MATH 560, Homework 4

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Solutions

Problem 1. (§2.3 Problem 12) Let V, W, and Z be vector spaces, and let $T: V \to W$ and $U: W \to Z$ be linear.

(a) Prove that if UT is injective, then T is injective. Must U also be injective?

(b) Prove that if UT is surjective, then U is surjective. Must T also be surjective?

(c) Prove that if U and T are bijective, then UT is also.

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Proof (Part (a)). Suppose that UT is injective. Then for distinct $v_1, v_2 \in V$ we have $UT(v_1) = z_1$ and $UT(v_2) = z_2$ where $z_1 \neq z_2$. This means we also have $U(w_1) = z_1$ and $U(w_2) = z_2$ with $w_1, w_2 \in W$ with $w_1 \neq w_2$ else otherwise we'd have that $z_1 = z_2$. Thus we have that T is injective. Also, we must have that U is injective. $\square$

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Proof (Part (b)). Suppose that UT is surjective. Then for any $z \in Z$ we have that $\exists v \in V$ so that $UT(v) = z$. Then $T(v) = w \in W$ and that $U(w) = z$. Since z was arbitrary, U is surjective. Also, T need not be surjective. $\square$

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Proof (Part (c)). Suppose that U and T are bijective. Then if we have $v_1, v_2 \in V$ distinct we also have $T(v_1) = w_1$ and $T(v_2) = w_2$ with $w_1 \neq w_2$ by the surjectivity and injectivity of T. Similarly we also have $U(w_1) = z_1$ and $U(w_2) = z_2$ with $z_1 \neq z_2$ by the injectivity and surjectivity of U. Thus $UT(v_1) = z_1$ and $UT(v_2) = z_2$ with arbitrary z_1 and z_2 and we conclude that UT is bijective. $\square$

Problem 2. (§2.3 Problem 17) Let V be a vector space. Determine all linear transformations $T: V \to V$ such that $T = T^2$. *Hint:* Note that $x = T(x) + (x - T(x))$ for every x in V, and show that $V = \{y | T(y) = y\} \oplus \mathcal{N}(T)$ (see the exercises of §1.3).

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Proof. For every $x \in V$ we have that $x = T(x) + (x - T(x))$. $T(x) \in \mathscr{R}(T)$ and since $T(x - T(x)) = T(x) - T^2(x) = T(x) - T(x) = 0$ we have that $x - T(x) \in \mathcal{N}(T)$. So $V = \mathscr{R}(T) + \mathcal{N}(T)$ so we know that for linear operators we have $V = \{x | T(x) = x\} \oplus \mathcal{N}(T)$. (This is from an exercise earlier in the book.) $\square$

Problem 3. (§2.4 Problem 13.) Let $\sim$ mean "is isomorphic to." Prove that $\sim$ is an equivalence relation on the class of vector spaces over $\mathbb{F}$.

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Proof. Let V, W, Z be vector spaces over $\mathbb{F}$. Let $\sim$ mean "is isomorphic to." Then

- We have $V \sim V$ verified by letting the identity map be the isomorphism.
- Suppose we have $V \sim W$. Then there exists an isomorphism $T\colon V \to W$ and thus $T^{-1}\colon W \to V$ is also an isomorphism. Thus $W \sim V$.
- Let $V \sim W$ and $W \sim Z$ by $T\colon V \to W$ and $U\colon W \to Z$. Then UT is bijective since compositions of bijections are bijective. Thus we have $UT\colon V \to Z$ is an isomorphism and $V \sim Z$.

So $\sim$ is an equivalence relation. $\square$

Problem 4. (§2.4 Problem 16.) Let B be an $n \times n$ invertible matrix. Define $\Phi\colon M_{n\times n}(\mathbb{F}) \to M_{n\times n}(\mathbb{F})$ by $\Phi(A) = B^{-1}AB$. Prove that Φ is an isomorphism.

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Proof. To show Φ is an isomorphism we will show that it is invertible. Consider Φ^{-1} defined by $\Phi^{-1}(A) = BAB^{-1}$. Then $\Phi^{-1}(\Phi(A)) = \Phi^{-1}(B^{-1}AB) = BB^{-1}ABB^{-1} = A$. Thus Φ is an isormorphism. $\square$

Problem 5. (§2.4 (Problem 17.) Let V and W be finite-dimensional vector spaces and $T: V \to W$ be an isomorphism. Let V_0 be a subspace of V.

(a) Prove that $T(V_0)$ is a subspace of W.

(b) Prove that $\dim(V_0) = \dim(T(V_0))$.

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Proof (Part (a)). We show three properties that prove $T(V_0)$ is a subspace of W.

- Since V_0 is a subspace, $0 \in V_0$ and we have $T(0) = 0$ and so $0 \in T(V_0)$.
- Let $u, w \in V_0$, then $u + w \in V_0$. Then $T(u + w) = T(u) + T(w)$ and since $T(u + w) \in T(V_0)$ then $T(u) + T(w) \in T(V_0$. Since u, w were arbitrary, we have that $T(V_0)$ is closed under addition.
- Let $u \in V_0$ and $a \in \mathbb{F}$ then, $au \in V_0$ so $T(au) = aT(u)$. We then have $T(au) \in T(V_0)$ and thus $aT(u) \in T(V_0$ so $T(V_0)$ is closed under scalar multiplication. Thus $T(V_0)$ is a subspace of W. ☐

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Proof (Part (b)). By the dimension theorem $\dim(V_0) = \dim(\mathscr{R}(T)) + \dim(\mathscr{N}(T))$. Since T is an isomorphism it is thus injective and $\mathscr{N}(T) = \{0\}$. Thus $\dim(V_0) = \dim(\mathscr{R}(T)) = \dim(T(V_0))$. ☐

Problem 6. (§2.4 Problem 24.) Let $T\colon V \to Z$ be a linear transformation of a vector space V onto a vector space Z. Define the mapping

$$\bar{T}\colon V/\mathcal{N}(T) \to Z \quad \text{by} \quad \bar{T}(v+\mathcal{N}(T)) = T(v)$$

for any coset $v+\mathcal{N}(T)$ in $V/\mathcal{N}(T)$.

(a) Prove that $\bar{T}$ is well-defined; that is, prove that if $v+\mathcal{N}(T) = v'+\mathcal{N}(T)$, then $T(v) = T(v')$.

(b) Prove that $\bar{T}$ is linear.

(c) Prove that $\bar{T}$ is an isomorphism.

(d) Prove that the diagram shown in Figure 2.3 commutes; that is, prove that $T = \bar{T}_\eta$.

$$\begin{tikzcd} V \arrow[r, dashed, "T"] \arrow[d, dashed, "\eta"'] & Z \\ V/\mathcal{N}(T) \arrow[ur, dashed, "\bar{T}"'] & \end{tikzcd}$$

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Proof (Part (a)). We have

$$\begin{aligned} v+\mathcal{N}(T) &= v'+\mathcal{N}(T) \\ \bar{T}(v+\mathcal{N}(T)) &= \bar{T}(v'+\mathcal{N}(T)) \\ T(v) &= T(v'). \end{aligned}$$

So $\bar{T}$ is well defined. ☐

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Proof (Part (b)). We have for $u+\mathcal{N}(T), v+\mathcal{N}(T) \in V/\mathcal{N}(T)$ and $a \in \mathbb{F}$

$$\begin{aligned} \bar{T}((u+\mathcal{N}(T))+a(v+\mathcal{N}(T))) &= \bar{T}(u+av+\mathcal{N}(T) \\ &= T(u+av) \\ &= T(u)+aT(v) \\ &= \bar{T}(u+\mathcal{N}(T))+\bar{T}(v+\mathcal{N}(T)). \end{aligned}$$

So $\bar{T}$ is linear. ☐

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Proof (Part (c)). To show $\bar{T}$ is an isomorphism we define $\bar{T}^{-1}$ by $\bar{T}^{-1}(v) = v+\mathcal{N}(T)$. Then for any $v+\mathcal{N}(T) \in V/\mathcal{N}(T)$ we have $\bar{T}^{-1}\bar{T}(v+\mathcal{N}(T)) = \bar{T}^{-1}(v) = v+\mathcal{N}(T)$. Since the inverse exists, $\bar{T}$ is an isomorphism. ☐

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Proof (Part (d)). We have for $v \in V$

$$\begin{aligned} \bar{T}\eta(v) &= \bar{T}(v+\mathcal{N}(T)) \\ &= T(v). \end{aligned}$$

So the diagram commutes. ☐

Problem 7. (§2.5 Problem 8.) Prove the following generalization of Theorem 2.23. Let $T\colon V \to W$ be a linear transformation from a finite-dimensional vector space V to a finite-dimensional vector space W. Let β and β' be ordered bases for V, and let γ and γ' be ordered bases for W. Then $[T]_{\beta'}^{\gamma'} = P^{-1}[T]_\beta^\gamma Q$, where Q is the matrix that changes β'-coordinates into β-coordinates and P is the matrix that changes γ'-coordinates into γ-coordinates.

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Proof. Take $v \in V$ and we have

$$[T]_{\beta'}^{\gamma'}[v]_{\beta'} = [Tv]_{\gamma'};$$

as well as

$$\begin{aligned} P^{-1}[T]_\beta^\gamma Q[v]_{\beta'} &= P[T]_\beta^\gamma[v]_\beta \\ &= P^{-1}[Tv]_\gamma \\ &= [Tv]_{\gamma'} \end{aligned}$$

Thus we have $[T]_{\beta'}^{\gamma'} = P^{-1}[T]_\beta^\gamma Q$. $\square$

Problem 8. (§2.5 Problem 9.) Prove that “is similar to" is an equivalence relation on $M_{n\times n}(\mathbb{F})$.

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Proof. We show $\sim$ meaning, “is similar to" is an equivalence relation by satisfying the three requirements.

- $A \sim A$ via $A = I^{-1}AI = I^{-1}A = A$.
- Let $A \sim B$ and thus we have $A = Q^{-1}BQ$ which implies that $QAQ^{-1} = B$. Then let $Q^{-1} = S$ and then $B = S^{-1}AS$ which means $B \sim A$.
- Let $A \sim B$ and $B \sim C$ then $A = Q^{-1}BQ$ and $B = S^{-1}CS$. So then we have $A = S^{-1}Q^{-1}CQS$ and then let $P = QS$ and thus $A = P^{-1}CP$ so that $A \sim C$. □

Problem 9. (§2.5 Problem 11.) Let V be a finite-dimensional vector space with ordered bases α, β, and γ.

(a) Prove that if Q and R are the change of coordinate matrices that change α-coordinates into β-coordinates and β-coordinates into γ-coordinates, respectively, then RQ is the change of coordinate matrix that changes α-coordinates into γ-coordinates.

(b) Prove that if Q changes α-coordinates into β-coordinates, then Q^{-1} changes β-coordinates into α-coordinates.

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Proof (Part (a)). We have that $Q[v]_\alpha = [v]_\beta$ and that $R[v]_\beta = [v]_\gamma$. Then $RQ[v]_\alpha = R[v]_\beta = [v]_\gamma$. □

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Proof (Part (b)). Take $Q[v]_\alpha = [v]_\beta$ and then $I[v]_\alpha = Q^{-1}Q[v]_\alpha = Q^{-1}[v]_\beta = [v]_\alpha$. □

Problem 10. (§2.5 Problem 13.) Let V be a finite dimensional vector space over a field $\mathbb{F}$, and let $\beta = \{x_1, x_2, \ldots, x_n\}$ be an ordered basis for V. Let Q be an $n \times n$ invertible matrix with entries from $\mathbb{F}$. Define

$$x'_j = \sum_{i=1}^{n} Q_{ij} x_i \quad \text{for } 1 \leq j \leq n,$$

and set $\beta' = \{x'_1, x'_2, \ldots, x'_n\}$. Prove that β' is a basis for V and hence that Q is the change of coordinate matrix changing β'-coordinates into β-coordinates.

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Proof. We have that $\beta' = \{x'_1, \ldots, x'_n\}$ with each x'_j defined by $\sum Q_{ij} x_i = x'_j$. Since Q is invertible, we have that Q is an isomorphism and thus is bijective. Since Q is bijective, it must be injective and surjective and thus x'_j are linearly independent and span V. Thus β' is a basis for V. $\square$