MATH 560, Homework 2

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September 7, 2017

Solutions

Problem 1. The singular value decomposition of a real $m \times n$ matrix is written

$$A = U\Sigma V^T$$

where $U^T U = I_{m\times m}$, $V^T V = I_{n\times n}$ and $\Sigma_{m\times n}$ has zero entries aside from the $n \times n$ block diagonal with entries $(\sigma_1, ..., \sigma_r)$. We will assume, without loss of generality, that $m \geq n$.

(a) Show exactly the structure of Σ as a matrix, populating this matrix with the r non-zero singular values.

(b) Show that the left singular vectors can be found by solving an $m \times m$ eigenvector problem. Explicitly construct this problem.

(c) Show that the right singular vectors can be found by solving an $n \times n$ eigenvector problem. Explicitly construct this problem.

(d) Show that these eigenvector problems are for symmetric matrices in each case.

(e) Show that the left singular vectors associated with non-zero singular values may be computed in terms of A, Σ and V. Write down the formula.

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Solution (Part (a)).

$$\Sigma = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ m-1 \\ m \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & \cdots & n-1 & n \end{array} \\ \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \end{bmatrix} \end{array}$$

With m rows n columns and the off diagonals all zero. ■

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Solution (Part (b)).

$$\begin{aligned} A &= U\Sigma V^T \\ AA^T &= U\Sigma V^T (V\Sigma^T U^T) \\ AA^T &= U(\Sigma\Sigma^T)U^T \end{aligned}$$

Which is an $m \times m$ eigenvalue problem. It gives us the following,

$$AV_i = \sigma_i U_i$$

Where the V_i and U_i are the i^{th} columns of the matrices. with U_i being the left singular vectors and V_i being the right singular vectors. ■

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Solution (Part (c)).

$$
\begin{aligned}
A &= U\Sigma V^T \\
A^T A &= (V\Sigma^T U^T)(U\Sigma V^T) \\
A^T A &= V(\Sigma^T\Sigma)V^T
\end{aligned}
$$

Which is an $n \times n$ eigenvalue problem. It gives us the following,

$$A^T U_i = \sigma_i V_i$$

Where the V_i and U_i are the i^{th} column vectors with U_i being the left singular vectors and V_i being the right singular vectors. ■

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Solution (Part (d)). Since $U^T U = I_{m\times m}$ and $V^T V = I_{n\times n}$ then we have that U and V are symmetric matrices. ■

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Solution (Part (e)).

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\begin{aligned}
A &= U\Sigma V^T \\
AV &= U\Sigma \\
\frac{1}{\det(\Sigma)}\Sigma AV &= U
\end{aligned}
$$

Which allows us to find U in terms of A, Σ and V. ■

Problem 2. This problem concerns finding bases for the four fundamental subspaces in terms of the SVD of a matrix.

(a) Reconstruct the argument in class to find a basis for $\mathscr{R}(A)$. What is the column rank?

(b) Reconstruct the argument in class to find a basis for $\mathscr{R}(A^T)$. What is the row rank?

(c) Find a basis for $\mathscr{N}(A)$. Prove that is is a basis. What is the dimension of the null space?

(d) Find a basis for $\mathscr{N}(A^T)$. Prove that this is a basis. What is the dimension of the left null space?

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Solution (Part (a)). The column rank is r. Since we have a basis $\{v_1, ..., v_n\}$ for $\mathbb{R}^n$ and $\{Av_1, ..., Av_n\}$ forms the range. But Av_i for $i = r, ...n$ is zero. Thus our basis for the range is $\{u_1, ..., u_r\}$. ■

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Solution (Part (b)). The column rank is r. Since we have a basis $\{u_1, ..., u_m\}$ for $\mathbb{R}^m$ and $\{A^T u_1, ..., A^T u_m\}$ forms the range. But Av_i for $i = r, ...m$ is zero. Thus our basis for the range is $\{v_1, ..., v_r\}$. ■

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Solution (Part (c)). The $\dim \mathscr{N}(A) = n - r$. Then with a basis $\{v_1, ...v_n\}$ for $\mathbb{R}^n$ we have that $\{v_{r+1}, ..., v_n\}$ is the basis for $\mathscr{N}(A)$ by the argument in part (a). ■

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Solution (Part (d)). The $\dim \mathscr{N}(A) = m - r$. Then with a basis $\{u_1, ...u_m\}$ for $\mathbb{R}^m$ we have that $\{u_{r+1}, ..., v_m\}$ is the basis for $\mathscr{N}(A^T)$ by the argument in part (b). ■

§1.6 Problem 35. Let W be a subspace of a finite-dimensional vector space V, and consider the basis $\{u_1, u_2, ..., u_k\}$ for W. Let $\{u_1, ..., u_k, u_{k+1}, ..., u_n\}$ be an extension of this basis to a basis for V.

(a) Prove that $\{u_{k+1} + W, u_{k+2} + W, ..., u_n + W\}$ is a basis for V/W.

(b) Derive a formula relating $\dim(V)$, $\dim(W)$, and $\dim(V/W)$.

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Proof (Part (a)). Consider the following,

$$a_{k+1}(u_{k+1} + W) + ... + a_n(u_n + W) = 0 + W.$$

Which implies $a_{k+1}u_{k+1} + ... + a_n u_n = 0$. But these vectors linearly independent, thus we would have that each a_i is 0. Finally, consider $x + W \in V/W$ be arbitrary and we have that $x = a_1 u_1 + ... + a_n u_n$ so that $x + W = (a_1 u_1 + ... + a_n u_n) + W = (a_{k+1} u_{k+1} ... a_n u_n + W$. Thus any arbitrary element is in the span of these linearly independent vectors. So we have $\{u_{k+1} + W, u_{k+2} + W, ..., u_n + W\}$ is a basis. □

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Proof (Part (b)). We have that $\dim(V) = n$, $\dim(W) = k$ and we know that $V/W = \text{span}\{u_{k+1} + W, ..., u_n + W\}$ Thus we have that

$$\dim(V/W) = \dim(V) - \dim(W).$$

□

§2.1 Problem 3. $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$. Prove that T is linear and find bases for both $\mathcal{N}(T)$ and $\mathcal{R}(T)$. The compute the nullity and rank of T, and verify the dimension theorem. Finally, use the appropriate theorems in this section to determine whether T is injective or surjective.

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Proof. Consider

$$\begin{aligned} T(a(x_1, x_2) + (y_1, y_2)) &= T(ax_1 + y_1, ax_2 + y_2) \\ &= (ax_1 + x_2 + y_1 + y_2, 0, 2ax_1 + 2y_1 - 2ax_2 - y_2) \\ &= (a(x_1 + x_2) + (y_1 + y_2), 0, a(2x_1 - x_2) + (2y_1 - y_2)) \\ &= aT(x_1, x_2) + T(y_1, y_2) \end{aligned}$$

So T is linear. To find the basis for $\mathcal{N}(T)$ we find what elements are mapped to the zero vector. Thus we need to satisfy

$$\begin{aligned} a_1 + a_2 &= 0 \\ 2a_1 - a_2 &= 0 \end{aligned}$$

Which implies that $a_1 = a_2 = 0$. So the basis for $\mathcal{N}(T)$ is $\{0\}$. A basis for $\mathcal{R}(T)$ is given by $\{(1,0,0),(0,0,1)\}$. nullity$(T) = 0$, rank$(T) = 2$ and we have $\dim(V) = 2 =$ nullity$(T)+$rank$(T) = 0+2$. Since nullity$(T) = 0$ we have that T is injective. But since $\dim(\mathbb{R}^3) >$ rank(T) we have that T is not surjective. $\square$

§2.1 Problem 4. $T\colon M_{2\times3}(F) \to M_{2\times2}(F)$ defined by

$$T\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{bmatrix}.$$

Prove that T is linear and find bases for both $\mathcal{N}(T)$ and $\mathcal{R}(T)$. The compute the nullity and rank of T, and verify the dimension theorem. Finally, use the appropriate theorems in this section to determine whether T is injective or surjective.

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Proof. To show that T is linear, we want to show $T(aA+B) = aT(A) + T(B)$. So we have,

$$\begin{aligned} T(aA+B) &= T\left(\begin{bmatrix} aA_{11}+B_{11} & aA_{12}+B_{12} & aA_{13}+B_{13} \\ aA_{21}+B_{21} & aA_{22}+B_{22} & aA_{23}+B_{23} \end{bmatrix}\right) \\ &= \begin{bmatrix} 2aA_{11}+B_{12}-aA_{12}-B_{12} & aA_{13}+B_{13}+2aA_{12}+2B_{12} \\ 0 & 0 \end{bmatrix} \\ &= a\begin{bmatrix} 2A_{11}-A_{12} & A_{13}+2A_{12} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B_{12}-B_{12} & B_{13}+2B_{12} \\ 0 & 0 \end{bmatrix} \\ &= aT(A) + T(B) \end{aligned}$$

So T is linear. A basis for $\mathcal{N}(T)$ is given by

$$\left\{ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

A basis for $\mathcal{R}(T)$ is given by

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

Then we have $\text{nullity}(T) = 4$ and $\text{rank}(T) = 2$. And $\dim(M_{2\times3}(\mathbb{F})) = 6 = \text{nullity}(T) + \text{rank}(T) = 4 + 2$. T is not injective since $\text{nullity}(T) \neq 0$ and not surjective since $\text{rank}(T) < \dim(M_{2\times2}(\mathbb{F}))$. $\square$

§2.1 Problem 11. Prove that there exists a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$. What is $T(8,11)$?

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Solution.

$$\begin{aligned} (8,11) &= a(1,1) + b(2,3) \\ \implies a &= 2, b = 3 \end{aligned}$$

Thus we have

$$\begin{aligned} T(8,11) &= 2T(1,1) + 3T(2,3) \\ &= 2(1,0,2) + 3(1,-1,4) \\ &= (5,-3,16) \end{aligned}$$ ■

§2.1 Problem 15. Recall the definition of $P(\mathbb{R})$ on page 10. Define

$$T\colon P(\mathbb{R}) \to P(\mathbb{R}) \text{ by } T(f(x)) = \int_0^x f(t)dt.$$

Prove that T is linear and injective, but not surjective.

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Proof. To show that T is linear we show that $T(af(x)+g(x)) = aT(f(x)) + T(g(x))$. So

$$\begin{aligned} T(af(x)+g(x)) &= \int_0^x (af(t)+g(t)dt \\ &= a\int_0^x f(t)dt + \int_0^x g(t)dt \qquad = aT(f(x)) + T(g(x)) \end{aligned}$$

by properties of integrals.

Suppose that $f(x) = a_0 + a_1x + \dots + a_nx^n \in \mathcal{N}(T)$. Thus

$$\begin{aligned} T(f(x)) &= \int_0^x (a_0 + \dots + a_nx^n)dt \\ &= a_0\int_0^x 1dt + \dots + a_n\int_0^x x^n dt \end{aligned}$$

Thus since no integrand evaluates to 0, we have that $a_i = 0\ \forall i$. So $\mathcal{N}(T) = \{0\}$. So T is injective.

Consider $c \in P(\mathbb{R})$. Then let $f(x) = a_0 + a_1x + \dots + a_nx^n \in P(\mathbb{R})$ Thus

$$\begin{aligned} c = T(f(x)) &= \int_0^x (a_0 + \dots + a_nx^n)dt \\ &= a_0\int_0^x 1dt + \dots + a_n\int_0^x x^n dt \\ c &= a_0x + \dots + a_nx^{n+1} \end{aligned}$$

Which has no solution. Thus T is not surjective since there exists an element of $P(\mathbb{R})$ not in $\mathscr{R}(T)$. $\square$

§2.1 Problem 17. Let V and W be finite-dimensional vector spaces and $T\colon V \to W$ be linear.

(a) Prove that if $\dim(V) < \dim(W)$, then T cannot be surjective.

(b) Prove that if $\dim(V) > \dim(W)$, then T cannot be injective.

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Proof (Part (a)). We have that

$$\dim(W) > \dim(V) \geq \operatorname{rank}(T)$$

Since $\operatorname{rank}(T)$ is less than $\dim(W)$, T is not surjective. □

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Proof (Part (b)). We have that

$$\operatorname{rank}(T) \leq \dim(W) < \dim(V)$$

So we have

$$\dim(V) - \operatorname{rank}(T) > 0$$

Which means that $\operatorname{nullity}(T) > 0$ by the dimension theorem. This means that T is not injective. □

§2.1 Problem 35. Let V be a finite-dimensional vector space and $T\colon V \to V$ be linear.

(a) Suppose that $V = \mathscr{R}(T) + \mathscr{N}(T)$. Prove that $V = \mathscr{R}(T) \oplus \mathscr{N}(T)$.

(b) Suppose that $\mathscr{R}(T) \cap \mathscr{N}(T) = \{0\}$. Prove that $V = \mathscr{R}(T) \oplus \mathscr{N}(T)$.

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Proof (Part (a)). Suppose that $V = \mathscr{R} + \mathscr{N}(T)$ and that we have $v \in \mathscr{R}(T) \cap \mathscr{N}(T)$. Then we have $T(v) = 0$ since $v \in \mathscr{N}(T)$, which means that $v = 0$ since $v \in \mathscr{R}(T)$. Thus $\mathscr{R}(T) \cap \mathscr{N}(T) = \{0\}$ and thus $V = \mathscr{R}(T) \oplus \mathscr{N}(T)$. ☐

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Proof (Part (b)). Suppose that $\mathscr{R}(T) \cap \mathscr{N}(T) = \{0\}$. Suppose we have $v \in V$ so that $T(v) \notin \mathscr{R}(T) + \mathscr{N}(T)$. Thus we know that $T(v) \neq 0$ since $0 \in \mathscr{R}(T) + \mathscr{N}(T)$. But then if $T(v) \neq 0$ then $T(v) \in \mathscr{R}(T)$ and we contradict $T(v) \notin \mathscr{R}(T) + \mathscr{N}(T)$. So $V = \mathscr{N}(T) \oplus \mathscr{R}(T)$. ☐

§2.1 Problem 40. Let V be a vector space and W be a subspace of V. Define the mapping $\eta\colon V \to V/W$ by $\eta(v) = v + W$ for $v \in V$.

(a) Prove that η is a linear transformation from V onto V/W and that $\mathcal{N}(\eta) = W$.

(b) Suppose that V is finite-dimensional. Use (a) and the dimension theorem to derive a formula relating $\dim(V)$, $\dim(W)$, and $\dim(V/W)$.

(c) Read the proof of the dimension theorem. Compare the method of solving (b) with the method of deriving the same result as outlined in Exercise 35 of Section 1.6.

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Proof (Part (a)). Let $u, v \in V$ and $a \in \mathbb{F}$. Then

$$\begin{aligned}\eta(av+u) &= (av+u)+W \\ &= (av+W)+(u+W) \\ &= a(v+W)+(u+W) \\ &= a\eta(v)+\eta(u)\end{aligned}$$

So η is linear. Then let $v + W \in V/W$ be arbitrary and note that $\eta(v) = v + W$ for $v \in V$ and thus η is surjective. □

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Proof (Part (b)). We have

$$\begin{aligned}\dim(V) &= \dim(\mathcal{R}(\eta)) + \dim(\mathcal{N}(\eta)) \\ &= \dim(V/W) + \dim(W) && \text{since } \eta \text{ is onto} \\ \implies \dim(V/W) &= \dim(V) - \dim(W)\end{aligned}$$

□

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Solution (Part (c)). (b) uses an onto linear transformation to allow us to utilize the dimension theorem. But Ex. 35 of §1.6 uses an argument which involves constructing bases for $\mathcal{R}(T)$ and $\mathcal{N}(T)$. ■