MATH 317, Homework 5

Colin Roberts

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Solutions

Problem 1. Define $f\colon (0,1) \to \mathbb{R}$ by $f(x) = \frac{1}{\sqrt{x}} - \sqrt{\frac{x+1}{x}}$. Can some $\widehat{f}(0)$ be defined to make $\widehat{f}\colon [0,1) \to \mathbb{R}$ continuous at 0? Justify.

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Proof. Yes this is possible. Define,

$$\hat{f}(x) = \begin{cases} 0 & \text{if } x = 0 \\ f(x) & \text{if } x \in (0,1) \end{cases}$$

We can show this by proving that $\lim_{x\to 0} f(x) = 0$ which can easily be done using L'hôpital's rule.

$$\begin{aligned} \lim_{x\to 0} f(x) &= \lim_{x\to 0} \frac{1-\sqrt{x+1}}{\sqrt{x}} \\ &= \lim_{x\to 0} \frac{\mathrm{d/dx}(1-\sqrt{x+1})}{\mathrm{d/dx}\sqrt{x}} \quad \text{By L'hôpital's rule} \\ &= \lim_{x\to 0} \frac{\frac{1}{2}(x+1)^{-1/2}}{\frac{1}{2}x^{-1/2}} \\ &= \lim_{x\to 0} \frac{\sqrt{x}}{\sqrt{x+1}} \\ &= 0 \end{aligned}$$

Thus we have that $\hat{f}$ is continuous at 0. □

Problem 2. Suppose $f: \mathbb{R} \to \mathbb{R}$ is continuous and that $f(r) = r^2$ for all $r \in \mathbb{Q}$. Determine $f(\sqrt(2))$ and justify your conclusion.

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Proof. I predict that $f(\sqrt{2}) = 2$. Since f is continuous over $\mathbb{R}$ and we know $f(r) = r^2$ $\forall r \in \mathbb{Q}$, if the limit as $r \to \sqrt{2}$ exists and is equal to 2 we are done. Fix $\epsilon > 0$ and let $0 < \delta < -\sqrt{2} + \sqrt{2+\epsilon}$. Then we have,

$$\begin{aligned}
|f(r) - 2| &= |r^2 - 2| \\
&= |r - \sqrt{2}||r + \sqrt{2}| \\
&\leq |r - \sqrt{2}||(r - \sqrt{2}) + 2\sqrt{2}| \\
&< \delta(\delta + 2\sqrt{2}) \\
&< (-\sqrt{2} + \sqrt{2+\epsilon})(-\sqrt{2} + \sqrt{2+\epsilon} + 2\sqrt{2}) \\
&= \epsilon
\end{aligned}$$

Thus we know that $f(\sqrt{2}) = 2$. $\square$

Problem 3. Give an example of a function $f: \mathbb{R} \to \mathbb{R}$ that is continuous and bounded but not uniformly continuous. Prove your claim.

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Proof. Let $f(x) = \cos\left(x^2\right)$. Here $f: \mathbb{R} \to [-1,1]$ is bounded and continuous. If f is uniformly continuous then $\forall \epsilon > 0 \quad \exists \delta > 0$ such that if $x, y \in \mathbb{R}$ and $|x-y| < \delta$ we have $|f(x) - f(y)| < \epsilon$. Define the sequences $\{x_n\} = \sqrt{n\pi}$, $\left\{y_n\right\} = \sqrt{n\pi + \pi}$. We have shown in class that $\lim_{x\to\infty} \sqrt{x - x_0} - \sqrt{x} = 0$ thus $|x_n - y_n| < \delta \ \ \forall n \in \mathbb{N}$ sufficiently large and thus we should have that $|f(x_n) - f(y_n)| < \epsilon$. Fix $\epsilon < 2$ then $\forall n \in \mathbb{N}$, $|f(x_n) - f(y_n)| = |\cos n\pi - \cos(n\pi + \pi)| = 2$ which contradicts $|f(x_n) - f(y_n)| < \epsilon$ for some $n \in \mathbb{N}$. Thus f is not uniformly continuous. $\square$

Problem 4. Let $E \subseteq \mathbb{R}$ be compact and nonempty. Prove that $\sup E \in E$ and $\inf E \in E$.

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Proof. Since $E \subseteq \mathbb{R}$ is compact, it must also be closed and bounded. Thus both $\sup E$ and $\inf E$ exist. Suppose, for a contradiction, that $\sup E \notin E$. Since E is closed, it contains all of its limit points. But since $\sup E \notin E \quad \exists \epsilon > 0$ such that $Q = (\sup E - \epsilon, \sup E + \epsilon)$, a neighborhood of $\sup E$ does not contain any points in E. But, by definition of the supremum, $\forall \epsilon > 0 \quad \exists e \in E$ such that $\sup E - \epsilon < e < \sup E$. Since $Q \cap E = \emptyset$, this contradicts the definition of the supremum and $\sup E \in E$.

The proof for $\inf E \in E$ is remarkably similar. Suppose, for a contradiction, that $\inf E \notin E$. Since E is closed, it contains all of its limit points. But since $\inf E \notin E \quad \exists \epsilon > 0$ such that $Q = (\inf E - \epsilon, \inf E + \epsilon)$, a neighborhod of $\inf E$ does not contain any points in E. But, by definition of the infemum, $\forall \epsilon > 0$ $\exists e \in E$ such that $\inf E + \epsilon > e > \inf E$. Since $Q \cap E = \emptyset$, this contradicts the definition of the infemum and $\inf E \in E$. □

Problem 5. Suppose that $f\colon [a,b] \to [a,b]$ is continuous. Prove that there is at least one fixed point in $[a,b]$ (that is, there exists at least one $x \in [a,b]$ such that $f(x) = x$).

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Proof. Define $g(x) = f(x) - x$. Suppose that $f(x) \neq x \quad \forall x \in [a,b]$. Thus, $g(x) \neq 0$ over the domain as well. Now consider $g(a) = f(a) - a$. The result of *Problem 4* tells us that since a is the smallest member of $[a,b]$ and $[a,b]$ is compact, $a = \inf\{[a,b]\}$. Thus, since $f(a) \neq a$, $g(a) = f(a) - a > 0$ since $\operatorname{im} f([a,b]) = [a,b]$. It is not possible that $f(x) < a$ for any $x \in [a,b]$ since a is the smallest member of the image set. Now, since $g(x)$ is defined by the addition of two continuous functions f and x on a connected domain, we have that the intermediate value theorem holds. Thus if $g(x) < x$ for some x, then $\exists y \in [a,b]$ such that $g(y) = 0$. Thus it must be that $g(x) > x \quad \forall x \in [a,b]$. But we also have that $g(b) = f(b) - b$. Again *Problem 4* states that b is the supremum of the domain and image of f and since $f(b) \neq b \quad f(b) < b$. But this is a contradiction as we said that $g(x) > 0$ for every x, and by the mean value theorem if $g(b) < 0$, $\exists y \in [a,b]$ such that $g(y) = 0$. And thus for some y, $f(y) = y$ and this contradicts our supposition. ☐

Problem 6. Let $f : [-4,0] \to \mathbb{R}$ by $f(x) = \frac{2x^2-18}{x+3}$ for$x \neq -3$ and $f(-3) = -12$. Show that f is continuous at -3.

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Proof. We want to show that f is continuous at -3 given $f(-3) = -12$. Fix $\epsilon > 0$ and let $\delta < \frac{\epsilon}{2}$. Then for $x \in [-4,0]$ and $|x-(-3)| < \delta$ we have,

$$\begin{aligned}
|f(x) - f(-12)| &= \left|\frac{2x^2-18}{x+3} + 12\right| \\
&= |2(x-3)+12| \\
&= 2|x+3| \\
&< 2\delta \\
&< \epsilon
\end{aligned}$$

Thus f is continuous at -3. $\square$

Problem 7. Let $f, g \colon D \to \mathbb{R}$ be uniformly continuous. Prove that $f+g \colon D \to \mathbb{R}$ is uniformly continuous.

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Proof. Since f is uniformly continuous, fix $\epsilon > 0$ and $\exists \delta_1 > 0$ such that for $x, y \in D$ where $|x-y| < \delta_1$ we have $|f(x) - f(y)| < \frac{\epsilon}{2}$. With the same ϵ, $\exists \delta_2$ such that if $|x-y| < \delta_2$ we have $|g(x) - g(y)| < \frac{\epsilon}{2}$. Thus if we let $\delta = \min\{\delta_1, \delta_2\}$ then we have,

$$\begin{aligned} |(f+g)(x) - (f+g)(y)| &= |f(x) - f(y) + g(x) - g(y)| \\ &\leq |f(x) - f(y)| + |g(x) - g(y)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Thus we have that $f+g$ is also uniformly continuous. □