Incompressible Fluid Flow:

Arnol'd's Geometrical Approach

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Definition

A **topological manifold** M **of dimension** m is a locally Euclidean space.

- Every open set on M are homeomorphic to open sets in $\mathbb{R}^m$.
- Homeomorphisms are continuous bijections with continuous inverse.

Example

$\mathbb{R}^m$ itself is a topological manifold. The homeomorphisms are trivial.

Example

The m-sphere, S^m is a topological manifold.

- There are many homeomorphisms mapping open sets of S^m to $\mathbb{R}^m$.
- Stereographic projection is gives us a two chart covering of S^m.

Example

Topological_Hydrodynamics/stereog

Figure: Stereographic projection of S^2.

Adding Smoothness

Semi_Riemannian/smooth_manifold.png

Figure: The key picture for smooth manifolds.

Semi_Riemannian/smooth_manifold.png

More Formally

- $\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \subset \mathbb{R}^m \to \mathbb{R}^m$, which means we can talk about derivatives.
- We require the *transition map* $\psi \circ \varphi^{-1}$ to be smooth on the overlap.
- Collect these *charts* into an *atlas*.
- These charts are *diffeomorphisms*

Some smooth manifolds:

- $\mathbb{R}^m$
- S^m
- Mobius Band
- Klein Bottle
- $\mathrm{GL}(m)$
- $\mathrm{SO}(m)$
- Products of smooth manifolds

Definition

- Fix M, N smooth manifolds. We say $f\colon M \to N$ is **smooth** if for any choice of coordinates on M (φ) and on N (ψ), the map $\hat{f} := \psi \circ f \circ \varphi^{-1}$ is smooth (usually C^∞).
- f is a **diffeomorphism** if $\hat{f}$ is smooth with smooth inverse (think smooth homeomorphism).

Coordinates

Topological_Hydrodynamics/latlonggrid.jpg

Figure: Longitude and latitude lines on a sphere as coordinates.

- Given $(M, \mathcal{A})$, we can pick coordinates from the atlas $\mathcal{A}$.
- Usually take $\mathbf{x} : U \subset M \to \mathbb{R}^m$ without mentioning U or the specific function $\mathbf{x}$.
- The component functions are $x^i : U \subset M \to \mathbb{R}$, for $i = 1, \ldots, m$.
- They induce coordinates on tangent space and cotangent space, $\frac{\partial}{\partial x^i}$ and dx^i respectively.

Tangent Space

Semi_Riemannian/tangent_space.png

Figure: The pictorial idea of a tangent space.

- Take a curve $\gamma\colon (-\epsilon, \epsilon) \to M$.
- Choose coordinates so $\gamma(0) = x \in M$.
- Consider
$$T_x M \ni v = \dot{\gamma}(0) := \left.\frac{d}{dt}\gamma(t)\right|_{t=0}.$$
- We let $T_x M$ be the vector space isomorphic to $\mathbb{R}^m$ of equivalence classes of velocity vectors to curves through x.
- Basis vectors are $\frac{\partial}{\partial x^i}$ for $i = 1, \ldots, m$.

Definition

A **Riemannian manifold** is a smooth manifold with an inner product in each tangent space.

Remark

Everything I'll be working with will be a Riemannian manifold. In fact, any smooth manifold admits a Riemannian structure.

Definition

A **Lie group** G is a smooth manifold G that is also a group. The group operation on G must also be smooth as a map from $G \to G$.

Example

- $\mathbb{R}^m$, group operation is vector addition.
- S^1, group operation is rotation.
- $T^m = \underbrace{S^1 \times \cdots S^1}_{m-\text{times}}$, group operation is that given by the direct product of S^1 with itself however many times.
- $\mathrm{SO}(m)$, group operation is matrix multiplication.

There are a few important reasons to care about Lie groups:

- We can better understand M by seeing how Lie groups act on it.
- Actions can allow us to see symmetries and flows on M.
- Lie groups show up everywhere in physics (mostly as symmetry groups and gauges).

Definition

A vector field on M is a smooth function (section) X that assigns a tangent vector at each point on M.

More formally, $X\colon M \to TM$ is smooth and $\pi \circ X = \mathrm{id}_M$

Definition

A flow on M is an additive $\mathbb{R}$ action on M. That is,

$$\varphi \colon M \times R \to M$$

so that

$$\varphi(p, 0) = p,$$

and

$$\varphi(\varphi(x, t), s) = \varphi(x, s + t).$$

This is to say, flows are a 1-parameter group of diffeomorphisms on M.

Example of A Flow

Topological_Hydrodynamics/flow.jpg

Figure: Flows of points are integral curves of a vector field.

Definition

The **Lie algebra $\mathfrak{g}$ of a Lie group** G is the tangent space at the identity, T_eG.

Example

- If $G = S^1$, $\mathfrak{g} \cong \mathbb{R}$.
- If $G = \mathrm{SO}(m)$, then $\mathfrak{so}(m) := \mathfrak{g} \cong \{A \in \mathrm{GL}(m) \mid A^T = -A\}$

Example

Take $\gamma\colon (-\epsilon, \epsilon) \to \mathrm{SO}(m)$ with $\gamma(0) = I$. Note that we have

$$\gamma(t)\gamma(t)^T = I.$$

If we differentiate with respect to t at $t = 0$, then

$$\gamma'(0)\gamma(0)^T + \gamma(0)\gamma'(0)^T = 0$$

$$\implies \gamma(0)^T + \gamma(0) = 0.$$

So $\mathfrak{so}(m)$ consists of the skew symmetric matrices.

- Defined for any smooth manifold.
- Easier definition for Lie groups.
  - (Finite dimensional) Lie groups are all submanifolds of $\mathrm{GL}(m)$.
  - The exponential map is the matrix exponential, i.e., we take

$$\exp\colon \mathfrak{g} \to G$$

by letting $A \in \mathfrak{g}$ and taking

$$\exp(A) = e^A.$$

Topological_Hydrodynamics/exponen

Figure: Pictorial representation of the exponential map.

The more general definition for the exponential map can be understood as follows:

- Pick a point $p \in M$ and consider a vector $v \in T_pM$. Then we can take

$$\exp_p(v) = \gamma_v(1)$$

  where γ_v is a curve on M passing through p with v as its tangent vector at p. Moreover, γ_v is a *geodesic*.

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- Given a smooth manifold M, the length functional

$$I\colon \text{Curves on } M \to \mathbb{R},$$

we can optimize this functional via a *variation*. T

- For a curve γ, we write

$$I = \int_{t_0}^{t_1} \|\dot{\gamma}(t)\|_g dt,$$

where

$$\|\dot{\gamma}(t)\|_g$$

is the length of the tangent vector to γ at time $t \in [t_0, t_1]$.

- The variation is taken with respect to an arbitrary curve c,

$$D[I;c] = \lim_{\epsilon \to 0} \frac{I[\gamma + \epsilon c] - I[\gamma]}{\epsilon}.$$

- In general, this gives us the *Euler-Lagrange equations*. For this specific case, we find the geodesic equations

$$\frac{d^2\gamma^i}{dt^2} + \Gamma^i_{jk} \frac{d\gamma^j}{dt} \frac{d\gamma^k}{dt} = 0,$$

where γ^i are the coordinates chosen for our curve γ. (i.e., $\gamma^i = x^i \circ \gamma$.)

There are two nice ways to think of geodesics:

- Optimal solutions to the length functional.
  - Can be understood as the paths taken by free particles on M.
- 'Straightest' lines on M.
  - Can be understood by requireing

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0$$

which says that geodesics are constant speed paths that do not "turn." (i.e., in a car coasting and not turning the steering wheel.)

Example (Geodesics on S^2)

- Choose spherical coordinates on S^2: $x^1 = \theta$, $x^2 = \phi$, giving $\Gamma^2_{22} = -\sin\theta\cos\theta$ and $\Gamma^2_{12} = \Gamma^2_{21} = 2\cot\theta$.
- Then, the geodesic equations read

$$\ddot{\theta} - (\dot{\phi})^2 \sin\theta\cos\theta = 0$$

$$\ddot{\phi} + 2\dot{\theta}\dot{\phi}\cot\theta = 0.$$

- More work yields

$$\frac{d\phi}{d\theta} = \frac{c}{\sin\theta\sqrt{\sin^2\theta - c^2}}.$$

Topological_Hydrodynamics/great_circle.png

Figure: Great circle on S^2. (i.e., paths that planes fly.

Question

How can we create a geodesic in $\mathrm{SO}(m)$?

- We take $X \in \mathfrak{so}(m)$ (skew symmetric),
- define

$$\gamma(t) := \exp(tX),$$

- then $\dot{\gamma}(0) = X$ and $\gamma(0) = I$.
- If we want a geodesic beginning at another point, we can take $A \in \mathrm{SO}(m)$ and note that

$$\tilde{\gamma}(t) = A \exp(tX)$$

satisfies $\tilde{\gamma}(0) = A$ and is a geodesic.

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Given a manifold M, and a Lie group G, we can define a (left) action

$$G \circlearrowright M$$

by taking

$$L \colon G \to \mathrm{Diff}(M)$$

and then we have for $g \in G$

$$L_g \colon M \to M.$$

Remark

Note that $\mathrm{Diff}(M)$ is (in a sense) an infinite dimensional Lie group. It's Lie algebra is the smooth vector fields on M.

Let $M = S^2$ and $G = \mathrm{SO}(3)$, and define a left action by embedding S^2 in R^3 as column vectors and writing elements of G as 3×3 matrices.
Take $p \in S^2$ as $p = (x_1, x_2, x_3)^T$ where $x_1^2 + x_2^2 + x_3^2 = 1$ and $A \in \mathrm{SO}(3)$ as

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$Ap = (x_1 \cos\theta - x_2 \sin\theta, x_1 \sin\theta + x_2 \cos\theta, x_3)^T.$$

Note $\|Ap\| = 1$ and that this is a rotation about the z-axis.

- The important Lie group for our case is the volume preserving diffeomorphism group on M.
- $\mathrm{SDiff}(M) := \{f \in \mathrm{Diff}(M) \mid f(\mu) = \mu\}$
- Think of this space as the configuration space of a volume of fluid.

Remark

To gain some intuition, imagine $\mathrm{SDiff}(S^2)$. If we fill S^2 with water, then $f \in \mathrm{SDiff}(S^2)$ means $f(S^2)$ still holds the same volume of water.

The Euler-Equations for an ideal incompressible fluid are

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla_{\mathbf{u}} \mathbf{u} = -\nabla p$$

$$\operatorname{div}(\mathbf{u}) = 0.$$

- Arnol'd follows the belief that motion of an inertial system is governed by least action.
- In other words, geodesics on configuration spaces are ideal trajectories.
- Makes the analogy with rigid body motion being described as geodesics on the group of rotations.
- Postulates that a similar result holds for fluids.

Theorem (Arnol'd)

Incompressible fluid flow in M corresponds to geodesic flow on the space of volume preserving diffeomorphisms $\mathrm{SDiff}(M)$.

- Take $B\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined by

$$\langle [X,Y], Z\rangle = \langle B(Z,Y), X\rangle.$$

- Let $\gamma\colon \mathbb{R} \to M$ be a geodesic (flow).
- g is a right invariant (Riemannian) metric on M.

Theorem (Arnol'd)

Let $X(t) := \dot{\gamma}(t)/\gamma(t)$, then X satisfies

$$\frac{d}{dt}X(t) = B(X(t), X(t)).$$

- Can turn PDEs into ODEs.
- ODE theory allows for proving existence, uniqueness, and well-posedness more easily.
- It gives us another interpretation of a system.

Can this framework be applied to more general fluid flow like the Navier-Stokes equation?

- Hunter-Saxton (waves in liquid crystals)

$$(u_t + uu_x)_x = \frac{1}{2}u_x^2.$$

- Camassa-Holm (shallow water waves)

$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}$$

- Discontinuous fluid flow from a Lie groupoid perspective (Vortex Sheets and DIffeomorphism Groupoids).
- Find other areas and equations where this can be done.
- Information manifolds and flows with relation to quantum mechanics.