{n} \leq b$. $\square$

Problem 3. Let X and Y be sets and let $A \subseteq X$, $B \subseteq Y$. Let $f : X \to Y$ be a function.

(i) Prove that $f(f^{-1}(B)) \subseteq B$.

(ii) Is it true that $B = f(f^{-1}(B))$? Either prove or give a counter example.

Proof (Part (i)). If we let A be the inverse image of B under f then we have for all $b \in B$, $f^{-1}(b) = \{a \in A | f(a) \in B\}$. Thus we have $f^{-1}(b) = a$ and we know that $f(a) \in B$. If we compose the functions, $f(f^{-1}(b)) = f(a) \in B$ and this means that $f(f^{-1}(B)) \subseteq B$. □

Proof (Part (ii)). Yes this is true! We already showed that $f(f^{-1}(B)) \subseteq B$ so we just need to show inclusion the other direction. Consider some $b \in B$. Since, $f^{-1}(b) = \{a \in A | f(a) \in B\}$ we have $f^{-1}(b) = a$. Since this means that $f(a) = b$ we know that $f(f^{-1}(b)) = f(a) = b$. Since b was arbitrary, $B \subseteq f(f^{-1}(B))$. Thus, $f(f^{-1}(B)) = B$. □

Note: If we consider f to be non-injective, then we have $f^{-1}(f(A)) \neq A$. But when f is injective, we have equality and the proof is fairly similar.

Problem 4. Prove $1^2 + 2^2 + ... + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all positive integers n.

Proof. First we check the base case, which is $n = 1$.

$$1^2 = \frac{1}{6}(1+1)(2(1)+1)$$

$$= \frac{1}{6}(2)(3)$$

$$\frac{6}{6} = 1$$

Which is correct. Next we assume that this is true for the n^{th} case, and test the $(n+1)^{th}$ case.

$$1^2 + 2^2 + ...n^2 + (n+1)^2 = \frac{1}{6}(n+1)(n+2)(2(n+1)+1)$$

$$\frac{1}{6}n(2n+1) + (n+1) = \frac{1}{6}(n+2)(2n+3)$$

$$\frac{1}{6}n(2n^2+n) + (n+1) = \frac{1}{6}(2n^2+7n+6)$$

$$\frac{1}{6}n(2n^2+n) + \frac{1}{6}(6n+6) = \frac{1}{6}(2n^2+7n+6)$$

$$\frac{1}{6}n(2n^2+7n+6) = \frac{1}{6}(2n^2+7n+6)$$

Which is equal. $\square$

Problem 5. Prove $1^3+2^3+...+n^3=(1+2+...+n)^2$ for all positive integers n.

Proof. For this problem, I will use the fact that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

First though, we need to establish our base case. For this example, the base case is $n=1$.

$$1^3=(1)^2$$

$$1=1$$

This is true. Next we assume that the case is true for n and prove the $(n+1)^{th}$ case.

$$1^3+2^3+...+n^3+(n+1)^3=(1+2+...+n+(n+1))^2$$

Let us just look at the left hand side of the equality first,

$$1^3+2^3+...+n^3+(n+1)^3=\left(\frac{n(n+1)}{2}\right)^2+(n+1)^3$$

$$=\frac{1}{4}(n^2(n+1)^2)+(n+1)(n+1)^2$$

$$=(\frac{1}{4}n^2+n+1)(n+1)^2 \tag{1}$$

Next, let's take the right hand side of the original expression and use the same property mentioned before,

$$(1+2+...+n+(n+1))^2=\left(\frac{(n+1)(n+2)}{2}\right)^2$$

$$=\frac{1}{4}(n+1)^2(n^2+4n+4)$$

$$=(\frac{1}{4}n^2+n+1)(n+1)^2 \tag{2}$$

We have shown that the left hand side reduces to *Eqn. 1* and the right reduces to *Eqn. 2* which are equal. ☐

Problem 6.

(a) Decide for which integers the inequality $2^n > n^2$ is true.

(b) Prove your claim in (a) by mathematical induction.

Solution (Part (a)). We want to find where the inequality is equal, since we know that n^2 grows slower than 2^n as soon as the last equality is achieved, any number following that will make the inequality true.

$$2^n = n^2$$

Is true for $n = 2, 4$. To show that $n > 4$ will make the equality true, we can begin with our base case of $n = 5$,

$$2^5 > 5^2$$

$$32 > 25$$

Which is also true. ■

Proof (Part (b)). We already showed the base case in the previous part. Now we can assume that this holds for the n^{th} case and then test the $(n+1)^{th}$ case.

$$(n+1)^2 < n^{n+1}$$

$$n^2 < n^{n+1} - 2n - 1$$

$$1 < n^{n-1} - \frac{2}{n} - \frac{1}{n^2}$$

Notice that since $n > 5$ the fractions $\frac{2}{n}$ and $\frac{1}{n^2}$ are both less than 1. Also since $n > 5$ and n^2 is monotone $\forall n \in \mathbb{N}$, we know that n^{n-1} is at the very least, $5^4 = 625$. It is very obvious that $625 - 1 - 1 = 623 > 1$ and since the two fractions are actually less than one, we have $n^{n-1} - \frac{2}{n} - \frac{1}{n^2} > 623 > 1$. □

Problem 7. For $n \in \mathbb{N}$, define

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{for} \quad k = 0, 1, \dots, n.$$

The *binomial theorem* asserts that

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

$$a^n + na^{n-1}b + \frac{1}{2}n(n-1)a^{n-2}b^2 + \dots + nab^{n-1} + b^n$$

(a) Verify the binomial theorem for $n = 1, 2$ and 3.
(b) Show $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ for $k = 1, 2, \dots, n$.
(c) Prove the binomial theorem using mathematical induction and part (b).

Solution (Part (a)). For $n = 1$, we have the left hand side,

$$(a+b)^1 = a + b$$

On the right hand side,

$$\binom{1}{0}a^1 + \binom{1}{1}b^1$$
$$= a + b = (a+b)^1$$

For $n = 2$, we have the left hand side,

$$(a+b)^2 = a^2 + 2ab + b^2$$

On the right hand side,

$$\binom{2}{0}a^2 + \binom{2}{1}a^1b^1 + \binom{2}{2}b^2$$
$$= a^2 + 2ab + b^2 = (a+b)^2$$

For $n = 3$, we have the left hand side,

$$(a+b)^3 = a + 3a^2b + 3ab^2 + b^3$$

On the right hand side,

$$\binom{3}{0}a^3 + \binom{3}{1}a^2b^1 + \binom{3}{2}a^1b^2 + \binom{3}{3}b^3$$
$$= a + 3a^2b + 3ab^2 + b^3 = (a+b)^3$$

Thus, we know that the binomial theorem is correct for $n = 1, 2$ and 3. ■

Solution (Part (b)). First let's see what we are aiming to get,

$$\binom{n+1}{k} = \frac{(n+1)!}{k!(n+1-k)!}$$

Which is the right hand side of our equation we are trying to show. On the left,

$$\binom{n}{k} + \binom{n}{k-1}$$

$$
\begin{aligned}
&= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n+1-k)!} \\
&= \frac{n!(n+1-k)}{k!(n+1-k)!} + \frac{k(n!)}{k!(n+1-k)!} \\
&= \frac{n!(n+1)}{k!(n+1-k)!} \\
&= \frac{(n+1)!}{k!(n+1-k)!}
\end{aligned}
\qquad \blacksquare
$$

Proof (Part (c)). For our base case we have shown that $(a+b)^1 = a+b = \binom{n}{0}a + \binom{1}{1}b$. Now we assume the statement is true for n, and begin induction with the $(n+1)$ step.

$$(a+b)^{n+1} = \binom{n+1}{0}a^{n+1} + \binom{n+1}{1}a^n b + \dots + \binom{n+1}{n+1}b^{n+1}$$

$$(a+b)(a+b)^n = \binom{n+1}{0}a^{n+1} + \binom{n+1}{1}a^n b + \dots + \binom{n+1}{n+1}b^{n+1}$$

$$(a+b)\left(\binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \dots + \binom{n}{n}b^n\right) = \binom{n+1}{0}a^{n+1} + \binom{n+1}{1}a^n b + \dots + \binom{n+1}{n+1}b^{n+1}$$

$$\binom{n}{0}a^{n+1}+\binom{n+1}{1}a^n b+\dots+\binom{n}{n+1}ab^n+\binom{n}{0}a^n b+\binom{n}{1}a^{n-1}b^2+\dots+\binom{n}{n}b^{n+1} = \binom{n+1}{0}a^{n+1}+\binom{n+1}{1}a^n b+\dots+\binom{n+1}{n+1}b^{n+1}$$

$$\binom{n}{1}a^n b + \dots + \binom{n}{n}ab^n + \binom{n}{0}a^n b + \dots + \binom{n}{n-1}ab^n = \binom{n+1}{1}a^n b + \dots + \binom{n+1}{n-1}ab^n$$

$$\left(\binom{n}{1} + \binom{n}{0}\right)a^n b + \dots + \left(\binom{n}{n} + \binom{n}{n-1}\right)ab^n = \binom{n+1}{1}a^n b + \dots + \binom{n+1}{n-1}ab^n$$

And using the fact from (b),

$$\binom{n+1}{1}a^n b + \dots + \binom{n+1}{n-1}ab^n = \binom{n+1}{1}a^n b + \dots + \binom{n+1}{n-1}ab^n \qquad \square$$

P.S. Sorry about the messiness on this part (c). I was getting a bit lazy and didn't want to go through breaking lines and stuff.