Clifford Analysis and a Noncommutative Gelfand Representation

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Overview

Section 1

Introduction

Subsection 1

Motivation

Electrical Impedance Tomography

Electrical Impedance Tomography (EIT) asks whether one can determine the conductivity of a medium based on measurements along the boundary.

Other questions

- Can we retrieve topological information from spaces of functions on a manifold M?
- Do these spaces also contain geometric information such as metric data?
- Can we determine enough about these spaces from partial information – say information only on the boundary?

Subsection 2

Preliminaries

- ***Clifford algebra*** originated in 1878 with William Kingdon Clifford's work that extends Hermann Grassmann's ***exterior algebra***.
- ***Clifford analysis*** arrived in the 1980's due to Hestenes, Sobczyk, Sommen, Brackx, and Delenghe in order to enrich Éllie Cartan's ***differential forms***.

Clifford algebras

Let V be a vector space over a field $\mathbb{F}$ with quadratic form Q.

- Define the tensor algebra

$$\mathcal{T}(V) \coloneqq \bigoplus_{j=0}^{\infty} V^{\otimes j} = \mathbb{F} \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots.$$

- Form the ***Clifford algebra*** via a quotient

$$C\ell(V, Q) \coloneqq \mathcal{T}(V) / \langle \mathbf{v} \otimes \mathbf{v} - Q(V) \rangle.$$

Geometric and exterior algebras

Let V be a vector space over a field $\mathbb{F}$ with quadratic form Q.

- Given a (pseudo) inner product g, we set $Q(\cdot) = g(\cdot,\cdot)$ and define a ***geometric algebra***

$$\mathcal{G} \coloneqq C\ell(V,g).$$

- The ***exterior algebra*** is given by

$$\bigwedge(V) \coloneqq C\ell(V,0).$$

Algebra structure

We define a multiplication in $\mathcal{G}$ by noting how the product $\otimes$ acts in the quotient.

- Given $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{G}$ we can take the product

$$\boldsymbol{u}\boldsymbol{v} = \underbrace{\boldsymbol{u} \cdot \boldsymbol{v}}_{\text{scalar}} + \underbrace{\boldsymbol{u} \wedge \boldsymbol{v}}_{\text{bivector}} .$$

  - The scalar part is symmetric: $\boldsymbol{u} \cdot \boldsymbol{v} = g(\boldsymbol{u}, \boldsymbol{v})$.
  - The bivector part is antisymmetric: $\boldsymbol{u} \wedge \boldsymbol{v} = -\boldsymbol{v} \wedge \boldsymbol{u}$.

Multivectors

- $\mathcal{G}$ is graded and of dimension 2^n.
  - There are $\binom{n}{r}$ elements of grade r called ***r-vectors***.
  - Those that are exterior products of r independent vectors are ***r-blades***.

    E.g., $\boldsymbol{A_r} = \boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_r$.
- The most general elements are ***multivectors*** and are given by

$$A = \sum_{r=0}^{n} \langle A \rangle_r,$$

where $\langle A \rangle_r \in \mathcal{G}^r$ extracts the grade r part of A.

Algebra Structure

Extend the multiplication from vectors to multivectors by

$$A_r B_s = \langle A_r B_s \rangle_{|r-s|} + \langle A_r B_s \rangle_{|r-s|+2} + \cdots + \langle A_r B_s \rangle_{r+s}$$

with

$$A_r \cdot B_s \coloneqq \langle A_r B_s \rangle_{|r-s|} \qquad A_r \wedge B_s \coloneqq \langle A_r B_s \rangle_{r+s}$$

$$A_r \rfloor B_s \coloneqq \langle A_r B_s \rangle_{s-r} \qquad A_r \lfloor B_s \coloneqq \langle A_r B_s \rangle_{r-s}$$

Reciprocals and reverses

Given any vector basis $\boldsymbol{v}_i$ we define the ***reciprocal vectors*** by $\boldsymbol{v}^i \cdot \boldsymbol{v}_j = \delta^i_j$. The ***reverse*** of a multivector is extended linearly from the action on r-blades by

$$\boldsymbol{A_r}^\dagger = (\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_r)^\dagger = \boldsymbol{v}_r \wedge \cdots \wedge \boldsymbol{v}_1.$$

Inner product and norm

We define the ***multivector inner product*** by

$$(A, B) \coloneqq \left\langle A^\dagger B \right\rangle$$

which is bilinear, symmetric, and positive definite if g is positive definite. Then we define the ***multivector norm*** by

$$|A| \coloneqq \sqrt{(A, A)}.$$

Adjoint

Note the reverse acts as an adjoint by

$$(CA, B) = (A, C^\dagger B) \tag{1}$$

$$(AC, B) = (A, BC^\dagger). \tag{2}$$

Pseudoscalars

Pseudoscalars are the grade-n elements. For example, $\boldsymbol{\mu} = \boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_n$. We define the ***unit pseudoscalar*** by

$$\boldsymbol{I} \coloneqq \frac{1}{|\boldsymbol{\mu}|}\boldsymbol{\mu}.$$

Blades and subspaces

If $|\boldsymbol{A_r}| = 1$, then $\boldsymbol{A_r}$ is a ***unit blade***.

All unit r-blades correspond to an r-dimensional subspace and can be identified with points in $\mathrm{Gr}(r, n)$.

Duality

Given any multivector A, we can take its ***dual***

$$A^{\perp} \coloneqq A\boldsymbol{I}^{-1}.$$

Note $A_r^{\perp} \in \mathcal{G}^{n-r}$, much like the Hodge star $\star$.

Quaternions and complex numbers

Claim: $\mathbb{H}$ arises naturally as the even subalgebra $\mathcal{G}_3^+$.

Claim: $\mathbb{C}$ arises naturally as the even subalgebra $\mathcal{G}_2^+$.

Take the standard basis $\boldsymbol{e}_1, \boldsymbol{e}_2$, and define $\boldsymbol{B}_{12} = \boldsymbol{e}_1 \boldsymbol{e}_2$ and note $\boldsymbol{B}_{12}^2 = -1$. Thus

$$(u_1 + v_1 \boldsymbol{B}_{12})(u_2 + v_2 \boldsymbol{B}_{12}) = u_1 u_2 - v_1 v_2 + (u_1 v_2 + u_2 v_1)\boldsymbol{B}_{12}.$$

Moreover, right multiplication of vectors by $\boldsymbol{B}_{12}$ rotates counter-clockwise by $\pi/2$.

Clifford algebra structure on manifolds

We let M be a smooth, compact, connected, and oriented n-dimensional Riemannian manifold with metric g (unless otherwise stated).

Idea: Form the Clifford algebras on tangent spaces.

- Each $C\ell(T_pM, g_p)$ is a ***geometric tangent space*** which we glue together to form

$$C\ell(TM, g) \coloneqq \bigsqcup_{p \in M} C\ell(T_pM, g_p).$$

- The space of ***(smooth) multivector fields*** is

$$\mathcal{G}(M) \coloneqq \{C^\infty\text{-smooth sections of } C\ell(TM, g)\}.$$

Section 2

Clifford analysis

Covariant derivative

On M we have the unique torsion free Levi-Civita connection ∇ and covariant derivative $\nabla_{\boldsymbol{u}}$.

indler, 2018 , $\nabla_{\boldsymbol{u}}$ can be extended to multivectors and it is grade preserving

$$\nabla_{\boldsymbol{u}} A_r = \langle \nabla_{\boldsymbol{u}} A_r \rangle_r.$$

- $\nabla_{\boldsymbol{u}}$ is compatible with dot and wedge since

$$\nabla_{\boldsymbol{u}}(A \cdot B) = (\nabla_{\boldsymbol{u}} A) \cdot B + A \cdot (\nabla_{\boldsymbol{u}} B)$$
$$\nabla_{\boldsymbol{u}}(A \wedge B) = (\nabla_{\boldsymbol{u}} A) \wedge B + A \wedge (\nabla_{\boldsymbol{u}} B).$$

Gradient

We define the ***gradient*** (or ***Dirac operator***) in some local basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ by

$$\boldsymbol{\nabla} = \sum_{i=1}^{n} \boldsymbol{v}^i \nabla_{\boldsymbol{v}_i}$$

Note that this has the algebraic properties of a vector in $\mathcal{G}(M)$.

Gradient

We define the ***gradient*** (or ***Dirac operator***) in some local basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ by

$$\nabla = \sum_{i=1}^{n} \boldsymbol{v}^i \nabla_{\boldsymbol{v}_i}$$

Note that this has the algebraic properties of a vector in $\mathcal{G}(M)$ and has the Leibniz rule

$$\nabla(AB) = \dot{\nabla}\dot{A}B + \dot{\nabla}A\dot{B}.$$

Note $\nabla^2 = \Delta$, the Laplace-Beltrami operator.

Subsection 1

Integration

Differential forms

We define the ***r*-dimensional directed measure**

$$dX_r \coloneqq \boldsymbol{v}_{j_1} \wedge \cdots \boldsymbol{v}_{j_r} dx^1 \cdots dx^n$$

where summation is implied over the increasing set of indices $1 \leq j_1 < \cdots < j_r \leq n$. This allows us to define an r-form α_r by

$$\alpha_r = A_r \cdot dX_k^\dagger$$

where $A_r = \frac{1}{r!}\alpha_{i_1 \cdots i_r} \boldsymbol{v}^{i_1} \wedge \cdots \wedge \boldsymbol{v}^{i_r}$. We call A_r the ***multivector equivalent*** of α_r.

Volume form

The ***volume form*** on M is given in local coordinates by

$$\mu = \sqrt{|g|}\,dx^1 \cdots dx^n = \boldsymbol{I}^{-1} \cdot dX_n$$

Hence, we can integrate scalar fields A_0 on M by

$$\int_M A_0^\perp \cdot dX_n = \int_M A_0 \mu.$$

Exterior algebra and calculus

- Given an r- and s-form α_r and β_s we have

$$\alpha_r + \beta_s = (A_r + B_s) \cdot dX_r^\dagger$$
$$\alpha_r \wedge \beta_s = (A_r \wedge B_s) \cdot dX_{r+s}^\dagger.$$

- The exterior derivative on multivector equivalents is

$$d\alpha_r = (\nabla \wedge A_r) \cdot dX_{r+1}^\dagger$$

- The Hodge star on multivector equivalents is

$$\star\alpha_r = (\boldsymbol{I}^{-1} A_r)^\dagger \cdot dX_{n-r}^\dagger$$

Multivector field inner product

- We define an inner product on multivector fields by

$$\ll A, B \gg \coloneqq \frac{1}{\mathrm{vol}(M)} \int_M (A, B)\mu$$

- This realizes the r-form inner product

$$\int_M \alpha_r \wedge \star\beta_r = \int_M \left\langle A_r^\dagger B_r \right\rangle \mu = \mathrm{vol}(M) \ll A, B \gg$$

- A_r and B_s are orthogonal when $r \neq s$ so this agrees with the grade direct sum $\oplus$ – we use the same notation for both.

Boundary

On the boundary ∂M, we have the boundary pseudoscalar $\boldsymbol{I}_\partial$ and the boundary normal $\boldsymbol{\nu} = \boldsymbol{I}_\partial^\perp$. Then

$$\mu_\partial \coloneqq \boldsymbol{I}_\partial^{-1} \cdot dX_{n-1}$$

and we define

$$\ll A, B \gg_\partial \coloneqq \frac{1}{\mathrm{vol}(M)} \int_{\partial M} (A, B) \mu_\partial .$$

Multivector valued integrals

We can define a multivector valued integral on an oriented submanifold R by taking $A \in \mathcal{G}(M)$ and computing

$$\int_R A\boldsymbol{I}_R \mu_R.$$

Theorem (Hestenes, Sobczyk, 1984)

Let $A, B \in \mathcal{G}(M)$, *then*

$$\begin{aligned}
\int_M \dot{A}\dot{\boldsymbol{\nabla}}\boldsymbol{I}\mu &= \int_{\partial M} A\boldsymbol{I}_\partial \mu_\partial \\
\int_M \boldsymbol{I}\boldsymbol{\nabla} B\mu &= \int_{\partial M} \boldsymbol{I}_\partial B\mu_\partial \\
\int_M \dot{A}\dot{\boldsymbol{\nabla}}\boldsymbol{I} B\mu &= (-1)^n \int_M A\boldsymbol{I}\boldsymbol{\nabla} B\mu + \int_{\partial M} A\boldsymbol{I}_\partial B\mu_\partial .
\end{aligned}$$

Theorem

We have the Green's formula for the gradient

$$\ll A, \boldsymbol{I}\nabla B \gg = (-1)^n \ll \nabla A, \boldsymbol{I}B \gg + \ll A, \boldsymbol{I}_\partial B \gg_\partial .$$

Proof. Fix $A^\dagger, B \in \mathcal{G}(M)$ and note that

$$\begin{aligned}\int_M A^\dagger \boldsymbol{I}\nabla B\mu &= (-1)^n \int_M \dot{A}^\dagger \dot{\nabla}\boldsymbol{I}B\mu + \int_{\partial M} A^\dagger \boldsymbol{I}_\partial B\mu_\partial \\ &= (-1)^n \int_M (\nabla A)^\dagger \boldsymbol{I}B\mu + \int_{\partial M} A^\dagger \boldsymbol{I}_\partial B\mu_\partial .\end{aligned}$$

Then, take the scalar part and divide by $\mathrm{vol}(M)$ to find

$$\ll A, \boldsymbol{I}\nabla B \gg = (-1)^n \ll \nabla A, \boldsymbol{I}B \gg + \ll A, \boldsymbol{I}_\partial B \gg_\partial .$$

Monogenic fields and gradients

- The space of ***monogenic fields*** is

$$\mathcal{M}(M) \coloneqq \{A \in \mathcal{G}(M) \mid \nabla A = 0\}.$$

- Let $f = u + v\boldsymbol{B} \in \mathcal{G}_2^+(\mathbb{R}^2)$ then $\nabla f = 0$ yields the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

- The ***gradients*** are

$$\nabla\mathcal{G}(M) \coloneqq \{\nabla A \mid A \in \mathcal{G}(M) \text{ and } A|_{\partial M} = 0\}.$$

For M a domain in $\mathbb{R}^n$ with $n \geq 2$, we have the vector valued field

$$E(x) \coloneqq \frac{1}{S_n} \frac{x}{|x|^n}$$

where S_n is the surface area of the unit ball. Note

$$\nabla E(x) = -\dot{E}(x)\dot{\nabla} = \delta(x).$$

We then define the ***Cauchy kernel*** by $G(x, x') \coloneqq E(x' - x)$.

Cauchy integral

If $A \in \mathcal{M}(M)$, then we have the ***Cauchy integral formula***

$$A(x) = (-1)^n \boldsymbol{I}^{-1} \int_{\partial M} G(x, x') \boldsymbol{I}_\partial (x') A(x') \mu_\partial (x').$$

This allows us to uniquely determine a monogenic field from boundary values $A|_{\partial M}$.

Lemma

Let $A \in \mathcal{M}(M)$ *be such that* $A|_{\partial M} = 0$. *Then* $A = 0$ *on all of* M.

Proof.

$$|A(x)| \leq \left| \int_{\partial M} G(x,x') \boldsymbol{I}_{\partial}(x') A(x') \mu_{\partial}(x') \right| \leq \int_{\partial M} \left| G(x,x') \boldsymbol{I}_{\partial}(x') A(x') \right| \mu_{\partial}(x') = 0.$$

Lemma

Fix a multivector field $A \in \mathcal{G}(M)$*. If*

$$\ll A, B \gg = 0$$

for all $B \in \mathcal{G}(M)$ *with* $B|_{\partial M} = 0$*, then* $A = 0$*.*

Proof sketch.

- Use mollifiers to smooth indicator functions χ_U on open subsets U to be supported only on closed ϵ neighborhood $\overline{U^\epsilon}$. Call these functions χ_U^ϵ.
- Write $A = \sum_J A_J \boldsymbol{V}^J$ with $\boldsymbol{V}^J = \boldsymbol{v}^{j_1} \wedge \cdots \wedge \boldsymbol{v}^{j_r}$. Then note

$$\ll A, A_J \boldsymbol{V}_J \chi_U^\epsilon \gg = 0$$

  implies $A_J = 0$ on U^ϵ for all J since $(\boldsymbol{V}^J, \boldsymbol{V}_K) = \delta_K^J$. Hence $A = 0$ on U^ϵ.
- Cover M in such U^ϵ and repeat the argument leaving the $A|_{\partial M}$ undetermined. But, by smoothness of A, $A = 0$ on M.

Theorem (Clifford-Hodge-Morrey Decomposition)

The space of multivector fields $\mathcal{G}(M)$ has the L^2-orthogonal decomposition

$$\mathcal{G}(M) = \mathcal{M}(M) \oplus \boldsymbol{I}\nabla\mathcal{G}(M).$$

Proof

- *Orthogonality:* Let $A \in \mathcal{M}(M)$ and $\boldsymbol{I}\nabla B \in \boldsymbol{I}\nabla\mathcal{G}(M)$ and note

$$\ll A, \boldsymbol{I}\nabla B \gg = (-1)^n \ll \nabla A, \boldsymbol{I}B \gg + \ll A, \boldsymbol{I}_\partial B \gg = 0,$$

  by the multivector Green's formula.

- Let $C \in \mathcal{G}(M)$ be in the orthogonal complement to $\boldsymbol{I}\nabla\mathcal{G}(M)$. Then, by the Cauchy integral formula, construct a monogenic field $\tilde{C}$ from $C|_{\partial M}$ and note $C = \tilde{C} + C_0$ where $C_0|_{\partial M} = 0$. Note

$$0 = \ll C, \boldsymbol{I}\nabla B \gg = \ll \nabla C_0, \boldsymbol{I}B \gg .$$

  By the previous lemmas, it must be that $C_0 = 0$. Hence the orthogonal complement to $\boldsymbol{I}\nabla\mathcal{G}(M)$ is $\mathcal{M}(M)$.

Comparing to Hodge-Morrey

The Hodge-Morrey decomposition reads

$$\mathcal{G}^r(M) = \mathcal{E}^r(M) \oplus \mathcal{C}^r(M) \oplus \mathcal{M}^r(M).$$

whereas the Clifford-Hodge-Morrey decomposition ignores specific grades

$$\mathcal{G}(M) = \mathcal{M}(M) \oplus \boldsymbol{I}\nabla\mathcal{G}(M).$$

Section 3

Gelfand theory

Open question

Motivation from belishev 2d and 3d papers. Mention we will work with M imbedded in $\mathbb{R}^n$

Subsurface spinor fields

- Let $\boldsymbol{B} \in \mathcal{G}(M)$ be a constant unit 2-blade, then an even multivector field f_+ satisfying
$$f_+ = \mathrm{P}_{\boldsymbol{B}} \circ f_+ \circ \mathrm{P}_{\boldsymbol{B}}$$
is a ***subsurface spinor field*** and we let $\mathcal{G}^+_{\boldsymbol{B}}(M)$ to denote the space such fields.
- We note that the space
$$\mathcal{A}_{\boldsymbol{B}(M)} = \{f_+ \in \mathcal{G}^+_{\boldsymbol{B}}(M) \mid \nabla f_+ = 0\}$$
is a commutative unital Banach algebra.

Functionals

We define the ***spinor dual*** $\mathcal{M}^*(M)$ as the continuous right $\mathcal{G}_n$-module homomorphisms

$$\mathcal{M}^*(M) \coloneqq \{l\colon \mathcal{M}^+(M) \to \mathcal{G}_n^+ \mid l(fs+g) = l(f)s + l(g),\ \forall f, g \in \mathcal{M}(M),\ s \in \mathcal{G}_n^+\}$$

and refer to the elements as ***spin functionals***. We provide $\mathcal{M}^*(M)$ with the weak-$\star$ topology so that every $x \in M$ corresponds to a continuous map on $\mathcal{M}^*(M)$.

Characters

Define the algebra $\mathbb{A}_{\boldsymbol{B}}$ to be the algebra generated by 1 and $\boldsymbol{B}$. Then, the ***spinor spectrum*** $\mathfrak{M}(M)$ is the set of algebra homomorphisms

$$\mathfrak{M}(M) \coloneqq \{\delta \in \mathcal{M}^*(M) \mid \delta(f) \in \mathbb{A}_{\boldsymbol{B}},\ \delta(fg) = \delta(f)\delta(g),\ \forall f, g \in \mathcal{A}_{\boldsymbol{B}}(M),\ \boldsymbol{B} \in \mathrm{Gr}(2, n)\}$$

and refer to the elements as ***spin characters***. Note that one example of such characters are point evaluations $\delta(f) = f(x^\delta)$.

z analogs and monogeneic polynomials

Take $\boldsymbol{e}_i$ to be an orthonormal basis for $\mathbb{R}^n$, let $\boldsymbol{B}_{ij} = \boldsymbol{e}_i\boldsymbol{e}_j$ and define the functions $z_{ij} = x_j - x_i\boldsymbol{B}_{ij}$ and note $z_{ij} \in \mathcal{A}_{\boldsymbol{B}_{ij}}(M)$.

Let σ be a permutation of $\{2, 3, \ldots, n\}$, then the homogeneous polynomial of degree j

$$p_{j_2\cdots j_n}(x) = \frac{1}{j!}\sum_{\text{permutations}} z_{1\sigma(1)}(x)\cdots z_{1\sigma(j)}(x)$$

is monogenic.

Collect these into the set of ***monogenic polynomials***

$$\mathcal{M}^{\mathcal{P}}(M) = \left\{\sum_{j=0}^{N}\left(\sum_{\substack{j_2\cdots j_n\\ j_2+\cdots j_n=j}} p_{j_2\cdots j_n}a_{j_2\cdots j_n}\right) \mid j_2+\cdots+j_n = j, N \in \mathbb{N},\ a_{j_2\cdots j_n} \in \mathcal{G}_n\right\}.$$

Lemma (Density)

The space $\mathcal{M}^{\mathcal{P}}(M)$ is dense in $\mathcal{M}(M)(\mathbb{B}_{R,w})$.

Proof sketch. Let $f \in \mathcal{M}(\mathbb{B}_{R,w})$ and use the Cauchy integral formula to define the coefficients by

$$a_{j_2\cdots j_n} = \int_{\partial B(w,R)} \frac{\partial^j G(w,y)}{\partial y_2^{j_2}\cdots\partial y_n^{j_n}} \boldsymbol{\nu}(y) f(y) \mu_\partial(y),$$

where each $a_{j_2\cdots j_n} \in \mathcal{G}_n^+$. Then

$$f(x) = \sum_{j=0}^{\infty} \left(\sum_{\substack{j_2\cdots j_n \\ j_2+\cdots j_n = j}} p_{j_2\cdots j_n}(x-w) a_{j_2\cdots j_n} \right),$$

converges pointwise for $x \in \mathbb{B}_{R,w}$ by [Ryan, 2004].

Idea

By linearity, we can note that for $\delta \in \mathfrak{M}(M)$

$$\delta(f(x)) = \sum_{j=0}^{\infty}\left(\sum_{\substack{j_2\cdots j_n \\ j_2+\cdots j_n=j}} \delta(p_{j_2\cdots j_n}(x-w))a_{j_2\cdots j_n}\right)$$

and on each monogenic polynomial

$$\delta(p_{j_2\ldots j_n}(x)) = \frac{1}{j!}\sum_{\text{permutations}} \delta\left((z_{1\sigma(1)}(x)\right)\cdots\delta\left(z_{1\sigma(j)}(x)\right)$$

by the multiplicativity of δ.

Lemma (Point evaluation)

Let $\delta \in \mathfrak{M}(\mathbb{B}_{R,w})$ and $z_{ij} \in \mathcal{A}_{\boldsymbol{B}_{ij}}(\mathbb{B}_{R,w})$, then $\delta(z_{ij}) = z_{ij}(x^\delta)$ for some $x^\delta \in \mathbb{R}^n$.

Proof sketch. Note that δ is an algebra homomorphism and thus

$$\delta(z_{ij}) = \alpha_{ij} + \beta_{ij}\boldsymbol{B}_{ij}.$$

Two key relationships $z_{ij}\boldsymbol{B}_{ji} = -z_{ji}$ and $z_{ij} + z_{kj} + z_{ik}\boldsymbol{B}_{kj}$ yield the relationships

$$\alpha_{ji} = -\beta_{ij} \qquad \alpha_{ij} = \alpha_{kj} \qquad \beta_{ij} = \beta_{ik} \qquad \alpha_{ik} = -\beta_{kj}.$$

Each set of constants α and β is thus given by n independent numbers and so it must be that $\delta(z_{ij}) = z_{ij}(x^\delta)$ for some $x^\delta \in \mathbb{R}^n$.

Lemma (Identification)

Let $f \in \mathcal{M}(\mathbb{B}_{R,w})$, *then* $\delta(f) = f(x^\delta)$ *for some* $x^\delta \in \mathbb{B}_{R,w}$.

Proof: Take $G_0 \in \mathcal{M}^+(\mathbb{B}_{R,w})$ by $G_0(x) = G(x, x_0)\mathbf{e}_1$ with $x_0 \notin \mathbb{B}_{R,w}$. Fix $\delta \in \mathfrak{M}(\mathbb{B}_{R,w})$ and note $\delta(G_0) = G_0(x^\delta)$. Take a sequence $x_n \to x^\delta$ with $x_n \notin \mathbb{B}_{R,w}$ and note that the sequence of functions $G_n(x) = G(x, x_n)\mathbf{e_1} \in \mathcal{M}(\mathbb{B}_{R,w})$. But

$$\lim_{n\to\infty} \delta(G_n) = \lim_{n\to\infty} G_n(x^\delta)$$

does not converge due to a singularity at x^δ. It must be that $x^\delta \in \mathbb{B}_{R,w}$ by continuity of δ.

Theorem (Noncommutative Gelfand representation)

For any $\delta \in \mathfrak{M}(\mathbb{B}_{R,w})$, *there is a point* $x^\delta \in \mathbb{B}_{R,w}$ *such that* $\delta(f) = f(x^\delta)$ *for any* $f \in \mathcal{M}(\mathbb{B}_{R,w})$. *Given the weak-*$\star$ *topology on* $\mathcal{M}^*(\mathbb{B}_{r,w})$, *the map*

$$\gamma: \mathfrak{M}(\mathbb{B}_{R,w}) \to \mathbb{B}_{R,w}, \quad \delta \mapsto x^\delta$$

is a homeomorphism.

Proof: The lemmas show that the map $\gamma : \mathfrak{M}(\mathbb{B}_{R,w}) \to \mathbb{B}_{R,w}$ is bijective. To see that this map is a homeomorphism, take a sequence $\delta_n \to \delta$ in $\mathfrak{M}(\mathbb{B}_{R,w})$ and note

$$\gamma(\delta_n) = x^{\delta_n}.$$

For $f \in \mathcal{M}(\mathbb{B}_{R,w})$ we have

$$f(\gamma(\delta_n)) = f(x^{\delta_n}) = \delta_n(f) = \gamma^{-1}(x^{\delta_n})(f).$$

Taking $n \to \infty$ we realize γ and γ^{-1} are continuous therefore γ is a homeomorphism.

Section 4

Future work

Calderón problem on manifolds

Question: Let (M, g) be an unknown Riemannian manifold with known boundary ∂M. Consider the forward problem

$$\begin{cases} \Delta\omega = 0 & \text{in } M \\ \iota^*\omega = \phi & \text{on } \partial M \end{cases}$$

Define the ***Dirichlet-to-Neumann map*** on forms by $\Lambda\phi = \iota^*(\star d\omega)$. Can we determine (M, g) from Λ?

Calderón problem on manifolds

This problem is equivalent to the electical impedance tomography problem in dimension 3. The problem has been solved in dimension $n = 2$ [Belishev, 2003] and in dimensions $n \geq 3$ when M is an analytic manifold [Krupchyk, Lassas, Uhlmann, 1989]. The smooth cases is still unsolved.

Calderón problem on manifolds

When M is dimension $n = 3$, the scalar potential u and magnetic bivector field b are two parts of a monogenic field $f = u + b$ due to Ohm's and Ampere's laws

$$-\nabla \wedge u = \nabla \rfloor b.$$

If Λ can provide us $b|_{\partial M}$, then we can reconstruct $\mathcal{M}^+(M)$. Perhaps we can show that $\mathcal{M}^+(M)$ recreates M up to homeomorphism. Moreover, we know the algebraic structure of each $\mathcal{A}_{\boldsymbol{B}}(M)$, can this be used to determine g up to isometry?

Other inverse problems

- Can the magnetic impedance tomography problem can provide some extra insight on the EIT problem?
- The Hodge-Morrey decomposition is an instrumental tool for boundary value problems that, for example, allows one to show that Λ determines the Betti numbers of M.
- Can the Clifford-Hodge-Morrey decomposition can allow us to work on other related boundary inverse problems?
- These problems could include spacetime problems where the metric g is of mixed signature.

Section 5

Conclusions

- We have utilized multivector fields to serve as a meaningful generalization of both the complex numbers and differential forms.
- This provides a new way to decompose fields on domains of $\mathbb{R}^n$ and this can likely be generalized to arbitrary compact orientable pseudo-Riemannian manifolds.
- Likewise, we have proven that the monogenic fields contain a wealth of topological information and this information is supported on the boundary by the Cauchy integral formula.

Data Assimilation

Over the past two years I have also worked with a team on developing new techniques for data assimilation. We have submitted an article titled “Model and Data Reduction for Data Assimilation: Particle Filters Employing Projected Forecasts and Data with Application to a Shallow Water Model" We are continuing to work to apply our scheme to new models such as the Modular Arbitrary-Order Ocean-Atmospheric Model (MAOOAM).