Hodge Theory, Gelfand Theory, and Clifford Analysis Applied to Tomography

Colin Roberts



Overview

1 Introduction

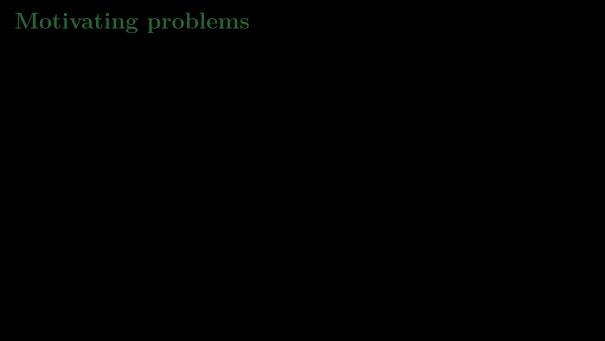
2 Clifford analysis

3 Hodge theory

4 Tomography

Section 1

Introduction



Motivating problems

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- The Calderón problem replaces the medium with a manifold M, conductivity with g, and replaces the voltage-to-current map with the Dirichlet-to-Neumann operator Λ .

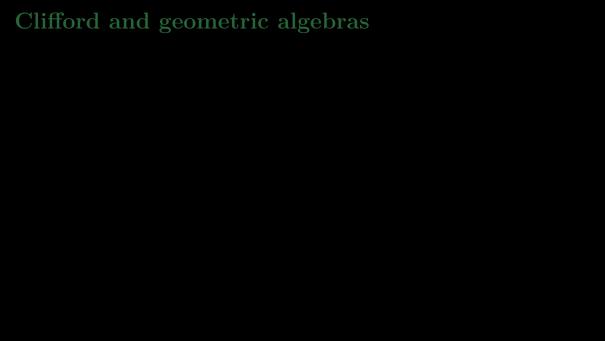
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- Can we access these functions from the boundary?

Subsection 1

Preliminaries



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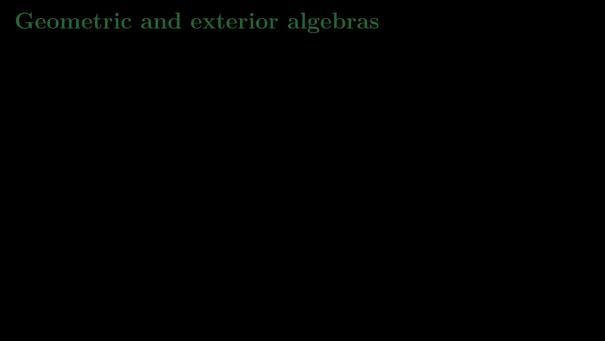
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■ The associated *Clifford algebra* is the quotient

$$C\ell(V, g) := \mathcal{T}(V)/\langle \mathbf{v} \otimes \mathbf{v} - g(\mathbf{v}, \mathbf{v}) \rangle.$$



Geometric and exterior algebras

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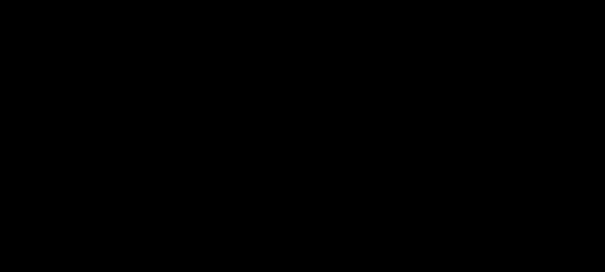
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lacktriangle The completely degenerate case is the exterior algebra

$$\bigwedge(V) \coloneqq C\ell(V,0).$$



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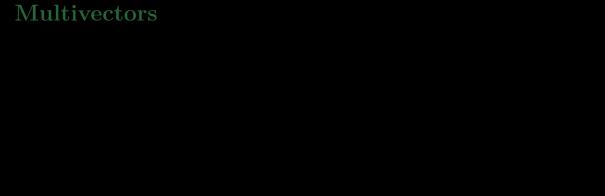
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- The scalar part is symmetric: $\mathbf{u} \cdot \mathbf{v} = g(\mathbf{u}, \mathbf{v})$.
- The bivector part is antisymmetric: $\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u}$.



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■ Since
$$\mathcal{G} = \bigoplus_{r=0}^{\infty} \mathcal{G}^r$$
 a general multivector is $A = \sum_{r=0}^{\infty} \langle A \rangle_r$.

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 \blacksquare The most important products for us are

$$A_r \wedge B_s := \langle A_r B_s \rangle_{r+s}$$
$$A_r \, \lrcorner \, B_s := \langle A_r B_s \rangle_{s-r}$$



Reciprocals and reverses

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- \blacksquare The reverse † is extended linearly from the action on r-blades

$${m A_r}^\dagger = ({m v}_1 \wedge \cdots \wedge {m v}_r)^\dagger = {m v}_r \wedge \cdots \wedge {m v}_1.$$



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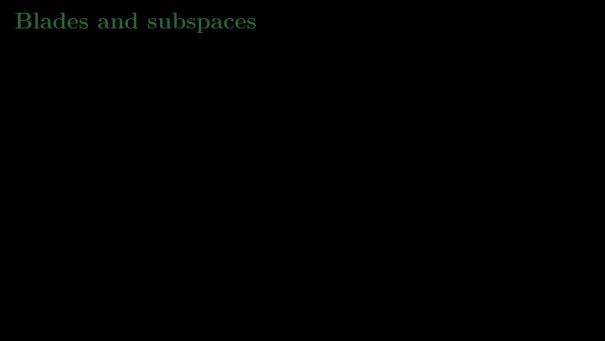
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- The projection of A into a subspace U_r by

$$P_{\mathbf{U_r}}(A) \coloneqq A \sqcup \mathbf{U_r} \mathbf{U_r}^{-1}.$$

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$$\boldsymbol{\mu} = \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_n.$$

■ We define the unit pseudoscalar (which corresponds to $V \subset V$) by

$$I \coloneqq \frac{1}{|\mu|} \mu.$$

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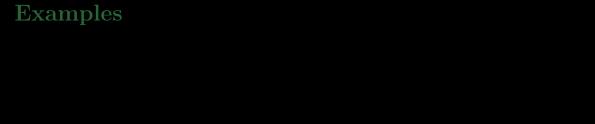
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■ The *Hodge star* \star_q of a multivector A is

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■ Dual exchanges products $(A \, \lrcorner \, B)^{\perp} = A \wedge B^{\perp}$.



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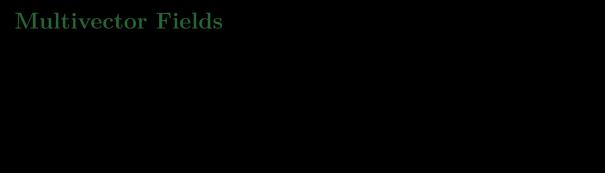
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 - Right multiplication of vectors by e_{12} rotates counter-clockwise by $\pi/2$.

Section 2

Clifford analysis



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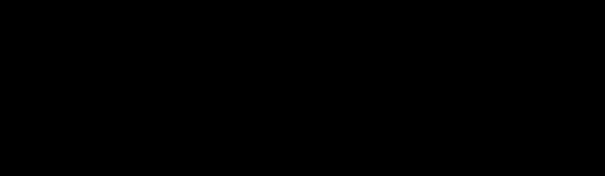
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- The (smooth) multivector fields $\mathfrak{X}(M)$ are the sections of $\mathcal{G}M$.
- Take same naming scheme and notation: $\mathfrak{X}^r(M)$, $\mathfrak{X}^+(M)$, etc.



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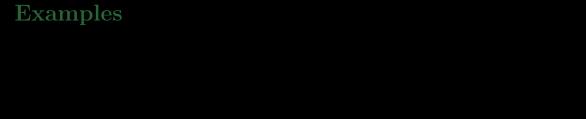
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- $lackbox{} \nabla$ acts as a vector in $\mathfrak{X}(M)$ with Leibniz rule $\nabla(AB) = \dot{\nabla}\dot{A}B + \dot{\nabla}A\dot{B}$.
- $\mathbf{\nabla}^{2}$ is the Laplace-Beltrami operator.



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Specifically,

$$\operatorname{curl}(\boldsymbol{v}) = (\boldsymbol{\nabla} \wedge \boldsymbol{v})^{\perp}$$



Differential forms

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- Any r-form α_r has a multivector equivalent A_r so $\alpha_r = A_r \, \lrcorner \, dX_r^{\dagger}$.
- Algebra and calculus descend to multivector equivalent:

$$\alpha_r \wedge \beta_s = (A_r \wedge B_s) \, \lrcorner \, dX_{r+s} \qquad \alpha_r \, \lrcorner \, \beta_s = (A_r \, \lrcorner \, B_s) \, \lrcorner \, dX_{r-s}$$

$$\underline{d\alpha_r = (\nabla \wedge A_r) \, \lrcorner \, dX_{r+1}^{\dagger}}_{\text{exterior derivative}} \qquad \underline{\delta\alpha_r = (-\nabla \, \lrcorner \, A_r) \, \lrcorner \, dX_{r-1}^{\dagger}}_{\text{codifferential}}$$

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■ For M this yields $d\mu = \sqrt{|g|} dx^1 \cdots dx^n$



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$$\langle\!\langle A, B \rangle\!\rangle_R := \int_R A * B d\mu_R$$



Green's formulas

■ From [Hestenes, Sobczyk, 1984] and [Booß- Bavnbek, Wojciechowski, 1993]

$$(\nabla A, B) = (-1)^n (A, \nabla B) + (A, B)_{\partial M}$$

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$$(\!(\boldsymbol{\nabla} A,B)\!) = (-1)^n (\!(A,\boldsymbol{\nabla} B)\!) + (\!(A,B)\!)_{\partial M}$$

Following from the above

$$\langle\!\langle \boldsymbol{\nabla} A, B \rangle\!\rangle = -\langle\!\langle A, \boldsymbol{\nabla} B \rangle\!\rangle + \langle\!\langle A, \boldsymbol{\nu} B \rangle\!\rangle.$$

Subsection 1

Monogenic fields



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Cauchy integral

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- There exists a vector-valued Cauchy kernel G_x where $\nabla G_x = \delta_x$.
- Given $A \in \mathcal{M}(M)$, the Cauchy integral is

$$A(x) = (-1)^{n-1} (A, G_x)_{\partial M}^{\perp}.$$

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■ Scalar part of the above is the double layer potential.

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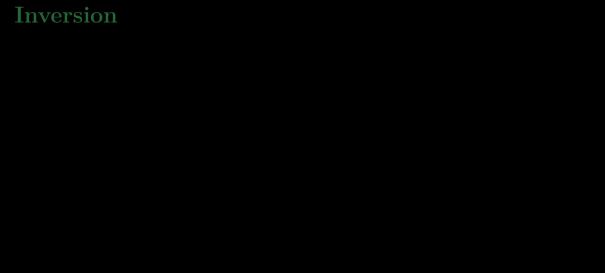
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- Cauchy integral is evaluation and an isomorphism from $\operatorname{tr} \mathcal{M}(M)$.



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■ In a region $M \subset \mathbb{R}^3$ take a vector field J,

$$\mathrm{BS}(\boldsymbol{J})(\boldsymbol{x}) = \left\langle (\boldsymbol{J}, G_{\boldsymbol{x}})^{\perp} \right\rangle_2 = \frac{1}{4\pi} \int_{N^3} \boldsymbol{J}(\boldsymbol{y}) \wedge \frac{\boldsymbol{x}' - \boldsymbol{x}}{|\boldsymbol{x}' - \boldsymbol{x}|^3} d\mu_{N^3}(\boldsymbol{x}').$$

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$$A(x) = (-1)^{n-1} (B, G_x)^{\perp}.$$

■ In a region $M \subset \mathbb{R}^3$ take a vector field J,

$$BS(\boldsymbol{J})(\boldsymbol{x}) = \left\langle (\boldsymbol{J}, G_{\boldsymbol{x}})^{\perp} \right\rangle_{2} = \frac{1}{4\pi} \int_{N^{3}} \boldsymbol{J}(\boldsymbol{y}) \wedge \frac{\boldsymbol{x}' - \boldsymbol{x}}{|\boldsymbol{x}' - \boldsymbol{x}|^{3}} d\mu_{N^{3}}(\boldsymbol{x}').$$

■ This is the Biot-Savart formula which recovers magnetic field from current.

Section 3

Hodge theory

Idea

Hodge theory relates analysis to topology. $\,$

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Put some pictures I've already made such as solid torus.

Product on cohomologies

Proposition

The contraction $\ \ \, \lrcorner \,$ is a product on cohomologies by:

- \blacksquare $\lrcorner: H^r(M) \times H^s(M) \to H^{s-r}(M);$
- $\blacksquare \ \ \lrcorner : H^r(M, \partial M) \times H^s(M, \partial M) \to H^{s-r}(M, \partial M);$
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■ This product is equivalent to the mixed cup product in [Shonkwiler, 2009].

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■ But, $\mathcal{M}(M) \neq \bigoplus_{j=1}^n \mathcal{M}^j(M)$.

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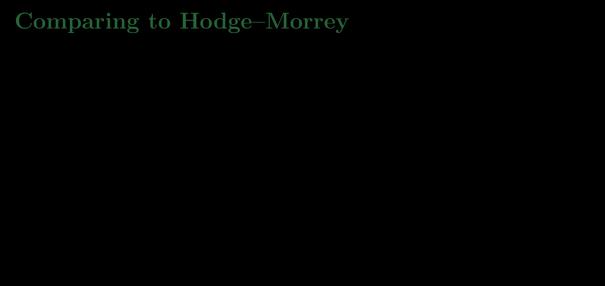
■ Define the *Dirac fields* $\nabla \mathfrak{X}(M)$ as

$$\nabla \mathfrak{X}(M) := {\nabla A \mid A \in \mathfrak{X}(M) \text{ and } A|_{\partial M} = 0};$$

Theorem: Clifford–Hodge Decomposition

The space of multivector fields $\mathfrak{X}(M)$ has the orthogonal decomposition

$$\mathfrak{X}(M)=\mathcal{M}(M)\oplusoldsymbol{
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Comparing to Hodge–Morrey

■ From Hodge–Morrey

$$\mathfrak{X}(M) = \sum_{r=0}^{\kappa} \underbrace{\mathcal{E}_D^r(M)}_{\operatorname{Im}(\nabla \wedge)} \oplus \underbrace{\mathcal{C}_N^r(M)}_{\operatorname{Im}(\nabla \cdot)} \oplus \underbrace{\mathcal{H}^r(M)}_{\operatorname{Ker}(\nabla)}.$$

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■ But the Clifford-Hodge-Morrey is not filtered by grades

$$\mathfrak{X}(M) = \mathcal{M}(M) \oplus \nabla \mathfrak{X}(M).$$

Section 4

Tomography



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- **Question:** Can we determine (M, σ) from Λ ?

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- Solved in dimensions $n \geq 3$ when M is an analytic manifold [Lassas, Taylor, Uhlmann: 2003].
- The smooth cases is still unsolved.

