

Incompressible Fluid Flow: Arnol'd's Geometrical Approach

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Definition

A **topological manifold** M of dimension m is a locally Euclidean space.

- Every open set on M are homeomorphic to open sets in \mathbb{R}^m .
- Homeomorphisms are continuous bijections with continuous inverse.



Example

\mathbb{R}^m itself is a topological manifold. The homeomorphisms are trivial.

Example

The m -sphere, S^m is a topological manifold.

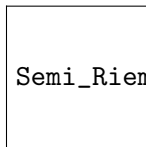
- There are many homeomorphisms mapping open sets of S^m to \mathbb{R}^m .
- Stereographic projection gives us a two chart covering of S^m .



Example

Topological_Hydrodynamics/stereog

Figure: Stereographic projection of S^2 .



Semi_Riemannian/smooth_manifold.png

Figure: The key picture for smooth manifolds.



Semi_Riemannian/smooth_manifold.png

More Formally

- $\psi \circ \varphi^{-1}: \varphi(U \cap V) \subset \mathbb{R}^m \rightarrow \mathbb{R}^m$, which means we can talk about derivatives.
- We require the *transition map* $\psi \circ \varphi^{-1}$ to be smooth on the overlap.
- Collect these *charts* into an *atlas*.
- These charts are *diffeomorphisms*



Some smooth manifolds:

- \mathbb{R}^m
- S^m
- Mobius Band
- Klein Bottle
- $GL(m)$
- $SO(m)$
- Products of smooth manifolds



Definition

- Fix M, N smooth manifolds. We say $f: M \rightarrow N$ is **smooth** if for any choice of coordinates on M (φ) and on N (ψ), the map $\hat{f} := \psi \circ f \circ \varphi^{-1}$ is smooth (usually C^∞).
- f is a **diffeomorphism** if \hat{f} is smooth with smooth inverse (think smooth homeomorphism).



Figure: Longitude and latitude lines on a sphere as coordinates.

- Given (M, \mathcal{A}) , we can pick coordinates from the atlas \mathcal{A} .
Usually take $U \subset M \cong \mathbb{R}^m$ without mentioning U or the specific function \mathbf{x} .
- The component functions are $x^i: U \subset M \rightarrow \mathbb{R}$, for $i = 1, \dots, m$.
- They induce coordinates on tangent space and cotangent space, $\frac{\partial}{\partial x^i}$ and dx^i respectively.

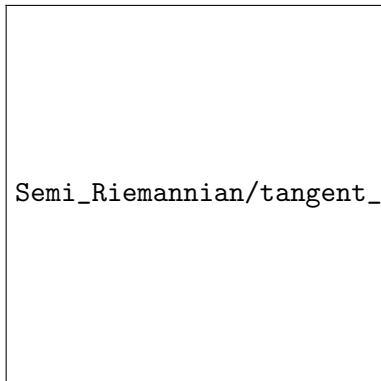


Figure: The pictorial idea of a tangent space.

- Take a curve $\gamma: (-\epsilon, \epsilon) \rightarrow M$.
- Choose coordinates so $\gamma(0) = x \in M$.
- Consider

$$T_x M \ni v = \dot{\gamma}(0) := \left. \frac{d}{dt} \gamma(t) \right|_{t=0}.$$

- We let $T_x M$ be the vector space isomorphic to \mathbb{R}^m of equivalence classes of velocity vectors to curves through x .
- Basis vectors are $\frac{\partial}{\partial x^i}$ for $i = 1, \dots, m$.



Definition

A **Riemannian manifold** is a smooth manifold with an inner product in each tangent space.

Remark

Everything I'll be working with will be a Riemannian manifold. In fact, any smooth manifold admits a Riemannian structure.



Definition

A **Lie group** G is a smooth manifold G that is also a group. The group operation on G must also be smooth as a map from $G \rightarrow G$.

Example

- \mathbb{R}^m , group operation is vector addition.
- S^1 , group operation is rotation.
- $T^m = \underbrace{S^1 \times \cdots S^1}_{m\text{-times}}$, group operation is that given by the direct product of S^1 with itself however many times.
- $\text{SO}(m)$, group operation is matrix multiplication.



There are a few important reasons to care about Lie groups:

- We can better understand M by seeing how Lie groups act on it.
- Actions can allow us to see symmetries and flows on M .
- Lie groups show up everywhere in physics (mostly as symmetry groups and gauges).



Definition

A vector field on M is a smooth function (section) X that assigns a tangent vector at each point on M .

More formally, $X: M \rightarrow TM$ is smooth and $\pi \circ X = \text{id}_M$



Definition

A flow on M is an additive \mathbb{R} action on M . That is,

$$\varphi: M \times \mathbb{R} \rightarrow M$$

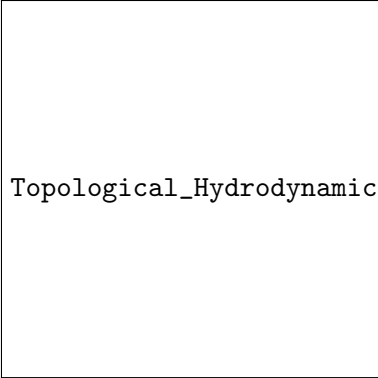
so that

$$\varphi(p, 0) = p,$$

and

$$\varphi(\varphi(x, t), s) = \varphi(x, s + t).$$

This is to say, flows are a 1-parameter group of diffeomorphisms on M .



Topological_Hydrodynamics/flow.jpg

Figure: Flows of points are integral curves of a vector field.



Definition

The **Lie algebra** \mathfrak{g} of a **Lie group** G is the tangent space at the identity, $T_e G$.

Example

- If $G = S^1$, $\mathfrak{g} \cong \mathbb{R}$.
- If $G = \mathrm{SO}(m)$, then $\mathfrak{so}(m) := \mathfrak{g} \cong \{A \in \mathrm{GL}(m) \mid A^T = -A\}$



Example

Take $\gamma: (-\epsilon, \epsilon) \rightarrow \text{SO}(m)$ with $\gamma(0) = I$. Note that we have

$$\gamma(t)\gamma(t)^T = I.$$

If we differentiate with respect to t at $t = 0$, then

$$\gamma'(0)\gamma(0)^T + \gamma(0)\gamma'(0)^T = 0$$

$$\implies \gamma'(0)^T + \gamma'(0) = 0.$$

So $\mathfrak{so}(m)$ consists of the skew symmetric matrices.



- Defined for any smooth manifold.
- Easier definition for Lie groups.
 - (Finite dimensional) Lie groups are all submanifolds of $GL(m)$.
 - The exponential map is the matrix exponential, i.e., we take

$$\exp: \mathfrak{g} \rightarrow G$$

by letting $A \in \mathfrak{g}$ and taking

$$\exp(A) = e^A.$$



Topological_Hydrodynamics/exponen

Figure: Pictorial representation of the exponential map.



The more general definition for the exponential map can be understood as follows:

- Pick a point $p \in M$ and consider a vector $v \in T_p M$. Then we can take

$$\exp_p(v) = \gamma_v(1)$$

where γ_v is a curve on M passing through p with v as its tangent vector at p . Moreover, γ_v is a *geodesic*.





- Given a smooth manifold M , the length functional

$$I: \text{Curves on } M \rightarrow \mathbb{R},$$

we can optimize this functional via a *variation*. T

- For a curve γ , we write

$$I = \int_{t_0}^{t_1} \|\dot{\gamma}(t)\|_g dt,$$

where

$$\|\dot{\gamma}(t)\|_g$$

is the length of the tangent vector to γ at time $t \in [t_0, t_1]$.



- The variation is taken with respect to an arbitrary curve c ,

$$D[I; c] = \lim_{\epsilon \rightarrow 0} \frac{I[\gamma + \epsilon c] - I[\gamma]}{\epsilon}.$$

- In general, this gives us the *Euler-Lagrange equations*. For this specific case, we find the geodesic equations

$$\frac{d^2 \gamma^i}{dt^2} + \Gamma_{jk}^i \frac{d\gamma^j}{dt} \frac{d\gamma^k}{dt} = 0,$$

where γ^i are the coordinates chosen for our curve γ . (i.e., $\gamma^i = x^i \circ \gamma$.)



There are two nice ways to think of geodesics:

- Optimal solutions to the length functional.
 - Can be understood as the paths taken by free particles on M .
- ‘Straightest’ lines on M .
 - Can be understood by requiring

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0$$

which says that geodesics are constant speed paths that do not “turn.” (i.e., in a car coasting and not turning the steering wheel.)



Example (Geodesics on S^2)

- Choose spherical coordinates on S^2 : $x^1 = \theta$, $x^2 = \phi$, giving $\Gamma_{22}^2 = -\sin \theta \cos \theta$ and $\Gamma_{12}^2 = \Gamma_{21}^2 = 2 \cot \theta$.
- Then, the geodesic equations read

$$\ddot{\theta} - (\dot{\phi})^2 \sin \theta \cos \theta = 0$$

$$\ddot{\phi} + 2\dot{\theta}\dot{\phi} \cot \theta = 0.$$

- More work yields

$$\frac{d\phi}{d\theta} = \frac{c}{\sin \theta \sqrt{\sin^2 \theta - c^2}}.$$

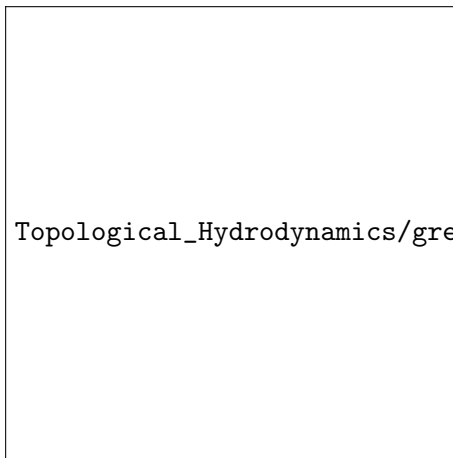


Figure: Great circle on S^2 . (i.e., paths that planes fly.

Question

How can we create a geodesic in $\mathrm{SO}(m)$?

- We take $X \in \mathfrak{so}(m)$ (skew symmetric),
- define

$$\gamma(t) := \exp(tX),$$

- then $\dot{\gamma}(0) = X$ and $\gamma(0) = I$.
- If we want a geodesic beginning at another point, we can take $A \in \mathrm{SO}(m)$ and note that

$$\tilde{\gamma}(t) = A \exp(tX)$$

satisfies $\tilde{\gamma}(0) = A$ and is a geodesic.





Given a manifold M , and a Lie group G , we can define a (left) action

$$G \curvearrowright M$$

by taking

$$L: G \rightarrow \text{Diff}(M)$$

and then we have for $g \in G$

$$L_g: M \rightarrow M.$$

Remark

Note that $\text{Diff}(M)$ is (in a sense) an infinite dimensional Lie group. It's Lie algebra is the smooth vector fields on M .



Let $M = S^2$ and $G = \text{SO}(3)$, and define a left action by embedding S^2 in R^3 as column vectors and writing elements of G as 3×3 matrices.

Take $p \in S^2$ as $p = (x_1, x_2, x_3)^T$ where $x_1^2 + x_2^2 + x_3^2 = 1$ and $A \in \text{SO}(3)$ as

$$A = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$Ap = (x_1 \cos \theta - x_2 \sin \theta, x_1 \sin \theta + x_2 \cos \theta, x_3)^T.$$

Note $\|Ap\| = 1$ and that this is a rotation about the z -axis.



- The important Lie group for our case is the volume preserving diffeomorphism group on M .
- $\text{SDiff}(M) := \{f \in \text{Diff}(M) \mid f(\mu) = \mu\}$
- Think of this space as the configuration space of a volume of fluid.

Remark

To gain some intuition, imagine $\text{SDiff}(S^2)$. If we fill S^2 with water, then $f \in \text{SDiff}(S^2)$ means $f(S^2)$ still holds the same volume of water.



The Euler-Equations for an ideal incompressible fluid are

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla_{\mathbf{u}} \mathbf{u} = -\nabla p$$

$$\operatorname{div}(\mathbf{u}) = 0.$$



- Arnol'd follows the belief that motion of an inertial system is governed by least action.
- In other words, geodesics on configuration spaces are ideal trajectories.
- Makes the analogy with rigid body motion being described as geodesics on the group of rotations.
- Postulates that a similar result holds for fluids.



Theorem (Arnol'd)

Incompressible fluid flow in M corresponds to geodesic flow on the space of volume preserving diffeomorphisms $\text{SDiff}(M)$.



- Take $B: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ defined by

$$\langle [X, Y], Z \rangle = \langle B(Z, Y), X \rangle.$$

- Let $\gamma: \mathbb{R} \rightarrow M$ be a geodesic (flow).
- g is a right invariant (Riemannian) metric on M .

Theorem (Arnol'd)

Let $X(t) := \dot{\gamma}(t)/\gamma(t)$, then X satisfies

$$\frac{d}{dt}X(t) = B(X(t), X(t)).$$



- Can turn PDEs into ODEs.
- ODE theory allows for proving existence, uniqueness, and well-posedness more easily.
- It gives us another interpretation of a system.



Can this framework be applied to more general fluid flow like the Navier-Stokes equation?



- Hunter-Saxton (waves in liquid crystals)

$$(u_t + uu_x)_x = \frac{1}{2}u_x^2.$$

- Camassa-Holm (shallow water waves)

$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}$$



- Discontinuous fluid flow from a Lie groupoid perspective (Vortex Sheets and Diffeomorphism Groupoids).
- Find other areas and equations where this can be done.
- Information manifolds and flows with relation to quantum mechanics.