Commutative Banach Algebras of Multivectors from the Scalar Dirichlet-to-Neumann Operator

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November 10, 2020

Abstract

The problem of determining an unknown Riemannian manifold given the Dirichlet-to-Neumann (DN) operator is known as the Calderón problem. One method of solving this problem in the two dimensional case is through the Boundary Control method. There, one uses the DN operator to construct a Banach algebra of holomorphic functions on the manifold. The Gelfand transform of this algebra is then homeomorphic to the manifold. In higher dimensions, we replace the complex field with a Clifford algebra and use the DN operator to determine a $\mathrm{Spin}(n)$ invariant space of monogenic multivector fields. Using a power series representation for monogenic fields, one decomposes the space of monogenics into products of commutative algebras of $(0+2)$-vector fields constant on translations of planes and monogenic in $\mathbb{R}^n$. Using this decomposition, we define spinor characters on the space of monogenic fields that correspond to Dirac measures on the manifold. The set of these Dirac measures is then homeomorphic to the underlying manifold with the Gelfand topology.

Replace vector space V with $\mathbb{V}$.

1 Introduction

In 1980, Alberto Calderón proposed an inverse problem in his paper *On an inverse boundary value problem* calderon_inverse_2006 [7] where he asks if one can determine the conductivity matrix of a medium from Cauchy data supplied on the boundary. In dimensions $n > 2$, this is equivalent to determining a Riemannian manfiold up to isometry from the scalar Dirichlet-to-Neumann (DN) operator feldman_calderproblem_nodate, salo_calderon_nodate, uhlmann_inverse_2014 [10, 15, 19]. The DN operator takes any given Dirichlet boundary values and outputs the corresponding Neumann data of a solution to Laplace's equation in order to generate the relevant Cauchy data.

One approach to reconstructing the Riemannian metric in dimension $n = 2$ appears in belishev_calderon_2003 [2], where the author uses the Boundary–Control (BC) method to determine the manifold up to conformal class. Add in a bunch of other citations to the BC method. The BC method takes an algebraic approach. Specifically, the DN operator determines the algebra of holomorphic functions on M and realizes M as homeomorphic to the Gelfand spectrum of this commutative algebra. The metric g is then recovered after providing M with a complex structure. In dimension $n = 2$, the Laplace-Beltrami operator is conformally invariant, and this result cannot be improved. An attempt to generalize this approach to dimension $n = 3$ can be found in by replacing the complex structure with a quaternionic structure but this has not lead to a complete solution belishev_algebras_2017, belishev_algebraic_2019 [3, 4]. It has been shown that when M is the 3-dimensional ball in $\mathbb{R}^3$, there is an associated space of harmonic quaternion fields that has a quaternion spectrum homeomorphic to the ball. But, a connection to the DN operator has not been made, and this method has also not been generalized to higher dimensions.

In this paper, I show that there exists a space of spin characters $\mathfrak{M}$ acting on a $\mathrm{Spin}(n)$ invariant space of monogenic multivector fields on the n-dimensional ball that is homeomorphic to the ball. We then observe that this space of monogenics is determined from the DN map, and thus recover the ball up to homeomorphism from the boundary data. This is summarized in two main theorems.

Theorem 1. *The set of multiplicative $\mathfrak{spin}(n)$-linear functionals on the* $\mathrm{Spin}(n)$ *invariant space of monogenic fields $\mathcal{M}$ on the n-dimensional ball $\mathbb{B}$ is homeomorphic to $\mathbb{B}$ with the Gelfand topology.*

Theorem 2. *The scalar DN operator determines the* $\mathrm{Spin}(n)$ *invariant space of monogenic fields on regions in $\mathbb{R}^n$.*

The second theorem can be extended to Riemannian manifolds quite readily.

We first introduce the Clifford algebra setting. Given a vector space with an inner product, we can create the graded Clifford algebra. In particular, we extend these Clifford algebras to Clifford algebra valued functions (or multivector fields) on regions $M \subset \mathbb{R}^n$. Inside the multivector fields sit the even graded multivectors consisting of scalars, bivectors, and other $2k$-vectors. In $\mathbb{R}^2$ with the Euclidean inner product, this space is isomorphic to the $\mathbb{C}$-algebra and so the functions valued in this even sub-Clifford algebra can be thought of as complex valued functions. Clifford analysis generalizes the notion of holomorphicity to monogenicity and we find that monogenic functions lie in the kernel of the Dirac operator $\boldsymbol{\nabla}$ just as $\mathbb{C}$-holomorphic functions lie in the kernel of the Wirtinger derivative $\frac{\partial}{\partial \bar{z}}$. Moreover, one has that $\boldsymbol{\nabla}$ is the square root Laplace-Beltrami operator $\Delta = \boldsymbol{\nabla}^2$. Even monogenic multivector fields are $\mathrm{Spin}(n)$ invariant and each grade is harmonic (in the kernel of Δ).

When M is the n-ball, we have that space of even monogenics $\mathcal{M}$ which can be generated by the algebras of even graded B-planar monogenic biparavector fields (each field constant on translations of the B-plane in $\mathbb{R}^n$). Those generating subalgebras are individually isomorphic to the algebra of holomorphic functions on the complex unit disk $\mathbb{D}$. On these spaces, one can define $\mathfrak{spin}(n)$-linear multiplicative functionals $\mathfrak{M}$, referred to as spin characters. Each spin character is equivalent to a Dirac measure on the n-ball which, with the Gelfand topology, provide a homeomorphic copy of the n-ball.

The space of $(0+2)$-vector monogenics is found from the DN operator in the following sense. The DN operator determines a Hilbert transform on multivector fields that allows one to determine the monogenic conjugate bivector field b corresponding to a scalar solution u to the Laplace equation $\Delta u = 0$ so that $f = u + b$ is monogenic. Haven't actually done this yet Considering all smooth boundary conditions generates the relevant space of monogenics, from which we determine the space of spin characters. Thus, the DN operator provides a means of constructing a homeomorphic of the n-ball.

2 Preliminaries

2.1 Clifford algebras

The complex algebra $\mathbb{C}$ can be generalized in a handful of ways. Some of which can be found through the use of Clifford algebras and, more specifically, in geometric algebras. We define the more general Clifford algebras first and realize geometric algebras as particularly nice Clifford algebras with a quadratic form arising from an inner product. Elements of a geometric algebra are known as multivectors and these multivectors carry a wealth of geometric information in their algebraic structure. $\mathbb{C}$ itself can be realized as a special subalgebra of biparavectors in the geometric algebra on $\mathbb{R}^2$ with the Euclidean inner product and the quaternions $\mathbb{H}$ are realized as an analogous algebra on $\mathbb{R}^3$. In particular, both $\mathbb{C}$ and $\mathbb{H}$ arise as the 2- and 3-dimensional even Clifford groups Γ^+ respectively.

Formally, we let (V, Q) be an n-dimensional vector space V over some field K with an arbitrary quadratic form Q. The tensor algebra is given by

$$\mathcal{T}(V) \coloneqq \bigoplus_{j=0}^{\infty} V^{\otimes j} = K \bigoplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots,$$

where the elements (tensors) inherit a multiplication $\otimes$ (the tensor product). From the tensor algebra $\mathcal{T}(V)$, we can quotient by the ideal generated by $v \otimes v - Q(v)$ to define *Clifford algebra* $C\ell(V, Q)$. That is,

$$C\ell(V, Q) = \mathcal{T}(V) \ / \ \langle v \otimes v - Q(v) \rangle.$$

To see how the tensor product descends to the quotient, we let $e_1, \ldots, e_n$ be an arbitrary basis for V, then we can consider the tensor product of basis elements $e_i \otimes e_j$ which induces a product in the quotient $C\ell(V,Q)$ which we refer to as the *Clifford multiplication*. In this basis, we write this product as concatenation e_ie_j and define the multiplication by

$$e_ie_j = \begin{cases} Q(e_i) & \text{if } i=j, \\ e_i \wedge e_j & \text{if } i \neq j, \end{cases}$$

where $\wedge$ is the typical exterior product satisfying $v \wedge w = -w \wedge v$ for all $v, w \in V$. As a consequence, the exterior algebra $\bigwedge(V)$ can be realized as a subalgebra of any Clifford algebra over V or as a Clifford algebra with a trivial quadratic form $Q = 0$.

Note that $C\ell(V,Q)$ is a $\mathbb{Z}$-graded algebra with elements of grade-0 up to elements of grade-n. We refer to grade-0 elements as scalars, grade-1 elements as vectors, grade-2 elements as *bivectors*, grade-k elements as *k-vectors*, and grade-n elements as *pseudoscalars*. We denote the space of k-vectors by $C\ell(V,Q)^k$. For each grade there is a basis of $\binom{n}{k}$ *k-blades* which are k-vectors of the form

$$A_k = \prod_{j=1}^{k} v_j, \text{ for } v_j \in V.$$

For example, if $\dim(V) = 3$, then there are $\binom{3}{2} = 3$ 2-blades that form a basis for the bivectors. One particular choice given our vector basis of V would be the following list of 2-blades

$$B_{12} = e_1 \wedge e_2, \quad B_{13} = e_1 \wedge e_3, \quad B_{23} = e_e \wedge e_3.$$

We refer to an $(n-1)$-blade as a *pseudovector* and it should be noted that every $(n-1)$-vector is a pseudovector. In other literature, some will refer to a k-blade as a *simple* or a *decomposable* k-vector.

In general, an element $A \in C\ell(V,Q)$ is written as a linear combination of basis elements of all possible grades and we refer to A as a *multivector*. To extract the grade-k components of A, we use the notation

$$\langle A \rangle_k$$

to denote the grade-k components of the multivector A. Any multivector A can then be given by

$$A = \sum_{k=0}^{n} \langle A \rangle_k$$

which shows the decomposition

$$C\ell(V,Q) = \bigoplus_{j=0}^{n} C\ell(V,Q)^j.$$

For example, $A \in C\ell(\mathbb{R}^3, \|\cdot\|)$ is given by

$$A = a + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \beta_{12} B_{12} + \beta_{13} B_{13} + \beta_{23} B_{23} + r e_1 \wedge e_2 \wedge e_3$$

in general, and we have

$$\langle A \rangle_0 = a, \quad \langle A \rangle_1 = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3, \quad \langle A \rangle_2 = \beta_{12} B_{12} + \beta_{13} B_{13} + \beta_{23} B_{23}, \quad \langle A \rangle_3 = r e_1 \wedge e_2 \wedge e_3.$$

If A contains only grade-k components, then we say that A is *homogeneous*. For example, when we refer to vectors we realize them as homogeneous grade-1 multivectors and likewise we realize bivectors as homogeneous grade-2 multivectors. We also refer elements in

$$C\ell(V,Q)^{0+2} = C\ell(V,Q) \oplus C\ell(V,Q)^2$$

as *biparavectors*.

The Clifford multiplication of vectors can be extended to multiplication of vectors with homogeneous grade-k multivectors. In particular, given a vector $v \in C\ell(V,Q)$ and a homogeneous grade-k multivector $A_k \in C\ell(V,Q)$, we have

$$aA_k = \langle aA_k \rangle_{k-1} + \langle aA_k \rangle_{k+1}, \tag{1}$$

eq:vector_r

which decomposes the multiplication into a grade lowering *interior product* and a grade raising *exterior product*. This allows us to extend the Clifford multiplication further. Given a homogeneous grade-s multivector B_s, we have

$$A_k B_s = \langle A_r B_s \rangle_{|r-s|} + \langle A_r B_s \rangle_{|r-s|+2} + \cdots + \langle A_r B_s \rangle_{r+s}. \tag{2}$$

eq:general_

This rule for multiplication then allows for the multiplication of two general multivectors in $C\ell(V,Q)$.

Some specific graded elements of the above product are worth noting here,

$$A_k \cdot B_s \coloneqq \langle A_k B_s \rangle_{|k-s|} \tag{3}$$

$$A_k \wedge B_s \coloneqq \langle A_k B_s \rangle_{k+s} \tag{4}$$

$$A_k \rfloor B_s \coloneqq \langle A_k B_s \rangle_{s-k} \tag{5}$$

eq:left_co

$$A_k \lfloor B_s \coloneqq \langle A_k B_s \rangle_{k-s}. \tag{6}$$

eq:right_c

These products are particularly emphasized as many helpful identities used in this paper are phrased using these notions. Another key reason behind the additional multiplication symbols $\rfloor$ and $\lfloor$ is to avoid needing to pay special attention to the specific grade of each multivector in a product. The product $\cdot$ on A_k and B_s depends on k and s and as such given by either $\rfloor$ or $\lfloor$ but one must know k and s in order to define this product exactly.

We also have the identities

$$A_r \cdot B_s = A_r \rfloor B_s \qquad \text{if } k \leq s \tag{7}$$

eq:left_co

$$A_r \cdot B_s = A_r \lfloor B_s \qquad \text{if } k \geq s. \tag{8}$$

eq:right_c

For homogeneous k-vectors A_k and B_k, the products above simplify to

$$A_k \lfloor B_k = A_k \rfloor B_k = A_k \cdot B_s. \tag{9}$$

dot_equiva

Using this notation, for a vector α we have

$$\alpha A_k = \alpha \rfloor A_k + \alpha \wedge A_k, \tag{10}$$

so the $\cdot$ and $\lfloor$ notation coincide for left multiplication by vectors. If we are given two k-blades $A_k = \alpha_1 \wedge \cdots \wedge \alpha_k$ and $B_k = \beta_1 \wedge \cdots \wedge \beta_k$ we have

$$A_k \cdot B_k = \det(\alpha_i \cdot \beta_j)_{i,j=1}^{k}, \tag{11}$$

eq:dot_pro

dot_equivalent_contraction

which is equivalent to $A_k \rfloor B_k$ and $A_k \lfloor B_k$ through 9 and this is extended to all k-vectors as is typically seen when constructing the inner product of k-vectors. If we are given two bivectors B and B', then we have another special multiplication

$$B \times B' \coloneqq \langle BB' \rangle_2 = \frac{1}{2}(BB' - B'B), \tag{12}$$

eq:bivecto

which is the grade preserving anti-symmetric portion of the product BB'.

As discussed, $C\ell(V,Q)$ is naturally a $\mathbb{Z}$-graded algebra but we also find that it carries a $\mathbb{Z}/2\mathbb{Z}$-grading as well. This additional grading can be realized by sorting k-vectors in $C\ell(V,Q)$ into the sets where k is even or odd. We say a k-vector is *even* (resp. *odd*) k is even (resp. odd) and in general if a multivector A is a sum of only even (resp. odd) grade elements we also refer to A as even (resp. odd). Taking note of the multiplication defined in eq:general_clifford_multiplication 2, one can see that the multiplication of even multivectors with another even multivectors outputs an even multivector. Thus, the even multivectors form closed subalgebra of $C\ell(V,Q)$ which we denote by $C\ell(V,Q)^+$.

esentation **Example 2.1.**

- Let $V = \mathbb{R}^2$ and let the quadratic form Q be given by the Euclidean norm $Q(\cdot) = \|\cdot\|$. Let e_1 and e_2 be the standard unit vectors and note that we have 1 as the basis scalar, and $B_{12} = e_1 \wedge e_2 = e_1 e_2$ as the basis pseudoscalar. Thus, a general multivector m and r can be written as

$$m = m_0 + m_1 e_1 + m_2 e_2 + m_{12} B_{12}, \qquad r = r_0 + r_1 e_1 + r_2 e_2 + r_{12} B_{12}.$$

We can then multiply mr and find

$$\langle mr \rangle_0 = m_0 r_0 + m_1 r_1 + m_2 r_2 - m_{12} r_{12},$$

$$\langle mr \rangle_1 = (m_0 r_1 + m_1 r_0 - m_2 r_{12} + m_{12} r_2) e_1 + (m_0 r_2 + m_2 r_0 + m_1 r_{12} - m_{12} r_1) e_2,$$

and

$$\langle mr \rangle_2 = (m_1 r_2 - m_2 r_1) B_{12}.$$

Most notably, we see that $B_{12}^2 = -1$ and this allows us to consider a biparavector

$$z = x + y B_{12}$$

as a representation of the complex number $\zeta = x + iy$ in $\mathcal{G}_n^{0+2}$. Thus, the even subalgebra of this Clifford algebra is indeed isomorphic to the complex numbers $\mathbb{C}$.

- If $V = \mathbb{R}^n$, with $n \geq 2$, and with the analogous Q, then there are natural copies of $\mathbb{C}$ contained inside of $C\ell(V,Q)$. In particular, we have the isomorphism

  $$\mathbb{C} \cong \{\lambda + \beta B \mid \lambda, \beta \in C\ell(V,Q)^0,\ B \in C\ell(V,Q)^2,\ B^2 = -1.\},$$

  which shows that complex numbers arise as biparavectors. Given the standard basis $e_1, \dots, e_n$ we have copies of $\mathbb{C}$ for each of the $\binom{n}{2}$ unit bivectors B_{jk} with $k = 2, \dots, n$ and $j < k$. Note that $B_{jk}B_{jk} = -1$ and we have the representation of $\mathbb{C}$ since

  $$\zeta = x + yB,$$

  behaves as a complex number $z = x + iy$.

uaternions **Example 2.2.** Let $V = \mathbb{R}^3$ and $Q(\cdot) = \|\cdot\|$. Then, let

$$B_{23} = e_2e_3, \quad B_{31} = e_3e_1, \quad B_{12} = e_1e_2,$$

and note that we can write a even multivector as

$$q = a + \beta_{23}B_{23} + \beta_{31}B_{31} + \beta_{12}B_{12}.$$

Note as well that

$$B_{23}^2 = B_{31}^2 = B_{12}^2 = -1,$$

and

$$B_{23}B_{31}B_{12} = +1.$$

In this case, this even subalgebra is extremely close to being a copy of the quaternion algebra $\mathbb{H}$. Indeed, one can arrive at a representation of the quaternions by taking

$$\boldsymbol{i} \leftrightarrow B_{23}, \quad \boldsymbol{j} \leftrightarrow -B_{31} = B_{13}, \quad \boldsymbol{k} \leftrightarrow B_{12},$$

and noting that we then have $ijk = -1$ as well as $i^2 = j^2 = k^2 = -1$. A more in depth explanation is provided in [9].

Once again, quaternions arise naturally as parabivectors since we can put

$$q = \alpha + u_1B_{23} - u_2B_{13} + u_3B_{12},$$

and recover the necessary arithmetic seen in $\mathbb{H}$.

In the case where V has a (pseudo) inner $(\cdot,\cdot)$, we can induce a quadratic form Q by $Q(v) = (v,v)$ and give rise to a Clifford algebra $C\ell(V,Q)$. This is a special case and we refer to this type of Clifford algebra as a *geometric algebra*. We generally put $\mathcal{G}(V)$ and assume the inner product will be given alongside or will be clear from context. For example, when $V = \mathbb{R}^n$ and we define Q from the Euclidean inner product, we have $C\ell(V,Q) = \mathcal{G}(\mathbb{R}^n)$ and moreover we put $\mathcal{G}(\mathbb{R}^n) = \mathcal{G}_n$. For more information on the topic of geometric algebras see the classical text [12] or the text [9] which also provides a wide range of applications to physics problems. Both these sources include much of the other necessary preliminaries I cover in the remainder of this section. Finally, the paper [8] proves many of the useful identities I claimed above.

2.1.1 Duality and pseudoscalars

eudoscalars

For the remainder of this paper we will be working with geometric algebras with a positive definite inner product g. Given access to an inner product we have a natural isomorphism between V and V^* by the Riesz representation. Namely, given an arbitrary basis e_i for V there exists the dual basis f_i for V^* such that $f_i(e_j) = \delta_{ij}$. This dual basis resides inside V itself in the following manner. There is then a unique map $\sharp \colon V^* \to V$ with $f \mapsto f^\sharp$ such that

$$f_i^\sharp \cdot e_j = \delta_{ij},$$

where δ_{ij} is the Kronecker delta symbol. In terms of the geometric algebra, we put $e^i := f_i^\sharp$ and can note that e^i is simply a vector in the geometric algebra. For an arbitrary basis $e_1, \ldots, e_n$ for V, the coefficients for the inner product g are given by $g_{ij} = e_i \cdot e_j$ and we can put $e^i = g^{ij} e_j$ where g^{ij} is the coefficients to matrix inverse of g_{ij}. There is inverse isomorphism $\flat \colon V \to V^*$ given by $e \mapsto e^\flat$ satisfying

$$e_i^\flat(e_j) = \delta_{ij}.$$

Given these identifications, there is no need to distinguish between the vector space V and its dual V^* as it suffices to consider V itself with reciprocal basis elements e^i with the application of the scalar product.

A volume element can be defined by $\mu = e_1 \wedge e_2 \wedge \cdots \wedge e_n = \sqrt{|g|} I$ where $\sqrt{|g|}$ is the square root of the determinant of the matrix g_{ij} and I is the unit pseudoscalar. It follows that the unit pseudoscalar is given by $I = \frac{1}{\sqrt{|g|}} e_1 \wedge e_2 \wedge \cdots e_n$. We can define μ^{-1} such that $\mu^{-1}\mu = 1 = \mu\mu^{-1}$ and analogously I^{-1}. One can equivalently put $e^j = (-1)^{j-1} e_1 \wedge e_2 \wedge \cdots \wedge \check{e}_j \wedge \cdots \wedge e_n \mu^{-1}$ and note that this gives $\mu^{-1} = e^n \wedge \cdots \wedge e^1$. Conveniently, the unit pseudoscalar satisfies the relation

$$IA_k = (-1)^{k(n-1)} A_k I.$$

Thus, I commutes with the even subalgebra, and anticommutes with the odd subalgebra. Moreso, the pseudoscalar allows one to exchange the interior and exterior products as

$$(A_k \wedge B_s) I = A_k \cdot (B_s I) \tag{13}$$

eq:wedge_t

for homogeneous k and s-vectors A_k and B_s. The above holds true if we replace I with I^{-1} when working in spaces where g is positive definite due to the fact that I^{-1} differs only by a sign. If $B_s = C_{n-s} I$ then,

$$(A_k \cdot B_s) I^{-1} = A_k \cdot (C_{n-s} I) = (A_k \wedge C_{n-s}) I = (A_k \wedge (B_s I)) I,$$

and in particular

$$(A_k \cdot B_s) I^{-1} = A_k \wedge (B_s I). \tag{14}$$

eq:dot_to_

This shows the duality between the inner and exterior products. The duality extends further to provide an isomorphism between the spaces of k-vectors and $(n-k)$-vectors. For any k-vector A_k, we can take $A_k I^{-1} = B_{n-k}$ to get the corresponding $(n-k)$-vector B_{n-k}. It is under this isomorphism one can see that all pseudovectors are $(n-1)$-blades.

ss_product **Example 2.3.** Consider $\mathcal{G}_3$ with the standard orthonormal vector basis $e_1, \dots, e_n$ and Euclidean inner product. Then, we can define the *cross product* of two vectors u and v by

$$u \times v = (u \wedge v)I^{-1}.$$

The special fact of $\mathcal{G}_3$ is that vectors and bivectors (pseudoscalars in 3-dimensions) are dual to one another. One can also note that the vector $w = u \times v$ is sometimes refered to as axial and in other cases the pseudovector $u \wedge v$ is referred to as axial.

The $\times$ symbol is now overloaded from the bivector definition we saw prior to this example. But, referring back to Example ex:quaternions 2.2, we can realize the cross product of vectors as the antisymmetric product of bivectors

$$(uI^{-1}) \times (vI^{-1}).$$

The necessary relationships for the cross product are seen clearly on the products of the basis blades B_{23}, B_{31}, and B_{12}. In particular, $e_1 = B_{23}I^{-1}$, $e_2 = B_{31}I^{-1}$, and $e_3 = B_{12}I^{-1}$.

2.1.2 Reverse, inverses, and the Clifford and spin groups

We had used the notation $^{-1}$ to denote the inverse for the pseudoscalar, but there are other invertible elements in a geometrical algebra. In particular, all blades are invertible. From this, we can construct a group of all invertible elements referred to as the *Clifford group* Γ for a geometric algebra $\mathcal{G}$ by

$$\Gamma := \left\{ \prod_{j=1}^{k} v_i \mid k \in \mathbb{Z}^+,\ \forall j\ : 1 \leq j \leq k\ :\ v_i \in \mathbb{R}^n \text{ such that } |v_i| \neq 0 \right\}.$$

We refer to elements of the Clifford group as *Clifford multivectors*. For any Clifford multivectors $A = v_1 \cdots v_k$ in the group Γ, we have that multiplicative inverse A^{-1} is given by $A^{-1} = v^k \dots v^1$ as we can see that $A^{-1}A = AA^{-1} = 1$ by construction. Of note is the fact that all scalars, vectors, pseudovectors, and pseudoscalars are always in the Clifford group and have multiplicative inverses. The inverse of a vector v is given by $\frac{v}{v \cdot v}$. It becomes useful to define the *reverse* $\dagger$ such that $A^\dagger = v_k \cdots v_1$. For a k-blade A_k, the reverse also satisfies the relationship

$$A_k^\dagger = (-1)^{k(k-1)/2} A_k. \tag{15}$$ eq:reverse_

One can then see that the inverse for the unit pseudoscalar is $I^{-1} = I^\dagger$ which is an identification I will often use. One can see that the inverse of an element of the Clifford group A is the reverse of the corresponding product of reciprocal vectors since $A^{-1} = (v^1 \cdots v^k)^\dagger$. Note as well that elements $s \in \Gamma^+$ act as rotations on $A \in \mathcal{G}_n$ given the conjugate action

$$A \mapsto sAs^{-1}.$$

In fact, all nonzero vectors $v \in \Gamma$ define a reflection in the hyperplane perpendicular to v via the same conjugation action above.

Following these realizations, one can see that the Clifford group contains important subgroups such as the orthogonal and special orthogonal groups as quotients

$$\mathrm{O}(n) \cong \Gamma/\mathbb{R} \qquad \text{and} \qquad \mathrm{SO}(n) \cong \Gamma/\mathbb{R}.$$

This motivatives the following definition.

Definition 2.1. The *Clifford norm* $\|\cdot\|$ for $s \in \Gamma$ is given by

$$\|s\|^2 := ss^\dagger.$$

Note that for vectors the Clifford norm corresponds with the norm induced from the inner product in that with a vector v we have $\|v\| = vv^\dagger = v \cdot v$. We also give the name *unit* to k-blades A_k with unit spinor norm $1 = \|A_k\|$. We can also see that

$$\|\mu\| = \sqrt{|g|}, \tag{16}$$ eq:pseudos

and so

$$\|I\| = 1.$$

With this, we have the *pin* and *spin groups*

$$\mathrm{Pin}(n) := \{s \in \Gamma \mid \|s\| = 1\}.$$
$$\mathrm{Spin}(n) := \{s \in \Gamma^+ \mid \|s\| = 1\}.$$

Moreover,

$$\mathrm{Pin}(n) \cong \Gamma/\mathbb{R}^+ \qquad \text{and} \qquad \mathrm{Spin}(n) \cong \Gamma^+/\mathbb{R}^+.$$

The spin group $\mathrm{Spin}(n)$ is a Lie group and its associated Lie algebra is denoted by $\mathfrak{spin}(n)$. In particular, the $\mathfrak{spin}(n)$ is isomorphic to the algebra of bivectors with the antisymmetric product $\times$. Then, for any bivector B, we have an element in the spin group given by

$$e^B = \sum_{j=0}^{\infty} \frac{B^n}{n!}.$$

Fundamentally, $\mathrm{Spin}(n)$ acts on the even subalgebra $\mathcal{G}_n^+$. A *spinor* ψ is an element that transforms under a left action of an element of $\mathrm{Spin}(n)$ to produce another spinor. In terms of geometric algebra, a spinor is simply an even multivector. Of note are the two cases we have had as examples before.

Example 2.4. Consider $\mathcal{G}_2$ and note that we have shown the algebra of spinors $\mathcal{G}_2^+$ is isomorphic to the complex numbers $\mathbb{C}$. Indeed, there is one unit 2-blade B_{12} in $\mathcal{G}_2$ to form the spin algebra $\mathfrak{spin}(2) \cong \mathbb{R}$ and as a consequence all unit norm elements in $\mathcal{G}_2^+$ can be written as

$$e^{\theta B_{12}} = \sum_{n=0}^{\infty} \frac{\theta B_{12}}{n!} = \cos(\theta) + B_{12}\sin(\theta),$$

where θB_{12} is a general bivector in $\mathcal{G}_2$. Hence, we arrive at $\mathrm{Spin}(2) \cong \mathrm{U}(1)$. Any element in $\mathbb{C}$ is also a scaled version of an element of the spin group $\mathrm{Spin}(2)$. Hence, we can use a spin representation for an element in $\mathbb{C}$ via $z = re^{\theta B_{12}} \in R \times \mathrm{Spin}(2)$. This special case shows that parabivectors in $\mathcal{G}_2$ have a unique spin representation and they are spinors as well.

Example 2.5. Consider $\mathcal{G}_3$ and note that we have shown the spinors $\mathcal{G}_3^+$ are isomorphic to the quaternion $\mathbb{H}$ algebra. We also realize $\mathbb{H}$ as scalar copies of elements of $\mathrm{Spin}(3) \cong \mathrm{Sp}(1)$. That is to say that $\mathbb{H} \cong \mathbb{R} \times \mathrm{Spin}(3)$. Indeed, since elements of $\mathcal{G}_3^+$ are simply biparavectors, the biparavectors once again admit a natural spin representation. Likewise,

2.1.3 Projection and rejection

There is a direct relationship between unit k-blades and k-dimensional subspaces. Indeed, each unit k-blade B_k ($\|B_k\| = 1$) corresponds to a k-dimensional subspace. That is, each point in $\mathrm{Gr}(k,n)$ corresponds to a unit k-blade. Since blades represent subspaces, they also give us a compact way of projecting multivectors into subspaces. In general, given an multivector A the projection onto the subspace spanned by B_k is

$$\mathrm{P}_{B_k}(A) \coloneqq (A \rfloor B_k) B_k^{-1}. \tag{17}$$ eq:project

By definition, we have

$$\mathrm{P}_{B_k}(A) \in \bigoplus_{j=0}^{k} \mathcal{G}_n^j = \mathcal{G}_n^{0+\cdots+k}$$

Specifically,

$$\mathrm{P}_{B_k}(\langle A \rangle_j) \in G_n^j,$$

shows the projection preserves grades.

G a vector v, the projection onto the subspace spanned by the k-blade A_k is given by the identity

$$(v \rfloor A_k) A_k^{-1} = (v \rfloor A_k) \rfloor A_k^{-1} = (v \cdot B_k) \cdot B_k^{-1}. \tag{18}$$ eq:vector_

and more enlightening is to take a projection of a vector v onto another vector u

$$(v \rfloor u) u^{-1} = (v \cdot u) \frac{u}{\|u\|^2},$$

which is the expected result.

define rejection then go back and use it in the B-planar proofs

2.2 Multivector fields

We want to generalize the setting of geometric algebra to include a smooth structure. One can take the work above for $\mathcal{G}_n$ and consider a C^∞-module structure as opposed to the $\mathbb{R}$-algebra structure in the proceeding section. For brevity, we utilize the same notation $\mathcal{G}_n$ for the C^∞-module and $\mathbb{R}$-algebra as the structure will be clear from context. The multivectors themselves can be realized as constant multivector fields. This smooth setting simply makes the coefficients of the global basis blades given by C^∞ functions as opposed to $\mathbb{R}$ scalars. In this case, we refer to a generic element in the C^∞-module $\mathcal{G}_n$ as a *multivector field*. If we wish to specificy a domain $\Omega \subset \mathbb{R}^n$ for the multivector fields, we put

$$\mathcal{G}(\Omega) \coloneqq \{f \colon \Omega \to \mathcal{G}_n \mid f \text{ is } C^\infty\text{-smooth}\},$$

where smoothness is meant in terms of the C^∞-module structure.

Perhaps the C^∞-module structure obfuscates the point slightly. Instead, one should think of the fields in $\mathcal{G}_n$ as multivector valued functions on $\mathbb{R}^n$. Taking this identification allows for an extended toolbox at our disposal. In particular, points in $\mathbb{R}^n$ are uniquely identified with constant vector fields in $\mathcal{G}_n^1$ and one can consider endomorphisms living in $\mathcal{G}_n$ (acting on

$\mathcal{G}_n^1$) as acting on the input of fields valued in $\mathcal{G}_n$ as well. Thus, there is not only an algebraic structure on the fields themselves, but on the point in which the field is evaluated. This is perhaps the key insight on why authors developed the so-called vector manifolds widely used in the geometric algebra landscape.

Example 2.6. Consider a multivector field f valued in $\mathcal{G}_n$. With $x \in \mathbb{R}^n$ be identified with the vector in $\mathcal{G}_n^k$ (really as a constant vector field), we output a multivector $f(x)$ at each point x. One may be interested in the restriction of f to a vector subspace of $\mathbb{R}^n$ which amounts to using projection on the input. For example, perhaps we wish to know how f behaves on the subspace generated by some k-blade. As such, it suffices to then study $f(\mathrm{P}_v(x))$. Likewise, we may want to study

We refer to smooth functions valued in the subalgebra Γ as *Clifford fields* and elements valued in Γ^+ as *spinor fields*. Once again, we reuse the notation Γ and Γ^+ to refer to the C^∞-module counterpart. These fields will be shown to carry a Banach algebra structure.

2.2.1 Directional derivative and gradient

Note that $\mathbb{R}^n$ has global coordinates and thus we can choose a global constant vector field basis $e_1, \dots, e_n$ and generate $\mathcal{G}_n$ from this basis and note that we will adopt the Einstein summation convention when needed. With respect to these fields, we have the *directional derivative* ∇_ω with $\omega = \omega^i e_i$. The *gradient* (or *Dirac operator*) is defined as $\boldsymbol{\nabla} = \sum_i e^i \nabla_{e_i}$ and it acts a grade-1 element in the algebra. Note then that $\omega \cdot \boldsymbol{\nabla} = \nabla_\omega$ defines the directional derivative via the gradient. The directional derivative is also grade preserving in that for a multivector A

$$\nabla_\omega \langle A \rangle_k = \langle \nabla_\omega A \rangle_k.$$

This structure defined above is typically referred to as *geometric calculus*. The setting for geometric calculus extends the setting of differential forms and reduces some of the complexity with tensor computations. Since $\boldsymbol{\nabla}$ is a grade-1 object, it acts on a homogeneous k-vector A_k by

$$\boldsymbol{\nabla} A_k = \langle \boldsymbol{\nabla} A_k \rangle_{k-1} + \langle \boldsymbol{\nabla} A_k \rangle_{k+1} \coloneqq \boldsymbol{\nabla} \cdot A_k + \boldsymbol{\nabla} \wedge A_k.$$

Thus, the gradient splits into two operators $\boldsymbol{\nabla}\cdot$ (or $\boldsymbol{\nabla}\lfloor$) and $\boldsymbol{\nabla}\wedge$. Here, $\boldsymbol{\nabla}\wedge$ can be identified with the exterior derivative d and $\boldsymbol{\nabla}\cdot$ can be identified with the codifferential δ on differential forms up to a sign (see schindler_geometric_2020 [17] There are more citations to use). This of course means the standard properties that apply to d and δ apply to $\boldsymbol{\nabla}\wedge$ and $\boldsymbol{\nabla}\cdot$. Namely, we have

$$(\boldsymbol{\nabla}\wedge)^2 = 0 \qquad (\boldsymbol{\nabla}\cdot)^2 = 0, \tag{19}$$

eq:differen

when acting on a homogeneous k-vector. Since eq:differential_properties 19 holds, the gradient operator gives rise to the grade preserving *Laplace-Beltrami operator*

$$\Delta = \boldsymbol{\nabla}\boldsymbol{\nabla} = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \wedge + \boldsymbol{\nabla} \wedge \boldsymbol{\nabla} \cdot,$$

which is manifestly coordinate invariant by definition. It also motivates the use of the physicist notation $\boldsymbol{\nabla}^2 = \Delta$, but we do not adopt this here. We refer to multivector fields f in the kernel of the Laplace-Beltrami operator *harmonic*.

2.2.2 Monogenic fields

Geometric calculus includes another definition for multivectors that is a big motivation for those who study Clifford analysis.

Definition 2.2. Let $f \in \mathcal{G}(M)$. Then we say that f is *monogenic* if $f \in \ker(\boldsymbol{\nabla})$.

Monogenic fields are of utmost importance as they have many beautiful properties. One should find them as a suitable generalization of the notion of complex holomorphicity. For example, in regions of Euclidean spaces, a monogenic field f can be completely determined by its Dirichlet boundary values through a generalized Cauchy integral formula. For any even monogenic field, the each of the graded components of f are harmonic.

We put $\mathcal{M}(\Omega)$ to refer to the *space of monogenic fields* on a region Ω. As a subset, we also have the *monogenic spinors* $\mathcal{M}^+(\Omega)$. Though these spaces do not form algebras in their own right, they do indeed form a vector space as sums of monogenic functions are monogenic due to the linearity of the gradient. Moreover, the monogenic spinors are invariant under multiplication from the Clifford group Γ^+ (realized as constant fields in $\Gamma^+(\Omega)$).

_invariant **Lemma 2.1.** *The space of monogenic spinors $\mathcal{M}^+(\Omega)$ is Γ^+ invariant.*

Proof. It is clear that Bf is ω-axial due to the grade preserving linearity of the covariant derivative.

To see that Bf is monogenic, we take $Bf = Bu - \beta$. Then,

$$\boldsymbol{\nabla}(Bf) = \boldsymbol{\nabla}(Bu) - \boldsymbol{\nabla}\beta,$$

where we have the graded components

$$\begin{aligned} \langle \boldsymbol{\nabla}(Bf) \rangle_1 &= (\boldsymbol{\nabla} \cdot Bu) - \boldsymbol{\nabla}\beta \\ \langle \boldsymbol{\nabla}(Bf) \rangle_3 &= (\boldsymbol{\nabla} \wedge Bu). \end{aligned}$$

Note that

$$\boldsymbol{\nabla} \cdot (Bu) = -\omega \times (\boldsymbol{\nabla} \wedge u) = -\omega \times (\omega \times \boldsymbol{\nabla}\beta) = -\omega(\nabla_\omega \beta) + \boldsymbol{\nabla}\beta = \boldsymbol{\nabla}\beta$$

by eq:axial_cauchy_riemann ?? and thus $\langle \boldsymbol{\nabla}(Bf) \rangle_1 = 0$.

For the grade-3 component,

$$\boldsymbol{\nabla} \wedge (Bu) = \omega \cdot (\boldsymbol{\nabla} \wedge B) I I^{-1} u = I^{-1} \nabla_\omega u = 0$$

since u is ω-axial. Thus we have $\boldsymbol{\nabla}(Bf) = 0$ is monogenic. □

This lemma is classical in the theory of the Dirac operator source? I think it may not be too hard to see..

2.3 Differential forms and integration

Introduce the directed measure and hodge duality. Then prove greens formula. Give an example of the 3-ball in spherical coordinates and surface integrals and stuff

Naturally, we would also like to be able to integrate multivectors. In order to do so, we appeal to the language of differential forms and build a path from multivectors to forms. Given the coordinate system x^i on $\mathbb{R}^n$, we form the basis of tangent vector fields $\partial_i = \frac{\partial}{\partial x^i}$ with the reciprocal 1-forms dx^i which are gradients of the coordinate functions. Thinking of 1-forms as linear functions on tangent vectors, we have $dx^i \partial_j = \delta^i_j$. The benefit of this definition is that the 1-forms dx^i carry a natural measure and we can form product measures via the exterior product. For example, we have $d\Sigma = e_i \wedge e_j dx^i dx^j$. Then,we have $(e^j \wedge e^i) \cdot d\Sigma = dx^i dx^j - dx^j dx^i$ which retains the antisymmetry of the differential forms.

Is dX_k really just a k-density? Good answers on stack exchange

In an n-dimensional space with a position dependent inner product g, we have the n-dimensional volume measure $d\Omega = \sqrt{|g|}dx^1 \dots dx^n$. If we then define $dX_n = e^n \wedge \cdots \wedge e^1 dx^1 \dots dx^n$ we then find that $d\Omega = I^\dagger \cdot dX_n$ as

$$I^\dagger \cdot dX_n = \sqrt{|g|}(e_n \wedge \cdots \wedge e_1) \cdot (e^n \wedge \cdots \wedge e^1) dx^1 \cdots dx^n.$$

Similarly, for $k < n$, we can define the k-dimensional volume measure as

$$dX_k = \frac{1}{k!}(e^{i_k} \wedge \cdots \wedge e^{i_1}) dx^{i_1} \cdots dx^{i_k}.$$

We can now write a k-form α_k as $\alpha_k = A_k \cdot dX_k$. In this sense, a differential form is made up of two essential components namely the multivector field and the k-dimensional volume measure. This decomposition is important when the underlying space has interesting topological or geometrical features. In $\mathbb{R}^n$, this distinction is less important.

For example, if we wish to write a 2-form α_2 we take $dX_2 = \frac{1}{2!} e^j \wedge e^i dx^i dx^j$ and $A_2 = a_{ij} e_i \wedge e_j$ to yield

$$\alpha_2 = A_2 \cdot dX_2 = \frac{a_{ij}}{2!}(e_i \wedge e_j) \cdot (e^j \wedge e^i) dx^i dx^j = \frac{a_{ij}}{2!}(dx^i dx^j - dx^j dx^i)$$

Thus, we arrive at an isomorphism between k-forms and k-vectors as a contraction with the k-dimensional volume measure dX_k since

$$\alpha_k = A_k \cdot dX_k.$$

Hence, we can see now how a differential form simply appends the measure attached to the underlying space. We can also see how this generalizes the musical isomorphisms $\sharp$ and $\flat$ by taking a vector field a and noting

$$a \cdot dX_1 = a^i e_i \cdot e^j dx^j = a^i dx^i, \tag{20}$$ eq:line_el

corresponds to the usual $\flat$ map on vector fields.

The exterior algebra of differential forms comes with an addition $+$ and exterior multiplication $\wedge$. We note that the sum of two k-forms α_k and β_k that $\alpha_k + \beta_k$ is also a k-form which we can see by letting $\alpha_k = A_k \cdot dX_k$ and $\beta_k = B_k \cdot dX_k$ and putting

$$\alpha_k + \beta_k = (A_k \cdot dX_k) + (B_k \cdot dX_k) = (A_k + B_k) \cdot dX_k,$$

due to the linearity of $\cdot$. If instead had an s form β_s then we have the exterior product

$$\alpha_k \wedge \beta_s = (A_k \wedge B_k) \cdot dX_{k+s},$$

where $dX_{k+s} = 0$ if $k+s > n$.

With differential forms one also has the exterior derivative d, the Hodge star $\star$, and the codifferential δ. Given we can write a differential k-form as $\alpha_k = A_k \wedge dX_k$, we wish to define d, $\star$, δ by their actions on the k-vector A_k. In particular, we have

$$d\alpha_k = (\boldsymbol{\nabla} \wedge A_k) \cdot dX_{k+1},$$

which realizes the exterior derivative as the grade raising component of the gradient $\boldsymbol{\nabla}$. Of course, for scalars, this returns the gradient as desired. The Hodge star inputs a k-form and outputs a $(n-k)$-form and we define $\star$ so that for two k-forms α_k and β_k we have $\alpha_k \wedge \star\beta_k = (A_k \cdot B_k^\dagger) d\Omega$. This is since

$$A_k \cdot B_k^\dagger = \langle A_K, B_K \rangle \sqrt{|g|},$$

where $\langle A_K, B_K \rangle$ is the typical inner product on k-vectors extended through to exterior algebra. Thus, a coordinate expression for $\star$ acting on multivectors is given by $B_k^\star = (I^{-1}B_k)^\dagger$ so that $\star\beta = (I^{-1}B_k)^\dagger \cdot dX_{n-k}$. Indeed, we have

$$\begin{aligned}
\alpha_k \wedge \star\beta_k &= (A_k^\dagger \wedge B_k^\star) \cdot dX_n \\
&= (A_k \wedge (I^{-1}B_k)^\dagger) \cdot dX_n \\
&= (A_k \wedge (B_k^\dagger I)) \cdot dX_n \\
&= (A_k \cdot B_k^\dagger) I^{-1} \cdot dX_n \\
&= A_k \cdot B_k^\dagger d\Omega,
\end{aligned}$$

since $I^{-1} = I^\dagger$ in spaces with g positive definite.

Cite Hestenes. Also I think this can be simplified using the Clifford conjugate in that δ is like the clifford conjugate of the codifferential or something. See [11] francis_construction_2005

Then, in the typical fashion we define the codifferential $\delta = (-1)^{n(k-1)+1} \star d \star$ when acting on k-forms. Then,

$$\begin{aligned}
\delta\alpha_k &= (-1)^{n(k-1)+1} \star d \star \alpha_k \\
&= (-1)^{n(k-1)+1} \star d[(I^{-1}A_k)^\dagger \cdot dX_{n-k}] \\
&= (-1)^{n(k-1)+1} \star [\boldsymbol{\nabla} \wedge (A_k^\dagger I)] \cdot dX_{n-k+1} \\
&= (-1)^{n(k-1)+1} \star [(\boldsymbol{\nabla} \cdot A_k^\dagger) I^{-1}] \cdot dX_{n-k+1} \\
&= (-1)^{n(k-1)+1} \star [
\end{aligned}$$

3 Algebras of multivector fields

3.1 Banach algebras of Clifford fields

Finish this section. I'm saying this here but it should go later on, but this should lead to the weak formulation for the laplace equation??? Does there exist an inner product instead of just a norm?

Letting Ω be a region in $\mathbb{R}^n$, recall that the space of monogenic fields $\mathcal{M}(\Omega)$ is not an algebra. However, $\mathcal{M}(\Omega)$ does contain algebras that are commutative Banach algebras. For example, we always have the following algebra.

The Clifford fields are given as functions $s\colon \Omega \to \Gamma$ for which we say $s \in \Gamma(\Omega)$. This space $\Gamma(\Omega)$ has a norm induced from the spinor norm in the L_2 sense by

$$\|s\|_{L_2} = \int_\Omega ss^\dagger d\Omega$$

This gives us a normed algebra of Clifford fields. One can see that we have the unit 1 in this algebra. We also have for multivectors $s, r \in \Gamma(\Omega)$ (constant Clifford fields)

$$\|sr\| = \|s\|\|r\|$$

since

$$\|sr\|^2 = (sr)(sr)^\dagger = srr^\dagger s^\dagger = s\|rr^\dagger\|^2 s^\dagger = \|s\|^2\|r\|^2.$$

It follows for non-constant C^∞-fields s and r

$$\|sr\|_{L_2} \leq \|s\|_{L_2}\|r\|_{L_2}.$$

This shows the algebra is uniform. Identifying the constant fields in the algebra $\Gamma(\Omega)$ with $\mathbb{R}^{2^n}$ we see that the algebra is also complete. Thus we have shown that the space $\Gamma(\Omega)$ is a (noncommutative) Banach algebra. The subspace $\mathcal{M}(\Omega) \cap \Gamma(\Omega)$ is thus a Banach algebra contained inside of $\mathcal{M}(\Omega)$. There are more algebras to discover.

Remark 3.1. It is worth noting that while the constant elements in $\Gamma(\Omega)$ form a group, the elements in general do not. That is to say that not every element in $\Gamma(\Omega)$ is invertible. This not only expected since functions in general are not invertible, but this is also a nonissue as the algebra structure of the Clifford fields is what remains important.

3.1.1 Planar monogenic fields

Generically, if I take some multivector A times a monogenic field f, Af need not be monogenic. This is exactly why $\mathcal{M}(\Omega)$ fails to be an algebra. But, there are certain types of monogenic fields in which this property is true. We describe a set of parabivectors that operate entirely on a plane given by a unit bivector B. These specific fields will be of great utility for the remainder of this paper.

Definition 3.1. Let f be a parabivector and B a unit 2-blade. Then f is a *B-planar field* if $f = \mathrm{P}_B \circ f \circ \mathrm{P}_B$.

We then refer to the *B-planar monogenic fields* f when f is both B-planar and monogenic. Planar monogenic fields will serve as a realization of complex valued functions since they carry over some additional nice properties and admit a nice representation.

Lemma 3.1. *Let f be a B-planar monogenic field, then:*

- *The directional derivatives in all directions other than in the B plane are zero;*

- *We have the representation $f = u + \beta B$ for a $u, \beta \in G_n^0$ and B the given unit bivector.*

Proof.

- Let v be a unit vector not in the B plane so that $\mathrm{P}_B(v) = 0$. Since f is B-planar, we know $f = f \circ \mathrm{P}_B$ which shows that $f(x + \epsilon v) = f(x)$. It follows that $\nabla_v f = 0$.
- Let $f = u + b$ for $u \in \mathcal{G}_n^0$ and $b \in G_n^2$. Then $f = \mathrm{P}_B(v) \circ f$ and so $\mathrm{P}_B(u + b) = u + b$. In particular, $\mathrm{P}_B = b$ and thus $b = \beta B$ for a scalar $\beta \in \mathcal{G}_n^0$.

$\square$

To get a geometric interpretation of B-planar fields we can note that they are constant on translations of the B-plane. It follows that

$$(\boldsymbol{\nabla} \wedge B) f = 0. \tag{21}$$

eq:exterio

In $\mathbb{R}^3$, for example, this amounts to fields constant along an axis $\omega = IB^{-1}$ perpendicular to B as

$$\boldsymbol{\nabla} \wedge B = \boldsymbol{\nabla} \wedge \omega I = \boldsymbol{\nabla} \cdot \omega = \nabla_\omega. \tag{22}$$

eq:omega_a

Rephrase this with rejection?

ex:complex_representation

Recall from Example 2.1 that multivectors in the form $\zeta = x + yB$ mimic the complex number ζ when B is a unit 2-blade since $B^2 = -1$. Planar monogenic fields are thus a direct analog of $\mathbb{C}$-holomorphic functions. Indeed, for simplicity take the orthonormal basis e_i and the blade $B = B_{12}$ and for scalar fields u and β put

$$f = u + \beta B_{12}$$

and note

$$\boldsymbol{\nabla} f = 0$$

yields the Cauchy-Riemann equations

$$\nabla_{e_1} u = \nabla_{e_2} \beta \qquad \text{and} \qquad \nabla_{e_2} u = -\nabla_{e_1} \beta.$$

Holomorphic functions form an algebra and we shall show the B-planar monogenic fields do as well.

We let

$$\mathcal{A}_B(\Omega) = \{f \mid f \text{ is } B\text{-planar and monogenic}\}$$

be the space of B-planar monogenic fields. For any 2-blade B in $\mathrm{Gr}(2, n)$, we have a copy of $\mathcal{A}_B(\Omega)$. Multiplication of two B-planar fields $f = u_f + \beta_f B$ and $g = u_g + \beta_g B$ is given by

$$fg = u_f u_g - \beta_f \beta_g + B(u_f b_g + u_g b_f) = gf. \tag{23}$$

eq:axial_m

Another property mimics $\mathbb{C}$-holomorphicity. Namely, scaling a holomorphic function by constant complex numbers remains holomorphic. We realize this for B-planar fields as Spin(2) invariance (really $\mathbb{R} \times \mathrm{Spin}(2)$ invariant). This corollary follows from Lemma 2.1 since Spin(2) is a subgroup of Γ^+

lem:clifford_inva

_monogenic **Corollary 3.1.** *Let $f = u + \beta B$ be an B-planar monogenic field and let $\zeta = x + yB$ for constant scalars x and y. Then ζf is a B-planar monogenic.*

Proof. Note that ζ is in $\Gamma^+(\Omega)$, and utilize Lemma lem:clifford_invariant 2.1. □

The point here is that we have now effectively found functions that can be scaled by B-planar constants ζ and remain monogenic.

With the above, we show the space $\mathcal{A}_B(\Omega)$ is closed under multiplication and is in fact abelian.

monogenics **Lemma 3.2.** *Let f and g be monogenic and B-planar. Then $fg = gf$, and fg is a B-planar monogenic.*

Proof.

- First, it is clear that $fg = gf$ by Equation eq:axial_multiplication 23.
- The product fg is B-planar since u_f, u_g, β_f, and β_g are all constant on translations of the B-plane, i.e. that $fg = fg \circ \mathrm{P}_B$. Due again to Equation eq:axial_multiplication 23 we have $fg = \mathrm{P}_B \circ fg$ as well.
- To see that the product is monogenic, we have

$$\boldsymbol{\nabla}(fg) = \boldsymbol{\nabla}(u_f u_g - b_f b_g + B(u_f b_g + u_g b_f)).$$

Then the grade-1 components are

$$\langle \boldsymbol{\nabla}(fg) \rangle_1 = \boldsymbol{\nabla} \wedge (u_f u_g - b_f b_g) + \boldsymbol{\nabla} \cdot B(u_f b_g + u_g b_f),$$

and note that we have

$$\begin{aligned} \boldsymbol{\nabla}(u_f u_g - b_f b_g) &= (\boldsymbol{\nabla} u_f) u_g + u_f(\boldsymbol{\nabla} u_g) - (\boldsymbol{\nabla} b_f) b_g - b_f(\boldsymbol{\nabla} b_g) \\ \boldsymbol{\nabla} \cdot B(u_f b_g + u_g b_f) &= (\boldsymbol{\nabla} \cdot B u_f) b_g + u_f(\boldsymbol{\nabla} \cdot B b_g) + b_f(\boldsymbol{\nabla} \cdot B u_g) + (\boldsymbol{\nabla} \cdot B b_f) u_g, \end{aligned}$$

and since f and g are both monogenic we have

$$\langle \boldsymbol{\nabla}(fg) \rangle_1 = (\boldsymbol{\nabla} \cdot B u_f - \boldsymbol{\nabla} b_f) b_g + (\boldsymbol{\nabla} \cdot B u_g - \boldsymbol{\nabla} b_g) b_f.$$

$$0 = \langle \boldsymbol{\nabla} B f \rangle_1 = \boldsymbol{\nabla} \cdot B u_f - \boldsymbol{\nabla} b_f$$

by Corollary cor:mult_by_i_monogenic 3.1 and likewise for $\langle \boldsymbol{\nabla} B g \rangle_1$. Thus,

$$\langle \boldsymbol{\nabla}(fg) \rangle_1 = 0.$$

The grade-3 components for the gradient are

$$\langle \boldsymbol{\nabla}(fg) \rangle_3 = \boldsymbol{\nabla} \wedge B(u_f b_g + u_g b_f),$$

and we can note that $\boldsymbol{\nabla} \wedge B = 0$ since u_f, b_g, u_g, and b_f are all B-planar.

□

From the above work, we realize that for each $\mathcal{A}_B(\Omega)$ we have a well defined multiplicative structure. But, we need to show that inverses also exist. Doing show realizes that $\mathcal{A}_B(\Omega)$ sits inside of Clifford fields $\Gamma^+(\Omega)$. This is clear as any constant field $\zeta = x+yB$ is invertible. Thus we arrive at the following corollary.

Corollary 3.2. *The space $\mathcal{A}_B$ is a commutative unital Banach algebra.*

Proof. Let f and g be B-planar monogenic fields. It is clear that the sum $f+g$ is a B-planar monogenic by the linearity of $\boldsymbol{\nabla}$ and the projection. Since $fg = gf$ is B-planar and monogenic we find that each $\mathcal{A}_B(\Omega)$ is an algebra. Since $\mathcal{A}_B(\Omega)$ is a commutative subalgebra of $\Gamma(\Omega)$ (really of $\Gamma^+(\Omega)$), it is also a commutative Banach algebra. $\square$

3.1.2 ω-axial fields

The authors in belishev_algebraic_2019,belishev_algebras_2019 [4, 5] give a thorough treatment of an analogous story but with quaternion fields. We show the relationship between the two stories in this section and we find them to be entirely equivalent. As in Example ex:quaternions 2.2, we can see these quaternion fields as parabivector fields. The authors work exclusively in 3-dimensions and quickly specialize to the fields which are ω-axial due to their rich algebraic structure. There, ω is a purely imaginary unit quaternion. Their harmonic ω-axial fields are equivalent to monogenic B-planar fields if we take the axis $\omega = BI^{-1}$. First, note we define ω-axial in the same way.

Definition 3.2. Let $A \in \mathcal{G}_3$ be a multivector field then A is *ω-axial* if $A(x+t\omega) = A(x+t\omega)$.

This definition allows us to perfectly coincide the notions of B-planar monogenic fields with ω-axial harmonic quaternion fields.

Proposition 3.1. *In $\mathbb{R}^3$, every B-planar monogenic field is in correspondence with an ω-axial harmonic quaternion field $h = \varphi + \psi\omega$.*

Proof. Let f be a B-planar monogenic field with $\tilde{\omega} = BI^{-1}$ and note that $f(x+t\tilde{\omega}) = f(x)$ since $\mathrm{P}_B(t\omega) = 0$. Thus, f is $\tilde{\omega}$-axial.

Given the quaternion multiplication is a left handed bivector multiplication (see Example ex:quaternions 2.2, we can replace the purely imaginary quaternion ω and get a vector in $\mathcal{G}_3^1$ by using the correspondence $\boldsymbol{i} \leftrightarrow e_1$, $\boldsymbol{j} \leftrightarrow e_2$, and $\boldsymbol{k} \leftrightarrow e_3$ we generate $\tilde{\omega} \in \mathcal{G}_3^1$. We then have the 2-blade $B = \tilde{\omega}I$ such that

$$\tilde{h} = \varphi + \psi B,$$

is the corresponding parabivector in $\mathcal{G}_3$. It's clear that $\mathrm{P}_B \circ \tilde{h} = \tilde{h}$. Likewise, since φ and ψ were constant on the axis given by ω, then by the previous work $\varphi \circ \mathrm{P}_B$ and $\psi \circ \mathrm{P}_B$ implies that $\tilde{h} \circ \mathrm{P}_B$ and so $\tilde{h}$ is a B-planar. Hence, setting $\varphi = u$ and $\psi = \beta$, we recover a unique f from a given h.

Then, if $h = \varphi + \psi\omega$ is harmonic, we know

$$\boldsymbol{\nabla}\psi = \omega \times \boldsymbol{\nabla}\varphi,$$

where we take the vector cross product $\times$. Based on Example ex:cross_product 2.3, we can see that corresponding B-planar field $f = u + \beta B$ yields the analogous equation

$$\boldsymbol{\nabla} u = \boldsymbol{\nabla} \cdot \beta B = (\boldsymbol{\nabla} \wedge \tilde{\omega})I = \tilde{\omega} \times \boldsymbol{\nabla}\beta.$$

Thus, the notions of an ω-axial harmonic quaternion field coincides with B-planar monogenic fields in $\mathbb{R}^3$ so long as $B = \tilde{\omega} I$. $\square$

The ω-axial fields do not generalize properly and this definition is solely a happy circumstance seen in $\mathbb{R}^3$ given the duality between vectors and bivectors. In higher dimensions, the notion of B-planar retains all the desired properties that let us define a notion of a Gelfand spectrum.

3.1.3 Spinor spectrum

This story no longer continues in higher dimensions and one can find the two and three dimensional cases to be happy accidents. Instead, now we must deal fully with the situation at hand to dissect the relevant algebras. At our disposal are the algebras $\mathcal{A}_B(\Omega)$ of B-planar monogenic fields. Take the case where the domain $\mathbb{B} \subset \mathbb{R}^n$ is the unit n-ball and moreover let $\mathbb{D}$ be the unit disk in $\mathbb{C} \cong \mathbb{R}^2$. By Gelfand, the maximal ideal space of the commutative Banach algebra $\mathcal{A}_B(\mathbb{B})$ is homeomorphic to the disk given the isomorphism mapping the blade $B \leftrightarrow i$ in the complex plane. Since the space $\mathcal{M}$ is no longer an algebra or even commutative, we are at a loss to determine maximal ideals. Instead, one can note that maximal ideals of a commutative Banach algebra correspond to the multiplicative linear functions. Using this identification, we carry on and describe functionals on the monogenics.

Definition 3.3. Define the *spinor dual* $\mathcal{M}^\times(\Omega)$ as

$$\mathcal{M}^\times(\Omega) \coloneqq \{l \in \mathcal{L}(\mathcal{M}(\Omega); \Gamma^+) \mid l(sf) = sl(f),\ \forall f \in \mathcal{M},\ s \in \mathfrak{spin}(n)\}$$

$\mathcal{M}^\times(\Omega)$ are the spinor valued functionals or *spin functionals*. Similarly, we have the definition for the spinor functionals that are multiplicative on the B-planar monogenics. In other words, spin characters are simply algebra homomorphisms from $\mathcal{A}_B(\Omega)$ to Γ^+.

Definition 3.4. The *spinor spectrum* is the set

$$\mathfrak{M}(\Omega) \coloneqq \{\mu \in \mathcal{M}^\times(\Omega) \mid \mu(fg) = \mu(f)\mu(g),\ \forall f, g \in \mathcal{A}_B,\ B \in \mathrm{Gr}(2, n)\},$$

and we refer to the elements as *spin characters*.

In the case where Ω itself is 2-dimensional and compact, we realize Γ^+ is isomorphic to $\mathbb{C}$ and we find that these match the typical definition for characters $\mu \in \mathfrak{M}(\Omega)$. These spin characters each amount to function evalation. Take $f \in \mathcal{M}(\Omega)$ and note that $f \in \mathcal{A}_B(\Omega)$ as well. f is then a holomorphic function when we identify $B \leftrightarrow i$ and as such the spin character μ acts by $\mu(f) = f(x_\mu)$ for some point $x_\mu \in \Omega$ showing the correspondence of points in Ω with spin characters in $\mathfrak{M}(\Omega)$. Hence, with the weak-$*$ topology, the space $\mathfrak{M}(\Omega)$ is homeomorphic to Ω.

There is the question now on what is the homeomorphism type of $\mathcal{A}_B(\Omega)$ for an arbitrary Ω and for a given B. Use 2d Belishev somehow? Describe the weak-$*$ topology here to use later.

4 Gelfand theory

4.1 Topology from monogenics

For the remainder of this section, we will consider the domain of interest to be the unit n-ball, $\mathbb{B}$. We seek to determine that the space $\mathfrak{M}(\mathbb{B})$ is homeomorphic to $\mathbb{B}$. Thinking of the Calderón problem, we may only have access to functions defined on $\mathbb{B}$ and not the whole of $\mathbb{B}$ itself. If one can recover the spin characters $\mathfrak{M}(\mathbb{B})$, we can utilize the following result.

Theorem 4.1. *For any $\mu \in \mathfrak{M}(\mathbb{B})$, there is a point $x^\mu \in \mathbb{B}$ such that $\mu(f) = f(x_\mu)$ for any $f \in \mathcal{M}(\mathbb{B})$. Given the weak-$*$ topology on $\mathfrak{M}(\mathbb{B})$, the map*

$$\gamma\colon \mathfrak{M}(\mathbb{B}) \to \mathbb{B}, \quad \mu \mapsto x^\mu$$

is a homeomorphism. The Gelfand transform

$$\widehat{\ }\colon \mathcal{M}(\mathbb{B}) \to C(\mathfrak{M}(\mathbb{B}); \Gamma^+), \quad \widehat{f}(\mu) \coloneqq \mu(f), \quad \mu \in \mathfrak{M}(\mathbb{B}),$$

is an isometry onto its image, so that $\mathfrak{M}(\mathbb{B})$ is isomorphic to $\widehat{\mathcal{M}(\mathbb{B})}$ as algebras.

We prove this theorem in two main parts. First, we can realize a power series representation for elements in $\mathcal{M}(\mathbb{B})$. This power series is constructed using specific B-planar monogenic fields. Finally, we constructively show a correspondence between $\mu \in \mathfrak{M}(\mathbb{B})$ with $x^\mu \in \mathbb{B}$.

4.1.1 Power series

One beautiful result in Clifford analysis is the celebrated generalization of the Cauchy integral formula for $\mathbb{C}$-holomorphic functions. Details of the Cauchy integral formula and Hilbert transform for multivector fields can be found in brackx_hilbert_2008 [6]. We have the fundamental solution to $\boldsymbol{\nabla}$ is a vector field given by

$$E(x) = \frac{1}{a_m}\frac{x}{|x|^m},$$

for $x \in \mathbb{R}^n$. That is to say that $\boldsymbol{\nabla} E(x) = \delta(x)$. For any region $\Omega \subset \mathbb{R}^n$ with boundary Σ, we define the *Cauchy kernel* for $x \in \mathbb{R}^n$ and $y \in \Sigma$ using the fundamental solution E as

$$C(y,x) = -\frac{1}{a_n}\nu(x_0)E(x-y),$$

where a_n is the surface area of the n-ball and $\nu(x_0)$ is the outward normal at x_0. The *Cauchy integral* for $\phi \in L_2(\Sigma)$ is then

$$\mathcal{C}[\phi](x) = \frac{1}{a_n}\int_\Sigma \frac{x_0 - x}{|x-x_0|^n}\nu(x_0)\phi(x_0)d\Sigma(x_0).$$

The Cauchy integral is indeed a monogenic function and note that for a scalar ϕ we have $\mathcal{C}[\phi] \in \mathcal{M}(\Omega)$ since it must be a parabivector as well.

do for an arbitrary basis for the remainder and fix all the notation up

Fix a basis $e_1, \ldots, e_n$ in $\mathbb{R}^n$ and we can define the functions $z^i_j = x^j - x^i e_i e_j$. Recall that for an orthonormal basis the reciprocal basis elements $e^i = e_i$. To further condense notation, we let $B_{ij} = e_i e_j$ be the 2-blade acting as the pseudoscalar for the $e_i e_j$-plane and likewise put $B^i_j = e^i e_j$ and $B^{ij} = e^i e^j$ as necessary. In the same vein, the functions z^i_j are very analogous to z in $\mathbb{C}$ but rather in the B^i_j plane. One can note

$$z^i_j = x^j - x^i B^i_j = e_j \, \mathrm{P}_{B_{ij}}(x).$$

It is worth noting that the z^i_j are monogenic and are B^i_j-planar.

Show this, or find where I show it later.

For sake of simplicity, we let $e_1, \ldots, e_n$ be an arbitrary basis for $\mathbb{R}^n$.

- Consider the function $z_{B_{\sigma(j)}}(x) = x_{\sigma(j)} - x_1 e^1 e_{\sigma(j)}$ for $\sigma \in \{2, \ldots, n\}$ a permutation. Note that $z_{B_{\sigma(j)}}$ is $B_{\sigma(j)}$-planar with $B_{\sigma(j)} = e^1 e_{\sigma(j)}$. Moreover, $z_{B_{\sigma(j)}}$ is monogenic as

$$\boldsymbol{\nabla} z_{B_{\sigma(j)}} = e_{\sigma(j)} - e_1 e^1 e_{\sigma(j)} = 0.$$

  We denote as well $B_{\sigma(j)} = e^1 e_{\sigma(j)}$.

- Let $f \in \mathcal{M}$. Then by Theorem 4 in ryan_left_1986 [14], we have the monogenic polynomials

$$P_{j_2 \cdots j_n}(x) = \frac{1}{j!} \sum_{\text{permutations}} z_{B_{\sigma(1)}}(x) \cdots z_{B_{\sigma(j)}}(x),$$

  which generate f as a power series as

$$f(x) = \sum_{j=0}^{\infty} \left(\sum_{j_2 \cdots j_{n_{j_2 + \cdots j_n = j}}} P_{j_2 \cdots j_n}(x) a_{j_2 \cdots j_n} \right),$$

  where

$$a_{j_2 \cdots j_n} = \frac{1}{\omega_n} \int_{\partial \Sigma} \nabla^{j_2}_{e_2} \cdots \nabla^{j_n}_{e_n} G(y) \nu(y) f(y) d\Sigma(y),$$

  where $G(y)$ is the Cauchy kernel.

  Relate this to the Cauchy kernel I defined.

4.1.2 Correspondence

The functions z^i_j play a crucial role in the above power series representation but they also play a key part in determining the behavior of the spin characters $\mu \in \mathfrak{M}$. If we are able to deduce the action $\mu(z^i_j)$, then we can extend this to any monogenic f via the power series representation. Note that for any $\mu \in \mathfrak{M}$ that $\mathbb{A}_B = \mu(\mathcal{A}_B)$ is a commutative subalgebra of $\mathfrak{spin}(n)$. In particular, for a constant $c \in \mathcal{A}_B$, $\mu(c) = c$ and so we retrieve $\mathbb{A}_B$ must be generated by scalars and the bivector B. Thus, $\mathbb{A}_B$ is an isomorphic copy of $\mathfrak{spin}(2)$ as the algebra of the B-plane. Note that μ will be constant on 2-blades.

Working with the same orthonormal basis and applying μ yields

$$\mu(z_j^i) = a_j^i + b_j^i B_{ij},$$

for some constants a_j^i and b_j^i. The z_j^i are not independent from one another. In fact, we have two key relationships in that

$$z_j^i B_{ij} = z_i^j. \tag{24}$$

eq:z_recip

Similarly, we have

$$z_j^i = B_{jk} z_j^k B_{kj} - z_i^k B_{ij}. \tag{25}$$

eq:z_relat

We simply compute the above,

$$\begin{aligned} B_{jk} z_j^k B_{kj} - z_i^k B_{ij} &= B_{jk}(x^j - x^k B_{kj}) B_{kj} - (x^i - x^k B_{ki}) B_{ij} \\ &= x^j - x^k B_{kj} - x^i B_{ij} + x^k B_{kj} \\ &= z_j^i. \end{aligned}$$

Thus, we can take μ of Equations 24 and 25. First,

$$\mu(z_i^j) = \mu(z_j^i B_{ij}) = \mu(z_j^i) B_{ij}$$

yields

$$a_i^j - b_i^j B_{ij} = -b_j^i + a_j^i B_{ij}$$

and so $a_i^j = b_j^i$ for all $i \neq j$. Next,

$$\mu(z_j^i) = \mu(B_{jk} z_j^k B_{kj} - z_i^k B_{ij}) = B_{jk} \mu z_j^k B_{kj} - \mu(z_i^k) B_{ij}$$

and so

$$a_j^i + b_j^i B_{ij} = B_{jk}(a_j^k + b_j^k B_{kj}) B_{kj} - (a_i^k + b_i^k B_{ki}) B_{ij} = a_j^k - a_i^k B_{ij}$$

yields $a_j^i = a_j^k$ and $a_i^k = -b_j^i$. In particular, for all z_j^i, we have the relationships

$$a_i^j = -b_j^i, \quad a_j^i = a_j^k, \quad a_i^k = -b_j^i, \quad \text{for } i \neq j \neq k.$$

More simply, we can note

$$a_i^\bullet = -b_\bullet^i \ \forall i \qquad \text{and} \qquad a_j^\bullet = a_j^\bullet \ \forall j.$$

Letting $\mu(z_j^i) = z_j^i(x_\mu)$ satisfies these requirements above since $z_j^i(x_\mu) = x_\mu^j - x_\mu^i B_{ij}$ for all $i \neq j$. Make the matrix argument and stuff then show that the point itself must also be in $\mathbb{B}$.

Finish this and note that this proves the theorem. Motivate the next section.

5 Calderón problem

Let $u^\phi \in \Omega^0(M)$ be a smooth 0-form (scalar function) that is a solution to the following Dirichlet boundary value problem

$$\begin{cases} \Delta u^\phi = 0 & \text{in } M \\ \iota^*(u) = \phi. \end{cases}, \tag{26}$$

where Δ refers to the Laplace-Beltrami operator on differential forms. For the Calderón problem, the manifold M and metric g are unknown and one seeks to determine as much as possible about (M, g) from measurements on the boundary. Due to the relationship between the EIT and Calderón problem, we use the notation ϕ for the Dirichlet boundary values since ϕ should be thought of as the prescribed voltage along the boundary.

For any given solution to the boundary value problem, there is the corresponding Neumann data $E = \iota^*(\star du)$. As with ϕ, the notation E is used as the Neumann data measured in the EIT problem corresponds to the electric field flux at the boundary. One attains the current J by multiplying with E by the boundary conductivity matrix. The set of both boundary conditions (ϕ, E) is the *Cauchy data* and the *Dirichlet-to-Neumann (DN) map* Λ is defined such that $\Lambda\phi = E$ and in particular this yields $\iota^*(\star du^\phi) = E$. Note that this map Λ is often referred to as the *scalar* DN map as $\Lambda \colon \Omega^0(\partial M) \to \Omega^{n-1}(\partial M)$ inputs a scalar Dirichlet condition. An extension of the DN map to forms can be found in [1, 18]. The Calderón problem for Riemannian manifolds is then to recover the pair (M, g) up to isometry from complete knowledge of the DN map Λ.

Denote by $\mathcal{H}(M) = \{u \in \Omega^0(M) \mid du = 0\}$ the space of harmonic 0-forms on M. From the DN map, one can define the *Hilbert transform* $T \colon \iota^*\mathcal{H}(M) \to \iota^*\mathcal{H}(M)$. This function acts on traces of harmonic forms by

$$T\phi = d\Lambda^{-1}\phi,$$

and is defined in [1]. The authors show benefit to defining the Hilbert transform as it provides the ability to generate so called conjugate forms. When the condition

$$\left(\Lambda + (-1)^n d\Lambda^{-1} d\right)\phi = 0, \tag{27}$$

is met, then there exists a *conjugate form* $\epsilon^\psi \in \Omega^{n-2}(M)$ with boundary trace $\psi = \iota^*\epsilon$ satisfying $Td\phi = d\psi$. As well, ϵ is also coclosed in that $\delta\epsilon = 0$.

Now, there exists a 2-form b^ψ such that $\star b^\psi = \epsilon$. Using the isomorphism between forms and multivectors, we can let U be the scalar field corresponding to u^ϕ and we can let B be the bivector field corresponding to b^ψ. We can add these to yield the paravector $F = U + B \in \mathcal{G}(M)$. Recall that a multivector field is monogenic if $\boldsymbol{\nabla} F = 0$. Applying this to the paravector F yields the equations

$$\boldsymbol{\nabla} \wedge U = -\boldsymbol{\nabla} \cdot B \qquad \text{and} \qquad \boldsymbol{\nabla} \wedge B = 0.$$

The conjugacy relation $du^\phi = \star d\epsilon^\psi$ is equivalent to having the multivector F be monogenic.

Lemma 5.1. *Given the forms u^ϕ and b^ψ conjugate as above, the corresponding paravector field*

$$F = U + B$$

is monogenic.

Proof. Let $\star b^\psi = \epsilon$ and note that

$$du = \star d\epsilon = \star d \star b^\psi.$$

Now, writing the multivector equivalent of the right hand side yields

$$\begin{aligned}(\boldsymbol{\nabla} \wedge B^\star)^\star &= [(\boldsymbol{\nabla} \cdot B^\dagger)I]^\star \\ &= [I^{-1}((\boldsymbol{\nabla} \cdot B^\dagger)I)]^\dagger \\ &= ((\boldsymbol{\nabla} \cdot B^\dagger)I)^\dagger I \\ &= \boldsymbol{\nabla} \cdot B^\dagger && \text{since } \dagger \text{ of a vector is trivial} \\ &= -\boldsymbol{\nabla} \cdot B. && \text{since } \dagger \text{ of a bivector is -1}\end{aligned}$$

Thus, we have $\boldsymbol{\nabla} \wedge U + \boldsymbol{\nabla} \cdot B = 0$. Since ϵ is coclosed we have

$$\begin{aligned}0 = \boldsymbol{\nabla} \cdot B^\star &= \boldsymbol{\nabla} \cdot (I^{-1}B)^\dagger \\ &= \boldsymbol{\nabla} \cdot (B^\dagger I) \\ &= (\boldsymbol{\nabla} \wedge B^\dagger)I \\ &= \boldsymbol{\nabla} \wedge B.\end{aligned}$$

Thus $\boldsymbol{\nabla} F = 0$ and F is monogenic. □

5.1 Calderon problem in geometric calculus

Indeed, the above work invites one to rephrase the problem in terms of geometric calculus. Instead, the classical problem is given as follows.

Question 5.1. Let M be an unknown Riemannian manifold with unknown metric g and with known boundary Σ. Let $u^\phi \in \mathcal{G}(M)$ be a scalar field satisfying the Dirichlet problem

$$\begin{cases}\Delta u^\phi = 0 & \text{in } M \\ u|_\Sigma = \phi.\end{cases}, \tag{28}$$

eq:dirichl

Define the Dirichlet to Neumann map as

$$\Lambda u^\phi = \mathrm{P}_\nu(\boldsymbol{\nabla} u^\phi),$$

where ν is the normal to Σ given by $I_\Sigma I$. Can one recover M and g from knowledge of Σ and Λ?

It is a well known fact that the inverse of the DN map is known up to a constant

5.2 EXTRA STUFF

Add about the 2D problem and generating algebras?

We should start with the boundary algebra and show that we can generate algebras inside. Use the maximum principle.

For this section, let $n = \dim(M) = 3$. Supposing that ϕ satisfies eq:conjugate_requirement 27 (I dropped this requirement for now) one can generate paravectors $f = u + b$ and define the space of *monogenic paravectors*

$$\mathcal{M} = \{f \mid \boldsymbol{\nabla} f = 0\}$$

The original requirement that $\Delta u^\phi = 0$ is obtained since f is monogenic. We can then generate an algebra from this set by

$$\mathcal{A} = \{fg \mid f, g \in \mathcal{M}\},$$

but, as mentioned in belishev_algebras_2019 [5], this algebra generated by these monogenic fields in $\mathcal{M}$ produce fields that are not monogenic. Indeed, this is a well known fact in Clifford analysis mentioned in schepper_introductory_nodate [16]. Fundamentally, however, this fact that the product of monogenics is no longer monogenics makes the direct approach in belishev_calderon_2003 [2] intractable. This issue comes down to the lack of commutivity of paravectors in dimensions higher than 2. However, for certain so-called axial fields, commutivity is regained. In fact, the construction of these fields was done in belishev_algebra [3] in order to create a closed commutative algebra of monogenic fields. These axial fields will relate directly to complex holomorphic functions.

In belishev_algebras_2017, belishev_algebras_2019 [3, 5], the definition of axial is defined for quaternion fields and the properties are discussed. It is evident from the Example ex:quaternions 2.2 that quaternion fields are analogous to paravector fields via the given identification. This identification is key in connecting the relevant algebras to the DN map. So we proceed by following the definitions in place.

6 Further questions

6.1 Spin fibration

maybe pose this as a question in relation to using the 2d belishev stuff.

The inner product for characters is what you use for fourier theory, maybe we can do something here with characters as maps to the grassmannian? Do these form some kind of orthogonal basis? Also, the Dirac operator and Laplacian are spin invariant! This is what they use the $\mathbb{H}$ module structure for!

A main question to answer now is how the B-planar algebras $\mathcal{A}_B$ relate to the space of monogenic functions $\mathcal{M}$. In particular, this question seems analogous to the invertibility of a 2-plane x-ray transform. Let f be a monogenic, can f be generated by B-planar monogenics? Noting that each unit 2-blade corresponds to a unique 2-plane in $\mathbb{R}^n$, we can realize every B as a point in $\mathrm{Gr}(2,n)$. Letting f_B be some B-planar axial monogenic, is

$$f = \int_{B \in \mathrm{Gr}(2,n)} a(B) f_B \, d\lambda,$$

where $a(B)$ is a scalar function on $\mathrm{Gr}(2,n)$ and $d\lambda$ is the Haar measure on $\mathrm{Gr}(2,n)$ monogenic? Moreover, can any monogenic f be constructed in this manner? First, we start with a lemma describing the form of f_B.

Lemma 6.1. *Let f be a monogenic (0+2)-vector and define $f_B \coloneqq \mathrm{P}_B(f(\mathrm{P}_B(x)))$. Then f_B is B-planar and monogenic.*

Proof. It is clear by definition that f_B is constant along translations of the B-plane and can be written as $u_B + \beta b_B$ and so f_B is B-planar. To see f_B is monogenic, let $e_1, \dots, e_n$ be a basis such that $B = e_1 e_2$ and $e_i \cdot B = 0$ for $i \neq 1,2$. Then note $\nabla_{e_i} f_B = 0$ when $i \neq 1,2$ as well leading to

$$\boldsymbol{\nabla} f_B = e^1 \nabla_{e_1} f_B + e^2 \nabla_{e_2} f_B$$

Recall that $f = u + b$ must satisfy

$$\boldsymbol{\nabla} \wedge u = \boldsymbol{\nabla} \cdot b \qquad \text{and} \qquad \boldsymbol{\nabla} \wedge b = 0.$$

Specifically,

$$e^1 \wedge \nabla_{e_1} u + e^2 \wedge \nabla_{e_2} u + \cdots + e^n \wedge \nabla_{e_n} = e^1 \cdot \nabla_{e_1} b + e^2 \cdot \nabla_{e_2} b + \cdots + e^n \cdot \nabla_{e_n} b$$

Clearly, $\boldsymbol{\nabla} \wedge b_B = 0$, thus we need only show

$$\boldsymbol{\nabla} \wedge u_B = \boldsymbol{\nabla} \cdot b_B.$$

In particular □

We can note that the B-planar monogenics are given by a power series $\sum_{n=0}^{\infty} a_n (x + yB)^n$ due to the isomorphism of algebras $\mathfrak{spin}(2) \cong \mathbb{C}$ This shouldn't be hard to show without appealing to this isomorphism. In particular, any B-planar monogenic is approximated arbitrarily closely by a homogeneous polynomial of degree n in the variables x and y. Moreover, 1 and $x + yB$ generate the B-planar monogenics. $\mathrm{Spin}(n)$ then acts on B. Okay, well maybe there's some nice way to talk about characters as mappings to the grassmannian instead of the circle? Should read more about characters and maybe they are really maps to spin group? They are for the 2d case. Structure space and stuff. Should probably rename some of these things I have.s

Countable basis for $\mathcal{M}$?

6.2 Generating axial monogenics

The following questions remain for a domain in $\mathbb{R}^3$.

Question 6.1. For what boundary values $\varphi \in C_\infty(\Sigma)$ can we generate axial monogenics?

Question 6.2. Do these boundary values exhaust the whole axial algebra $\mathcal{A}_\omega$?

Fix an axis ω which defines the blade $B = \omega I$ and thus defines the B-plane in $\mathbb{R}^3$. Then, let $f = u + \beta B$ be an ω-axial monogenic. We can then determine the boundary values for f on Σ by orthogonal projection onto the B-plane. That is, we care only about the components of f perpendicular to the axis ω and hence we take for $\zeta \in \Sigma$

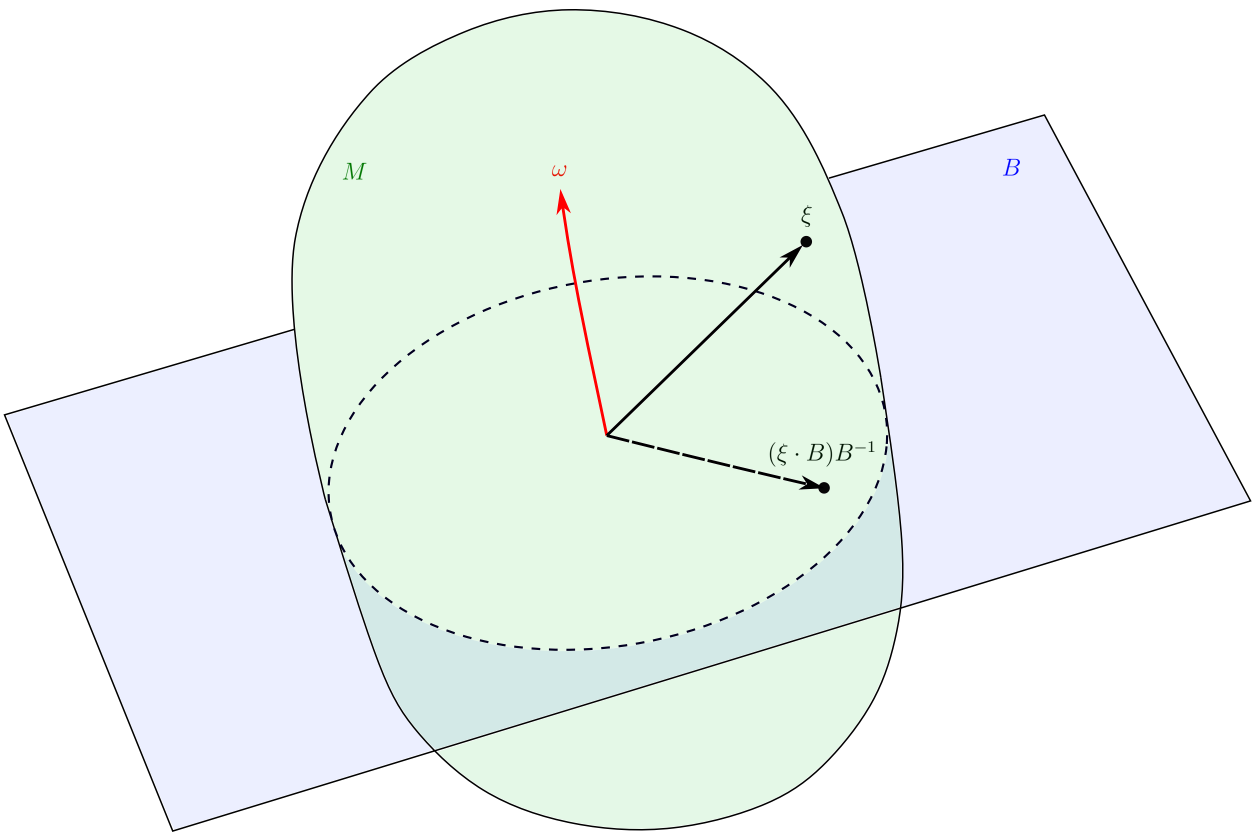
$$\zeta^{\perp} = \omega\omega \wedge \zeta = (x \cdot B)B^{-1}.$$

showing the relationship between projection onto a plane and being orthogonal to an axis in $\mathbb{R}^3$. Specifically, this means that the relationship $f(x) = f(x + t\omega)$ can be written as

$$f(x) = f((x \cdot B)B^{-1}),$$

in that we only care about the portion of x along the plane given by B. Thus, for $\xi \in \Sigma$ we have

$$f(\xi) = f((\xi \cdot B)B^{-1}).$$



So boundary values of axial monogenics are axial and...?.

Example 6.1. Consider the 3-dimensional example with $M = B_3$ and $\Sigma = S^2$. Let e_1, e_2, e_3 be a global orthonormal basis and let $g_{ij} = \delta_{ij}$. Then let $B = e_1 \wedge e_2$. Then the paravector field $f(x^1, x^2, x^3) = x^1 + x^2 B$ is e_3-axial. Clearly we can see that $f(x^1, x^2, x^3 + t) = f(x^1, x^2, x^3)$ for any t. f is also monogenic as one can show

$$\boldsymbol{\nabla} f = e_1 + (e_2 \wedge e_3)I = e_1 - e_1 = 0.$$

Indeed, this f is none other than the complex function $f(z) = z$ with B taking the role of the imaginary unit i.

Let $x = x^1e_1 + x^2e_2 + x^3e_3$. Then,

$$B(x \cdot B) = (e_1e_2)(x^1e_2 - x^2e_1) = x^1e_1 + x^2e_2.$$

Thus, for $\xi \in S^2$, we have $f(\xi) = \xi^1 + \xi^2 B$.

If we consider now every ω-axial monogenic can be written as a power series, if we can construct z we should be done...?

It is clear that we can define a monogenic field $f = u + b$ via the Cauchy integral, but we then require $\nabla_\omega f = 0$. Let $f = \mathcal{C}[\varphi](x)$, then we must have

$$\nabla_\omega \langle \mathcal{C}[\varphi](x)\rangle_0 = 0 \qquad \text{and} \qquad \nabla_\omega \langle \mathcal{C}[\varphi](x)\rangle_2 = 0.$$

The first condition yields

$$0 = \int_\Sigma \frac{(\nu(\zeta)\cdot x)(\omega \cdot x)}{|x-\zeta|^2}\phi(\zeta)d\Sigma(\zeta).$$

Theorem 6.1. *For any $\omega \in Gr(1,3)$ we have that $\mathcal{A}_\omega \subset \mathcal{M}$.*

Proof. This seems to be saying that we need boundary values in some hardy space or something. They defined this conjugacy thing as G. Fix a unit vector ω. We want to show that for any $f = u + b \in \mathcal{A}_\omega$ that $\iota^* u = \phi$ satisfies eq:conjugacy_requirement ??. That is,

$$G\phi = (\Lambda - d\Lambda^{-1}d)\phi = 0.$$

Note that ϕ is the trace of a harmonic function, so this operator is well defined. Note that the equation

$$\Lambda\psi = d\phi$$

has a solution □

7 Radon transform and integral geometry

I feel like there is some way to go from projection onto subspaces as a map to grassmannians and reconstructing the manifold. It's like a morse function type of thing. Radon transforms also come to mind.

8 Relation to the BC Method

Describe how this process can lead to the BC method in dimension $n = 2$

9 Conclusion

A Appendix

Put axial condition for cauchy integral and some other quick proofs in here.

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