MATH 571, Homework 3

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Solutions

Problem 1. Use van Kampen's theorem to prove that the n-sphere S^n has trivial fundamental group for $n \geq 2$

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Proof. Choose a base point x_0 on S^n other than $\{(0,0,\ldots,0,1)\}$ or $\{(0,0,\ldots,0-1\}$. Then let $\mathcal{N} = S^n \setminus \{(0,0,\ldots,0,-1)\}$ and $\mathcal{S} = S^n \setminus \{(0,0,\ldots,0,1)\}$ be two open subsets of S^n such that $\mathcal{N} \cup \mathcal{S} = S^n$ and note $\mathcal{N} \cup \mathcal{S} = S^n \setminus \{(0,0,\ldots,0,\pm 1)\} \simeq S^{n-1}$ is path connected. For the base case, $n = 2$, we have that $\pi_1(\mathcal{N}) \cong \pi_1(\mathcal{S}) \cong \{e\}$ are trivial groups since $\mathcal{N} \simeq \mathcal{S} \simeq B^2$. Then for any $w \in \pi_1(\mathcal{N} \cap \mathcal{S})$, note that $i_{\mathcal{N}\mathcal{S}}(w) i_{\mathcal{S}\mathcal{N}}(w)^{-1} \simeq C_{x_0}$ is the constant path since any loops are contractible in both $\mathcal{N}$ and $\mathcal{S}$. So $\pi_1(\mathcal{N} \cap .\mathcal{S}) = \{e\}$. Finally, by Van Kampen's theorem, we have $\pi_1(S^2) \cong \pi_1(\mathcal{N}) * \pi_1(\mathcal{S}/\{e\} \cong \{e\}$ and so the fundamental group of S^2 is trivial.

Now, suppose this is true for $n-1$, and consider the case for S^n. Now $\mathcal{N} \simeq \mathcal{S} \simeq B^n$ and so $\pi_1(\mathcal{N}) \cong \pi_1(\mathcal{S}) \cong \{e\}$. Also, $\mathcal{N} \cap \mathcal{S} \simeq S^{n-1}$, and so $\pi_1(S^{n-1})$ is trivial by our induction hypothesis and hence $\pi_1(\mathcal{N} \cap \mathcal{S}) \cong \{e\}$. Finally, Van Kampen's theorem then gives us $\pi_1(S^n) \cong (\{e\} * \{e\})/\{e\} \cong \{e\}$, showing that $\pi_1(S^n)$ is trivial. □

Problem 2. Let M be an n-dimensional manifold, with $n \geq 3$. Let $p \in M$ be any point in the manifold M. There is a nice relationship between the fundamental groups $\pi_1(M)$ and $\pi_1(M \setminus \{p\})$ — how are they related? Prove your answer is correct.

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Proof. We claim that $\pi_1(M) \cong \pi_1(M \setminus \{p\})$. To see this, let $A_1 = M \setminus \{p\}$ and choose $\epsilon > 0$ so that $A_2 = B^n_\epsilon(p) \subseteq M$. Then we have

$$\pi_1(M) \cong (\pi_1(A_1) * \pi_1(A_2))/N,$$

where N is the normal subgroup generated by elements of the form $i_{21}(w) i_{12}(w)^{-1}$ with $w \in \pi_1(A_1 \cap A_2)$. Note that $\pi_1(A_2) \cong \{e\}$ is trivial, and that N is also trivial since any loop in $A_1 \cap A_2$ is homotopy equivalent to a trivial loop. This gives

$$\begin{aligned}\pi_1(M) &\cong (\pi_1(A_1) * \pi_1(A_2))/N \\ &\cong \pi_1(A_1) \cong \pi_1(M \setminus \{p\}).\end{aligned}$$

□

Problem 3.

(a) Problem 8 on page 53 of Hatcher: "Compute the fundamental group of the space obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus." Let's call this identification space X. I want you to use van Kampen's theorem to compute $\pi_1(X)$.

(b) Write this identification space X as a product $X = Y \times Z$ (where neither Y nor Z are just a single point), and use this to give an alternate computation of $\pi_1(X)$.

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Proof.

(a) Let x_0 be our base point and let $U_1 = S^1 \times S^1$ be the first two tori and $U_2 = S^1 \times S^1$ be the second. Then let $O = \{N_x : x \in U_1 \cap U_2\}$ be a collection of neighborhoods of points in the intersection of the two tori and let $A_1 = U_1 \cup O$ and $A_2 = U_2 \cup O$ so that A_1 and A_2 are open in X. Then $\cap A_2 = S^1 \times \{x_0\}$ and so $A_1 \cap A_2$ is path connected. Then $\pi_1(A_1) \cong \langle a, b_1 | aba^{-1}b^{-1} \rangle$ and $\pi_1(A_2) \cong \langle b_2, c \rangle$ and $\pi_1(A_1 \cap A_2) \cong \langle b \rangle \cong \mathbb{Z}$. Then for any $w \in \pi_1(A_1 \cap A_2)$ we have that $i_{21}(w) = b_1$ and $i_{12}(w)^{-1} = b_2^{-1}$ so that $b_1 b_2^{-1} = 1$. So now, letting N be the group generated by all elements of the form $I_{21}(w) I_{12}^{-1}$ with $w \in \pi_1(A_1 \cap A_2)$ we have that

$$
\begin{aligned}
\pi_1(X) &\cong (\pi_1(A_1) * \pi_1(A_2))/N \\
&\cong \langle a, b_1, b_2, c | ab_1a^{-1}b_1^{-1}, b_2cb_2^{-1}c^{-1}, b_1b_2^{-1} \rangle \\
&\cong \langle a, b, c | aba^{-1}b^{-1}, bcb^{-1}c^{-1} \rangle \qquad \text{letting } b_1 = b_2 = b \\
&\cong (\mathbb{Z} * \mathbb{Z}) \oplus \mathbb{Z}.
\end{aligned}
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To see this visually, a loop (denoted as b above) in the intersection $A_1 \cap A_2$ is a loop that commutes with all other loops, but the other two loops (denoted as a and c above) come from a wedge of two circles and thus will not commute with each other.

(b) We let $Y = S^1 \wedge S^1$ and $Z = S^1$ and we have that $X = Y \times Z$. We then have that $\pi_1(X) = \pi_1(Y) \times \pi_1(Z)$ and so $\pi_1(Y) = \mathbb{Z} * \mathbb{Z}$ and $\pi_1(Z) = \mathbb{Z}$, hence $\pi_1(X) = (\mathbb{Z} * \mathbb{Z}) \oplus \mathbb{Z}$. □