Approaching Geometric Inverse Problems using Clifford Analysis

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ABSTRACT

APPROACHING GEOMETRIC INVERSE PROBLEMS USING CLIFFORD ANALYSIS

The geometrical inverse problem of determining an unknown Riemannian manifold M from the Dirichlet-to-Neumann (DN) map on the boundary is known as the Calderón problem and, in dimension two, this problem is solvable up to conformal deformation. One technique is to us the DN map to the algebra of holomorphic functions on the manifold which, by Gelfand, this set is then homeomorphic to the manifold. In higher dimensions, one can replace the complex line bundle with differential forms, but another natural choice is to consider a geometric (or Clifford) algebra bundle whose sections are multivector fields. The graded algebra of multivector fields comes with a natural differential operator known as the gradient (or Dirac) operator that replaces the Hodge-Dirac operator in the exterior algebra of forms. Fields in the kernel of the gradient are monogenic, and these monogenic fields are a natural generalization of holomorphic functions. In this work, we prove a version of the Hodge-Morrey decomposition which splits the space of multivector fields into an orthogonal sum of monogenic fields and gradients. We define a suitable notion of a spectrum of the space of monogenic fields and prove the spectrum to be homeomorphic to the underlying manifold with the weak-$*$ topology.

ACKNOWLEDGEMENTS

Fill in acknowledgements here.

DEDICATION

I would like to dedicate this thesis to my dog fluffy.

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Chapter 1

Introduction

1.1 Introduction

In 1980, Alberto Calderón proposed an inverse problem in his paper *On an inverse boundary value problem* [11] where he asks if one can determine the electrical conductivity matrix of some Ohmic medium from knowledge of voltage and current measurements on the boundary of the given domain. This problem goes under the name of Electrical Impedance Tomography (EIT). Physically, the EIT problem is a static 3-dimensional boundary value inverse problem wherein the practicioner has access to voltage to a discrete subset of the boundary of a body and make noisy measurements of the outgoing current flux along the same discrete subset. In other words, a partial and noisy version of the voltage-to-current map is known.

One notes that the voltage-to-current map inputs a scalar potential (the Dirichlet data) on the boundary which, if the conductivity matrix was known, would allow one to determine the potential on the interior by solving a second order elliptic partial differential equation with coefficients defined by the conductivity. In an Ohmic material, the current field is induced by from the electric field (the gradient of the potential) and the conductivity, and hence for a fixed conductivity, utilizing different Dirichlet data can induce different current fields in the body. The practicioner has the ability to measure the outgoing current flux along the boundary (the Neumann data) and thus, they have access to the voltage-to-current map.

This problem can be generalized naturally into dimensions ≥ 2 as a geometric inverse problem. One replaces the medium with a manifold and the conductivity becomes the Riemannian metric. Thus, the second order elliptic equation from before amounts to finding scalar fields in the kernel of the Laplace-Beltrami operator. This leads to a small caveat that in dimension 2 since the Laplace-Beltrami operator is conformally invariant. At any rate, the EIT problem is equivalent to determining an unknown Riemannian manifold up to isometry from the classical Dirichlet-to-

Neumann (DN) map which inputs a scalar field and outputs the outward normal component of the derivative of the solution [14, 23, 27].

There are a handful of approaches to solving this problem, but it remains unsolved. In order to make progress, theorists have allowed themselves access to larger sets of data, for example, complete knowledge of a generalized DN map on differential forms [20, 26, 2, 19]. In dimension 2, the smooth problem has been solved up to conformal invariance and in dimension ≥ 3, the problem has been solved for analytic manifolds [21]. Another approach that is unique to a manifold of dimension 2 appears in [3]. In this paper, Belishev determines the algebra of holomorphic functions from the DN map and realizes the spectrum of this algebra homeomorphic to the underlying manifold by Gelfand. The metric g is then recovered up to conformal class by extracting the complex structure from this algebra as well. An attempt to generalize this approach to dimension $n = 3$ can be found in by replacing the complex structure with a quaternionic structure but this has not lead to a complete solution [5, 6]. It has been shown that the 3-dimensional round ball can be determined up to homeomorphism from a quaternionic spectrum. Belishev and Vakulenko ask whether this can be extended to higher dimensions and to other spaces. An answer to this question is provided by theorem 3.3.1.

In this work, I first introduce the geometric algebras $\mathcal{G}$ as special cases of more general Clifford algebras in section 2.1. Following this, I take a smooth, oriented, Riemannian manifold M and construct a Clifford algebra bundle whose sections lie in the space $\mathcal{G}(M)$ and are referred to as multivector fields in section 2.2. The graded algebraic structure of $\mathcal{G}(M)$ expands upon the exterior algebra of forms $\Omega(M)$, and moreover, there exists a natural differential structure via the gradient operator $\boldsymbol{\nabla}$, which finds similarities to the Hodge-Dirac operator $d + \delta$. The space $\mathcal{G}(M)$ proves to be more rich than $\Omega(M)$ since one can realize $\Omega(M)$ as a trivial case of some $\mathcal{G}(M)$. Moreover, it is quite natural to look at multivector fields that consist of many differently graded elements at once. For example, multivector fields that lie in the kernel of $\boldsymbol{\nabla}$ are called monogenic and these fields share many of the same properties as holomorphic functions on $\mathbb{C}$ including, but not limited to, a Cauchy integral formula eq. (3.11). However, unlike holomorphic functions, the space of monogenic fields $\mathcal{M}(M)$ is not, in general, commutative or an algebra.

A useful version of a Green's formula is shown in theorem 2.2.2, and this allows us to prove a multivector version of the Hodge-Morrey decomposition that we realize in the following theorem.

Theorem 3.1.1 (Monogenic Hodge Decomposition)**.** *The space of multivector fields $\mathcal{G}(M)$ has the L^2-orthogonal decomposition*

$$\mathcal{G}(M) = \mathcal{M}(M) \oplus \boldsymbol{I}^{-1}\boldsymbol{\nabla}\mathcal{G}(M). \tag{1.1}$$

The space $\mathcal{M}(M)$ is a right module over the constant multivectors $\mathcal{G}$. For the special case of the Euclidean geometric algebra $\mathcal{G}_n$, I define a space of module homomorphisms from $\mathcal{M}(M)$ to $\mathcal{G}_n$ and refer to these morphisms $\mathcal{G}_n$-functionals. Inside $\mathcal{M}(M)$ lie commutative subalgebras $\mathcal{A}_{\boldsymbol{B}}(M)$ that are analogs of $\mathbb{C}$ and on these algebras we can define $\mathcal{G}_n$-characters as the $\mathcal{G}_n$-functionals that are also algebra morphisms on each $\mathcal{A}_{\boldsymbol{B}}(M)$ into $\mathcal{G}_n$. The space of $\mathcal{G}_n$-characters, $\mathfrak{M}(M)$, with the weak-$*$ topology, is shown to be homeomorphic to M in the special case where M is a region of $\mathbb{R}^n$ and inherits the Euclidean metric. This is summarized in the following theorem.

Theorem 3.3.1. *For any $\delta \in \mathfrak{M}(M)$, there is a point $x^\delta \in M$ such that $\delta(f) = f(x^\delta)$ for any $f \in \mathcal{M}(M)$ a monogenic field. Given the weak-$*$ topology on $\mathcal{M}^\times(M)$, the map*

$$\gamma \colon \mathfrak{M}(M) \to M, \quad \delta \mapsto x^\delta$$

is a homeomorphism.

Owing to the original intention of this work, I consider physical and geometric inverse boundary value problems related to the Calderón problem. For example, I discuss the electric and magnetic impedance tomography problems and their statements in terms of multivector fields section 4.1. Relationships between the two problems are established, and one finds that the Ohmic property of a medium couples together the scalar potential u and the magnetic bivector field b into a single monogenic field. Given knowledge of the electrostatic and magnetostatic version of the DN map alongside this new relationship, can one determine the underlying conductivity? Likewise, in higher dimensions, do theorems 3.1.1 and 3.3.1 provide new tools for solving the Calderón or other related inverse problems? Finally, there also exists a Hilbert transform in two guises via [2, 9]. Are these two notions equivalent? Does either add any more useful information for solving boundary inverse problems?

Chapter 2

Preliminaries

2.1 Clifford and geometric algebras

The complex algebra $\mathbb{C}$ can be generalized in a handful of ways. Some of which can be found through the use of Clifford algebras and, more specifically, in geometric algebras. We define the more general Clifford algebras first and realize geometric algebras as particularly nice Clifford algebras with a quadratic form arising from an inner product. Elements of a geometric algebra are known as multivectors and these multivectors carry a wealth of geometric information in their algebraic structure. $\mathbb{C}$ itself can be realized as a special subalgebra of parabivectors in the geometric algebra on $\mathbb{R}^2$ with the Euclidean inner product and the quaternions $\mathbb{H}$ are realized as an analogous algebra on $\mathbb{R}^3$. In particular, both $\mathbb{C}$ and $\mathbb{H}$ arise as the 2- and 3-dimensional even Clifford groups Γ^+ respectively.

First, we present a review of Clifford algebras and the relevant notions needed for this work. Those who feel they are familiar with both Clifford and geometric algebras may wish to skim through this subsection and visit **??** to review the notation used throughout this manuscript.

Formally, we let (V, Q) be an n-dimensional vector space V over some field K with an arbitrary quadratic form Q. The tensor algebra is given by

$$\mathcal{T}(V) \coloneqq \bigoplus_{j=0}^{\infty} V^{\otimes j} = K \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots , \tag{2.1}$$

where the elements (tensors) inherit a multiplication $\otimes$ (the tensor product). From the tensor algebra $\mathcal{T}(V)$, we can quotient by the ideal generated by $\boldsymbol{v} \otimes \boldsymbol{v} - Q(\boldsymbol{v})$ to create a new algebra.

Definition 2.1.1. The *Clifford algebra* $C\ell(V, Q)$ is the quotient algebra

$$C\ell(V, Q) = \mathcal{T}(V) \;/\; \langle \boldsymbol{v} \otimes \boldsymbol{v} - Q(\boldsymbol{v}) \rangle. \tag{2.2}$$

To see how the tensor product descends to the quotient, we let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ be an arbitrary basis for V, then we can consider the tensor product of basis elements $\boldsymbol{v}_i \otimes \boldsymbol{v}_j$ which induces a product in the quotient $C\ell(V, Q)$ which we refer to as the *Clifford multiplication*. In this basis, we write this product as concatenation $\boldsymbol{v_i v_j}$ and define the multiplication by

$$\boldsymbol{v}_i \boldsymbol{v}_j = \begin{cases} Q(\boldsymbol{v}_i) & \text{if } i = j, \\ \boldsymbol{v}_i \wedge \boldsymbol{v}_j & \text{if } i \neq j, \end{cases} \tag{2.3}$$

where $\wedge$ is the typical exterior product satisfying $\boldsymbol{v} \wedge \boldsymbol{w} = -\boldsymbol{w} \wedge \boldsymbol{v}$ for all $\boldsymbol{v}, \boldsymbol{w} \in V$. As a consequence, the exterior algebra $\bigwedge(V)$ can be realized as a subalgebra of any Clifford algebra over V or as a Clifford algebra with a trivial quadratic form $Q = 0$.

In the case where V has a (pseudo) inner product g, we can induce a quadratic form Q by $Q(\boldsymbol{v}) = g(\boldsymbol{v}, \boldsymbol{v})$ and give rise to a special type of Clifford algebra which motivates the following definition.

Definition 2.1.2. Let V be a vector space with an (pseudo) inner product $g(\cdot, \cdot)$. Then taking $Q(\cdot) = g(\cdot, \cdot)$, the Clifford algebra $C\ell(V, Q)$ is called a *geometric algebra*.

In general, we put $\mathcal{G}$ and assume the inner product and vector space will be arbitrary, given alongside, or will be clear from context. For example, when $V = \mathbb{R}^n$ we have the standard orthonormal basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ which allows us to neatly define the quadratic form Q from the Euclidean inner product which has coefficients δ_{ij} with respect to this basis. Since we frequently utilize this geometric algebra, we put $\mathcal{G}_n \coloneqq C\ell(\mathbb{R}^n, |\cdot|)$ to simplify notation. In broader generality, we do not need to have a definite inner product. For example, we can take an inner product where p vectors square to negative values and q vectors square to positive values which is of interest for those studying curved spacetime. Vectors whose square is negative are *temporal* and those whose square is positive are *spatial*. We put $\mathcal{G}_{p,q}$ for a geometric algebra with p temporal vectors and q spatial vectors where, in particular, p vectors square to -1 and q vectors square to 1.. The factor p will return in various different calculations.

Geometric algebras are an old and widely studied topic. For more information, see the classical text [16] or the more modern text [13] which also provides a wide range of applications to physics problems. Both these sources include much of the other necessary preliminaries I cover in the remainder of this section. Finally, the paper [12] proves many of the useful identities and notation used throughout this paper.

2.1.1 Multivectors and grading

Note that $C\ell(V,Q)$ is a $\mathbb{Z}$-graded algebra with elements of grade-0 up to elements of grade-n. We refer to grade-0 elements as scalars, grade-1 elements as vectors, grade-2 elements as *bivectors*, grade-r elements as *r-vectors*, and grade-n elements as *pseudoscalars*. For example, the pseudoscalar $\boldsymbol{\mu} = \boldsymbol{v}_1 \wedge \boldsymbol{v}_2 \wedge \cdots \wedge \boldsymbol{v}_n$ is an n-vector we will frequently return to. We denote the space of r-vectors by $C\ell(V,Q)^r$. For each grade there is a basis of $\binom{n}{r}$ *r-blades* which are r-vectors of the form

$$\boldsymbol{A_r} = \bigwedge_{j=1}^{r} \boldsymbol{v}_j, \text{ for linearly independent } \boldsymbol{v}_j \in V, \tag{2.4}$$

and we use a boldface of both the character and its subscript to specify that a r-vector is a r-blade and we note that vectors (since they are 1-blades) will not use this subscript. Instead, a vector $\boldsymbol{v}$ may use a non-boldfaced subscript to reference an index. Briefly, take for example the case where $\dim(V) = 3$, then there are $\binom{3}{2} = 3$ 2-blades that form a basis for the bivectors and one particular choice of a bivector basis would be the following list of 2-blades

$$\boldsymbol{B}_{12} = \boldsymbol{v}_1 \wedge \boldsymbol{v}_2, \quad \boldsymbol{B}_{13} = \boldsymbol{v}_1 \wedge \boldsymbol{v}_3, \quad \boldsymbol{B}_{23} = \boldsymbol{v}_2 \wedge \boldsymbol{v}_3. \tag{2.5}$$

We will repeatedly use the notation $\boldsymbol{B}_{ij} \coloneqq \boldsymbol{v}_i \wedge \boldsymbol{v}_j$ and the underlying basis will be clear from context. We refer to an $n-1$-vector as a *pseudovector* and it should be noted that every $n-1$-vector is a blade (see section 2.1.3). In other literature, some will refer to a r-blade as a *simple* or a *decomposable* r-vector.

In general, an element $A \in C\ell(V,Q)$ is written as a linear combination of basis elements of all possible grades and we refer to A as a *multivector*. To extract the grade-r components of A, we use the *grade projection* for which we have the notation

$$\langle A \rangle_r \in C\ell(V,Q)^r \tag{2.6}$$

to denote the grade-r components of the multivector A (i.e., $\langle A \rangle_r \in C\ell(V,Q)^r$). For the scalar component we put $\langle A \rangle$ and we can note we have the cyclic property

$$\langle AB \cdots CD \rangle = \langle DAB \cdots C \rangle \tag{2.7}$$

Any multivector A can then be given by

$$A = \sum_{r=0}^{n} \langle A \rangle_r \tag{2.8}$$

which shows the decomposition via the $\mathbb{Z}$-grading

$$C\ell(V,Q) = \bigoplus_{j=0}^{n} C\ell(V,Q)^j. \tag{2.9}$$

If A contains only components of a single grade, then we say that A is *homogeneous* and if the components are grade-r we put A_r and refer to A_r as a *homogeneous r-vector* or simply an *r-vector*. For example, when we refer to vectors we realize them as 1-vectors and likewise we realize bivectors as 2-vectors. Also of interest will be the elements in

$$C\ell(V,Q)^{0+2} = C\ell(V,Q) \oplus C\ell(V,Q)^2 \tag{2.10}$$

which we refer to as *surface spinors*.

The Clifford multiplication of vectors defined in 2.3 can be extended to multiplication of vectors with homogeneous r-vectors. In particular, given a vector $\boldsymbol{v} \in C\ell(V,Q)$ and a homogeneous

r-vector $A_r \in C\ell(V, Q)$, we have

$$\boldsymbol{v}A_r = \langle \boldsymbol{v}A_r \rangle_{r-1} + \langle \boldsymbol{v}A_r \rangle_{r+1}, \tag{2.11}$$

which decomposes the multiplication into a grade lowering *interior product* and a grade raising *exterior product*. This allows us to extend the Clifford multiplication further. Given an s-vector B_s, we have

$$A_r B_s = \langle A_r B_s \rangle_{|r-s|} + \langle A_r B_s \rangle_{|r-s|+2} + \cdots + \langle A_r B_s \rangle_{r+s}. \tag{2.12}$$

This rule for multiplication then allows for the multiplication of two general multivectors in $C\ell(V, Q)$. For this multiplication, specific grades of the product are worth noting.

$$A_r \cdot B_s \coloneqq \langle A_r B_s \rangle_{|r-s|} \tag{2.13}$$

$$A_r \wedge B_s \coloneqq \langle A_r B_s \rangle_{r+s} \tag{2.14}$$

$$A_r \rfloor B_s \coloneqq \langle A_r B_s \rangle_{s-r} \tag{2.15}$$

$$A_r \lfloor B_s \coloneqq \langle A_r B_s \rangle_{r-s}. \tag{2.16}$$

Finally, we have a special product for bivectors called the *commutator product* given by

$$A_2 \times B_2 \coloneqq \langle A_2 B_2 \rangle_2 \equiv \frac{1}{2}(A_2 B_2 - B_2 A_2). \tag{2.17}$$

These products are particularly emphasized as many helpful identities used in this paper are phrased using these notions. For example,

$$A_r \rfloor B_s = (-1)^{r(s-1)} B_s \lfloor A_r \tag{2.18}$$

$$A_r \wedge B_s = (-1)^{rs} B_s \wedge A_r. \tag{2.19}$$

Proofs for the identities used throughout can be found in [12]. Taking eqs. (2.11), (2.14) and (2.15) into mind, we see that the grade lowering interior product can be written as

$$\langle \boldsymbol{v} A_r \rangle_{r-1} \equiv \boldsymbol{v} \rfloor A_r \equiv \boldsymbol{v} \cdot A_r \tag{2.20}$$

and the grade raising exterior product can be written as

$$\langle \boldsymbol{v} A_r \rangle_{r+1} \equiv \boldsymbol{v} \wedge A_r. \tag{2.21}$$

Finally, to suppress needless additional parentheses later on, we assert that the above products take precedence over the geometrical product in order of operation. For example, for multivectors A, B, and C, we must take

$$A \cdot BC \equiv (A \cdot B)C, \tag{2.22}$$

and extend this to the other products defined in eqs. (2.14) to (2.17) as well.

We can also define an inner product on multivector fields that captures that mimics Euclidean inner product on structure of $\mathbb{R}^{2^n}$, i.e., treating each of the basis blades as independent vectors in $\mathbb{R}^{2^n}$.

Definition 2.1.3. Let $A, B \in \mathcal{G}$, then the *multivector inner product* is given by

$$(A, B) \coloneqq \left\langle A^\dagger B \right\rangle. \tag{2.23}$$

This product is bilinear, symmetric, positive definite, and satisfies

$$(A, B) = (A^\dagger, B^\dagger) \tag{2.24}$$

so long as g is positive definite. The product $(\cdot, \cdot)$ is a natural extension of the inner product g on the n-dimensional space V to the 2^n-dimensional $\mathcal{G}$. To see this, take an orthonormal basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}n$ and construct the basis of blades by defining an index set $J = \{j_1, j_2, \ldots, j_r\}$ for

$0 < j_1 < j_2 < \cdots < j_r \leq n$. The set of all such J for all $0 \leq r \leq n$ allows us to define the basis blades $\boldsymbol{E}_J = \boldsymbol{e}_{j_1} \cdots \boldsymbol{e}_{j_r}$ for which we can define any multivector A by $A = \sum_J A_J \boldsymbol{E}_J$ where A_J are scalar coefficients. The inner product then returns

$$(A, B) = \sum_J A_J B_J. \tag{2.25}$$

The symmetry, definiteness, and bilinearity become apparent. However, if g is not positive definite, then there are vectors called *null vectors* such that $(\boldsymbol{v}, \boldsymbol{v}) = 0$. This can be realized in the spacetime algebra (see section 2.1.5).

With this inner product, we have a notion of an adjoint.

Proposition 2.1.1. *Take* $A, B, C \in \mathcal{G}$ *then*

$$(CA, B) = (A, C^\dagger B) \tag{2.26}$$

$$(AC, B) = (A, BC^\dagger) \tag{2.27}$$

Proof. First,

$$(CA, B) = \left\langle (CA)^\dagger B \right\rangle = \left\langle A^\dagger C^\dagger B \right\rangle = (A, C^\dagger B), \tag{2.28}$$

and

$$(AC, B) = \left\langle (AC)^\dagger B \right\rangle = \left\langle C^\dagger A^\dagger B \right\rangle = (A, BC^\dagger), \tag{2.29}$$

both by eq. (2.7) □

Also, we have the induced norm.

Definition 2.1.4. The *multivector norm* $|\cdot|$ for $A \in \mathcal{G}$ is given by

$$|A| \coloneqq \sqrt{(A, A)}. \tag{2.30}$$

As discussed, $C\ell(V,Q)$ is naturally a $\mathbb{Z}$-graded algebra but we also find that it carries a $\mathbb{Z}/2\mathbb{Z}$-grading as well. Some would then refer to $C\ell(V,Q)$ as an *superalgebra* . This additional grading can be realized by sorting r-vectors in $C\ell(V,Q)$ into the sets where r is even or odd. We say a r-vector is *even* (resp. *odd*) if r is even (resp. odd) and in general if a multivector A is a sum of only even (resp. odd) grade elements we also refer to A as even (resp. odd). Taking note of the multiplication defined in 2.12, one can see that the multiplication of even multivectors with another even multivectors outputs an even multivector and that motivates the following.

Definition 2.1.5. The *even subalgebra* $C\ell(V,Q)^+ \subset C\ell(V,Q)$ is the subalgebra of even grade multivectors

$$C\ell(V,Q)^+ := C\ell(V,Q)^0 \oplus C\ell(V,Q)^2 \oplus C\ell(V,Q)^4 \oplus \cdots . \tag{2.31}$$

The split between even and odd subspaces of $C\ell(V,Q)$ makes the space $C\ell(V,Q)$ into a *superalgebra*. Though, one should note that the space of odd grade multivectors, $C\ell(V,Q)^-$, is not an algebra in its own right, it is a $C\ell(V,Q)^+$-module. We can then take the even part of a multivector A by $\langle A\rangle_+$ and the odd part by $\langle A\rangle_-$ and note

$$A = \langle A\rangle_+ + \langle A\rangle_-. \tag{2.32}$$

In the same vein, we will denote an even multivector by A_+ and an odd multivector by A_-. The even subalgebra is an extremely important entity that arises throughout physics due to its encapsulation of spinors which we touch on next.

2.1.2 Multivector operations and the Clifford and spin groups

For the remainder of this paper, let us focus solely on geometric algebras $\mathcal{G}$. Given access to an (pseudo) inner product we have a natural isomorphism between V and V^* by the Riesz representation. Namely, given an arbitrary basis $\boldsymbol{v}_i$ for V there exists the corresponding dual basis f_i for V^* such that $f_i(\boldsymbol{v}_j) = \delta_{ij}$. In geometric algebra, this notion is somewhat superfluous as we can realize the dual basis inside V itself in the following manner. Note that there is a unique map

$\sharp \colon V^* \to V$ for which $f \mapsto \boldsymbol{f}^\sharp$ such that

$$\boldsymbol{f}_{\boldsymbol{i}}^\sharp \cdot \boldsymbol{v}_j = \delta_{ij}. \tag{2.33}$$

Hence, if we simply put $\boldsymbol{v}^i := \boldsymbol{f}^\sharp{}_i$ we can note that $\boldsymbol{v}^i$ is simply a vector in the geometric algebra.

Definition 2.1.6. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ be an arbitrary basis of V generating $\mathcal{G}$. Then we have the *reciprocal basis* $\boldsymbol{v}^1, \boldsymbol{v}^2, \ldots, \boldsymbol{v}^n$ satisfying

$$\boldsymbol{v}^i \cdot \boldsymbol{v}_j = \delta^i_j, \tag{2.34}$$

and we refer to each $\boldsymbol{v}^i$ as a *reciprocal vector*.

In terms of the inner product g, we have that the coefficients are given by $g_{ij} = \boldsymbol{v}_i \cdot \boldsymbol{v}_j$ and thus we have an explicit definition for the reciprocal vectors by putting $\boldsymbol{v}^i = g^{ij}\boldsymbol{v}_j$ where g^{ij} is the coefficients to the matrix inverse $(g_{ij})^{-1}$ and we assume the Einstein summation convention.

The inverse to this isomorphism is $\flat \colon V \to V^*$ which is given by $\boldsymbol{v} \mapsto v^\flat$ satisfying

$$v_i^\flat(\boldsymbol{v}_j) = \delta_{ij}. \tag{2.35}$$

Given these identifications, there is no need to distinguish between the vector space V and its dual V^* as it suffices to consider V itself with reciprocal vectors $\boldsymbol{v}^{\boldsymbol{i}}$ with the application of the scalar product. For reference, the maps $\sharp$ and $\flat$ are the *musical isomorphisms* .

For a geometric algebra with a positive definite inner product, all blades have an inverse and hence form a group. With a pseudo inner product, the invertible elements are not quite as broad. To this end, we can construct a group of all invertible elements referred to as the *Clifford group* $\Gamma(\mathcal{G})$ for an arbitrary geometric algebra $\mathcal{G}$ by

$$\Gamma(\mathcal{G}) := \left\{ \prod_{j=1}^{k} \boldsymbol{v}_j \mid k \in \mathbb{Z}^+,\ \forall j : 1 \le j \le k :\ \boldsymbol{v}_i \in V \text{ such that } g(\boldsymbol{v}_i, \boldsymbol{v}_i) \neq 0 \right\}. \tag{2.36}$$

We refer to elements of the Clifford group as *Clifford multivectors*. Note that Clifford multivectors are not necessarily blades since the product used in the construction is not the exterior product $\wedge$. For any Clifford multivectors $A = \boldsymbol{v}_1 \cdots \boldsymbol{v}_k$ in the group Γ, we have that multiplicative inverse A^{-1} is given by $A^{-1} = \boldsymbol{v}^k \ldots \boldsymbol{v}^1$ as we can see that $A^{-1}A = AA^{-1} = 1$ by construction. Another note is that all scalars, vectors, pseudovectors, and pseudoscalars are always in the Clifford group and have multiplicative inverses. The inverse of a vector $\boldsymbol{v}$ is given by $\frac{\boldsymbol{v}}{\boldsymbol{v}\cdot\boldsymbol{v}}$. The form of the inverse motivates the utility of the *reverse* operator $\dagger$ defined so that $A^{\dagger} = \boldsymbol{v}_k \cdots \boldsymbol{v}_1$. For a r-blade A_r, the reverse also satisfies the relationship

$$A_r^{\dagger} = (-1)^{r(r-1)/2} A_r \tag{2.37}$$

as well as

$$(AB)^{\dagger} = B^{\dagger} A^{\dagger}. \tag{2.38}$$

One can see that the multiplicative inverse of an element of the Clifford group A is the reverse of the corresponding product of reciprocal vectors since $A_r^{-1} = (\boldsymbol{v}^1 \cdots \boldsymbol{v}^k)^{\dagger}$. When we take $V = \mathbb{R}^n$ with the Euclidean inner product, we can note that elements $s \in \Gamma^{+}(\mathcal{G}_n)$ act as rotations on multivectors $A \in \mathcal{G}_n$ through a conjugate action

$$A \mapsto sAs^{-1}. \tag{2.39}$$

In fact, all nonzero vectors $\boldsymbol{v} \in \Gamma(\mathcal{G}_n)$ define a reflection in the hyperplane perpendicular to $\boldsymbol{v}$ via the same conjugation action above. This allows one can realize that all rotations are even products of reflections.

Following these realizations, one can see that the Clifford group $\Gamma(\mathcal{G})$ contains important subgroups such as the orthogonal and special orthogonal groups as quotients

$$\mathrm{O}(V) \cong \Gamma(\mathcal{G})/(\mathbb{R} \setminus 0) \qquad \text{and} \qquad \mathrm{SO}(V) \cong \Gamma^{+}(\mathcal{G})/(\mathbb{R} \setminus 0), \tag{2.40}$$

where $\mathbb{R} \setminus 0$ is the multiplicative group of real numbers. We give the name *unit* to r-blades $\boldsymbol{A_r}$ with unit Clifford norm $1 = |\boldsymbol{A_r}|$. Finally, this allows us to arrive at a definition for the classical pin and spin groups.

Definition 2.1.7. The *pin* and *spin* groups $\mathrm{Pin}(V)$ and $\mathrm{Spin}(V)$ are defined to be

$$\mathrm{Pin}(V) \coloneqq \{s \in \Gamma(\mathcal{G}) \mid |s| = 1\}. \tag{2.41a}$$

$$\mathrm{Spin}(V) \coloneqq \{s \in \Gamma^+(\mathcal{G}) \mid |s| = 1\}. \tag{2.41b}$$

Our focus will be the case where we take $\mathcal{G} = \mathcal{G}_n$ for which we put $\mathrm{Spin}(n)$, but these statements can often be more broadly generalized. Moreover, we can realize this group as a quotient of the Clifford group $\Gamma(\mathcal{G}_n)$ by

$$\mathrm{Spin}(V) \cong \Gamma^+(\mathcal{G})/\mathbb{R}_+, \tag{2.42}$$

where $\mathbb{R}_+$ is the multiplicative group of positive real numbers. The spin group $\mathrm{Spin}(V)$ is a Lie group usually derived via a short exact sequence of groups

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}(V) \to \mathrm{SO}(V) \to 1. \tag{2.43}$$

Here, we have given a more concrete realization of the spin group as special elements inside a geometric algebra. The Lie algebra of the spin group is denoted by $\mathfrak{spin}(V)$ and $\mathfrak{spin}(n)$ when referencing $\mathrm{Spin}(n)$. This algebra typically characterized as the tangent space of $\mathrm{Spin}(V)$ at the identity. However, through this approach, we realize that $\mathfrak{spin}(V)$ is isomorphic to the algebra of bivectors with the antisymmetric product $\times$. Then, for any bivector B, we can generate an element in the spin group given via the exponential

$$e^B = \sum_{j=0}^{\infty} \frac{B^n}{n!}. \tag{2.44}$$

Fundamentally, the even subalgebra $\mathcal{G}^+$ is invariant under the action of $\mathrm{Spin}(V)$ since all elements in both sets are of even grade. This definition follows.

Definition 2.1.8. Let $\mathcal{G}$ be a geometric algebra with an inner product of arbitrary signature, then we define a *spinor* to be an element of $\mathcal{G}^+$.

Morally, this definition is telling us $\psi \in \mathcal{G}^+$ is an element that transforms under a left action of an element of $\mathrm{Spin}(V)$ to produce another spinor which leaves us with a convenient definition in that a spinor is simply an even multivector. Or, in other words, we realize that $\mathcal{G}^+$ is really a left $\mathrm{Spin}(V)$ module. Likewise, it motivates the name of surface spinor for the multivectors consisting of only grade-0 and grade-2 elements. For more on the topic, see [18].

2.1.3 Pseudoscalars and duality

Pseudoscalars are a deeply useful aspect of geometric algebra and we will now cover some of their utility. First and foremost, these pseudoscalars grant us a means of determining volumes. This will be a necessary notion in order to define integration in section 2.2.4.

Definition 2.1.9. Let $\mathcal{G}$ be a geometric algebra, then the *volume element* in the arbitrary basis $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ is

$$\boldsymbol{\mu} = \boldsymbol{v}_1 \wedge \boldsymbol{v}_2 \wedge \cdots \wedge \boldsymbol{v}_n. \tag{2.45}$$

It is worth noting that all volume elements and pseudoscalars are invertible in any geometric algebra.

We also want to note that the volume element here fits our intuition and indeed we find

$$|\boldsymbol{\mu}| = \sqrt{\det(g)}. \tag{2.46}$$

Since pseudoscalars are generated by a single element (recall there are $\binom{n}{n}$ independent grade-n elements), we should realize that the volume element is simply a scalar copy of a pseudoscalar that is unital.

Definition 2.1.10. Let $\boldsymbol{\mu}$ be the volume element, then we have the *unit pseudoscalar*

$$\boldsymbol{I} := \frac{1}{|\boldsymbol{\mu}|}\boldsymbol{\mu}. \tag{2.47}$$

As is clear by the definition above, we must have that

$$|\boldsymbol{I}| = 1. \tag{2.48}$$

The unit pseudoscalar satisfies a useful relationship when swapping the left for right multiplication with an r-vector by

$$\boldsymbol{I}A_r = (-1)^{r(n-1)}A_r\boldsymbol{I}. \tag{2.49}$$

Thus, $\boldsymbol{I}$ always commutes with the even subalgebra and the commutation property with the odd subalgebra depends on the dimension. Then, we can note

$$\boldsymbol{I}^2 = (-1)^{n(n-1)/2+p}, \tag{2.50}$$

which lets us see that the inverse is given by

$$\boldsymbol{I}^{-1} = (-1)^{n(n-1)/2+p}\boldsymbol{I}, \tag{2.51}$$

which is an identification that we will often use. Formulas throughout are usually given in their most general context and substitution is done only when working with specialized algebras. From here, one notices that when g is positive definite we have no temporal vectors and $p = 0$ which means $\boldsymbol{I}^\dagger = \boldsymbol{I}^{-1}$.

Note that for a homogeneous r-rector A_r we have that $A_r^\perp$ is an $n-r$-vector. Indeed, if we take an invertible r-blade $\boldsymbol{A_r}$, then we can find the $\boldsymbol{A_r}$*-subspace dual* of a multivector B by

$$B \rfloor \boldsymbol{A_r}^{-1}.$$

The notions of duality here give us geometrical insight. Taking an s-blade $\boldsymbol{B_s}$ we can note:

- If $s > r$, the $\boldsymbol{A_r}$-subspace dual of $\boldsymbol{B_s}$ vanishes.
- If $s = r$, the $\boldsymbol{A_r}$-subspace dual of $\boldsymbol{B_s}$ is a scalar and is zero if $\boldsymbol{B_s}$ contains a vector orthogonal to $\boldsymbol{A_r}$.
- If $s < r$, the $\boldsymbol{A_r}$-subspace dual of $\boldsymbol{B_s}$ represents the orthogonal complement of the subspace corresponding to $\boldsymbol{B_s}$ in the subspace corresponding to $\boldsymbol{A_r}$.

Since the pseudoscalar is a blade representing the entire vector space, this allows one to create dual elements within the entire vector space.

Definition 2.1.11. Given a multivector B, we define the *dual* of B to be

$$B^{\perp} \coloneqq B \rfloor \boldsymbol{I}^{-1} \equiv B\boldsymbol{I}^{-1}. \tag{2.52}$$

The dual allows one to exchange interior and exterior products in the following way.

$$(A \wedge B)^{\perp} = A \rfloor B^{\perp} \tag{2.53}$$

$$(A \rfloor B)^{\perp} = A \wedge B^{\perp} \tag{2.54}$$

This shows the natural duality between the inner and exterior products and their interpretations as subspace operations. The duality extends further to provide an isomorphism between the spaces of r-vectors and $n-r$-vectors since for any r-vector A_r we have $A_r^{\perp}$ is an $n-r$-vector. It is under this isomorphism one can realize that all pseudovectors are $n-1$-blades. Furthermore, for multivectors A and B,

$$(AB)^{\perp} = AB^{\perp} \tag{2.55}$$

For those familiar with the Hodge star operator, $\star$, this should feel familiar. This is discussed in section 2.2.3.

Remark 2.1.1. If we consider $\mathcal{G}_3$, we can realize the cross product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ by

$$\boldsymbol{u} \boldsymbol{\times} \boldsymbol{v} \coloneqq (\boldsymbol{u} \wedge \boldsymbol{v})^{\perp} \equiv \boldsymbol{u} \rfloor \boldsymbol{v}^{\perp} \equiv (\boldsymbol{u}^{\perp}) \times (\boldsymbol{v}^{\perp}), \tag{2.56}$$

where we use the bold notation for $\boldsymbol{\times}$ to distinguish between the bivector commutator product $\times$ defined in eq. (2.17). The special fact of $\mathcal{G}_3$ that is abused in a standard multivariate calculus course is that vectors and bivectors are dual to one another. In fact, the first equality is exactly this pedagogical reasoning; the cross product returns a vector perpendicular to the subspace spanned by the two input vectors and is zero when the two inputs are linearly dependent. One can also note that the vector $\boldsymbol{w} = \boldsymbol{u} \boldsymbol{\times} \boldsymbol{v}$ is sometimes refered to as axial and in other cases the pseudovector $\boldsymbol{u} \wedge \boldsymbol{v}$ is referred to as axial. The similar product notation of $\times$ and $\boldsymbol{\times}$ now becomes transparent.

2.1.4 Blades and subspaces

Each invertible unit r-blade $\boldsymbol{U_r}$ ($|\boldsymbol{U_r}| = 1$) corresponds to a r-dimensional subspace and can be identified with a point in the Grassmannian of r-dimensional subspaces in an n-dimensional vector space, $\mathrm{Gr}(r,n)$. We will often allude to this identification directly by referring to a subspace via a reference to a unit blade, e.g., the subspace $\boldsymbol{U_r}$. Extending the dual to act on the unit r-blades that make up $\mathrm{Gr}(r,n)$, one realizes that $\mathrm{Gr}(r,n)^{\perp} = \mathrm{Gr}(n-r,n)$ shows the spaces are in bijection. Moreover, given a subspace $\boldsymbol{U_r}$, we can complete the vector space by

$$\boldsymbol{U_r} \wedge \boldsymbol{U_r}^{\perp} = \boldsymbol{I}. \tag{2.57}$$

We can also note that any invertible blade $\boldsymbol{A_r}$ is simply a scaling of some unit blade so that $\boldsymbol{A_r} = \alpha \boldsymbol{U_r}$. This interpretation also proves to be a wonderfully geometrical perspective on the products defined in eqs. (2.13) to (2.16). For example, we see that there are a handful of reasons to adopt the additional multiplication symbols $\rfloor$ and $\lfloor$.

- The products $\rfloor$ and $\lfloor$ allow us to avoid needing to pay special attention to the specific grade of each multivector in a product. The product $\cdot$ on A_r and B_s depends on k and s and as such given by either $\rfloor$ or $\lfloor$ but one must know k and s in order to define this product exactly.

- We gain geometrical insight on the structure of r-blades in terms of their corresponding subspaces. Let $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ be nonzero blades with $r, s \geq 1$ then

  - $\boldsymbol{A_r} \rfloor \boldsymbol{B_s} = 0$ iff $\boldsymbol{A_r}$ contains a nonzero vector orthogonal to $\boldsymbol{B_s}$.
  - If $r < s$ then if $\boldsymbol{A_r} \rfloor \boldsymbol{B_s} \neq 0$ then the result is a $s - r$-blade representing the orthogonal complement of $\boldsymbol{A_r}$ in $\boldsymbol{B_s}$.
  - If $\boldsymbol{A_r}$ is a subspace of $\boldsymbol{B_s}$ then $\boldsymbol{A_r}\boldsymbol{B_s} = \boldsymbol{A_r} \rfloor \boldsymbol{B_s}$.
  - If $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ are orthogonal, then $\boldsymbol{A_r}\boldsymbol{B_s} = \boldsymbol{A_r} \wedge \boldsymbol{B_s}$.

We also have the equivalences

$$A_r \cdot B_s \equiv A_r \rfloor B_s \qquad \text{if } k \leq s \tag{2.58}$$

$$A_r \cdot B_s \equiv A_r \lfloor B_s \qquad \text{if } k \geq s. \tag{2.59}$$

For homogeneous r-vectors A_r and B_r, the products above simplify to

$$A_r \cdot B_r \equiv A_r \lfloor B_r \equiv A_r \rfloor B_r. \tag{2.60}$$

In fact, if we are given two r-blades $\boldsymbol{A_r} = \boldsymbol{a_1} \wedge \cdots \wedge \boldsymbol{a_r}$ and $\boldsymbol{B_r} = \boldsymbol{b_1} \wedge \cdots \wedge \boldsymbol{b_r}$ we have the

$$\boldsymbol{A_r} \cdot \boldsymbol{B_r}^{\dagger} = \det(\boldsymbol{a_i} \cdot \boldsymbol{b_j})_{i,j=1}^{r} = \boldsymbol{A_r}^{\dagger} \cdot \boldsymbol{B_r}, \tag{2.61}$$

which is the typical extension of the inner product g to an inner product on $\bigwedge^r(V)$ through linearity.

Given the direct relationship between unit r-blades and r-dimensional subspaces we can also form a compact way of projecting multivectors into subspaces in a manner closely related to the subspace dual.

Definition 2.1.12. Given an multivector B, the *projection* onto the subspace $\boldsymbol{A_r}$ is

$$\mathsf{P}_{\boldsymbol{A_r}}(B) \coloneqq B \rfloor \boldsymbol{A_r} \boldsymbol{A_r}^{-1} \equiv (B \rfloor \boldsymbol{A_r}) \rfloor \boldsymbol{A_r}^{-1} \tag{2.62}$$

Following this definition, one can see that

$$\mathsf{P}_{\boldsymbol{A_r}}(B) \in \bigoplus_{j=0}^{r} \mathcal{G}^j = \mathcal{G}^{0+\cdots+r}, \tag{2.63}$$

since the subspace $\boldsymbol{A_r}$ is r-dimensional and moreover the operation preserves grades since

$$\mathsf{P}_{\boldsymbol{A_r}}(\langle B \rangle_j) \in \mathcal{G}^j. \tag{2.64}$$

For example, given vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ we retrieve the familiar statement

$$\mathsf{P}_{\boldsymbol{u}}(\boldsymbol{v}) = (\boldsymbol{v} \cdot \boldsymbol{u}) \frac{\boldsymbol{u}}{|\boldsymbol{u}|^2}. \tag{2.65}$$

A dual notion exists as well; we can project onto the subspace perpendicular to $\boldsymbol{A_r}$.

Definition 2.1.13. Given a multivector B, the *rejection* from the subspace $\boldsymbol{A_r}$ is

$$\mathsf{R}_{\boldsymbol{A_r}}(B) \coloneqq B \wedge \boldsymbol{A_r} \boldsymbol{A_r}^{-1} \equiv (B \wedge \boldsymbol{A_r}) \lfloor \boldsymbol{A_r}^{-1}. \tag{2.66}$$

Note that this operation is also grade preserving. In the case we have a blade $\boldsymbol{C_k}$ with $k < r$ and $k < n - r$, we can note

$$\boldsymbol{C_k} = \mathsf{P}_{\boldsymbol{A_r}}(\boldsymbol{C_k}) + \mathsf{R}_{\boldsymbol{A_r}}(\boldsymbol{C_k}). \tag{2.67}$$

Another useful result follows.

Proposition 2.1.2. *Let $\mathcal{G}$ come with a positive definite g, let $\boldsymbol{A_{n-1}}$ unit pseudovector, and let $\boldsymbol{B_r}$ be an r-vector for $r \leq n-1$. Then,*

$$|\boldsymbol{B_r}| = \left|\mathsf{P}_{\boldsymbol{A_{n-1}}}(\boldsymbol{B_r})\right| + \left|\mathsf{P}_{\boldsymbol{A_{n-1}}}(\boldsymbol{B_r}^{\perp})^{\perp}\right|. \tag{2.68}$$

Proof. Take an orthonormal basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ for $\mathcal{G}$. Let J be an increasing index set of length r, i.e., $J = \{j_1, j_2, \dots, j_r\}$ with $1 \leq j_1 < j_2 < \cdots < j_r < n$ and define $\boldsymbol{E}_J = \boldsymbol{e}_{j_1}\boldsymbol{e}_{j_2}\cdots\boldsymbol{e}_{j_r}$ with $r \leq n-1$. Then note that any unit pseudovector $\boldsymbol{A_{n-1}}$ is a blade and so we can put $\boldsymbol{A_{n-1}} = \boldsymbol{\nu}^{\perp}$ for some unit vector $\boldsymbol{\nu}$. Note that if J contains k then $\mathsf{P}_{\boldsymbol{A_{n-1}}}(\boldsymbol{E}_J)$ is zero and otherwise this product is the identity since $\boldsymbol{E}_J$ lies in the subspace of $\boldsymbol{A_{n-1}}$. Next,

$$\begin{aligned}
\mathsf{P}_{\boldsymbol{A_{n-1}}}(\boldsymbol{E}_J^{\perp})^{\perp} &= \left(\boldsymbol{E}_J^{\perp} \rfloor \boldsymbol{A_{n-1}}\boldsymbol{A_{n-1}}^{-1}\right)^{\perp} && (2.69)\\
&= \boldsymbol{E}_J^{\perp} \rfloor \boldsymbol{A_{n-1}}(\boldsymbol{\nu}\boldsymbol{I}^{-1})^{-1}\boldsymbol{I}^{-1} && (2.70)\\
&= (-1)^{n-1}\boldsymbol{E}_J^{\perp} \rfloor (\boldsymbol{\nu}^{\perp})\boldsymbol{\nu} && (2.71)\\
&= (-1)^{n-1}(\boldsymbol{E}_J^{\perp} \wedge \boldsymbol{\nu})^{\perp}\boldsymbol{\nu} && (2.72)\\
&= (-1)^{(r+1)(n-1)}(\boldsymbol{\nu} \wedge \boldsymbol{E}_J^{\perp})^{\perp}\boldsymbol{\nu} && (2.73)\\
&= (-1)^{(n-1)(n+2r+2)/2}(\boldsymbol{\nu} \rfloor \boldsymbol{E}_J)\boldsymbol{\nu}. && (2.74)
\end{aligned}$$

Note that if J does not contain k, then $\boldsymbol{\nu} \rfloor \boldsymbol{E}_J = 0$ and otherwise the product is the identity up to sign.

Since this holds for $\boldsymbol{E}_J$, we note that we can write $B = \sum_J B_J \boldsymbol{E}_J$ so that B is a sum of these basis blades. By linearity of P, we have proven the proposition. □

The projection and rejection provide us a way to revisit the geometric notions of the interior and exterior products. In particular, we note that

$$B \rfloor \boldsymbol{A_r} = \mathsf{P}_{\boldsymbol{A_r}}(B)\boldsymbol{A_r} \tag{2.75}$$

$$B \wedge \boldsymbol{A_r} = \mathsf{R}_{\boldsymbol{A_r}}(B)\boldsymbol{A_r}. \tag{2.76}$$

Both the notion of projection and rejection prove to be useful and behave nicely with the dual by

$$\mathsf{P}_{\boldsymbol{A_r}^{\perp}}(B) = \mathsf{R}_{\boldsymbol{A_r}}(B). \tag{2.77}$$

Finally, the exterior product of orthogonal blades gives us a direct sum of subspaces in the following sense. Let $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ be orthogonal so that $\boldsymbol{A_r} \wedge \boldsymbol{B_s} = \boldsymbol{A_r}\boldsymbol{B_s}$, then we can note that if $k < r$ and $k < s$ we have

$$\mathsf{P}_{\boldsymbol{A_r}\wedge\boldsymbol{B_s}}(\boldsymbol{C_k}) = \mathsf{P}_{\boldsymbol{A_r}}(\boldsymbol{C_k}) + \mathsf{P}_{\boldsymbol{B_s}}(\boldsymbol{C_k}). \tag{2.78}$$

Perhaps it is most enlightening for the reader to revisit eqs. (2.67) and (2.78) replacing $\boldsymbol{C_k}$ with a vector $\boldsymbol{v}$ since a vector will always prove to be a representative for a “small enough" subspace.

2.1.5 Motivating example

Rather than a sequence of multiple examples, it will prove to be far more illuminating to construct one large example for which most of the preliminaries to this point can be used in a meaningful way. As such, we shall not rule out the utility of geometric algebras with pseudo inner products. The classical example is the *spacetime algebra* defined by taking $V = \mathbb{R}^4$ with a vector basis $\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3$ satisfying

$$\boldsymbol{\gamma}_0 \cdot \boldsymbol{\gamma}_0 = -1 \tag{2.79a}$$

$$\boldsymbol{\gamma}_0 \cdot \boldsymbol{\gamma}_i = 0 \qquad i = 1,2,3 \tag{2.79b}$$

$$\boldsymbol{\gamma}_i \cdot \boldsymbol{\gamma}_j = \delta_{ij}, \qquad i,j = 1,2,3. \tag{2.79c}$$

Where $\boldsymbol{\gamma}_0$ is temporal since its square is negative and $\boldsymbol{\gamma}_i$ for $i = 1,2,3$ are all spatial since their squares are positive. For this basis, we can denote the matrix for this inner product $\eta = \mathrm{diag}(-+++)$ (often called the *Minkowski metric*) and define Q from η. Then, we have for a spacetime vector $\boldsymbol{v} = v_0\boldsymbol{\gamma}_0 + v_1\boldsymbol{\gamma}_1 + v_2\boldsymbol{\gamma}_2 + v_3\boldsymbol{\gamma}_3$,

$$|\boldsymbol{v}| = \boldsymbol{v} \cdot \boldsymbol{v} = -v_0^2 + \sum_{i=1}^{3} v_i^2, \tag{2.80}$$

which defines the algebra $\mathcal{G}_{1,3}$ as the spacetime algebra. The reader may now wish to, for example, revisit section 2.1.2 with $\mathcal{G}_{p,q}$ in mind in order to see a realization of the groups $\mathrm{SO}(p,q)$, $\mathrm{Spin}(p,q)$, and the spacetime spinors.

As the naming above suggests, the geometric algebra of Euclidean space, $\mathcal{G}_3$, should naturally inside of the spacetime algebra. Note that we have the *spatial pseudoscalar* $\boldsymbol{I}_S \coloneqq \boldsymbol{\gamma}_1 \wedge \boldsymbol{\gamma}_2 \wedge \boldsymbol{\gamma}_3$, which, allowing for an extension of our notion of projection to the whole algebra, allows us to put

$$\mathsf{P}_{\boldsymbol{I}_S}(\mathcal{G}_{1,3}) \equiv \mathsf{R}_{\boldsymbol{\gamma}_0}(\mathcal{G}_{1,3}) = \mathcal{G}_3. \tag{2.81}$$

Perhaps one could refer to this mapping as the *static map* as we project only onto the spatial subspace and, via duality, reject the temporal subspace. It is also worth noting that this static map is not just producing an isomorphic copy of $\mathcal{G}_3$, but a a copy of $\mathcal{G}_3$ directly. Now, in $\mathcal{G}_3$, we can specify an arbitrary multivector A by

$$A = a_0 + a_1\boldsymbol{\gamma}_1 + a_2\boldsymbol{\gamma}_2 + a_3\boldsymbol{\gamma}_3 + a_{12}\boldsymbol{B}_{12} + a_{13}\boldsymbol{B}_{13} + a_{23}\boldsymbol{B}_{23} + a_{123}\boldsymbol{\gamma}_1 \wedge \boldsymbol{\gamma}_2 \wedge \boldsymbol{\gamma}_3, \tag{2.82}$$

and so the grade projections read

$$\langle A \rangle = a_0 \tag{2.83a}$$

$$\langle A \rangle_1 = a_1\boldsymbol{\gamma}_1 + a_2\boldsymbol{\gamma}_2 + a_3\boldsymbol{\gamma}_3 \tag{2.83b}$$

$$\langle A \rangle_2 = a_{12}\boldsymbol{B}_{12} + a_{13}\boldsymbol{B}_{13} + a_{23}\boldsymbol{B}_{23} \tag{2.83c}$$

$$\langle A \rangle_3 = a_{123}\boldsymbol{\gamma}_1 \wedge \boldsymbol{\gamma}_2 \wedge \boldsymbol{\gamma}_3. \tag{2.83d}$$

Then, we can write a even multivector as

$$q = q_0 + q_{23}\boldsymbol{B}_{23} + q_{31}\boldsymbol{B}_{31} + q_{12}\boldsymbol{B}_{12}. \tag{2.84}$$

Note as well that

$$\boldsymbol{B}_{23}^2 = \boldsymbol{B}_{31}^2 = \boldsymbol{B}_{12}^2 = -1 \tag{2.85a}$$

$$\boldsymbol{B}_{23}\boldsymbol{B}_{31}\boldsymbol{B}_{12} = +1, \tag{2.85b}$$

which is typical for spatial bivectors. In this case, one may notice that this even subalgebra is extremely close to being a copy of the quaternion algebra $\mathbb{H}$. One can arrive at a representation of the quaternions by taking

$$\boldsymbol{i} \leftrightarrow \boldsymbol{B}_{23}, \quad \boldsymbol{j} \leftrightarrow -\boldsymbol{B}_{31} = \boldsymbol{B}_{13}, \quad \boldsymbol{k} \leftrightarrow \boldsymbol{B}_{12}, \tag{2.86}$$

and noting that we then have $\boldsymbol{ijk} = -1$ as well as $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1$. A more in depth explanation is provided in [13]. Thus, we realize a quaternion as a spinor q and a purely imaginary quaternion is simply the grade-2 portion of the spinor $\langle q \rangle_2$. We also realize $\mathbb{H}$ as scalar copies of elements of $\mathrm{Spin}(3) \cong \mathrm{Sp}(1)$. That is to say that $\mathbb{H} \cong \mathbb{R} \times \mathrm{Spin}(3)$. Indeed, since elements of $\mathcal{G}_3^+$ are simply surface spinors, the surface spinors admit a natural spin representation.

But we are not done here, and we can project down one dimension further by

$$\mathsf{P}_{\boldsymbol{\gamma}_1 \wedge \boldsymbol{\gamma}_2}(\mathcal{G}_3) = \mathcal{G}_2. \tag{2.87}$$

To see this in action, we let $\boldsymbol{v} = v_1\boldsymbol{\gamma}_1 + v_2\boldsymbol{\gamma}_2 + v_3\boldsymbol{\gamma}_3$

$$\mathsf{P}_{\boldsymbol{\gamma}_1\wedge\boldsymbol{\gamma}_2} = \mathsf{P}_{\boldsymbol{B}_{12}}(\boldsymbol{v}) = (v_1\boldsymbol{\gamma}_1 + v_2\boldsymbol{\gamma}_2 + v_3\boldsymbol{\gamma}_3)\rfloor\boldsymbol{B}_{12}\boldsymbol{B}_{12}^{-1} \tag{2.88a}$$

$$= (v_1\boldsymbol{\gamma}_2 - v_2\boldsymbol{\gamma}_1)\boldsymbol{B}_{12}^{-1} \tag{2.88b}$$

$$= v_1\boldsymbol{\gamma}_1 + v_2\boldsymbol{\gamma}_2. \tag{2.88c}$$

Then, arbitrary multivectors A and B can be specified by

$$A = a_0 + a_1\boldsymbol{\gamma}_1 + a_2\boldsymbol{\gamma}_2 + a_{12}\boldsymbol{B}_{12}, \qquad B = b_0 + b_1\boldsymbol{\gamma}_1 + b_2\boldsymbol{\gamma}_2 + b_{12}\boldsymbol{B}_{12}. \tag{2.89}$$

We can then take the product AB to yield

$$\langle AB\rangle_0 = a_0b_0 + a_1b_1 + a_2b_2 - a_{12}b_{12} \tag{2.90a}$$

$$\langle AB\rangle_1 = (a_0b_1 + a_1b_0 - a_2b_{12} + a_{12}b_2)\boldsymbol{\gamma}_1 + (a_0b_2 + a_2b_0 + a_1b_{12} - a_{12}b_1)\boldsymbol{\gamma}_2 \tag{2.90b}$$

$$\langle AB\rangle_2 = (a_1b_2 - a_2b_1)\boldsymbol{B}_{12}. \tag{2.90c}$$

Most notably, we see that $\boldsymbol{B}_{12}^2 = -1$ and this allows us to consider a spinor

$$z = x + y\boldsymbol{B}_{12} \tag{2.91}$$

which is exactly a representation of the complex number $\zeta = x + iy$ in $\mathcal{G}_2^{0+2} = \mathcal{G}_2^+$. Thus, the even subalgebra of this geometric algebra is indeed isomorphic to the complex numbers $\mathbb{C}$. Indeed, there is one unit 2-blade $\boldsymbol{B}_{12}$ in $\mathcal{G}_2$ to form the spin algebra $\mathfrak{spin}(2) \cong \mathbb{R}$ and as a consequence all unit norm elements in $\mathcal{G}_2^+$ can be written as

$$e^{\theta\boldsymbol{B}_{12}} = \sum_{n=0}^{\infty}\frac{\theta\boldsymbol{B}_{12}}{n!} = \cos(\theta) + \boldsymbol{B}_{12}\sin(\theta), \tag{2.92}$$

where $\theta \boldsymbol{B}_{12}$ is a general bivector in $\mathcal{G}_2$ when $\theta \in \mathbb{R}$ is arbitrary. Hence, we arrive at $\mathrm{Spin}(2) \cong \mathrm{U}(1)$. Any element in $\mathbb{C}$ is also a scaled version of an element of the spin group $\mathrm{Spin}(2)$. Hence, we can use a spin representation for an element in $\mathbb{C}$ via $z = re^{\theta \boldsymbol{B}_{12}} \in \mathbb{R} \times \mathrm{Spin}(2)$. This special case shows that spinors in $\mathcal{G}_2$ have a unique spin representation.

But, the above work is not necessary special to the starting point of $\mathcal{G}_{1,3}$ or $\mathcal{G}_3$. In fact, if we take $\mathcal{G}_n$ for $n \geq 2$, then there are natural copies of $\mathbb{C}$ contained inside of $\mathcal{G}_n$. In particular, we have the isomorphism

$$\mathbb{C} \cong \{x + y\boldsymbol{B} \mid x, y \in \mathcal{G}_n^0,\ \boldsymbol{B} \in \mathrm{Gr}(2, n).\}, \tag{2.93}$$

which shows that complex numbers arise as surface spinors via the representation

$$\zeta = x + y\boldsymbol{B}, \tag{2.94}$$

since $\boldsymbol{B}^2 = -1$. Given the standard basis $\boldsymbol{e_1}, \dots, \boldsymbol{e_n}$ we have copies of $\mathbb{C}$ for each of the $\binom{n}{2}$ unit bivectors $\boldsymbol{B}_{jk}$ with $k = 2, \dots, n$ and $j < k$.

2.2 Geometric manifolds

We want to generalize the setting of geometric algebra to include a smooth structure. For instance, we can consider a manifold M (likely with boundary ∂M) with a metric structure and develop a geometric algebra at each tangent space to this manifold (e.g., following [24]). We refer to this as the *geometric tangent space* and put $C\ell(T_xM, g_x)$.

Definition 2.2.1. A manifold M with a pseudo-Riemannian metric g is a *geometric manifold* if each tangent space is a geometric tangent space.

On geometric manifolds we will be able to attach multivector fields and compute their derivatives as well as integrate. This leads us to the realm of geometric calculus and Cifford analysis. Geometric calculus is intimately related to both the vector calculus in $\mathbb{R}^3$ and differential forms. It has the added advantage of notational convenience and clarity as we have seen with geometric algebra and its subspace operations. In the beginning of section 2.1 we realize as well that the exterior algebra is contained inside any Clifford algebra and, to this end, geometric calculus will contain the calculus of differential forms.

Forms are a useful language for proving general theorems about boundary value problems [25], and so we will retrieve all of these theorems for our own utility. Given that we have increased geometrical intuition on different graded elements of a geometric algebra, we can realize that we can work with multivector equivalents of forms instead of concentrating on forms of a specific grade. For example, in section 4.1.1 we see that one can think of the electromagnetic field as a multivector consisting of elements of various degree as opposed to the usual field strength 2-form [28]. In fact, under certain other restrictions such as those present in Ohmic materials, we find there are surface spinors that fall into the kernel of a Dirac-type operator.

This Dirac-type operator, $\boldsymbol{\nabla}$, is the grade-1 derivative operator studied in Clifford analysis. Fundamentally, this operator generalizes the Wirtinger derivative for complex functions to multivectors and, as such, generalizes the notion of a $\mathbb{C}$-holomorphic function to that of a monogenic function (see). Happily, we even retain a Taylor series representation (see lemma 3.3.1) for functions in the kernel of $\boldsymbol{\nabla}$ due to a generalized form of the Cauchy integral formula. This Cauchy integral formula has been applied elsewhere (see [9]). The Cauchy integral also provides a direct correspondence between smooth functions defined on the boundary ∂M of a manifold M.

2.2.1 Multivector fields

In order to develop fields on a geometric manifold we must first create the relevant bundle structure. There is a natural bundle associated to a geometric manifold given by by gluing together each of the tangent geometric algebras. The *geometric algebra bundle* of a geometric manifold (M, g) is the space

$$\bigsqcup_{x \in M} C\ell(T_x M, g_x). \tag{2.95}$$

Given this bundle, the fields follow.

Definition 2.2.2. A *(smooth) multivector field* is a (C^∞-smooth) section of the geometric algebra bundle. We put $\mathcal{G}(M)$ as the *space of multivector fields on* M.

Note that the we will assume that all multivector fields are C^∞-smooth and drop this additional modifier when speaking of any type of multivector field. The above definition above is very general and we may not find ourselves working over arbitrary geometric manifolds. For example, we highlight a specific use case by letting M be a connected region of $\mathbb{R}^n$. For brevity, we will put $\mathcal{G}_n(M)$ to denote we are working over a region $M \subseteq \mathbb{R}^n$. In this case, the multivectors themselves are realized as constant multivector fields which allows us to say $\mathcal{G}_n \subset \mathcal{G}_n(M)$. This smooth setting simply makes the coefficients of the global basis blades given by C^∞ functions as opposed to $\mathbb{R}$ scalars. Hence, $\mathcal{G}_n(M)$ is simply the C^∞-module equivalent of $\mathcal{G}_n$.

Perhaps the C^∞-module structure obfuscates the point slightly, but the notion of a smooth section does not. One should think of the fields in $\mathcal{G}_n(M)$ as multivector valued functions on $M \subset \mathbb{R}^n$. Taking this identification allows for an extended toolbox at our disposal. In particular, points in M are uniquely identified with constant vector fields in $\mathcal{G}_n^1$ and one can consider endomorphisms living in $\mathcal{G}_n$ (acting on $\mathcal{G}_n^1$) as acting on the input of fields in $\mathcal{G}_n(M)$ as well (see remark 2.2.1). Thus, there is not only an algebraic structure on the fields themselves, but on the point in which the field is evaluated. This is perhaps the key insight on why authors developed the so-called vector manifolds widely used in the geometric algebra landscape. Fundamentally, this is true in all local coordinates for an arbitrary manifold M, but it is not a global phenomenon since, for example, not all manifolds admit everywhere smooth nonzero constant vector fields. Just take the 2-sphere, $M = S^2$, and note the hairy ball theorem.

Remark 2.2.1. If we consider a multivector field $f \in \mathcal{G}_n(\mathbb{R}^n)$. With $x \in \mathbb{R}^n$ being identified with the vector $\boldsymbol{x} \in \mathcal{G}_n^1$, we can safely put $f(\boldsymbol{x})$. One may be interested in the restriction of the input of f to a subspace $\boldsymbol{U_r}$ which yields $f(\mathsf{P}_{\boldsymbol{U_r}}(\boldsymbol{x}))$.

As noted throughout section 2.1, there are spaces of multivectors inside $\mathcal{G}$ of interest and each of these extends to their field counterpart. Construction of each is done pointwise and made global through the relevant bundle. Let us list the relevant spaces of fields.

- The *r-vector fields*,

$$\mathcal{G}^r(M) := \left\{ \text{smooth sections of } \bigsqcup_{x \in M} C\ell(T_xM, g_x)^r \right\}; \tag{2.96}$$

- The *spinor fields*,

$$\mathcal{G}^+(M) := \left\{ \text{smooth sections of } \bigsqcup_{x \in M} C\ell(T_xM, g_x)^+ \right\}; \tag{2.97}$$

- The *surface spinor fields*,

$$\mathcal{G}^{0+2}(M) := \left\{ \text{smooth sections of } \bigsqcup_{x \in M} C\ell(T_xM, g_x)^{0+2} \right\}; \tag{2.98}$$

Our operations from section 2.1 carry over. We simply define all the products seen in eqs. (2.13) to (2.16) to act pointwise in each geometric tangent space. Previously we referred to r-blades as special r-vectors. Thus, we realize an r-blade field $\boldsymbol{A_r} \in \mathcal{G}^r(M)$ assumes the same form of eq. (2.4) where the vectors $\boldsymbol{v}_j$ are to be understood as vector fields for which all $\boldsymbol{v}_j(x)$ are linearly independent in T_xM at the point x.

Given local coordinates x^i on M containing the point p, the tangent vectors in a neighborhood about p are induced from the coordinates by $\frac{\partial}{\partial x^i}$. However, this choice of basis may be canonical, but it is not arbitrary. Instead, at each point we can simply choose an arbitrary local vector basis $\boldsymbol{v}_i$ and let the components of the metric be given in this basis by $g_{ij(x)} = \boldsymbol{v}_i(x) \cdot \boldsymbol{v}_j(x)$. From here, we can suppress the pointwise notion and instead just put $g_{ij} = \boldsymbol{v}_i \cdot \boldsymbol{v}_j$ locally. This allows us to work notationally with bases in a global manner without any reference to coordinates, so long as we assume the understanding is clear – these vector bases do only exist locally. If explicit computations are to be carried out, one can just take the canonical basis so that $\boldsymbol{v}_i = \frac{\partial}{\partial x^i}$. Thus, locally we have the reciprocal basis $\boldsymbol{v}^i = g^{ij}\boldsymbol{v}_j$, the reverse $\dagger$, dual $\perp$, projection P, and rejection R that act on multivector fields pointwise in $C\ell(T_xM, g_x)$ and, if the need arises, all computations can be done in local coordinates.

2.2.2 Geometric calculus

On M we have the unique torsion free Levi-Civita connection ∇ for which we can define the covariant derivative $\nabla_{\boldsymbol{u}}$ for a vector field $\boldsymbol{u}$. The covariant derivative is extended to act on multivector fields following [24]. We can note that $\nabla_{\boldsymbol{u}}$ is a grade preserving differential operator so that

$$\nabla_{\boldsymbol{u}}\langle A_r\rangle_r = \langle \nabla_{\boldsymbol{u}}\langle A_r\rangle_r\rangle_r, \tag{2.99}$$

and it is a dot-compatible and wedge-compatible operator since

$$\nabla_{\boldsymbol{u}}(A\cdot B) = (\nabla_{\boldsymbol{u}}A)\cdot B + A\cdot(\nabla_{\boldsymbol{u}}B) \tag{2.100}$$

$$\nabla_{\boldsymbol{u}}(A\wedge B) = (\nabla_{\boldsymbol{u}}A)\wedge B + A\wedge(\nabla_{\boldsymbol{u}}B) \tag{2.101}$$

Definition 2.2.3. Let $\boldsymbol{v}_i$ be an arbitrary basis, then the *gradient* (or *Dirac operator*) $\boldsymbol{\nabla}$ is defined by

$$\boldsymbol{\nabla} = \sum_i \boldsymbol{v}^i \nabla_{\boldsymbol{v}_i}. \tag{2.102}$$

The space of multivector fields $\mathcal{G}(M)$ along with $\boldsymbol{\nabla}$ is usually referred to as geometric calculus. One should note that $\boldsymbol{\nabla}$ is acts as a grade-1 element. Thus, the gradient splits into two operators,

$$\boldsymbol{\nabla}\rfloor\colon \mathcal{G}_n^r(M) \to \mathcal{G}_n^{r-1}(M), \tag{2.103}$$

$$\boldsymbol{\nabla}\wedge\colon \mathcal{G}_n^r(M) \to \mathcal{G}_n^{r+1}(M), \tag{2.104}$$

which satisfy the properties

$$(\boldsymbol{\nabla}\wedge)^2 = 0, \tag{2.105}$$

$$(\boldsymbol{\nabla}\rfloor)^2 = 0, \tag{2.106}$$

when acting on a homogeneous r-vector. Since 2.105 holds, the gradient operator gives rise to the grade preserving *Laplace-Beltrami operator*

$$\Delta = \boldsymbol{\nabla}^2 = \boldsymbol{\nabla}\rfloor \circ \boldsymbol{\nabla} \wedge + \boldsymbol{\nabla} \wedge \circ \boldsymbol{\nabla}\rfloor,$$

which is manifestly coordinate invariant by definition. It also motivates the use of the physicist notation $\boldsymbol{\nabla}^2 = \Delta$, but we do not adopt this here. We refer to multivector fields f in the kernel of the Laplace-Beltrami operator *harmonic multivector fields* or simply as *harmonic*.

Note that since Euclidean space $\mathbb{R}^n$ has global orthonormal coordinates $\boldsymbol{e}_i$ we can choose a global constant vector field basis since we identified $\mathcal{G}_n^1$ with $\mathcal{G}(\mathbb{R}^n)^1$. With respect to these fields, we have the that $\nabla_{\boldsymbol{u}}$ reduces to the directional derivative. Note then that $\boldsymbol{u} \cdot \boldsymbol{\nabla} = \nabla_{\boldsymbol{u}}$ defines the directional derivative via the gradient. In fact, given a subspace $\boldsymbol{U_r}$, one could even describe a derivative in $\boldsymbol{U_r}$ by $\mathsf{P}_{\boldsymbol{U_r}}(\boldsymbol{\nabla})$.

There exists a Leibniz rule for $\boldsymbol{\nabla}$ as well given by

$$\boldsymbol{\nabla}(AB) = \boldsymbol{\nabla}AB + \dot{\boldsymbol{\nabla}}A\dot{B}, \tag{2.107}$$

where we use the overdot to signify which multivector field we are taking derivatives of. The Clifford product, however, does not change.

2.2.3 Differential forms

The language of differential forms [15] rests neatly inside geometric calculus. We will develop the relationship between multivectors and forms which will serve as a link between the two notions so that researchers with interest in Clifford analysis can communicate with those who study forms. In order to do so, we appeal to the language of differential forms and build a relationship between multivector fields and forms through measures. Forms have their appeal in global understanding via their properties through integration (e.g., Stokes' and Green's theorems) and the exterior calculus along with de Rham cohomology will provide us a larger toolbox.

Given coordinates $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ on M we have the local basis tangent vector fields $\boldsymbol{v}_i = \frac{\partial \boldsymbol{x}}{\partial x_i}$ with the corresponding 1-forms dx^i that are each local sections of T^*M and are the exterior derivatives (or gradients) of the coordinate functions. 1-forms are linear functionals on tangent vectors and in these coordinates we have $dx^i(\boldsymbol{v}_i) = \delta^i_j$ and one can thus take a pairing of 1-form fields and vector fields and integrate over 1-dimensional submanifolds. The benefit of this definition is that the 1-forms dx^i carry a natural measure and we can form product measures via the exterior product $\wedge$.

On M, we let $\Omega(M)$ be the exterior algebra of smooth form fields on M, and let $\Omega^r(M)$ be the space of smooth r-form fields on M. Then we have the Riemannian volume measure $\mu \in \Omega^n(M)$ given in local coordinates by

$$\mu = \sqrt{|g|} dx^1 \ldots dx^n. \tag{2.108}$$

Definition 2.2.4. The *r-dimensional directed measure* dX_r is given in local coordinates by

$$dX_r \coloneqq \boldsymbol{v}_{i_1} \wedge \cdots \wedge \boldsymbol{v}_{i_r} dx^{i_1} \cdots dx^{i_r}. \tag{2.109}$$

For example, along a 2-dimensional submanifold we have the 2-dimensional directed measure

$$dX_2 = \boldsymbol{v}_i \wedge \boldsymbol{v}_j dx^i dx^j \tag{2.110}$$

and we can note that

$$(\boldsymbol{v}^i \wedge \boldsymbol{v}^j) \cdot dX_2^\dagger = dx^i dx^j - dx^j dx^i \tag{2.111}$$

is completely antisymmetric and provides us a surface measure we can integrate; this is a differential 2-form. We then find that

$$\mu = \boldsymbol{I}^{-1} \cdot dX_n = \boldsymbol{I}^{-1} dX_n = \boldsymbol{I}^{-1\dagger} \cdot dX_n^\dagger = 1^\perp \cdot dX_n, \tag{2.112}$$

where $\boldsymbol{I}$ is the unit pseudoscalar field defined on M with respect to g. The last of the equalities above is quite important. It seeks to tell us that, morally, many of our familiar statements about integrals will involve the dual.

We can now write a r-form $\alpha_r = \alpha_{i_1 \cdots i_r} dx^{i_1} \wedge \cdots dx^{i_r}$ as

$$\alpha_r = A_r \cdot dX_k^\dagger, \tag{2.113}$$

where

$$A_r = \frac{1}{r!} \alpha_{i_1 \cdots i_r} \boldsymbol{v}^{i_1} \wedge \cdots \wedge \boldsymbol{v}^{i_r}. \tag{2.114}$$

We refer to A_r as the *multivector equivalent* of α_r and note that by eq. (2.112) that the multivector equivalent to μ is $\boldsymbol{I}^{-1\dagger}$. This provides an isomorphism between r-forms and r-vectors via a contraction with the r-dimensional volume directed measure. In this sense, a differential form is made up of two essential components namely the multivector field and the r-dimensional directed measure. Hence, we can see now how a differential form simply appends the measure attached to the underlying space. We can also see how this generalizes the musical isomorphism $\flat$ by taking a vector field $\boldsymbol{v}$ and noting

$$\boldsymbol{v} \cdot dX_1 = v_i \boldsymbol{v}_i \cdot \boldsymbol{v}^j dx^j = v_i dx^i. \tag{2.115}$$

The exterior algebra of differential forms comes with an addition $+$ and exterior multiplication $\wedge$. We note that the sum of two r-forms α_r and β_r is also a r-form which we can see reduces to addition on the multivector equivalents A_r and B_r by

$$\alpha_r + \beta_r = (A_r \cdot dX_r^\dagger) + (B_r \cdot dX_r^\dagger) = (A_r + B_r) \cdot dX_r^\dagger, \tag{2.116}$$

due to the linearity of $\cdot$. If instead had an s form β_s then we have the exterior product

$$\alpha_r \wedge \beta_s = (A_r \wedge B_s) \cdot dX_{r+s}^\dagger, \tag{2.117}$$

where $dX_{r+s} = 0$ if $r + s > n$.

With differential forms one also has the exterior derivative d giving rise to the exterior calculus. On the multivector equivalents we have

$$d\alpha_r = (\boldsymbol{\nabla} \wedge A_r) \cdot dX_{r+1}^{\dagger}, \tag{2.118}$$

which realizes the exterior derivative as the grade raising component of the gradient $\boldsymbol{\nabla}$. Of course, for scalar fields, this returns the gradient as desired. We will find $\boldsymbol{\nabla}\rfloor$ can be identified with the codifferential δ up to a sign.

2.2.4 Integration

Given a r-dimensional submanifold $R \subset M$ with a r-form α_r defined on R, we can integrate this r-form. However, we want to phrase this in terms of the the multivector equivalents. First, we will do this for scalar valued integrals.

Scalar valued integrals

Let μ_R be the volume measure for the submanifold R. Given R is a submanifold of M, for any $x \in R$ we have tangent space T_xR which is a subspace of T_xM. Hence, we can put $\boldsymbol{I}_R(x)^{-1\dagger}$ to be the multivector equivalent of μ_R by

$$\mu_R = \boldsymbol{I}_R^{-1\dagger} \cdot dX_r^{\dagger} = \boldsymbol{I}_R^{-1} \cdot dX_r. \tag{2.119}$$

We should think of $\boldsymbol{I}_R^{-1\dagger}$ as representing the subspace $T_xR \subset T_xM$ and note that we think of $\boldsymbol{I}_R^{-1\dagger}$ as a unit pseudoscalar field defined on R.

An s-vector field A_s on R is said to be *tangent to* R if

$$A_s = \mathsf{P}_{\boldsymbol{I}_R}(A_s) \tag{2.120}$$

so that for any $x \in R$ that $A_s = \mathrm{P}_{\boldsymbol{I}_{R(x)}}(A_s(x))$. Immediately we can conclude that we must have $s \leq r$ or this projection is zero (see section 2.1.4). We may, for example, wish to integrate scalar fields A_0 over R and in this case we can put $A_r = A_0\boldsymbol{I}_R^{-1}$ and contract with dX_r to create a tangent r-form on R by

$$\alpha_r = A_r \cdot dX_r^\dagger = A_0\mu_R \tag{2.121}$$

which can be integrated as

$$\int_K \alpha = \int_K A_0\mu_R. \tag{2.122}$$

This of course applies to scalar fields on M itself, for which we can take $A_n = A_0\boldsymbol{I}^{-1}$. Then this form can be integrated by

$$\int_M \alpha_n = \int_M A_0\mu. \tag{2.123}$$

There is also the normal space N_xR that is everywhere orthogonal to T_xR with respect to g on M. This yields the normal $n-r$-blade field $\boldsymbol{\nu}_R = \boldsymbol{I}_R^\perp$. Since R is a submanifold of M, we have the inclusion $\iota\colon R \to M$ and the induced pullback on forms $\iota^*\colon \Omega(M) \to \Omega(R)$.

Proposition 2.2.1. *Let α_s be an s-form defined on M and let $\iota\colon R \to M$ be the inclusion of the submanifold R into M. Then the pullback ι^* on the multivector equivalent A_s is given by*

$$\iota^*\alpha_s = \mathsf{P}_{\boldsymbol{I}_R}(A_s) \cdot dX_s. \tag{2.124}$$

Proof. Note that by definition we have

$$(\iota^*\alpha_s)_x(\boldsymbol{v}_1, \dots, \boldsymbol{v}_r) = (\alpha_s)_x(d\iota_x\boldsymbol{v}_1, \dots, d\iota_x\boldsymbol{v}_r),$$

for arbitrary vector fields $\boldsymbol{v}_1, \dots, \boldsymbol{v}_s$ and at all $x \in R$. Then, since ι is inclusion, we have

$$d\iota_x = \mathsf{P}_{\boldsymbol{I}_{R(x)}},$$

at each point $x \in R$ and hence

$$\iota^* \alpha_s = \alpha_s \circ \mathsf{P}_{\boldsymbol{I}_R}.$$

For all $\boldsymbol{v}_i$ we can put

$$\boldsymbol{v}_i = \mathsf{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_i) + \mathsf{R}_{\boldsymbol{I}_R}(\boldsymbol{v}_i),$$

and note for the multivector equivalent

$$(\mathsf{P}_{\boldsymbol{I}_R}(A_s) \cdot dX_s)(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_s) = (\mathsf{P}_{\boldsymbol{I}_R}(A_s) \cdot dX_s)(\mathsf{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_1) + \mathsf{R}_{\boldsymbol{I}_R}(\boldsymbol{v}_1), \ldots, \mathsf{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_s) + \mathsf{R}_{\boldsymbol{I}_R}(\boldsymbol{v}_s)) \tag{2.125}$$

$$= (\mathsf{P}_{\boldsymbol{I}_R}(A_s) \cdot dX_s)(\mathsf{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_1), \ldots, \mathsf{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_s)), \tag{2.126}$$

since $\mathsf{P}_{\boldsymbol{I}_R}(A_s)$ is supported only on R. Then, if $s \leq r$,

$$\begin{aligned}
\iota^* \alpha_s &= (A_s \cdot dX_s)(\mathsf{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_1), \ldots, \mathsf{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_s)) \\
&= ((\mathsf{P}_{\boldsymbol{I}_R}(A_s) + \mathsf{R}_{\boldsymbol{I}_R}(A_s)) \cdot dX_s)(\mathsf{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_1), \ldots, \mathsf{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_s)) \\
&= (\mathsf{P}_{\boldsymbol{I}_R}(A_s) \cdot dX_s)(\mathsf{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_1), \ldots, \mathsf{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_s)),
\end{aligned}$$

and by eq. (2.125) we have our intended result. If $s > r$, then

$$\iota^* \alpha_s = 0 = \mathsf{P}_{\boldsymbol{I}_R}(A_s)$$

which proves the proposition. □

The above seems to motivate the choice of [25] to put $\boldsymbol{t}_R = \iota^*$ to refer to the tangential part of a differential form. The normal part of a form is $\mathbf{n}_R \alpha_s = \alpha_s - \boldsymbol{t}_R \alpha_s$. The following corollary is immediate given eqs. (2.67) and (2.77).

Corollary 2.2.1. *Let α_s be an s-form with $s < r$ and $s < n - r$ and multivector equivalent A_s. Then*

$$\mathbf{n}_R \alpha_s = \alpha_s - \mathrm{P}_{\boldsymbol{I}_R}(A_s) \cdot dX_s^\dagger = \mathsf{R}_{I_R}(A_s) \cdot dX_s^\dagger. \tag{2.127}$$

This is pertinent when we take M to be a manifold with boundary ∂M. In this case we let $\boldsymbol{I}_\partial$ denote the tangent $n-1$-blade and build boundary measure via

$$\mu_\partial \coloneqq \boldsymbol{I}_\partial^{-1} \cdot dX_{n-1}. \tag{2.128}$$

The normal space is 1-dimensional and we put $\boldsymbol{\nu}$ to refer to the boundary normal space. It is common to compute the flux of a vector field $\boldsymbol{v}$ through ∂M by integrating $\mathsf{P}_{\boldsymbol{\nu}}(\boldsymbol{v})$ over the boundary. However, the the vector field $\mathsf{P}_{\boldsymbol{\nu}}(\boldsymbol{v})$ is the multivector equivalent of a 1-form. Hence, what we should have is a pseudovector $\mathsf{P}_{\boldsymbol{I}_\partial}(\boldsymbol{v}^\perp)$ which is the equivalent to the $n-1$-form

$$\mathsf{P}_{\boldsymbol{I}_\partial}(\boldsymbol{v}^\perp) \cdot dX_{n-1}^\dagger = (-1)^p \boldsymbol{v} \cdot \boldsymbol{\nu} \mu_\partial. \tag{2.129}$$

This tells us that the flux is determined both by the vector field $\boldsymbol{v}$ and the local geometry of ∂M captured by μ_∂. A proof follows.

Proposition 2.2.2. *Then the flux of a vector field $\boldsymbol{v}$ through ∂M is*

$$\int_{\partial M} \mathsf{P}_{\boldsymbol{I}_\partial}(\boldsymbol{v}^\perp) \cdot dX_{n-1}^\dagger = (-1)^p \int_{\partial M} \boldsymbol{v} \cdot \boldsymbol{\nu} \mu_\partial, \tag{2.130}$$

where p is the number of temporal vectors in $\mathcal{G}(M)$.

Proof. Take

$$\begin{aligned}
\mathsf{P}_{\boldsymbol{I}_\partial}(\boldsymbol{v}^\perp) &= \boldsymbol{v}^\perp \rfloor \boldsymbol{I}_\partial \boldsymbol{I}_\partial^{-1} \\
&= (\boldsymbol{v}^\perp \wedge \boldsymbol{\nu})^\perp \boldsymbol{I}_\partial^{-1} \\
&= (-1)^{n-1} (\boldsymbol{\nu} \rfloor \boldsymbol{v})^{\perp\perp} \boldsymbol{I}_\partial^{-1} \\
&= (-1)^{\frac{1}{2}(n+1)(n-1)+p} \boldsymbol{v} \cdot \boldsymbol{\nu} \boldsymbol{I}_\partial^{-1} \\
&= (-1)^p \boldsymbol{v} \cdot \boldsymbol{\nu} \boldsymbol{I}_\partial^{-1\dagger}.
\end{aligned}$$

Hence

$$\mathsf{P}_{\boldsymbol{I}_\partial}(\boldsymbol{v}^\perp) \cdot dX_{n-1}^\dagger = (-1)^p \boldsymbol{v} \cdot \boldsymbol{\nu} \mu_\partial.$$

□

For smooth r-forms α_r and β_r, we have an L^2-inner product

$$\int_M \alpha_r \wedge \star \beta_r \tag{2.131}$$

where $\star$ is the Hodge star. By definition, the Hodge star acts on r-forms by returning a Hodge dual $n-r$-form so that on the multivector equivalents we have

$$\alpha_r \wedge \star \beta_r = \left\langle A_r B_r^\dagger \right\rangle \mu = (A_r, B_R)\mu \tag{2.132}$$

as well as

$$\alpha_r \wedge \star \alpha_r = |A_r|^2 \mu. \tag{2.133}$$

For the action of $\star$ on the multivector equivalents we will put $B_r^\star$ for which we have

$$B_r^\star = (\boldsymbol{I}^{-1} B_r)^\dagger \tag{2.134}$$

and we can quickly verify that

$$(A_r \wedge B_r^\star) \cdot dX_n^\dagger = (A_r \cdot B_r^\dagger)\boldsymbol{I}^{-1\dagger} \cdot dX_n^\dagger = \langle A_r B_r^\dagger \rangle \mu. \tag{2.135}$$

This allows us to define can now define an L^2 inner product on multivector fields.

Definition 2.2.5. Let A and B be multivector fields. Then the *multivector field inner product* is defined by

$$\ll A, B \gg := \frac{1}{\operatorname{vol}(M)} \int_M (A, B)\mu. \tag{2.136}$$

If $\ll A, B \gg = 0$, then we say A and B are orthogonal. Once again, this is only a true inner product when g is positive definite. We put $\ll \cdot, \cdot \gg_\partial$ to represent the inner product on the boundary manifold. Note that if we take a homogeneous r-vector field A_r and s-vector field B_s that if $s \neq r$, the multivector field inner product is zero. Hence, the orthogonal direct sum with respect to the L^2 multivector inner product agrees with the grade based direct sum. It will suffice to use the symbol $\oplus$ for both. One should view this as a slight extension to the r-form inner product that garners the ability to consider the inner product of elements that are not necessarily homogeneous in grade. The following proposition confirms this.

Proposition 2.2.3. *Given two r-forms, the r-form inner product is equal to the multivector inner product on their corresponding multivector equivalents up to the constant* $\operatorname{vol}(M)$.

Proof. Let α_r and β_r be r-forms with multivector equivalents A_r and B_r respectively. Then

$$\int_M \alpha_r \wedge \star\beta = \int_M A_r \cdot B_r^\dagger \mu = \int_M \langle A_r^\dagger B_r \rangle \mu = \operatorname{vol}(M) \ll A_r, B_r \gg,$$

by the definition of the Hodge star. □

Multivector valued integrals

The integrals defined before allow us to encapsulate integration via differential forms, but geometric calculus allows for an extension to multivector valued integrals. Examples using kernel functions are prevalent in physics. Take for instance, determining a magnetic field from a charge distribution or the Biot-Savart law to determine a magnetic field from a current distribution. No drastic changes are needed to our previous formulation.

Let $A \in \mathcal{G}(M)$ be a multivector field and take a submanifold $R \subset M$. Then, we can define a multivector valued integral by

$$\int_R A\boldsymbol{I}_R \mu_R. \tag{2.137}$$

The benefit here is more pronounced in section 2.2.6. For example, this notion allows us to define multivector fields via integration. It will lead us to eq. (3.11).

2.2.5 Stokes' and Green's formula

With forms, we have a compact form of Stokes' theorem given by

$$\int_M d\alpha_{n-1} = \int_{\partial M} \iota^* \alpha_{n-1}, \tag{2.138}$$

for sufficiently smooth $n-1$-forms α_{n-1}. This theorem can be applied to submanifolds R of M as well, just with $r-1$-forms. For example, if $M \subset \mathbb{R}^3$ is a 2-dimensional submanifold of $\mathbb{R}^3$, then one retrieves the Stokes' theorem in vector calculus. sections 2.2.3 and 2.2.4 allows us to determine this in terms of the multivector equivalents. We have the multivector version of Stokes' theorem given by

$$\int_M (\boldsymbol{\nabla} \wedge A_{n-1}) \cdot dX_n = \int_{\partial M} \mathsf{P}_{\boldsymbol{I}_\partial}(A_{n-1}) \cdot dX_{n-1}. \tag{2.139}$$

But this has another, more physical, interpretation. Let us consider the dual relationship by taking vector field $\boldsymbol{v}$ and noting that $\boldsymbol{v}^\perp$ is an pseudovector for which Stokes' theorem can be applied. Hence,

$$\int_M (\boldsymbol{\nabla} \wedge \boldsymbol{v}^\perp) \cdot dX_n = \int_{\partial M} \mathsf{P}_{\boldsymbol{I}_\partial}(\boldsymbol{v}^\perp) \cdot dX_{n-1}, \tag{2.140}$$

which realizes the divergence theorem

$$\int_M \boldsymbol{\nabla} \cdot \boldsymbol{v} \mu = \int_{\partial M} \boldsymbol{v} \cdot \boldsymbol{\nu} \mu_\partial. \tag{2.141}$$

Based on Stokes' theorem and the product rule for the exterior derivative, we also have Green's formula

$$\int_M d\alpha_{r-1} \wedge \star\beta_r = \int_M \alpha_{r-1} \wedge \star\delta\beta_r + \int_{\partial M} \iota^*(\alpha_{r-1} \wedge \star\beta_r) \tag{2.142}$$

This equation motivates the definition of *codifferential* δ as the adjoint to d under the r-form inner product. In the case of a closed manifold M, $\partial M = \emptyset$ and the boundary integral vanishes, we see that δ is adjoint to d.

Definition 2.2.6. The adjoint operator $\boldsymbol{\nabla}\wedge^*$ to $\boldsymbol{\nabla}\wedge$ on r-vectors is given by

$$\boldsymbol{\nabla}\wedge^* = (-1)^{r-1}(\boldsymbol{\nabla}\rfloor A_r^\dagger). \tag{2.143}$$

This leads to the Hodge-Dirac operator $d+\delta$. One should compare this operator to $\boldsymbol{\nabla}$ and notice the subtle differences in the dependence on the manifold dimension and degree of the multivector via both the $(-1)^{r-1}$ term and the application of the reverse $\dagger$.

Proposition 2.2.4. *On multivector equivalents A_{r-1} and B_r, we have Green's formula*

$$\ll \boldsymbol{\nabla} \wedge A_{r-1}, B_r \gg = \ll A_{r-1}, \boldsymbol{\nabla} \wedge^* B_r \gg + (-1)^p \int_{\partial M} (A_{r-1} \rfloor B_r^\dagger) \cdot \boldsymbol{\nu} \mu_\partial. \tag{2.144}$$

Proof. First, we have

$$\int_M d(\alpha_{r-1} \wedge \star\beta_r) = \underbrace{\int_M d\alpha_{r-1} \wedge \star\beta_r}_{1} + \underbrace{(-1)^{r-1} \int_M \alpha_{r-1} \wedge d \star \beta_r}_{2}, \tag{2.145}$$

by the Leibniz rule. By Stokes' theorem,

$$\int_M d(\alpha_{r-1} \wedge \star\beta_r) = \underbrace{\int_{\partial M} \iota^*(\alpha_{r-1} \wedge \star\beta_r)}_{3} . \tag{2.146}$$

For underbrace 1,

$$\int_M d\alpha_{r-1} \wedge \star\beta_r = \int_M (\boldsymbol{\nabla} \wedge A_{r-1}) \cdot B_r^\dagger \mu = \ll \boldsymbol{\nabla} \wedge A_{r-1}, B_r \gg . \tag{2.147}$$

For underbrace 2,

$$(-1)^{r-1} \int_M \alpha_{r-1} \wedge d \star \beta_r = \int_M A_{r-1} \wedge (\boldsymbol{\nabla} \wedge B_r^\star) \cdot dX_n^\dagger \tag{2.148}$$

$$= (-1)^{r-1+n(n-1)} \int_M [A_{r-1} \wedge (\boldsymbol{\nabla} \wedge (B_r^\perp)^\dagger)] \cdot dX_n^\dagger \tag{2.149}$$

$$= (-1)^{r-1+\xi} \int_M A_{r-1} \wedge (\boldsymbol{\nabla} \rfloor B_r)^\perp \cdot dX_n^\dagger \tag{2.150}$$

$$= (-1)^{r-1+\xi} \int_M A_{r-1} \rfloor (\boldsymbol{\nabla} \rfloor B_r) \mu \tag{2.151}$$

$$= \ll A_{r-1}, \boldsymbol{\nabla} \wedge^* B_r \gg . \tag{2.152}$$

For underbrace 3,

$$\int_{\partial M} \iota^*(\alpha_{r-1} \wedge \star\beta_r) = \int_{\partial M} \mathsf{P}_{\boldsymbol{I}_\partial}(A_{r-1} \wedge B_r^\star) \cdot dX_{n-1}^\dagger \tag{2.153}$$

$$= (-1)^\xi \int_{\partial M} \mathsf{P}_{\boldsymbol{I}_\partial}(A_{r-1} \wedge B_r^\perp) \cdot dX_{n-1}^\dagger \tag{2.154}$$

$$= (-1)^\xi \int_{\partial M} \mathsf{P}_{\boldsymbol{I}_\partial}((A_{r-1} \rfloor B_r)^\perp) \cdot dX_{n-1}^\dagger \tag{2.155}$$

$$= (-1)^{\xi+p} \int_{\partial M} (A_{r-1} \rfloor B_r) \cdot \boldsymbol{\nu} \mu_\partial \tag{2.156}$$

with the final equality by proposition 2.2.2. □

Stokes' theorem and Green's formula are essential in determining the L^2-orthogonal decomposition of the space of differential r-forms $\Omega^r(M)$. The applications thereof provide general existence and uniqueness results for boundary value problems. An analogy of this result can be found next in section 2.2.6.

2.2.6 Fundamental theorem of geometric calculus

The containment of the exterior algebra inside a geometric algebra motivates us to push both Stokes' theorem and Green's formula to further limits. Green's formula is derived via Stokes' theorem and both solely make use of the exterior derivative, its adjoint, and the scalar valued Clifford inner product. As it turns out, there is a more general version of Stokes' theorem based on the gradient $\boldsymbol{\nabla}$. This theorem turns out to take advantage of the multivector-valued nature of directed integration. Moreover, we pose no restrictions that require single graded elements and we realize that multiple versions exist to the fact that $\boldsymbol{\nabla}$ can act on both sides of a multivector.

Theorem 2.2.1 (Fundamental theorems of geometric calculus)**.** *Let* $A, B \in \mathcal{G}(M)$*. Then*

$$\int_M \dot{A}\dot{\boldsymbol{\nabla}}\boldsymbol{I}\mu = \int_{\partial M} A\boldsymbol{I}_\partial \mu_\partial \tag{2.157}$$

$$\int_M \boldsymbol{I}\boldsymbol{\nabla}B\mu = \int_{\partial M} \boldsymbol{I}_\partial B\mu_\partial \tag{2.158}$$

$$\int_M \dot{A}\dot{\boldsymbol{\nabla}}\boldsymbol{I}B\mu = (-1)^n \int_M A\boldsymbol{I}\boldsymbol{\nabla}B\mu + \int_{\partial M} A\boldsymbol{I}_\partial B\mu_\partial. \tag{2.159}$$

Finally,

$$\int_M \dot{\mathsf{L}}(\dot{\boldsymbol{\nabla}}\boldsymbol{I})\mu = \int_{\partial M} \mathsf{L}(\boldsymbol{I}_\partial)\mu_\partial, \tag{2.160}$$

holds for linear functions L *of pseudovectors.*

The above theorem is proved in a handful of texts, but originates via Hestene's work in our go-to reference [17]. One may question the inclusion of the unit pseudoscalar in equations eqs. (2.157) to (2.159) and whether they can be pulled outside of the integral. The answer is no, unless M is a region of $\mathbb{R}^n$ since, in that case, the pseudoscalar is constant. In fact, this is used explicitly in the Cauchy integral formula in complex analysis for which we will describe the generalization found in eq. (3.11). Note that eq. (2.159) is close to describing a multivector valued form of a Green's formula. This is wholeheartedly allowing us to consider consequences of the actions of $\boldsymbol{\nabla}$ on both sides of a multivector. In fact, taking the scalar part of eq. (2.159) will lead us to the following result.

Theorem 2.2.2 (Multivector Green's formula)**.** *We have the Green's formula for the gradient*

$$\ll \boldsymbol{I}^\dagger A, \boldsymbol{\nabla} B \gg = (-1)^n \ll \boldsymbol{\nabla} A, \boldsymbol{I} B \gg + \ll A, \boldsymbol{I}_\partial B \gg_\partial . \tag{2.161}$$

Proof. Fix $A^\dagger, B \in \mathcal{G}(M)$ and note eq. (2.159) of theorem 2.2.1 yields

$$\int_M A^\dagger \boldsymbol{I} \boldsymbol{\nabla} B \mu = (-1)^n \int_M \dot{A}^\dagger \dot{\boldsymbol{\nabla}} \boldsymbol{I} B \mu + \int_{\partial M} A^\dagger \boldsymbol{I}_\partial B \mu_\partial \tag{2.162}$$

$$\int_M A^\dagger \boldsymbol{I} \boldsymbol{\nabla} B \mu = (-1)^n \int_M (\boldsymbol{\nabla} A)^\dagger \boldsymbol{I} B \mu + \int_{\partial M} A^\dagger \boldsymbol{I}_\partial B \mu_\partial . \tag{2.163}$$

Now, if we take the scalar part of the above equation and dividing by $\mathrm{vol}(M)$ we have

$$\ll A, \boldsymbol{I} \boldsymbol{\nabla} B \gg = (-1)^n \ll \boldsymbol{\nabla} A, \boldsymbol{I} B \gg + \ll A, \boldsymbol{I}_\partial B \gg_\partial . \tag{2.164}$$

□

Remark 2.2.2. The position of $\boldsymbol{I}$ and $\boldsymbol{I}_\partial$ in the above computation is an artifact of choice. Recall proposition 2.1.1 and note, for example, that

$$\ll A, \boldsymbol{I} \boldsymbol{\nabla} B \gg = \ll \boldsymbol{I}^\dagger A, \boldsymbol{\nabla} B \gg . \tag{2.165}$$

Another benefit to this formulation is there was no mention of dependence on the properties of the metric g. So, theorem 2.2.2 holds in spaces with temporal vectors. However, one should remark that we do lose the definiteness of the inner product.

Chapter 3

Analysis of multivector fields

3.1 Spaces of fields

3.1.1 Monogenic fields

Multivectors in the kernel of $\boldsymbol{\nabla}$ are of fundamental importance in geometric calculus and these multivectors are the motivation for Clifford analysis much like elements in the kernel of Δ give rise to harmonic analysis.

Definition 3.1.1. Let $A \in \mathcal{G}(M)$. Then we say that A is *monogenic* if $\boldsymbol{\nabla} A = 0$.

Monogenic fields are of utmost importance as they have many beautiful properties. One should find them as a suitable generalization of the notion of complex holomorphicity. For example, in regions of Euclidean spaces, a monogenic field A can be completely determined by its Dirichlet boundary values through a generalized Cauchy integral formula and for a spinor field A_+ each of the graded components are harmonic. We put

$$\mathcal{M}(M) \coloneqq \{A \in \mathcal{G}(M) \mid \boldsymbol{\nabla} A = 0\} \tag{3.1}$$

to refer to elements of this set as *monogenic fields* on M. As subspaces we also have the *monogenic r-vectors* $\mathcal{M}^r(M)$, *monogenic spinors* $\mathcal{M}^+(M)$, and the *monogenic surface spinors* $\mathcal{M}^{0+2}(M)$.

Remark 3.1.1. The definition for $\mathcal{M}^r(M)$ is multivector equivalent to space of harmonic fields,

$$\mathcal{H}^r(M) \coloneqq \{\alpha_r \in \Omega^r(M) \mid d\alpha_r = 0,\ \delta\alpha_r = 0\}. \tag{3.2}$$

We will avoid the term harmonic fields since we reference multivector fields in the kernel of Δ as harmonic.

It will be pertinent in section 3.3 to speak of function algebras. Hence, one could consider if the space $\mathcal{M}(M)$ is, in general, an algebra. While it is clear that the sum of two monogenic fields is also a monogenic field, it is not necessarily true that the product of two monogenic fields is monogenic. Hence, these spaces do not always form algebras in their own right. Yet, there are algebras inside of $\mathcal{M}(M)$ (see section 3.2).

Elaborating further, let M be 2-dimensional, then the space of monogenic spinors $\mathcal{M}^+(M)$ is indeed an algebra. In fact, taking $\mathcal{G}_2(\mathbb{R}^2)$ we can note that monogenic spinors are exactly the complex holomorphic functions via the identification in **??**. Take the coordinates x, y and the standard orthonormal basis $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. Then if $f = u + v\boldsymbol{B}_{12} \in \mathcal{G}_2(\mathbb{R}^2)$ we can note that $\boldsymbol{\nabla} f = 0$ yields the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{3.3}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{3.4}$$

Though the two dimensional case is special, there will be nontrivial algebras living inside each $\mathcal{M}(M)$ for manifolds of dimension > 3. One should also note that the space $\mathcal{M}(M)$ is a right $\mathcal{G}$-module since for $a \in \mathcal{G}$ and $A \in \mathcal{M}(M)$

$$\boldsymbol{\nabla}(As) = \boldsymbol{\nabla} As + \dot{\boldsymbol{\nabla}} A\dot{s} = 0. \tag{3.5}$$

This of course is sensible to write, but with the caveat that the constant vector fields may not be smooth on M.

Cauchy integral

One beautiful result in Clifford analysis and geometric caclulus is the celebrated generalization of the Cauchy integral formula for $\mathbb{C}$-holomorphic functions. Details and proofs can be found in our standard texts [13, 17] as well as many others. Briefly, let the smooth, compact, oriented, n-dimensional manifold M with a positive definite g be isometrically imbedded into $\mathbb{R}^n$. Then, there exists a Green's function

$$E(x) := \frac{1}{S_n} \frac{x}{|x|^n} \tag{3.6}$$

satisfying the equation

$$\boldsymbol{\nabla} E(x) = -\dot{E(x)}\dot{\boldsymbol{\nabla}} = \delta(x), \tag{3.7}$$

where $\delta(x)$ is the Dirac delta distribution and where $x \in \mathbb{R}^n$ for $n \geq 2$. All this to say that $E(x)$ is the fundamental solution to the gradient operator. Let $A \in \mathcal{M}(M)$, then we can note

$$\int_{\partial M} E \boldsymbol{I}_\partial A \mu_\partial = (-1)^{n-1} \int_M \dot{E} \dot{\boldsymbol{\nabla}} \boldsymbol{I} A \mu + \int_M E \boldsymbol{I} \boldsymbol{\nabla} A \mu \tag{3.8}$$

$$= (-1)^{n-1} \int_M \dot{E} \dot{\boldsymbol{\nabla}} \boldsymbol{I} A \mu \tag{3.9}$$

$$= (-1)^n \int_M \delta(x) \boldsymbol{I} A, \tag{3.10}$$

This allows for us to define the *Cauchy kernel* by $G(x, x') := E(x' - x)$ and with this we arrive at the *Cauchy integral formula*

$$A(x) = (-1)^n \boldsymbol{I}^{-1} \int_{\partial M} G(x, x') \boldsymbol{I}_\partial(x') A(x') \mu_\partial(x'). \tag{3.11}$$

Hence, we have a method for uniquely determining a monogenic field A from the boundary values $A|_{\partial M}$. This Cauchy integral formula is a fundamental and powerful result in the world of Clifford analysis (see, for example [8, 10, 7]).

Now, take M to be a manifold that is not imbedded into $\mathbb{R}^n$ but with otherwise equivalent properties as before. Then, our goal is to construct a Cauchy kernel function G on M. Since M is compact, we can take an arbitrary finite open cover $\{U_i\}_{i=1}^N$ lying in the atlas of M. Taking our previous work, we realize that for each coordinate patch of M, we have a well defined Green's function G_i on each U_i. Take a smooth partition of unity subordinate to the open cover $\{\rho_i\}_{i=1}^N$. Using this partition of unity, we define a global vector field $G \in \mathcal{G}(M)$. Hence, the local behavior of G can be extended throughout all of M and eqs. (3.7), (3.8) and (3.11) all hold for M. One may see that eq. (3.11) is written in a handful of slightly different ways. If M is a region of $\mathbb{R}^n$, then $\boldsymbol{I}$ is constant and can be taken inside the integral.

Based on the Cauchy integral, we can determine another property of the monogenic fields. In particular, we have the following lemma.

Lemma 3.1.1. *Let $A \in \mathcal{M}(M)$ be such that $A|_{\partial M} = 0$. Then $A = 0$ on all of M.*

Proof. Since A is monogenic, we can write

$$A(x) = (-1)^n \boldsymbol{I}^{-1} \int_{\partial M} G(x,x')\boldsymbol{I}_\partial(x')A(x')\mu_\partial(x'). \tag{3.12}$$

Then,

$$|A| = \left| \int_{\partial M} G(x,x')\boldsymbol{I}_\partial A(x')\mu_\partial(x') \right| \leq \int_{\partial M} |G(x,x')\boldsymbol{I}_\partial A(x')|\, \mu_\partial = 0. \tag{3.13}$$

Hence, $A = 0$ on M. □

3.1.2 Hodge-type decompositions

For Riemannian manifolds with definite g, we have distinguished subspaces of $\mathcal{G}(M)$. These spaces provide an essential utility in solving boundary value problems.

Definition 3.1.2. Let $\mathcal{G}(M)$ be the space of multivector fields on a smooth manifold M, then we have the *Dirichlet fields*

$$\mathcal{G}_D(M) \coloneqq \{A \in \mathcal{G}(M) \mid \mathsf{P}_{I_\partial}(A) = 0\}, \tag{3.14}$$

and the *Neumann fields*

$$\mathcal{G}_N(M) \coloneqq \{A \in \mathcal{G}(M) \mid \mathsf{P}_{I_\partial}(A^\perp) = 0\}. \tag{3.15}$$

Let us define the following spaces of multivectors that mimic their differential forms counterpart.

Definition 3.1.3. We have

- the *gradients*,

$$\boldsymbol{\nabla}\mathcal{G}(M) \coloneqq \{\boldsymbol{\nabla} A \mid A \in \mathcal{G}(M) \text{ and } A|_{\partial M} = 0\}; \tag{3.16}$$

- the *exact fields*,

$$\mathcal{E}(M) \coloneqq \{\boldsymbol{\nabla} \wedge A \mid A \in \mathcal{G}_D(M)\}; \tag{3.17}$$

- the *co-exact fields*,

$$\mathcal{C}(M) \coloneqq \{\boldsymbol{\nabla} \rfloor A \mid A \in \mathcal{G}_N(M)\}; \tag{3.18}$$

- the *Dirichlet harmonic fields*,

$$\mathcal{M}_D(M) \coloneqq \mathcal{M}(M) \cap \mathcal{G}_D(M); \tag{3.19}$$

- the *Neumann harmonic fields*,

$$\mathcal{M}_N(M) \coloneqq \mathcal{M}(M) \cap \mathcal{G}_N(M). \tag{3.20}$$

We then use superscripts to denote the associated r-vector subspace. For instance, we may put

$$\mathcal{M}_D^r(M)^\perp = \mathcal{M}_N^r(M), \tag{3.21}$$

which can be noted in [2], for example. Likewise, we can use the superscripts $+$ or $-$ to denote the spaces of even and odd grading respectively and remark

$$\boldsymbol{\nabla}\mathcal{G}^+(M) \subset \mathcal{G}^-(M) \qquad \text{and} \qquad \boldsymbol{\nabla}\mathcal{G}^-(M) \subset \mathcal{G}^+(M). \tag{3.22}$$

Notice that boundary behavior of these different spaces are important and if the manifold does not have boundary, they can be ignored to realize the correct definitions. Then, under the scalar valued multivector inner product, we find the orthogonal direct sum decomposition

$$\mathcal{G}^r(M) = \mathcal{E}^r(M) \oplus \mathcal{C}^r(M) \oplus \mathcal{M}^r(M), \tag{3.23}$$

known as the Hodge-Morrey decomposition.

Definition 3.1.4. Within the space of monogenic fields we have

$$\mathcal{M}_{\text{ex}}(M) \coloneqq \mathcal{M}(M) \cap \mathcal{E}(M), \tag{3.24}$$

$$\mathcal{M}_{\text{co}}(M) \coloneqq \mathcal{M}(M) \cap \mathcal{C}(M). \tag{3.25}$$

Further, we have two decompositions of the space of harmonic fields

$$\mathcal{M}^r(M) = \mathcal{M}^r_D(M) \oplus \mathcal{M}^r_{\text{co}}(M), \tag{3.26}$$

$$\mathcal{M}^r(M) = \mathcal{M}^r_N(M) \oplus \mathcal{M}^r_{\text{ex}}(M), \tag{3.27}$$

which are the Friedrichs decompositions.

So, this is all to say that monogenic fields of a single grade are already well studied, but now we can study monogenic fields of mixed grades. For example, it is a very reasonable question to ask whether the Hodge-Morrey decomposition extends to

$$\mathcal{G}(M) \stackrel{?}{=} \mathcal{E}(M) \oplus \mathcal{C}(M) \oplus \mathcal{M}(M) \tag{3.28}$$

under the multivector field inner product. This is, in fact, not true. While it is clear that the following spaces have a grade-based L^2 orthogonal decomposition,

$$\mathcal{G}(M) = \bigoplus_{j=1}^{n} \mathcal{G}^j(M) \tag{3.29}$$

$$\mathcal{E}(M) = \bigoplus_{j=1}^{n} \mathcal{E}^j(M) \tag{3.30}$$

$$\mathcal{C}(M) = \bigoplus_{j=1}^{n} \mathcal{C}^j(M), \tag{3.31}$$

we have the failure for the space of monogenic fields in that

$$\mathcal{M}(M) \neq \bigoplus_{j=1}^{n} \mathcal{M}^j(M). \tag{3.32}$$

However, rephrasing this in terms of the gradient brings new light. First, an essential lemma.

Lemma 3.1.2. *Fix a multivector field $A \in \mathcal{G}(M)$. If*

$$\ll A, B \gg = 0 \tag{3.33}$$

for all $B \in \mathcal{G}(M)$ with $B|_{\partial M} = 0$, then $A = 0$.

Proof. Fix $\epsilon > 0$ and consider an open coordinate patch $U \subset M$ such that the set U^ϵ containing all points within a distance ϵ to U also has its closure $\overline{U^\epsilon} \subset M$. In local coordinates x_i on U^ϵ, let χ_U be the indicator function in so that $\chi_U(x) = 1$ if $x \in U$ and is otherwise zero. Note that since χ_U is not smooth so $\chi_U \notin \mathcal{G}(M)$. To ammend this, take the standard mollifier [1, §C.4] $\eta_\epsilon(x)$ and note that the convolution $\chi_U^\epsilon = \chi_U * \eta_\epsilon$ is smooth and supported on U^ϵ for all $\epsilon > 0$.

In the same local coordinates, denote the tangent vector fields by $\boldsymbol{v}_i$ and note that we have the blade basis elements $\boldsymbol{V}_J = \boldsymbol{v}_{j_1} \wedge \cdots \wedge \boldsymbol{v}_{j_r}$ where $0 < j_1 < j_2 < \cdots < j_r \leq n$. Then, note that $\chi_U^\epsilon \boldsymbol{V}_J$ is a smooth r-blade supported on U^ϵ.

Fix $A \in \mathcal{G}$ such that $\ll A, B \gg = 0$ for all $B \in \mathcal{G}(M)$ with $B|_{\partial M} = 0$. Note that locally we can write $A = \sum_J A_J \boldsymbol{V}^J$, where $\boldsymbol{V}^J$ is built by reciprocal elements by $\boldsymbol{V}^J = \boldsymbol{v}^{j_1} \wedge \cdots \wedge \boldsymbol{v}^{j_r}$ and where the A_J are scalar coefficients. Note that $\boldsymbol{V}^{J\dagger} \boldsymbol{V}_K = \delta_K^J$ [17, eq. (3.19)] and thus

$$\ll A, A_J \boldsymbol{V}_J \chi_U^\epsilon \gg = \int_{U^\epsilon} |A_J|^2 \chi_U^\epsilon \mu = 0. \tag{3.34}$$

Hence, $A_J = 0$ on U^ϵ. Note that J was arbitrary and thus it is clear that $A = 0$ on U^ϵ.

Cover M in such sets U^ϵ and note that for any such set, we will find $A = 0$ by the same process leaving the value of A along the boundary of M undetermined. But, by smoothness of A, if $A = 0$ on the interior of M, it must be that $A = 0$ on ∂M as well, and thus $A = 0$ identically. $\square$

Theorem 3.1.1 (Clifford-Hodge Decomposition)**.** *The space of multivector fields $\mathcal{G}(M)$ has the L^2-orthogonal decomposition*

$$\mathcal{G}(M) = \mathcal{M}(M) \oplus \boldsymbol{I}\boldsymbol{\nabla}\mathcal{G}(M). \tag{3.35}$$

Proof. Let $A \in \mathcal{M}(M)$ and $\boldsymbol{I}\boldsymbol{\nabla}B \in \boldsymbol{I}\boldsymbol{\nabla}\mathcal{G}(M)$ then note

$$\ll A, \boldsymbol{I}\boldsymbol{\nabla}B \gg = \ll \boldsymbol{\nabla}A, \boldsymbol{I}B \gg +(-1)^n \ll A, \boldsymbol{I}_\partial B \gg = 0 \tag{3.36}$$

by theorem 2.2.2. Thus the spaces $\mathcal{M}(M)$ and $\boldsymbol{I}\boldsymbol{\nabla}\mathcal{G}(M)$ are orthogonal.

Next, let $C \in \mathcal{G}(M)$ be in the orthogonal complement of $\boldsymbol{I}\boldsymbol{\nabla}\mathcal{G}(M)$. Then, by the Cauchy integral formula eq. (3.11), we can construct a monogenic field $\tilde{C}$ from $C|_{\partial M}$ and we have

$$C = \tilde{C} + C_0, \tag{3.37}$$

where $C_0 \in \mathcal{G}(M)$ and $C_0|_{\partial M} = 0$. Then, take $\boldsymbol{I}\boldsymbol{\nabla}B \in \boldsymbol{I}\boldsymbol{\nabla}\mathcal{G}(M)$ and

$$0 = \ll C, \boldsymbol{I}\boldsymbol{\nabla}B \gg = \ll \boldsymbol{\nabla}C_0, \boldsymbol{I}B \gg . \tag{3.38}$$

Thus, by lemma 3.1.2, it must be that C_0 is monogenic and by lemma 3.1.1, it must be that $C_0 = 0$. Therefore, the orthogonal complement to $\boldsymbol{I}\boldsymbol{\nabla}\mathcal{G}(M)$ is $\mathcal{M}(M)$. □

Remark 3.1.2. One could also put

$$\mathcal{G}(M) = \boldsymbol{I}^\dagger\mathcal{M}(M) \oplus \boldsymbol{\nabla}\mathcal{G}(M) \tag{3.39}$$

using the adjoint.

The space $\mathcal{M}(M)$ is quite a bit more rich than the other spaces. For example, the field $x_1 + x_2\boldsymbol{B}_{12}$ is monogenic but the individual graded components are not. Fundamentally, this is due to the mixing of grades that we pick up when considering multivectors (e.g., in eq. (3.3)). Since the gradient of a multivector consists of a grade raising and lowering component, we will have an interaction between, for example, r, $r-2$, and $r+2$-vectors. This leads to the following proposition.

Lemma 3.1.3. *The space of monogenic fields is decomposed into even and odd components by*

$$\mathcal{M}(M) = \mathcal{M}^{+}(M) \oplus \mathcal{M}^{-}(M). \tag{3.40}$$

Proof. Let $A \in \mathcal{M}(M)$ and let $A_{+} = \langle A \rangle_{+}$ denote the even grade components of A and let $A_{-} = \langle A \rangle_{-}$ denote the odd components of A. Then it is clear that

$$\ll A_{+}, A_{-} \gg = 0. \tag{3.41}$$

Then,

$$\boldsymbol{\nabla} A_{+} \in \mathcal{G}^{-}(M) \qquad \text{and} \qquad \boldsymbol{\nabla} A_{-} \in \mathcal{G}^{+}(M), \tag{3.42}$$

hence $\boldsymbol{\nabla} A = \boldsymbol{\nabla} A_{+} + \boldsymbol{\nabla} A_{-}$ and since $\boldsymbol{\nabla} A = 0$ it must be that $\boldsymbol{\nabla} A_{+} = 0$ and $\boldsymbol{\nabla} A_{-} = 0$. Together with eq. (3.41) proves the result. □

The following corollary is immediate from theorem 3.1.1 and lemma 3.1.3.

Corollary 3.1.1. *When n is odd we have*

$$\mathcal{G}^{+}(M) = \mathcal{M}^{+}(M) \oplus \boldsymbol{I}\boldsymbol{\nabla}\mathcal{G}^{+}(M) \tag{3.43}$$

$$\mathcal{G}^{-}(M) = \mathcal{M}^{-}(M) \oplus \boldsymbol{I}\boldsymbol{\nabla}\mathcal{G}^{-}(M) \tag{3.44}$$

$$\tag{3.45}$$

and when n is odd

$$\mathcal{G}^+(M) = \mathcal{M}^+(M) \oplus \boldsymbol{I}\boldsymbol{\nabla}\mathcal{G}^-(M) \tag{3.46}$$

$$\mathcal{G}^-(M) = \mathcal{M}^-(M) \oplus \boldsymbol{I}\boldsymbol{\nabla}\mathcal{G}^+(M) \tag{3.47}$$

$$\tag{3.48}$$

3.2 Algebras of fields

3.2.1 Subsurface fields

The space $\mathcal{M}(M)$ is a vector space due but it is not, in general, an algebra. For instance, if M is dimension $n = 2$, then $\mathcal{M}^+(M)$ is an algebra due to the commutivity of $\mathcal{M}^+(M)$. Yet, the $\mathcal{M}(M)$ does contain algebras that are commutative Banach algebras given essentially by eq. (2.93). For this section, take M imbedded in $\mathbb{R}^n$ with the Euclidean metric.

But, there are certain types of monogenic fields in which this property is true. We describe a set of parabivectors that operate entirely on a 2-dimensional submanifolds defined by an unit bivector field $\boldsymbol{B}$. These specific fields will be of great utility for the remainder of this paper.

Definition 3.2.1. Let $\boldsymbol{B} \in \mathcal{G}(M)$ be a constant unit bivector field. Then an even multivector field f_+ satisfying

$$f_+ = \mathsf{P}_{\boldsymbol{B}} \circ f \circ \mathsf{P}_{\boldsymbol{B}} \tag{3.49}$$

is a *subsurface spinor field* and we put $\mathcal{G}^+_{\boldsymbol{B}}(M)$ to denote the space of such fields.

Note that in $\mathbb{R}^n$, we have $\mathrm{Gr}(2,n)$ choices of a constant $\boldsymbol{B} \in \mathcal{G}(M)$. Choosing a specific point in $\mathbb{R}^n$ allows us to construct a plane R passing through that point with $\boldsymbol{B}$ the defining the tangent space at each point of that surface. Thus, we define a unique plane in $\mathbb{R}^n$ by choice of point in the product manifold $\mathbb{R}^n \times \mathrm{Gr}(2,n)$.

We can note that $\mu_R = \boldsymbol{B}^{-1} \cdot dX_2$ as constructed in eq. (2.119) and a multivector valued integral of a field g can be evaluated by

$$\int_R g \boldsymbol{B} \mu_R. \tag{3.50}$$

By definition, we have made f_+ constant on translations of R by the precomposition $f_+ = f_+ \circ \mathsf{P}_{\boldsymbol{B}}$. It follows that

$$(\dot{\boldsymbol{\nabla}} \wedge \boldsymbol{B})\dot{f} = 0. \tag{3.51}$$

In $\mathbb{R}^3$, for example, this amounts to fields constant along an axis ω such that $\omega^\perp = \boldsymbol{B}$ then

$$(\dot{\boldsymbol{\nabla}} \wedge \boldsymbol{B})f = (\dot{\boldsymbol{\nabla}} \wedge \omega^\perp)f = (\dot{\boldsymbol{\nabla}} \cdot \omega)f = \nabla_\omega f. \tag{3.52}$$

Hence, the intrinsic gradient on any plane R defined by choosing a point in $\mathbb{R}^n \times \mathrm{Gr}(2,n)$ can be found via projection of $\boldsymbol{\nabla}$ onto R by

$$\boldsymbol{\nabla}_R f \coloneqq \mathsf{P}_{\boldsymbol{B}}(\dot{\boldsymbol{\nabla}})\dot{f}. \tag{3.53}$$

Monogenic subsurface spinor fields satisfy $\boldsymbol{\nabla} f_+ = 0$. Note that, by definition, we have defined f_+ so that $\nabla_{\boldsymbol{v}} f_+ = 0$ for vector fields in the normal space to R. This leads to the fact that

$$\boldsymbol{\nabla} f_+ = \boldsymbol{\nabla}_R f_+. \tag{3.54}$$

Hence, f_+ if monogenic if $\boldsymbol{\nabla}_R f_+ = 0$. Monogenic subsurface spinor fields will serve as a realization of complex holomorphic functions since they carry over some additional nice properties and admit a nice representation. We can always put $f_+ = u + \beta\boldsymbol{B}$ for $u, \beta \in \mathcal{G}^0(M)$ since we required the postcomposition $f_+ = \mathsf{P}_{\boldsymbol{B}} \circ f_+$ and so f_+ must be a C^∞-linear combination of even elements defined by the subspace $\boldsymbol{B}$.

Recall from Example 2.1.5 that multivectors in the form $z = x + y\boldsymbol{B}$ mimic the complex number z when $\boldsymbol{B}$ is a unit 2-blade since $\boldsymbol{B}^2 = -1$. Monogenic subsurface spinor fields are thus a direct analog of $\mathbb{C}$-holomorphic functions. As noted before, we also have the Cauchy-Riemann equations by eq. (3.3).

Let us choose some unit bivector $\boldsymbol{B}$ and define the space of monogenic subsurface spinors by

$$\mathcal{A}_{\boldsymbol{B}}(M) = \{f_+ \in \mathcal{G}_{\boldsymbol{B}}^+ \mid \boldsymbol{\nabla} f_+ = 0\}. \tag{3.55}$$

Note that multiplication of two fields $f = u_f + \beta_f B$ and $g = u_g + \beta_g B$ in $\mathcal{A}_{\boldsymbol{B}}(M)$ is commutative and given pointwise by

$$fg = u_f u_g - \beta_f \beta_g + \boldsymbol{B}(u_f \beta_g + u_g \beta_f) = gf. \tag{3.56}$$

In fact, products of monogenic subsurface spinors $\mathcal{A}_{\boldsymbol{B}}(M)$ forms a commutative Banach algebra.

Proposition 3.2.1. *The space $\mathcal{A}_{\boldsymbol{B}}(\Omega)$ is a commutative unital Banach algebra.*

Proof. Note that the space $\mathcal{A}_{\boldsymbol{B}}(M)$ inherits the norm $\|\cdot\|$ and the associative algebra structure from the space $\mathcal{G}(M)$ since $\mathcal{A}_{\boldsymbol{B}}(M) \subset \mathcal{G}(M)$. Taking $f, g \in \mathcal{A}_{\boldsymbol{B}}(M)$ we can note that at each point

$$(fg, fg) = \langle (fg)^\dagger fg \rangle = \langle g^\dagger f^\dagger fg \rangle = \langle gg^\dagger f^\dagger f \rangle = (gg^\dagger, f^\dagger f), \tag{3.57}$$

and by the Cauchy-Schwarz inequality

$$|fg|^2 = (fg, fg) \leq (f^\dagger f, f^\dagger f)(gg^\dagger, gg^\dagger) = |f|^2 |g|^2. \tag{3.58}$$

Therefore, it follows that $\|fg\| \leq \|f\| \|g\|$. Clearly, $\|1\| = 1$. We can then note that we have shown that for any $f, g \in \mathcal{A}_{\boldsymbol{B}}(M)$ that $fg \in \mathcal{G}_{\boldsymbol{B}}^+(M)$ and moreover that $fg = gf$ by eq. (3.56). Given commutivity, we realize that the product fg is monogenic by

$$\boldsymbol{\nabla}(fg) = \boldsymbol{\nabla} fg + \dot{\boldsymbol{\nabla}} f \dot{g} = \boldsymbol{\nabla} gf = 0. \tag{3.59}$$

$\square$

The set of algebras $\mathcal{A}_{\boldsymbol{B}}(M)$ parameterized by $\mathrm{Gr}(2,n)$ are sufficiently intriguing. For a brief moment, if M is a region of $\mathbb{R}^n$ with a Euclidean metric, we have a natural choice for an element in $\mathcal{A}_{\boldsymbol{B}}(M)$. Specifically, take two orthogonal unit vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, then define $\boldsymbol{B} = \boldsymbol{v}\boldsymbol{w}$. Then, note that the field

$$z = \boldsymbol{w}\mathsf{P}_{\boldsymbol{B}}(x) \tag{3.60}$$

is a subsurface spinor. By definition, it is clear that $z = z \circ \mathsf{P}_{\boldsymbol{B}}$ and since $\boldsymbol{w}$ is in the subspace $\boldsymbol{B}$, we can note $z = \mathsf{P}_{\boldsymbol{B}} \circ z$. Moreover, z is monogenic since

$$\boldsymbol{\nabla} z = \boldsymbol{\nabla}\left(\boldsymbol{w}(x\rfloor\boldsymbol{B})\boldsymbol{B}^{-1}\right) \tag{3.61}$$

$$= \boldsymbol{\nabla}\left((x\cdot\boldsymbol{v})\boldsymbol{w}\boldsymbol{v} + (x\cdot\boldsymbol{w})\ \right) \tag{3.62}$$

$$= \boldsymbol{v}\boldsymbol{w}\boldsymbol{v} + \boldsymbol{w} = 0, \tag{3.63}$$

using, the fact that $\boldsymbol{\nabla}(x\cdot\boldsymbol{v}) = \boldsymbol{v}$, which is shown in [13, eq. (6.5)]. We can define a function z for any choice of $\boldsymbol{B}$ and construct new functions from these. The notation z should make one think of z in complex analysis and, in much the same vein, we can develop a power series using such z.

3.3 Gelfand theory

3.3.1 $\mathcal{G}_n$-spectrum

We turn our focus to the geometric content of the algebras $\mathcal{A}_B(M)$ of monogenic subsurface spinor fields. Take the case where the domain $\mathbb{D}$ be the unit disk in $\mathbb{C} \cong \mathbb{R}^2$. By Gelfand, the maximal ideal space of the commutative Banach algebra $\mathcal{A}_B(\mathbb{D})$ is homeomorphic to the disk since the algebra $\mathcal{A}_B(\mathbb{D})$ is exactly the algebra of holomorphic functions in $\mathbb{D}$. Naively attempting to generalize this notion leads on to consider the maximal ideal space of $\mathcal{M}(M)$ or $\mathcal{M}^+(M)$, but no such maximal ideal space can be determined. Instead, one can note that maximal ideals of a commutative Banach algebra $\mathcal{A}$ correspond to the algebra morphisms $\mathcal{A} \to \mathbb{C}$. Using this as our guiding intuition, we carry on and describe the relevant morphisms of the monogenic fields.

Definition 3.3.1. Define the $\mathcal{G}_n$*-dual* $\mathcal{M}^{\times}(M)$ as the continuous right $\mathcal{G}_n$-module homomorphisms

$$\mathcal{M}^{\times}(M) \coloneqq \{l \colon \mathcal{M}(M) \to \mathcal{G}_n \mid l(fs+g) = l(f)s + l(g),\ \forall f,g \in \mathcal{M}(M),\ r,s \in \mathcal{G}_n\}, \quad (3.64)$$

and refer to elements of $\mathcal{M}^{\times}(M)$ are $\mathcal{G}_n$*-functionals*.

Similarly, we will now define the $\mathcal{G}_n$-functionals that are multiplicative, and therefore constitute algebra morphisms, on the monogenic subsurface spinor fields. In other words, spin characters are simply algebra homomorphisms from $\mathcal{A}_{\boldsymbol{B}}(M)$ to $\mathcal{G}_n^+$.

Definition 3.3.2. The $\mathcal{G}_n$*-spectrum* $\mathfrak{M}(M)$ is the set of algebra homomorphisms

$$\mathfrak{M}(M) \coloneqq \{\delta \in \mathcal{M}^{\times}(M) \mid \delta(fg) = \delta(f)\delta(g),\ \forall f,g \in \mathcal{A}_{\boldsymbol{B}}(M),\ \boldsymbol{B} \in \mathrm{Gr}(2,n)\},$$

and we refer to the elements as $\mathcal{G}_n$*-characters*.

One choice of $\mathcal{G}_n$-characters is point evaluation. Take $\delta(f) = f(x^\delta)$ for some $x^\delta \in M$. We find that these characters exhaust $\mathfrak{M}(M)$. Along with this, we define *weak-$*$* topology on $\mathfrak{M}(M)$, which is defined to be the coarsest topology so that every $x \in M$ corresponds to a continuous map on $\mathcal{M}^{\times}(M)$.

3.3.2 Topology from monogenics

We seek to determine that the space $\mathfrak{M}(M)$ is homeomorphic to M in the case that M is n-dimensional, smooth, oriented, imbedded manifold in $\mathbb{R}^n$ inheriting the Euclidean metric.

Theorem 3.3.1. *For any $\delta \in \mathfrak{M}(M)$, there is a point $x^\delta \in M$ such that $\delta(f) = f(x^\delta)$ for any $f \in \mathcal{M}(M)$ a monogenic field. Given the weak-$*$ topology on $\mathcal{M}^{\times}(M)$, the map*

$$\gamma \colon \mathfrak{M}(M) \to M, \quad \delta \mapsto x^\delta$$

is a homeomorphism.

We prove this theorem in three main components. First, we can realize a power series representation for elements in a ball $\mathbb{B}$ and denote this set as $\mathcal{M}^{\mathcal{P}}(\mathbb{B})$ which is dense in $\mathcal{M}(\mathbb{B})$. This power series is constructed using specific monogenic subsurface spinor fields. Finally, we constructively show a correspondence between $\delta \in \mathfrak{M}(\mathbb{B})$ with $x^\delta \in \mathbb{B}$. Then, we can take a cover M generated by unions of balls to complete the proof.

Taylor series

Fix an orthonormal basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ for $\mathbb{R}^n$ and define the functions $z_{ij} = x_j - x_i \boldsymbol{e}_i \boldsymbol{e}_j$. Recall that for an orthonormal basis the reciprocal basis elements $\boldsymbol{e}^i = \boldsymbol{e}_i$. To further condense notation, we define $\boldsymbol{B}_{ij} \coloneqq \boldsymbol{e}_i \boldsymbol{e}_j$ for $i \neq j$ as necessary. The functions z_{ij} are the analogs to z in $\mathbb{C}$ but specifically in the $\boldsymbol{B}_{ij}$ plane. We then note

$$z_{ij} = x_j - x_i \boldsymbol{B}_{ij} = \boldsymbol{e}_j \mathsf{P}_{\boldsymbol{B}_{ij}}(x), \tag{3.65}$$

for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. Hence, this is but a special case of eq. (3.60) and we note that each z_{ij} is monogenic and hence belongs in $\mathcal{A}_{\boldsymbol{B}_{ij}}$ so long as $i \neq j$. One can quickly verify z_{ii} is not monogenic. These functions find their use in a power series representation for monogenic fields f. Specifically, let σ be a permutation of the set $\{2, 3, \dots, n\}$, then we have the polynomials

$$p_{j_2 \dots j_n}(x) = \frac{1}{j!} \sum_{\text{permutations}} z_{1\sigma(1)}(x) \cdots z_{1\sigma(j)}(x), \tag{3.66}$$

each of which is monogenic, linearly independent ([22, Proposition 1]) and formed by products of elements $z_{ij} \in \mathcal{A}_{\boldsymbol{B}_{ij}}$. We put

$$\mathcal{M}^{\mathcal{P}}(M) = \left\{ \sum_{j=0}^{\infty} \left(\sum_{\substack{j_2 \cdots j_n \\ j_2 + \cdots j_n = j}} p_{j_2 \cdots j_n} a_{j_2 \cdots j_n} \right) \mid j_2 + \cdots + j_n = j,\ 0 \leq j < \infty,\ a_{j_2 \cdots j_n} \in \mathcal{G}_n \right\} \tag{3.67}$$

to refer to the set of *monogenic polynomials*.

Lemma 3.3.1. *The space $\mathcal{M}^{\mathcal{P}}(\mathbb{B}_{R,w})$ is dense in $\mathcal{M}(\mathbb{B}_{R,w})$.*

Proof. We can center a ball of radius R at w to get the monogenic polynomials $p_{j_2\cdots j_n}(x-w)$. Then, let $f \in \mathcal{M}(\mathbb{B}_{R,w})$ and define the coefficients $a_{j_2\cdots j_n} \in \mathcal{G}_n$ by

$$a_{j_2\cdots j_n} = \int_{\partial B(w,R)} \frac{\partial^j G(w,y)}{\partial y_2^{j_2}\cdots \partial y_n^{j_n}} \boldsymbol{\nu}(y) f(y) \mu_\partial(y), \tag{3.68}$$

where G is the Cauchy kernel and we have used the Cauchy integral formula with the fact $\boldsymbol{I}$ is constant. By [22, Theorem 4], we have

$$f(x) = \sum_{j=0}^{\infty} \left(\sum_{\substack{j_2\cdots j_n \\ j_2+\cdots j_n = j}} p_{j_2\cdots j_n}(x-w) a_{j_2\cdots j_n} \right), \tag{3.69}$$

which converges pointwise to f for points $x \in \mathbb{B}_{R,w}$. $\square$

We have found that all monogenic fields are generated as power series of homogeneous polynomials in the variables z_{ij}. Thus, we have a form for which we can determine the action of a $\mathcal{G}_n$-character on a monogenic field. Specifically, take the series for $f(x)$ above and note for $\delta \in \mathfrak{M}(\mathbb{B}_{R,w})$ that

$$\delta(f(x)) = \sum_{j=0}^{\infty} \sum_{j=0}^{\infty} \left(\sum_{\substack{j_2\cdots j_n \\ j_2+\cdots j_n = j}} \delta(p_{j_2\cdots j_n}(x-w)) a_{j_2\cdots j_n} \right) \tag{3.70}$$

and on each monogenic polynomial

$$\delta(p_{j_2\cdots j_n}(x)) = \frac{1}{j!} \sum_{\text{permutations}} \delta\left(\left(z_{1\sigma(1)}(x)\right) \cdots \delta\left(z_{1\sigma(j)}(x)\right)\right), \tag{3.71}$$

by definition. Hence, we now need to determine the action of δ on the variables z_{ij}.

Correspondence

The functions z_{ij} played a crucial role in the above power series representation and they also play a key part in determining the behavior of the $\mathcal{G}_n$-characters on monogenic fields Deducing the action of $\delta(z_{ij})$ will allow us to extend this to any monogenic f via lemma 3.3.1.

Lemma 3.3.2. *Let $\delta \in \mathfrak{M}(\mathbb{B}_{R,w})$ and $z_{ij} \in \mathcal{A}_{B_{ij}}$ then $\delta(z_{ij}) = z_{ij}(x^\delta)$ for some $x^\delta \in \mathbb{R}^n$.*

Proof. Let $\delta \in \mathfrak{M}(\mathbb{B}_{R,w})$ and note that $\delta(1) = 1$ since δ is an algebra homomorphism. Hence, let $\boldsymbol{B}$ be an arbitrary unit 2-blade then

$$\delta(\alpha + \beta\boldsymbol{B}) = \delta(\alpha) + \delta(\beta\boldsymbol{B}) = \alpha\delta(1) + \delta(1)\beta\boldsymbol{B} = \alpha + \beta\boldsymbol{B}. \tag{3.72}$$

by definition. Applying δ to z_{ij} yields

$$\delta(z_{ij}) = \alpha_{ij} + \beta_{ij}\boldsymbol{B}_{ij}, \tag{3.73}$$

for some constants α_{ij} and β_{ij}. Then, note that we have two key relationships

$$z_{ij}\boldsymbol{B}_{ji} = -z_{ji} \tag{3.74}$$

$$z_{ij} = z_{kj} + z_{ik}\boldsymbol{B}_{kj}. \tag{3.75}$$

Applying δ to eq. (3.74)

$$\delta(z_{ij}\boldsymbol{B}_{ji}) = \delta(z_{ij})\boldsymbol{B}_{ji} = -\delta(z_{ji}) \tag{3.76}$$

yields

$$(\alpha_{ij} + \beta_{ij}\boldsymbol{B}_{ij})\boldsymbol{B}_{ji} = \beta_{ij} + \alpha_{ij}\boldsymbol{B}_{ji} = -\alpha_{ji} - \beta_{ji}\boldsymbol{B}_{ji}. \tag{3.77}$$

Hence, $\alpha_i^j = -\beta_j^i$ for all $i \neq j$.

Applying δ to eq. (3.75)

$$\delta(z_{ij}) = \delta(z_{kj} + z_{ik}\boldsymbol{B}_{kj}) = \delta(z_{kj}) + \delta(z_{ik})\boldsymbol{B}_{kj} \tag{3.78}$$

yields

$$a_{ij} + b_{ij}\boldsymbol{B}_{ij} = \alpha_{kj} + \beta_{kj}\boldsymbol{B}_{kj} + (\alpha_{ik} + \beta_{ik}\boldsymbol{B}_{ik})B_{kj} = \alpha_{kj} + \beta_{ik}B_{ij} + (\alpha_{ik} + \beta_{kj})\boldsymbol{B}_{kj} \tag{3.79}$$

yields the relationships $\alpha_{ij} = \alpha_{kj}$, $\beta_{ij} = \beta_{ik}$, and $\alpha_{ik} = -\beta_{kj}$.

These relationships allow us to achieve our proof. Briefly, picture α_{ij} and β_{ij} as components of the $n \times n$ matrices α and β where we index row by column. Note that α and β both have zero diagonal since the functions z_i^i are not monogenic. The relationship $\alpha_{ji} = -\beta_{ij}$ for $i \neq j$ then shows that $\alpha = -\beta^\top$. Then we have $\alpha_{ij} = \alpha_{kj}$ for $i \neq j \neq k$ shows that α is constant along rows and hence β is constant along columns. This is consistent with $\alpha = -\beta^\top$, $\beta_j^i = \beta_k^i$, and the final relationship $\alpha_{ik} = -\beta_{kj}$. The matrices α and β are thus uniquely determined by n numbers. Moreover, treating $\delta(z_{ij}) = z_{ij}(x_\delta)$ for some $x^\delta \in \mathbb{R}^n$ satisfies the relationships granted above. Thus, we simply find the x^δ such that we retrieve the desired components for α and β. □

Lemma 3.3.3. *Let $f \in \mathcal{M}(\mathbb{B}_{R,w})$, then $\delta(f) = f(x^\delta)$ for some $x^\delta \in \mathbb{B}_{R,w}$.*

Proof. To see that $x^\delta \in \mathbb{B}_{R,w}$, take $G_0 \in \mathcal{M}(\mathbb{B}_{R,w})$ by $G_0(x) \coloneqq G(x - x_0)$ where G is the Green's function in eq. (3.6) and take some $x' \notin \mathbb{B}_{R,w}$. Fix $\delta \in \mathfrak{M}(\mathbb{B}_{R,w})$, then,

$$\delta(G_0) = G_0(x^\delta). \tag{3.80}$$

If $x^\delta \notin \mathbb{B}_{R,w}$, then take a sequence $x_n \to x^\delta$ and note that the sequence of functions $G_n(x) \coloneqq G(x - x_n) \in \mathcal{M}(\mathbb{B}_{R,w})$ and the sequence converges to a monogenic function $G_\infty = G(x - x^\delta)$ but

$$\lim_{n\to\infty} \delta(G_n) = \lim_{n\to\infty} G_n(x^\delta), \tag{3.81}$$

does not converge. Hence, it must be that $x^{\delta} \in \mathbb{B}_{R,w}$ via continuity of δ. $\square$

Take an arbitrary open cover of M with balls $\mathbb{B}_{R,w}$. Via this cover, we extend the lemmas to prove theorem 3.3.1.

Proof. This theorem is not yet proven, but the previous lemmas prove the theorem in the case where $M = \mathbb{B}_{R,w}$. $\square$

Chapter 4

Inverse problems

4.1 Tomography

There is an application in mind with the toolbox we have developed. This is the Calderón problem. This physical inverse problem is due to Alberto Calderón who asked how much information of a domain can we determine from measurements along the boundary of the domain. To conduct this experiment physically, one applies a voltage along subsets of the boundary of a given domain and the user measures the outgoing current flux. It is this set of information, the boundary ∂M, the input voltage ϕ, and the measured flux j that is accessible to the user. From this information, can one determine the conductivity of the interior M? This is the Electrical Impedance Tomography (EIT) problem.

Other forms of this problem exist. For example, magnetic impedance tomography , ultrasound tomography , and magnetic resonance imaging are all examples of tomography. Fundamentally, these problems exist to determine the interior structure of materials that we do not wish to, or, cannot destroy to determine more. To make an approach to these problems in general, we can consider geometrical analogs. For example, in EIT (at least in dimensions $n > 2$, one can do away with the notion of the conductivity by replacing the matrix with an intrinsic Riemannian metric.

Tomography is useful, yet, challenging practice for which there are unanswered questions. For example, it has yet to been proved that the smooth EIT problem with complete boundary measurements even has a solution. One may ask just how much information is necessary to solve the EIT, or related tomography problems. This line of thought has lead researchers to consider generalizations using differential forms . Using forms, there is less restriction on the types of functions we use to perform tomography and, moreover, what information we allow ourselves to know along the boundary.

4.1.1 Forward problems

Electrostatics

Let M be a smooth, compact, oriented, Euclidean, 3-dimensional region in $\mathbb{R}^3$ with boundary ∂M; M plays the role of the domain we wish to perform EIT on. Take σ to be a symmetric positive definite matrix to play the role of a conductivity. If σ can be diagonalized as an scalar field times an identity matrix, we say that M is constructed of *isotropic* material, otherwise M is made of *anisotropic* material. We have access to the boundary ∂M and we to this end, we make choices of a static scalar potential (voltage) ϕ to apply along ∂M. This applied voltage induces the potential u in the interior of M. Since M is Euclidean, we have the freedom to choose a global basis for which the metric coefficients satisfy $g_{ij} = \delta_{ij}$. Thus, we construct the M as a geometric manifold, where each geometric tangent space is Euclidean $C\ell(T_xM, |\cdot|)$, so that we are working with multivector fields in $\mathcal{G}_n(M)$. Finally, we posit that M is built from an electrically conductive Ohmic material. Succinctly, the scalar potential u and the current $\boldsymbol{j}$ satisfy Ohm's law

$$-\sigma \boldsymbol{\nabla} \wedge u = \boldsymbol{j} \tag{4.1}$$

on the entirety of M. We also put $\boldsymbol{E} \coloneqq \boldsymbol{\nabla} \wedge u$ as the electric field.

Inside M there must be no free charges that can accumulate and we arrive at the following conservation law

$$\int_{\partial M} \boldsymbol{j} \cdot \boldsymbol{\nu} \mu_\partial = \int_{\partial M} \mathsf{P}_{\boldsymbol{I}_\partial}(\boldsymbol{j}^\perp) \cdot dX_{n-1} = 0 \tag{4.2}$$

due to proposition 2.2.2. Via Stokes' theorem through eqs. (2.139) and (2.140) we arrive at the conclusion that

$$\boldsymbol{\nabla} \cdot \boldsymbol{j} = 0. \tag{4.3}$$

Thus, for the scalar potential we have

$$\boldsymbol{\nabla} \cdot (\sigma \boldsymbol{\nabla} \wedge u) = 0, \tag{4.4}$$

as an equivalent condition to eq. (4.2). A more thorough analysis can be found in [14].

Taking some arbitary basis, conductivity matrix assumes the components σ_{ij} for $i, j = 1, 2, 3$. Via [27] in dimension $n > 2$, we can realize that the conductivity matrix can be replaced with an intrinsic Riemannian metric with the components in this basis given by

$$g_{ij} = (\det \sigma^{k\ell})^{\frac{1}{n-2}} (\sigma^{ij})^{-1}, \quad \sigma^{ij} = (\det g_{k\ell})^{\frac{1}{2}} (g_{ij})^{-1}. \tag{4.5}$$

It is worth noting that these cannot hold in dimension $n = 2$. Due to eq. (4.5), we can remove the extrinsic need of σ with an intrinsic g on the Clifford bundle structure. That is, we are working with $\mathcal{G}(M)$ where each geometric tangent space is given by $C\ell(T_xM, g_x)$. Hence, Ohm's law is given as

$$-\boldsymbol{\nabla} \wedge u = \boldsymbol{j}. \tag{4.6}$$

Then by eq. (4.2), we find the scalar potential is harmonic

$$\Delta u = 0 \quad \text{in } M. \tag{4.7}$$

Hence, this yields the Dirichlet boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } M \\ u = \phi & \text{on } \partial M. \end{cases} \tag{4.8}$$

It is a well known fact that this problem is uniquely solvable (e.g., see [25, Theorem 3.4.6]).

Magnetostatics

Tomography can be performed using magnetic fields as well. In this case, we consider the boundary value problem for the magnetic vector field $\boldsymbol{h}$ by

$$\begin{cases} \Delta \boldsymbol{h} = 0, \ \boldsymbol{\nabla} \rfloor \boldsymbol{h} = 0 \quad \text{in } M \\ \boldsymbol{\nu} \times \boldsymbol{h} = \boldsymbol{j}^{\boldsymbol{I}_\partial}, \end{cases} \tag{4.9}$$

where $\boldsymbol{j}^{\boldsymbol{I}_\partial}$ is the tangential component of the boundary current $\boldsymbol{j}^{\boldsymbol{I}_\partial} := \mathsf{P}_{\boldsymbol{I}_\partial}$, which we can refer to as the surface current. The equation

$$\boldsymbol{\nabla} \rfloor \boldsymbol{h} = 0 \tag{4.10}$$

is Gauss's law for magnetism. It becomes quite clear there is a direct relationship between the electric and magnetic impedance tomography problems. We shall examine this further later. Note that this problem is not uniquely solvable ([25, Theorem 3.5.6]) as the solution is determined up to a field in $\mathcal{M}^1_D(M)$ and we can choose to take $\boldsymbol{h}$ to be orthogonal to $\mathcal{M}^1_D(M)$ under the scalar valued Clifford inner product (see, for instance, [4]).

Let us examine this problem locally on ∂M. Let $\boldsymbol{e}_1$,$\boldsymbol{e}_2$, and $\boldsymbol{\nu}$ constitute a right-handed local orthonormal basis around a point $x \in \partial M$. Hence, the local pseudoscalar is $\boldsymbol{I} = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{\nu}$ and thus the boundary pseudoscalar is given by $\boldsymbol{I}_\partial = \boldsymbol{e}_1\boldsymbol{e}_2$ by definition since $\boldsymbol{\nu} = \boldsymbol{I}_\partial \boldsymbol{I}^{-1}$. Then let $\boldsymbol{h} = h_1\boldsymbol{e}_1 + h_2\boldsymbol{e}_2 + h_{\boldsymbol{\nu}}\boldsymbol{\nu}$. Then,

$$\boldsymbol{\nu} \times \boldsymbol{h} = h_1\boldsymbol{e}_2 - h_2\boldsymbol{e}_1 = \mathsf{P}_{\boldsymbol{I}_\partial}(\boldsymbol{h})\boldsymbol{I}_\partial = \boldsymbol{h} \rfloor \boldsymbol{I}_\partial = \boldsymbol{j}^{\boldsymbol{I}_\partial}. \tag{4.11}$$

From eq. (4.11), one can deduce that there are a few geometrical insights. The foremost is that the surface current $\boldsymbol{j}^{\boldsymbol{I}_\partial}$ is simply rotated $\pi/2$ from the projection (or pullback) of $\boldsymbol{h}$ into the boundary.

Via Maxwell's equations, we note Ampere's law

$$\boldsymbol{\nabla} \times \boldsymbol{h} = \boldsymbol{j}, \tag{4.12}$$

Via remark 2.1.1, we see

$$\boldsymbol{\nabla} \rfloor \boldsymbol{h}^{\perp}, \tag{4.13}$$

is equivalent and this leads us to define $\boldsymbol{b} \coloneqq \boldsymbol{h}^{\perp}$ as the *magnetic bivector field*. In eq. (4.9), we can note that

$$\boldsymbol{\nabla} \rfloor \boldsymbol{h} = \boldsymbol{\nabla} \wedge \boldsymbol{b} = 0 \tag{4.14}$$

and moreover

$$\Delta \boldsymbol{h} = \boldsymbol{\nabla} \rfloor (\boldsymbol{\nabla} \wedge \boldsymbol{h}) = \boldsymbol{\nabla} \rfloor (\boldsymbol{\nabla} \rfloor \boldsymbol{b}) \boldsymbol{I} = (-1)^{3n(n-1)/2+p} \left(\boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \rfloor \boldsymbol{b}) \right)^{\perp}. \tag{4.15}$$

Finally, with another application of remark 2.1.1, we find eq. (4.9) can be written equivalently as

$$\begin{cases} \Delta \boldsymbol{b} = 0, \;\; \boldsymbol{\nabla} \wedge \boldsymbol{b} = 0 & \text{in } M \\ \boldsymbol{\nu} \rfloor \boldsymbol{b} = \boldsymbol{j}^{\boldsymbol{I}_\partial}, \end{cases} \tag{4.16}$$

in terms of the magnetic bivector field $\boldsymbol{b}$. By analogous logic, this boundary value problem is uniquely solvable up to some element of $\mathcal{M}_N^2(M)$. The statement on the boundary can be given equivelently in a few ways by eq. (2.56) seen in remark 2.1.1, e.g.

$$\boldsymbol{\nu} \rfloor \boldsymbol{b} = \boldsymbol{b} \times \boldsymbol{I}_\partial. \tag{4.17}$$

From eqs. (4.12) and (4.17), we find

$$\mathsf{P}_{\boldsymbol{I}_\partial}(\boldsymbol{\nabla} \rfloor \boldsymbol{b}) = \mathsf{P}_{\boldsymbol{I}_\partial}(\boldsymbol{b} \times \boldsymbol{I}_\partial). \tag{4.18}$$

Electromagnetostatics

One can seek to combine the problems above into a single multivector formulation. Note that a combination of Ohm's law (eq. (4.1)) and Ampere's law (eq. (4.12)) yields the expression

$$-\boldsymbol{\nabla} \wedge u = \boldsymbol{j} = \boldsymbol{\nabla} \rfloor \boldsymbol{b}. \tag{4.19}$$

Combined with Gauss's law (eq. (4.10)) in the form $\boldsymbol{\nabla} \wedge \boldsymbol{b} = 0$, we can note that the spinor field $u + \boldsymbol{b} \in \mathcal{G}^+(M)$ is left monogenic since

$$\boldsymbol{\nabla}(u + \boldsymbol{b}) = \boldsymbol{\nabla} \wedge u + \boldsymbol{\nabla} \rfloor \boldsymbol{b} + \boldsymbol{\nabla} \wedge \boldsymbol{b} = 0. \tag{4.20}$$

The Dirichlet problem for the scalar potential (eq. (4.8)) and the magnetic field (**??**) both find unique solutions (once again, up to a component in $\mathcal{M}_N^2(M)$).

Generalization to forms

This problem can be cast in a new light by considering harmonic r-forms instead of a harmonic 0-form u. Given some $\varphi \in \Omega^r(\partial M)$, we have the boundary value problem

$$\begin{cases} \Delta \alpha_r = 0, & \text{in } M \\ \iota^* \alpha_r = \varphi, \quad \iota^*(\delta \alpha_r) = 0 & \text{on } \partial M. \end{cases} \tag{4.21}$$

As stated in [2], there exists a solution α_r to this problem up to a monogenic Dirichlet field λ_D.

Note that the operator Λ is often referred to as the *scalar* DN map since the input is the scalar field ϕ whereas a more general operator on differential r-forms has been described in [2, 26]. There, we begin with equation eq. (4.21). The DN map is extended to r-forms by

$$\Lambda \varphi = \iota^*(\star d\alpha_r). \tag{4.22}$$

In terms of the multivector equivalent A_r, we find

$$\iota^*(\star d\alpha_r) = \mathsf{P}_{\boldsymbol{I}_\partial}((\boldsymbol{\nabla} \wedge A_r)^\star) \cdot dX^\dagger_{n-r-1} = \tag{4.23}$$

One should note that in the case of a scalar potential

$$\Lambda_{\mathrm{Cl}}\phi = \Lambda\phi \tag{4.24}$$

Calderón problem. Let Ω be an unknown Riemannian manifold with unknown metric g and with known boundary Σ and known DN operator Λ. Can one recover Ω and the spatial inner product g from knowledge of Σ and Λ?

4.1.2 Multivector tomography

Electrical impedance tomography

In the realm of EIT, the Dirichlet data ϕ amounts to an input voltage along the boundary and by Ohm's law $\boldsymbol{j} = \boldsymbol{\nabla} \wedge u$ provides us the current. For any given solution to the boundary value problem, there is the corresponding Neumann data is the outward normal derivative of the solution u, $\nabla_{\boldsymbol{\nu}}\phi$. In this case, all vectors are spatial and since $\boldsymbol{\nu}$ is unital, $\boldsymbol{\nu} = \boldsymbol{\nu}^{-1}$ which allows us to note

$$\nabla_{\boldsymbol{\nu}}\phi = \boldsymbol{\nu}\rfloor(\boldsymbol{\nabla} \wedge \phi) = (\boldsymbol{\nabla} \wedge \phi) \cdot \boldsymbol{\nu} = \mathsf{P}_{\boldsymbol{\nu}}(\boldsymbol{\nabla} \wedge \phi)\boldsymbol{\nu}, \tag{4.25}$$

with the last equality by eq. (2.75). The sole difference in interpration lies in the fact that the projection $\mathsf{P}_{\boldsymbol{\nu}}(\boldsymbol{\nabla} \wedge u)$ is vector valued whereas $\nabla_{\boldsymbol{\nu}}\phi$ is scalar valued. Since the span of $\boldsymbol{\nu}$ is one dimensional, the difference is only in taking the whole outward component of $\boldsymbol{\nabla} \wedge \phi$ itself or the coefficient thereof. This motivates the so called Voltage-to-Current (VC) operator or *Dirichlet-to-Neumann (DN) map*

$$\Lambda_{\mathrm{Cl}}\phi = \mathsf{P}_{\boldsymbol{\nu}}(\boldsymbol{\nabla} \wedge u), \tag{4.26}$$

and we put $\Lambda_{\mathrm{Cl}}\phi = \boldsymbol{j}^{\boldsymbol{\nu}}$ as the normal component of the boundary current $\boldsymbol{j}|_{\partial M}$. he inverse problem is to determine g from complete knowledge of Λ_{Cl}.

Generalizations

There are two notable related questions that can be stated in terms of multivectors. First, the most natural boundary value problems are

$$\begin{cases} \Delta A = 0 & \text{in } M, \\ A|_{\partial M} = B|_{\partial M} & \text{on } \partial M, \end{cases} \tag{4.27}$$

and

$$\begin{cases} \boldsymbol{\nabla} A = 0 & \text{in } M, \\ A|_{\partial M} = B|_{\partial M} & \text{on } \partial M. \end{cases} \tag{4.28}$$

It should be noted that we have

$$A|_{\partial M} = \mathsf{P}_{\boldsymbol{I}_\partial}(A) + \mathsf{R}_{\boldsymbol{I}_\partial}(A) = \mathsf{P}_{\boldsymbol{I}_\partial}(A) + \mathsf{P}_{\boldsymbol{\nu}}(A) \tag{4.29}$$

in order to consider all boundary values for a multivector.

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