Hodge Theory, Gelfand Theory, and Clifford Analysis Applied to Tomography

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Overview

Section 1

Introduction

Motivating problems

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- The ***Calderón problem*** replaces the medium with a manifold M, conductivity with g, and replaces the voltage-to-current map with the Dirichlet-to-Neumann operator Λ.

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- Do these functions also contain metric data?
- Can we access these functions from the boundary?

Subsection 1

Preliminaries

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- The associated ***Clifford algebra*** is the quotient

$$C\ell(V, g) \coloneqq \mathcal{T}(V)/\langle \mathbf{v} \otimes \mathbf{v} - g(\mathbf{v}, \mathbf{v})\rangle.$$

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- The completely degenerate case is the ***exterior algebra***

$$\bigwedge(V) := C\ell(V, 0).$$

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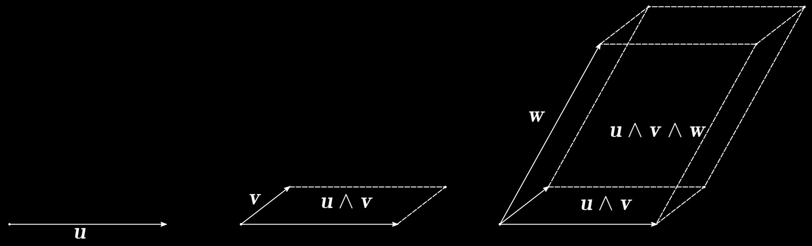
  - Elements of the even grade subalgebra, $\mathcal{G}^+$, are called ***spinors***.
- Since $\mathcal{G} = \bigoplus_{r=0}^{n} \mathcal{G}^r$ a general ***multivector*** is $A = \sum_{r=0}^{n} \langle A \rangle_r$.



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- The most important products for us are

$$A_r \wedge B_s := \langle A_r B_s \rangle_{r+s}$$

$$A_r \lrcorner B_s := \langle A_r B_s \rangle_{s-r}$$

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- The *reverse* $\dagger$ is extended linearly from the action

$$(\boldsymbol{v}_1 \boldsymbol{v}_2 \cdots \boldsymbol{v}_r)^\dagger = \boldsymbol{v}_r \cdots \boldsymbol{v}_2 \boldsymbol{v}_1.$$

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- The *projection* of A into a subspace $\boldsymbol{U_r}$ by

$$\mathrm{P}_{\boldsymbol{U_r}}(A) \coloneqq A \lrcorner \boldsymbol{U_r} \boldsymbol{U_r}^{-1}.$$

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$$\boldsymbol{I} := \frac{1}{|\boldsymbol{\mu}|}\boldsymbol{\mu}.$$

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- Dual exchanges products $(A \lrcorner B)^{\perp} = A \wedge B^{\perp}$.

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- *Complex algebra* $\mathbb{C}$ is isomorphic to $\mathcal{G}_{0,2}^+$.
  - Standard basis $\boldsymbol{e}_1, \boldsymbol{e}_2$, and $\boldsymbol{e}_{12} := \boldsymbol{e}_1\boldsymbol{e}_2$. Then $\boldsymbol{e}_{12}^2 = -1$.
  - Right multiplication of vectors by $\boldsymbol{e}_{12}$ rotates counter-clockwise by $\pi/2$.

Section 2

Clifford analysis

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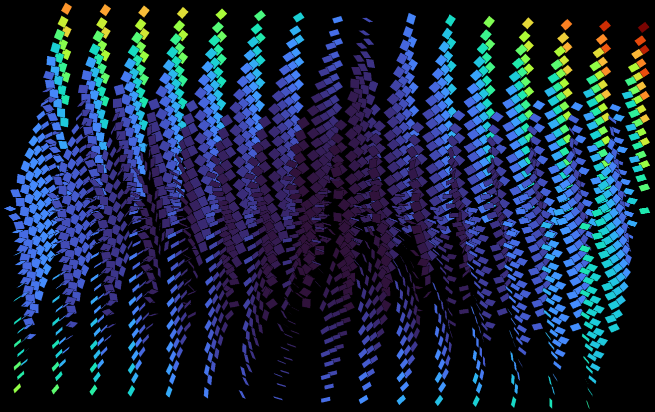
- Take same naming scheme and notation: $\mathfrak{X}^r(M)$, $\mathfrak{X}^+(M)$, etc.

Scalar field

$$\langle p_{(1,0)} + p_{(0,1)} + p_{(1,1)} + p_{(2,0)} + p_{(0,2)} + p_{(3,0)} + p_{(0,3)} + p_{(2,1)}(1 + \mathbf{e}_{31}) + p_{(1,2)}(1 + \mathbf{e}_{31}) \rangle$$

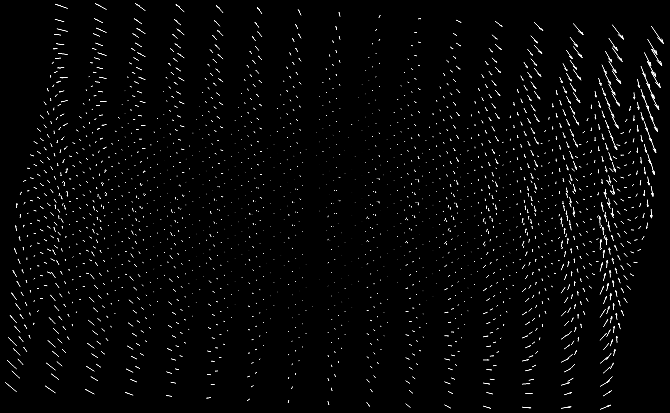
Bivector field

$$\left\langle p_{(1,0)} + p_{(0,1)} + p_{(1,1)} + p_{(2,0)} + p_{(0,2)} + p_{(3,0)} + p_{(0,3)} + p_{(2,1)}(1 + \mathbf{e}_{31}) + p_{(1,2)}(1 + \mathbf{e}_{31}) \right\rangle_2$$



Vector field

$$\langle p_{(1,0)} + p_{(0,1)} + p_{(1,1)} + p_{(2,0)} + p_{(0,2)} + p_{(3,0)} + p_{(0,3)} + p_{(2,1)}(1 + \boldsymbol{e}_{31}) + p_{(1,2)}(1 + \boldsymbol{e}_{31}) \rangle_2^{\perp}$$



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- $\boldsymbol{\nabla}^2$ is the Laplace-Beltrami operator.

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- For $A_+ \in \mathfrak{X}^+(\mathbb{R}^2)$ if $\boldsymbol{\nabla} A_+ = 0$ then A_+ is a **holomorphic function**.
- For a vector field $\boldsymbol{v} \in \mathfrak{X}^1(\mathbb{R}^3)$ we have

$$\boldsymbol{\nabla}\boldsymbol{v} = \underbrace{\boldsymbol{\nabla} \lrcorner \boldsymbol{v}}_{\textbf{divergence}} + \underbrace{\boldsymbol{\nabla} \wedge \boldsymbol{v}}_{\textbf{curl}}.$$

where

$$\operatorname{curl}(\boldsymbol{v}) = (\boldsymbol{\nabla} \wedge \boldsymbol{v})^{\perp}$$

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- Algebra and calculus descend to multivector equivalent:

$$\alpha_r \wedge \beta_s = (A_r \wedge B_s) \lrcorner dX_{r+s}^\dagger \qquad \alpha_r \lrcorner \beta_s = (A_r \lrcorner B_s) \lrcorner dX_{r-s}^\dagger$$

$$\underbrace{d\alpha_r = (\boldsymbol{\nabla} \wedge A_r) \lrcorner dX_{r+1}^\dagger}_{\textbf{exterior derivative}} \qquad \underbrace{\delta\alpha_r = (-\boldsymbol{\nabla} \lrcorner A_r) \lrcorner dX_{r-1}^\dagger}_{\textbf{codifferential}}$$

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- For M this yields $d\mu = \sqrt{|g|}\, dx^1 \cdots dx^n$

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- Define the *multivector field inner product on R* by

$$\langle\!\langle\, A, B \,\rangle\!\rangle_R \ := \int_R A * B d\mu_R$$

Green’s formulas

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- From [Hestenes, Sobczyk: 1984] and [Booß- Bavnbek, Wojciechowski: 1993]

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- Following from the above

$$\langle\!\langle \boldsymbol{\nabla} A, B \rangle\!\rangle = -\langle\!\langle A, \boldsymbol{\nabla} B \rangle\!\rangle + \langle\!\langle A, \boldsymbol{\nu} B \rangle\!\rangle.$$

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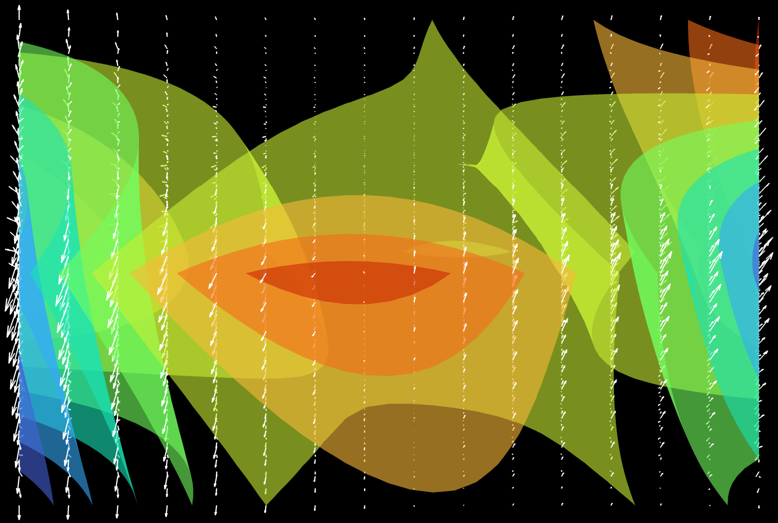
- Field $p_{(1,0)} + p_{(0,1)} + \cdots$ is monogenic (or quaternion harmonic).



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- [Calderbank: 1995], this map is an isomorphism.

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$$A(\boldsymbol{x}) = \frac{1}{S_n} \int_{\partial M} \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^n} \boldsymbol{\nu}(\boldsymbol{x}') A(\boldsymbol{x}') \, d\mu_{\partial M}(\boldsymbol{x}').$$

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Consider fields on a region $M \subset \mathbb{R}^n$:

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$$A(\boldsymbol{x}) = \frac{1}{S_n} \int_{\partial M} \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^n} \boldsymbol{\nu}(\boldsymbol{x}') A(\boldsymbol{x}') \, d\mu_{\partial M}(\boldsymbol{x}').$$

- **Scalar part of the above is the double layer potential.**

Inversion

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$$\mathrm{BS}(\boldsymbol{J})(\boldsymbol{x}) = \left\langle \llparenthesis \boldsymbol{J}, G_{\mathbf{x}} \rrparenthesis^{\perp} \right\rangle_2 = \frac{1}{4\pi} \int_{N^3} \boldsymbol{J}(\boldsymbol{y}) \wedge \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \, d\mu_{N^3}(\mathbf{x}').$$

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- **This is the Biot–Savart operator.** [Cantarella, et al.: 2001]

z-polynomials

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Proposition

Monogenic fields can be written as a power series in z_{ij} and the coefficients are computed with a Cauchy integral.

Section 3

Hodge theory

Idea

Hodge theory relates analysis to topology.

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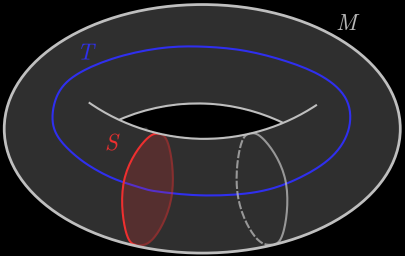
- **Theorem (Hodge Isomorphisms).**

$$H^r(M) \cong \mathcal{M}_N^r(M) \qquad H^r(M, \partial M) \cong \mathcal{M}_D^r(M, \partial M).$$



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The contraction $\lrcorner$ is a product on cohomologies by:

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- $\lrcorner \colon H^r(M, \partial M) \times H^s(M, \partial M) \to H^{s-r}(M, \partial M)$;
- $H^r(M) \lrcorner H^s(M, \partial M)$ is trivial;
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- This product is equivalent to the mixed cup product in [Shonkwiler, 2009].

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Hodge, Morrey, Friedrichs found decompositions of the space of forms.

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- But, $\mathcal{M}(M) \neq \bigoplus_{j=1}^n \mathcal{M}^j(M)$.

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- Define the ***Dirac fields*** $\boldsymbol{\nabla}\mathfrak{X}(M)$ as

$$\boldsymbol{\nabla}\mathfrak{X}(M) := \{\boldsymbol{\nabla} A \mid A \in \mathfrak{X}(M) \text{ and } A|_{\partial M} = 0\};$$

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Theorem: Clifford–Hodge Decomposition

The space of multivector fields $\mathfrak{X}(M)$ has the orthogonal decomposition

$$\mathfrak{X}(M) = \mathcal{M}(M) \oplus \boldsymbol{\nabla}\mathfrak{X}(M).$$

Comparing to Hodge–Morrey

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- From Hodge–Morrey

$$\mathfrak{X}(M) = \sum_{r=0}^{n} \underbrace{\mathcal{E}_D^r(M)}_{\mathrm{im}(\boldsymbol{\nabla}\wedge)} \oplus \underbrace{\mathcal{C}_N^r(M)}_{\mathrm{im}(\boldsymbol{\nabla}\,\lrcorner)} \oplus \underbrace{\mathcal{M}^r(M)}_{\ker(\boldsymbol{\nabla})}.$$

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- But the Clifford–Hodge is not filtered by grades

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Section 4

Tomography

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$$\Lambda_E \colon \boldsymbol{t}\mathfrak{X}^0(M) \to \boldsymbol{t}\mathfrak{X}^0(M) \quad \text{by} \quad \Lambda_E \phi = \boldsymbol{\nu} \lrcorner \boldsymbol{\nabla} \wedge u = \frac{\partial u}{\partial \boldsymbol{\nu}}.$$

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- **Question:** Can we determine (M, g) from Λ_E?

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- Define the *magnetic DN operator* $\Lambda_B \colon \boldsymbol{n}\mathfrak{X}^2(M) \to \boldsymbol{n}\mathfrak{X}^2(M)$ by $\Lambda_B(\boldsymbol{\nu} \wedge \boldsymbol{J}) = \boldsymbol{\nu} \wedge \boldsymbol{\nabla} \lrcorner B$.

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- **Question:** What can we get from Λ_B?

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- The smooth cases is still unsolved.

Electromagnetic tomography

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- Can we recover the space of monogenic fields from the DN operators?
- If the space of monogenic fields contained geometric data, this would be a step forward in the Calderón problem...

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- Applying Hodge isomorphisms...

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We have $\ker \Lambda_E \cong H^r(M)$ and $\ker \Lambda_B \cong H^r(M, \partial M)$.

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- $\Lambda_E \times \Lambda_B$ is equivalent to **complete DN operator** [Shonkwiler, Sharafutdinov: 2013].

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Section 5

Gelfand theory

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- We will prove this is true for arbitrary regions.

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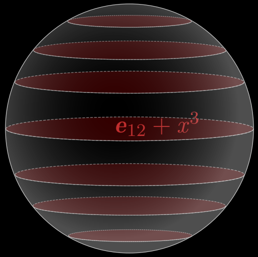

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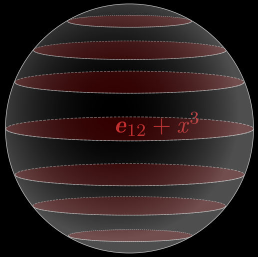

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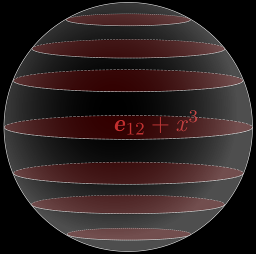

- The ***algebra of monogenic subsurface spinors*** is $\mathcal{A}_{\boldsymbol{B}}(O)$.
- **Algebra is a commutative Banach algebra.**

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- **We show these are the only elements in the spectrum.**

Idea

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By linearity, we can note that for $\delta \in \mathfrak{M}(M)$

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by the multiplicativity of δ. Then $z_{ij} \in \mathcal{A}_{\boldsymbol{e}_{ij}}(O)$.

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Lemma: Identification

Let $A_+ \in \mathcal{M}^+(M)$, then $\delta(A_+) = A_+(x_\delta)$ for some $x_\delta \in M$.

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Theorem: Clifford-algebraic Gelfand theorem

With the weak-$*$ topology on $\mathfrak{M}(M)$, the map

$$\gamma\colon \mathfrak{M}(M) \to M, \quad \delta \mapsto x_\delta$$

is a homeomorphism. The Gelfand transform $\widehat{A_+}(\delta) = \delta[A_+]$ is an isometric isomorphism so $\mathcal{M}^+(M) \cong \widehat{\mathcal{M}^+(M)}$.

Section 6

Further results, open questions, conclusion

A Stone–Weierstrass theorem

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- Using continuation from a z_{ij}:

Lemma

The space $\overline{\mathcal{M}^+(M)}$ separates points.

A Stone–Weierstrass theorem

- Using continuation from a z_{ij}:

Lemma

The space $\overline{\mathcal{M}^+(M)}$ separates points.

- Using [Laville, Ramadanoff: 1996]:

Theorem: Stone–Weierstrass

$\vee\overline{\mathcal{M}^+(M)}$ is dense in $C(M;\mathcal{G}^+)$.

Sheaf theory

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- Using unique continuation:

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Theorem

The sheaf $\mathcal{M}_M$ is Hausdorff and the map $\pi\colon \mathcal{M}_M \to M$ is a local homeomorphism.

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Sheaf theory

- Using unique continuation:

Theorem

The sheaf $\mathcal{M}_M$ is Hausdorff and the map $\pi\colon \mathcal{M}_M \to M$ is a local homeomorphism.

- Can one find a component of $\mathcal{M}_M$ that is homeomorphic to M?
- This would be helpful in using technique of [Lassas, Uhlmann: 2001].

Future work and open questions

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- The DN operator determines $\mathrm{tr}\mathcal{M}^+(M)$.
- The map $\mathrm{tr}\colon \vee\mathcal{M}^+(M) \to \mathrm{tr}\vee\mathcal{M}^+(M)$ is an isometric isomorphism of algebras.
- The space $\mathcal{M}^+(M)$ determines the metric structure of M up to isometry.

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- Many of these approaches use the Hilbert transform which is also used by Belishev, Sharafutdinov, and Shonkwiler to study the Calderón problem.
- The Hilbert transform is also a classical part of Clifford analysis.
- [Lassas, Uhlmann: 2001] use sheaf theory to solve the analytic Calderón problem.
- [Santacesaria: 2019] proposes another method for solving the Calderón problem using Clifford algebras and complex geometric optics.

Conclusion

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- Able to describe DN operators and extract homological information and boundary values of monogenic fields.
- Monogenic fields are uniquely determined by boundary traces via Cauchy integral.
- Monogenic fields are able to tell us the topology of the manifold they are defined on.

Thank you!