MATH 560, Homework 6

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Solutions

Problem 1. Verify that the Fourier vectors are eigenvectors of circulant matrices. What are the eigenvalues?

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Proof. Let F be an $n\times n$ circulant matrix given by

$$F=\begin{bmatrix} c_1 & c_n & c_{n-1} & \cdots & c_2 \\ c_2 & c_1 & c_n & \cdots & c_3 \\ c_3 & c_2 & c_1 & \cdots & c_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_1 \end{bmatrix}.$$

Then consider the Fourier vector $v_j=(1,z^j,...,z^{(n-1)j})^T$. Then

$$Fv_j=\begin{bmatrix} c_1 & c_n & c_{n-1} & \cdots & c_2 \\ c_2 & c_1 & c_n & \cdots & c_3 \\ c_3 & c_2 & c_1 & \cdots & c_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_1 \end{bmatrix}\begin{bmatrix} 1 \\ z^j \\ \vdots \\ z^{(n-2)j} \\ z^{(n-1)j} \end{bmatrix}=\begin{bmatrix} c_1+c_2z^j+...+c_2z^{(n-1)j} \\ c_2+c_1z^j+...+c_3z^{(n-1)j} \\ \vdots \\ c_n+c_{n-2}z^j+...+c_{n-1}z^{(n-1)j} \\ c_{n-1}+c_{n-2}z^j...+c_1z^{(n-1)j} \end{bmatrix}=\lambda_j\begin{bmatrix} 1 \\ z^j \\ \vdots \\ z^{(n-2)j} \\ z^{(n-1)j} \end{bmatrix}.$$

Where $\lambda_j=c_1+c_nz^j+c_{n-1}z^{2j}+...+c_2z^{(n-1)j}$ are the eigenvalues. □

Problem 2. **(§5.2 Problem 14 (b))** Find the general solution to each system of differential equations.

$$x_1' = 8x_1 + 10x_2$$
$$x_2' = -5x_1 - 7x_2$$

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Solution. If we let $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $x'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}$, and $A = \begin{bmatrix} 8 & 10 \\ -5 & -7 \end{bmatrix}$. Then $x'(t) = Ax(t)$. We then diagonalize A. So

$$\det\left(\begin{bmatrix} 8-\lambda & 10 \\ -5 & -7-\lambda \end{bmatrix}\right) = \lambda^2 - \lambda - 6 \qquad = (\lambda-3)(\lambda+2).$$

So we have eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -2$. Next we get the eigenvectors as follows.

$$\begin{bmatrix} 5 & 10 \\ -5 & -7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3a \\ 3b \end{bmatrix}$$
$$\implies 5a + 10b = 3a$$
$$-5a - 5b = 3b.$$

Which gives us the eigenvector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ corresponding to $\lambda_1 = 3$. Then we find the next eigenvector for $\lambda_2 = -2$.

$$\begin{bmatrix} 5 & 10 \\ -5 & -7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2a \\ -2b \end{bmatrix}$$
$$\implies 5a + 10b = -2a$$
$$-5a - 5b = -2b.$$

Which gives us the eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ corresponding to $\lambda_2 = -2$. This gives us that $Q = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$ and $Q^{-1} = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}$. So now we can write $Q^{-1}x'(t) = DQ^{-1}x(t)$ for D a diagonal matrix. Which yields equations

$$y_1' = 3y_1$$
$$y_2' = -2y_1.$$

This gives us $y_1(t) = c_1e^{3t}$ and $y_2(t) = c_2e^{-2t}$. Then $Qy(t) = x(t)$ so

$$\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1e^{3t} \\ c_2e^{-2t} \end{bmatrix} = \begin{bmatrix} -2c_1e^{3t} - c_2e^{-2t} \\ c_1^{3t} - c_2e^{-2t} \end{bmatrix} = x(t).$$

This is our general solution for $x(t)$. ■

Problem 3. (§5.2 Problem 18.)

(a) Prove that if T and U are simultaneously diagonalizable operators, then T and U commute.

(b) Show that if A and B are simultaneously diagonalizable matrices, then A and B commute.

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Proof (a). First, let's show that if we have two diagonal $n \times n$ matrices A and B then $AB = BA$. Let

$$A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$
$$B = \begin{bmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_n \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} \lambda_1\gamma_1 & & \\ & \ddots & \\ & & \lambda_n\gamma_n \end{bmatrix} = \begin{bmatrix} \gamma_1\lambda_1 & & \\ & \ddots & \\ & & \gamma_n\lambda_n \end{bmatrix} = BA.$$

Now, let T and U be simultaneously diagonalizable. Thus for some basis β, $[T]_\beta$ and $[U]_\beta$ are diagonal. Then

$$\begin{aligned} [T]_\beta[U]_\beta &= [U]_\beta[T]_\beta \quad \text{since both are diagonal} \\ \implies [TU]_\beta &= [UT]_\beta \\ \implies TU &= UT. \end{aligned}$$
□

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Proof (b). Since A and B are simultaneously diagonalizable, we have that $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal for some Q. Then

$$\begin{aligned} (Q^{-1}AQ)(Q^{-1}BQ) &= (Q^{-1}BQ)(Q^{-1}BQ)(Q^{-1}AQ) \quad \text{since both are diagonal} \\ \implies Q^{-1}ABQ &= Q^{-1}BAQ \\ ABQ &= BAQ \\ AB &= BA. \end{aligned}$$
□

Problem 4. (§5.2 Problem 19.) Let T be a diagonalizable linear operator on a finite-dimensional vector space, and let m be any positive integer. Prove that T and T^m are simultaneously diagonalizable.

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Proof. Since T is diagonal we have that for some basis β that $[T]_\beta = Q^{-1}[T]_\beta Q$ and $Q^{-1}[T]_\beta Q$ is diagonal. Then

$$(Q^{-1}[T]_\beta Q)^m = Q^{-1}[T]_\beta^m Q.$$

Hence $[T]_\beta^m$ is simultaneously diagonalizable. $\square$

Problem 5. (§5.4 Problem 2.) For each of the following linear operators T on the vector space V, determine whether the given subspace W is a T-invariant subspace of V.

(a) $V = P_3(\mathbb{R})$, $T(f(x)) = f'(x)$, and $W = P_2(\mathbb{R})$

(b) $V = P(\mathbb{R})$, $T(f(x)) = xf(x)$, and $W = P_2(\mathbb{R})$

(c) $V = \mathbb{R}^3$, $T(a,b,c) = (a+b+c, a+b+c, a+b+c)$, and $W = \{(t,t,t) | t \in \mathbb{R}\}$

(d) $V = C([0,1])$, $T(f(t)) = \left[\int_0^1 f(x)dx\right] t$, and $W = \{f \in V | f(t) = at + b \text{ for } a \text{ and } b\}$

(e) $V = \mathbf{M}_{2\times 2}(\mathbb{R})$, $T(A) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A$, and $W = \{A \in V | A^t = A\}$

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Proof (a). Let $f(x) = a_0 + a_1 x + a_2 x^2 \in P_2(\mathbb{R}) = W$. Then

$$T(f(x)) = a_1 + 2a_2 x \in P_2(\mathbb{R}).$$

Hence W is T invariant. □

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Proof (b). Let $f(x) = a_0 + a_1 x + a_2 x^2 \in P_2(\mathbb{R}) = W$. Then

$$T(f(x)) = a_0 x + a_1 x^2 + a_2 x^2 \notin P_2(\mathbb{R}).$$

Hence W is not T invariant. □

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Proof (c). Let $(a,a,a) \in W$. Then

$$T(a,a,a) = (3a, 3a, 3a) \in W.$$

Hence W is T invariant. □

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Proof (d). Let $f(t) = at + b \in W$. Then

$$\begin{aligned} T(f(t)) = \left[\int_0^1 ax + bdx\right] t &= \left(\frac{a}{2}x^2 + bx\right)\Big|_0^1 t \\ &= \left(\frac{a}{2} + b\right) t \in W. \end{aligned}$$

Hence W is T invariant. □

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Proof (e). Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix} \in W$. Then

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{12} & A_{22} \\ A_{11} & A_{12} \end{bmatrix} \notin W.$$

Hence W is not T invariant. □

Problem 6. (§5.4 Problem 3.) Let T be a linear operator on a finite-dimensional vector space V. Prove that the following subspaces are T-invariant.

(a) $\{0\}$ and V

(b) $\mathcal{N}(T)$ and $\mathcal{R}(T)$

(c) E_λ for any eigenvalue λ of T

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Proof (a). We have that $T(0) = 0$ so then $\{0\}$ is surely invariant. Then since T is an operator and by definition $T \colon V \to V$, we have that for any $v \in V$ that $T(v) = w \in V$. So both $\{0\}$ and V are T invariant. □

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Proof (b). Let $v \in \mathcal{N}(T)$, then $T(v) = 0 \in \mathcal{N}(T)$. Thus $\mathcal{N}(T)$ is T invariant. Next let $u \neq 0 \in \mathcal{R}(T)$. Then if $T(u) \notin \mathcal{R}(T)$ we have that $T(u) \in \mathcal{N}(T)$ and $T(u) = 0$. Since $V = \mathcal{R}(T) \oplus \mathcal{N}(T)$, we have that $u = 0$ which contradicts $u \neq 0$ and thus $T(u) \in \mathcal{R}(T)$. If $u = 0$ then $T(u) = 0$ and $0 \in \mathcal{R}(T)$ and thus we have that $\mathcal{R}(T)$ is T invariant. □

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Proof (c). Let $v \in E_\lambda$. Then $T(v) = \lambda v \in E_\lambda$. So E_λ is T invariant. □

Problem 7. (§5.4 Problem 5.) Let T be a lienar operator on a vector space V. Prove that the intersection of any collection of T-invariant subspaces of V is a T-invariant subspace of V.

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Proof. Let U and W be T-invariant subspaces of V. Then let $v \in U \cap W$ and consider $T(v)$. Since $v \in U \cap W$ then $v \in U$ and $v \in W$ and thus since both are T-invariant, $T(v) \in U$ and $T(v) \in W$. Thus $T(v) \in U \cap W$ and thus $U \cap W$ is T-invariant. $\square$

Problem 8. (§5.4 Problem 11.) Let T be a linear operator on a vector space V, and let v be a nonzero vector in V, and let W be the T-cyclic subspace of V generated by v. Prove that

(a) W is T-invariant

(b) Any T-invariant subspace of V containing v also contains W.

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Proof (a). For $w \in W = \text{span}(\{v, T(v), T^2(v), \ldots\})$, we have that $w = \lambda_1 v + \lambda_2 T(v) + \lambda_3 T^2(v) + \ldots$ so then $T(w) = \lambda_1 T(v) + \lambda_2 T^2(v) + \ldots \in W$. So W is T invariant. □

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Proof (b). Let U be a T invariant subspace with $v \in U$. Thus

$$\begin{aligned}
& T(v) \in U \\
\iff\ & T(T(v)) \in U && \text{since } U \text{ is } T \text{ invariant} \\
\iff\ & T^2(v) \in U \\
\iff\ & T^3(v) \in U && \text{since } U \text{ is } T \text{ invariant} \\
& \vdots \\
\iff\ & T^n(v) \in U && \text{for all } n \in \mathbb{N} \text{ since } T^{n-1}(v) \in U.
\end{aligned}$$

□

Problem 9. (§5.4 Problem 17.) Let A be an $n \times n$ matrix. Prove that

$$\dim(\operatorname{span}(\{I_n, A, A^2, \ldots\})) \leq n.$$

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Proof. Let $T\colon V \to V$ which is realized by $[T]_\beta = A \in M_{n\times n}(\mathbb{F})$ for $\dim(V) = n$. We have from Theorem 5.22 that for a W T-cyclic subspace of V generated by a nonzero vector v that a T-invariant subspace of dimension k has a basis $\{v, T(v), T^2(v), \ldots, T^{k-1}(v)\}$. Then $\{I_n[v]_\beta, A[v]_\beta, \ldots, A^{k-1}[v]_\beta\}$ is the largest linearly independent set of vectors for W by how we defined $A = [T]_\beta$. Note that

$$\dim(\operatorname{span}(\{I_n[v]_\beta, A[v]_\beta, \ldots, A^{k-1}[v]_\beta\})) = k = \dim(W).$$

This implies that

$$\dim(\operatorname{span}(\{I_n, A, \ldots, A^{k-1}\})) = k = \dim(W),$$

and any other matrix A^m for $m \geq k$ would make the set linearly dependent if added. Since W is a subspace of V we have that $\dim(W) \leq n$ which implies that

$$\dim(\operatorname{span}(\{I_n, A, \ldots, A^{k-1}\})) \leq n. \qquad \square$$

Problem 10. (§5.4 Problem 18.) Let A be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0.$$

(a) Prove that A is invertible if and only if $a_0 \neq 0$.

(b) Prove that if A is invertible, then

$$A^{-1} = (-1/a_0)[(-1)^n A^{n-1} + a_{n-1} A^{n-2} + \dots + a_1 I_n].$$

(c) Use (b) to compute A^{-1} for

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix}$$

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Proof (a). For the forward direction, let A is invertible. Then no eigenvalues $t = 0$. Thus the characteristic polynomial does not have a factor of t (i.e., a zero root) which means that $a_0 \neq 0$. For the converse direction, let $a_0 \neq 0$ and thus $t = 0$ is not a root. Thus no eigenvalue is zero which means that A is invertible. □

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Proof (b). By Cayley-Hamilton we have

$$\begin{aligned} (-1)^n A^n + a_{n-1} A^{n-1} + \dots + a_1 A + a_0 I = 0 \\ a_0^{-1}((-1)^n A^n + a_{n-1} A^{n-1} + \dots + a_1 A) = I \\ a_0^{-1}((-1)^n A^{n-1} + a_{n-2} A^{n-1} + \dots + a_1 I) = A^{-1}. \end{aligned}$$

□

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Proof (c). We have that

$$\det(A - tI) = \left| \begin{bmatrix} 1-t & 2 & 1 \\ 0 & 2-t & 3 \\ 0 & 0 & -1-t \end{bmatrix} \right| = (1-t)(2-t)(-1-t) = -t^3 + 2t^2 + t - 2.$$

Then

$$A^{-1} = \frac{1}{2}(-A^2 + 2A + I).$$

So we have

$$\begin{aligned} A^{-1} &= \frac{1}{2}\left(-\begin{bmatrix} 1 & 6 & 6 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & -1 & -2 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & -1 \end{bmatrix}. \end{aligned}$$

Which is indeed the inverse of A. □