Multivector boundary value inverse problems

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Chapter 1

Introduction

1.1 Clifford and geometric algebras

The complex algebra $\mathbb{C}$ can be generalized in a handful of ways. Some of which can be found through the use of Clifford algebras and, more specifically, in geometric algebras. We define the more general Clifford algebras first and realize geometric algebras as particularly nice Clifford algebras with a quadratic form arising from an inner product. Elements of a geometric algebra are known as multivectors and these multivectors carry a wealth of geometric information in their algebraic structure. $\mathbb{C}$ itself can be realized as a special subalgebra of parabivectors in the geometric algebra on $\mathbb{R}^2$ with the Euclidean inner product and the quaternions $\mathbb{H}$ are realized as an analogous algebra on $\mathbb{R}^3$. In particular, both $\mathbb{C}$ and $\mathbb{H}$ arise as the 2- and 3-dimensional even Clifford groups Γ^+ respectively. reword this paragraph

First, we present a review of Clifford algebras and the relevant notions needed for this work. Those who feel they are familiar with both Clifford and geometric algebras may wish to skim through this subsection and visit section 1.1.5 to review the notation used throughout this manuscript.

Formally, we let (V, Q) be an n-dimensional vector space V over some field K with an arbitrary quadratic form Q. The tensor algebra is given by

$$\mathcal{T}(V) \coloneqq \bigoplus_{j=0}^{\infty} V^{\otimes j} = K \bigoplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots , \tag{1.1}$$

where the elements (tensors) inherit a multiplication $\otimes$ (the tensor product). From the tensor algebra $\mathcal{T}(V)$, we can quotient by the ideal generated by $\boldsymbol{v} \otimes \boldsymbol{v} - Q(\boldsymbol{v})$ to create a new algebra.

Definition 1.1.1. The *Clifford algebra* $C\ell(V, Q)$ is the quotient algebra

$$C\ell(V, Q) = \mathcal{T}(V) \ / \ \langle \boldsymbol{v} \otimes \boldsymbol{v} - Q(\boldsymbol{v}) \rangle. \tag{1.2}$$

To see how the tensor product descends to the quotient, we let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ be an arbitrary basis for V, then we can consider the tensor product of basis elements $\boldsymbol{v}_i \otimes \boldsymbol{v}_j$ which induces a product in the quotient $C\ell(V,Q)$ which we refer to as the *Clifford multiplication*. In this basis, we write this product as concatenation $\boldsymbol{v_i v_j}$ and define the multiplication by

$$\boldsymbol{v}_i\boldsymbol{v}_j = \begin{cases} Q(\boldsymbol{v}_i) & \text{if } i=j, \\ \boldsymbol{v}_i \wedge \boldsymbol{v}_j & \text{if } i \neq j, \end{cases} \tag{1.3}$$

where $\wedge$ is the typical exterior product satisfying $\boldsymbol{v}\wedge\boldsymbol{w} = -\boldsymbol{w}\wedge\boldsymbol{v}$ for all $\boldsymbol{v},\boldsymbol{w}\in V$. As a consequence, the exterior algebra $\bigwedge(V)$ can be realized as a subalgebra of any Clifford algebra over V or as a Clifford algebra with a trivial quadratic form $Q=0$.

In the case where V has a (pseudo) inner product g, we can induce a quadratic form Q by $Q(\boldsymbol{v}) = g(\boldsymbol{v},\boldsymbol{v})$ and give rise to a special type of Clifford algebra which motivates the following definition.

Definition 1.1.2. Let V be a vector space with an (pseudo) inner product $g(\cdot,\cdot)$. Then taking $Q(\cdot) = g(\cdot,\cdot)$, the Clifford algebra $C\ell(V,Q)$ is called a *geometric algebra*.

In general, we put $\mathcal{G}$ and assume the inner product and vector space will be arbitrary, given alongside, or will be clear from context. For example, when $V=\mathbb{R}^n$ we have the standard orthonormal basis $\boldsymbol{e}_1,\dots,\boldsymbol{e}_n$ which allows us to neatly define the quadratic form Q from the Euclidean inner product which has coefficients δ_{ij} with respect to this basis. Since we frequently utilize this geometric algebra, we put $\mathcal{G}_n \coloneqq C\ell(\mathbb{R}^n, |\cdot|)$ to simplify notation.

Geometric algebras are an old and widely studied topic. For more information, see the classical text [6] or the more modern text [4] which also provides a wide range of applications to physics problems. Both these sources include much of the other necessary preliminaries I cover in the remainder of this section. Finally, the paper [3] proves many of the useful identities and notation used throughout this paper.

1.1.1 Multivectors and grading

fix all vector indices to be not bold

Note that $C\ell(V,Q)$ is a $\mathbb{Z}$-graded algebra with elements of grade-0 up to elements of grade-n. We refer to grade-0 elements as scalars, grade-1 elements as vectors, grade-2 elements as *bivectors*, grade-r elements as *r-vectors*, and grade-n elements as *pseudoscalars*. For example, the pseudoscalar $\boldsymbol{\mu} = \boldsymbol{v}_1\wedge\boldsymbol{v}_2\wedge\cdots\wedge\boldsymbol{v}_n$ is an n-vector we will frequently return to. We denote the space of r-vectors by $C\ell(V,Q)^r$. For each grade there is a basis of $\binom{n}{r}$ *r-blades* which are r-vectors of the form

$$\boldsymbol{A_r} = \bigwedge_{j=1}^{r} \boldsymbol{v}_j, \text{ for linearly independent } \boldsymbol{v}_j \in V, \tag{1.4}$$

and we use a boldface of both the character and its subscript to specify that a r-vector is a r-blade and we note that vectors (since they are 1-blades) will not use this subscript. Instead, a vector $\boldsymbol{v}$ may use a non-boldfaced subscript to reference an index. Briefly, take for example the case where $\dim(V)=3$, then there are $\binom{3}{2}=3$ 2-blades that form a basis for the bivectors and one particular choice of a bivector basis would be the following list of 2-blades

$$\boldsymbol{B}_{12} = \boldsymbol{v}_1\wedge\boldsymbol{v}_2, \quad \boldsymbol{B}_{13} = \boldsymbol{v}_1\wedge\boldsymbol{v}_3, \quad \boldsymbol{B}_{23} = \boldsymbol{v}_2\wedge\boldsymbol{v}_3. \tag{1.5}$$

We will repeatedly use the notation $\boldsymbol{B}_{ij} \coloneqq \boldsymbol{v}_i \wedge \boldsymbol{v}_j$ and the underlying basis will be clear from context. We refer to an $n-1$-vector as a *pseudovector* and it should be noted that every $n-1$-vector is a blade (see section 1.1.3). In other literature, some will refer to a r-blade as a *simple* or a *decomposable* r-vector. citations

In general, an element $A \in C\ell(V,Q)$ is written as a linear combination of basis elements of all possible grades and we refer to A as a *multivector*. To extract the grade-r components of A, we use the *grade projection* for which we have the notation

$$\langle A \rangle_r \in C\ell(V,Q)^r \tag{1.6}$$

to denote the grade-r components of the multivector A (i.e., $\langle A \rangle_r \in C\ell(V,Q)^r$). Any multivector A can then be given by

$$A = \sum_{r=0}^{n} \langle A \rangle_r \tag{1.7}$$

which shows the decomposition via the $\mathbb{Z}$-grading

$$C\ell(V,Q) = \bigoplus_{j=0}^{n} C\ell(V,Q)^j. \tag{1.8}$$

If A contains only components of a single grade, then we say that A is *homogeneous* and if the components are grade-r we put A_r and refer to A_r as a *homogeneous r-vector* or simply an *r-vector*. For example, when we refer to vectors we realize them as 1-vectors and likewise we realize bivectors as 2-vectors. Also of interest will be the elements in

$$C\ell(V,Q)^{0+2} = C\ell(V,Q) \oplus C\ell(V,Q)^2 \tag{1.9}$$

which we refer to as *parabivectors*.

The Clifford multiplication of vectors defined in 1.3 can be extended to multiplication of vectors with homogeneous r-vectors. In particular, given a vector $\boldsymbol{v} \in C\ell(V,Q)$ and a homogeneous r-vector $A_r \in C\ell(V,Q)$, we have

$$\boldsymbol{v}A_r = \langle \boldsymbol{v}A_r \rangle_{r-1} + \langle \boldsymbol{v}A_r \rangle_{r+1}, \tag{1.10}$$

which decomposes the multiplication into a grade lowering *interior product* and a grade raising *exterior product*. This allows us to extend the Clifford multiplication further. Given an s-vector B_s, we have

$$A_rB_s = \langle A_rB_s \rangle_{|r-s|} + \langle A_rB_s \rangle_{|r-s|+2} + \cdots + \langle A_rB_s \rangle_{r+s}. \tag{1.11}$$

This rule for multiplication then allows for the multiplication of two general multivectors in $C\ell(V,Q)$. For this multiplication, specific grades of the product are worth noting.

$$A_r \cdot B_s \coloneqq \langle A_rB_s \rangle_{|r-s|} \tag{1.12}$$

$$A_r \wedge B_s \coloneqq \langle A_rB_s \rangle_{r+s} \tag{1.13}$$

$$A_r \rfloor B_s \coloneqq \langle A_rB_s \rangle_{s-r} \tag{1.14}$$

$$A_r \lfloor B_s := \langle A_r B_s \rangle_{r-s}. \tag{1.15}$$

Finally, we have a special product for bivectors called the *commutator product* given by

$$A_2 \times B_2 := \langle A_2 B_2 \rangle_2 \equiv \frac{1}{2}(A_2 B_2 - B_2 A_2). \tag{1.16}$$

These products are particularly emphasized as many helpful identities used in this paper are phrased using these notions. Taking eqs. (1.10), (1.13) and (1.14) into mind, we see that the grade lowering interior product can be written as

$$\langle \boldsymbol{v} A_r \rangle_{r-1} \equiv \boldsymbol{v} \rfloor A_r \equiv \boldsymbol{v} \cdot A_r \tag{1.17}$$

and the grade raising exterior product can be written as

$$\langle \boldsymbol{v} A_r \rangle_{r+1} \equiv \boldsymbol{v} \wedge A_r. \tag{1.18}$$

Finally, to suppress needless additional parentheses later on, we assert that the above products take precedence over the geometrical product in order of operation. For example, for multivectors A, B, and C, we must take

$$A \cdot BC \equiv (A \cdot B)C, \tag{1.19}$$

and extend this to the other products defined in eqs. (1.13) to (1.16) as well.

As discussed, $C\ell(V,Q)$ is naturally a $\mathbb{Z}$-graded algebra but we also find that it carries a $\mathbb{Z}/2\mathbb{Z}$-grading as well. Some would then refer to $C\ell(V,Q)$ as an *superalgebra* . This additional grading can be realized by sorting r-vectors in $C\ell(V,Q)$ into the sets where r is even or odd. We say a r-vector is *even* (resp. *odd*) if r is even (resp. odd) and in general if a multivector A is a sum of only even (resp. odd) grade elements we also refer to A as even (resp. odd). Taking note of the multiplication defined in 1.11, one can see that the multiplication of even multivectors with another even multivectors outputs an even multivector and that motivates the following.

Definition 1.1.3. The *even subalgebra* $C\ell(V,Q)^+ \subset C\ell(V,Q)$ is the subalgebra of even grade multivectors

$$C\ell(V,Q)^+ := C\ell(V,Q)^0 \oplus C\ell(V,Q)^2 \oplus C\ell(V,Q)^4 \oplus \cdots . \tag{1.20}$$

The even subalgebra is an extremely important entity that arises throughout physics due to its encapsulation of spinors. We touch on this next.

1.1.2 Multivector operations and the Clifford and spin groups

For the remainder of this paper, let us focus solely on geometric algebras $\mathcal{G}$. Given access to an (pseudo) inner product we have a natural isomorphism between V and V^* by the Riesz representation. Namely, given an arbitrary basis $\boldsymbol{v}_i$ for V there exists the corresponding dual basis f_i for V^* such that $f_i(\boldsymbol{v}_j) = \delta_{ij}$. In geometric algebra, this notion is somewhat superfluous as we can realize the dual basis inside V itself in the following manner. Note that there is a unique map $\sharp\colon V^* \to V$ for which $f \mapsto \boldsymbol{f}^\sharp$ such that

$$\boldsymbol{f_i^\sharp} \cdot \boldsymbol{v}_j = \delta_{ij}. \tag{1.21}$$

Hence, if we simply put $\boldsymbol{v}^i \coloneqq \boldsymbol{f^\sharp}_i$ we can note that $\boldsymbol{v}^i$ is simply a vector in the geometric algebra.

Definition 1.1.4. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ be an arbitrary basis of V generating $\mathcal{G}$. Then we have the *reciprocal basis* $\boldsymbol{v}^1, \boldsymbol{v}^2, \dots, \boldsymbol{v}^n$ satisfying

$$\boldsymbol{v}^i \cdot \boldsymbol{v}_j = \delta^i_j, \tag{1.22}$$

and we refer to each $\boldsymbol{v}^i$ as a *reciprocal vector*.

In terms of the inner product g, we have that the coefficients are given by $g_{ij} = \boldsymbol{v}_i \cdot \boldsymbol{v}_j$ and thus we have an explicit definition for the reciprocal vectors by putting $\boldsymbol{v}^i = g^{ij}\boldsymbol{v}_j$ where g^{ij} is the coefficients to the matrix inverse $(g_{ij})^{-1}$ and we assume the Einstein summation convention.

The inverse to this isomorphism is $\flat\colon V \to V^*$ which is given by $\boldsymbol{v} \mapsto v^\flat$ satisfying

$$v_i^\flat(\boldsymbol{v}_j) = \delta_{ij}. \tag{1.23}$$

Given these identifications, there is no need to distinguish between the vector space V and its dual V^* as it suffices to consider V itself with reciprocal vectors $\boldsymbol{v^i}$ with the application of the scalar product. For reference, the maps $\sharp$ and $\flat$ are the *musical isomorphisms* . sources

For a geometric algebra with a positive definite inner product, all blades have an inverse and hence form a group. With a pseudo inner product, the invertible elements are not quite as broad. To this end, we can construct a group of all invertible elements referred to as the *Clifford group* $\Gamma(\mathcal{G})$ for an arbitrary geometric algebra $\mathcal{G}$ by give an example later

$$\Gamma(\mathcal{G}) \coloneqq \left\{ \prod_{j=1}^k \boldsymbol{v}_j \mid k \in \mathbb{Z}^+,\ \forall j\ : 1 \leq j \leq k\ :\ \boldsymbol{v}_i \in V \text{ such that } g(\boldsymbol{v}_i, \boldsymbol{v}_i) \neq 0 \right\}. \tag{1.24}$$

We refer to elements of the Clifford group as *Clifford multivectors*. Note that Clifford multivectors are not necessarily blades since the product used in the construction is not the exterior product $\wedge$. For any Clifford multivectors $A = \boldsymbol{v}_1 \cdots \boldsymbol{v}_k$ in the group Γ, we have that multiplicative inverse A^{-1} is given by $A^{-1} = \boldsymbol{v}^k \dots \boldsymbol{v}^1$ as we can see that $A^{-1}A = AA^{-1} = 1$ by construction. Another note is that all scalars, vectors, pseudovectors, and pseudoscalars are always in the Clifford group and have multiplicative inverses. The inverse of a vector $\boldsymbol{v}$ is given by $\frac{\boldsymbol{v}}{\boldsymbol{v}\cdot\boldsymbol{v}}$. The form of the inverse motivates the utility of the *reverse* operator $\dagger$ defined so that $A^\dagger = \boldsymbol{v}_k \cdots \boldsymbol{v}_1$. For a r-blade A_r, the reverse also satisfies the relationship

$$A_r^\dagger = (-1)^{r(r-1)/2} A_r. \tag{1.25}$$

One can see that the multiplicative inverse of an element of the Clifford group A is the reverse of the corresponding product of reciprocal vectors since $A_r^{-1} = (\boldsymbol{v}^1 \cdots \boldsymbol{v}^k)^\dagger$. When we take $V=\mathbb{R}^n$ with the Euclidean inner product, we can note that elements $s \in \Gamma^+(\mathcal{G}_n)$ act as rotations on multivectors $A \in \mathcal{G}_n$ through a conjugate action

$$A \mapsto sAs^{-1}. \tag{1.26}$$

In fact, all nonzero vectors $\boldsymbol{v} \in \Gamma(\mathcal{G}_n)$ define a reflection in the hyperplane perpendicular to $\boldsymbol{v}$ via the same conjugation action above. This allows one can realize that all rotations are even products of reflections.

Following these realizations, one can see that the Clifford group $\Gamma(\mathcal{G}_n)$ contains important subgroups such as the orthogonal and special orthogonal groups as quotients

$$\mathrm{O}(n) \cong \Gamma(\mathcal{G}_n)/(\mathbb{R}\setminus 0) \qquad \text{and} \qquad \mathrm{SO}(n) \cong \Gamma^+(\mathcal{G}_n)/(\mathbb{R}\setminus 0), \tag{1.27}$$

where $\mathbb{R}\setminus 0$ is the multiplicative group of real numbers. This motivates the following definition.

Definition 1.1.5. The *Clifford norm* $\|\cdot\|$ for $s \in \Gamma(\mathcal{G})$ is given by

$$\|s\|^2 \coloneqq ss^\dagger. \tag{1.28}$$

Note that when the inner product is positive definite the Clifford norm is indeed a norm but can fail to be a norm in spaces with mixed signature (see eq. (1.60)). Also, note that for vectors the Clifford norm corresponds with the norm induced from the inner product in that with a vector $\boldsymbol{v}$ we have $\|\boldsymbol{v}\| = \boldsymbol{v}\boldsymbol{v}^\dagger = \boldsymbol{v}\cdot\boldsymbol{v}$. We also give the name *unit* to r-blades $\boldsymbol{A_r}$ with unit Clifford norm $1=\|\boldsymbol{A_r}\|$. Finally, this allows us to arrive at a definition for the classical pin and spin groups.

source

Definition 1.1.6. The *pin* and *spin* groups $\mathrm{Pin}(V)$ and $\mathrm{Spin}(V)$ are defined to be

$$\mathrm{Pin}(V) \coloneqq \{s \in \Gamma(\mathcal{G}) \mid \|s\| = 1\}. \tag{1.29a}$$
$$\mathrm{Spin}(V) \coloneqq \{s \in \Gamma^+(\mathcal{G}) \mid \|s\| = 1\}. \tag{1.29b}$$

Our focus will be the case where we take $\mathcal{G}=\mathcal{G}_n$ for which we put $\mathrm{Spin}(n)$, but these statements can often be more broadly generalized. Moreover, we can realize this group as a quotient of the Clifford group $\Gamma(\mathcal{G}_n)$ by

$$\mathrm{Spin}(n) \cong \Gamma^+(\mathcal{G}_n)/\mathbb{R}_+, \tag{1.30}$$

where $\mathbb{R}_+$ is the multiplicative group of positive real numbers. The spin group $\mathrm{Spin}(V)$ is a Lie group usually derived via a short exact sequence of groups

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}(V) \to \mathrm{SO}(V) \to 1. \tag{1.31}$$

Here, we have given a more concrete realization of the spin group as special elements inside a geometric algebra. The Lie algebra of the spin group is denoted by $\mathfrak{spin}(V)$ and $\mathfrak{spin}(n)$ when referencing $\mathrm{Spin}(n)$. This algebra typically characterized as the tangent space of $\mathrm{Spin}(V)$ at the identity. However, through this approach, we realize that $\mathfrak{spin}(V)$ is isomorphic to the algebra of bivectors with the antisymmetric product $\times$. Then, for any bivector B, we can

provide a citation.

generate an element in the spin group given via the exponential

$$e^B = \sum_{j=0}^{\infty} \frac{B^n}{n!}. \tag{1.32}$$

Fundamentally, the even subalgebra $\mathcal{G}^+$ is invariant under the action of $\mathrm{Spin}(V)$ since all elements in both sets are of even grade. This definition follows.

Definition 1.1.7. Let $\mathcal{G}$ be a geometric algebra with an inner product of arbitrary signature, then we define a *spinor* to be an element of $\mathcal{G}^+$.

Morally, this definition is telling us $\psi \in \mathcal{G}^+$ is an element that transforms under a left action of an element of $\mathrm{Spin}(V)$ to produce another spinor which leaves us with a convenient definition in that a spinor is simply an even multivector. For more on the topic, see [7].

1.1.3 Pseudoscalars and duality

Pseudoscalars are a deeply useful aspect of geometric algebra and we will now cover some of their utility. First and foremost, these pseudoscalars grant us a means of determining volumes. This will be a necessary notion in order to define integration in section 1.2.4.

Definition 1.1.8. Let $\mathcal{G}$ be a geometric algebra, then the *volume element* in the arbitrary basis $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ is

$$\boldsymbol{\mu} = \boldsymbol{v}_1 \wedge \boldsymbol{v}_2 \wedge \cdots \wedge \boldsymbol{v}_n. \tag{1.33}$$

It is worth noting that all volume elements and pseudoscalars are invertible in any geometric algebra.

We also want to note that the volume element here fits our intuition and indeed we find

$$\|\boldsymbol{\mu}\| = \sqrt{\det(g)}. \tag{1.34}$$

Since pseudoscalars are generated by a single element (recall there are $\binom{n}{n}$ independent grade-n elements), we should realize that the volume element is simply a scalar copy of a pseudoscalar that is unital.

Definition 1.1.9. Let $\boldsymbol{\mu}$ be the volume element, then we have the *unit pseudoscalar*

$$\boldsymbol{I} \coloneqq \frac{1}{\|\boldsymbol{\mu}\|}\boldsymbol{\mu}. \tag{1.35}$$

As is clear by the definition above, we must have that

$$\|\boldsymbol{I}\| = 1. \tag{1.36}$$

The unit pseudoscalar satisfies a useful relationship when swapping the left for right multiplication with an r-vector by

$$\boldsymbol{I} A_r = (-1)^{r(n-1)} A_r \boldsymbol{I}. \tag{1.37}$$

Thus, $\boldsymbol{I}$ commutes with the even subalgebra, and anticommutes with the odd subalgebra. We can also see that the inverse for unit pseudoscalar is $\boldsymbol{I}^{-1} = \boldsymbol{I}^{\dagger}$ which is an identification that we will often use. Formulas throughout are usually given in their most general context and substitution is done only when working with specialized algebras.

Note that for a homogeneous r-rector A_r we have that $A_r^{\perp}$ is an $n-r$-vector. Indeed, if we take an invertible r-blade $\boldsymbol{A_r}$, then we can find the $\boldsymbol{A_r}$*-subspace dual* of a multivector B by

$$B \rfloor \boldsymbol{A_r}^{-1}.$$

The notions of duality here give us geometrical insight. Taking an s-blade $\boldsymbol{B_s}$ we can note:

- If $s > r$, the $\boldsymbol{A_r}$-subspace dual of $\boldsymbol{B_s}$ vanishes.
- If $s = r$, the $\boldsymbol{A_r}$-subspace dual of $\boldsymbol{B_s}$ is a scalar and is zero if $\boldsymbol{B_s}$ contains a vector orthogonal to $\boldsymbol{A_r}$.
- If $s < r$, the $\boldsymbol{A_r}$-subspace dual of $\boldsymbol{B_s}$ represents the orthogonal complement of the subspace corresponding to $\boldsymbol{B_s}$ in the subspace corresponding to $\boldsymbol{A_r}$.

Since the pseudoscalar is a blade representing the entire vector space, this allows one to create dual elements within the entire vector space.

Definition 1.1.10. Given a multivector B, we define the *dual* of B to be

$$B^{\perp} \coloneqq B \rfloor \boldsymbol{I}^{-1} \equiv B\boldsymbol{I}^{-1}. \tag{1.38}$$

The dual allows one to exchange interior and exterior products in the following way.

$$(A \wedge B)^{\perp} = A \rfloor B^{\perp} \tag{1.39}$$

$$(A \rfloor B)^{\perp} = A \wedge B^{\perp} \tag{1.40}$$

This shows the natural duality between the inner and exterior products and their interpretations as subspace operations. The duality extends further to provide an isomorphism between the spaces of r-vectors and $n-r$-vectors since for any r-vector A_r we have $A_r^{\perp}$ is an $n-r$-vector. It is under this isomorphism one can realize that all pseudovectors are $n-1$-blades. Furthermore, for multivectors A and B,

$$(AB)^{\perp} = AB^{\perp} \tag{1.41}$$

For those familiar with the Hodge star operator, $\star$, this should feel familiar. This is discussed in **??**.

Remark 1.1.1. If we consider $\mathcal{G}_3$, we can realize the cross product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ by

$$\boldsymbol{u} \boldsymbol{\times} \boldsymbol{v} \coloneqq (\boldsymbol{u} \wedge \boldsymbol{v})^{\perp} \equiv (\boldsymbol{u}^{\perp}) \times (\boldsymbol{v}^{\perp}), \tag{1.42}$$

where we use the bold notation for $\boldsymbol{\times}$ to distinguish between the bivector commutator product $\times$ defined in eq. (1.16). The special fact of $\mathcal{G}_3$ that is abused in a standard multivariate calculus course is that vectors and bivectors are dual to one another. In fact, the first equality is exactly this pedagogical reasoning; the cross product returns a vector perpendicular to the subspace spanned by the two input vectors and is zero when the two inputs are linearly dependent. One can also note that the vector $\boldsymbol{w} = \boldsymbol{u} \boldsymbol{\times} \boldsymbol{v}$ is sometimes refered to as axial and in other cases the pseudovector $\boldsymbol{u} \wedge \boldsymbol{v}$ is referred to as axial. The similar product notation of $\times$ and $\boldsymbol{\times}$ now becomes transparent.

1.1.4 Blades and subspaces

Each invertible unit r-blade $\boldsymbol{U_r}$ ($\|\boldsymbol{U_r}\| = 1$) corresponds to a r-dimensional subspace and can be identified with a point in the Grassmannian of r-dimensional subspaces in an n-dimensional vector space, $\mathrm{Gr}(r,n)$. We will often allude to this identification directly by referring to a subspace via a reference to a unit blade, e.g., the subspace $\boldsymbol{U_r}$. Extending the dual to act on the unit r-blades that make up $\mathrm{Gr}(r,n)$, one realizes that $\mathrm{Gr}(r,n)^{\perp} = \mathrm{Gr}(n-r,n)$ shows the spaces are in bijection. Moreover, given a subspace $\boldsymbol{U_r}$, we can complete the vector space by

$$\boldsymbol{U_r} \wedge \boldsymbol{U_r}^{\perp} = \boldsymbol{I}. \tag{1.43}$$

We can also note that any invertible blade $\boldsymbol{A_r}$ is simply a scaling of some unit blade so that $\boldsymbol{A_r} = \alpha \boldsymbol{U_r}$. This interpretation also proves to be a wonderfully geometrical perspective on the products defined in eqs. (1.12) to (1.15). For example, we see that there are a handful of reasons to adopt the additional multiplication symbols $\rfloor$ and $\lfloor$ [3].

- The products $\rfloor$ and $\lfloor$ allow us to avoid needing to pay special attention to the specific grade of each multivector in a product. The product $\cdot$ on A_r and B_s depends on k and s and as such given by either $\rfloor$ or $\lfloor$ but one must know k and s in order to define this product exactly.

- We gain geometrical insight on the structure of r-blades in terms of their corresponding subspaces. Let $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ be nonzero blades with $r,s \geq 1$ then

  - $\boldsymbol{A_r} \rfloor \boldsymbol{B_s} = 0$ iff $\boldsymbol{A_r}$ contains a nonzero vector orthogonal to $\boldsymbol{B_s}$.
  - If $r < s$ then if $\boldsymbol{A_r} \rfloor \boldsymbol{B_s} \neq 0$ then the result is a $s-r$-blade representing the orthogonal complement of $\boldsymbol{A_r}$ in $\boldsymbol{B_s}$.
  - If $\boldsymbol{A_r}$ is a subspace of $\boldsymbol{B_s}$ then $\boldsymbol{A_r}\boldsymbol{B_s} = \boldsymbol{A_r} \rfloor \boldsymbol{B_s}$.
  - If $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ are orthogonal, then $\boldsymbol{A_r}\boldsymbol{B_s} = \boldsymbol{A_r} \wedge \boldsymbol{B_s}$.

  We also have the equivalences

$$A_r \cdot B_s \equiv A_r \rfloor B_s \qquad \text{if } k \leq s \tag{1.44}$$

$$A_r \cdot B_s \equiv A_r \lfloor B_s \qquad \text{if } k \geq s. \tag{1.45}$$

For homogeneous r-vectors A_r and B_r, the products above simplify to

$$A_r \cdot B_r \equiv A_r \lfloor B_r \equiv A_r \rfloor B_r. \tag{1.46}$$

In fact, if we are given two r-blades $\boldsymbol{A_r} = \boldsymbol{a_1} \wedge \cdots \wedge \boldsymbol{a_r}$ and $\boldsymbol{B_r} = \boldsymbol{b_1} \wedge \cdots \wedge \boldsymbol{b_r}$ we have the

$$\boldsymbol{A_r} \cdot \boldsymbol{B_r}^{\dagger} = \det(\boldsymbol{a_i} \cdot \boldsymbol{b_j})_{i,j=1}^{r} = \boldsymbol{A_r}^{\dagger} \cdot \boldsymbol{B_r}^{\dagger}, \tag{1.47}$$

which is the typical extension of the inner product g to an inner product on $\bigwedge^r(V)$ through linearity.

Given the direct relationship between unit r-blades and r-dimensional subspaces we can also form a compact way of projecting multivectors into subspaces in a manner closely related to the subspace dual.

Definition 1.1.11. Given an multivector B, the *projection* onto the subspace $\boldsymbol{A_r}$ is

$$\mathrm{P}_{\boldsymbol{A_r}}(B) \coloneqq B \rfloor \boldsymbol{A_r} \boldsymbol{A_r}^{-1} \equiv (B \rfloor \boldsymbol{A_r}) \rfloor \boldsymbol{A_r}^{-1} \tag{1.48}$$

Following this definition, one can see that

$$\mathrm{P}_{\boldsymbol{A_r}}(B) \in \bigoplus_{j=0}^{r} \mathcal{G}^j = \mathcal{G}^{0+\cdots+r}, \tag{1.49}$$

since the subspace $\boldsymbol{A_r}$ is r-dimensional and moreover the operation preserves grades since

$$\mathrm{P}_{\boldsymbol{A_r}}(\langle B \rangle_j) \in \mathcal{G}^j. \tag{1.50}$$

For example, given vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ we retrieve the familiar statement

$$\mathrm{P}_{\boldsymbol{u}}(\boldsymbol{v}) = (\boldsymbol{v} \cdot \boldsymbol{u}) \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|^2}. \tag{1.51}$$

A dual notion exists as well; we can project onto the subspace perpendicular to $\boldsymbol{A_r}$.

Definition 1.1.12. Given a multivector B, the *rejection* from the subspace $\boldsymbol{A_r}$ is

$$\mathrm{R}_{\boldsymbol{A_r}}(B) \coloneqq B \wedge \boldsymbol{A_r} \boldsymbol{A_r}^{-1} \equiv (B \wedge \boldsymbol{A_r}) \lfloor \boldsymbol{A_r}^{-1}. \tag{1.52}$$

Note that this operation is also grade preserving. In the case we have a blade $\boldsymbol{C_k}$ with $k < r$ and $k < n - r$, we can note

$$\boldsymbol{C_k} = \mathrm{P}_{\boldsymbol{A_r}}(\boldsymbol{C_k}) + \mathrm{R}_{\boldsymbol{A_r}}(\boldsymbol{C_k}). \tag{1.53}$$

This provides us a way to revisit the geometric notions of the interior and exterior products. In particular, we note that

$$B \rfloor \boldsymbol{A_r} = \mathrm{P}_{\boldsymbol{A_r}}(B) \boldsymbol{A_r} \tag{1.54}$$

$$B \wedge \boldsymbol{A_r} = \mathrm{R}_{\boldsymbol{A_r}}(B) \boldsymbol{A_r}. \tag{1.55}$$

Both the notion of projection and rejection prove to be useful and behave nicely with the dual by

$$\mathrm{P}_{\boldsymbol{A_r}^{\perp}}(B) = \mathrm{R}_{\boldsymbol{A_r}}(B), \tag{1.56}$$

and

$$\mathrm{P}_{\boldsymbol{A_r}}(B^{\perp}) = \mathrm{R}_{\boldsymbol{A_r}}(B)^{\perp}. \tag{1.57}$$

Finally, the exterior product of orthogonal blades gives us a direct sum of subspaces in the following sense. Let $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ be orthogonal so that $\boldsymbol{A_r} \wedge \boldsymbol{B_s} = \boldsymbol{A_r}\boldsymbol{B_s}$, then we can note that if $k < r$ and $k < s$ we have

prove both of these statements?

$$\mathrm{P}_{\boldsymbol{A_r} \wedge \boldsymbol{B_s}}(\boldsymbol{C_k}) = \mathrm{P}_{\boldsymbol{A_r}}(\boldsymbol{C_k}) + \mathrm{P}_{\boldsymbol{B_s}}(\boldsymbol{C_k}). \tag{1.58}$$

Perhaps it is most enlightening for the reader to revisit eqs. (1.53) and (1.58) replacing $\boldsymbol{C_k}$ with a vector $\boldsymbol{v}$ since a vector will always prove to be a representative for a “small enough” subspace.

1.1.5 Motivating example

Rather than a sequence of multiple examples, it will prove to be far more illuminating to construct one large example for which most of the preliminaries to this point can be used in a meaningful way. As such, we shall not rule out the utility of geometric algebras with pseudo inner products. The classical example is the *spacetime algebra* defined by taking $V = \mathbb{R}^4$ with a vector basis $\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3$ satisfying

$$\boldsymbol{\gamma}_0 \cdot \boldsymbol{\gamma}_0 = -1 \tag{1.59a}$$
$$\boldsymbol{\gamma}_0 \cdot \boldsymbol{\gamma}_i = 0 \qquad i = 1,2,3 \tag{1.59b}$$
$$\boldsymbol{\gamma}_i \cdot \boldsymbol{\gamma}_j = \delta_{ij}, \qquad i,j = 1,2,3. \tag{1.59c}$$

We refer to $\boldsymbol{\gamma}_0$ as *temporal* since its square is negative and $\boldsymbol{\gamma}_i$ for $i = 1,2,3$ as *spatial* since its square is positive. For this basis, we can denote the matrix for this inner product $\eta = \operatorname{diag}(-+++)$ (often called the *Minkowski metric*) and define Q from η. Then, we have for a spacetime vector $\boldsymbol{v} = v_0\boldsymbol{\gamma}_0 + v_1\boldsymbol{\gamma}_1 + v_2\boldsymbol{\gamma}_2 + v_3\boldsymbol{\gamma}_3$,

$$\|\boldsymbol{v}\| = \boldsymbol{v} \cdot \boldsymbol{v} = -v_0^2 + \sum_{i=1}^{3} v_i^2. \tag{1.60}$$

Then we put $\mathcal{G}_{1,3}$ to represent the spacetime algebra and, in broader generality, we put $\mathcal{G}_{p,q}$ for a geometric algebra with p temporal vectors and q spatial vectors. The factor p will return in various different calculations. The reader may wish to, for example, revisit section 1.1.2 with $\mathcal{G}_{p,q}$ in mind in order to see a realization of the groups $\mathrm{SO}(p,q)$, $\mathrm{Spin}(p,q)$, and the spacetime spinors.

As the naming above suggests, the geometric algebra of Euclidean space, $\mathcal{G}_3$, should naturally inside of the spacetime algebra. Note that we have the *spatial pseudoscalar* $\boldsymbol{I}_S := \boldsymbol{\gamma}_1 \wedge \boldsymbol{\gamma}_2 \wedge \boldsymbol{\gamma}_3$, which, allowing for an extension of our notion of projection to the whole algebra, allows us to put

$$\mathrm{P}_{\boldsymbol{I}_S}(\mathcal{G}_{1,3}) \equiv \mathrm{R}_{\boldsymbol{\gamma}_0}(\mathcal{G}_{1,3}) = \mathcal{G}_3. \tag{1.61}$$

Perhaps one could refer to this mapping as the *static map* as we project only onto the spatial subspace and, via duality, reject the temporal subspace. It is also worth noting that this static map is not just producing an isomorphic copy of $\mathcal{G}_3$, but a a copy of $\mathcal{G}_3$ directly. Now, in $\mathcal{G}_3$, we can specify an arbitrary multivector A by

$$A = a_0 + a_1\boldsymbol{\gamma}_1 + a_2\boldsymbol{\gamma}_2 + a_3\boldsymbol{\gamma}_3 + a_{12}\boldsymbol{B}_{12} + a_{13}\boldsymbol{B}_{13} + a_{23}\boldsymbol{B}_{23} + a_{123}\boldsymbol{\gamma}_1 \wedge \boldsymbol{\gamma}_2 \wedge \boldsymbol{\gamma}_3, \tag{1.62}$$

and so the grade projections read

$$\langle A\rangle_0 = a_0 \tag{1.63a}$$
$$\langle A\rangle_1 = a_1\boldsymbol{\gamma}_1 + a_2\boldsymbol{\gamma}_2 + a_3\boldsymbol{\gamma}_3 \tag{1.63b}$$
$$\langle A\rangle_2 = a_{12}\boldsymbol{B}_{12} + a_{13}\boldsymbol{B}_{13} + a_{23}\boldsymbol{B}_{23} \tag{1.63c}$$
$$\langle A\rangle_3 = a_{123}\boldsymbol{\gamma}_1 \wedge \boldsymbol{\gamma}_2 \wedge \boldsymbol{\gamma}_3. \tag{1.63d}$$

Then, we can write a even multivector as

$$q = q_0 + q_{23}\boldsymbol{B}_{23} + q_{31}\boldsymbol{B}_{31} + q_{12}\boldsymbol{B}_{12}. \tag{1.64}$$

Note as well that

$$\boldsymbol{B}_{23}^2 = \boldsymbol{B}_{31}^2 = \boldsymbol{B}_{12}^2 = -1 \tag{1.65a}$$
$$\boldsymbol{B}_{23}\boldsymbol{B}_{31}\boldsymbol{B}_{12} = +1, \tag{1.65b}$$

which is typical for spatial bivectors. In this case, one may notice that this even subalgebra is extremely close to being a copy of the quaternion algebra $\mathbb{H}$. One can arrive at a representation of the quaternions by taking

$$\boldsymbol{i} \leftrightarrow \boldsymbol{B}_{23}, \quad \boldsymbol{j} \leftrightarrow -\boldsymbol{B}_{31} = \boldsymbol{B}_{13}, \quad \boldsymbol{k} \leftrightarrow \boldsymbol{B}_{12}, \tag{1.66}$$

and noting that we then have $\boldsymbol{ijk} = -1$ as well as $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1$. A more in depth explanation is provided in [4]. Thus, a purely we realize a quaternion as a parabivector q and a purely imaginary quaternion is simply the grade-2 portion of the parabivector q. We also realize $\mathbb{H}$ as scalar copies of elements of $\mathrm{Spin}(3) \cong \mathrm{Sp}(1)$. That is to say that $\mathbb{H} \cong \mathbb{R} \times \mathrm{Spin}(3)$. Indeed, since elements of $\mathcal{G}_3^+$ are simply parabivectors, the parabivectors admit a natural spin representation.

But we are not done here, and we can project down one dimension further by

$$\mathrm{P}_{\boldsymbol{\gamma}_1 \wedge \boldsymbol{\gamma}_2}(\mathcal{G}_3) = \mathcal{G}_2. \tag{1.67}$$

To see this in action, we let $\boldsymbol{v} = v_1\boldsymbol{\gamma}_1 + v_2\boldsymbol{\gamma}_2 + v_3\boldsymbol{\gamma}_3$

$$\mathrm{P}_{\boldsymbol{\gamma}_1 \wedge \boldsymbol{\gamma}_2} = \mathrm{P}_{\boldsymbol{B}_{12}}(\boldsymbol{v}) = (v_1\boldsymbol{\gamma}_1 + v_2\boldsymbol{\gamma}_2 + v_3\boldsymbol{\gamma}_3) \rfloor \boldsymbol{B}_{12}\boldsymbol{B}_{12}^{-1} \tag{1.68a}$$
$$= (v_1\boldsymbol{\gamma}_2 - v_2\boldsymbol{\gamma}_1)\boldsymbol{B}_{12}^{-1} \tag{1.68b}$$
$$= v_1\boldsymbol{\gamma}_1 + v_2\boldsymbol{\gamma}_2. \tag{1.68c}$$

Then, arbitrary multivectors A and B can be specified by

$$A = a_0 + a_1\boldsymbol{\gamma}_1 + a_2\boldsymbol{\gamma}_2 + a_{12}\boldsymbol{B}_{12}, \qquad B = b_0 + b_1\boldsymbol{\gamma}_1 + b_2\boldsymbol{\gamma}_2 + b_{12}\boldsymbol{B}_{12}.$$

We can then take the product AB to yield

$$\langle AB \rangle_0 = a_0b_0 + a_1b_1 + a_2b_2 - a_{12}b_{12} \tag{1.69a}$$
$$\langle AB \rangle_1 = (a_0b_1 + a_1b_0 - a_2b_{12} + a_{12}b_2)\boldsymbol{\gamma}_1 + (a_0b_2 + a_2b_0 + a_1b_{12} - a_{12}b_1)\boldsymbol{\gamma}_2 \tag{1.69b}$$
$$\langle AB \rangle_2 = (a_1b_2 - a_2b_1)\boldsymbol{B}_{12}. \tag{1.69c}$$

Most notably, we see that $\boldsymbol{B}_{12}^2 = -1$ and this allows us to consider a parabivector

$$z = x + y\boldsymbol{B}_{12} \tag{1.70}$$

which is exactly a representation of the complex number $\zeta = x + iy$ in $\mathcal{G}_2^{0+2} = \mathcal{G}_2^+$. Thus, the even subalgebra of this geometric algebra is indeed isomorphic to the complex numbers $\mathbb{C}$. Indeed, there is one unit 2-blade $\boldsymbol{B}_{12}$ in $\mathcal{G}_2$ to form the spin algebra $\mathfrak{spin}(2) \cong \mathbb{R}$ and as a consequence all unit norm elements in $\mathcal{G}_2^+$ can be written as

$$e^{\theta \boldsymbol{B}_{12}} = \sum_{n=0}^{\infty} \frac{\theta \boldsymbol{B}_{12}}{n!} = \cos(\theta) + \boldsymbol{B}_{12}\sin(\theta), \tag{1.71}$$

where $\theta \boldsymbol{B}_{12}$ is a general bivector in $\mathcal{G}_2$ when $\theta \in \mathbb{R}$ is arbitrary. Hence, we arrive at $\mathrm{Spin}(2) \cong \mathrm{U}(1)$. Any element in $\mathbb{C}$ is also a scaled version of an element of the spin group Spin(2). Hence, we can use a spin representation for an element in $\mathbb{C}$ via $z = re^{\theta \boldsymbol{B}_{12}} \in \mathbb{R} \times \mathrm{Spin}(2)$. This special case shows that parabivectors in $\mathcal{G}_2$ have a unique spin representation and they are spinors as well since the whole of the even subalgebra consists of parabivectors.

But, the above work is not necessary special to the starting point of $\mathcal{G}_{1,3}$ or $\mathcal{G}_3$. In fact, if we take $\mathcal{G}_n$ for $n \geq 2$, then there are natural copies of $\mathbb{C}$ contained inside of $\mathcal{G}_n$. In particular, we have the isomorphism

$$\mathbb{C} \cong \{\lambda + \beta \boldsymbol{B} \mid \lambda, \beta \in \mathcal{G}_n^0,\ \boldsymbol{B} \in \mathrm{Gr}(2,n).\}, \tag{1.72}$$

which shows that complex numbers arise as parabivectors via the representation

$$\zeta = x + y\boldsymbol{B}, \tag{1.73}$$

since $\boldsymbol{B}^2 = -1$. Given the standard basis $\boldsymbol{e_1}, \dots, \boldsymbol{e_n}$ we have copies of $\mathbb{C}$ for each of the $\binom{n}{2}$ unit bivectors $\boldsymbol{B}_{jk}$ with $k = 2, \dots, n$ and $j < k$.

Talk about bivectors, spinors, and rotors. Rotations and what not. Euler angles. Would all be good to put in here. Rotations in the complex plane.

1.2 Geometric manifolds

We want to generalize the setting of geometric algebra to include a smooth structure. For instance, we can consider a manifold M (likely with boundary ∂M) with a metric structure and develop a geometric algebra at each tangent space to this manifold (e.g., following [8]). We refer to this as the *geometric tangent space* and put $C\ell(T_xM, g_x)$.

Definition 1.2.1. A manifold M with a pseudo-Riemannian metric g is a *geometric manifold* if each tangent space is a geometric tangent space.

On geometric manifolds we will be able to attach multivector fields and compute their derivatives as well as integrate. This leads us to the realm of geometric calculus and Cifford analysis. Geometric calculus is intimately related to both the vector calculus in $\mathbb{R}^3$ and differential forms. It has the added advantage of notational convenience and clarity as we have seen with geometric algebra and its subspace operations. In the beginning of section 1.1 we realize as well that the exterior algebra is contained inside any Clifford algebra and, to this end, geometric calculus will contain the calculus of differential forms.

Forms are a useful language for proving general theorems about boundary value problems [9], and so we will retrieve all of these theorems for our own utility. Given that we have increased geometrical intuition on different graded elements of a geometric algebra, we can realize that we can work with multivector equivalents of forms instead of concentrating on forms of a specific grade. For example, in we see that one can think of the electromagnetic field as a multivector consisting of elements of various degree as opposed to the usual field strength 2-form . In fact, under certain other restrictions such as those present in Ohmic materials, we find there are parabivectors that fall into the kernel of a Dirac-type operator.

reference electromagnetic stuff later on

cite a typical electromagnetic paper

This Dirac-type operator, $\boldsymbol{\nabla}$, is the grade-1 derivative operator studied in Clifford analysis. Fundamentally, this operator generalizes the Wirtinger derivative for complex functions to multivectors and, as such, generalizes the notion of a $\mathbb{C}$-holomorphic function to that of a monogenic function (see). Happily, we even retain a Taylor series representation (see) for functions in the kernel of $\boldsymbol{\nabla}$ due to a generalized form of the Cauchy integral formula. This Cauchy integral formula has been applied elsewhere (see [2]). The Cauchy integral also acts as an isomorphism between smooth functions defined on the boundary ∂M of a manifold M.

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1.2.1 Multivector fields

In order to develop fields on a geometric manifold we must first create the relevant bundle structure. There is a natural bundle associated to a geometric manifold given by by gluing together each of the tangent geometric algebras. The *geometric algebra bundle* of a geometric manifold (M, g) is the space

$$\bigsqcup_{x \in M} C\ell(T_x M, g_x). \tag{1.74}$$

Given this bundle, the fields follow.

Definition 1.2.2. A *(smooth) multivector field* is a (C^∞-smooth) section of the geometric algebra bundle. We put $\mathcal{G}(M)$ as the *space of multivector fields on* M.

Note that the we will assume that all multivector fields are C^∞-smooth and drop this additional modifier when speaking of any type of multivector field. The above definition above is very general and we may not find ourselves working over arbitrary geometric manifolds. For example, we highlight a specific use case by letting M be a connected region of $\mathbb{R}^n$. For brevity, we will put $\mathcal{G}_n(M)$ to denote we are working over a region $M \subseteq \mathbb{R}^n$. In this case, the multivectors themselves are realized as constant multivector fields which allows us to say $\mathcal{G}_n \subset \mathcal{G}_n(M)$. This smooth setting simply makes the coefficients of the global basis blades given by C^∞ functions as opposed to $\mathbb{R}$ scalars. Hence, $\mathcal{G}_n(M)$ is simply the C^∞-module equivalent of $\mathcal{G}_n$.

Perhaps the C^∞-module structure obfuscates the point slightly, but the notion of a smooth section does not. One should think of the fields in $\mathcal{G}_n(M)$ as multivector valued functions on $M \subset \mathbb{R}^n$. Taking this identification allows for an extended toolbox at our disposal. In particular, points in M are uniquely identified with constant vector fields in $\mathcal{G}_n^1$ and one can consider endomorphisms living in $\mathcal{G}_n$ (acting on $\mathcal{G}_n^1$) as acting on the input of fields in $\mathcal{G}_n(M)$ as well (see remark 1.2.1). Thus, there is not only an algebraic structure on the fields themselves, but on the point in which the field is evaluated. This is perhaps the key insight on why authors developed the so-called vector manifolds widely used in the geometric algebra landscape. Fundamentally, this is true in all local coordinates for an arbitrary manifold M, but it is not a global phenomenon since not all manifolds admit everywhere nonzero constant vector fields.

Remark 1.2.1. If we consider a multivector field $f \in \mathcal{G}_n(\mathbb{R}^n)$. With $x \in \mathbb{R}^n$ being identified with the vector $\boldsymbol{x} \in \mathcal{G}_n^1$, we can safely put $f(\boldsymbol{x})$. One may be interested in the restriction of the input of f to a subspace $\boldsymbol{U_r}$ which yields $f(\mathrm{P}_{\boldsymbol{U_r}}(\boldsymbol{x}))$.

As noted throughout section 1.1, there are spaces of multivectors inside $\mathcal{G}$ of interest and each of these extends to their field counterpart. Construction of each is done pointwise and made global through the relevant bundle. Let us list the relevant spaces of fields.

- The *r-vector fields*,

$$\mathcal{G}^r(M) := \left\{ \text{smooth sections of } \bigsqcup_{x\in M} C\ell(T_xM, g_x)^r \right\}; \tag{1.75}$$

- The *spinor fields*,

$$\mathcal{G}^+(M) := \left\{ \text{smooth sections of } \bigsqcup_{x\in M} C\ell(T_xM, g_x)^+ \right\}; \tag{1.76}$$

- The *parabivector fields*,

$$\mathcal{G}^{0+2}(M) := \left\{ \text{smooth sections of } \bigsqcup_{x\in M} C\ell(T_xM, g_x)^{0+2} \right\}; \tag{1.77}$$

Our operations from section 1.1 must carry over. To that end, we simply define all the products seen in eqs. (1.12) to (1.15) to act pointwise in each geometric tangent space. Previously we referred to r-blades as special r-vectors. Thus, we realize an r-blade field $\boldsymbol{A_r} \in \mathcal{G}^r(M)$ assumes the same form of eq. (1.4) where the vectors $\boldsymbol{v}_j$ are to be understood as vector fields for which all $\boldsymbol{v}_j(x)$ are linearly independent in T_xM at the point x.

Given local coordinates x^i on M containing the point p, the tangent vectors in a neighborhood about p are induced from the coordinates by $\frac{\partial}{\partial x^i}$. However, this choice of basis may be canonical, but it is not arbitrary. Instead, at each point we can simply choose an arbitrary local vector basis $\boldsymbol{v}_i$ and let the components of the metric be given in this basis by $g_{ij(x)} = \boldsymbol{v}_i(x)\cdot\boldsymbol{v}_j(x)$. From here, we can suppress the pointwise notion and instead just put $g_{ij} = \boldsymbol{v}_i\cdot\boldsymbol{v}_j$ locally. This allows us to work notationally with bases in a global manner without any reference to coordinates, so long as we assume the understanding is clear – these vector bases do only exist locally. If explicit computations are to be carried out, one can just take the canonical basis so that $\boldsymbol{v}_i = \frac{\partial}{\partial x^i}$. Thus, locally we have the reciprocal basis $\boldsymbol{v}^i = g^{ij}\boldsymbol{v}_j$, the reverse $\dagger$, dual $\perp$, projection P, and rejection R that act on multivector fields pointwise in $C\ell(T_xM, g_x)$.

1.2.2 Geometric calculus

On M we have the unique torsion free Levi-Civita connection ∇ for which we can define the covariant derivative $\nabla_{\boldsymbol{u}}$ for a vector field $\boldsymbol{u}$. The covariant derivative is extended to act on multivector fields following [8]. We can note that $\nabla_{\boldsymbol{u}}$ is a grade preserving differential operator so that

$$\nabla_{\boldsymbol{u}}\langle A_r\rangle_r = \langle \nabla_{\boldsymbol{u}}\langle A_r\rangle_r\rangle_r, \tag{1.78}$$

and it is a dot-compatible and wedge-compatible operator since

$$\nabla_{\boldsymbol{u}}(A\cdot B) = (\nabla_{\boldsymbol{u}}A)\cdot B + A\cdot(\nabla_{\boldsymbol{u}}B) \tag{1.79}$$

$$\nabla_{\boldsymbol{u}}(A\wedge B) = (\nabla_{\boldsymbol{u}}A)\wedge B + A\wedge(\nabla_{\boldsymbol{u}}B) \tag{1.80}$$

Definition 1.2.3. Let $\boldsymbol{v}_i$ be an arbitrary basis, then the *gradient* (or *Dirac operator*) $\boldsymbol{\nabla}$ is defined by

$$\boldsymbol{\nabla} = \sum_i \boldsymbol{v}^i \nabla_{\boldsymbol{v}_i}. \tag{1.81}$$

The space of multivector fields $\mathcal{G}(M)$ along with $\boldsymbol{\nabla}$ is usually referred to as geometric calculus. One should note that $\boldsymbol{\nabla}$ is acts as a grade-1 element. Thus, the gradient splits into two operators,

$$\boldsymbol{\nabla}\rfloor \colon \mathcal{G}_n^r(M) \to \mathcal{G}_n^{r-1}(M), \tag{1.82}$$
$$\boldsymbol{\nabla}\wedge \colon \mathcal{G}_n^r(M) \to \mathcal{G}_n^{r+1}(M), \tag{1.83}$$

which satisfy the properties

$$(\boldsymbol{\nabla}\wedge)^2 = 0, \tag{1.84}$$
$$(\boldsymbol{\nabla}\rfloor)^2 = 0, \tag{1.85}$$

when acting on a homogeneous r-vector. Since 1.84 holds, the gradient operator gives rise to the grade preserving *Laplace-Beltrami operator*

$$\Delta = \boldsymbol{\nabla}^2 = \boldsymbol{\nabla}\rfloor \circ \boldsymbol{\nabla} \wedge + \boldsymbol{\nabla} \wedge \circ \boldsymbol{\nabla}\rfloor,$$

which is manifestly coordinate invariant by definition. It also motivates the use of the physicist notation $\boldsymbol{\nabla}^2 = \Delta$, but we do not adopt this here. We refer to multivector fields f in the kernel of the Laplace-Beltrami operator *harmonic multivector fields* or simply as *harmonic*.

Note that since Euclidean space $\mathbb{R}^n$ has global orthonormal coordinates $\boldsymbol{e}_i$ we can choose a global constant vector field basis since we identified $\mathcal{G}_n^1$ with $\mathcal{G}(\mathbb{R}^n)^1$. With respect to these fields, we have the that $\nabla_{\boldsymbol{u}}$ reduces to the directional derivative. Note then that $\boldsymbol{u} \cdot \boldsymbol{\nabla} = \nabla_{\boldsymbol{u}}$ defines the directional derivative via the gradient. In fact, given a subspace $\boldsymbol{U_r}$, one could even describe a derivative in $\boldsymbol{U_r}$ by $\mathrm{P}_{\boldsymbol{U_r}}(\boldsymbol{\nabla})$.

1.2.3 Differential forms

The language of differential forms rests neatly inside geometric calculus. We will develop the relationship between multivectors and forms which will serve as a link between the two notions so that researchers with interest in Clifford analysis can communicate with those who study forms. In order to do so, we appeal to the language of differential forms and build a relationship between multivector fields and forms through measures. Forms have their appeal in global understanding via their properties through integration (e.g., Stokes' and Green's theorems) and the exterior calculus along with de Rham cohomology will provide us a larger toolbox.

Given coordinates x^i on M we have the local basis tangent vector fields $\boldsymbol{v}_i = \frac{\partial}{\partial x^i}$ with the corresponding 1-forms dx^i that are each local sections of T^*M and are the exterior derivatives (or gradients) of the coordinate functions. Typically, 1-forms are viewed as linear functionals on tangent vectors and in these coordinates we have $dx^i(\partial_j) = \delta^i_j$ and one can thus take a pairing of 1-form fields and vector fields and integrate over 1-dimensional submanifolds. The benefit of this definition is that the 1-forms dx^i carry a natural measure and we can form product measures via the exterior product $\wedge$.

Let M be an n-dimensional pseudo Riemannian manifold with metric g, let $\Omega(M)$ be the exterior algebra of smooth form fields on M, and let $\Omega^r(M)$ be the space of smooth r-form fields on M. Then we have the Riemannian volume measure $\omega \in \Omega^n(M)$ given in local coordinates by

$$\omega = \sqrt{|g|}dx^1 \dots dx^n. \tag{1.86}$$

Definition 1.2.4. The *r-dimensional directed measure* dX_r is given in local coordinates by

$$dX_r \coloneqq \boldsymbol{v}_{i_1} \wedge \cdots \wedge \boldsymbol{v}_{i_r} dx^{i_1} \cdots dx^{i_r}. \tag{1.87}$$

For example, along a 2-dimensional submanifold we have the 2-dimensional directed measure

$$dX_2 = \boldsymbol{v}_i \wedge \boldsymbol{v}_j dx^i dx^j \tag{1.88}$$

and we can note that

$$(\boldsymbol{v}^i \wedge \boldsymbol{v}^j) \cdot dX_2^\dagger = dx^i dx^j - dx^j dx^i \tag{1.89}$$

is completely antisymmetric and provides us a surface measure we can integrate; this is a differential 2-form. We then find that

$$\omega = \boldsymbol{I}^{-1} \cdot dX_n = \boldsymbol{I}^{-1\dagger} \cdot dX_n^\dagger = 1^\perp \cdot dX_n, \tag{1.90}$$

where $\boldsymbol{I}$ is the unit pseudoscalar field defined on M with respect to g. The last of the equalities above is quite important. It seeks to tell us that, morally, we will tend integrate duals.

We can now write a r-form $\alpha_r = \alpha_{i_1 \cdots i_r} dx^{i_1} \wedge \cdots dx^{i_r}$ as

$$\alpha_r = A_r \cdot dX_k^\dagger, \tag{1.91}$$

where

$$A_r = \frac{1}{r!} \alpha_{i_1 \cdots i_r} \boldsymbol{v}^{i_\cdot} \wedge \cdots \boldsymbol{v}^{i_r}. \tag{1.92}$$

We refer to A_r as the *multivector equivalent* of α_r and note that by eq. (1.90) that the multivector equivalent to ω is $I^{-1\dagger}$. This provides an isomorphism between r-forms and r-vectors via a contraction with the r-dimensional volume directed measure. In this sense, a differential form is made up of two essential components namely the multivector field and the r-dimensional directed measure. Hence, we can see now how a differential form simply appends the measure attached to the underlying space. We can also see how this generalizes the musical isomorphism $\flat$ by taking a vector field $\boldsymbol{v}$ and noting

$$\boldsymbol{v} \cdot dX_1 = v_i \boldsymbol{v}_i \cdot \boldsymbol{v}^j dx^j = v_i dx^i. \tag{1.93}$$

The exterior algebra of differential forms comes with an addition $+$ and exterior multiplication $\wedge$. We note that the sum of two r-forms α_r and β_r is also a r-form which we can see reduces to addition on the multivector equivalents A_r and B_r by

$$\alpha_r + \beta_r = (A_r \cdot dX_r^\dagger) + (B_r \cdot dX_r^\dagger) = (A_r + B_r) \cdot dX_r^\dagger, \tag{1.94}$$

due to the linearity of $\cdot$. If instead had an s form β_s then we have the exterior product

$$\alpha_r \wedge \beta_s = (A_r \wedge B_s) \cdot dX^\dagger_{r+s}, \tag{1.95}$$

where $dX_{r+s} = 0$ if $r+s > n$.

With differential forms one also has the exterior derivative d giving rise to the exterior calculus. On the multivector equivalents we have

$$d\alpha_r = (\boldsymbol{\nabla} \wedge A_r) \cdot dX^\dagger_{r+1}, \tag{1.96}$$

which realizes the exterior derivative as the grade raising component of the gradient $\boldsymbol{\nabla}$. Of course, for scalar fields, this returns the gradient as desired. We will find $\boldsymbol{\nabla}\rfloor$ can be identified with the codifferential δ up to a sign.

1.2.4 Integration

Given a r-dimensional submanifold $R \subset M$ with a r-form α_r defined on R, we can integrate this r-form. However, we want to phrase this in terms of the the multivector equivalents. First, let ω_R be the volume measure for the submanifold R. Given R is a submanifold of M, for any $x \in R$ we have tangent space T_xR which is a subspace of T_xM. Hence, we can put $\boldsymbol{I}_R(x)^{-1\dagger}$ to be the multivector equivalent of ω_R by

$$\omega_R = \boldsymbol{I}_R^{-1\dagger} \cdot dX^\dagger_r = \boldsymbol{I}_R^{-1} \cdot dX_r. \tag{1.97}$$

We should think of $\boldsymbol{I}_R^{-1\dagger}$ as representing the subspace $T_xR \subset T_xM$ and note that we think of $\boldsymbol{I}_R^{-1\dagger}$ as a unit pseudoscalar field defined on R.

An s-vector field A_s on R is said to be *tangent to* R if

$$A_s = \mathrm{P}_{\boldsymbol{I}_R}(A_s) \tag{1.98}$$

so that for any $x \in R$ that $A_s = \mathrm{P}_{\boldsymbol{I}_R(x)}(A_s(x))$. Immediately we can conclude that we must have $s \leq r$ or this projection is zero (see section 1.1.4). We may, for example, wish to integrate scalar fields A_0 over R and in this case we can put $A_r = A_0\boldsymbol{I}_R^{-1}$ and contract with dX_r to create a tangent r-form on R by

$$\alpha_r = A_r \cdot dX_r = A\omega_R \tag{1.99}$$

which can be integrated as

$$\int_K \alpha = \int_K A_0\omega_R. \tag{1.100}$$

This of course applies to scalar fields on M itself, for which we can take $A_n = A_0\boldsymbol{I}^{-1}$. Then this form can be integrated by

$$\int_M \alpha_n = \int_M A_0\omega. \tag{1.101}$$

There is also the normal space N_xR that is everywhere orthogonal to T_xR with respect to g on M. This yields the normal $n-r$-blade field $\boldsymbol{\nu}_R = \boldsymbol{I}_R^\perp$. Since R is a submanifold of M, we have the inclusion $\iota\colon R \to M$ and the induced pullback on forms $\iota^*\colon \Omega(M) \to \Omega(R)$.

Proposition 1.2.1. *Let α_s be an s-form defined on M and let $\iota\colon R \to M$ be the inclusion of the submanifold R into M. Then the pullback ι^* on the multivector equivalent A_s is given by*

$$\iota^*\alpha_s = \mathrm{P}_{\boldsymbol{I}_R}(A_s) \cdot dX_s. \tag{1.102}$$

Proof. Note that by definition we have

$$(\iota^*\alpha_s)_x(\boldsymbol{v}_1, \dots, \boldsymbol{v}_r) = (\alpha_s)_x(d\iota_x \boldsymbol{v}_1, \dots, d\iota_x \boldsymbol{v}_r),$$

for arbitrary vector fields $\boldsymbol{v}_1, \dots, \boldsymbol{v}_s$ and at all $x \in R$. Then, since ι is inclusion, we have

$$d\iota_x = \mathrm{P}_{\boldsymbol{I}_{R(x)}},$$

at each point $x \in R$ and hence

$$\iota^*\alpha_s = \alpha_s \circ \mathrm{P}_{\boldsymbol{I}_R}.$$

For all $\boldsymbol{v}_i$ we can put

$$\boldsymbol{v}_i = \mathrm{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_i) + \mathrm{R}_{\boldsymbol{I}_R}(\boldsymbol{v}_i),$$

and note for the multivector equivalent

$$(\mathrm{P}_{\boldsymbol{I}_R}(A_s) \cdot dX_s)(\boldsymbol{v}_1, \dots, \boldsymbol{v}_s) = (\mathrm{P}_{\boldsymbol{I}_R}(A_s) \cdot dX_s)(\mathrm{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_1) + \mathrm{R}_{\boldsymbol{I}_R}(\boldsymbol{v}_1), \dots, \mathrm{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_s) + \mathrm{R}_{\boldsymbol{I}_R}(\boldsymbol{v}_s)) \tag{1.103}$$

$$= (\mathrm{P}_{\boldsymbol{I}_R}(A_s) \cdot dX_s)(\mathrm{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_1), \dots, \mathrm{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_s)), \tag{1.104}$$

since $\mathrm{P}_{\boldsymbol{I}_R}(A_s)$ is supported only on R. Then, if $s \leq r$,

$$\begin{aligned}
\iota^*\alpha_s &= (A_s \cdot dX_s)(\mathrm{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_1), \dots, \mathrm{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_s)) \\
&= ((\mathrm{P}_{\boldsymbol{I}_R}(A_s) + \mathrm{R}_{\boldsymbol{I}_R}(A_s)) \cdot dX_s)(\mathrm{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_1), \dots, \mathrm{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_s)) \\
&= (\mathrm{P}_{\boldsymbol{I}_R}(A_s) \cdot dX_s)(\mathrm{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_1), \dots, \mathrm{P}_{\boldsymbol{I}_R}(\boldsymbol{v}_s)),
\end{aligned}$$

and by eq. (1.103) we have our intended result. If $s > r$, then

$$\iota^*\alpha_s = 0 = \mathrm{P}_{\boldsymbol{I}_R}(A_s)$$

which proves the proposition. □

The above seems to motivate the choice of [9] to put $\boldsymbol{t}_R = \iota^*$ to refer to the tangential part of a differential form. The normal part of a form is $\boldsymbol{n}_R\alpha_s = \alpha_s - \boldsymbol{t}_R\alpha_s$. The following corollary is immediate given eqs. (1.53) and (1.56).

Corollary 1.2.1. *Let α_s be an s-form with normal part $\boldsymbol{n}_R\alpha_s$, then on the multivector equivalent A_s we have*

$$\boldsymbol{n}_R\alpha_p = \mathrm{P}_{\boldsymbol{\nu}_R}(A_s) \cdot dX_s = \mathrm{R}_{I_R}(A_s) \cdot dX_s. \tag{1.105}$$

This is pertinent when we take M to be a manifold with boundary ∂M. In this case we let $\boldsymbol{I}_\partial$ denote the tangent $n-1$-blade and build boundary measure via

$$\omega_\partial \coloneqq \boldsymbol{I}_\partial^{-1} \cdot dX_{n-1}. \tag{1.106}$$

The normal space is 1-dimensional and we put $\boldsymbol{\nu}$ to refer to the boundary normal space. It is common to compute the flux of a vector field $\boldsymbol{v}$ through ∂M by integrating $\mathrm{P}_{\boldsymbol{\nu}}(\boldsymbol{v})$ over the boundary. However, the the vector field $\mathrm{P}_{\boldsymbol{\nu}}(\boldsymbol{v})$ is the multivector equivalent of a 1-form. Hence, what we should have is a pseudovector $\mathrm{P}_{\boldsymbol{I}_\partial}(\boldsymbol{v}^\perp)$ which is the equivalent to the $n-1$-form

$$\mathrm{P}_{\boldsymbol{I}_\partial}(\boldsymbol{v}^\perp) \cdot dX_{n-1}^\dagger = (-1)^p \boldsymbol{v} \cdot \boldsymbol{\nu} \omega_\partial. \tag{1.107}$$

This tells us that the flux is determined both by the vector field $\boldsymbol{v}$ and the local geometry of ∂M captured by ω_∂. A proof follows.

Lemma 1.2.1. *Then the flux of a vector field $\boldsymbol{v}$ through ∂M is*

$$\int_{\partial M} \mathrm{P}_{\boldsymbol{I}_\partial}(\boldsymbol{v}^\perp) \cdot dX_{n-1}^\dagger = (-1)^p \int_{\partial M} \boldsymbol{v} \cdot \boldsymbol{\nu} \omega_\partial, \tag{1.108}$$

where p is the number of temporal vectors in $\mathcal{G}(M)$.

Proof. Take

$$\begin{aligned}
\mathrm{P}_{\boldsymbol{I}_\partial}(\boldsymbol{v}^\perp) &= \boldsymbol{v}^\perp \rfloor \boldsymbol{I}_\partial \boldsymbol{I}_\partial^{-1} \\
&= (\boldsymbol{v}^\perp \wedge \boldsymbol{\nu})^\perp \boldsymbol{I}_\partial^{-1} \\
&= (-1)^{n-1} (\boldsymbol{\nu} \wedge \boldsymbol{v}^\perp)^\perp \boldsymbol{I}_\partial^{-1} \\
&= (-1)^{n-1} (\boldsymbol{\nu} \rfloor \boldsymbol{v})^{\perp\perp} \boldsymbol{I}_\partial^{-1} \\
&= (-1)^{\frac{1}{2}(n+2)(n-1)+p} \boldsymbol{v} \cdot \boldsymbol{\nu} \boldsymbol{I}_\partial^{-1} \\
&= (-1)^p \boldsymbol{v} \cdot \boldsymbol{\nu} \boldsymbol{I}_\partial^{-1\dagger}.
\end{aligned}$$

Hence

$$\mathrm{P}_{\boldsymbol{I}_\partial}(\boldsymbol{v}^\perp) \cdot dX_{n-1}^\dagger = (-1)^s \boldsymbol{v} \cdot \boldsymbol{\nu} \omega_\partial.$$

□

For smooth r-forms α_r and β_r, we have an L^2-inner product

$$\int_M \alpha_r \wedge \star \beta_r \tag{1.109}$$

where $\star$ is the Hodge star. By definition, the Hodge star acts on r-forms by returning a Hodge dual $n-r$-form so that on the multivector equivalents we have

$$\alpha_r \wedge \star \beta_r = (A_r \cdot B_r^\dagger)\omega \tag{1.110}$$

as well as

$$\alpha_r \wedge \star \alpha_r = \|A_r\| \omega, \tag{1.111}$$

where $\|A_r\|$ is the pointwise Clifford norm. For the action of $\star$ on the multivector equivalents we will put $B_r^\star$.

Proposition 1.2.2. *We have that $B_r^\star$ is given by*

$$B_r^\star = (-1)^{r(n-r)}(B_r^\perp)^\dagger. \tag{1.112}$$

Proof. Indeed,

$$\begin{aligned}
\alpha_r \wedge \star\beta_r &= (A_r \wedge B_r^\star) \cdot dX_n^\dagger \\
&= (-1)^{r(n-r)}(A_r \wedge (B_r^\perp)^\dagger) \cdot dX_n^\dagger \\
&= (-1)^{r(n-r)}(B_r^\perp \wedge A_r^\dagger)^\dagger \cdot dX_n^\dagger \\
&= (A_r^\dagger \wedge B_r^\perp)^\dagger \cdot dX_n^\dagger \\
&= (A_r^\dagger \cdot B_r)^\perp \cdot dX_n \\
&= A_k \cdot B_k^\dagger \omega,
\end{aligned}$$

with the final equality by eq. (1.47). □

1.2.5 Stokes' and Green's theorem

Given our definition of the Hodge star on multivector equivalents, we can now define an L^2-inner product on multivector fields.

Definition 1.2.5. Let A_r and B_s be a r- and s-vector fields. Then the *multivector field inner product* is defined by

$$\ll A_r, B_s \gg := \int_M \langle A_r B_s^\dagger \rangle_0 \omega. \tag{1.113}$$

Note that when $s \neq r$, this inner product is zero.

Corollary 1.2.2. *Given two r-forms, the r-form inner product is equal to the multivector inner product on their corresponding multivector equivalents.*

Proof. Let α_r and β_r be r-forms with multivector equivalents A_r and B_r respectively. Then

$$\int_M \alpha_r \wedge \star\beta = \int_M A_r \cdot B_r^\dagger \omega = \ll A_r, B_r \gg,$$

by the proof of proposition 1.2.2. □

On forms, we have a compact form of Stokes' theorem

$$\int_M d\alpha_{n-1} = \int_{\partial\Omega} \iota^* \alpha_{n-1},$$

for sufficiently smooth $n-1$-forms α_{n-1}.

Then, in terms of the multivector equivalents, Stokes' theorem reads

$$\int_M (\boldsymbol{\nabla} \wedge A_{n-1}) \cdot dX_n = \int_{\partial M} \mathrm{P}_{\boldsymbol{I}_\partial}(A_{n-1}) \cdot dX_{n-1}. \tag{1.114}$$

But this has another, more physical, interpretation, in particular with the dual relationship. If we take a vector field $\boldsymbol{v}$, then

$$\int_M (\boldsymbol{\nabla} \wedge \boldsymbol{v}^\perp) \cdot dX_n = \int_{\partial M} \mathrm{P}_{\boldsymbol{I}_\partial}(\boldsymbol{v}^\perp) \cdot dX_{n-1}, \tag{1.115}$$

which implies

$$\int_M \boldsymbol{\nabla} \cdot \boldsymbol{v}\omega = \int_{\partial M} \boldsymbol{v} \cdot \boldsymbol{\nu}\omega_\partial \tag{1.116}$$

1.3 Spaces of fields

1.3.1 Monogenic fields

Multivectors in the kernel of $\boldsymbol{\nabla}$ are of fundamental importance in geometric calculus and these multivectors are the motivation for Clifford analysis much like elements in the kernel of Δ give rise to harmonic analysis.

Definition 1.3.1. Let $A \in \mathcal{G}(M)$. Then we say that A is *monogenic* if $A \in \ker(\boldsymbol{\nabla})$.

Monogenic fields are of utmost importance as they have many beautiful properties. One should find them as a suitable generalization of the notion of complex holomorphicity. For example, in regions of Euclidean spaces, a monogenic field f can be completely determined by its Dirichlet boundary values through a generalized Cauchy integral formula and for a spinor field each of the graded components of f are harmonic. We put

$$\mathcal{M}(M) \coloneqq \{A \in \mathcal{G}(M) \mid \boldsymbol{\nabla} A = 0\}$$

to refer to elements of this set as *monogenic fields* on M. As subspaces we also have the *monogenic r-vectors* $\mathcal{M}^r(M)$, *monogenic spinors* $\mathcal{M}^+(M)$, and the *monogenic parabivectors* $\mathcal{M}^{0+2}(M)$.

Remark 1.3.1. The definition for $\mathcal{M}^r(M)$ is multivector equivalent to space of harmonic fields,

$$\mathcal{H}^r(M) \coloneqq \{\alpha_r \in \Omega^r(M) \mid d\alpha_r = 0,\ \delta\alpha_r = 0\}. \tag{1.117}$$

We will avoid the term harmonic fields since we reference multivector fields in the kernel of Δ as harmonic.

It will be pertinent in to speak of function algebras. Hence, one could consider if the space $\mathcal{M}(M)$ is, in general, an algebra. While it is clear that the sum of two monogenic fields is also a monogenic field, it is not necessarily true that the product of two monogenic fields is monogenic. Hence, these spaces do not form algebras in their own right, they do indeed form a vector space as sums of monogenic functions are monogenic due to the linearity of the gradient.

reference later section

To the contrary, let M be 2-dimensional, then the space of monogenic spinors $\mathcal{M}^+(M)$ is indeed an algebra. In fact, taking $\mathcal{G}_2(\mathbb{R}^2)$ we can note that monogenic spinors are exactly the complex holomorphic functions via the identification in section 1.1.5. Take the coordinates

x, y and the standard basis $\boldsymbol{e}_i$, then if $f = u + v\boldsymbol{B}_{12} \in \mathcal{G}_2(\mathbb{R}^2)$ we can note that $\boldsymbol{\nabla} f = 0$ yields the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1.118}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1.119}$$

Though the two dimensional case is special, there will be nontrivial algebras living inside each $\mathcal{M}(M)$ for manifolds of dimension > 3. Also, in all dimensions, the gradient is invariant under actions from the spin group.

Lemma 1.3.1. *Let $s \in \operatorname{Spin}(n)$ then $\boldsymbol{\nabla} \circ s = s \circ \boldsymbol{\nabla}$.*

This lemma is classical in the theory of the Dirac operator, Clifford analysis, and harmonic analysis so we omit a proof. One can see [7], for example. The following corollary is immediate.

Corollary 1.3.1. *The space of monogenic spinors $\mathcal{M}^+(M)$ is $\operatorname{Spin}(n)$ invariant.*

.

revisit these lemma and corollary for spin(V) not just spin(n)

1.3.2 Hodge-type decompositions

Let us define the following spaces of multivectors that mimic their differential forms counterpart.

- The *exact fields*,
$$\mathcal{E}(M) \coloneqq \{\boldsymbol{\nabla} \wedge A \mid A \in \mathcal{G}(M),\ \ \mathrm{P}_{\boldsymbol{\nu}}(A) = 0\}; \tag{1.120}$$
- The *co-exact fields*,
$$\mathcal{C}(M) \coloneqq \{\boldsymbol{\nabla} \rfloor A \mid A \in \mathcal{G}(M),\ \ \mathrm{R}_{\boldsymbol{\nu}}(A) = 0\}; \tag{1.121}$$
- The *Dirichlet harmonic fields*,
$$\mathcal{M}_D(M) \coloneqq \{A \in \mathcal{M}(M) \mid \ \mathrm{P}_{\boldsymbol{\nu}}(A) = 0\}; \tag{1.122}$$
- The *Neumann harmonic fields*,
$$\mathcal{M}_N(M) \coloneqq \{A \in \mathcal{M}(M) \mid \ \mathrm{R}_{\boldsymbol{\nu}}(A) = 0\}. \tag{1.123}$$

We then use superscripts to denote the associated r-vector subspace. Notice that the exact and coexact fields satisfy not only a differential condition, but a boundary condition as well. Then, under the r-form inner product, we find the orthogonal direct sum decomposition

$$\mathcal{G}^r(M) = \mathcal{E}^r(M) \oplus \mathcal{C}^r(M) \oplus \mathcal{M}^r(M), \tag{1.124}$$

known as the Hodge-Morrey decomposition. Within the space of harmonic fields we have

$$\mathcal{M}_{\mathrm{ex}}(M) \coloneqq \{A \in \mathcal{M}(M) \mid A = \boldsymbol{\nabla} \wedge B\}, \tag{1.125}$$
$$\mathcal{M}_{\mathrm{co}}(M) \coloneqq \{A \in \mathcal{M}(M) \mid A = \boldsymbol{\nabla} \rfloor B\}. \tag{1.126}$$

Further, we have two decompositions of the space of harmonic fields

$$\mathcal{M}^r(M) = \mathcal{M}^r_D(M) \oplus_{L^2} \mathcal{M}^r_{\mathrm{co}}(M), \tag{1.127}$$
$$\mathcal{M}^r(M) = \mathcal{M}^r_N(M) \oplus_{L^2} \mathcal{M}^r_{\mathrm{ex}}(M), \tag{1.128}$$

which are the Friedrichs decompositions.

Monogenics of a single grade are already studied. but now we can study monogenics of mixed grades! It is a very reasonable question to ask whether the Hodge-Morrey decomposition extends to

$$\mathcal{G}(M) \stackrel{?}{=} \mathcal{E}(M) \oplus \mathcal{C}(M) \oplus \mathcal{M}(M) \tag{1.129}$$

under the multivector field inner product. This is, in fact, not true. While it is clear that

$$\mathcal{G}(M) = \bigoplus_{j=1}^{n} \mathcal{G}^j(M) \tag{1.130}$$

$$\mathcal{E}(M) = \bigoplus_{j-1}^{n} \mathcal{E}^j(M) \tag{1.131}$$

$$\mathcal{C}(M) = \bigoplus_{j-1}^{n} \mathcal{C}^j(M), \tag{1.132}$$

we have the following failure

$$\mathcal{M}(M) \neq \bigoplus_{j=1}^{n} \mathcal{M}^j(M). \tag{1.133}$$

1.3.3 Integral transforms

Chapter 2

Calderón problem

One application for the work we have done thus far is with the Calderón inverse problem. This problem stems from a physical question asked by Alberto Calderón in which he asked how much information of a body can we determine from measurements along the boundary of the body. More specifically, by applying a voltage to the boundary and measuring the outgoing current flux, can we determine the conductivity of the body? This is the Electrical Impedance Tomography (EIT) problem.

The more modern formulation of this problem casts it as a geometrical inverse problem by replacing the boundary data with the Dirichlet-to-Neumann map (i.e., the voltage-to-current map) and replacing the conductivity with a Riemannian metric. This has lead to generalizations using differential forms.

sources

2.1 Forward problem

Classically we want to find harmonic fields

Classical EIT

In the EIT problem, we begin with a region M with unknown symmetric positive definite conductivity matrix σ. We apply a static scalar potential ϕ on ∂M which produces the scalar potential u in the interior. We assume M is flat so that we can choose $g_{ij} = \delta_{ij}$ globally; or, in other words we are working over $\mathcal{G}_n(M)$ where each geometric tangent space is Euclidean $C\ell(T_xM, |\cdot|)$. Then, we posit that M is built from Ohmic material in order to satisfy Ohm's law

$$-\sigma \boldsymbol{\nabla} \wedge u = \boldsymbol{j} \tag{2.1}$$

where $\boldsymbol{j}$ is the current density vector field. The conservation law

$$\int_{\partial M} \boldsymbol{j} \cdot \boldsymbol{\nu} \omega_\partial = \int_{\partial M} \mathrm{P}_{\boldsymbol{I}_\partial}(\boldsymbol{j}^\perp) \cdot dX_{n-1} = 0, \tag{2.2}$$

implies $\boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$ via Stokes' theorem. Hence, we arrive at the condition on the scalar potential

$$\boldsymbol{\nabla} \cdot (\sigma \boldsymbol{\nabla} \wedge u) = 0. \tag{2.3}$$

In other words, the conservation law in eq. (2.2) is equivalent to the interior of M being charge free. This is derived more fully in [5].

Taking some arbitary basis, conductivity matrix assumes the components σ_{ij} for $i,j = 1,2,3$. Via [11] in dimension $n > 2$, we can realize that the conductivity matrix can be replaced with an intrinsic Riemannian metric with the components in this basis given by

$$g_{ij} = (\det \sigma^{k\ell})^{\frac{1}{n-2}}(\sigma^{ij})^{-1}, \quad \sigma^{ij} = (\det g_{k\ell})^{\frac{1}{2}}(g_{ij})^{-1}. \tag{2.4}$$

It is worth noting that these cannot hold in dimension $n = 2$. Due to eq. (2.4), we can remove the extrinsic need of σ with an intrinsic g on the Clifford bundle structure. That is, we are working with $\mathcal{G}(M)$ where each geometric tangent space is given by $C\ell(T_xM, g_x)$. Hence, Ohm's law is given as

$$-\boldsymbol{\nabla} \wedge u = \boldsymbol{j}. \tag{2.5}$$

Then by eq. (2.2), we find the scalar potential is harmonic

$$\Delta u = 0 \quad \text{in } M. \tag{2.6}$$

Via Maxwell's equations, it follows that

$$-\sigma \boldsymbol{\nabla} \wedge u^{\phi} = \boldsymbol{\nabla} \cdot b, \tag{2.7}$$

where b is the magnetic bivector field.

Generalization to forms

This problem can be cast in a new light by considering harmonic r-forms instead of a harmonic 0-form u. Given some $\varphi \in \Omega^r(\partial M)$, we have the boundary value problem

$$\begin{cases} \Delta\alpha_r = 0, & \text{in } M \\ \iota^*\alpha_r = \varphi, \quad \iota^*(\delta\alpha_r) = 0 & \text{on } \partial M. \end{cases} \tag{2.8}$$

As stated in [1], there exists a solution α_r to this problem up to a Dirichlet harmonic field λ_D.

2.1.1 Inverse problem

Ohm's law

and we arrive at $\Delta u = 0$ for the scalar potential and $\Delta \mathbf{A} = \mathbf{J}$ for the magnetic vector potential. In terms of the magnetic field bivector, we have $\boldsymbol{\nabla} \cdot b = \mathbf{J}$ and once again by Ohm's law we have $-\boldsymbol{\nabla} \wedge u^{\phi} = \boldsymbol{\nabla} \cdot b$. This leads us to consider the parabivector field $f = u + b$. We can note that f is (spatially) monogenic since

$$\boldsymbol{\nabla} f = 0 \iff -\boldsymbol{\nabla} \wedge u^{\phi} = \boldsymbol{\nabla} \cdot b \text{ and } \boldsymbol{\nabla} \wedge b = 0,$$

is satisfied. We see now that the fact that the body Ω is ohmic gives us a necessary coupling between the scalar potential and the magnetic field. The classical forward problem in terms of geometric calculus is given by the following scenario. We have an ohmic M and we find the electrostatic potential u satisfying the Dirichlet problem

$$\begin{cases} \Delta u^\phi = 0 & \text{in } M \\ u^\phi|_{\partial M} = \phi & \text{on } \partial M. \end{cases} . \tag{2.9}$$

In the realm of EIT, the Dirichlet data ϕ amounts to an input voltage along the boundary and by Ohm's law $\mathbf{J} = \boldsymbol{\nabla} \wedge u^\phi$ provides us the current. For any given solution to the boundary value problem, there is the corresponding Neumann data $\mathbf{J}^\perp = \mathrm{P}\, \nu \boldsymbol{\nabla} u^\phi$ where ν is the normal to the boundary Σ defined by $\nu = I_\Sigma I$ for the oriented boundary pseudoscalar I_Σ. This motivates the so called Voltage-to-Current (VC) operator $\phi \mapsto \mathbf{J}^\perp$. In general, we refer to set of both boundary conditions $(\phi, \mathbf{J}^\perp)$ $\forall \phi$ as the *Cauchy data* and define the *Dirichlet-to-Neumann (DN) operator* Λ such that $\Lambda\phi = \mathbf{J}^\perp$. This mimics the VC operator in EIT. With our notation from before we have

$$\Lambda\phi = \mathrm{P}\, \nu \boldsymbol{\nabla} u^\phi = \mathbf{J}^\perp.$$

Note that this operator Λ is often referred to as the *scalar* DN operator since the input is the scalar field ϕ whereas a more general operator on differential k-forms has been described in [1, 10]. The inverse problem follows.

Calderón problem. Let Ω be an unknown Riemannian manifold with unknown metric g and with known boundary Σ and known DN operator Λ. Can one recover Ω and the spatial inner product g from knowledge of Σ and Λ?

With the DN operator, we can reconstruct the boundary four current J. On Σ, we have the gradient $\boldsymbol{\nabla}_\Sigma$ inherited from $\boldsymbol{\nabla}$ on Ω. In particular, we have the relationship

$$\boldsymbol{\nabla}_\Sigma \phi = \mathrm{P}\, I_\Sigma \boldsymbol{\nabla} \phi,$$

which is accessible with our knowledge of ϕ and Σ. The boundary current is then

$$\mathbf{J}|_\Sigma = \boldsymbol{\nabla}_\Sigma \phi + \Lambda(\phi).$$

Though we do not have access to u^ϕ directly, we do know that $\Delta u^\phi = \rho$ and as such we have the boundary four current by

$$J|_\Sigma = \Delta u^\phi|_\Sigma \gamma_0 + \mathbf{J}|_\Sigma$$

as well as the interior four current $J = \mathbf{J}$ since the interior is free of charges. Defining the the four vector potential as before, we arrive at the extra equation $\Delta\mathbf{A} = \mathbf{J}$ in Ω. Once again define the magnetic bivector field $b = \boldsymbol{\nabla} \wedge \mathbf{A}$ and we note that Ohm's law implies $\boldsymbol{\nabla} \cdot b = -\boldsymbol{\nabla} \wedge u^\phi$ in Ω and so the parabivector field $f = u^\phi + b$ is spatially monogenic since we also have $\boldsymbol{\nabla} \wedge b = 0$. This all holds assuming that we can solve the electromagnetic Neumann boundary value problem

$$\begin{cases} \Delta A = \mathbf{J} & \text{in } \Omega \\ A = A_\Sigma & \text{on } \Sigma \end{cases}$$

Show that we can determine the magnetic potential A_Σ on the boundary. This may also show that the two notions of the DN operator are equivalent. That'd be nice.

If we show there is always a unique monogenic conjugate b for any harmonic u then this must be what we are doing here. Is this gauranteed by the Cauchy integral?

Though briefly we mentioned Ω as a Riemannian manifold, we now take Ω to be a region in $\mathbb{R}^n$ for brevity. Using the DN operator, one can define a *Hilbert transform* by

$$T\phi = d\Lambda^{-1}\phi,$$

as in [1]. It has yet to be shown that this definition coincides with the definition in [2], but there is reason to believe they are related. The classical Hilbert transform on $\mathbb{C}$ inputs a harmonic function and outputs another harmonic function v such that $u+iv$ is holomorphic. Essentially, this translates into finding a conjugate bivector field b to u^ϕ such that $u^\phi + b$ is monogenic. First, we require ϕ satisfies

This statement should come from the lagrangian perspective hopefully.

$$\left(\Lambda + (-1)^n d\Lambda^{-1} d\right)\phi = 0, \tag{2.10}$$

where d is the exterior derivative on forms. They show how to find the image of this, perhaps I can show what the kernel is. As shown earlier in Section **??**, d amounts to $\boldsymbol{\nabla}\wedge$ on the multivector field constituent of a form. When condition 2.10 is met, there exists a *conjugate form* $\epsilon \in \Omega^{n-2}(M)$. As well, ϵ is also coclosed in that $\delta\epsilon = 0$. To retrieve the constituent $(n-2)$-vector E, we just note $\epsilon = E \cdot dX_k$. Given Hodge duality, we have a 2-form β such that $\star\beta = \epsilon$ and the corresponding bivector $b^\star = E$. Combining the fields u^ϕ and b into the parabivector $f = u^\phi + b \in \mathcal{G}_n^{0+2}(\Omega)$. We then note that f is monogenic if and only if

$$\boldsymbol{\nabla} \wedge u = -\boldsymbol{\nabla} \cdot b \qquad \text{and} \qquad \boldsymbol{\nabla} \wedge b = 0.$$

Lemma 2.1.1. *Given the fields u^ϕ and b as above, the corresponding parabivector field*

$$f = u^\phi + b$$

is monogenic.

Proof. Let $\star\beta^\psi = \epsilon$ as before and note that

$$du^\phi = \star d\epsilon = \star d \star \beta^\psi, \tag{2.11}$$

as shown in Theorem 5.1 in [1]. The multivector equivalent of the right hand side of Equation [**?**] yields

$$\begin{aligned}
(\boldsymbol{\nabla} \wedge b^\star)^\star &= [(\boldsymbol{\nabla} \cdot b^\dagger)I]^\star \\
&= [I^{-1}((\boldsymbol{\nabla} \cdot b^\dagger)I)]^\dagger \\
&= ((\boldsymbol{\nabla} \cdot b^\dagger)I)^\dagger I \\
&= \boldsymbol{\nabla} \cdot b^\dagger && \text{since } \dagger \text{ of a vector is trivial} \\
&= -\boldsymbol{\nabla} \cdot b. && \text{since } \dagger \text{ of a bivector is -1}
\end{aligned}$$

Perhaps I should just show this property in the differntial forms section. Thus, we have $\boldsymbol{\nabla} \wedge u + \boldsymbol{\nabla} \cdot b = 0$. Since ϵ is coclosed we have

$$\begin{aligned} 0 = \boldsymbol{\nabla} \cdot b^\star &= \boldsymbol{\nabla} \cdot (I^{-1} b)^\dagger \\ &= \boldsymbol{\nabla} \cdot (b^\dagger I) \\ &= (\boldsymbol{\nabla} \wedge b^\dagger) I \\ \implies \quad 0 &= \boldsymbol{\nabla} \wedge b. \end{aligned}$$

Perhaps I should just show this property in the differntial forms section. Thus $\boldsymbol{\nabla} f = 0$ and F is monogenic. □

We have shown that conjugate forms give rise to monogenic fields. We now seek to determine for what boundary conditions ϕ we have at our disposal. Let $E^{\parallel} := \mathrm{P}\, I_\Sigma E$, with I_Σ the boundary pseudoscalar satisfying $\nu I_\Sigma = I$. Hence by Equation **??** we have $E^{\parallel} = \mathrm{R}_\nu(E)$ then in investigating the requirement from Equation 2.10 we find the multivector equivalent

$$(\Lambda + (-1)^n (\boldsymbol{\nabla} \wedge) \Lambda^{-1} (\boldsymbol{\nabla} \wedge))\phi = E^{\perp} + (-1)^n T E^{\parallel}$$

so we arrive at the fact that we must have

$$E^{\perp} = (-1)^{n-1} T E^{\parallel}.$$

In other words,

$$T\, \mathrm{R}_\nu(E) = (-1)^{n-1}\, \mathrm{P}\, \nu E.$$

Thus, the Hilbert transform maps tangential components of $\boldsymbol{\nabla} u^\phi = E$ to nontangential boundary components on the boundary.

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