The Calderón Problem

on Riemannian Manifolds

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Outline

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- Rephrase the problem in a geometrical way.
- Prove the problem in 2 dimensions using the boundary control method.
- What can and can't we do to generalize this method?

Section 1

Introduction

Subsection 1

Calderón Problem

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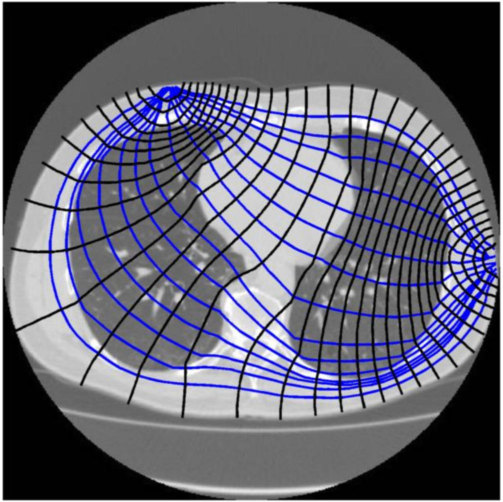
- This is the Electrical Impedence Tomography problem.
- Originally his motivation was for oil prospecting.
- This problem sparked interest due to its usefulness in geophysical and medical imaging.



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- Can we determine the conductivity matrix γ from Λ?

Section 2

The Calderón Problem on Riemannian Manifolds

Subsection 1

Preliminaries

Geometry

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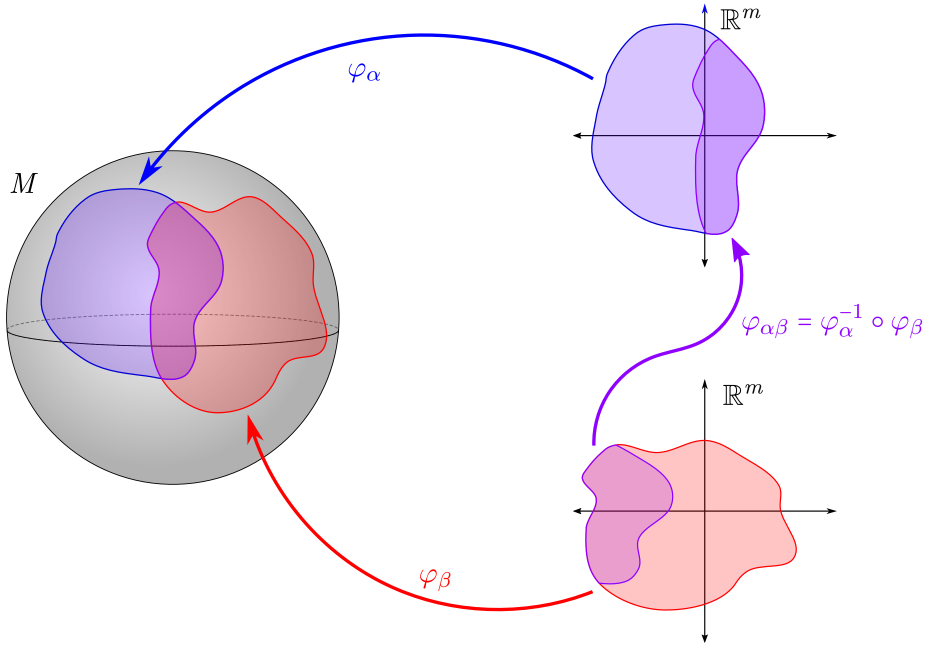
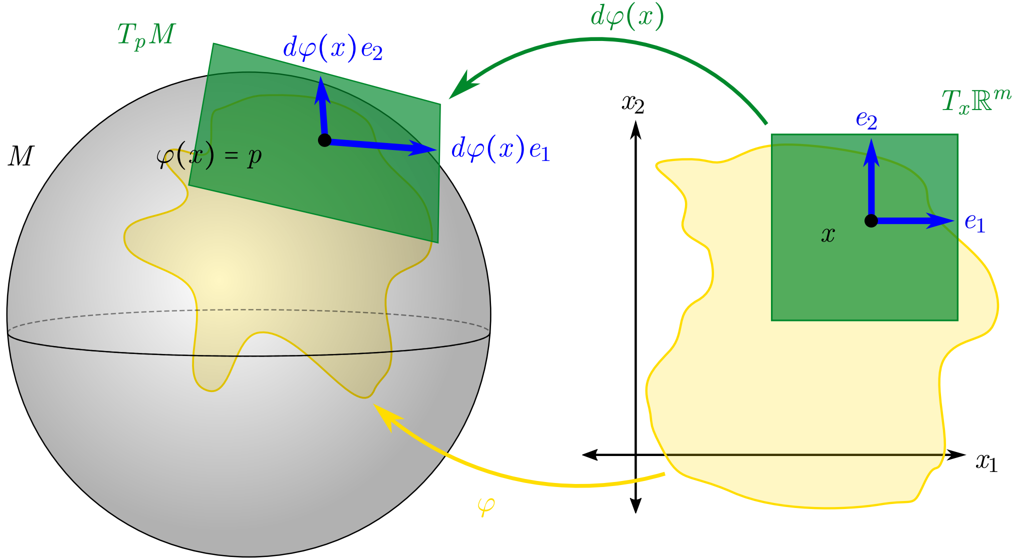
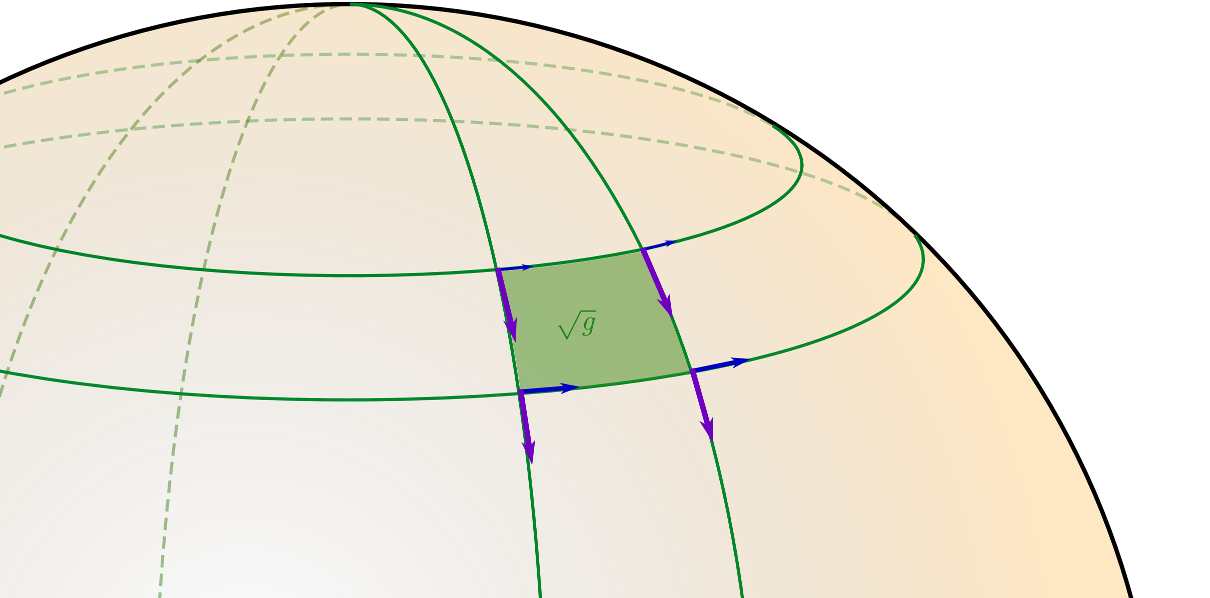
- ***Hodge Star***: Attached to the exterior algebra when we also have a Riemannian metric. Gives an isomorphism between k and $n-k$-forms.

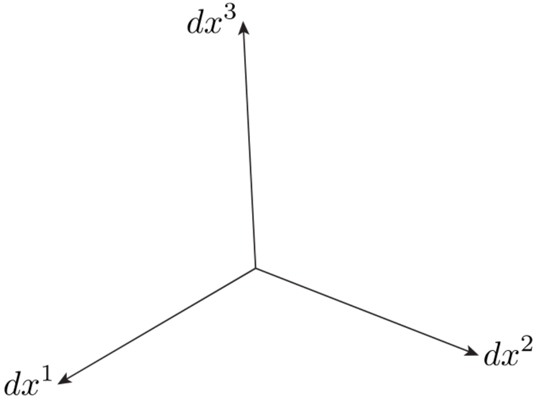


M
φ_α
$\mathbb{R}^m$
$\varphi_{\alpha\beta} = \varphi_\alpha^{-1} \circ \varphi_\beta$
$\mathbb{R}^m$
φ_β

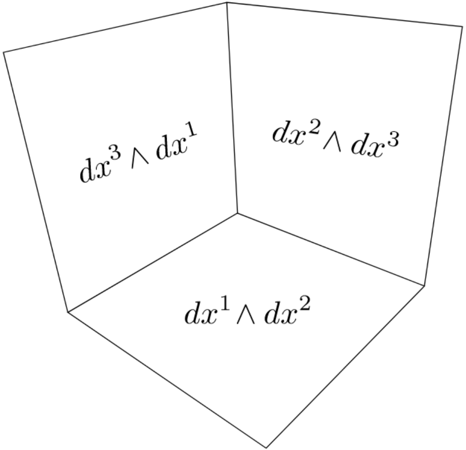

T_pM
$d\varphi(x)e_2$
$d\varphi(x)$
$T_x\mathbb{R}^m$
M
$\varphi(x) = p$
$d\varphi(x)e_1$
x_2
e_2
e_1
x
x_1
φ


$\sqrt{g}$

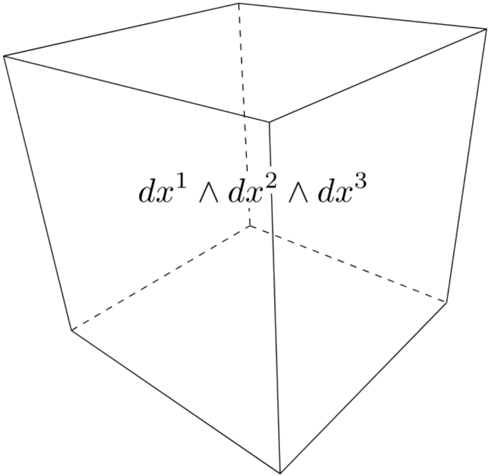
1-Forms



2-Forms



3-Forms



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- ***Dirac Operator***: $D = d + \delta$.
- ***Laplace-Beltrami Operator***: $\Delta = d\delta + \delta d = D^2$ and in coordinates

$$\Delta f = \frac{1}{\sqrt{|g|}} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial}{\partial x^i} \sqrt{|g|} g^{ij} \frac{\partial}{\partial x^j} f$$

Subsection 2

Raphrasing EIT Problem in a Geometrical Language

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- Recover g from knowing Λ.

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Expected and Current Results

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So, g is a function of n variables that needs to be determined by the kernel λ which is $2n - 2$ variables.

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- $n \geq 3$ is overdetermined.

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Indeed, let $\tilde{g} = e^{2\phi} g$, then

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When $n = 2$, the extra term cancels.

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- Can only know the total impedence between the two electrodes.

Isotropic Case

For g isotropic and $n \geq 3$ one can determine g from Λ.

(*Sylvester-Uhlmann 1987*)

2 Dimensional Anisotropic

- Can recover g up to conformal class and can't do better.
- Proven by Lassas and Uhlmann in *On Determining the Riemannian manifold from the Dirichlet to Neumann map*.

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- Can determine the boundary C^∞-jet of g in Lee and Uhlmann's *Determining anisotropic real-analytic conductivities by boundary measurements.*
- For smooth manifolds, the anisotropic problem is open. The goal is to recover the metric up to isometry.

Section 3

Boundary Control Method in 2 Dimensions

Theorem

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Two 2-dimensional compact orientable manifolds with single common boundary are conformally equivalent iff their DN-maps coincide.

Belishev's *The Calderón Problem for Two-Dimensional Manifolds by the BC-Method.*

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- Gelfand transform relates an algebra $\mathcal{A}$ to the algebra of continuous functions on the spectrum of that algebra, $C(\mathrm{spec}\mathcal{A})$.
- This gives us a way to realize Ω from functions defined on Ω that we have access to.

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- Show that this algebra is generic. (Lemma 2)
- Represent the trace algebra with the DN map. (Lemma 3)
- Construct the manifold. (Theorem)

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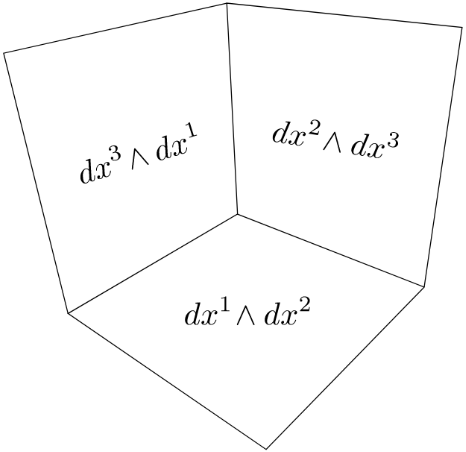
- The pullback $\iota^*\colon T^*\Omega \to T^*\partial\Omega$.
- Define Λ by $\iota^*(\star du)$ for a harmonic u.
- Λ maps boundary k-forms to boundary $n-k-1$ forms.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- $\dim \operatorname{Ran} \left[\Lambda + d\Lambda^{-1} d\right] = \beta_1(\Omega).$

Corollary

Λ completely determines the topology of Ω.

Proof

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- Since Ω is a single connected component, $\beta_0(\Omega) = 1$.
- We have $\beta_1(\Omega)$ from before.
- Since Ω is a surface with boundary, $\beta_2(\Omega) = 0$.
- Since Ω is two dimensional, $\beta_n(\Omega) = 0$ for $n \geq 3$.

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- $Dw = 0$ gives us the Cauchy-Riemann equations
- We call u and v conjugate by CREs.

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- Define $\mathcal{H} = d\Lambda^{-1}$.

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- This is analogous to having the Hodge star on a surface.

Subsection 2

Lemma 2

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- $\mathcal{M}$ with this topology is called the ***spectrum*** $\mathrm{spec}\mathcal{A}$.

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- A generic algebra is (spatially) isomorphic to its Gelfand transform.

Lemma 2

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The algebra of holomorphic functions $\mathcal{A}(\Omega)$ is generic.

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- The lemma shows that $\epsilon\colon \Omega \to \mathrm{spec}\mathcal{A}(\Omega)$ is a homeomorphism, so we have determined Ω up to homeomorphism.

What's Left?

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We can only have hope access to the trace algebra $\mathcal{A}(\partial\Omega)$. So we need to determine this to reach our goal.

Subsection 3

Lemma 3

Trace Algebra

Trace Algebra

- The trace algebra $\mathcal{A}(\partial\Omega) \coloneqq \iota^* \mathcal{A}(\Omega)$ is isometrically isomorphic to $\mathcal{A}(\Omega)$ since

$$\|w\|_{\mathcal{A}(\Omega)} = \|\iota^* w\|_{\mathcal{A}(\partial\Omega)}$$

  and since a holomorphic function is uniquely determined by its boundary values.

Lemma 3

We have the representation

$$\mathcal{A}(\partial\Omega) = \mathrm{clos}_{C(\partial\Omega)}\{f + ih\},$$

where h is conjugate to f by $\mathcal{H}$.

Subsection 4

Proof of the Main Theorem

Construction of the Manifold

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- Step 1: We know $g|_{\partial\Omega}$ by Lee and Uhlmann, and thus we know $\mathcal{H}$ and $C^\infty(\partial\Omega)$. This allows us to recover the trace algebra $\mathcal{A}(\partial\Omega)$ using the representation in Lemma 3.

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- Step 5: Equip Ω with a metric g conforming to this complex structure.

Section 4

Generalizing This Method

First Issue

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- No complex structure in higher dimensions.

Resolution

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- The tools of Clifford analysis allow us to recover a notion of holomorphicity known as ***monogenicity***.
- We can recover a similar algebra (Hardy space) of monogenic functions.

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- Geometric product can be extended to multivectors.

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$$x + iy \iff 1x + e_1 \wedge e_2 y,$$

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  where $e_1 \wedge e_2$ is the bivector (and psuedoscalar).
- $\mathbb{H}$ lives inside of $\mathrm{C}\ell(\mathbb{R}^3, \cdot)$ as the even subalgebra.

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- We can replace $d + \delta$ with the Dirac operator D

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- Monogenic objects in even subalgebra have components that are harmonic.
- There are Cauchy integral and Hilbert transform type operators for D in arbitrary dimension.

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- The spectral theory for Belishev's solution required a commutative Banach algebra.
- The spectral theory for noncommutative Banach algebras is not as developed.
- There is still work to do here to find some way around this.

Section 5

Conclusion

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- Ideal results are still not yet obtained.

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- It relies heavily on complex analysis and the spectral theory for commutative Banach algebras.
- These issues remain if we try to naively generalize this approach.

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- We can still construct the same algebra of holomorphic functions.
- The relevant algebras are noncommutative for dimensions ≥ 3.
- There are possibly other tools at our disposal that may be able to replace the loss of commutivity.

Thank you!