

# Riemannian Geometry

for Dummies

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# Section 1

## **Introduction**

Riemannian geometry is the study of a *smooth manifold*  $M$  along with a *Riemannian metric*  $g$ .

The point of Riemannian geometry is to generalize the differentiable and metric structure of  $\mathbb{R}^n$ .

We generalize to spaces that have interesting topology and geometry.

This will require us to rethink some notions we found “easy”  
in  $\mathbb{R}^n$ .

But we will gain a very general framework for working with differentiable objects.

## Section 2

### **Motivation**



Why study this in the first place?

Example: Partial differential equations (PDEs) on spaces that are not flat.

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- Fluid flow on Earth;
- Electrical Impedence Tomography (EIT);
- General relativity.

Example: Optimization in interesting spaces.

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- Grassmannians;

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- Grassmannians;
- Flags.



## Section 3

### **Preliminaries**

add more math text before/after pics so that people see some notation. More examples.

## Subsection 1

### **Smooth Manifolds**

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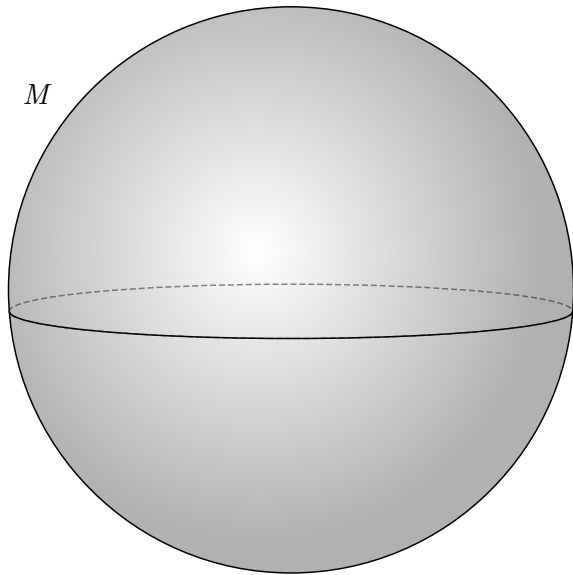
- Start with a topological space  $M$ ;
- Look at open sets  $U$  that cover  $M$ ;
- Construct local coordinates  $\varphi$ ;

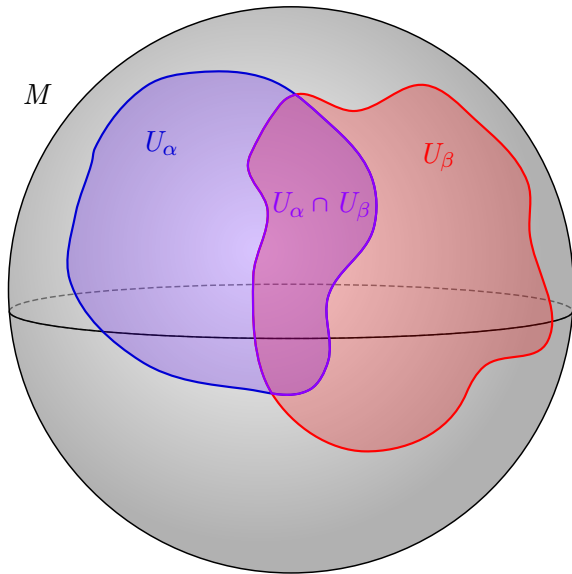
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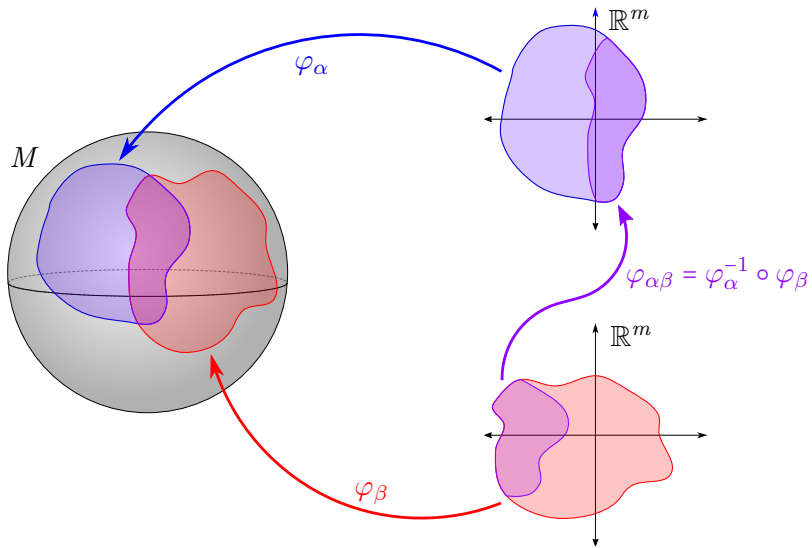
- Start with a topological space  $M$ ;
- Look at open sets  $U$  that cover  $M$ ;
- Construct local coordinates  $\varphi$ ;
- Show coordinate transition functions are smooth.

Define the sphere as the set of points in  $\mathbb{R}^3$ ... then say we'll mostly use this as an example so keep it in mind



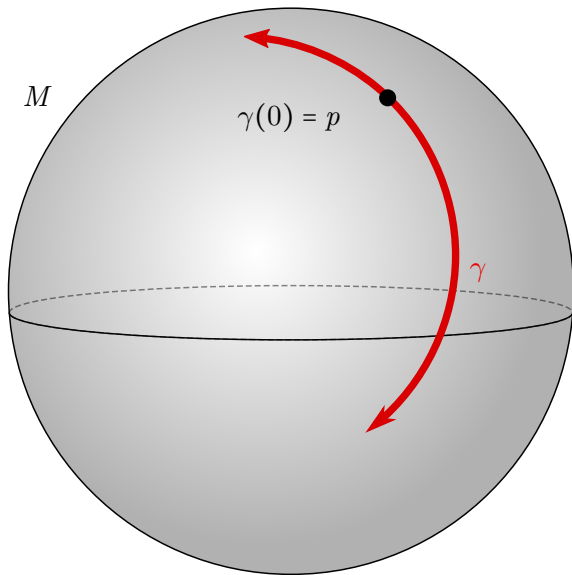


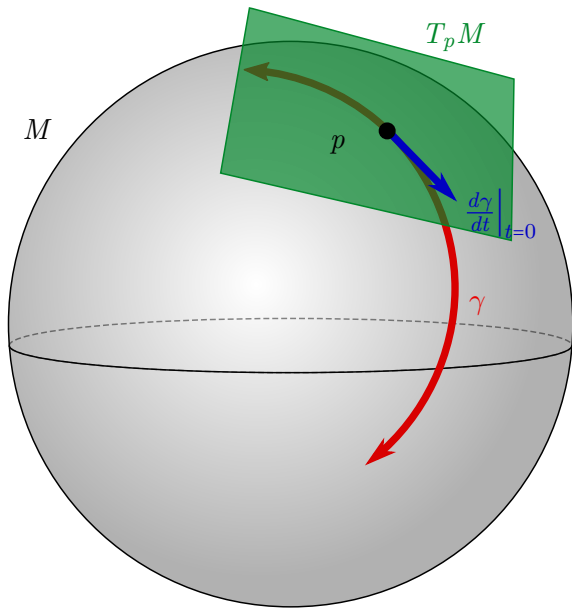


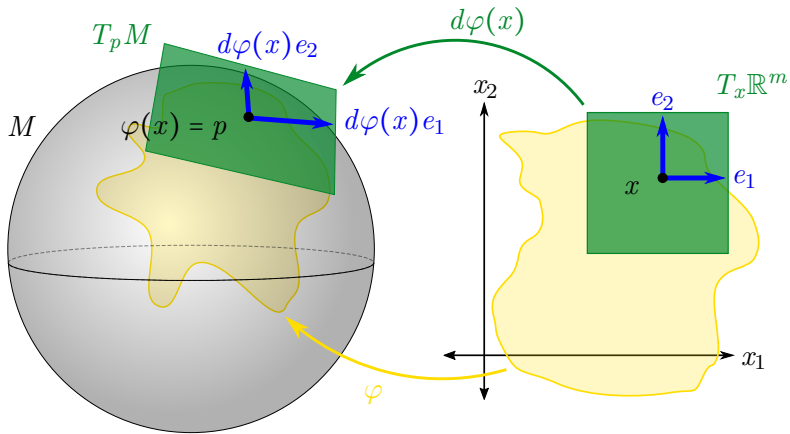


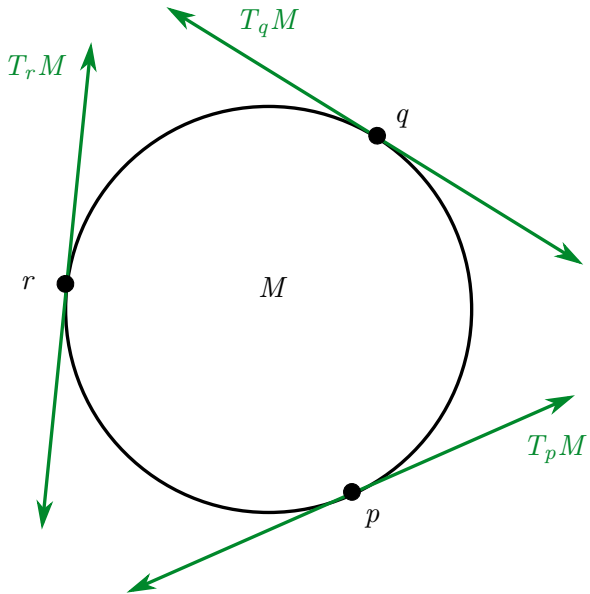
## Subsection 2

### **Vector Fields**

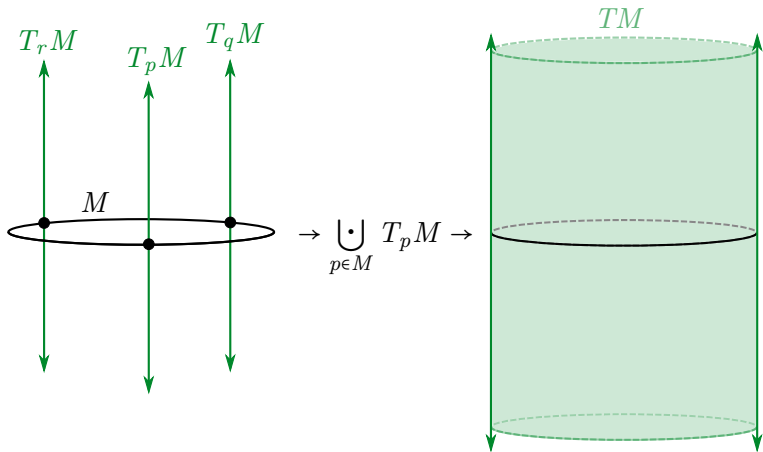


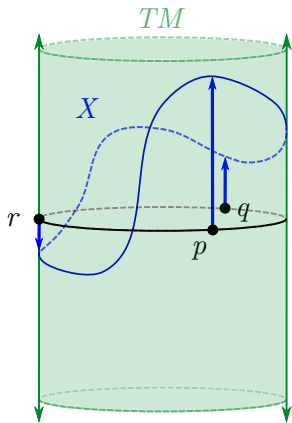
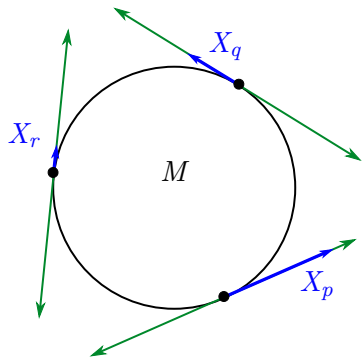








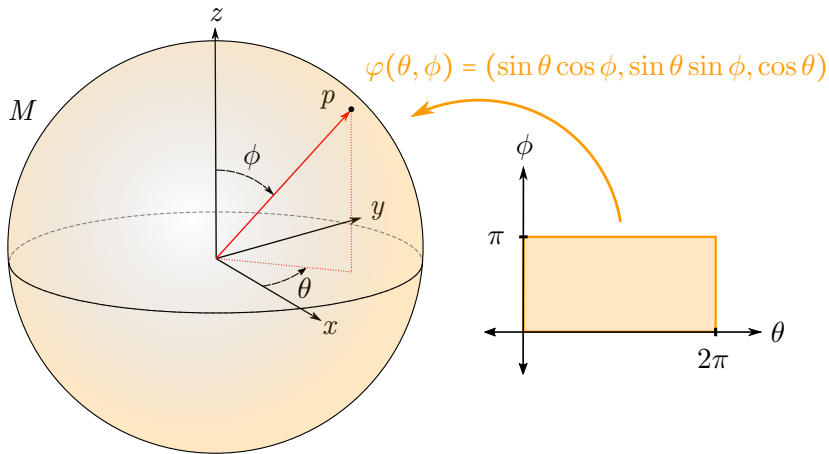


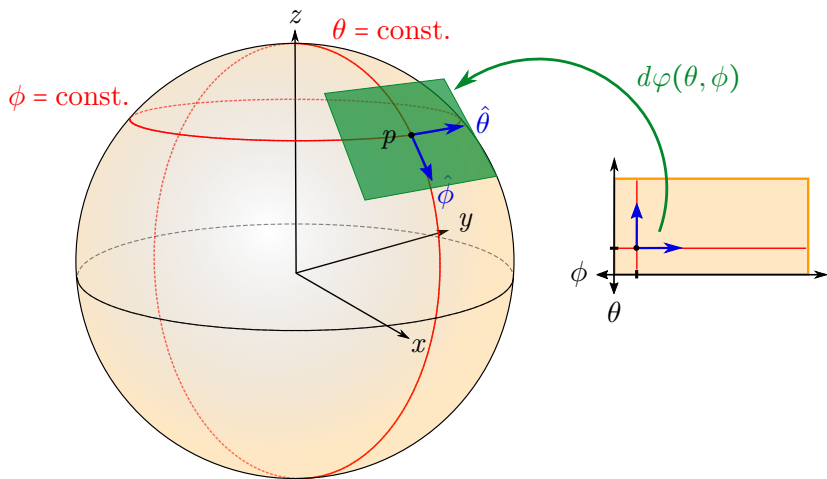


## Subsection 3

### **Example**

Example of charts and maps for sphere as well as a vector field. Do spherical coordinates





## Section 4

# Riemannian Geometry

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- Build an inner product on the tangent space  $T_pM$ ;
- Have the inner product vary smoothly as we vary the point  $p$ ;
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- Extract geometrical and analytical qualities of the underlying manifold  $M$ .

## Subsection 1

### **Riemannian Metric**

$$g_{ij}(x) = \varphi^*(x) e_i \cdot \varphi^*(x) e_j = (\partial_i \varphi) \cdot (\partial_j \varphi) = \sum_{k=1}^m \frac{\partial \varphi_k}{\partial x_i} \frac{\partial \varphi_k}{\partial x_j}$$

connection covariant derivative interpretation, covariant derivative in spherical coordinates, second covariant matrix and covariant laplacian from hessian? Compatability with riemannian metric picture. Show a geodesic in the coordinates between cities or something?

From minimization of length/energy. Both are good to mention. Geodesic equation

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0$$

equivalent to

$$\ddot{x}^l + \dot{x}^j \dot{x}^k \Gamma_{jk}^l = 0$$

which is saying that the only "acceleration" of the curve comes from the geometry it lies on. When flat space,  $\Gamma_{jk}^l = 0$  and we have  $\ddot{x} = 0$ .

## Section 5

# **Applications**



navier stokes on the Earth. Riemannian metric as a

conductivity?

## Section 6

# Conclusions