

Clifford Analysis and a Noncommutative Gelfand Representation

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Section 1

Introduction

Electrical Impedance Tomography

Electrical Impedance Tomography (EIT) asks whether one can determine the conductivity of a medium based on measurements along the boundary.

Other questions

- Can we retrieve topological information from spaces of functions on a manifold M ?
- Do these spaces also contain geometric information such as metric data?
- Can we determine enough about these spaces from partial information – say information only on the boundary?

Subsection 1

Preliminaries

- *Clifford algebra* originated in 1878 with William Kingdon Clifford's work that extends Hermann Grassmann's *exterior algebra*.
- *Clifford analysis* arrived in the 1980's due to Hestenes, Sobczyk, Sommen, Brackx, and Delenghe in order to enrich Élie Cartan's *differential forms*.

Clifford algebras

Let V be a vector space over a field \mathbb{F} with quadratic form Q .

- Define the tensor algebra

$$\mathcal{T}(V) := \bigoplus_{j=0}^{\infty} V^{\otimes j} = \mathbb{F} \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots$$

- Form the *Clifford algebra* via a quotient

$$Cl(V, Q) := \mathcal{T}(V) / \langle \mathbf{v} \otimes \mathbf{v} - Q(\mathbf{v}) \rangle.$$

Geometric and exterior algebras

Let V be a vector space over a field \mathbb{F} with quadratic form Q .

- Given a (pseudo) inner product g , we set $Q(\cdot) = g(\cdot, \cdot)$ and define a *geometric algebra*

$$\mathcal{G} := Cl(V, g).$$

- The *exterior algebra* is given by

$$\bigwedge(V) := Cl(V, 0).$$

Algebra structure

We define a multiplication in \mathcal{G} by noting how the product \otimes acts in the quotient.

- Given $\mathbf{u}, \mathbf{v} \in \mathcal{G}$ we can take the product

$$\mathbf{u}\mathbf{v} = \underbrace{\mathbf{u} \cdot \mathbf{v}}_{\text{scalar}} + \underbrace{\mathbf{u} \wedge \mathbf{v}}_{\text{bivector}}.$$

- The scalar part is symmetric: $\mathbf{u} \cdot \mathbf{v} = g(\mathbf{u}, \mathbf{v})$.
- The bivector part is antisymmetric: $\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u}$.

Multivectors

- \mathcal{G} is graded and of dimension 2^n .
 - There are $\binom{n}{r}$ elements of grade r called *r -vectors*.
 - Those that are exterior products of r independent vectors are *r -blades*.
E.g., $\mathbf{A}_r = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r$.
- The most general elements are *multivectors* and are given by

$$A = \sum_{r=0}^n \langle A \rangle_r,$$

where $\langle A \rangle_r \in \mathcal{G}^r$ extracts the grade r part of A .

Algebra Structure

Extend the multiplication from vectors to multivectors by

$$A_r B_s = \langle A_r B_s \rangle_{|r-s|} + \langle A_r B_s \rangle_{|r-s|+2} + \cdots + \langle A_r B_s \rangle_{r+s}$$

and define

$$A_r \cdot B_s := \langle A_r B_s \rangle_{|r-s|}$$

$$A_r \wedge B_s := \langle A_r B_s \rangle_{r+s}$$

$$A_r \rfloor B_s := \langle A_r B_s \rangle_{s-r}$$

$$A_r \lrcorner B_s := \langle A_r B_s \rangle_{r-s}$$

Reciprocals and reverses

- Given any vector basis \mathbf{v}_i , define the *reciprocal vectors* by $\mathbf{v}^i \cdot \mathbf{v}_j = \delta_j^i$.
- The *reverse* of a multivector is extended linearly from the action on r -blades by

$$\mathbf{A}_r^\dagger = (\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r)^\dagger = \mathbf{v}_r \wedge \cdots \wedge \mathbf{v}_1.$$

Inner product and norm

- Define the *multivector inner product* by

$$(A, B) := \langle A^\dagger B \rangle$$

which is bilinear, symmetric, and positive definite if g is positive definite.

- Define the *multivector norm* by

$$|A| := \sqrt{(A, A)}.$$

Adjoint

Note the reverse acts as an adjoint by

$$(CA, B) = (A, C^\dagger B) \tag{1}$$

$$(AC, B) = (A, BC^\dagger). \tag{2}$$

Pseudoscalars

- *Pseudoscalars* are the grade- n elements. For example, the volume element

$$\boldsymbol{\mu} = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_n.$$

- We define the *unit pseudoscalar* by

$$\boldsymbol{I} := \frac{1}{|\boldsymbol{\mu}|} \boldsymbol{\mu}.$$

Blades and subspaces

- If g is positive definite, then all blades are invertible [Chisholm: 2012].
- If $|\mathbf{A}_r| = 1$, then \mathbf{A}_r is a *unit blade*.
- Unit r -blades correspond to an r -dimensional subspace and can be identified with points in $\text{Gr}(r, n)$.

Duality

- Given any multivector A , we can take its *dual*

$$A^\perp := A\mathbf{I}^{-1}.$$

- Note $A_r^\perp \in \mathcal{G}^{n-r}$, like the Hodge star \star .

Projection and rejection

- The *projection* of B into a subspace \mathbf{A}_r by

$$P_{\mathbf{A}_r}(B) := B \rfloor \mathbf{A}_r \mathbf{A}_r^{-1}$$

- The *rejection* by

$$R_{\mathbf{A}_r}(B) := B \wedge \mathbf{A}_r \mathbf{A}_r^{-1}.$$

- Both are grade preserving.

Quaternions and complex numbers

■ **Claim:** \mathbb{H} arises naturally as the even subalgebra \mathcal{G}_3^+ .

■ **Claim:** \mathbb{C} arises naturally as the even subalgebra \mathcal{G}_2^+ .

- Take the standard basis $\mathbf{e}_1, \mathbf{e}_2$, and define $\mathbf{B}_{12} = \mathbf{e}_1 \mathbf{e}_2$ and note $\mathbf{B}_{12}^2 = -1$.

Thus,

$$(u_1 + v_1 \mathbf{B}_{12})(u_2 + v_2 \mathbf{B}_{12}) = u_1 u_2 - v_1 v_2 + (u_1 v_2 + u_2 v_1) \mathbf{B}_{12}.$$

- Moreover, right multiplication of vectors by \mathbf{B}_{12} rotates counter-clockwise by $\pi/2$.

Clifford algebra structure on manifolds

- We let M be a smooth, compact, connected, and oriented n -dimensional Riemannian manifold with metric g (unless otherwise stated).
- Idea: Form the Clifford algebras on tangent spaces.
 - Each $Cl(T_p M, g_p)$ is a *geometric tangent space* which we glue together to form

$$Cl(TM, g) := \bigsqcup_{p \in M} Cl(T_p M, g_p).$$

- The space of *(smooth) multivector fields* is

$$\mathcal{G}(M) := \{C^\infty\text{-smooth sections of } Cl(TM, g)\}.$$

Section 2

Clifford analysis

Covariant derivative

On M we have the unique torsion free Levi-Civita connection ∇ and covariant derivative $\nabla_{\mathbf{u}}$.

- $\nabla_{\mathbf{u}}$ can be extended to multivectors and it is grade preserving [Schindler: 2018],

$$\nabla_{\mathbf{u}} A_r = \langle \nabla_{\mathbf{u}} A_r \rangle_r.$$

- $\nabla_{\mathbf{u}}$ is compatible with dot and wedge since

$$\nabla_{\mathbf{u}}(A \cdot B) = (\nabla_{\mathbf{u}} A) \cdot B + A \cdot (\nabla_{\mathbf{u}} B)$$

$$\nabla_{\mathbf{u}}(A \wedge B) = (\nabla_{\mathbf{u}} A) \wedge B + A \wedge (\nabla_{\mathbf{u}} B).$$

Gradient

- We define the *gradient* (or *Dirac operator*) in some local basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ by

$$\nabla = \sum_{i=1}^n \mathbf{v}^i \nabla_{\mathbf{v}_i}$$

- ∇ has the algebraic properties of a vector in $\mathcal{G}(M)$ and Leibniz rule

$$\nabla(AB) = \dot{\nabla} \dot{A} B + \dot{\nabla} A \dot{B}.$$

- Note $\nabla^2 = \Delta$, the Laplace-Beltrami operator.

Differential forms

We define the *r -dimensional directed measure*

$$dX_r := \mathbf{v}_{j_1} \wedge \cdots \mathbf{v}_{j_r} dx^{j_1} \cdots dx^{j_r}$$

where summation is implied over the increasing set of indices $1 \leq j_1 < \cdots < j_r \leq n$.

This allows us to define an r -form α_r by

$$\alpha_r = A_r \cdot dX_k^\dagger$$

where $A_r = \frac{1}{r!} \alpha_{i_1 \dots i_r} \mathbf{v}^{i_1} \wedge \cdots \wedge \mathbf{v}^{i_r}$. We call A_r the *$\text{multivector equivalent}$* of α_r .

Volume form

The *volume form* on M is given in local coordinates by

$$\mu = \sqrt{|g|} dx^1 \cdots dx^n = \mathbf{I}^{-1} \cdot dX_n$$

Hence, we can integrate scalar fields A_0 on M by

$$\int_M A_0^\perp \cdot dX_n = \int_M A_0 \mu.$$

Exterior algebra and calculus

- Given an r - and s -form α_r and β_s we have

$$\begin{aligned}\alpha_r + \beta_s &= (A_r + B_s) \cdot dX_r^\dagger \\ \alpha_r \wedge \beta_s &= (A_r \wedge B_s) \cdot dX_{r+s}^\dagger.\end{aligned}$$

- The exterior derivative on multivector equivalents is

$$d\alpha_r = (\nabla \wedge A_r) \cdot dX_{r+1}^\dagger$$

- The Hodge star on multivector equivalents is

$$\star\alpha_r = (I^{-1}A_r)^\dagger \cdot dX_{n-r}^\dagger$$

Multivector field inner product

- We define an inner product on multivector fields by

$$\ll A, B \gg := \frac{1}{\text{vol}(M)} \int_M (A, B) \mu$$

- This realizes the r -form inner product

$$\int_M \alpha_r \wedge \star \beta_r = \int_M \langle A_r^\dagger B_r \rangle \mu = \text{vol}(M) \ll A, B \gg$$

- A_r and B_s are orthogonal when $r \neq s$ so this agrees with the grade direct sum \oplus – we use the same notation for both.

Boundary

On the boundary ∂M , we have the boundary pseudoscalar \mathbf{I}_∂ and the boundary normal $\boldsymbol{\nu} = \mathbf{I}_\partial^\perp$. Then

$$\mu_\partial := \mathbf{I}_\partial^{-1} \cdot dX_{n-1}$$

and we define

$$\ll A, B \gg_\partial := \frac{1}{\text{vol}(M)} \int_{\partial M} (A, B) \mu_\partial.$$

Multivector valued integrals

We can define a multivector valued integral on an oriented submanifold R by taking $A \in \mathcal{G}(M)$ and computing

$$\int_R A \mathbf{I}_R \mu_R.$$

Theorem (Hestenes, Sobczyk, 1984)

Let $A, B \in \mathcal{G}(M)$, then

$$\begin{aligned}\int_M \dot{A} \dot{\nabla} \mathbf{I} \mu &= \int_{\partial M} A \mathbf{I} \partial \mu \partial \\ \int_M \mathbf{I} \nabla B \mu &= \int_{\partial M} \mathbf{I} \partial B \mu \partial \\ \int_M \dot{A} \dot{\nabla} \mathbf{I} B \mu &= (-1)^n \int_M A \mathbf{I} \nabla B \mu + \int_{\partial M} A \mathbf{I} \partial B \mu \partial.\end{aligned}$$

Theorem

We have the Green's formula for the gradient

$$\ll A, \mathbf{I} \nabla B \gg = (-1)^n \ll \nabla A, \mathbf{I} B \gg + \ll A, \mathbf{I} \partial B \gg_{\partial} .$$

Proof. Fix $A^{\dagger}, B \in \mathcal{G}(M)$ and note that

$$\begin{aligned} \int_M A^{\dagger} \mathbf{I} \nabla B \mu &= (-1)^n \int_M \dot{A}^{\dagger} \dot{\nabla} \mathbf{I} B \mu + \int_{\partial M} A^{\dagger} \mathbf{I} \partial B \mu_{\partial} \\ &= (-1)^n \int_M (\nabla A)^{\dagger} \mathbf{I} B \mu + \int_{\partial M} A^{\dagger} \mathbf{I} \partial B \mu_{\partial} . \end{aligned}$$

Then, take the scalar part and divide by $\text{vol}(M)$ to find

$$\ll A, \mathbf{I} \nabla B \gg = (-1)^n \ll \nabla A, \mathbf{I} B \gg + \ll A, \mathbf{I} \partial B \gg_{\partial} .$$

Monogenic fields and gradients

- The space of *monogenic fields* is

$$\mathcal{M}(M) := \{A \in \mathcal{G}(M) \mid \nabla A = 0\}.$$

- Let $f = u + v\mathbf{B} \in \mathcal{G}_2^+(\mathbb{R}^2)$ then $\nabla f = 0$ yields the Cauchy-Riemann equations

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}.\end{aligned}$$

- The *gradients* are

$$\nabla \mathcal{G}(M) := \{\nabla A \mid A \in \mathcal{G}(M) \text{ and } A|_{\partial M} = 0\}.$$

For M a domain in \mathbb{R}^n with $n \geq 2$, we have the vector valued field

$$E(x) := \frac{1}{S_n} \frac{x}{|x|^n}$$

where S_n is the surface area of the unit ball. Note

$$\nabla E(x) = -\dot{E}(x)\dot{\nabla} = \delta(x).$$

We then define the *Cauchy kernel* by $G(x, x') := E(x' - x)$.

Cauchy integral

If $A \in \mathcal{M}(M)$, then we have the *Cauchy integral formula*

$$A(x) = (-1)^n \mathbf{I}^{-1} \int_{\partial M} G(x, x') \mathbf{I}_{\partial}(x') A(x') \mu_{\partial}(x').$$

This allows us to uniquely determine a monogenic field from boundary values $A|_{\partial M}$.

Lemma

Let $A \in \mathcal{M}(M)$ be such that $A|_{\partial M} = 0$. Then $A = 0$ on all of M .

Proof.

$$|A(x)| \leq \left| \int_{\partial M} G(x, x') \mathbf{I}_{\partial}(x') A(x') \mu_{\partial}(x') \right| \leq \int_{\partial M} |G(x, x') \mathbf{I}_{\partial}(x') A(x')| \mu_{\partial}(x') = 0.$$

Lemma

Fix a multivector field $A \in \mathcal{G}(M)$. If

$$\ll A, B \gg = 0$$

for all $B \in \mathcal{G}(M)$ with $B|_{\partial M} = 0$, then $A = 0$.

Proof sketch.

- Use mollifiers to smooth indicator functions χ_U on open subsets U to be supported only on closed ϵ neighborhood $\overline{U^\epsilon}$. Call these functions χ_U^ϵ .
- Write $A = \sum_J A_J \mathbf{V}^J$ with $\mathbf{V}^J = \mathbf{v}^{j_1} \wedge \dots \wedge \mathbf{v}^{j_r}$. Then note

$$\langle\langle A, A_J \mathbf{V}_J \chi_U^\epsilon \rangle\rangle = 0$$

implies $A_J = 0$ on U^ϵ for all J since $(\mathbf{V}^J, \mathbf{V}_K) = \delta_K^J$. Hence $A = 0$ on U^ϵ .

- Cover M in such U^ϵ and repeat the argument leaving the $A|_{\partial M}$ undetermined. But, by smoothness of A , $A = 0$ on M .

Theorem (Clifford-Hodge-Morrey Decomposition)

The space of multivector fields $\mathcal{G}(M)$ has the L^2 -orthogonal decomposition

$$\mathcal{G}(M) = \mathcal{M}(M) \oplus \mathbf{I}\nabla\mathcal{G}(M).$$

Proof

■ *Orthogonality:* Let $A \in \mathcal{M}(M)$ and $\mathbf{I}\nabla B \in \mathbf{I}\nabla\mathcal{G}(M)$ and note

$$\ll A, \mathbf{I}\nabla B \gg = (-1)^n \ll \nabla A, \mathbf{I}B \gg + \ll A, \mathbf{I}\partial B \gg = 0,$$

by the multivector Green's formula.

- Let $C \in \mathcal{G}(M)$ be in the orthogonal complement to $\mathbf{I}\nabla\mathcal{G}(M)$. Then, by the Cauchy integral formula, construct a monogenic field \tilde{C} from $C|_{\partial M}$ and note $C = \tilde{C} + C_0$ where $C_0|_{\partial M} = 0$. Note

$$0 = \langle\langle C, \mathbf{I}\nabla B \rangle\rangle = \langle\langle \nabla C_0, \mathbf{I}B \rangle\rangle .$$

By the previous lemmas, it must be that $C_0 = 0$. Hence the orthogonal complement to $\mathbf{I}\nabla\mathcal{G}(M)$ is $\mathcal{M}(M)$.

Comparing to Hodge-Morrey

The Hodge-Morrey decomposition reads

$$\mathcal{G}^r(M) = \mathcal{E}^r(M) \oplus \mathcal{C}^r(M) \oplus \mathcal{M}^r(M).$$

whereas the Clifford-Hodge-Morrey decomposition ignores specific grades

$$\mathcal{G}(M) = \mathcal{M}(M) \oplus \mathbf{I}\nabla\mathcal{G}(M).$$

Section 3

Gelfand theory

Open question

ulenko, 2017 Prove a noncommutative Gelfand representation using quaternion fields for a ball \mathbb{B} in \mathbb{R}^3 .

- We will prove an analogous result for an arbitrary \mathbb{B} in \mathbb{R}^n .
- This approach can hopefully be used to prove the analogous result for any smooth orientable Riemannian manifold M .
- For now, we take M imbedded in \mathbb{R}^n .

Subsurface spinor fields

- Let $\mathbf{B} \in \mathcal{G}(M)$ be a constant unit 2-blade, then an even multivector field f_+ satisfying

$$f_+ = P_{\mathbf{B}} \circ f_+ \circ P_{\mathbf{B}}$$

is a *subsurface spinor field* and we let $\mathcal{G}_{\mathbf{B}}^+(M)$ to denote the space such fields.

- We note that the space

$$\mathcal{A}_{\mathbf{B}(M)} = \{f_+ \in \mathcal{G}_{\mathbf{B}}^+(M) \mid \nabla f_+ = 0\}$$

is a commutative unital Banach algebra.

Functionals

We define the *spinor dual* $\mathcal{M}^*(M)$ as the continuous right \mathcal{G}_n -module homomorphisms

$$\mathcal{M}^*(M) := \{l: \mathcal{M}^+(M) \rightarrow \mathcal{G}_n^+ \mid l(fs + g) = l(f)s + l(g), \forall f, g \in \mathcal{M}(M), s \in \mathcal{G}_n^+\}$$

and refer to the elements as *spin functionals*. We provide $\mathcal{M}^*(M)$ with the weak-* topology so that every $x \in M$ corresponds to a continuous map on $\mathcal{M}^*(M)$.

Characters

Define the algebra $\mathbb{A}_{\mathbf{B}}$ to be the algebra generated by 1 and \mathbf{B} . Then, the *spinor spectrum* $\mathfrak{M}(M)$ is the set of algebra homomorphisms

$$\mathfrak{M}(M) := \{ \delta \in \mathcal{M}^*(M) \mid \delta(f) \in \mathbb{A}_{\mathbf{B}}, \delta(fg) = \delta(f)\delta(g), \forall f, g \in \mathcal{A}_{\mathbf{B}}(M), \mathbf{B} \in \text{Gr}(2, n) \}$$

and refer to the elements as *spin characters*. Note that one example of such characters are point evaluations $\delta(f) = f(x^\delta)$.

z analogs and monogenic polynomials

Take \mathbf{e}_i to be an orthonormal basis for \mathbb{R}^n , let $\mathbf{B}_{ij} = \mathbf{e}_i \mathbf{e}_j$ and define the functions $z_{ij} = x_j - x_i \mathbf{B}_{ij}$ and note $z_{ij} \in \mathcal{A}_{\mathbf{B}_{ij}}(M)$.

Let σ be a permutation of $\{2, 3, \dots, n\}$, then the homogeneous polynomial of degree j

$$p_{j_2 \dots j_n}(x) = \frac{1}{j!} \sum_{\text{permutations}} z_{1\sigma(1)}(x) \cdots z_{1\sigma(j)}(x)$$

is monogenic.

Collect these into the set of *monogenic polynomials*

$$\mathcal{M}^{\mathcal{P}}(M) = \left\{ \sum_{j=0}^N \left(\sum_{\substack{j_2 \dots j_n \\ j_2 + \dots + j_n = j}} p_{j_2 \dots j_n} a_{j_2 \dots j_n} \right) \mid j_2 + \dots + j_n = j, N \in \mathbb{N}, a_{j_2 \dots j_n} \in \mathcal{G}_n \right\}.$$

Lemma (Density)

The space $\mathcal{M}^{\mathcal{P}}(M)$ is dense in $\mathcal{M}(M)(\mathbb{B}_{R,w})$.

Proof sketch. Let $f \in \mathcal{M}(\mathbb{B}_{R,w})$ and use the Cauchy integral formula to define the coefficients by

$$a_{j_2 \dots j_n} = \int_{\partial B(w,R)} \frac{\partial^j G(w,y)}{\partial y_2^{j_2} \dots \partial y_n^{j_n}} \nu(y) f(y) \mu_{\partial}(y),$$

where each $a_{j_2 \dots j_n} \in \mathcal{G}_n^+$. Then

$$f(x) = \sum_{j=0}^{\infty} \left(\sum_{\substack{j_2 \dots j_n \\ j_2 + \dots + j_n = j}} p_{j_2 \dots j_n}(x-w) a_{j_2 \dots j_n} \right),$$

converges pointwise for $x \in \mathbb{B}_{R,w}$ by [Ryan, 2004].

Idea

By linearity, we can note that for $\delta \in \mathfrak{M}(M)$

$$\delta(f(x)) = \sum_{j=0}^{\infty} \left(\sum_{\substack{j_2 \dots j_n \\ j_2 + \dots + j_n = j}} \delta(p_{j_2 \dots j_n}(x - w)) a_{j_2 \dots j_n} \right)$$

and on each monogenic polynomial

$$\delta(p_{j_2 \dots j_n}(x)) = \frac{1}{j!} \sum_{\text{permutations}} \delta((z_{1\sigma(1)}(x)) \cdots \delta(z_{1\sigma(j)}(x))$$

by the multiplicativity of δ .

Lemma (Point evaluation)

Let $\delta \in \mathfrak{M}(\mathbb{B}_{R,w})$ and $z_{ij} \in \mathcal{A}_{\mathbf{B}_{ij}}(\mathbb{B}_{R,w})$, then $\delta(z_{ij}) = z_{ij}(x^\delta)$ for some $x^\delta \in \mathbb{R}^n$.

Proof sketch. Note that δ is an algebra homomorphism and thus

$$\delta(z_{ij}) = \alpha_{ij} + \beta_{ij} \mathbf{B}_{ij}.$$

Two key relationships $z_{ij} \mathbf{B}_{ji} = -z_{ji}$ and $z_{ij} + z_{kj} + z_{ik} \mathbf{B}_{kj}$ yield the relationships

$$\alpha_{ji} = -\beta_{ij} \quad \alpha_{ij} = \alpha_{kj} \quad \beta_{ij} = \beta_{ik} \quad \alpha_{ik} = -\beta_{kj}.$$

Each set of constants α and β is thus given by n independent numbers and so it must be that $\delta(z_{ij}) = z_{ij}(x^\delta)$ for some $x^\delta \in \mathbb{R}^n$.

Lemma (Identification)

Let $f \in \mathcal{M}(\mathbb{B}_{R,w})$, then $\delta(f) = f(x^\delta)$ for some $x^\delta \in \mathbb{B}_{R,w}$.

Proof: Take $G_0 \in \mathcal{M}^+(\mathbb{B}_{R,w})$ by $G_0(x) = G(x, x_0)\mathbf{e}_1$ with $x_0 \notin \mathbb{B}_{R,w}$. Fix $\delta \in \mathfrak{M}(\mathbb{B}_{R,w})$ and note $\delta(G_0) = G_0(x^\delta)$. Take a sequence $x_n \rightarrow x^\delta$ with $x_n \notin \mathbb{B}_{R,w}$ and note that the sequence of functions $G_n(x) = G(x, x_n)\mathbf{e}_1 \in \mathcal{M}(\mathbb{B}_{R,w})$. But

$$\lim_{n \rightarrow \infty} \delta(G_n) = \lim_{n \rightarrow \infty} G_n(x^\delta)$$

does not converge due to a singularity at x^δ . It must be that $x^\delta \in \mathbb{B}_{R,w}$ by continuity of δ .

Theorem (Noncommutative Gelfand representation)

For any $\delta \in \mathfrak{M}(\mathbb{B}_{R,w})$, there is a point $x^\delta \in \mathbb{B}_{R,w}$ such that $\delta(f) = f(x^\delta)$ for any $f \in \mathcal{M}(\mathbb{B}_{R,w})$. Given the weak- $$ topology on $\mathcal{M}^*(\mathbb{B}_{r,w})$, the map*

$$\gamma: \mathfrak{M}(\mathbb{B}_{R,w}) \rightarrow \mathbb{B}_{R,w}, \quad \delta \mapsto x^\delta$$

is a homeomorphism.

Proof: The lemmas show that the map $\gamma: \mathfrak{M}(\mathbb{B}_{R,w}) \rightarrow \mathbb{B}_{R,w}$ is bijective. To see that this map is a homeomorphism, take a sequence $\delta_n \rightarrow \delta$ in $\mathfrak{M}(\mathbb{B}_{R,w})$ and note

$$\gamma(\delta_n) = x^{\delta_n}.$$

For $f \in \mathcal{M}(\mathbb{B}_{R,w})$ we have

$$f(\gamma(\delta_n)) = f(x^{\delta_n}) = \delta_n(f) = \gamma^{-1}(x^{\delta_n})(f).$$

Taking $n \rightarrow \infty$ we realize γ and γ^{-1} are continuous therefore γ is a homeomorphism.

Section 4

Future work

Calderón problem on manifolds

Question: Let (M, g) be an unknown Riemannian manifold with known boundary ∂M . Consider the forward problem

$$\begin{cases} \Delta\omega = 0 & \text{in } M \\ \iota^*\omega = \phi & \text{on } \partial M \end{cases}$$

Define the *Dirichlet-to-Neumann map* on forms by $\Lambda\phi = \iota^*(\star d\omega)$. Can we determine (M, g) from Λ ?

Calderón problem on manifolds

- This problem is equivalent to the electrical impedance tomography problem in dimension 3.
- The problem has been solved in dimension $n = 2$ [Belishev, 2003].
- Solved in dimensions $n \geq 3$ when M is an analytic manifold [Krupchyk, Lassas, Uhlmann, 1989]. The smooth cases is still unsolved.

Calderón problem on manifolds

When M is dimension $n = 3$, the scalar potential u and magnetic bivector field b are two parts of a monogenic field $f = u + b$ due to Ohm's and Ampere's laws

$$-\nabla \wedge u = \nabla \lrcorner b.$$

If Λ can provide us $b|_{\partial M}$, then we can reconstruct $\mathcal{M}^+(M)$. Perhaps we can show that $\mathcal{M}^+(M)$ recreates M up to homeomorphism. Moreover, we know the algebraic structure of each $\mathcal{A}_B(M)$, can this be used to determine g up to isometry?

Other inverse problems

- Can the magnetic impedance tomography problem can provide some extra insight on the EIT problem?
- The Hodge-Morrey decomposition is an instrumental tool for boundary value problems that, for example, allows one to show that Λ determines the Betti numbers of M [Belishev, Sharafutdinov: 2008].
- Can the Clifford-Hodge-Morrey decomposition can allow us to work on other related boundary inverse problems?
- These problems could include spacetime problems where the metric g is of mixed signature.

Section 5

Conclusions

- We have utilized multivector fields to serve as a meaningful generalization of both the complex numbers and differential forms.
- This provides a new way to decompose fields on domains of \mathbb{R}^n and this can likely be generalized to arbitrary compact orientable pseudo-Riemannian manifolds.
- Likewise, we have proven that the monogenic fields contain a wealth of topological information and this information is supported on the boundary by the Cauchy integral formula.

Data Assimilation

Over the past two years I have also worked with a team on developing new techniques for data assimilation. We have submitted an article titled “Model and Data Reduction for Data Assimilation: Particle Filters Employing Projected Forecasts and Data with Application to a Shallow Water Model” We are continuing to work to apply our scheme to new models such as the Modular Arbitrary-Order Ocean-Atmospheric Model (MAOOAM).