MATH 517, Homework 4

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September 22, 2017

Solutions

Problem 1. (Rudin 3.10) Suppose that the coefficients of the power series $\sum a_n z^n$ are integers, infinitely many of which are nonzero. Prove that the radius of convergence of the power series is at most 1.

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Proof. Suppose we have that $R > 1$. Then we can have $|z| > 1$. Then since infinitely many terms of a_n are nonzero we have that for infinitely many terms $|a_n z^n| > 1$ since the smallest nonzero integer is 1. But this means that $\sum a_n z^n$ does not converge. It is possible to have convergence with $|z| \leq 1$ if we have $\sum a_n$ converges. So the radius of convergence is at most 1. $\square$

Problem 2. (Rudin 3.23) Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges. (Note that this new sequence lives in $\mathbb{R}$.)

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Proof. Fix $\epsilon > 0$. Since $\{p_n\}$ is Cauchy we have for $n, m > N_1 \in \mathbb{N}$ that $d(p_n, p_m) < \frac{\epsilon}{2}$. Similarly for $\{q_n\}$ we have $n, m > N_2$ so that $d(p_n, p_m) < \frac{\epsilon}{2}$. Let $N = \max(N_1, N_2)$ and for $n, m > N$ we have

$$\begin{aligned} |d(p_n, q_n) - d(p_m, q_m)| &= |d(p_n, q_n) - d(q_n, p_m) + d(q_n, p_m) - d(p_m, q_m)| \\ &\leq |d(p_n, p_m) + d(q_m, q_n)| && \text{since } d(x, y) \leq d(x, z) + d(y, z) \\ &< \epsilon \end{aligned}$$

Thus $\{d(p_n, q_n)\} \in \mathbb{R}$ is also Cauchy, and since $\mathbb{R}$ is complete, we have that $\{d(p_n, q_n)\}$ converges. □

Problem 3. (Rudin 3.24) Let X be a metric space.

(a) Call two Cauchy sequences $\{p_n\}, \{q_n\}$ in X *equivalent* if

$$\lim_{n\to\infty} d(p_n, q_n) = 0$$

Prove that this is an equivalence relation.

(b) Let X^* be the set of all equivalence classes from (a). If $P \in X^*$ has representative $\{p_n\}$ and $Q \in X^*$ has representative $\{q_n\}$, define

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n),$$

which exists by the previous exercise. Show that $\Delta(P, Q)$ is independent of the choice of representatives for the equivalence classes P and Q, so it is a well-defined distance on X^*.

(c) Prove that the metric space (X^*, Δ) is complete (i.e., every Cauchy sequence in X^* converges).

(d) For each $p \in X$, there is a Cauchy sequence all of whose terms are p; let P_p be the equivalence class of this sequence. Prove that

$$\Delta(P_p, P_q) = d(p, q)$$

for each $p, q \in X$. In other words, the map $\phi\colon X \to X^*$ is an *isometry* onto its image (an isometry is a distance-preserving map).

(e) Prove that $\phi(X)$ is dense in X^*, and that $\phi(X) = X^*$ if X is complete. From (d), we can identify X and $\phi(X)$, and se we see X as densely embedded in the complete metric space X^*. X^* is called the *completion* of X.

[*Remark:* To formally construct $\mathbb{R}$, we can simply define $\mathbb{R} = \mathbb{Q}^*$, the completion of the metric space $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|$. As mentioned in class, completing $\mathbb{Q}$ with respect to the p-adic metric d_p produces the field $\mathbb{Q}_p$ of p-adic numbers.]

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Proof (Part (a)). First we have that $\{p_n\} \sim \{p_n\}$ since $d(p_n, p_n) = 0$ for every n. Next, let $\{q_n\}$ be a sequence so that $\{p_n\} \sim \{q_n\}$. Then $\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(q_n, p_n) = 0$ and thus $\{q_n\} \sim \{p_n\}$. Finally let $\{r_n\}$ be a sequence so we have $\{p_n\} \sim \{q_n\}$ and $\{q_n\} \sim \{r_n\}$. Then $d(p_n, r_n) \le d(p_n, q_n) + d(q_n, r_n)$ and $\lim_{n\to\infty}(d(p_n, q_n) + d(p_n, r_n)) = 0$ and thus $\lim_{n\to\infty} d(p_n, r_n) = 0$. Thus $\{p_n\} \sim \{r_n\}$. $\square$

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Proof (Part (b)). Consider different representatives $\{p'_n\}$ and $\{q'_n\}$ for P and Q respectively. Then

$$\begin{aligned}
\lim_{n\to\infty}(d(p'_n, q_n) - d(q_n, q'_n)) \le \lim_{n\to\infty} d(p'_n, q'_n) &\le \lim_{n\to\infty}(d(p'_n, p_n) + d(p_n, q'_n)) \\
\iff \lim_{n\to\infty} d(p'_n, q_n.) \le \lim_{n\to\infty} d(p'_n, q'_n) &\le \lim_{n\to\infty} d(p_n, q'_n) \\
\iff \lim_{n\to\infty}(d(q_n, p_n) - d(p_n, p'_n)) \le \lim_{n\to\infty} d(p'_n, q'_n) &\le \lim_{n\to\infty}(d(p_n, q_n) + d(q_n, q'_n)) \\
\iff \lim_{n\to\infty} d(p_n, q_n) \le \lim_{n\to\infty} d(p'_n, q'_n) &\le \lim_{n\to\infty} d(p_n, q_n)
\end{aligned}$$

So we have that $\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(p'_n, q'_n)$. $\square$

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Proof (Part (c)). Let $\{P_i\}$ be a Cauchy sequence in X^* where $\{P_i\}$ is a representative of the equivalence class of a sequence $\{p_{i_j}\} \in X$. Since $\{P_i\}$ is Cauchy, $\forall \epsilon > 0$ we have that for $n, m > N \in \mathbb{N}$ that

$$\Delta(P_n, P_m) < \epsilon.$$

Then consider $\Delta(P_i, \hat{0})$ where $\hat{0}$ is the equivalence class of sequences converging to zero. Then we have that this is a sequence of real numbers and note

$$\Delta(P_n, \hat{0}) - \Delta(P_m, \hat{0}) \leq \Delta(P_n, P_m) < \epsilon.$$

Thus we have that $\{\Delta(P_i, \hat{0})\}$ is a Cauchy sequence in $\mathbb{R}$ and so it must converge to a limit $|L| \in \mathbb{R}$. It's also worth noting that L can be less than 0 but $\{\Delta(P_i, \hat{0})\} \geq 0$ and thus can't converge to a negative value. Now consider $\hat{L}$ which is the equivalence class of sequences converging to L and we have that

$$\begin{aligned}
\Delta(P_n, \hat{L}) &\leq \Delta(P_n, P_m) + \Delta(P_m, \hat{L}) \\
&< \epsilon + \Delta(P_m, \hat{L}) \\
&= \epsilon + 2|L| && \text{if the sign of } \hat{L} \text{ is incorrect.}
\end{aligned}$$

But if that is the case then if we had $-\hat{L}$ then

$$\begin{aligned}
\Delta(P_n, \hat{L}) &\leq \Delta(P_n, P_m) + \Delta(P_m, \hat{L}) \\
&< \epsilon + \Delta(P_m, -\hat{L}) \\
&= \epsilon.
\end{aligned}$$

Thus either $\pm|\hat{L}|$ is the limit for $\{P_i\}$ and we have that X^* is complete. ☐

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Proof (Part (d)). We have that P_p can be represented by $\{p_n\}$ which is a constant sequence with $p_n = p$ for every n. Similarly we have P_q is represented by $\{q_n\}$ with $q_n = q$ for every n. Thus $\Delta(P_p, P_q) = \lim_{n\to\infty} d(p_n, q_n) = d(p, q)$. ☐

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Proof (Part (e)). Let $\phi(X) \subseteq X$ and suppose we have a point P in X^*. Fix $\epsilon > 0$. Then for some $Q \in X^*$ we have that $\{q_i\}$ is a representative of Q and we also have that $\{p_i\}$ is a representative for P. Then we let $q_i = p_i + \frac{\epsilon}{2}$ and note that $\Delta(P, Q) = \lim_{i\to\infty} d(p_i, q_i) < \epsilon$. Note, these choices of P and Q are allowed since X^* contains a equivalence classes of sequences that converge to any real number. Since P was arbitrary and the representative for P and Q does not matter, we have that P was a limit point of X^*. Thus $\phi(X)$ is dense in X^*.

Finally, let X be complete. So we have that every Cauchy sequence in X converges in X. Since $\phi(X)$ contains all Cauchy sequences in X, $\phi(X)$ must contain all of its limit points. For example, let P be a limit point of $\phi(X)$ then P is represented by a Cauchy sequence in $\{p\} \in X$ and $\phi(\{p\}) = P$ and we have that $P \in \phi(X)$. And thus since $\phi(X)$ is dense in X^* and contains all of its limit points, necessarily $\phi(X) = X^*$. ☐

Problem 4. (Rudin 4.16) For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ be the floor function; i.e., $\lfloor x \rfloor$ is the largest integer less than or equal to x. Let $(x) :- x - \lfloor x \rfloor$ be the fractional part of x. For which x is $\lfloor x \rfloor$ continuous? For which x is (x) continuous?

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Solution. For $x \in (n, n+1)$ with $n \in \mathbb{N}$, we have that $\lfloor x \rfloor$ is continuous. It is also the case that for $x \in (n, n+1)$ that (x) is continuous. ■