MATH 570, Homework 9

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October 26, 2017

Solutions

Problem 1. Describe the homomorphism $f_*\colon \pi_1(S^1, 1) \to \pi_1(S^1, f(1))$, i.e., $f_*\colon \mathbb{Z} \to \mathbb{Z}$, induced by the following maps $f\colon S^1 \to S^1$, where we're thinking of the circle in complex coordinates ($S^1 = \{e^{i\theta} \in \mathbb{C} \mid 0 \le \theta < 2\pi\}$). Your answer should be a definition of of $f_*(m) \in \mathbb{Z}$ for each input $m \in \mathbb{Z}$. You do not need to justify your answer.

(a) $f(e^{i\theta}) = e^{i\theta}$.

(b) $f(e^{i\theta}) = e^{-i\theta}$.

(c) $f(e^{i\theta}) = \begin{cases} e^{i\theta} & \text{if } 0 \le \theta \le \pi, \\ e^{i(2\pi-\theta)} & \text{if } \pi \le \theta \le 2\pi. \end{cases}$

(d) $f(e^{i\theta}) = e^{in\theta}$, for some fixed $n \in \mathbb{Z}$.

(e) $f(e^{i\theta}) = e^{i(\theta+\pi)}$.

(a) This induces $f_*(m) = m$.

(b) This induces $f_*(m) = -m$.

(c) This induces $f_*(m) = 0$.

(d) This induces $f_*(m) = nm$.

(e) This induces $f_*(m) = m$.

Problem 2. Draw or define an abstract simplicial complex whose geometric realization is homeomorphic to

(a) a torus,

(b) a Klein bottle, and

(c) a projective plane.

(a) Torus

(b) Klein Bottle

(c) Projective Plane ($\mathbb{RP}^2$)

Problem 3. Construct a connected graph X (i.e. a connected 1-dimensional CW complex X) and continuous maps $f, g\colon X \to X$ such that $f \circ g = \mathrm{Id}_X$ but f and g do not induce isomorphisms on $pi_1(X)$.

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Proof. Let X be a topological space and define X by picking $p \in S^1$ and taking $X = \coprod_{i\in\mathbb{N}} S^1/\{p\}$, so X is a countable wedge sum of circles (identified at a single vertex p). Then consider $f\colon X \to X$ that maps the ith loop to the $(i-1)$th loop. We have that $\pi_1(X,p) = \langle a_1, a_2, \dots \rangle$ is the free group with $\mathbb{N}$ amount of generators (i.e., a_i for $i \in \mathbb{N}$). Then define $f\colon X \to X$ that maps the ith loop to the $(i-1)$th loop and the first loop to the point p and $g\colon X \to X$ that maps the ith loop to the $(i+1)$th loop. Then $f_*(a_i) = (a_{i+1})$ and $g_*(a_i) = (a_{i-1})$ if $i \geq 2$ and $g(a_1) = p$. Then we have that niether f or g induce isomorphisms on $\pi_1(X,p)$ since f is not surjective and g is not injective. Yet, $f \circ g$ induces the identity on $\pi_1(X,p)$. $\square$

Problem 4. Let $f\colon X \to Y$ be a continuous injective function with X compact and Y Hausdorff. Prove that X and $f(X)$ are homeomorphic.

Here are two proofs. The second is shorter and clearer. (Thanks to: Brenden and Tanner)

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Proof. Since f is injective we have that f is a bijection between X and $f(X)$. Thus, we just need to show that f^{-1} is continuous. So, let $U \in X$ be open and then consider $f(U) \in Y$. Since f is injective, any $x \in U$ is mapped to a unique point $f(x) = y \in f(X)$. Since Y Hausdorff, we have that there exist disjoint neighborhoods about y and any $p \in f(X) \setminus f(U)$. Note that X compact and f continuous implies that $f(X)$ is also compact. Let $N_p(y)$ be an open set containing y such that it is disjoint from the open set $V_p \in f(X) \setminus f(U)$ containing the point p. Note then that this collection $\{V_p\}_{p \in f(X) \setminus f(U)}$ with the subspace topology (meaning each $V_p = O_p \cap f(X) \setminus f(U)$ for some O_p open in X) is a cover of $f(X) \setminus f(U)$ and has a finite subcover given by the collection $\{V_{p_i}\}_{i=1,\dots,n}$ since $f(X)$ is compact. Finally we have that $\cap_{i=1}^{n} N_{p_i}$ is an open set containing y in $f(U)$ disjoint from $f(X) \setminus f(U)$ and this implies that $f(U)$ is open in $f(X)$. Thus, X and $f(X)$ are homeomorphic. $\square$

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Proof. Since f maps onto $f(X)$ it suffices to show that f is a closed map. Let $C \subseteq X$ be closed. Then since X is compact, C is compact. By continuity, $f(C) \subseteq f(X)$ is compact. Since Y is Hausdorff, $f(X)$ is Hausdorff. Since compact subsets of Hausdorff spaces are closed, it follows that $f(C)$ is closed in $f(X)$. Therefore f is a closed map, which shows that $X \cong f(X)$. $\square$