Von Neumann Algebras in Mathematics and Physics

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Motivation

The last ten years or so have seen a considerable synthesis in mathematics and mathematical physics. In this talk I will be concerned only with those topics appearing in Fig. 1, all connected by having something to do with braids.

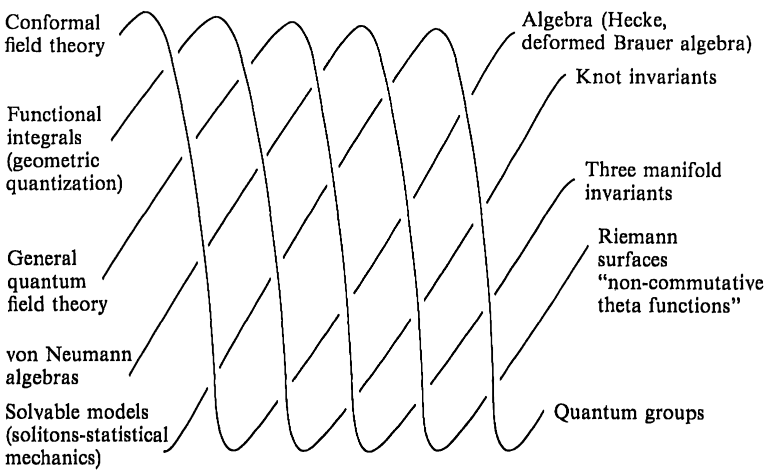

Fig. 1

Many very different themes could be used for a talk such as this one, but I have chosen von Neumann algebras because they are what led me into this circle of ideas. Thus the presentation will be historical rather than logical.

A *von Neumann algebra* M is a $*$-algebra of bounded operators on a Hilbert space $(\mathscr{H}, \langle\ ,\ \rangle)$ which contains the identity and is closed in the weak operator topology, i.e. if a_n is a net of operators in M and $\langle a_n\xi, \eta\rangle \to \langle a\xi, \eta\rangle$ for some a and all ξ and η in $\mathscr{H}$, then a is in M. Most of the interest is when M is infinite dimensional so it should be pointed out at the outset that a finite dimensional von Neumann algebra is just a direct sum of matrix algebras, each acting with a certain multiplicity on $\mathscr{H}$.

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These algebras were first introduced by Murray and von Neumann in [MvN1]. Their motivations were many, not the least being a beautiful result of von Neumann in [vN] which showed that von Neumann algebras could equally be defined as *commutants* of self-adjoint sets (notation: if S is a set of bounded operators, S', the commutant of S, is by definition the set of all bounded operators x such that $xs = sx$ for all s in S). This means that von Neumann algebras are the algebras of *symmetries* for any structure belonging to Hilbert space, such as geometric configurations of subspaces or unitary group representations. Let me discuss some other motivations for looking at von Neumann algebras.

a) Unitary Group Representations

The abstract theory of representations of a compact group is very complete but as soon as the group is not compact, many different phenomena occur. A unitary representation no longer decomposes simply as a direct sum of irreducibles, which is reflected in a possibly exotic structure of the commutant of the group. This already happens for, say, the regular representation of the free group on two generators. Of course a semisimple Lie group G is an example of a "type I" group where there is a satisfactory reduction to irreducible representations, but even here the restriction of a discrete series representation of G to a lattice Γ in G will resemble the regular representation of Γ. So even in arithmetic questions (say $G = PSL_2(\mathbb{R})$, $\Gamma = PSL_2(\mathbb{Z})$), "exotic" von Neumann algebras occur quite naturally. For infinite dimensional groups such as loop groups the situation is even more interesting.

b) Abstract Algebra

The abstract theory of finite dimensional semisimple algebras over a field is also a complete theory. When the algebra becomes infinite dimensional, general theory must be replaced by a heterogeneous collection of examples unless some analysis is introduced. Over $\mathbb{C}$ semisimplicity is implied (in finite dimensions) by the existence of a $*$ operation and to say that an algebra is closed in the weak topology is the strongest reasonable condition on it. So von Neumann algebras should be the best behaved abstract family of infinite dimensional algebras. Although the theory is considerably richer than in finite dimensions there do exist non-trivial general results and there is a significant class of von Neumann algebras (those approximable by finite dimensional algebras) for which a complete classification exists.

c) Unbounded Operators

At first sight the important operator d/dx appears unnatural on Hilbert space since it is not defined on all vectors (in $L^2(\mathbb{R})$). However the opposite is true. Provided d/dx is given the right *domain* its graph is a closed subspace and hence natural to Hilbert space. In general an operator is called pre-closed if it is densely defined and the closure of its graph is the graph of an operator. All interesting linear operators seem to be pre-closed.[1]

The trouble with domains comes when one tries to add and multiply unbounded operators. For this reason one would hope to replace unbounded operators by bounded ones. Thus the relations $[P, Q] = \mathrm{id}$ can be handled in the Weyl form with two unitary groups $U(t)$ and $V(t)$ with $U(t)V(s) = e^{2\pi ist}V(s)U(t)$. In general one could use the von Neumann algebra of all bounded operators having the same symmetries as the unbounded ones.

d) Quantum Theory

The very language of quantum mechanics suggests von Neumann algebras. States are vectors in a Hilbert space. Observables are self-adjoint operators and numerical information about observables for systems in states is given by scalar products. Thus it is quite natural to consider a von Neumann algebra of observables associated with any subsystem of a quantum system. Certainly von Neumann was thinking along these lines. Such an approach was indeed adopted as an "algebraic" approach to quantum field theory by Haag and Kastler [HK] who postulated von Neumann algebras associated with regions of space time and satisfying certain causality, positivity and Lorentz covariance conditions. Although it is very abstract and difficult to produce mathematical examples, some things can be deduced from such a general theory and new results in von Neumann algebras have recently added to the possibilities.

1. Factors and Their Types

The spectral theorem shows that abelian von Neumann algebras are abstractly of the form $L^\infty(X, \mu)$ acting with some multiplicity on a Hilbert space. Attention turns immediately to *factors* which are by definition von Neumann algebras with trivial centre. The simplest example of such a factor is $\mathscr{B}(\mathscr{H})$, the algebra of all bounded operators. One of the fundamental discoveries of Murray and von Neumann was that of factors not abstractly isomorphic to any $\mathscr{B}(\mathscr{H})$.

The first example was that of the commutant M of the left regular representation of any discrete group Γ all of whose (non-identity) conjugacy classes are infinite (such as free groups F_n or $PSL_n(\mathbb{Z})$ for $n \geq 2$). One can show that this von Neumann algebra is generated by the right regular representation and that it is a factor. Also if $\xi \in l^2(\Gamma)$ is the characteristic function of the identity then the linear functional $\mathrm{tr}(x) = \langle x\xi, \xi \rangle$ defines a *trace* on M (i.e. $\mathrm{tr}(xy) = \mathrm{tr}(yx)$). There is no such functional on $\mathscr{B}(\mathscr{H})$ if $\dim \mathscr{H} = \infty$.

An infinite dimensional factor admitting such a trace is called a *type* II_1 *factor* and a factor of the form $\mathscr{B}(\mathscr{H}) \otimes$ (a type II_1 factor) is called a type II_∞ factor if $\dim \mathscr{H} = \infty$. A factor which is neither of type I ($\cong \mathscr{B}(\mathscr{H})$) nor of type II is called a type III factor.

[1] I can only think of one exception – the "derivations" in Fox's free differential calculus are not pre-closed on l^2 of the free group.

A II_1 factor shares some of the nice features of a finite dimensional matrix algebra (e.g. it is simple). Its most seductive feature is continuous dimensionality. If one looks at the numbers trace(p) (trace normalized so that trace(1) $= 1$), where $p \in M_n(\mathbb{C})$ are projections, one obtains the numbers m/n for $m = 0, 1, \dots, n$, where m is of course the rank of p. In a II_1 factor one obtains the whole unit interval $[0, 1]$.

2. GNS Construction

An important construction is the Gelfand-Naimark-Segal construction. One begins with a $*$-algebra A and a linear functional (state) $\varphi : A \to \mathbb{C}$ with $\varphi(a^*a) \geq 0$. Define $\langle\ ,\ \rangle$ on A by $\langle a, b\rangle = \varphi(b^*a)$. Quotienting if necessary A becomes a pre-Hilbert space and its completion is written $\mathscr{H}_\varphi$. Under mild conditions A will act on $\mathscr{H}_\varphi$ by left multiplication and the von Neumann algebra it generates is said to result from the GNS construction from φ. It should be thought of as the completion of A with respect to φ. Examples of all types of factors can now be obtained, using $A = \otimes_{i=1}^{\infty} M_2(\mathbb{C})$. If $h_i \in M_2(\mathbb{C})$ are positive matrices of trace 1 then the formula $\varphi(\otimes_{i=1}^{\infty} x_i) = \prod_{i=1}^{\infty} \mathrm{trace}(h_i x_i)$ defines a state, said to be a product state. The result of the GNS construction is then always a factor. It is of type I if $h_i = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, of type II_1 if $h_i = (1/2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and of type III if $h_i = (1+\lambda)^{-1}\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ for $0 < \lambda < 1$. In the last case the state is known as the Powers state after R. Powers [Pow] who proved that the factors for different λ are mutually non-isomorphic.

3. Modular Theory

The Tomita-Takesaki theory shows that to every weakly continuous state φ on a von Neumann algebra M there is a one parameter group σ_t of automorphisms of M characterized by the KMS condition $\varphi(xy) = \varphi(\sigma_i(y)x)$. It allows one to subdivide the type III factors into type III_λ, $\lambda \in [0, 1]$ where the Powers factors are of type III_λ, $\lambda \in (0, 1)$. Types III_0 and III_1 can be obtained from product states by suitable choices of the h_i's. Generically a factor is of type III_1. For details we refer to Connes' Helsinki congress talk [Co1], or [Ta].

4. Hyperfiniteness

A von Neumann algebra M is *hyperfinite* if there is an increasing sequence of finite dimensional $*$-subalgebras whose union is weakly dense in M. Thus our infinite tensor product factors are hyperfinite, by construction, but Murray and von Neumann showed that the free group II_1 factor is not hyperfinite, nor are the examples coming from lattices in semisimple Lie groups.

All hyperfinite factors are known. Here is a table of them.

Table 1. Hyperfinite factors

Type I_n $n = 1, 2, \ldots, \infty$	One for each n Proof: elementary
Type II_1	A unique factor, denoted R. Uniqueness proved by Murray and von Neumann in [MvN2]
Type II_∞	A unique factor. Uniqueness proved by Connes in [Co2]
Type III_0	One for each ergodic non-transitive flow. Proved by Krieger [Kr] and Connes [Co2]
Type III_λ $0 < \lambda < 1$	Powers factors are the unique examples. Uniqueness proved by Connes [Co2]
Type III_1	A unique example, first analysed by Araki and Woods in [AW]. Uniqueness proved by Haagerup [Ha] and Connes [Co3]

In quantum field theory it is expected that the von Neumann algebra of observables localized in a nice region is a hyperfinite type III_1 factor.

The classification of hyperfinite (also called "injective") factors is a great achievement. Among other things it paves the way for the study of subfactors, to which I now turn.

5. Index for Subfactors

The representation theory of a II_1 factor is very simple. There is a single parameter – a positive real number (or ∞), $\dim_M(\mathscr{H})$, which measures the size of $\mathscr{H}$ compared to the Hilbert space $L^2(M)$ obtained from M by the GNS construction using the trace (for which $\dim_M(L^2(M)) = 1$). Thus if $N \subset M$ are II_1 factors we define the index of N in M to be the real number (≥ 1):

$$[M : N] = \dim_N(L^2(M)) .$$

Examples:

(i) $[N \otimes M_n(\mathbb{C}) : N] = n^2$
(ii) $[M : M^G] = |G|$ if G is a group of outer automorphisms of M.

The next theorem shows that this is an interesting notion.

Theorem [Jo1]. a) *If* $[M : N] < 4$ *then there is an* $n \in \mathbb{Z}$, $n \geq 3$ *for which* $[M : N] = 4\cos^2 \pi/n$.

b) *All values of the index of part* a) *are realized, as is any real number* ≥ 4, *by subfactors of the hyperfinite* II_1 *factor.*

This theorem may be proved by iterating a certain *basic construction* which associates to $N \subset M$ an extension $\langle M, e_N \rangle$ of M where e_N projects from $L^2(M)$ to $L^2(N)$. One obtains a tower M_i with $N = M_0$, $M = M_1$ and $M_n = \langle M, e_1, \ldots, e_{n-1} \rangle$, e_j being orthogonal projection from $L^2(M_j)$ to $L^2(M_{j-1})$.

The e_i's satisfy the following properties

1) $e_i^2 = e_i = e_i^*$
2) $e_i e_j = e_j e_i$ if $|i-j| \geq 2$
3) $e_i e_{i\pm 1} e_i = \tau e_i \qquad (\tau = [M : N]^{-1})$
4) $\mathrm{tr}(we_{n+1}) = \tau\,\mathrm{tr}(w)$ if w is a word on $1, e_1, \ldots, e_n$.

The hermitian form $\mathrm{tr}(x^*y)$ on the algebra generated by the e_i's must be positive definite and this forces $[M : N]$ to be $4\cos^2 \pi/n$. We will see later how the same form arises in connection with surgery on three-manifolds and this result about degeneracies of the form is precisely what allows a simple explicit formula for some new three-manifold invariants!

The construction of examples of subfactors proceeds by the explicit construction of a sequence of e_i's and a trace satisfying 1)–4) above. The II_1 factor is then obtained by the GNS construction and the subfactor is that generated by $e_2, e_3, \ldots$. In the case of index > 4 these subfactors have non-trivial centralizer. For more details on what happens in index > 4 see Popa's talk in this volume.

6. Commuting Squares

The tower M_i arising from $N \subset M$ defines two towers of finite dimensional algebras which we will call the centralizer towers. They are the commutants $A_i = M' \cap M_i$ and $B_i = N' \cap M_i$. Clearly $A_i \subset B_i$ and they satisfy the "commuting square" condition (first used by Popa in a different context) that the orthogonal projections from B_{i+1} to A_{i+1} and B_i commute and $B_i \cap A_{i+1} = A_i$.

This condition allows one to control the von Neumann algebra inclusion of the GNS closures of $\cup A_i$ and $\cup B_i$, which would be impossible without the condition.

In fact commuting squares give subfactors automatically by iterating the basic construction and the study of commuting squares constitutes a new and intriguing problem in finite dimensional linear algebra. A machine for producing examples from quantum groups has been developed by Wenzl. Other examples abound – see [Su], [HJ] and [HS], but the general structure of commuting squares is quite unclear.

7. Finite Depth, Classification Results

Actions of finite groups are completely classified on the hyperfinite type II_1 factor ([Jo7]) and one might hope for an extension of these results to finite index subfactors. This is unlikely since one may take any finite set of automorphisms $\alpha_1, \alpha_2, \ldots, \alpha_n$ and form the subfactor

$$\left\{ \left. \begin{pmatrix} x & & & \\ & \alpha_1(x) & & \mathrm{O} \\ \mathrm{O} & & \ddots & \\ & & & \alpha_n(x) \end{pmatrix} \right| x \in R \right\}$$

of $R \otimes M_n(\mathbb{C})$. This subfactor remembers too much about the group generated by $\alpha_1, \alpha_2, \ldots, \alpha_n$ and although if this group is amenable we are in a good situation ([O1]), if it is not there will be no nice classification.

On the other hand there is a class of subfactors, first stressed by Ocneanu, of R for which classification is a well-posed problem. A subfactor $N \subset M$ is said to be of *finite depth* if the dimensions of the centres of the centralizer towers A_i and B_i are bounded. It is then easy to see that the centralizer towers exhibit periodicity beyond a certain point. The index of a finite depth subfactor is given by the square of the norm of the matrix, with integer entries, describing the stabilized inclusion for $A_i \subset A_{i+1}$. See [GHJ].

Popa has shown in [Po] that the stabilized commuting square of centralizer towers is a complete invariant for subfactors of R of finite depth. Another version of the result is claimed by Ocneanu who has a more elaborate and computable version of the invariant.

If the index is less than 4 a complete classification is possible. Coxeter-Dynkin diagrams arise out of the combinatorics of the centralizer towers and according to Ocneanu subfactors of index < 4 are classified by Dynkin diagrams of types A_n, D_{2n}, E_6 and E_8, there being one for each A_n and D_{2n} and two for each of E_6 and E_8. The index of the subfactor is $4\cos^2 \pi/n$, n being the Coxeter number (see [O2, GHJ]). Popa has extended this to index 4, where infinite depth and extended Dynkin diagrams occur (see [GHJ] and Popa's talk in these proceedings).

Popa has given a deep generalization of his result to cases of infinite depth provided certain asymptotic behavior of the combinatorics can be controlled. Combined with ideas of Wasserman in [Was] this gives new results about compact group actions. See Popa's paper in these proceedings.

8. Statistical Mechanical Models

The abstract algebra presented by the e_i relations 1), 2) and 3) was used by Temperley and Lieb [TL, Ba] to show the equivalence of the ice-type and self-dual Potts models of statistical mechanics. The relations are satisfied by certain matrices which combine to give the row-to-row transfer matrices of the models. The same abstract algebra occurs in the models of Andrews, Baxter and Forrester [ABF] but the parameter is now in the discrete series $4\cos^2 \pi/n$. Pasquier used the ADE Coxeter graphs to get more models ([Pa]).

The ice-type model is a vertex model where the interactions between the elementary components of a system take place at the vertices of a graph. The Potts model is a spin model where interactions are on the edges and the ABF and Pasquier models are IRF models with many-spin interactions around faces of a planar graph.

Many more elaborate models can be obtained from quantum groups and there are corresponding algebraic relations generalizing the e_i ones. One may use these models to construct subfactors including the Wenzl ones but apparently others as well ([Jo2, D+]).

The relation of subfactors to the solvability of the model, if any, remains unclear.

9. Bimodules, Hypergroups, Paragroups, Quantized Groups ...

The combinatorics of the centralizer towers are very rich and attempts are being made to extract the data in them. The most ambitious project is Ocneanu's. He uses bimodules and intertwiners and obtains a structure with many properties of an IRF model of statistical mechanics, where the "faces" become closed paths around induction-restriction diagrams. See his notes in these proceedings.

Sunder has used a less detailed structure called a hypergroup which is only supposed to contain the combinatorial structure of tensor products of bimodules ([Su]).

The idea of using bimodules was first stressed by Connes. See [Co4, Jo3].

10. Braid Groups

The braid group B_n on n strings may be presented on $n-1$ generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ with relations $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ and $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| \geq 2$ (see [Bi]). These relations bear a resemblance to the e_i ones and one may represent the braid group by sending σ_i to $te_i - (1-e_i)$ if $2+t+t^{-1} = \tau^{-1}$. The discrete series corresponds to $t = e^{2\pi i/n}$, σ_i unitary, and the continuous part to $t > 0$, σ_i self-adjoint.

The presence of such braid group representations is a pervasive feature of all the generalizations to do with quantum groups and solvable models. The e_i representations correspond to the 2-dimensional representation of $U_q(sl_2)$. It is not known whether these representations, either collectively or individually, are faithful for $n > 3$. Lawrence has found these representations as natural actions of braid groups on homology groups – see [La]. See also Varchenko's talk in these proceedings.

11. Hecke Algebras

If $H < G$ are groups, the Hecke algebra is the commutant of the representation of G on the vector space of functions on G/H. In the case where G is $GL_n(\mathbb{F}_q)$ and H is the upper triangular matrices, this algebra admits a presentation on generators $g_1, g_2, \ldots, g_{n-1}$ with relations

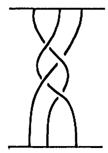
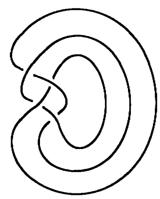
$$g_i^2 = (q-1)g_i + q, \quad g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad g_i g_j = g_j g_i \text{ if } |i-j| \geq 2.$$

For $|q| \neq 1$ or 0 this algebra is isomorphic to the group algebra of the symmetric group S_n (see [Bo]).

Clearly the braid group can be represented in the Hecke algebra in the obvious way. In fact these representations contain the e_i ones (with $q = t$) as direct summands. Under the isomorphism with $\mathbb{C}S_n$, the e_i representations correspond to Young diagrams with at most 2 rows. (See [Jo4].)

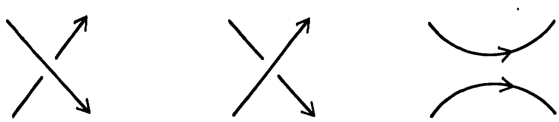
12. Knot Polynomials

Braids may be closed by tying the top to the bottom to form links (see below).

The braid $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$:  Its closure: 

It is clear that the closure of a braid $\alpha \in B_n$ does not change if α is conjugated in B_n, nor if α is embedded in B_{n+1} and multiplied by $\sigma_n^{\pm 1}$. These two "Markov moves" generate the equivalence relation of having the same closure. It follows from this and relation 4) in the e_i algebra that a normalized version of the trace of the algebra element representing a braid will be an invariant of the closure of the braid. For a link L this invariant turns out to be a polynomial $V_L(t)$ in the variable t (or $\sqrt{t}$) and can be calculated, though not always rapidly, from the following "skein relation".

Skein Relation ([Cnw]). If L_+, L_- and L_0 are links identical except near one crossing where they are as below



L_+ L_- L_0

then

$$\frac{1}{t}V_{L_+} - tV_{L_-} = (\sqrt{t} - 1/\sqrt{t})V_{L_0}\,.$$

The right-handed trefoil has $V(t) = t + t^3 - t^4$ and the left-handed one has $V(t) = 1/t + 1/t^3 - 1/t^4$. The Alexander polynomial has a similar relation and this prompted many people to develop a 2-variable polynomial called the HOMFLY polynomial $P_L(l, m)$ having arbitrary coefficients in its skein relation ([Jo5, F+, Li1]).

Kauffman found an explicit formula for $V_L(t)$ in [Ka1] from an arbitrary knot diagram which was used by him, Murasugi and Thistlethwaite to prove some old conjectures of Tait about alternating knots ([Ka1, Mu, Th]). He also found another two-variable polynomial generalization of $V(t)$, called the Kauffman polynomial $F(a, x)$. It does not contain the Alexander polynomial.

The polynomials are very useful in calculating the minimal number of crossings for a diagram of a knot and the minimal number of strings for a closed braid representation of a knot ([LT, Mo, FW]).

13. The R-Matrix, Powers State Picture

A special representation of the relations 1)–4) of §5 was discovered by Pimsner and Popa in [PP]. If $e \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ is defined by $e = \tau e_{11} \otimes e_{22} + \sqrt{\tau(1-\tau)}(e_{12} \otimes e_{21} + e_{21} \otimes e_{12}) + (1-\tau)e_{22} \otimes e_{11}$, then if we let e_i on $(\mathbb{C}^2)^{\otimes n}$ be defined between the i^{th} and $(i+1)^{th}$ tensor components as e and the identity on the others, one finds that 1), 2) and 3) are satisfied. Moreover if φ_λ is the Powers state (giving a type III_λ factor), then its restriction to the algebra generated by $1, e_1, \ldots, e_n$ defines a trace satisfying 4). This gives a useful way of calculating $V_L(t)$, with $t = \lambda$. This representation, without φ_λ, was used also in [TL] where the e_i's give the transfer matrix for an ice-type model.

At this point a remarkably rapid development in the understanding of the polynomials was made possible by the existence of the theory of quantum groups. Fadeev's Leningrad group, particularly Sklyanin [Sk] and Kulish and Reshetikhin [KR] had uncovered a new structure related to the ice-type model for which the relevant Lie algebra was $sl_2(\mathbb{C})$. For background see [Fa]. This had been generalized by Jimbo [Ji1] and Drinfeld [Dr] to produce analogues of the braiding matrices obtained from the e_i representation I have just described, one for each finite dimensional representation of every simple Lie algebra. The analogue of the Powers state was soon found and the following picture was established by Reshetikhin [Re] and Rosso [Ro]:

Let $\mathscr{G}$ be a simple finite dimensional complex Lie algebra and let $\mathscr{L}$ be the class of links with distinguished components $C_1, C_2, \ldots, C_n$. Then to each way of assigning finite dimensional representations of $\mathscr{G}$ to $C_1, \ldots, C_n$ there is a polynomial invariant of isotopy for links in $\mathscr{L}$.

The polynomial $V_L(t)$ corresponds to $\mathscr{G} = sl_2$ and the assignment of its 2-dimensional representation to all components. The HOMFLY and Kauffman polynomials are obtained in a similar way from sl_n and the symplectic (or orthogonal) algebras respectively.

In this picture the geometric operations of cabling (and satellites in general) can be understood in terms of tensor products of representations ([MSt]).

Explicit formulae on any (not necessarily braided) picture of a link may be given. This was first done for HOMFLY in [Jo6] and generalized to the Kauffman polynomial in [Tu1].

14. Positivity of the Markov Trace

Ocneanu's approach to the HOMFLY polynomial was a direct generalization of my construction of $V_L(t)$, by defining a trace on the Hecke algebra by the property $\mathrm{tr}(wg_{n+1}) = z\,\mathrm{tr}(w)$ if w is a word on $g_1, g_2, \ldots, g_n$, and where z is a new variable. Subfactors occur for the values of (q, z) allowing a $*$-algebra structure on the Hecke algebra for which the trace is positive. This set of values was determined by Ocneanu, and Wenzl constructed the subfactors and calculated their indices ([F+, We1]). It is convenient to use the variables $\tau = q/(1+q)^2$ and $\eta = (1+z)/(1+q)$. Then the "positivity spectrum" (Fig. 2) is as follows:

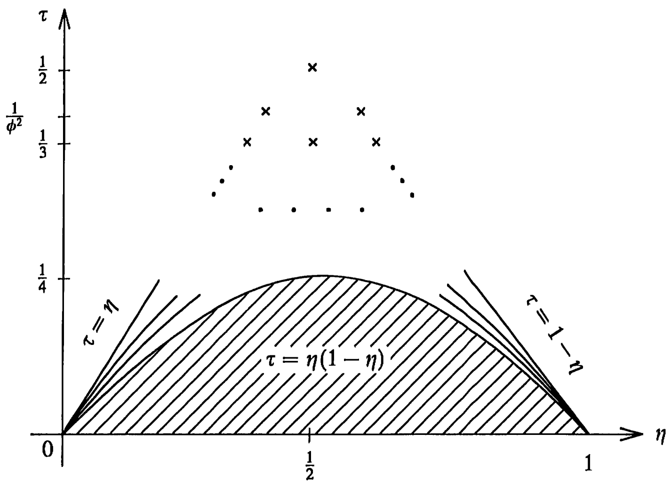


Fig. 2

The discrete points are the intersections in $[0, 1] \times [0, 1]$ of the curves $\eta = P_k(\tau)/P_{k-1}(\tau)$ with the curves $\eta = \tau P_{k'-2}(\tau)/P_{k'-1}(\tau)$ as indicated, where $P_k(\eta)$ are the polynomials given by $P_{k+1} = P_k - \eta P_{k-1}$, $P_0 = 1$, $P_1 = 1$, and $k + k' = l + 3$, $k = 2, 3, \dots, l + 1$, l indicating the horizontal row on which the point is situated. The index of the subfactor at the point labelled (l, k) is $\sin^2 k\pi/(l+3)/\sin^2 \pi/(l+3)$ ([Wel]).

15. Brauer and BMW Algebras

A new algebra was developed in [BW] and [Muk] to play the role for the Kauffman polynomial that the Hecke algebra plays for the HOMFLY polynomial. The idea, inspired by Kauffman, is to add objects like $|\dots| \asymp |\dots|$ to the braid group generators. Calling these generators E_i and the usual braid group generators G_i, the BMW algebra with parameters a, x, has presentation:

$$G_iG_{i+1}G_i = G_{i+1}G_iG_{i+1}, \quad G_iG_j = G_jG_i \text{ if } |i - j| \geq 2,$$
$$G_i + G_i^{-1} = x(1 + E_i), \quad E_iG_i = G_iE_i = aE_i, \quad E_i^2 = (a + a^{-1} - x)x^{-1}E_i,$$
$$E_iG_{i\pm1}^{\pm1}E_i = a^{\mp1}E_i, \quad E_iG_{i\pm1}G_i = E_iE_{i\pm1}.$$

These relations are best understood in terms of diagrams. The third is used because of the Kauffman polynomial. Using Markov traces and the basic construction of subfactor analysis Wenzl determined the structure of the BMW algebra for generic values of the parameters and a lot about it for special values. Its dimension is $1 \cdot 3 \cdots (2n + 1)$, n being the number of G_i's as above. It has the Hecke algebra as a natural quotient and the e_i algebra as a natural subalgebra.

Brauer in [Br] defined algebras of the same dimension as a model for the commutant of the symplectic and orthogonal groups in tensor powers of the vector representation. Wenzl in [Wez] solved the problem of the generic structure of the Brauer algebra essentially by viewing it as a specialization of the BMW algebra. Wenzl has also obtained subfactors from the BMW algebra using positivity considerations.

The commutant of the quantum group $U_q(sl_n)$ on the tensor powers of the n-dimensional representation is the Hecke algebra ([Ji2]). The BMW algebra does the same job for quantum groups of types B and C (see [We3]).

16. Conformal Field Theory

There are many similarities between subfactor theory and conformal field theory. I cite the discrete and continuous values of the central charge c ([FQS]), the ADE classification for $c < 1$ ([CIZ]), the "fusion rules" corresponding to the combinatorics of the centralizer towers ([GW]) and the existence of a tractable subclass called rational CFT's corresponding to finite depth subfactors. But the most compelling evidence for a connection is that the braid group representations which occur canonically in the subfactor picture for index < 4 occur also for the $SU(2)$ case of one of the main models of CFT – the Wess-Zumino-Witten model. Following [KZ], Tsuchiya and Kanie calculated the monodromy of the n-point functions in the holomorphic sector of this model and found the e_i-braid group representations (for spin 1/2). Thus the subfactors themselves could be constructed out of the WZW theory but in a somewhat indirect way.

Moore and Seiberg showed in [MSe] that the braid group situation was just the genus zero case of a theory that works in arbitrary genus for any conformal field theory. To do this they checked that the defining relations ([Waj]) for the mapping class group do indeed follow from their axioms for CFT.

Conformal field theories are often obtained as continuum limits of critical 2-dimensional classical statistical mechanical systems. It is not clear that there is any deep relationship between the appearances of the braid group in solvable models and in the CFT of the continuum limit.

17. Algebraic Quantum Field Theory, Superselection Sectors

Several people ([Fr, FRS, Lo]) have noticed that the braiding and Markov trace structure is inherent in any (not necessarily conformal) low dimensional QFT. In the framework of algebraic QFT ([HK]), Doplicher, Haag and Roberts in [DHR] introduced superselection sectors as representations of the observable algebra which are equivalent to the vacuum representation when restricted to the algebra of the causal complement of some bounded region. Haag duality is the property that the algebras of such bounded regions and their causal complements are each other's commutants. This duality is supposed to hold in the vacuum sector. A simple calculation shows that superselection sectors give rise to endomorphisms of the local observable algebra. Using geometric properties of space-time, implementers of the endomorphism give rise to unitary braid group representations where the endomorphism takes σ_i to σ_{i+1}. The Markov trace may be obtained as a simple weak limit.

The square root of the index of the subfactor which is the image of the endomorphism was called the "statistical dimension" of the sector in [DHR], but because they were considering 4 dimensions this dimension was necessarily an integer in their theory.

One of the problems with all this work is that there do not seem to be any concrete examples where the subfactors and braid group representations have been calcualted to the satisfaction of an expert in von Neumann algebras.

18. Loop Groups and Subfactors

The loop group $LSU(2)$ has a discrete series of "positive energy" projective unitary reprseentations labelled by a level l and a spin j, $0 \leq 2j \leq l$. Inspired by [TK] and in an attempt to understand and implement the ideas of superselection sectors, and to provide naturally occuring examples of subfactors, A. Wassermann and I have been working on the best known example, WZW theory, especially for $LSU(2)$. We interpret the superselection sectors as being the discrete series of the loop group for a fixed level. Given an interval I in the circle S^1 (I^c will denote the complementary interval), let $L_I G$ be the group of loops supported in I. The von Neumann algebra $(L_I G)''$ corresponds to the local algebra. Haag duality (which we have proved) says that for the vacuum sector (spin $=0$), $(L_I G)' = (L_{I^c} G)''$. These von Neumann algebras are type III_1 factors in any sector and we have shown that, if the level is fixed, the representations of $L_I G$ for any spin are unitarily equivalent.

We conjecture that $[(L_I G)' : (L_{I^c} G)''] = \sin^2\{(2j+1)\pi/(l+2)\}/\sin^2(\pi/(l+2))$ and in general that the subfactor is the tensor product of a Wenzl subfactor and the hyperfinite type III_1 factor. The vertex operators of Tsuchiya and Kanie, if made to become unbounded operator valued distributions on S^1, should provide explicit intertwiners between the representations of $L_I G$ for fixed level and different spin.

19. Witten's Interpretation of $V_L(t)$ and Its Generalizations

Motivated by his own work, ideas of Atiyah and Segal and the relations between subfactors, knots and CFT, Witten proposed the following formula for a link L with components $L_1, L_2, \ldots, L_n$ ([Wi1]).

$$V_L(e^{2\pi i/k+2}) = \int_A [\mathscr{D}A] e^{ik\int_{S^3} \mathrm{tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)} \prod_{i=1}^{n} W(L_i)$$

where A runs over all $su(2)$-valued 1-forms on S^3 (modulo the gauge group) and $W(L_i)$ is the trace of the parallel transport using A (in the 2-dimensional representation) along the closed curve L_i. A measure $[\mathscr{D}A]$ with the appropriate properties has not been shown to exist so this formula must be taken in the context of Witten's topological QFT ([A]), which predicts enough formal properties of such an integral for it to be calculated (i.e. identified with $V_L(t)$). This can be thought of as the solvability of this particular topological QFT.

The generalizations are now apparent. $SU(2)$ can be replaced by any compact Lie group G and one may choose any finite dimensional representation of G per component of L. This reproduces the ingredients of the picture that emerged from quantum groups in §13.

More significantly S^3 may be replaced by any closed 3-manifold so that Witten's theory predicts the existence of invariants for links in 3-manifolds and gives explicit formulae for calculating them from a surgery description of the link in the 3-manifold. These formulae have been checked using the Kirby calculus ([FR]) at least for $SU(2)$ in [RT] and [KM] using quantum groups. In [KM] certain explicit evaluations are given in terms of classical invariants. In [Li2] an elementary formula for these invariants occurs using no more than cabling and $V_L(t)$. The key ingredient in the proof is the degeneracy of the trace on the e_i algebra of §5. One wonders whether the 3-manifold invariants may be obtained directly from subfactors.

20. Topological Quantum Field Theory

Witten has developed a formalism in which a quantum field theory in $d+1$ dimensions assigns a "Hilbert space" to any d-dimensional manifold Σ with extra structure and every time this manifold is the boundary of a $d+1$ dimensional manifold M with compatible extra structure there is a vector in the Hilbert space of the boundary. In the simplest case $M = \Sigma \times [0,1]$ the vector is supposed to define an operator giving the time evolution of a system from time 0 to time 1. The Hilbert spaces and vectors are supposed to satisfy certain important and powerful axioms. If the "extra structure" is little more or less than an orientation one talks of *topological* quantum field theory. See [A] where examples are given connected with Donaldson and Floer theory.

In the case of the theory given by the Chern-Simons action $\mathrm{tr}(\mathrm{Ad}\,A + (2/3)A^3)$, $d = 2$ and Witten identifies the Hilbert space corresponding to a surface Σ as being the (finite dimensional) vector space of conformal blocks for the corresponding Wess-Zumino-Witten theory with the same gauge group. This is crucial for his calculations as it allows him to assert that, for the $SU(n)$ theory the vector space corresponding to the sphere with 4 marked points is 2-dimensional. This, together with the "gluing" axioms of topological QFT allows the formal calculation of the functional integral via skein theory.

The Hilbert space for a surface Σ is also deduced via a "geometric quantization" approach using a complex structure on Σ. That the Hilbert space should be independent of the complex structure is interpreted as implying the existence of a flat connection on certain natural bundles over Teichmüller space. See the talk by Tsuchiya in these proceedings.

Witten's surgery formula comes from identifying a basis for the Hilbert space of a torus with the vectors obtained by realizing the torus as the boundary of a solid torus containing one simple homotopically non-trivial closed curve to which an irreducible representation of the compact group is assigned. The action of the diffeomorphisms in this basis (all that is required for surgery formulae given the gluing axioms) is precisely that of $SL(2,\mathbb{Z})$ on the characters of the relevant affine Lie algebra at the given level.

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