

MATH 272, HOMEWORK 3
DUE FEBRUARY 17TH

Problem 1. Show that for any smooth (more than twice differentiable) fields $f(x, y, z)$ and $\vec{V}(x, y, z)$ that

- (a) $\vec{\nabla} \times (\vec{\nabla} f) = \vec{0}$;
- (b) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{V}) = 0$.

Problem 2. Let

$$\vec{U}(x, y, z) = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{V}(x, y, z) = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix},$$

be vector fields.

- (a) Explain why there exists no potential function $\phi(x, y, z)$ for the vector field \vec{U} .
- (b) Explain why there does exist a potential function $\phi(x, y, z)$ for the field \vec{V} .
- (c) Compute the potential function for \vec{V} .

Problem 3. Consider the two dimensional scalar field $T(x, y) = x + y$ that describes the temperature on the square plate Ω given by the set $0 \leq x, y \leq 1$. Compare the two answers you get!

- (a) Compute the integral

$$\int_{\Omega} T d\Omega.$$

- (b) Let $\vec{\gamma}$ be the curve that traverses the boundary of the square plate in the counterclockwise direction. Compute

$$\int_{\vec{\gamma}} T d\vec{\gamma}.$$

Problem 4. Let $f(x, y, z) = 2xy + e^{xz} + \sin(y)$ be a scalar field. Integrate f over the triangular prism Ω defined by taking the half triangle of the unit square in the xy -plane satisfying $x \leq y$ and with height 4 above the xy -plane.

Problem 5. Consider $f(x, y) = 3x^4 + x^3 - 18x^2y^2 - 3xy^2 + 3y^4$.

- (a) Show $\Delta f = 0$.
- (b) Find the surface normal to the graph of f .

Problem 6. Parameterize the following either implicitly or explicitly. In Cartesian coordinates, find the parameterization of the normal vector as well.

- (a) The plane perpendicular to the vector $\vec{v} = \hat{x} + \hat{y} + \hat{z}$ passing through the point $(1, 1, 1)$.
- (b) The upper half of the unit circle in \mathbb{R}^2 .
- (c) The surface of the unit sphere in \mathbb{R}^3 .

Problem 7. Consider the following vector field

$$\vec{E}(x, y, z) = \begin{pmatrix} \frac{x}{(x^2+y^2+z^2)^{3/2}} \\ \frac{y}{(x^2+y^2+z^2)^{3/2}} \\ \frac{z}{(x^2+y^2+z^2)^{3/2}} \end{pmatrix},$$

which models the electric field of an proton (in units of of charge $q = 1$) placed at the origin.

- (a) Show that $\vec{E}(x, y, z) = -\vec{\nabla}\phi(x, y, z)$ where $\phi(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$. We refer to $\phi(x, y, z)$ as the electrostatic potential (or voltage).
- (b) Let Ω be a box with side lengths two centered at the origin. Compute the total flux of \vec{E} through the surface of the box Σ . That is,

$$\int_{\Sigma} \vec{E} \cdot \hat{n} d\Sigma.$$

- (c) Does the total flux depend on the size or shape of the box?
- (d) Using the provided argument, one can compute

$$\int_{\Omega} \vec{\nabla} \cdot \vec{E} d\Omega.$$

- Compute $\vec{\nabla} \cdot \vec{E}$ and note that this is zero everywhere except at $(x, y, z) = (0, 0, 0)$.
- Note that the two integrals in this problem are equal. This is known as the *divergence theorem* and it is a special case of a more general theorem called *Stokes' theorem* which generalizes the fundamental theorem of calculus. Is it true that $\vec{\nabla} \cdot \vec{E} = 0$ everywhere?