MATH 560, Homework 8

Colin Roberts

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Solutions

Problem 1. (§6.2 Problem 10.) Let W be a finite-dimensional subspace of an inner product space V. Prove that there exists a projection T on W along $W^\perp$ that satisfies $\mathcal{N}(T) = W^\perp$. In addition, prove that $\|T(v)\| \le \|v\|$ for all $v \in V$. *Hint:* Use Theorem 6.6 and Exercise 10 of Section 6.1.

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Proof (Problem.). Let $v \in V$ be equal to $v = w + w'$ with $w \in W$ and $w' \in W^\perp$. By Theorem 6.6 we have, $V = W \oplus W^\perp$, and it follows that w and w' are uniquely defined. Then we define $T\colon V \to V$ by $Tv = w$. Then if $u \in W^\perp$ we have $Tu = 0$. Lastly, for the same arbitrary v we have

$$\begin{aligned} \|T(v)\| &= \langle Tv, Tv\rangle \\ &= \langle w, w\rangle. \end{aligned}$$

Note that if $v \in W$ then $w' = 0$ and $\|T(v)\| = \|v\|$ else $w' \neq 0$ and we have that $\|T(v)\| < \|v\|$. It then follows that $\|T(v)\| \le \|v\|$. $\square$

Problem 2. (§6.3 Problem 2(c).) For each of the following inner product spaces V (over $\mathbb{F}$) and linear transformations $g\colon V \to \mathbb{F}$, find a vector y such that $g(x) = \langle x, y\rangle$ for all $x \in V$.

$$V = P_2(\mathbb{R}) \text{ with } \langle f, h\rangle = \int_0^1 f(t)h(t)\,dt,\ g(f) = f(0) + f'(1)$$

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Proof. Consider $f(t) = a_0 + a_1 t + a_2 t^2$ and $h(t) = b_0 + b_1 t + a_2 t^2$. Note that $f'(t) = a_1 + 2a_2 t$ and $f(0) = a_0$ and $f'(1) = a_1 + 2a_2$. Then

$$\begin{aligned}
\langle f, h\rangle &= \int_0^1 (a_0 + a_1 t + a_2 t^2)(b_0 + b_1 t + b_2 t^2)\,dt \\
&= \int_0^1 (a_0 b_0 + a_0 b_1 t + a_0 b_2 t^2 + a_1 b_0 t + a_1 b_1 t^2 + a_1 b_2 3^2 + a_2 b_0 t^2 + a_2 b_1 t^3 + a_2 b_2 t^4)\,dt \\
&= \left[a_0 b_0 t + \frac{1}{2} a_0 b_1 t^2 + \frac{1}{3} a_0 b_2 t^3 + \frac{1}{2} a_1 b_0 t^2 + \frac{1}{3} a_1 b_1 t^3 + \frac{1}{4} a_1 b_2 t^4 + \frac{1}{3} a_2 b_0 t^3 + \frac{1}{4} a_2 b_1 t^4 + \frac{1}{5} a_2 b_2 t^5 \right]_0^1 \\
&= a_0 b_0 + \frac{1}{2} a_0 b_1 + \frac{1}{3} a_0 b_2 + \frac{1}{2} a_1 b_0 + \frac{1}{3} a_1 b_1 + \frac{1}{4} a_1 b_2 + \frac{1}{3} a_2 b_0 + \frac{1}{4} a_2 b_1 + \frac{1}{5} a_2 b_2.
\end{aligned}$$

Setting this equal to $f(0) + f'(1) = a_0 + a_1 + 2a_2$ yields

$$\begin{aligned}
a_0 + a_1 + 2a_2 &= a_0 b_0 + \frac{1}{2} a_0 b_1 + \frac{1}{3} a_0 b_2 + \frac{1}{2} a_1 b_0 + \frac{1}{3} a_1 b_1 + \frac{1}{4} a_1 b_2 + \frac{1}{3} a_2 b_0 + \frac{1}{4} a_2 b_1 + \frac{1}{5} a_2 b_2 \\
&= a_0 \left(b_0 + \frac{1}{2} b_1 + \frac{1}{3} b_2 \right) + a_1 \left(\frac{1}{2} b_0 + \frac{1}{3} b_1 + \frac{1}{4} b_2 \right) + a_2 \left(\frac{1}{3} b_0 + \frac{1}{4} b_1 + \frac{1}{5} b_2 \right),
\end{aligned}$$

and we find $b_0 = 33$, $b_1 = -204$, and $b_2 = 210$. So $h(t) = 33 - 204t + 210t^2$. □

Problem 3. (§6.3 Problem 4.) Complete the proof of Theorem 6.11.

Let V be an inner product space, and let T and U be linear operators on V. Then

(a) $(T+U)^* = T^* + U^*$;

(b) $(cT)^* = \overline{c}T^*$ for any $c \in \mathbb{F}$;

(c) $(TU)^* = U^*T^*$;

(d) $T^{**} = T$;

(e) $I^* = I$.

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Proof (a). Done in the text. □

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Proof (b). We have

$$\begin{aligned} \langle x, (cT)^* y\rangle &= \langle (cT)x, y\rangle \\ &= \langle c(Tx), y\rangle \\ &= c\langle Tx, y\rangle \\ &= c\langle x, T^* y\rangle \\ &= \langle x, \overline{c}T^* y\rangle. \end{aligned}$$

□

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Proof (c). We have

$$\begin{aligned} \langle x, (TU)^* y\rangle &= \langle (TU)x, y\rangle \\ &= \langle T(Ux), y\rangle \\ &= \langle Ux, T^* y\rangle \\ &= \langle x, U^* T^* y\rangle. \end{aligned}$$

□

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Proof (d). Done in the text. □

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Proof (e). We have

$$\begin{aligned} \langle x, y\rangle &= \langle Ix, y\rangle \\ &= \langle x, I^* y\rangle \\ &= \langle x, Iy\rangle && \text{by the first line.} \end{aligned}$$

□

Problem 4. (§6.3 Problem 9.) Prove that if $V = W \oplus W^{\perp}$ and T is the projection on W along $W^{\perp}$, then $T = T^*$. *Hint:* Recall that $\mathcal{N}(T) = W^{\perp}$.

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Proof. Since T is the projection on W along $W^{\perp}$ and $V = W \oplus W^{\perp}$, we have for $v \in V$ that $v = w + w'$ with $w \in W$ and $w' \in W^{\perp}$. Then note that $Tv = w$. It follows that for $v_1, v_2 \in V$ we have

$$\begin{aligned}
\langle Tv_1, v_2\rangle &= \langle T(w_1 + w_1'), w_2 + w_2'\rangle && w_1, w_2 \in W \text{ and } w_1', w_2' \in W^{\perp}, \\
&= \langle w_1, w_2 + w_2'\rangle \\
&= \langle w_1, w_2\rangle + \langle w_1, w_2'\rangle \\
&= \langle w_1, w_2\rangle && \text{since } w_2' \in W^{\perp}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\langle v_1, Tv_2\rangle &= \langle w_1 + w_1', T(w_2 + w_2')\rangle \\
&= \langle w_1 + w_1', w_2\rangle \\
&= \langle w_1, w_2\rangle + \langle w_1', w_2\rangle \\
&= \langle w_1, w_2\rangle && \text{since } w_1' \in W^{\perp}.
\end{aligned}$$

Hence, $T = T^*$. □

Problem 5. (§6.3 Problem 13.) Let T be a linear on a finite-dimensional vector space V. Prove the following results.

(a) $\mathcal{N}(T^*T) = \mathcal{N}(T)$. Deduce that $\text{rank}(T^*T) = \text{rank}(T)$.

(b) $\text{rank}(T) = \text{rank}(T^*)$. Deduce from (a) that $\text{rank}(TT^*) = \text{rank}(T)$.

(c) For any $n \times n$ matrix A, $\text{rank}(A^*A) = \text{rank}(AA^*) = \text{rank}(A)$.

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Proof (a). Let $v \in \mathcal{N}(T)$. Then

$$\begin{aligned} \langle T^*Tv, v\rangle &= \langle Tv, Tv\rangle \\ &= 0 && \text{since } v \in \mathcal{N}(T). \end{aligned}$$

Thus a vector $v \in \mathcal{N}(T)$ is also in $\mathcal{N}(T^*T)$. Now let $v' \neq 0 \in \mathcal{R}(T)$, i.e., $v' \notin \mathcal{N}(T)$, then

$$\langle T^*Tv', v'\rangle = \langle Tv', Tv'\rangle \neq 0 \qquad \text{since } v' \in \mathcal{R}(T).$$

Thus if $v' \notin \mathcal{N}(T)$ then $v' \notin \mathcal{N}(T^*T)$. By both of the above, we have $\mathcal{N}(T) = \mathcal{N}(T^*T)$.

It follows that $\text{rank}(T^*T) = \text{rank}(T)$ by above, or by noting the dimension theorem. In other words,

$$\begin{aligned} \dim(V) &= \text{rank}(T) + \text{nullity}(T) \\ &= \text{rank}(T^*T) + \text{nullity}(T^*T) \\ \implies \text{rank}(T^*T) &= \text{rank}(T) && \text{since nullity}(T^*T) = \text{nullity}(T). \end{aligned}$$ □

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Proof (b). Consider $v \neq 0 \in V$ and we have, by definition,

$$\langle Tv, v\rangle = \langle v, T^*v\rangle.$$

If $v \in \mathcal{R}(T)$ then it follows that $v \in \mathcal{R}(T^*)$ else the above equality would not hold. Similarly, if $v \in \mathcal{N}(T)$ then, necessarily, $v \in \mathcal{N}(T^*)$, else, again, the above equality does not hold. It follows immediately that $\text{rank}(T) = \text{rank}(T^*)$.

By part (a) and the above proof, we have that $\text{rank}(T^*T) = \text{rank}(T) = \text{rank}(T^*)$. Then for $v \in V$ we have $\langle T^*Tv, T*v\rangle = \langle Tv, TT^*v\rangle$. It follows that $\text{rank}(TT^*) = \text{rank}(T)$. □

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Proof (c). Fix a basis β for V. Then let $A = [T]_\beta$. Then since T was arbitrary and the choice of basis does not matter, we have that $\text{rank}(A^*A) = \text{rank}(AA^*) = \text{rank}(A)$ by parts (a) and (b). □

Problem 6. (§6.4 Problem 13.) An $n \times n$ real matrix A is said to be a *Gramian* matrix if there exists a real (square) matrix B such that $A = B^T B$. Prove that A is a Gramian matrix if and only if A is symmetric and all of its eigenvalues are nonnegative. *Hint:* Apply Theorem 6.17 to $T = L_A$ to obtain an orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ of eigenvectors with the associated eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Define the linear operator U by $U(v_i) = \sqrt{\lambda_i}\, v_i$.

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Proof. First, suppose that A is Gramian. Thus $A = B^T B$, and it follows immediately that A is self adjoint since A is real and $(B^T B)^T = B^T B$. This means that we have a basis $\beta = \{v_1, v_2, \ldots, v_n\}$ of orthonormal eigenvectors of A by Theorem 6.17. Let $T = L_A$ and then $[T]_\beta v_i = \lambda_i v_i$. In other words, $Av_i = \lambda_i v_i = B^T B v_i$. Then consider any eigenvalue λ_r and we have

$$\begin{aligned}
\lambda_r &= \langle Av_r, v_r\rangle \\
&= \langle B^T B v_r, v_r\rangle \\
&= \langle B v_r, B v_r\rangle \\
&= \|B v_r\|^2 = \|B\|^2 \geq 0.
\end{aligned}$$

Hence, all the eigenvalues are nonnegative.

For the converse, suppose that A is symmetric and all of its eigenvalues are nonnegative. A symmetric implies that $A^T = A$ and that we have a basis $\beta = \{v_1, v_2, \ldots, v_n\}$ of orthonormal eigenvectors and each eigenvalue λ_i is real. By supposition, these eigenvalues are also nonnegative. Define an operator U such that $U(v_i) = \sqrt{\lambda_i}\, v_i$ and then note that $A = Q^{-1} U^2 Q$ where Q changes bases from the original basis of A to the basis of orthonormal eigenvectors. Then

$$\begin{aligned}
\langle Av_i, v_i\rangle &= \langle U^2 v_i, v_i\rangle \\
&= \langle U v_i, U^T v_i\rangle \\
&= \langle v_i, (U^T)^2 v_i\rangle \\
&= \langle (U^T)^2 v_i, v_i\rangle && \text{since every eigenvalue is real.}
\end{aligned}$$

Hence $U = U^T$ and we have that $A = Q^{-1} U^2 Q$. Letting $UQ = B$ we have $B^T B = A$ and thus A is Gramian. □