MATH 517, Homework 11

Colin Roberts

November 16, 2017

Solutions

Problem 1. Find the minimum of the function

$$f(x_1, x_2, \dots, x_n) = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

on the hypersurface $S = \{(x_1, x_2, \dots, x_n) \mid x_1 x_2 \dots x_n = c, \text{each } x_i \geq 0\}$, where $C > 0$ is a constant. Use your answer to deduce the *arithmetic-geometric mean inequality*: for $a_1, a_2, \dots, a_n \geq 0$,

$$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

:

Proof. Define $g\colon \mathbb{R}^n \to \mathbb{R}$ by $g(\vec{x}) = x_1 x_2 \cdots x_n - C$ and note that $S = g(0)$. Then

$$\nabla g(\vec{x}) = \begin{bmatrix} x_2 x_3 x_4 x_5 \cdots x_n \\ x_1 x_3 x_4 x_5 \cdots x_n \\ x_1 x_2 x_3 x_5 \cdots x_n \\ \vdots \\ x_1 x_2 x_3 x_4 \cdots x_{n-1} \end{bmatrix}$$

$$\nabla f(\vec{x}) = \begin{bmatrix} 1/n \\ 1/n \\ 1/n \\ \vdots \\ 1/n \end{bmatrix}.$$

Now to minimize this function with the constraint we want

$$\nabla f(\vec{x}) = \lambda \nabla g(\vec{x}),$$

which only has a solution when $\vec{x} = (a, a, \dots, a)$, which results in

$$\begin{bmatrix} a^{n-1} \\ a^{n-1} \\ \vdots \\ a^{n-1} \end{bmatrix} = \lambda \begin{bmatrix} 1/n \\ 1/n \\ \vdots \\ 1/n \end{bmatrix},$$

meaning $\lambda = na^{n-1}$. Of course we also have that $a^n = C$ so we know $a = C^{1/n}$.

To see that this is in fact a minimum and not a maximum or saddle point , note that there exists a lower bound 0 for the function f restricted to S since all $x_i \geq 0$. To see there is no maximum, we note that we can make x_1 arbitrarily large and adjust all other x_i so that $x_1 \cdots x_n = C$ but $\frac{x_1 + \cdots + x_n}{n}$ is not bounded from above. Since the lagrange multiplier method finds these extrema points, we know that we must find a local minimum for f on S.

So now we let $a_1, a_2, \dots, a_n \geq 0$ and note that $a_1 a_2 \cdots a_n = C$ for some C, and we know that when $C^{1/n} = a_1 = a_2 = \cdots = a_n$ we minimize the arithmetic mean. It follows that for not all $a_i = a_j$ we have

$$\begin{aligned} \frac{C^{1/n} + \cdots + c^{1/n}}{n} &\leq \frac{a_1 + \cdots + a_n}{n} \\ nC^{1/n} &\leq a_1 + \cdots + a_n \\ C^{1/n} &\leq \frac{a_1 + \cdots + a_n}{n} \\ (a_1 a_2 \cdots a_n)^{1/n} &\leq \frac{a_1 + \cdots + a_n}{n}. \end{aligned}$$

Now if $a = a_1 = a_2 = \cdots = a_n$ we have

$$(a \cdots a)^{1/n} = a = \frac{n}{n} a = \frac{a + \cdots + a}{n}.$$

For the other direction, suppose for a contradiction that we have that for $\vec{a}'$

$$(a'_1 \cdots a'_n)^{1/n} = \frac{a'_1 + \cdots + a'_n}{n},$$

with not all $a'_i = a'_j$ but still satisfying $(a'_1 a'_2 \cdots a'_n)^{1/n} = (a_1 a_2 \cdots a_n)^{1/n} = C$. By the lagrange multiplier method above, we have that $\vec{a}'$ is not a solution, and is thus not a local minimum of $f(\vec{x})$. This means that

$$\begin{aligned} f(\vec{a}) &< f(\vec{a}') \\ \frac{a + \cdots + a}{n} &< \frac{a'_1 + \cdots + a'_n}{n} \\ (a_1 a_2 \cdots a_n)^{1/n} &< \frac{a'_1 + \cdots + a'_n}{n} && \text{by the equality shown above.} \end{aligned}$$

Hence, we only have equality if $a = a_1 = a_2 = \cdots = a_n$. $\square$

Problem 2. Let A be a symmetric $n \times n$ matrix and define $Q\colon \mathbb{R}^n \to \mathbb{R}$ by $Q(\vec{x}) = \langle \vec{x}, A\vec{x} \rangle$ (the Q is for ***Q**uadratic form*). Suppose the restriction of Q to the unit sphere attains a maximum or a minimum at the point $\vec{v}$. Show that $A\vec{v} = \lambda\vec{v}$ for some $\lambda \in \mathbb{R}$.

:

Proof. So we define $g\colon \mathbb{R}^n \to \mathbb{R}$ by $g(\vec{x}) = x_1^2 + x_2^2 + \cdots x_n^2 - 1$ and note that the unit sphere $S^n = g(0)$. Then we have that

$$\begin{aligned}
\nabla Q(x) &= \begin{bmatrix} \frac{\partial}{\partial x_1} \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j \\ \frac{\partial}{\partial x_2} \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j \\ \vdots \\ \frac{\partial}{\partial x_n} \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j \end{bmatrix} \\
&= \begin{bmatrix} \sum_{i=1}^n a_{i1} x_i + \sum_{j=1}^n a_{1j} x_j \\ \sum_{i=1}^n a_{i2} x_i + \sum_{j=1}^n a_{2j} x_j \\ \vdots \\ \sum_{i=1}^n a_{in} x_i + \sum_{j=1}^n a_{nj} x_j \end{bmatrix} \\
&= 2 \begin{bmatrix} \sum_{i=1}^n a_{i1} x_i \\ \sum_{i=1}^n a_{i2} x_i \\ \vdots \\ \sum_{i=1}^n a_{in} x_i \end{bmatrix} && \text{since } A \text{ is symmetric, } a_{ij} = a_{ji} \\
&= 2A\vec{x}.
\end{aligned}$$

We also have

$$\nabla g(\vec{x}) = \begin{bmatrix} 2x_1 \\ 2x_2 \\ \vdots \\ 2x_n \end{bmatrix}.$$

Then, it follows

$$\begin{aligned}
\nabla Q(\vec{x}) &= \lambda \nabla g(\vec{x}) \\
2A\vec{x} &= 2\lambda\vec{x} \\
A\vec{x} &= \lambda\vec{x},
\end{aligned}$$

and we know that $\vec{v}$ is an extremal point, so it must be that

$$A\vec{v} = \lambda\vec{v} \qquad \square$$

by the work above.