MATH 570, Homework 5

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September 21, 2017

Solutions

Problem 1.

(a) Prove that a topological space X is disconnected if and only if there exists a surjective continuous function from X to the discrete space $\{0,1\}$.

(b) Prove that if X is path-connected and $f\colon X \to Y$ is continuous, then $f(X)$ is path-connected.

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Proof (Part (a)). For the forward direction, suppose that X is disconnected. Thus we can say that there exists $A \subset X$ with $A \neq \emptyset$ which is open and closed in X. Then $X \setminus A$ is also open and closed. Let $f\colon X \to \{0,1\}$ with $f(A) = 0$ and $f(X \setminus A) = 1$. Then $f^{-1}(0) = A$ and $f^{-1}(1) = X \setminus A$. Thus we have that f is continuous since $\{0\}, \{1\}$ are open in $\{0,1\}$ and $A, X \setminus A$ are open.

For the reverse direction, suppose we have $f\colon X \to \{0,1\}$ is continuous and surjective. Thus $f^{-1}(0) \subseteq X$ and $f^{-1}(1) \subseteq X$ are nonempty and open due to surjectivity and continuity respectively. Thus $X \setminus f^{-1}(0)$ and $X \setminus f^{-1}(1)$ are nonempty, open and closed in X. Thus X is disconnected. □

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Proof (Part (b)). Suppose F is path connected and $f\colon X \to Y$ is continuous. Let $\gamma\colon [0,1] \to X$ be an arbitrary path connected x_1, x_2 (i.e., $\gamma(0) = x_1$ and $\gamma(1) = x_2$). Then $f \circ \gamma$ is a path in $f(x)$ with $f \circ \gamma(0) = f(x_1)$ and $f \circ \gamma(1) = f(x_2)$. This is a continuous function since composition of continuous functions is continuous. Since x_1, x_2 were arbitrary we have that $f(x_1)$ and $f(x_2)$ are arbitrary points in $f(X)$ and thus $f(X)$ is path connected. □

Problem 2. Prove Lemma 4.27 in our book, which says that if X is a topological space, then $A \subseteq X$ (with the subspace topology) is compact if and only if every cover of A by open subsets of X has a finite subcover.

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Proof. For the forward direction, let $A \subseteq \cup_{\alpha \in A} U_\alpha$ with each U_α open in X. Since we are supposing A is compact, there exists a finite collection of U_α so that $A \subseteq \cup_{i=1}^n U_{\alpha_i}$. So A has a finite subcover for an arbitrary cover given by a collection of open subsets in X.

For the reverse direction, suppose that we have an arbitrary open cover $\{U_\alpha\}_{\alpha \in A}$ of A. Since $A \subseteq X$, these $U_\alpha \subseteq X$ and thus we have a finite subcover of A given by $A \subseteq \cup_{i=1}^n U_{\alpha_i}$. Which means that $A = \cup_{i=1}^n A \cap U_{\alpha_i}$ is a finite open cover for A and we have that A is compact since our original cover was arbitrary. $\square$

Problem 3. Solutions to this problem are in our book – feel free to learn and use those solutions!

(a) Let X be a Hausdorff space and let $A, B \subseteq X$ be disjoint compact subsets. Prove that there exist disjoint open sets $U, V \subseteq X$ with $A \subseteq U$ and $B \subseteq V$.

(b) Prove that every compact subset A of a Hausdorff space X is closed.

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Proof (Part (a)). I will use the proof from the book, but just rewritten slightly. First, consider $B = \{q\}$ and then we have for all $p \in A$ that there exists open subsets $p \in U_p \subseteq X$ and $q \in V_p \subseteq X$ with $U_p \cap V_p = \emptyset$. Then $\cup_{p\in A} U_p$ is an open cover of A and so we have $\mathbb{U} = \cup_{i=1}^{n} U_{p_i}$ is a finite open subcover. Then $\mathbb{V} = \cap_{i=1}^{n} V_{p_i}$ is disjoint from $\mathbb{U}$ with $A \subseteq \mathbb{U}$ and $\{q\} \in \mathbb{V}$.

Now, suppose that $B \subseteq X$ is compact and disjoint from A. Then for each $q \in B$ we have open subsets created as in the above paragraph which we denote $\mathbb{U}_q, \mathbb{V}_q$ with $A \subseteq \mathbb{U}_q$ and $q \in \mathbb{V}_q$. Since B is compact, we have $\mathbb{B} = \cup_{i=1}^{m} \mathbb{V}_{q_i}$ covers B and $\mathbb{A} = \cap_{i=1}^{m} \mathbb{U}_{q_i}$ is a cover of A disjoint from $\mathbb{B}$. ☐

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Proof (Part (b)). Let A be a compact subset of a Hausdorff space X. Suppose that $\exists p \in X \setminus A$ which is a limit point of A. So for every neighborhood of p, $N(p)$, we have $N(p) \cap A \neq \emptyset$. But this means that points in A are not distinct from p since X is Hausdorff and each pair of distinct points can be contained in disjoint open sets. This contradicts $p \in X \setminus A$ and thus $p \in A$ and A must be closed since p was an arbitrary limit point.

Problem 4. Define $id\colon S^1 \to S^1$ by $id(p) = p$, and define $g\colon S^1 \to S^1$ by $g(p) = -p$. Find a homotopy $F\colon S^1 \times I \to S^1$ from id to g.

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Solution. We have $\mathrm{id}(p) = p = e^{i\theta}$ and $g(p) = -p = e^{i(\theta+\pi)}$. Then let $F\colon S^1 \times I \to S^1$ be given by $F(\theta, t) = e^{i(\theta+\pi t)}$. Then $F(\theta, 0) = \mathrm{id}(p)$ and $F(\theta, 1) = g(p)$. ■

Problem 5. Let $n > 1$. Prove that $\mathbb{R}^n$ is not homeomorphic to any open subset of $\mathbb{R}$.

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Proof. Suppose that our open subset $U \subseteq \mathbb{R}$ is multiple disjoint open intervals. Then U is not connected, but $\mathbb{R}^n$ is. Thus there cannot be a homeomorphism. Since a single open interval is homeomorphic to $\mathbb{R}$, it suffices to show that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}$. Suppose, for a contradiction, we have a homeomorphism $h \colon \mathbb{R}^n \to \mathbb{R}$. Then we also have a homeomorphism on the set $\mathbb{R}^n \setminus \{\vec{p}\}$ given by $h \colon \mathbb{R}^n \setminus \{\vec{p}\} \to \mathbb{R} \setminus \{h(\vec{p})\}$. But $\mathbb{R} \setminus \{h(\vec{p})\}$ is not connected and $\mathbb{R}^n \setminus \{\vec{p}\}$ is. Thus we contradict h being a homeomorphism and we have that $\mathbb{R}^n$ for $n > 1$ is not homeomorphic to any open subset of $\mathbb{R}$. $\square$

Problem 6. Let A be an infinite set, and let $\mathbb{R}^A$ denote the Cartesian product of A copies of $\mathbb{R}$ (namely $\mathbb{R}^A = \prod_{\alpha \in A} X_\alpha$ where $X_\alpha = \mathbb{R}$ for all $\alpha \in A$). Consider $\mathbb{R}^A$ equipped with two different topologies: $(\mathbb{R}^{\mathbb{A}}, \text{product})$ with the product topology, and $(\mathbb{R}^A, \text{box})$ with the box topology, as defined on page 63 of our book.
Show that $(\mathbb{R}^A, \text{box})$ equipped with the maps $\pi_\alpha^{\text{box}} \colon (\mathbb{R}^A, \text{box}) \to X_\alpha = \mathbb{R}$ defined via $\pi_\alpha^{\text{box}}((x_\alpha)_{\alpha \in A}) = x_\alpha$ is not the categorical product of A copies of $\mathbb{R}$ in the category of topological spaces, as follows (and *not* by using Corollary 3.39). Suppose for a contradiction $(\mathbb{R}^A, \text{box})$ satisfied the universal property on page 213. Choose W to be the actual categorical product $(\mathbb{R}^A, \text{product})$ equipped with the maps $\pi_\alpha^{\text{prod}} \colon (\mathbb{R}^A, \text{box}) \to X_\alpha = \mathbb{R}$ similarly defined via $\pi_\alpha^{\text{prod}}((x_\alpha)_{\alpha \in A}) = x_\alpha$. Show that there is no continuous f making the necessary diagrams commute.
Hint: Note that $(0,1)^A$ is open in $(\mathbb{R}^A, box)$. Can you explain why $(0,1)^A$ is not open in $\mathbb{R}^A, product)$?
Remark: When showing that an object is not a categorical product, it is often a good idea to choose "test object" W to be the actual categorical product.

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Proof. We have the following diagram:

$$\begin{tikzcd} & (\mathbb{R}^A, \text{prod}) \arrow[dl, "\pi_\alpha^{\text{prod}}"'] \arrow[d, dashed, "f"] \\ X_\alpha & (\mathbb{R}^A, \text{box}) \arrow[l, "\pi_\alpha^{\text{box}}"] \end{tikzcd}$$

We have that $(\mathbb{R}^A, \text{prod})$ is a product in the category of topological spaces and thus π_α^{prod} is continuous. Note if this diagram commutes then $\pi_\alpha^{\text{box}} \circ f = \pi_\alpha^{\text{prod}}$ is continuous. Due to how π_α^{box} and π_α^{prod} are defined, $f \colon (\mathbb{R}^A, \text{prod}) \to (\mathbb{R}^A, \text{box})$ is given by $(V, \text{prod}) \mapsto (V, \text{box})$ for $U \subset \mathbb{R}^A$. But note that if $V = (0,1)^A$ then $f^{-1}((0,1)^A, \text{box}) = ((0,1)^A, \text{prod})$ is not open since the product topology is generated by a base where all but finitely many $U_\alpha = X_\alpha$, and this is not the case. Thus f is not continuous and this diagram does not commute. $\square$