MATH 272, Homework 6, *Solutions*

Due March 24th

Problem 1. Let

$$\vec{\gamma}(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \\ t \end{pmatrix}, \quad f(x,y,z) = x^2 + y^2 - 2z^2, \quad \vec{\boldsymbol{V}}(x,y,z) = \begin{pmatrix} x-y \\ y+x \\ z \end{pmatrix}.$$

Compute derivatives of the following composite functions.

(a) $f(\vec{\gamma}(t))$.

(b) $\vec{\boldsymbol{V}}(\vec{\gamma}(t))$.

(c) $f(\vec{\boldsymbol{V}}(x,y,z))$.

Solution 1.

(a) We are considering the composite function $f \circ \vec{\gamma} \colon \mathbb{R} \to \mathbb{R}$. Hence, our result for the derivative must be a linear function $(f \circ \vec{\gamma})' \colon \mathbb{R} \to \mathbb{R}$. Specifically, this means that at any time t, we have a 1×1-matrix as the derivative. Following our nose, we can use the chain rule

$$(f \circ \vec{\gamma})' = \vec{\boldsymbol{\nabla}} f(\vec{\gamma}(t))\dot{\vec{\gamma}}(t).$$

Note that here we think of $\vec{\boldsymbol{\nabla}} f$ as the 1×3 *row* vector (which is often called a covector). Why is that? Well, recall that $f \colon \mathbb{R}^3 \to \mathbb{R}$ and hence the derivative is a linear function $f' = \vec{\boldsymbol{\nabla}} f \colon \mathbb{R}^3 \to \mathbb{R}$. Hence, $\vec{\boldsymbol{\nabla}} f$ must be a matrix that multiplies by a column vector (a 3×1-matrix) and gives us a number. This must mean that $\vec{\boldsymbol{\nabla}} f$ is a 1×3-matrix. Now,

$$\vec{\boldsymbol{\nabla}} f = \begin{pmatrix} 2x & 2y & -4z \end{pmatrix},$$

and

$$\vec{\boldsymbol{\nabla}} f(\vec{\gamma}(t)) = \begin{pmatrix} 2\cos(t) & 2\sin(t) & -4t \end{pmatrix}.$$

Then,

$$\dot{\vec{\gamma}}(t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 1 \end{pmatrix}.$$

Thus,

$$\begin{aligned} \vec{\boldsymbol{\nabla}} f(\vec{\gamma}(t))\dot{\vec{\gamma}}(t) &= \begin{pmatrix} 2\cos(t) & 2\sin(t) & -4t \end{pmatrix} \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 1 \end{pmatrix} \\ &= -2\cos(t)\sin(t) + 2\cos(t)\sin(t) - 4t \\ &= -4t. \end{aligned}$$

(b) Now $\vec{\boldsymbol{V}} \circ \vec{\boldsymbol{\gamma}} \colon \mathbb{R} \to \mathbb{R}^3$ and so we are expecting a 3×1-matrix result. In this case, it will be given by

$$\left(\vec{\boldsymbol{V}} \circ \vec{\gamma}\right)' = [J]_{\vec{\boldsymbol{V}}}(\vec{\gamma}(t))\dot{\vec{\gamma}}(t).$$

We compute the derivative of $\vec{\boldsymbol{V}}$ as the Jacobian

$$[J]_{\vec{\boldsymbol{V}}}(x,y,z) = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is constant. This means that

$$[J]_{\vec{\boldsymbol{V}}}(\vec{\gamma}(t)) = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We already computed $\dot{\vec{\boldsymbol{\gamma}}}$, and thus

$$\begin{aligned} [J]_{\vec{\boldsymbol{V}}}(\vec{\gamma}(t))\dot{\vec{\gamma}}(t) &= \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -\sin(t) - \cos(t) \\ -\sin(t) + \cos(t) \\ 1 \end{pmatrix}. \end{aligned}$$

(c) Finally, note $f \circ \vec{\boldsymbol{V}} \colon \mathbb{R}^3 \to \mathbb{R}$, which means we expect a 1×3 matrix for $(f \circ \vec{\boldsymbol{V}})'$. We have that

$$(f \circ \vec{\boldsymbol{V}})' = \vec{\boldsymbol{\nabla}} f(\vec{\boldsymbol{V}}(x,y,z))[J]_{\vec{\boldsymbol{V}}}(x,y,z),$$

where again we think of $\vec{\boldsymbol{\nabla}} f$ as a covector. Now, this yields

$$\begin{aligned} \vec{\boldsymbol{\nabla}} f(\vec{\boldsymbol{V}}(x,y,z))[J]_{\vec{\boldsymbol{V}}}(x,y,z) &= \begin{pmatrix} 2x & 2y & -4z \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2x + 2y & -2x + 2y & -4z \end{pmatrix}. \end{aligned}$$

Problem 2. Show that for any smooth (more than twice differentiable) fields $f(x, y, z)$ and $\vec{\boldsymbol{V}}(x, y, z)$ that

(a) $\vec{\boldsymbol{\nabla}} \times \left(\vec{\boldsymbol{\nabla}} f\right) = \vec{\mathbf{0}}$;

(b) $\vec{\boldsymbol{\nabla}} \cdot \left(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{V}}\right) = 0$.

Solution 2.

(a) We have that

$$\vec{\boldsymbol{\nabla}} f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}.$$

Taking the curl yields

$$\vec{\boldsymbol{\nabla}} \times \left(\vec{\boldsymbol{\nabla}} f\right) = \begin{pmatrix} \frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \\ \frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} \\ \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 f}{\partial z \partial y} - \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x} \\ \frac{\partial^2 f}{\partial y \partial x} - \frac{\partial f}{\partial x \partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

since partial derivatives commute for any smooth scalar field.

(b) First, the curl is

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{V}} = \begin{pmatrix} \frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \\ \frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} \\ \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \end{pmatrix},$$

and we can take the divergence

$$\begin{aligned} \vec{\boldsymbol{\nabla}} \cdot \left(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{V}}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) \\ &= 0, \end{aligned}$$

again since partial derivatives commute.

Problem 3. Let

$$\vec{\boldsymbol{U}}(x,y,z) = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \qquad \text{and} \qquad \vec{\boldsymbol{V}}(x,y,z) = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix},$$

be vector fields.

(a) Explain why there exists no potential function $\phi(x,y,z)$ for the vector field $\vec{\boldsymbol{U}}$.

(b) Explain why there does exist a potential function $\phi(x,y,z)$ for the field $\vec{\boldsymbol{V}}$.

(c) Compute the potential function for $\vec{\boldsymbol{V}}$.

Solution 3.

(a) There exists a potential function if the curl of $\vec{\boldsymbol{U}}$ is zero. So, taking the curl we find

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{U}} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix},$$

which is nonzero. Thus, there cannot be a potential function for $\vec{\boldsymbol{U}}$.

(b) Likewise, taking the curl for $\vec{\boldsymbol{V}}$ we get

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{U}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, there must be a potential function for $\vec{\boldsymbol{V}}$.

(c) To compute the potential $\phi(x,y,z)$, we integrate V_1 with respect to x, V_2 with respect to y, and V_3 with respect to z. This yields

$$\begin{aligned} \phi(x,y,z) &= \int 2x dx = x^2 + \psi_1(y,z) \\ \phi(x,y,z) &= \int 2y dy = y^2 + \psi_2(x,z) \\ \phi(x,y,z) &= \int 2z dz = z^2 + \psi_3(x,y). \end{aligned}$$

Since these are all equal, we must have that

$$\phi(x,y,z) = x^2 + y^2 + z^2 + C,$$

where C is a constant.

Problem 4. Parameterize the following either implicitly or explicitly. In Cartesian coordinates, find the parameterization of the normal vector as well.

(a) The plane perpendicular to the vector $\vec{\boldsymbol{v}} = \hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}$ passing through the point $(1, 1, 1)$.

(b) The upper half of the unit circle in $\mathbb{R}^2$.

(c) The surface of the unit sphere in $\mathbb{R}^3$.

Solution 4.

(a) Based on the vector perpendicular to the plane $\vec{\boldsymbol{v}}$, we are looking for a plane given by

$$0 = a(x - x_0) + b(y - y_0) + c(z - z_0),$$

where we have

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Similarly, we wish to have the plane pass through the point $(1, 1, 1)$ hence

$$(x_0, y_0, z_0) = (1, 1, 1).$$

Thus, our implicit equation for this plane is

$$0 = (x - 1) + (y - 1) + (z - 1).$$

We could also give an explicit equation for the plane as the graph of a function. Specifically, we have from the above work

$$z = -x - y + 3.$$

Thus, as a graph we would take the plane to be given by the points

$$(x, y, -x - y + 3).$$

One could also find two linearly independent vectors perpendicular to $\vec{\boldsymbol{v}}$ and based at $(1, 1, 1)$ and take their span.

(b) Implicitly, the unit circle is the set of all points a distance 1 from the origin. Thus, we are looking for (x, y) pairs that satisfy

$$\sqrt{x^2 + y^2} = 1.$$

Note that we could also write this as

$$x^2 + y^2 = 1.$$

Then, to receive the upper semi circle, we simply neglect values of $y < 0$ to get the implicit description

$$x^2 + y^2 = 1 \qquad \text{with } y \in [0, 1].$$

Explicitly, we can solve for y in terms of x from the previous work to get

$$y = \pm\sqrt{1 - x^2}.$$

Taking $y = \sqrt{1 - x^2}$ we know $y \geq 0$, and this gives us the upper half of the unit circle as the graph of a function.

Or, we could parameterize this as a curve to get another implicit description. We know the curve $\vec{\gamma}(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$ lie on the unit circle for all times t. Then, if we restrict $t \in [0, \pi]$, this gives us just the upper half.

(c) Similarly to (b), the surface of the unit sphere is the set of points a distance 1 from the origin and so we can write this as an implicit equation

$$x^2 + y^2 + z^2 = 1.$$

Explicitly, we could solve for z from the above work to get two graphs

$$z = \pm\sqrt{1 - x^2 - y^2},$$

which if we combine, gives us an explicit description of the surface of the unit sphere.

One could also arrive at a different explicit description of the unit sphere by using spherical coordinates. Take ϕ to the angle from the z-axis of a point on the sphere and θ to be the polar angle from the xz-plane and we have that the points on the surface of the unit sphere are given by

$$\begin{pmatrix} \cos\theta\sin\phi \\ \sin\theta\sin\phi \\ \cos\phi \end{pmatrix},$$

where $\theta \in [0, 2\pi)$ and $\phi \in [-\pi, \pi]$.

Problem 5. In cylindrical coordinates (either implicitly or explicitly), parameterize the following objects.

(a) A cylinder with radius 3 and height 5 along with end-caps.

(b) An infinite cone with a vertex angle of $\pi/4$.

(c) A helical curve with constant radius 1 and pitch 1.

(d) A hyperboloid of one sheet.

Solution 5. We will find the natural paramaterizations of these shapes are natural (and thus explicit) in these coordinates.

(a) If we have a cylinder of radius 3, then we have that $\rho = 3$. If the height is 5, we can just take $z \in [0, 5]$. The end caps can be described by the points satisfying $\rho < 3$ and $z = 0$ for the bottom cap as well as $\rho < 3$ and $z = 5$ for the top cap. This is an explicit description.

(b) Here, we can take the look at a cone from a side profile and notice that we get $\rho = Cz$ where C is a constant.

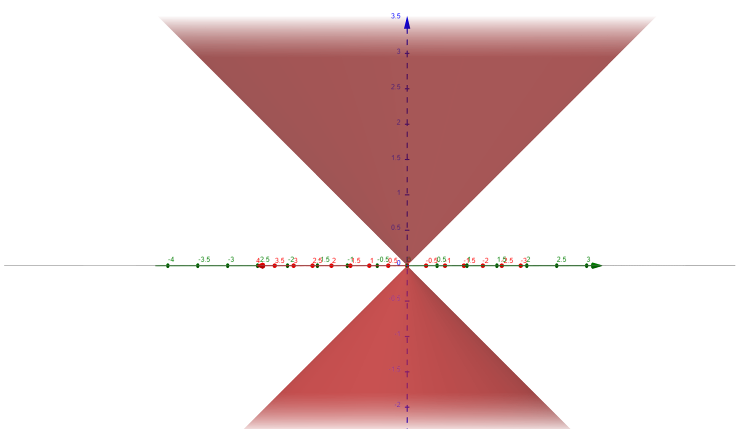

Figure 1: Side profile of a double cone.

If the angle of the vertex is to be $\pi/4$, then the slope of the line we see in the side profile for the ρz-plane should make an angle with the z-axis of $\pi/8$. This happens when $C = 2$, thus we take $\rho = 2z$.

(c) If we give a curve in cylindrical coordinates, we want

$$\vec{\gamma}(t) = \begin{pmatrix} \rho(t) \\ \theta(t) \\ z(t) \end{pmatrix}.$$

Since the radius is 1, $\rho(t) = 1$. Pitch of 1 means that for every full revolution (i.e., θ increases by 2π), we have z increases by 1. Thus $z = \frac{\theta}{2\pi}$. Indeed, this means we are actually free to choose $\theta(t)$ as any increasing function (since we don't want to double back). The simplest is choosing $\theta = t$ and thus we arrive at $\rho(t) = 1,\ \theta(t) = t,\ z(t) = \frac{t}{2\pi}$.

Recall that a hyperboloid of one sheet is given by

$$x^2 + y^2 - z^2 = C,$$

where $C > 0$. Note that $\rho^2 = x^2 + y^2$, and thus

$$\rho = \pm\sqrt{z^2 + C}.$$

One can also note that for $C = 1$, $\frac{\rho}{z} = \tanh(t)$ for $t \in (-\infty, \infty)$ and $\theta \in [0, 2\pi)$. This gives the relationship $\rho(t) = \cosh(t)$ and $z(t) = \sinh(t)$ which is gives us the relationship between hyperbolic (co)sine and the trigonometric (co)sine. Namely, this hyperbola cross section is the *unit* hyperbola and is closely related to the unit circle (see the simple shift in the sign of z for the implicit equation). One could then take $C = -1$, to get the related two-sheeted hyperboloid.