Commutative Banach Algebras of Multivectors from the Scalar Dirichlet-to-Neumann Operator

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Abstract

The problem of determining an unknown Riemannian manifold given the Dirichlet-to-Neumann (DN) operator is known as the Calderón problem. One method of solving this problem in the two dimensional case is through the Boundary Control method. There, one uses the DN operator to construct a Banach algebra of holomorphic functions on the manifold. The Gelfand transform of this algebra is then homeomorphic to the manifold. In higher dimensions, we replace the complex field with a Clifford algebra and use the DN operator to determine a $\mathrm{Spin}(n)$ invariant space of monogenic multivector fields. Using a power series representation for monogenic fields, one decomposes the space of monogenics into products of commutative algebras of $(0+2)$-vector fields constant on translations of planes and monogenic in $\mathbb{R}^n$. Using this decomposition, we define spinor characters on the space of monogenic fields that correspond to Dirac measures on the manifold. The set of these Dirac measures is then homeomorphic to the underlying manifold with the Gelfand topology.

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1 Preliminaries

The complex algebra $\mathbb{C}$ can be generalized in a handful of ways. Some of which can be found through the use of Clifford algebras and, more specifically, in geometric algebras. We define the more general Clifford algebras first and realize geometric algebras as particularly nice Clifford algebras with a quadratic form arising from an inner product. Elements of a geometric algebra are known as multivectors and these multivectors carry a wealth of geometric information in their algebraic structure. $\mathbb{C}$ itself can be realized as a special subalgebra of parabivectors in the geometric algebra on $\mathbb{R}^2$ with the Euclidean inner product and the quaternions $\mathbb{H}$ are realized as an analogous algebra on $\mathbb{R}^3$. In particular, both $\mathbb{C}$ and $\mathbb{H}$ arise as the 2- and 3-dimensional even Clifford groups Γ^+ respectively.

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1.1 Clifford and geometric algebras

First, we present a review of Clifford algebras and the relevant notions needed for this work. Those who feel they are familiar with both Clifford and geometric algebras may wish to skim through this subsection and visit section 1.2 to review the notation used throughout this manuscript.

Formally, we let (V, Q) be an n-dimensional vector space V over some field K with an arbitrary quadratic form Q. The tensor algebra is given by

$$\mathcal{T}(V) \coloneqq \bigoplus_{j=0}^{\infty} V^{\otimes j} = K \bigoplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots , \tag{1}$$

where the elements (tensors) inherit a multiplication $\otimes$ (the tensor product). From the tensor algebra $\mathcal{T}(V)$, we can quotient by the ideal generated by $\boldsymbol{v}\otimes\boldsymbol{v}-Q(\boldsymbol{v})$ to create a new algebra.

Definition 1.1. The *Clifford algebra* $C\ell(V,Q)$ is the quotient algebra

$$C\ell(V,Q)=\mathcal{T}(V)\;/\;\langle \boldsymbol{v}\otimes\boldsymbol{v}-Q(\boldsymbol{v})\rangle. \tag{2}$$

To see how the tensor product descends to the quotient, we let $\boldsymbol{v_1},\dots,\boldsymbol{v_n}$ be an arbitrary basis for V, then we can consider the tensor product of basis elements $\boldsymbol{v_i}\otimes\boldsymbol{v_j}$ which induces a product in the quotient $C\ell(V,Q)$ which we refer to as the *Clifford multiplication*. In this basis, we write this product as concatenation $\boldsymbol{v_i}\boldsymbol{e_j}$ and define the multiplication by

$$\boldsymbol{v_i}\boldsymbol{v_j}=\begin{cases}Q(\boldsymbol{v_i}) & \text{if } i=j,\\ \boldsymbol{v_i}\wedge\boldsymbol{v_j} & \text{if } i\neq j,\end{cases} \tag{3}$$

where $\wedge$ is the typical exterior product satisfying $\boldsymbol{v}\wedge\boldsymbol{w}=-\boldsymbol{w}\wedge\boldsymbol{v}$ for all $\boldsymbol{v},\boldsymbol{w}\in V$. As a consequence, the exterior algebra $\bigwedge(V)$ can be realized as a subalgebra of any Clifford algebra over V or as a Clifford algebra with a trivial quadratic form $Q=0$.

In the case where V has a (pseudo) inner product g, we can induce a quadratic form Q by $Q(\boldsymbol{v})=g(\boldsymbol{v},\boldsymbol{v})$ and give rise to a special type of Clifford algebra which motivates the following definition.

Definition 1.2. Let V be a vector space with an (pseudo) inner product $g(\cdot,\cdot)$. Then taking $Q(\cdot)=g(\cdot,\cdot)$, the Clifford algebra $C\ell(V,Q)$ is called a *geometric algebra*.

In general, we put $\mathcal{G}$ and assume the inner product and vector space will be arbitrary, given alongside, or will be clear from context. For example, when $V=\mathbb{R}^n$ we have the standard orthonormal basis $\boldsymbol{e_1},\dots,\boldsymbol{e_n}$ which allows us to neatly define the quadratic form Q from the Euclidean inner product which has coefficients δ_{ij} with respect to this basis. For this, we have the geometric algebra $\mathcal{G}(\mathbb{R}^n)$ and moreover we put $\mathcal{G}_n:=\mathcal{G}(\mathbb{R}^n)$ to simplify notation.

Geometric algebras are an old and widely studied topic. For more information, see the classical text [3] or the more modern text [2] which also provides a wide range of applications to physics problems. Both these sources include much of the other necessary preliminaries I cover in the remainder of this section. Finally, the paper [1] proves many of the useful identities and notation used throughout this paper.

1.1.1 Multivectors and grading

Note that $C\ell(V,Q)$ is a $\mathbb{Z}$-graded algebra with elements of grade-0 up to elements of grade-n. We refer to grade-0 elements as scalars, grade-1 elements as vectors, grade-2 elements as *bivectors*, grade-r elements as r*-vectors*, and grade-n elements as *pseudoscalars*. For example, the pseudoscalar $\boldsymbol{\mu}=\boldsymbol{v_1}\wedge\boldsymbol{v_2}\wedge\cdots\wedge\boldsymbol{v_n}$ is an n-vector we will frequently return to. We denote the space of r-vectors by $C\ell(V,Q)^r$. For each grade there is a basis of $\binom{n}{r}$ r*-blades* which are r-vectors of the form

$$\boldsymbol{A_r}=\prod_{j=1}^{k}\boldsymbol{v_j},\ \text{for linearly independent } \boldsymbol{v_j}\in V, \tag{4}$$

and we use a boldface to specify that a r-vector is a r-blade and we note that vectors are simply 1-blades. Briefly, take for example the case where $\dim(V) = 3$, then there are $\binom{3}{2} = 3$ 2-blades that form a basis for the bivectors and one particular choice of a bivector basis would be the following list of 2-blades

$$\boldsymbol{B_{12}} = \boldsymbol{v_1} \wedge \boldsymbol{v_2}, \quad \boldsymbol{B_{13}} = \boldsymbol{v_1} \wedge \boldsymbol{v_3}, \quad \boldsymbol{B_{23}} = \boldsymbol{v_2} \wedge \boldsymbol{v_3}. \tag{5}$$

{eq:3_dim_

We will repeatedly use the notation $\boldsymbol{B_{ij}} \coloneqq \boldsymbol{v_i} \wedge \boldsymbol{v_j}$ and the underlying basis will be clear from context. We refer to an $(n-1)$-blade as a *pseudovector* and it should be noted that every $(n-1)$-vector is a pseudovector as seen in section 1.1.3. In other literature, some will refer to a r-blade as a *simple* or a *decomposable* r-vector.

citations

In general, an element $A \in C\ell(V,Q)$ is written as a linear combination of basis elements of all possible grades and we refer to A as a *multivector*. To extract the grade-r components of A, we use the notation

$$\langle A \rangle_r \tag{6}$$

to denote the grade-r components of the multivector A (i.e., $\langle A \rangle_r \in C\ell(V,Q)^r$). Any multivector A can then be given by

$$A = \sum_{r=0}^{n} \langle A \rangle_r \tag{7}$$

which shows the decomposition via the $\mathbb{Z}$-grading

$$C\ell(V,Q) = \bigoplus_{j=0}^{n} C\ell(V,Q)^j. \tag{8}$$

If A contains only components of a single grade, then we say that A is *homogeneous* and if the components are grade-r we put A_r and refer to A_r as a *homogeneous r-vector* or simply an *r-vector*. For example, when we refer to vectors we realize them as 1-vectors and likewise we realize bivectors as 2-vectors. Also of interest will be the elements in

$$C\ell(V,Q)^{0+2} = C\ell(V,Q) \oplus C\ell(V,Q)^2 \tag{9}$$

which we refer to as *parabivectors*.

The Clifford multiplication of vectors defined in eq:clifford_multiplication 3 can be extended to multiplication of vectors with homogeneous r-vectors. In particular, given a vector $\boldsymbol{v} \in C\ell(V,Q)$ and a homogeneous r-vector $A_r \in C\ell(V,Q)$, we have

$$\boldsymbol{v}A_r = \langle \boldsymbol{v}A_r \rangle_{r-1} + \langle \boldsymbol{v}A_r \rangle_{r+1}, \tag{10}$$

{eq:vector_

which decomposes the multiplication into a grade lowering *interior product* and a grade raising *exterior product*. This allows us to extend the Clifford multiplication further. Given an s-vector B_s, we have

$$A_rB_s = \langle A_rB_s \rangle_{|r-s|} + \langle A_rB_s \rangle_{|r-s|+2} + \cdots + \langle A_rB_s \rangle_{r+s}. \tag{11}$$

{eq:genera

This rule for multiplication then allows for the multiplication of two general multivectors in $C\ell(V,Q)$. For this multiplication, specific grades of the product are worth noting.

$$A_r \cdot B_s \coloneqq \langle A_rB_s \rangle_{|r-s|} \tag{12}$$

{eq:dot}

$$A_r \wedge B_s \coloneqq \langle A_r B_s \rangle_{r+s} \tag{13}$$ {eq:wedge}

$$A_r \rfloor B_s \coloneqq \langle A_r B_s \rangle_{s-r} \tag{14}$$ {eq:left_c

$$A_r \lfloor B_s \coloneqq \langle A_r B_s \rangle_{r-s}. \tag{15}$$ {eq:right_

Finally, we have a special product for bivectors called the *commutator product* given by

$$A_2 \times B_2 \coloneqq \langle A_2 B_2 \rangle_2 \equiv \frac{1}{2}(A_2 B_2 - B_2 A_2). \tag{16}$$ {eq:commuta

These products are particularly emphasized as many helpful identities used in this paper are phrased using these notions. Taking eqs. (10), (13) and (14) into mind, we see that we have the grade lowering interior product can be written as

$$\langle \boldsymbol{v} A_r \rangle_{r-1} \equiv \boldsymbol{v} \rfloor A_r \equiv \boldsymbol{v} \cdot A_r \tag{17}$$

and the grade raising exterior product can be written as

$$\langle \boldsymbol{v} A_r \rangle_{r+1} \equiv \boldsymbol{v} \wedge A_r. \tag{18}$$

Finally, to suppress needless additional parentheses later on, we assert that above products take precedence over the geometrical product in order of operation. For example, for multivectors A, B, and C, we must take

$$A \cdot BC \equiv (A \cdot B)C, \tag{19}$$

and extend this to the other products defined in eqs. (13) to (16) as well.

As discussed, $C\ell(V,Q)$ is naturally a $\mathbb{Z}$-graded algebra but we also find that it carries a $\mathbb{Z}/2\mathbb{Z}$-grading as well. This additional grading can be realized by sorting r-vectors in $C\ell(V,Q)$ into the sets where r is even or odd. We say a r-vector is *even* (resp. *odd*) if r is even (resp. odd) and in general if a multivector A is a sum of only even (resp. odd) grade elements we also refer to A as even (resp. odd). Taking note of the multiplication defined in eq:general_clifford_multiplication 11, one can see that the multiplication of even multivectors with another even multivectors outputs an even multivector and that motivates the following.

Definition 1.3. The *even subalgebra* $C\ell(V,Q)^+ \subset C\ell(V,Q)$ is the subalgebra of even grade multivectors

$$C\ell(V,Q)^+ \coloneqq C\ell(V,Q)^0 \oplus C\ell(V,Q)^2 \oplus C\ell(V,Q)^4 \oplus \cdots . \tag{20}$$

The even subalgebra is an extremely important entity that arises throughout physics due to its encapsulation of spinors. We touch on this next.

1.1.2 Reverse, inverses, and the Clifford and spin groups

spin_groups

For the remainder of this paper, let us focus solely on geometric algebras $\mathcal{G}$. For a geometric algebra with a positive definite inner product, all blades have an inverse and hence form a group. With a pseudo inner product, the invertible elements are not quite as broad. To

give an example later

this end, we can construct a group of all invertible elements referred to as the *Clifford group* $\Gamma(\mathcal{G})$ for an arbitrary geometric algebra $\mathcal{G}$ by

$$\Gamma(\mathcal{G}) := \left\{ \prod_{j=1}^{k} \boldsymbol{v_j} \mid k \in \mathbb{Z}^+,\ \forall j : 1 \leq j \leq k \ : \ \boldsymbol{v_i} \in V \text{ such that } g(\boldsymbol{v_i}, \boldsymbol{v_i}) \neq 0 \right\}. \tag{21}$$

We refer to elements of the Clifford group as *Clifford multivectors.* Note that Clifford multivectors are not necessarily blades since the product used in the construction is not the exterior product $\wedge$. For any Clifford multivectors $A = \boldsymbol{v_1} \cdots \boldsymbol{v_k}$ in the group Γ, we have that multiplicative inverse A^{-1} is given by $A^{-1} = \boldsymbol{v^k} \ldots \boldsymbol{v^1}$ as we can see that $A^{-1}A = AA^{-1} = 1$ by construction. Another note is that all scalars, vectors, pseudovectors, and pseudoscalars are always in the Clifford group and have multiplicative inverses. The inverse of a vector $\boldsymbol{v}$ is given by $\frac{\boldsymbol{v}}{\boldsymbol{v} \cdot \boldsymbol{v}}$. The form of the inverse motivates the utility of the *reverse* operator $\dagger$ defined so that $A^\dagger = \boldsymbol{v_k} \cdots \boldsymbol{v_1}$. For a r-blade A_r, the reverse also satisfies the relationship

$$A_r^\dagger = (-1)^{r(r-1)/2} A_r. \tag{22}$$

{eq:revers

One can see that the multiplicative inverse of an element of the Clifford group A is the reverse of the corresponding product of reciprocal vectors since $A_r^{-1} = (\boldsymbol{v^1} \cdots \boldsymbol{v^k})^\dagger$. When we take $V = \mathbb{R}^n$ with the Euclidean inner product, we can note that elements $s \in \Gamma^+(\mathcal{G}_n)$ act as rotations on multivectors $A \in \mathcal{G}_n$ through a conjugate action

$$A \mapsto sAs^{-1}. \tag{23}$$

In fact, all nonzero vectors $\boldsymbol{v} \in \Gamma(\mathcal{G}_n)$ define a reflection in the hyperplane perpendicular to $\boldsymbol{v}$ via the same conjugation action above.

Following these realizations, one can see that the Clifford group $\Gamma(\mathcal{G}_n)$ contains important subgroups such as the orthogonal and special orthogonal groups as quotients

$$\mathrm{O}(n) \cong \Gamma(\mathcal{G}_n)/(\mathbb{R} \setminus 0) \qquad \text{and} \qquad \mathrm{SO}(n) \cong \Gamma^+(\mathcal{G}_n)/(\mathbb{R} \setminus 0), \tag{24}$$

where $\mathbb{R} \setminus 0$ is the multiplicative group of real numbers. This motivates the following definition.

Definition 1.4. The *Clifford norm* $\|\cdot\|$ for $s \in \Gamma(\mathcal{G})$ is given by

$$\|s\|^2 := ss^\dagger. \tag{25}$$

Note that when the inner product is positive definite the Clifford norm is indeed a norm but can fail to be a norm in spaces with mixed signature (see eq. (56)). Also, note that for vectors the Clifford norm corresponds with the norm induced from the inner product in that with a vector $\boldsymbol{v}$ we have $\|\boldsymbol{v}\| = \boldsymbol{v}\boldsymbol{v}^\dagger = \boldsymbol{v} \cdot \boldsymbol{v}$. We also give the name *unit* to r-blades $\boldsymbol{A_r}$ with unit spinor norm $1 = \|\boldsymbol{A_r}\|$. Finally, this allows us to arrive at a definition for the classical pin and spin groups.

Definition 1.5. The *pin* and *spin* groups $\mathrm{Pin}(V)$ and $\mathrm{Spin}(V)$ are defined to be

$$\mathrm{Pin}(V) := \{s \in \Gamma(\mathcal{G}) \mid \|s\| = 1\}. \tag{26a}$$
$$\mathrm{Spin}(V) := \{s \in \Gamma^+(\mathcal{G}) \mid \|s\| = 1\}. \tag{26b}$$

Our focus will be the case where we take $\mathcal{G} = \mathcal{G}_n$ for we put $\mathrm{Spin}(n)$, but these statements can often be more broadly generalized. Moreover, we can realize this group as a quotient of the Clifford group $\Gamma(\mathcal{G}_n)$ by

$$\mathrm{Spin}(n) \cong \Gamma^+/\mathbb{R}_+, \tag{27}$$

where $\mathbb{R}_+$ is the multiplicative group of positive real numbers. The spin group $\mathrm{Spin}(V)$ is a Lie group usually derived via a short exact sequence of groups

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}(V) \to \mathrm{SO}(V) \to 1. \tag{28}$$

Here, we have given a more concrete realization of the spin group as special elements inside a geometric algebra. The Lie algebra of the spin group is denoted by $\mathfrak{spin}(V)$ and $\mathfrak{spin}(n)$ when referencing $\mathrm{Spin}(n)$. This algebra typically characterized as the tangent space of $\mathrm{Spin}(V)$ at the identity. However, through this approach, we realize that $\mathfrak{spin}(V)$ is isomorphic to the algebra of bivectors with the antisymmetric product $\times$. Then, for any bivector B, we can generate an element in the spin group given via the exponential

provide a citation.

$$e^B = \sum_{j=0}^{\infty} \frac{B^n}{n!}. \tag{29}$$

Fundamentally, the even subalgebra $\mathcal{G}^+$ is invariant under the action of $\mathrm{Spin}(V)$ since all elements in both sets are of even grade. This definition follows.

Definition 1.6. Let $\mathcal{G}$ be a geometric algebra with an inner product of arbitrary signature, then we define a *spinor* to be an element of $\mathcal{G}^+$.

Morally, this definition is telling us $\psi \in \mathcal{G}^+$ is an element that transforms under a left action of an element of $\mathrm{Spin}(V)$ to produce another spinor which leaves us with a convenient definition in that a spinor is simply an even multivector. For more on the topic, see [4].

1.1.3 Duality and pseudoscalars

Given access to an inner product we have a natural isomorphism between V and V^* by the Riesz representation. Namely, given an arbitrary basis $\boldsymbol{v_i}$ for V there exists the corresponding dual basis f_i for V^* such that $f_i(\boldsymbol{v_j}) = \delta_{ij}$. In geometric algebra, this notion is somewhat superfluous as we can realize the dual basis inside V itself in the following manner. Note that there is a unique map $\sharp \colon V^* \to V$ for which $f \mapsto \boldsymbol{f}^\sharp$ such that

$$\boldsymbol{f_i^\sharp} \cdot \boldsymbol{v_j} = \delta_{ij}. \tag{30}$$

Hence, if we simply put $\boldsymbol{v^i} \coloneqq \boldsymbol{f_i^\sharp}$ we can note that $\boldsymbol{v^i}$ is simply a vector in the geometric algebra.

Definition 1.7. Let $\boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_n}$ be an arbitrary vector basis of V generating $\mathcal{G}$. Then we have the *reciprocal basis* $\boldsymbol{v^1}, \boldsymbol{v^2}, \dots, \boldsymbol{v^n}$ satisfying

$$\boldsymbol{v^i} \cdot \boldsymbol{v_j} = \delta^i_j, \tag{31}$$

and we refer to each $\boldsymbol{v^i}$ as a *reciprocal vector*.

In terms of the inner product g, we have that the coefficients are given by $g_{ij} = \boldsymbol{v_i} \cdot \boldsymbol{v_j}$ and thus we have an explicit definition for the reciprocal vectors by putting $\boldsymbol{v^i} = g^{ij}\boldsymbol{v_j}$ where g^{ij} is the coefficients to matrix inverse of g_{ij} and we assume the Einstein summation convention.

There is of course an inverse to this isomorphism that we call $\flat \colon V \to V^*$ which is given by $\boldsymbol{v} \mapsto v^\flat$ satisfying

$$v_i^\flat(\boldsymbol{v_j}) = \delta_{ij}.$$

Given these identifications, there is no need to distinguish between the vector space V and its dual V^* as it suffices to consider V itself with reciprocal vectors $\boldsymbol{v^i}$ with the application of the scalar product.

Pseudoscalars are a deeply useful aspect of geometric algebra and we will now cover some of their utility. First and foremost, these pseudoscalars grant us a means of determining volumes. This will be a necessary notion in order to define integration in section 1.4.2.

Definition 1.8. Let $\mathcal{G}$ be a geometric algebra with an inner product with definite signature. Then the *volume element* in the arbitrary basis $\boldsymbol{v_1}, \dots, \boldsymbol{v_n}$ is

$$\boldsymbol{\mu} = \boldsymbol{v_1} \wedge \boldsymbol{v_2} \wedge \cdots \wedge \boldsymbol{v_n}. \tag{32}$$

It is worth noting that volume elements are invertible since they are blades in a geometric algebra with definite inner product. Similarly, any pseudoscalar is invertible in this case.

We also want to note that the volume element here fits our intuition and indeed we find

$$\|\boldsymbol{\mu}\| = \sqrt{\det(g)}. \tag{33}$$ {eq:pseudo

Since pseudoscalars are generated by a single element (recall there are $\binom{n}{n}$ independent grade-n elements), we should realize that the volume element is simply a scalar copy of a pseudoscalar that is unital.

Definition 1.9. Let $\boldsymbol{\mu}$ be the volume element, then we have the *unit pseudoscalar*

$$\boldsymbol{I} \coloneqq \frac{1}{\|\boldsymbol{\mu}\|}\boldsymbol{\mu}. \tag{34}$$

As is clear by the definition above, we must have that

$$\|\boldsymbol{I}\| = 1. \tag{35}$$

The unit pseudoscalar satisfies a useful relationship when swapping the left for right multiplication with an r-vector by

$$\boldsymbol{I}A_k = (-1)^{k(n-1)}A_k\boldsymbol{I}. \tag{36}$$

Thus, I commutes with the even subalgebra, and anticommutes with the odd subalgebra. We can also see that the inverse for unit pseudoscalar is $\boldsymbol{I}^{-1} = \boldsymbol{I}^\dagger$ which is an identification that we will often use. Formulas throughout are usually given in their most general context and substitution is done only when working with specialized algebras.

Note that for a homogeneous r-rector A_r we have that $A_r^\perp$ is an $n-r$-vector. Indeed, if we take an invertible r-blade $\boldsymbol{A_r}$, then we can find the $\boldsymbol{A_r}$*-subspace dual* of a multivector B by

$$B \rfloor \boldsymbol{A_r}^{-1}.$$

The notions of duality here give us geometrical insight. Taking an s-blade $\boldsymbol{B_s}$ we can note:

- If $s > r$, the $\boldsymbol{A_r}$-subspace dual of $\boldsymbol{B_s}$ vanishes.
- If $s = r$, the $\boldsymbol{A_r}$-subspace dual of $\boldsymbol{B_s}$ is a scalar and is zero if $\boldsymbol{B_s}$ contains a vector orthogonal to $\boldsymbol{A_r}$.
- If $s < r$, the $\boldsymbol{A_r}$-subspace dual of $\boldsymbol{B_s}$ represents the orthogonal complement of the subspace corresponding to $\boldsymbol{B_s}$ in the subspace corresponding to $\boldsymbol{A_r}$.

Since the pseudoscalar is a blade representing the entire vector space, this allows one to create dual elements within the entire vector space.

Definition 1.10. Given a multivector B, we define the *dual* of B to be

$$B^{\perp} := B \rfloor \boldsymbol{I}^{-1} \equiv A\boldsymbol{I}^{-1}. \tag{37}$$

The dual allows one to exchange interior and exterior products in the following way.

$$(A \wedge B)^{\perp} = A \rfloor B^{\perp} \tag{38}$$ {eq:wedge_

$$(A \rfloor B)^{\perp} = A \wedge B^{\perp} \tag{39}$$ {eq:dot_to

This shows the natural duality between the inner and exterior products and their interpretations as subspace operations. The duality extends further to provide an isomorphism between the spaces of r-vectors and $n-r$-vectors since for any r-vector A_r we have $A_r^{\perp}$ is an $n-r$-vector. It is under this isomorphism one can realize that all pseudovectors are $n-1$-blades.

oss_product **Remark 1.1.** If we consider $\mathcal{G}_3$, we can realize that the cross product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ by

$$\boldsymbol{u} \boldsymbol{\times} \boldsymbol{v} := (\boldsymbol{u} \wedge \boldsymbol{v})^{\perp} \equiv \boldsymbol{u} \boldsymbol{\times} \boldsymbol{v} = (\boldsymbol{u}^{\perp}) \times (\boldsymbol{v}^{\perp}), \tag{40}$$

where we use the bold notation for $\boldsymbol{\times}$ to distinguish between the bivector commutator product $\times$ defined in eq. (16). The special fact of $\mathcal{G}_3$ that is abused in a standard multivariate calculus course is that vectors and bivectors are dual to one another. One can also note that the vector $\boldsymbol{w} = \boldsymbol{u} \boldsymbol{\times} \boldsymbol{v}$ is sometimes refered to as axial and in other cases the pseudovector $\boldsymbol{u} \wedge \boldsymbol{v}$ is referred to as axial. The similar product notation of $\times$ and $\boldsymbol{\times}$ now becomes transparent.

1.1.4 Blades and subspaces

Each unit r-blade $\boldsymbol{A_r}$ ($\|\boldsymbol{A_r}\| = 1$) corresponds to a r-dimensional subspace and can be identified with a point in $\mathrm{Gr}(r,n)$. We will often allude to this identification directly by referring to a subspace via a reference to a blade, e.g., the subspace $\boldsymbol{A_r}$. This proves to be a wonderfully geometrical perspective on the products defined in eqs. (12) to (15). For example, we see that there are a handful of reasons to adopt the additional multiplication symbols $\rfloor$ and $\lfloor$ chisolm_geometric_2012 [1].

- The products $\rfloor$ and $\lfloor$ allow us to avoid needing to pay special attention to the specific grade of each multivector in a product. The product $\cdot$ on A_r and B_s depends on k and s and as such given by either $\rfloor$ or $\lfloor$ but one must know k and s in order to define this product exactly.

- We gain geometrical insight on the structure of r-blades in terms of their corresponding subspaces. Let $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ be nonzero blades with $r, s \geq 1$ then
  - $\boldsymbol{A_r} \rfloor \boldsymbol{B_s} = 0$ iff $\boldsymbol{A_r}$ contains a nonzero vector orthogonal to $\boldsymbol{B_s}$.
  - If $r < s$ then if $\boldsymbol{A_r} \rfloor \boldsymbol{B_s} \neq 0$ then the result is a $s - r$-blade representing the orthogonal complement of $\boldsymbol{A_r}$ in $\boldsymbol{B_s}$.
  - If $\boldsymbol{A_r}$ is a subspace of $\boldsymbol{B_s}$ then $\boldsymbol{A_r B_s} = \boldsymbol{A_r} \rfloor \boldsymbol{B_s}$.
  - If $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ are orthogonal, then $\boldsymbol{A_r B_s} = \boldsymbol{A_r} \wedge \boldsymbol{B_s}$.

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We also have the equivalences

$$A_r \cdot B_s \equiv A_r \rfloor B_s \qquad \text{if } k \leq s \tag{41}$$

{eq:left_c

$$A_r \cdot B_s \equiv A_r \lfloor B_s \qquad \text{if } k \geq s. \tag{42}$$

{eq:right_

For homogeneous r-vectors A_r and B_r, the products above simplify to

$$A_r \lfloor B_r \equiv A_r \rfloor B_r \equiv A_r \cdot B_r. \tag{43}$$

{dot_equiva

In fact, if we are given two r-blades $\boldsymbol{A_r} = \boldsymbol{a_1} \wedge \cdots \wedge \boldsymbol{a_r}$ and $B_r = \boldsymbol{b_r} \wedge \cdots \wedge \boldsymbol{b_r}$ we have the

$$A_k \cdot B_k^\dagger = \det(\alpha_i \cdot \beta_j)_{i,j=1}^k, \tag{44}$$

{eq:dot_pro

which is the typical extension of the inner product g to an inner product on $\bigwedge^r(V)$ through linearity.

Given the direct relationship between unit r-blades and r-dimensional subspaces we can also form a compact way of projecting multivectors into subspaces in a manner closely related to the subspace dual.

Definition 1.11. Given an multivector B, the *projection* onto the subspace $\boldsymbol{A_r}$ is

$$\mathrm{P}_B(\boldsymbol{A_r}) \coloneqq B \rfloor \boldsymbol{A_r} \boldsymbol{A_r}^{-1} \equiv (B \rfloor \boldsymbol{A_r}) \rfloor \boldsymbol{A_r}^{-1} \tag{45}$$

{eq:project

Following this definition, one can see that

$$\mathrm{P}_{\boldsymbol{A_r}}(B) \in \bigoplus_{j=0}^{r} \mathcal{G}^j = \mathcal{G}^{0+\cdots+r}, \tag{46}$$

since the subspace $\boldsymbol{A_r}$ is r-dimensional and moreover the operation preserves grades since

$$\mathrm{P}_{\boldsymbol{A_r}}(\langle B \rangle_j) \in \mathcal{G}^j. \tag{47}$$

For example, given vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ we retrieve the familiar statement

$$\mathrm{P}_{\boldsymbol{u}}(\boldsymbol{v}) = (\boldsymbol{v} \cdot \boldsymbol{u}) \frac{u}{\|u\|^2}. \tag{48}$$

A dual notion exists as well; we can project onto the subspace perpendicular to $\boldsymbol{A_r}$.

Definition 1.12. Given a multivector B, the *rejection* from the subspace $\boldsymbol{A_r}$ is

$$\mathrm{R}_{\boldsymbol{A_r}}(B) := B \wedge \boldsymbol{A_r} \boldsymbol{A_r}^{-1} = (B \wedge \boldsymbol{A_r}) \lfloor \boldsymbol{A_r}^{-1}. \tag{49}$$ {eq:rejecti

Note that this operation is also grade preserving. In the case we have a blade $\boldsymbol{C_k}$ with $k < r$ and $k < n - r$, we can note

$$\mathrm{P}_{\boldsymbol{A_r}}(\boldsymbol{C_k}) + \mathrm{R}_{\boldsymbol{A_r}}(\boldsymbol{C_k}) = \boldsymbol{C_k}. \tag{50}$$ {eq:projec

This provides us a way to revisit the geometric notions of the interior and exterior products. In particular, we note that

$$B \rfloor \boldsymbol{A_r} = \mathrm{P}_{\boldsymbol{A_r}}(B) \boldsymbol{A_r} \tag{51}$$
$$B \wedge \boldsymbol{A_r} = \mathrm{R}_{\boldsymbol{A_r}}(B) \boldsymbol{A_r}. \tag{52}$$

Both the notion of projection and rejection prove to be useful and behave nicely with the dual by

$$\mathrm{P}_{\boldsymbol{A_r}^{\perp}}(B) = \mathrm{R}_{\boldsymbol{A_r}}(B). \tag{53}$$ {eq:projec

Finally, the exterior product of orthogonal blades gives us a direct sum of subspaces in the following sense. Let $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ be orthogonal so that $\boldsymbol{A_r} \wedge \boldsymbol{B_s} = \boldsymbol{A_r} \boldsymbol{B_s}$, then we can note that if $k < r$ and $k < s$ we have

$$\mathrm{P}_{\boldsymbol{A_r} \wedge \boldsymbol{B_s}}(\boldsymbol{C_k}) = \mathrm{P}_{\boldsymbol{A_r}}(\boldsymbol{C_k}) + \mathrm{P}_{\boldsymbol{B_s}}(\boldsymbol{C_k}). \tag{54}$$ {eq:projec

Perhaps it is most enlightening for the reader to revisit eqs. (50) and (54) replacing $\boldsymbol{C_k}$ with a vector $\boldsymbol{v}$ since a vector will always prove to be a representative for a “small enough” subspace.

1.2 Motivating example

ng_example

Rather than a sequence of multiple examples, it will prove to be far more illuminating to construct one large example for which most of the preliminaries to this point can be used in a meaningful way. As such, we shall not rule out the utility of geometric algebras with pseudo inner products. The classical example is the *spacetime algebra* defined by taking $V = \mathbb{R}^4$ with a vector basis $\boldsymbol{\gamma_0}, \boldsymbol{\gamma_1}, \boldsymbol{\gamma_2}, \boldsymbol{\gamma_3}$ satisfying

$$\boldsymbol{\gamma_0} \cdot \boldsymbol{\gamma_0} = -1 \tag{55a}$$
$$\boldsymbol{\gamma_0} \cdot \boldsymbol{\gamma_i} = 0 \qquad i = 1, 2, 3 \tag{55b}$$
$$\boldsymbol{\gamma_i} \cdot \boldsymbol{\gamma_j} = \delta_{ij}, \qquad i, j = 1, 2, 3. \tag{55c}$$

We refer to $\boldsymbol{\gamma_0}$ as *temporal* and $\boldsymbol{\gamma_i}$ for $i = 1, 2, 3$ as *spatial*. For this basis, we can denote the matrix for this inner product $\eta = \mathrm{diag}(- + ++)$ (often called the *Minkowski metric*) and define Q from η. Then, we have for a spacetime vector $\boldsymbol{v} = v_0 \boldsymbol{\gamma_0} + v_1 \boldsymbol{\gamma_1} + v_2 \boldsymbol{\gamma_2} + v_3 \boldsymbol{\gamma_3}$ we have

$$\|\boldsymbol{v}\| = \boldsymbol{v} \cdot \boldsymbol{v} = -a_0^2 + \sum_{i=1}^{3} a_i^2. \tag{56}$$ {eq:spaceti

Then we put $\mathcal{G}_{1,3}$ to represent the spacetime algebra and, in broader generality, we put $\mathcal{G}_{p,q}$ for a geometric algebra with p temporal vectors and q spatial vectors. The reader may wish to, for example, revisit section 1.1.2 with $\mathcal{G}_{p,q}$ in mind in order to see a realization of the groups $\mathrm{SO}(p,q)$, $\mathrm{Spin}(p,q)$, and the spacetime spinors.

As the naming above suggests, the geometric algebra of Euclidean space, $\mathcal{G}_3$, should naturally inside of the spacetime algebra. Note that we have the spatial pseudoscalar $\boldsymbol{\gamma_1} \wedge \boldsymbol{\gamma_2} \wedge \boldsymbol{\gamma_3}$, which, allowing for an extension of our notion of projection to the whole algebra, allows us to put

$$\mathrm{P}_{\boldsymbol{\gamma_1}\wedge\boldsymbol{\gamma_2}\wedge\boldsymbol{\gamma_3}}(\mathcal{G}_{1,3}) \equiv \mathrm{R}_{\boldsymbol{\gamma_0}}(\mathcal{G}_{1,3}) = \mathcal{G}_3. \tag{57}$$

Perhaps one could refer to this mapping as the *static map* as we project only onto the spatial subspace and, via duality, reject the temporal subspace. It is also worth noting that this static map is not only an isomorphism, but a direct equivalence of algebras. Now, in $\mathcal{G}_3$, we can specify an arbitrary multivector A by

$$A = a_0 + a_1\boldsymbol{\gamma_1} + a_2\boldsymbol{\gamma_2} + a_3\boldsymbol{\gamma_3} + a_{12}\boldsymbol{B_{12}} + a_{13}\boldsymbol{B_{13}} + a_{23}\boldsymbol{B_{23}} + a_{123}\boldsymbol{\gamma_1}\wedge\boldsymbol{\gamma_2}\wedge\boldsymbol{\gamma_3}, \tag{58}$$

and so the grade projections read

$$\langle A\rangle_0 = a_0 \tag{59a}$$
$$\langle A\rangle_1 = a_1\boldsymbol{\gamma_1} + a_2\boldsymbol{\gamma_2} + a_3\boldsymbol{\gamma_3} \tag{59b}$$
$$\langle A\rangle_2 = a_{12}\boldsymbol{B_{12}} + a_{13}\boldsymbol{B_{13}} + a_{23}\boldsymbol{B_{23}} \tag{59c}$$
$$\langle A\rangle_3 = a_{123}\boldsymbol{\gamma_1}\wedge\boldsymbol{\gamma_2}\wedge\boldsymbol{\gamma_3}. \tag{59d}$$

Then, we can write a even multivector as

$$q = q_0 + q_{23}\boldsymbol{B_{23}} + q_{31}\boldsymbol{B_{31}} + q_{12}\boldsymbol{B_{12}}. \tag{60}$$

Note as well that

$$\boldsymbol{B_{23}}^2 = \boldsymbol{B_{31}}^2 = \boldsymbol{B_{12}}^2 = -1 \tag{61a}$$
$$\boldsymbol{B_{23}}\boldsymbol{B_{31}}\boldsymbol{B_{12}} = +1, \tag{61b}$$

which is typical for spatial bivectors. In this case, one may notice that this even subalgebra is extremely close to being a copy of the quaternion algebra $\mathbb{H}$. One can arrive at a representation of the quaternions by taking

$$\boldsymbol{i} \leftrightarrow \boldsymbol{B_{23}}, \quad \boldsymbol{j} \leftrightarrow -\boldsymbol{B_{31}} = \boldsymbol{B_{13}}, \quad \boldsymbol{k} \leftrightarrow \boldsymbol{B_{12}}, \tag{62}$$

and noting that we then have $\boldsymbol{ijk} = -1$ as well as $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1$. A more in depth explanation is provided in [2]. Thus, a purely we realize a quaternion as a parabivector q and a purely imaginary quaternion is simply the grade-2 portion of the parabivector q. We also realize $\mathbb{H}$ as scalar copies of elements of $\mathrm{Spin}(3) \cong \mathrm{Sp}(1)$. That is to say that $\mathbb{H} \cong \mathbb{R} \times \mathrm{Spin}(3)$. Indeed, since elements of $\mathcal{G}_3^+$ are simply parabivectors, the parabivectors admit a natural spin representation.

But we are not done here, and we can project down one dimension further by

$$\mathrm{P}_{\boldsymbol{\gamma_1}\wedge\boldsymbol{\gamma_2}}(\mathcal{G}_3) = \mathcal{G}_2.$$

To see this in action, we let $\boldsymbol{v} = v_1\boldsymbol{\gamma_1} + v_2\boldsymbol{\gamma_2} + v_3\boldsymbol{\gamma_3}$

$$\mathrm{P}_{\boldsymbol{\gamma_1}\wedge\boldsymbol{\gamma_2}} = \mathrm{P}_{\boldsymbol{B_{12}}}(\boldsymbol{v}) = (v_1\boldsymbol{\gamma_1} + v_2\boldsymbol{\gamma_2} + v_3\boldsymbol{\gamma_3})\rfloor\boldsymbol{B_{12}}\boldsymbol{B_{12}}^{-1} \tag{63a}$$
$$= (v_1\boldsymbol{\gamma_2} - v_2\boldsymbol{\gamma_1})\boldsymbol{B_{12}}^{-1} \tag{63b}$$
$$= v_1\boldsymbol{\gamma_1} + v_2\boldsymbol{\gamma_2}. \tag{63c}$$

Then, arbitrary multivectors A and B can be specified by

$$A = a_0 + a_1\boldsymbol{\gamma_1} + a_2\boldsymbol{\gamma_2} + a_{12}\boldsymbol{B_{12}}, \qquad B = b_0 + b_1\boldsymbol{\gamma_1} + b_2\boldsymbol{\gamma_2} + b_{12}\boldsymbol{B_{12}}.$$

We can then take the product AB to yield

$$\langle AB\rangle_0 = a_0b_0 + a_1b_1 + a_2b_2 - a_{12}b_{12} \tag{64a}$$
$$\langle AB\rangle_1 = (a_0b_1 + a_1b_0 - a_2b_{12} + a_{12}b_2)\boldsymbol{\gamma_1} + (a_0b_2 + a_2b_0 + a_1b_{12} - a_{12}b_1)\boldsymbol{\gamma_2} \tag{64b}$$
$$\langle AB\rangle_2 = (a_1b_2 - a_2b_1)\boldsymbol{B_{12}}. \tag{64c}$$

Most notably, we see that $\boldsymbol{B_{12}}^2 = -1$ and this allows us to consider a parabivector

$$z = x + y\boldsymbol{B_{12}} \tag{65}$$

which is exactly a representation of the complex number $\zeta = x + iy$ in $\mathcal{G}_2^{0+2} = \mathcal{G}_2^+$. Thus, the even subalgebra of this geometric algebra is indeed isomorphic to the complex numbers $\mathbb{C}$. Indeed, there is one unit 2-blade B_{12} in $\mathcal{G}_2$ to form the spin algebra $\mathfrak{spin}(2) \cong \mathbb{R}$ and as a consequence all unit norm elements in $\mathcal{G}_2^+$ can be written as

$$e^{\theta\boldsymbol{B_{12}}} = \sum_{n=0}^{\infty}\frac{\theta\boldsymbol{B_{12}}}{n!} = \cos(\theta) + \boldsymbol{B_{12}}\sin(\theta), \tag{66}$$

where $\theta\boldsymbol{B_{12}}$ is a general bivector in $\mathcal{G}_2$ when $\theta \in \mathbb{R}$ is arbitrary. Hence, we arrive at $\mathrm{Spin}(2) \cong \mathrm{U}(1)$. Any element in $\mathbb{C}$ is also a scaled version of an element of the spin group $\mathrm{Spin}(2)$. Hence, we can use a spin representation for an element in $\mathbb{C}$ via $z = re^{\theta\boldsymbol{B_{12}}} \in \mathbb{R}\times\mathrm{Spin}(2)$. This special case shows that parabivectors in $\mathcal{G}_2$ have a unique spin representation and they are spinors as well since the whole of the even subalgebra consists of parabivectors.

But, the above work is not necessary special to the starting point of $\mathcal{G}_{1,3}$ or $\mathcal{G}_3$. In fact, if we take $\mathcal{G}_n$ for $n \geq 2$, then there are natural copies of $\mathbb{C}$ contained inside of $C\ell(V,Q)$. In particular, we have the isomorphism

$$\mathbb{C} \cong \{\lambda + \beta\boldsymbol{B} \mid \lambda, \beta \in \mathcal{G}_n^0,\ \boldsymbol{B} \in \mathrm{Gr}(2,n).\}, \tag{67}$$

which shows that complex numbers arise as parabivectors via the representation

$$\zeta = x + y\boldsymbol{B}, \tag{68}$$

since $\boldsymbol{B}^2 = -1$. Given the standard basis $\boldsymbol{e_1}, \dots, \boldsymbol{e_n}$ we have copies of $\mathbb{C}$ for each of the $\binom{n}{2}$ unit bivectors B_{jk} with $k = 2, \dots, n$ and $j < k$.

Talk about bivectors, spinors, and rotors. Rotations and what not. Euler angles. Would all be good to put in

1.3 Multivector fields

We want to generalize the setting of geometric algebra to include a smooth structure. One can take the work above for $\mathcal{G}_n$ and consider a C^∞-module structure as opposed to the $\mathbb{R}$-algebra structure in the proceeding section. For brevity, we put $\mathcal{G}_n(\mathbb{R}^n)$ for the C^∞-module and $\mathcal{G}_n$ for the $\mathbb{R}$-algebra. The multivectors themselves can be realized as constant multivector fields so that $\mathcal{G}_n \subset \mathcal{G}_n(\mathbb{R}^n)$. This smooth setting simply makes the coefficients of the global basis blades given by C^∞ functions as opposed to $\mathbb{R}$ scalars. In this case, we refer to a generic element in the C^∞-module $\mathcal{G}_n$ as a *multivector field*. We take $\Omega \subset \mathbb{R}^n$ as a connected region in $\mathbb{R}^n$ for the entirety of this paper and we put

$$\mathcal{G}_n(\Omega) \coloneqq \{f \colon \Omega \to \mathcal{G}_n \mid f \text{ is } C^\infty\text{-smooth}\},$$

where smoothness is meant in terms of the C^∞-module structure.

Perhaps the C^∞-module structure obfuscates the point slightly. Instead, one should think of the fields in $\mathcal{G}_n(\Omega)$ as multivector valued functions on $\Omega \subset \mathbb{R}^n$. Taking this identification allows for an extended toolbox at our disposal. In particular, points in Ω are uniquely identified with constant vector fields in $\mathcal{G}_n^1$ and one can consider endomorphisms living in $\mathcal{G}_n$ (acting on $\mathcal{G}_n^1$) as acting on the input of fields in $\mathcal{G}_n(\Omega)$ as well. Thus, there is not only an algebraic structure on the fields themselves, but on the point in which the field is evaluated. This is perhaps the key insight on why authors developed the so-called vector manifolds widely used in the geometric algebra landscape.

Example 1.1. Consider a multivector field f valued in $\mathcal{G}_n(\mathbb{R}^n)$. With $x \in \mathbb{R}^n$ being identified with the vector in $\mathcal{G}_n^1$, we output a multivector $f(x) \in \mathcal{G}_n$ at each point x. One may be interested in the restriction of f to a vector subspace of $\mathbb{R}^n$ which amounts to using projection on the input. For example, perhaps we wish to know how f behaves on the subspace corresponding to some r-blade $\boldsymbol{A_r}$. As such, it suffices to then study $f(\mathrm{P}_{\boldsymbol{A_r}}(x))$.

We refer to smooth fields valued in $\mathcal{G}_n^+$ as *spinor fields* and put $\mathcal{G}_n^+(\Omega)$ to refer to the C^∞-module counterpart. These fields will be shown to carry a Banach algebra structure.

1.3.1 Directional derivative and gradient

Note that $\mathbb{R}^n$ has global coordinates and thus we can choose a global constant vector field basis $e_1, \dots, e_n$ and we generate $\mathcal{G}_n$ from this basis. Note that we will adopt the Einstein summation convention when needed. With respect to these fields, we have the *directional derivative* ∇_ω with $\omega = \omega^i e_i$. The *gradient* (or *Dirac operator*) is defined as $\boldsymbol{\nabla} = \sum_i e^i \nabla_{e_i}$ and it acts a grade-1 element in the algebra. Note then that $\omega \cdot \boldsymbol{\nabla} = \nabla_\omega$ defines the directional derivative via the gradient. The directional derivative is also grade preserving in that for a multivector A

$$\nabla_\omega \langle A \rangle_r = \langle \nabla_\omega A \rangle_r. \tag{69}$$

This structure defined above is typically referred to as *geometric calculus*. The setting for geometric calculus extends the setting of differential forms and reduces some of the complexity with tensor computations. Since $\boldsymbol{\nabla}$ is a grade-1 object, it acts on a homogeneous r-vector A_r by

$$\boldsymbol{\nabla} A_r = \langle \boldsymbol{\nabla} A_r \rangle_{r-1} + \langle \boldsymbol{\nabla} A_r \rangle_{r+1} \coloneqq \boldsymbol{\nabla} \rfloor A_r + \boldsymbol{\nabla} \wedge A_r. \tag{70}$$

Thus, the gradient splits into two operators,

$$\boldsymbol{\nabla}\rfloor \colon \mathcal{G}_n^r(\Omega) \to \mathcal{G}_n^{r-1}(\Omega), \tag{71}$$
$$\boldsymbol{\nabla}\wedge \colon \mathcal{G}_n^r(\Omega) \to \mathcal{G}_n^{r+1}(\Omega), \tag{72}$$

which satisfy the properties

$$(\boldsymbol{\nabla}\wedge)^2 = 0, \tag{73}$$
$$(\boldsymbol{\nabla}\rfloor)^2 = 0, \tag{74}$$

{eq:differ

when acting on a homogeneous r-vector. Since eq:differential_properties 73 holds, the gradient operator gives rise to the grade preserving *Laplace-Beltrami operator*

$$\Delta = \boldsymbol{\nabla}^2 = \boldsymbol{\nabla}\rfloor \circ \boldsymbol{\nabla} \wedge + \boldsymbol{\nabla} \wedge \circ \boldsymbol{\nabla}\rfloor,$$

which is manifestly coordinate invariant by definition. It also motivates the use of the physicist notation $\boldsymbol{\nabla}^2 = \Delta$, but we do not adopt this here. We refer to multivector fields f in the kernel of the Laplace-Beltrami operator *harmonic*.

1.3.2 Monogenic fields

Geometric calculus includes another definition for multivectors that is a big motivation for those who study Clifford analysis.

Definition 1.13. Let $f \in \mathcal{G}_n(\Omega)$. Then we say that f is *monogenic* if $f \in \ker(\boldsymbol{\nabla})$.

Monogenic fields are of utmost importance as they have many beautiful properties. One should find them as a suitable generalization of the notion of complex holomorphicity. For example, in regions of Euclidean spaces, a monogenic field f can be completely determined by its Dirichlet boundary values through a generalized Cauchy integral formula. For any even monogenic field, the each of the graded components of f are harmonic.

We put

$$\mathcal{M}(\Omega) \coloneqq \{f \in \mathcal{G}_n(\Omega) \mid \boldsymbol{\nabla} f = 0\}$$

to refer to elements of this set as *monogenic fields* on Ω. As a subset, we also have the *monogenic spinors* $\mathcal{M}^+(\Omega)$, which are simply the even monogenic fields and the *monogenic parabivectors* $\mathcal{M}^{0+2}(\Omega)$. Though these spaces do not form algebras in their own right, they do indeed form a vector space as sums of monogenic functions are monogenic due to the linearity of the gradient. Moreover, the monogenic spinors are invariant under multiplication from the Clifford group Γ^+.

_invariant

Lemma 1.1. *Let $s \in \mathrm{Spin}(n)$ then $\boldsymbol{\nabla} \circ s = s \circ \boldsymbol{\nabla}$. In particular, the space of monogenic spinors $\mathcal{M}^+(\Omega)$ is $\mathrm{Spin}(n)$ invariant.*

This lemma is classical in the theory of the Dirac operator, Clifford analysis, and harmonic analysis so we omit a proof. One can see janssens_special_nodate [4], for example.

1.4 Differential forms

diff_forms

It has become clear that geometric algebra and geometric calculus combine into a single toolbox of multivector field analysis that is useful for vector space algebra and the calculus of $\mathbb{R}^n$. Conveniently, the language of differential forms rests neatly inside this toolbox as well. As such, we will also develop a means of integrating multivector fields. In this subsection we connect the two together into a single framework and note the additional benefits geometric algebra and calculus provide over forms. In order to do so, we appeal to the language of differential forms and build a relationship between multivector fields and forms through measures. Forms have their appeal in global understanding via their properties through integration (e.g., Stokes' and Green's theorems) and their utility extends to boundary value problems schwarz_hodge_1995 [6].

mention that we tend to just think of forms of a single degree but multi-vectors will combine the various degrees with some meaning

Given that there exists a global coordinate system x^i on $\mathbb{R}^n$, we can place this set of coordinates on any region Ω. Then, we form the basis of tangent vector fields $\partial_i = \frac{\partial}{\partial x^i}$ with the reciprocal 1-forms dx^i that are each global sections of $T^*\Omega$ and are the exterior derivatives (or gradients) of the coordinate functions. Typically, 1-forms are viewed as linear functionals on tangent vector fields and in these coordinates we have $dx^i(\partial_j) = \delta^i_j$. The benefit of this definition is that the 1-forms dx^i carry a natural measure and we can form product measures via the exterior product $\wedge$. For example, for a 2-dimensional surface Σ we have the *directed measure* $d\Sigma = e_i \wedge e_j dx^i dx^j$ and we can note that $(e^i \wedge e^j) \cdot d\Sigma^\dagger = dx^i dx^j - dx^j dx^i$ is completely antisymmetric and provides us a surface measure we can integrate; this is a differential 2-form.

In an n-dimensional space with a position dependent inner product g, we have the n-dimensional volume directed measure $d\Omega = \sqrt{|g|}dx^1 \dots dx^n$. If we then define $dX_n = e^n \wedge \cdots \wedge e^1 dx^1 \dots dx^n$ we then find that

Now I should show why it's $dX_n^\dagger$ due to the r-vector inner product. Also only some dX_k are blades? Worth mentioning?

$$d\Omega = I^\dagger \cdot dX_n.$$

Here I is the pseudoscalar field defined on Ω with respect to g. Similarly, for $k < n$, we can define the k-dimensional volume measure as

$$dX_k = \frac{1}{k!}(e^{i_k} \wedge \cdots \wedge e^{i_1}) dx^{i_1} \cdots dx^{i_k}.$$

We can now write a k-form α_k as $\alpha_k = A_k \cdot dX_k$. In this sense, a differential form is made up of two essential components namely the multivector field and the k-dimensional volume directed measure. For example, if we wish to write a 2-form α_2 we take $dX_2 = \frac{1}{2!} e^j \wedge e^i dx^i dx^j$ and $A_2 = a_{ij} e_i \wedge e_j$ to yield

$$\alpha_2 = A_2 \cdot dX_2 = \frac{a_{ij}}{2!}(e_i \wedge e_j) \cdot (e^j \wedge e^i) dx^i dx^j = \frac{a_{ij}}{2!}(dx^i dx^j - dx^j dx^i)$$

Thus, we arrive at an isomorphism between k-forms and k-vectors as a contraction with the k-dimensional volume directed measure dX_k since

$$\alpha_k = A_k \cdot dX_k.$$

Hence, we can see now how a differential form simply appends the measure attached to the underlying space. We can also see how this generalizes the musical isomorphisms $\sharp$ and $\flat$ by taking a vector field a and noting

$$a \cdot dX_1 = a^i e_i \cdot e^j dx^j = a^i dx^i, \tag{75}$$

{eq:line_el

corresponds to the usual $\flat$ map on vector fields.

1.4.1 Exterior algebra and calculus

The exterior algebra of differential forms comes with an addition $+$ and exterior multiplication $\wedge$. We note that the sum of two k-forms α_k and β_k that $\alpha_k + \beta_k$ is also a k-form which we can see by letting $\alpha_k = A_k \cdot dX_k$ and $\beta_k = B_k \cdot dX_k$ and putting

$$\alpha_k + \beta_k = (A_k \cdot dX_k) + (B_k \cdot dX_k) = (A_k + B_k) \cdot dX_k,$$

due to the linearity of $\cdot$. If instead had an s form β_s then we have the exterior product

$$\alpha_k \wedge \beta_s = (A_k \wedge B_k) \cdot dX_{k+s},$$

where $dX_{k+s} = 0$ if $k + s > n$.

With differential forms one also has the exterior derivative d giving rise to the calculus of forms. Given we can write a differential k-form as $\alpha_k = A_k \wedge dX_k$, In particular, we have

$$d\alpha_k = (\boldsymbol{\nabla} \wedge A_k) \cdot dX_{k+1},$$

which realizes the exterior derivative as the grade raising component of the gradient $\boldsymbol{\nabla}$. Of course, for scalar fields, this returns the gradient as desired.

Here, $\boldsymbol{\nabla}\wedge$ can be identified with the exterior derivative d and $\boldsymbol{\nabla}\rfloor$ can be identified with the codifferential δ on differential forms up to a sign schindler_geometric_2020 [5]. This of course means the standard properties that apply to d and δ apply to $\boldsymbol{\nabla}\wedge$ and $\boldsymbol{\nabla}\rfloor$.

1.4.2 Integration on chains

bmanifolds

rewrite this in terms of chains and currents?

Given a k-dimensional submanifold of $K \subset \Omega$ with a k-form α_k defined on K, we can integrate the k-form. Using the multivector equivalents leads us to the k-dimensional directed measure dK for the submanifold K. Given K is a submanifold of Ω, for any $x \in K$ we have tangent space T_xK which corresponds to a tangent k-blade $I_K(x)$. We put I_K as the smooth k-blade field everywhere tangent to K. Then we have the directed volume measure on K given by

$$dK = I_K^\dagger \cdot dX_k.$$

For a tangent k-vector field A_k on K, we must have for any $x \in K$ that $f = \mathrm{P}_{I_K} \circ f$ so that these fields lie purely tangent to K. In particular, we can always put $A_k = AI_k^\dagger$ for a scalar field A. These fields can contract with the directed measure dX_k to create a k-form on K by $\alpha_k = A_k \cdot dX_k = AdK$ which can be integrated as

$$\int_K \alpha = \int_K AdK.$$

Hence, on Ω itself, we can decompose top grade forms by taking

$$\alpha_n = A_n \cdot dX_n = AI^\dagger \cdot dX_n$$

for a scalar field A satisfying $A_n = AI^\dagger$. Then this form can be integrated by

$$\int_\Omega \alpha_n = \int_\Omega A d\Omega.$$

There is also the normal space $N_x K$ that is everywhere orthogonal (with respect to g on Ω) to $T_x K$. In particular, we have the normal $(n-k)$-blade field $\nu = I_K^\dagger I$. Note that for a unit k-blade I_K we have $I_K^{-1} = I_K^\dagger$ and we see $I_K \nu = I$. Since K is a submanifold of Ω we have the inclusion $\iota \colon K \to \Omega$ and the induced pullback on forms ι^* which is equivalent to the tangent projection operator $\boldsymbol{t}_K$ seen in schwarz_hodge_1995 [6]. Given a p-form α_p defined on Ω, we have that $\boldsymbol{t}_K \alpha_p = \alpha_p \circ \mathrm{P}_{I_K}$. Specifically, $\alpha_p = A_p \cdot dX_p$ we have

I should probably work through this to show it's true

$$\boldsymbol{t}_K \alpha_p = A_p \cdot (dX_p \circ \mathrm{P}_{I_K}) = \mathrm{P}_{I_K}(A_p) \cdot dX_p = \mathrm{R}_\nu(A_p) \cdot dX_p.$$

The normal projection $\boldsymbol{n}_K$ is then $\boldsymbol{n}_K \alpha_p = \alpha_p - \boldsymbol{t}_K \alpha_p$ and moreover

$$\boldsymbol{n}_K \alpha_p = \mathrm{R}_{I_K}(A_p) \cdot dX_p = \mathrm{P}_\nu(A_p) \cdot dX_p.$$

Not sure this is true. but `https://en.wikipedia.org/wiki/Geometric_algebra` talks about projection and rejection.

This is pertinent when we take the submanifold $\Sigma = \partial\Omega$. There, I_Σ yields the directed measure

$$d\Sigma \coloneqq I_\Sigma^\dagger \cdot dX_{n-1}.$$

The normal space is 1-dimensional and ν is the unit normal vector to the boundary. The pullback coincides with projection into the tangent space given by I_Σ. Then, for 1-forms $\alpha = \cdot dX_1$ it is apparent that $\boldsymbol{t}_\Sigma \alpha = \mathrm{P}\, I_\Sigma a \cdot dX_1$ and $\boldsymbol{n}_\Sigma \alpha = \mathrm{R}_{I_\Sigma}(a) \cdot dX_1$ by Equations eq:projection+re ?? and eq:projection_rejection_vector ??. One can then find the flux of a vector field through Σ arises as an $(n-1)$-form $\mathrm{P}\, \nu a I^{-1} \cdot dX_{n-1}$. Once again we see that the flux is determined both by the vector field a and the local geometry of Σ captured by $d\Sigma$ in the following way. Note that $\nu^{-1} = \nu$ since $\|\nu\| = 1$ everywhere on Σ and so $\mathrm{P}\, \nu a I^{-1} = a \cdot \nu\nu I^{-1} = a \cdot \nu I_\Sigma^\dagger$ which gives us the corresponding form $a \cdot \nu d\Sigma$ and the total flux of a through Σ is then

$$\int_\Sigma (\mathrm{P}\, \nu a I^{-1}) \cdot dX_{n-1} = \int_\Sigma a \cdot \nu d\Sigma.$$

1.4.3 k-form inner product

Show that this relates back to the spinor norm.

For smooth k-forms $\alpha_k = A_k \cdot dX_k$ and $\beta_k = B_k \cdot dX_k$, we have an inner product

$$\langle \alpha_k, \beta_k \rangle = \int_\Omega \alpha_k \wedge \star \beta_k$$

where $\star$ is the Hodge star. The Hodge star on k-forms inputs a k-form and outputs a a specific dual $(n-k)$-form so that we always have $\alpha_k \wedge \star\beta_k = (A_k \cdot B_k^\dagger)d\Omega$ as we note Equation eq:dot_product 44. Thus, we can realize how $\star$ acts on multivector representative. We let $\star\beta_k = B_k^\star \cdot dX_{n-k}$ by $B_k^\star = (I^{-1}B_k)^\dagger$. Indeed, we have

$$\begin{aligned}\alpha_k \wedge \star\beta_k &= (A_k \wedge B_k^\star) \cdot dX_n \\ &= A_k \cdot B_k^\dagger d\Omega.\end{aligned}$$

1.4.4 Stokes' and Green's theorem

For regions Ω with boundary Σ, we have a compact form of Stokes' theorem

$$\int_\Omega d\alpha_{n-1} = \int_\Sigma \boldsymbol{t}_\Sigma \alpha_{n-1},$$

for sufficiently smooth $(n-1)$-forms α_{n-1}. Taking the multivector equivalent $\alpha_{n-1} = A_{n-1} \cdot dX_{n-1}$ we retrieve *Stokes' theorem* as

$$\int_\Omega (\boldsymbol{\nabla} \wedge A_{n-1}) \cdot dX_n = \int_\Sigma \mathrm{P}\, I_\Sigma A_{n-1} \cdot dX_{n-1} = \int_\Sigma \mathrm{R}_\nu(A_{n-1}) \cdot dX_{n-1}$$

Let $v = A_{n-1}I$ denote the vector field dual to the pseudovector A_{n-1}, then we have the more recognizable (Helmholtzian?) form of Stokes' theorem

$$\int_\Omega \boldsymbol{\nabla} \cdot v d\Omega = \int_\Sigma v \cdot \nu d\Sigma.$$

Show this work?

This provides a compact relationship for those who choose to work with vector fields and those who choose to work with forms. Pause and reflect here about what Green's theorem and Stokes' theorem are really saying. Use DUAL notation!

Finally,

More on integration and do Ohms law as an example of some of this stuff. Do all of hodge decomposition and stuff?

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