MATH474 Homework #6

Colin Roberts

December 4, 2015

1. The goal of this problem is to determine which surfaces of revolution have constant Gaussian curvature. Suppose the surface of revolution Σ has parameterization

$$\vec{\mathbf{x}}(u,v) = (\phi(v)\cos[u], \phi(v)\sin[u], \psi(v)),$$

where the template curve $\alpha(s) = (\phi(s), 0, \psi(s))$ is parameterized by arclength (which of course means that $(\phi')^2 + (\psi')^2 = 1$). The goal is to solve for ϕ and ψ so that Σ has constant Gaussian curvature K.

(a) Prove that ϕ and ψ satisfy

$$\phi''(v) + K\phi(v) = 0 \text{ and } \psi(v) = \int_0^v \sqrt{1-(\phi'(t))^2}dt$$

Solution(a). We know K in terms of E, G and their derivatives from Gauss when $F = 0$.

$$K = \frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right)$$

So we just need to find E and G then their derivatives and make sure $F = 0$.

$$\vec{\mathbf{x}}_u = (-\phi(v)\sin u, \phi(v)\cos u, 0)$$

$$\vec{\mathbf{x}}_v = (\phi'(v)\cos u, \phi'(v)\sin u, \psi'(v))$$

$$\implies E = \phi^2 \text{ and } G = (\phi')^2 + (\psi')^2 = 1 \text{ and } F = 0$$

If we plug these into the equation above, then we receive the following:

$$K = \frac{-1}{2\sqrt{EG}}\left(\left(\frac{2\phi\phi'}{\sqrt{\phi^2}}\right)_v + 0\right)$$

$$K = \frac{-(\phi')_v}{\phi} = \frac{-\phi''}{\phi}$$

$$\implies \phi'' + K\phi = 0$$

For the next part, we start with,

$$(\phi')^2 + (\psi')^2 = 1$$

Which gives,

$$\psi' = \sqrt{1-(\phi')^2}$$

Which if we integrate yields,

$$\psi(v) = \int_0^v \sqrt{1-(\phi'(t))}dt$$

Which is the expression we wanted.

(b)Now assume $K = 1$ and show that, assuming the initial condition $\phi'(0) = 0$, the solutions of the equations from (a) are

$$\phi(v) = C\cos[v] \text{ and } \phi(v) = \int_0^v \sqrt{1 - C^2\sin^2 t}dt,$$

where C is a constant. Obviously, $\psi(v)$ is not defined for all v; find the domain of $\psi(v)$ (which depends on C) and sketch the curve $\alpha(s) = (\phi(s), 0, \psi(s))$ for $C < 1, C = 1$ and $C > 1$ (feel free to use Mathematica, Maple, Matlab, Wolfram Alpha, etc. for this). Show that only the $C = 1$ surface can be rotated around the z-axis to get a compact regular surface.

(c) Now assume $K = -1$ and show that ψ and ϕ satisfy one of the following set of equations:

$$\phi(v) = C\cosh v \text{ and } \psi(v) = \int_0^v \sqrt{1 - C^2\sinh^2 t}dt \tag{1}$$

$$\phi(v) = C\sinh v \text{ and } \psi(v) = \int_0^v \sqrt{1 - C^2\cosh^2 t}dt \tag{2}$$

$$\phi(v) = \exp v \text{ and } \psi(v) = \int_0^v \sqrt{1 - C^2\exp 2t}dt \tag{3}$$

In each case, determine the domain of $\psi(v)$ and sketch the resulting surface.

(d) Finally, assume $K = 0$. Prove that the only solutions are the cylinder, the cone, and the plane.

2. Show that if $F = 0$ then the Gaussian curvature K of a surface Σ is given by

$$K = -\frac{1}{2\sqrt{EG}}\left[\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right) + \frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right].$$

3. Let Σ be an oriented regular surface and let $\alpha(s)$ be an arclength parameterized curve on σ. Since α lies on Σ, we know that $\alpha' = T(s) \in T_{\alpha(s)}\Sigma$, and in particular $T(s)$ is perpendicular to the surface normal $N_\Sigma(s)$.
The *Darboux frame* of α is defined to be the triple of vectors

$$(T(s), V(s) = N_\Sigma(s) \times T(s), N_\Sigma(s)).$$

Like the Frenet frame, this frame's derivatives give information about the local geometry of α, but now that information relates also to how α lies in Σ.

(a) Show that the Darboux frame satisfies a system of equations vaguely similar to the Frenet equations:

$$\begin{array}{rccc} T' = & & \alpha(s)V(s) & +N_\Sigma(s) \\ V' = & -\alpha(s)T & & +c(s)N_\Sigma(s) \\ N'_\Sigma = & -b(s)T & -c(s)V(s) & \end{array}$$

for some coefficient functions $a(s)$, $b(s)$, and $c(s)$, which interpret in the following parts.

(b) Show that $c(s) = -\langle N_\Sigma, V\rangle$. In particular, α is a line of curvature if and only if $c(s) = 0$.
The function $-c(s)$ is called the *geodesic torsion* for obvious reasons.

(c) Show that $b(s)$ is the normal curvature κ_n of α.

(d) Show that $\alpha(s)$ is the geodesic curvature κ_g of α.

4. Let Σ be the hyperboloid of revolution

$$\vec{\mathbf{x}}(u,v) = (\cosh v \cos u, \cosh v \sin u, \sinh v),$$

which can also be described implicitly by the equation $x^2 + y^2 - z^2 = 1$. Suppose $\alpha(s)$ is a geodesic on Σ which makes the angle $\phi(s)$ with the $\vec{\mathbf{x}}_u$ direction at the point $\alpha(s) = \vec{\mathbf{x}}(u(s), v(s))$ and that the angle ϕ satisfies

$$\cos(\phi(s)) = \frac{1}{\cosh(v(s))}.$$

Show that the geodesic α spirals asymptotically into the circle $v = 0$.