Clifford Analysis and a Noncommutative Gelfand Representation

Colin Roberts



Overview

- 1 Introduction
- 2 Clifford analysis
- 3 Gelfand theory
- 4 Future work
- 5 Conclusions

Section 1

Introduction

Motivating problems

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- The *Calderón problem* replaces the medium with a manifold M, conductivity with g, and replaces the voltage-to-current map with the Dirichlet-to-Neumann operator Λ .



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- Do these functions also contain geometric information such as metric data?
- \blacksquare How much can we learn about M if our data is supported only on the boundary?

Subsection 1

Preliminaries

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- Clifford analysis arrived in the 1980's due to Hestenes, Sobczyk, Sommen, Brackx, and Delenghe in order to enrich Élie Cartan's differential forms. See: [Hestenes, Sobczyk: 1984] and [Doran, Lasenby: 2003].



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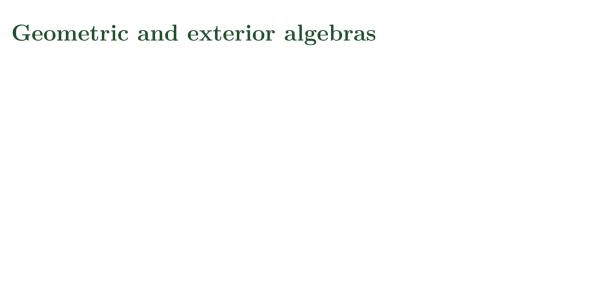
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■ Form the *Clifford algebra* via a quotient

$$C\ell(V,Q) \coloneqq \mathcal{T}(V)/\langle \mathbf{v} \otimes \mathbf{v} - Q(\mathbf{v}) \rangle.$$



Geometric and exterior algebras

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- The bivector part is antisymmetric: $\boldsymbol{u} \wedge \boldsymbol{v} = -\boldsymbol{v} \wedge \boldsymbol{u}$.



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 - Elements of the even grade subalgebra, \mathcal{G}^+ , are called *spinors*.
- The most general elements are *multivectors* and are given by

$$A = \sum_{r=0}^{n} \langle A \rangle_r,$$

where $\langle A \rangle_r \in \mathcal{G}^r$ extracts the grade r part of A. So $\mathcal{G} = \bigoplus_{i=1}^n \mathcal{G}^r$.

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■ The most important products are

$$A_r \cdot B_s \coloneqq \langle A_r B_s \rangle_{|r-s|} \qquad \qquad A_r \wedge B_s \coloneqq \langle A_r B_s \rangle_{r+s}$$

$$A_r \rfloor B_s \coloneqq \langle A_r B_s \rangle_{s-r} \qquad \qquad A_r \lfloor B_s \coloneqq \langle A_r B_s \rangle_{r-s}$$

Reciprocals and reverses

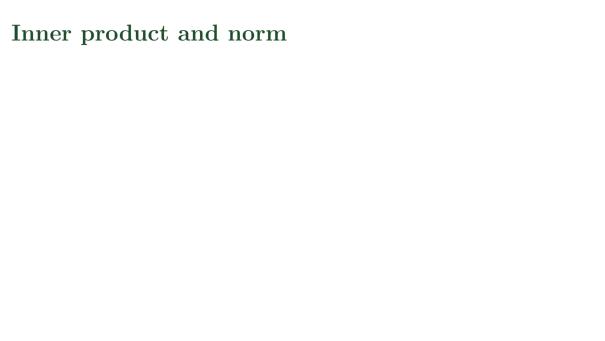
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- The reverse of a multivector is extended linearly from the action on r-blades by

$$\mathbf{A_r}^{\dagger} = (\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r)^{\dagger} = \mathbf{v}_r \wedge \cdots \wedge \mathbf{v}_1.$$



Inner product and norm

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■ Define the *multivector norm* by

$$|A| \coloneqq \sqrt{(A,A)}.$$

Adjoint

Note the reverse acts as an adjoint by

$$(CA, B) = (A, C^{\dagger}B)$$

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 \blacksquare We define the *unit pseudoscalar* by

$$I \coloneqq \frac{1}{|\boldsymbol{\mu}|} \boldsymbol{\mu}.$$



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- Unit r-blades correspond to r-dimensional subspaces so they correspond to points in Gr(r, n).



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■ Note $A_r^{\perp} \in \mathcal{G}^{n-r}$, like the Hodge star \star .

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 - Take the standard basis e_1 , e_2 , and define $B_{12} = e_1e_2$ and note $B_{12}^2 = -1$. Thus,

$$(u_1 + v_1 \mathbf{B}_{12})(u_2 + v_2 \mathbf{B}_{12}) = u_1 u_2 - v_1 v_2 + (u_1 v_2 + u_2 v_1) \mathbf{B}_{12}.$$

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■ Right multiplication by B_{12} rotates counter-clockwise by $\pi/2$.

Section 2

Clifford analysis

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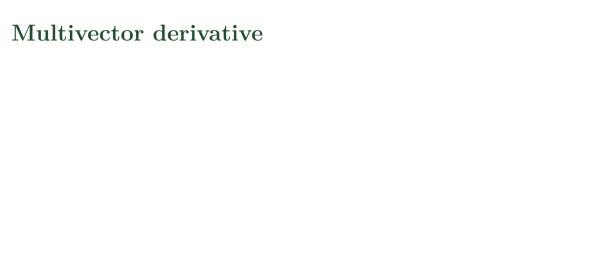
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■ Retain the same naming scheme as before.



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 \blacksquare ∇_u is compatible with dot and wedge since

$$\nabla_{\mathbf{u}}(A \cdot B) = (\nabla_{\mathbf{u}}A) \cdot B + A \cdot (\nabla_{\mathbf{u}}B)$$
$$\nabla_{\mathbf{u}}(A \wedge B) = (\nabla_{\mathbf{u}}A) \wedge B + A \wedge (\nabla_{\mathbf{u}}B).$$



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■ Note $\nabla^2 = \Delta$, the Laplace-Beltrami operator.



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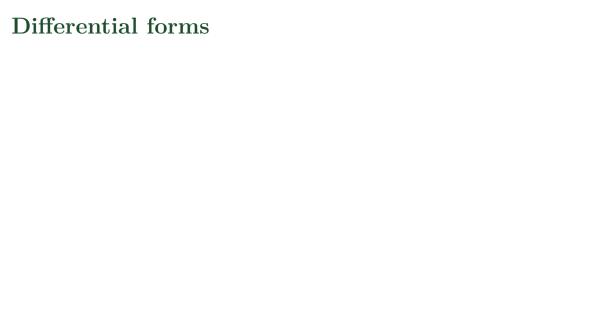
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■ Specifically,

$$\operatorname{curl}(\boldsymbol{v}) = (\nabla \wedge \boldsymbol{v})^{\perp}$$



Differential forms

■ Define the r-dimensional directed measure

$$dX_r \coloneqq \mathbf{v}_{j_1} \wedge \dots \wedge \mathbf{v}_{j_r} dx^{j_1} \dots dx^{j_r}$$

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■ Define an r-form a_r by

$$a_r = A_r \cdot dX_r^{\dagger}$$

where $A_r = \frac{1}{r!} a_{i_1 \cdots i_r} \mathbf{v}^{i_1} \wedge \cdots \wedge \mathbf{v}^{i_r}$.

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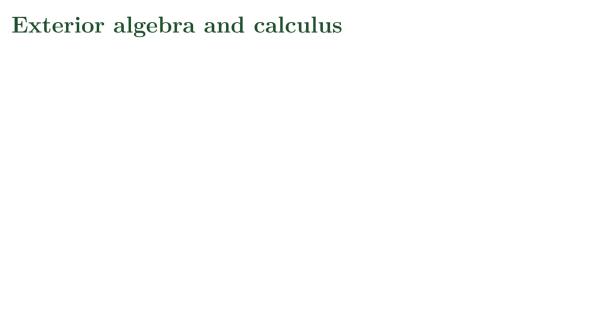
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■ Refer to A_r the multivector equivalent of a_r .



Exterior algebra and calculus

■ Given r-forms a_r , b_r , and an s-form c_s , we have

$$a_r + b_r = (A_r + B_r) \cdot dX_r^{\dagger}, \qquad a_r \wedge c_s = (A_r \wedge C_s) \cdot dX_{r+s}^{\dagger}.$$

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■ The Hodge star on multivector equivalents is

$$\star a_r = (\boldsymbol{I}^{-1} A_r)^{\dagger} \cdot dX_{n-r}^{\dagger}$$



Volume form

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$$\mu = \sqrt{|g|} dx^1 \cdots dx^n = \mathbf{I}^{-1} \cdot dX_n$$

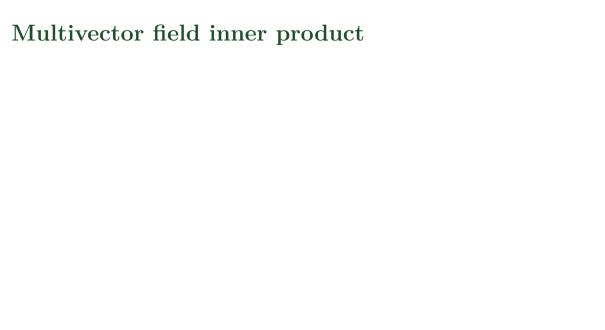
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■ We integrate scalar fields A_0 on M by

$$\int_{M} A_0^{\perp} \cdot dX_n = \int_{M} A_0 \mu.$$



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■ $\langle\langle A_r, B_s \rangle\rangle$ when $r \neq s$ so the L^2 -direct sum agrees with the grade based direct sum.



Boundary

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■ The boundary volume form is

$$\mu_{\partial} \coloneqq \boldsymbol{I}_{\partial}^{-1} \cdot dX_{n-1}$$

and we define

$$\ll A, B \gg_{\partial} \coloneqq \frac{1}{\operatorname{vol}(M)} \int_{\partial M} (A, B) \mu_{\partial}.$$

Multivector valued integrals

We can define a multivector valued integral on an oriented submanifold R by taking $A \in \mathcal{G}(M)$ and computing

$$\int_R A \mathbf{I}_R \mu_R.$$

Theorem (Hestenes, Sobczyk, 1984)

Let
$$A, B \in \mathcal{G}(M)$$
, then

$$\int_{M} \dot{A} \dot{\nabla} \mathbf{I} \mu = \int_{\partial M} A \mathbf{I}_{\partial} \mu_{\partial}$$

 $\int_{M} \mathbf{I} \nabla B \mu = \int_{\partial M} \mathbf{I}_{\partial} B \mu_{\partial}$

 $\int_{\mathcal{M}} \dot{A} \dot{\nabla} \mathbf{I} B \mu = (-1)^n \int_{\mathcal{M}} A \mathbf{I} \nabla B \mu + \int_{\partial \mathcal{M}} A \mathbf{I}_{\partial} B \mu_{\partial}.$

Theorem

We have the Green's formula for the gradient

 $\ll A, \mathbf{I} \nabla B \gg = (-1)^n \ll \nabla A, \mathbf{I} B \gg + \ll A, \mathbf{I}_{\partial} B \gg_{\partial}.$



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■ Define the *gradients*

$$\nabla \mathcal{G}(M) \coloneqq \{ \nabla A \mid A \in \mathcal{G}(M) \text{ and } A|_{\partial M} = 0 \}.$$



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■ Note,

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■ Define the Cauchy kernel by G(x, x') = E(x' - x).



Cauchy integral

■ Let $A \in \mathcal{M}(M)$, then we have the Cauchy integral formula

$$A(x) = (-1)^n \mathbf{I}^{-1} \int_{\partial M} G(x, x') \mathbf{I}_{\partial}(x') A(x') \mu_{\partial}(x').$$

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■ This uniquely determines a monogenic field from boundary values.

Lemma

Let $A \in \mathcal{M}(M)$ be such that $A|_{\partial M} = 0$. Then A = 0 on all of M.

Lemma

Fix a multivector field
$$A \in \mathcal{G}(M)$$
. If

for all $B \in \mathcal{G}(M)$ with $B|_{\partial M} = 0$, then A = 0.

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Theorem (Clifford-Hodge-Morrey Decomposition)

The space of multivector fields $\mathcal{G}(M)$ has the L^2 -orthogonal decomposition

The space of manifector fields
$$g(m)$$
 has the B -orthogonal accomposition

 $\mathcal{G}(M) = \mathcal{M}(M) \oplus \mathbf{I} \nabla \mathcal{G}(M).$



Proof.

• Orthogonality: Let $A \in \mathcal{M}(M)$ and $\mathbf{I} \nabla B \in \mathbf{I} \nabla \mathcal{G}(M)$ and note

 $\ll A, \mathbf{I} \nabla B \gg = (-1)^n \ll \nabla A, \mathbf{I} B \gg + \ll A, \mathbf{I}_{\partial} B \gg = 0,$

by the multivector Green's formula.

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- lacktriangle Use the Cauchy integral formula, construct a monogenic field \tilde{C} from $C|_{\partial M}$ and note $C = \tilde{C} + C_0$ where $C_0|_{\partial M} = 0$.
- Note, $0 = \langle C, \mathbf{I} \nabla B \rangle = \langle \nabla C_0, \mathbf{I} B \rangle$.

- By the previous lemmas, it must be that $C_0 = 0$. Hence the orthogonal
- complement to $\mathbf{I}\nabla\mathcal{G}(M)$ is $\mathcal{M}(M)$.

Comparing to Hodge-Morrey

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■ The Hodge-Morrey decomposition reads

$$\Omega^{r}(M) = \underbrace{\mathcal{E}_{D}^{r}(M)}_{\operatorname{Im}(\nabla \wedge)} \oplus \underbrace{\mathcal{C}_{N}^{r}(M)}_{\operatorname{Im}(\nabla \cdot)} \oplus \underbrace{\mathcal{H}^{r}(M)}_{\operatorname{Ker}(\nabla)}.$$

via [Schwarz: 1995].

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via [Schwarz: 1995].

■ Whereas the Clifford-Hodge-Morrey decomposition ignores specific grades

$$\mathcal{G}(M) = \mathcal{M}(M) \oplus \mathbf{I} \nabla \mathcal{G}(M).$$

Section 3

Gelfand theory

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- Belishev and Vakulenko as whether this is true in higher dimensions.
- We prove an analogous result for an arbitrary \mathbb{B} in \mathbb{R}^n .
- \blacksquare This approach can hopefully be used to prove the analogous result for any smooth orientable Riemannian manifold M.

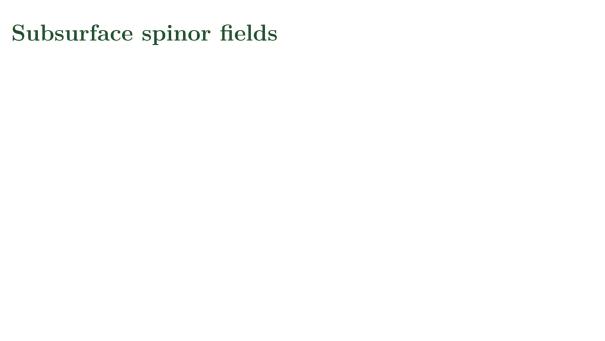
The boundary control (BC) method is implemented in [Belishev: 2003] in the following manner.

■ Determine the algebra $\mathcal{A}(M)$ of holomorphic functions on M from continuous function algebra on the boundary $\mathcal{A}(\partial M)$ using Λ .

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- \blacksquare Thus, we can find a g that is conformal with the complex structure.



Subsurface spinor fields

■ Let $\mathbf{B} \in \mathcal{G}(M)$ be a constant unit 2-blade, then $f_+ \in \mathcal{G}^+(M)$ satisfying

$$f_+ = \mathbf{P}_{\boldsymbol{B}} \circ f_+ \circ \mathbf{P}_{\boldsymbol{B}}$$

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■ The space of monogenic subsurface spinors

$$\mathcal{A}_{\mathbf{B}}(M) = \{ f_+ \in \mathcal{G}_{\mathbf{B}}^+(M) \mid \nabla f_+ = 0 \}$$

is a commutative unital Banach algebra.



Functionals

■ Define the *spinor dual* $\mathcal{M}^*(M)$ as the continuous right \mathcal{G}_n^+ -module homomorphisms

$$\mathcal{M}^*(M) \coloneqq \{l: \mathcal{M}^+(M) \to \mathcal{G}_n^+ \mid l(fs+g) = l(f)s + l(g), \ \forall f, g \in \mathcal{M}(M), \ s \in \mathcal{G}_n^+ \}$$

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and refer to the elements as $spin\ functionals$.

■ Assert the weak-* topology on $\mathcal{M}^*(M)$ so that every $x \in M$ corresponds to a continuous map on $\mathcal{M}^+(M)$.



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- One example of such characters are point evaluations $\delta(f) = f(x^{\delta})$.
- We show these are the only elements in the spectrum.

■ Take the standard basis for \mathbb{R}^n , and consider $M = \mathbb{B}_{R,w}$.

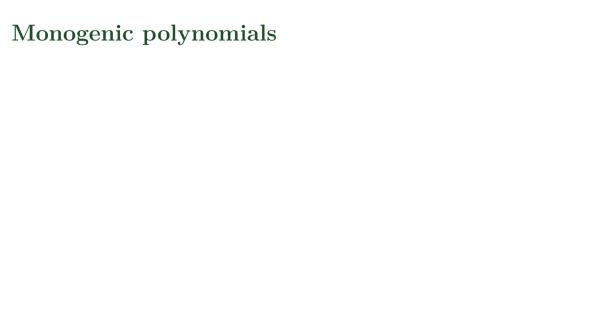
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■ Note $z_{ij} \in \mathcal{A}_{\boldsymbol{B}_{ij}}(\mathbb{B}_{R,w})$.



Monogenic polynomials

■ Let σ be a permutation of $\{2, 3, ..., n\}$, then

$$p_{j_2\cdots j_n}(x) = \frac{1}{j!} \sum_{\text{permutations}} z_{1\sigma(1)}(x)\cdots z_{1\sigma(j)}(x)$$

is a monogenic homogeneous polynomial of degree j.

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■ Collect these into the set of monogenic polynomials

$$\mathcal{M}^{\mathcal{P}}(\mathbb{B}_{R,w}) = \left\{ \sum_{j=0}^{N} \left(\sum_{\substack{j_2 \dots j_n \\ j_2 + \dots + j_n = j}} p_{j_2 \dots j_n} a_{j_2 \dots j_n} \right) \mid j_2 + \dots + j_n = j, N \in \mathbb{N}, \ a_{j_2 \dots j_n} \in \mathcal{G}_n^+ \right\}.$$

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Proof sketch.

■ Let $f \in \mathcal{M}^+(\mathbb{B}_{R,w})$ and use the Cauchy integral formula to define the coefficients $a_{i_2...i_n} \in \mathcal{G}_n^+$ by

$$a_{j_2\cdots j_n} = \int_{\partial \mathbb{B}_{R,w}} \frac{\partial^j G(w,y)}{\partial y_2^{j_2}\cdots \partial y_n^{j_n}} \boldsymbol{\nu}(y) f(y) \mu_{\partial}(y),$$

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■ Then

$$f(x) = \sum_{j=0}^{\infty} \left(\sum_{\substack{j_2 \dots j_n \\ j_2 + \dots + j_n = j}} p_{j_2 \dots j_n} (x - w) a_{j_2 \dots j_n} \right),$$

converges pointwise for $x \in \mathbb{B}_{R,w}$ by [Ryan, 2004].



Idea

■ By linearity, we can note that for $\delta \in \mathfrak{M}(\mathbb{B}_{R,w})$

$$\delta(f(x)) = \sum_{j=0}^{\infty} \left(\sum_{\substack{j_2 \dots j_n \\ j_2 + \dots + j_n = j}} \delta(p_{j_2 \dots j_n}(x-w)) a_{j_2 \dots j_n} \right)$$

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■ On each monogenic polynomial

$$\delta(p_{j_2...j_n}(x)) = \frac{1}{j!} \sum_{\text{permutations}} \delta((z_{1\sigma(1)}(x)) \cdots \delta(z_{1\sigma(j)}(x)))$$

by the multiplicativity of δ .

Let $\delta \in \mathfrak{M}(\mathbb{B}_{R,w})$ and $z_{ij} \in \mathcal{A}_{\mathbf{B}_{ij}}(\mathbb{B}_{R,w})$, then $\delta(z_{ij}) = z_{ij}(x^{\delta})$ for some $x^{\delta} \in \mathbb{R}^n$.

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■ The set of constants α and β are determined by n independent numbers, so we can say $\delta(z_{ij}) = z_{ij}(x^{\delta})$ for some $x^{\delta} \in \mathbb{R}^n$.

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- Note,

$$\lim_{n\to\infty}\delta(G_n)=\lim_{n\to\infty}G_n(x^{\delta})$$

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so this sequence not converge due to a singularity at x^{δ} .

■ Hence, it must be that $x^{\delta} \in \mathbb{B}_{R,w}$ by continuity of δ .

Theorem (Noncommutative Gelfand representation)

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For any
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, there is a point $x^{\delta} \in \mathbb{B}_{R,w}$ such that $\delta(f) = f(x^{\delta})$ for any

is a homeomorphism.

 $f \in \mathcal{M}(\mathbb{B}_{R,w})$. Given the weak-* topology on $\mathcal{M}^*(\mathbb{B}_{r,w})$, the map

 $\gamma: \mathfrak{M}(\mathbb{B}_{R,w}) \to \mathbb{B}_{R,w}, \quad \delta \mapsto x^{\delta}$



Proof.

■ The lemmas show that $\gamma:\mathfrak{M}(\mathbb{B}_{R,w})\to\mathbb{B}_{R,w}$ is bijective.

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- To see that γ is a homeomorphism, take a sequence $\delta_n \to \delta$ in $\mathfrak{M}(\mathbb{B}_{R,w})$.
- For $f \in \mathcal{M}^+(\mathbb{B}_{R,w})$ we have

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■ Taking $n \to \infty$ shows γ and γ^{-1} are continuous so γ is a homeomorphism.

Section 4

Future work



■ Let (M, g) be an unknown Riemannian manifold with known boundary ∂M . Consider the forward problem

$$\begin{cases} \Delta \omega = 0 & \text{in } M \\ \iota^* \omega = \phi & \text{on } \partial M \end{cases}$$

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- Define the *Dirichlet-to-Neumann map* on forms by $\Lambda \phi = \iota^*(\star d\omega)$.
- Question: Can we determine (M, g) from Λ ?



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- Solved in dimensions $n \ge 3$ when M is an analytic manifold [Lassas, Taylor, Uhlmann: 2003].
- The smooth cases is still unsolved.

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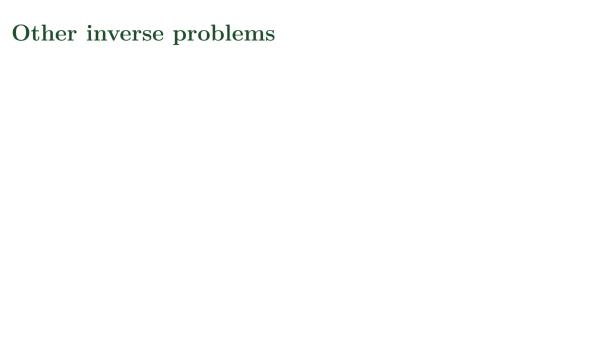
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- If Λ can provide us $b|_{\partial M}$, then we can possibly reconstruct $\mathcal{M}^+(M)$.
- Given the algebraic structure of each $\mathcal{A}_{B}(M) \subset \mathcal{M}^{+}(M)$, can this be used to determine q?



Other inverse problems

■ Can the magnetic impedance tomography problem can provide some extra insight on the EIT problem?

Other inverse problems

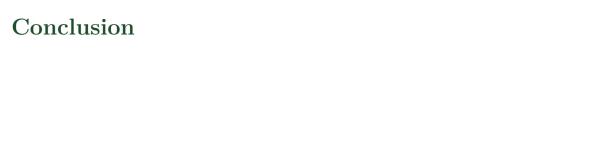
- Can the magnetic impedance tomography problem can provide some extra insight on the EIT problem?
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Other inverse problems

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- The Hodge-Morrey decomposition is an instrumental tool for boundary value problems that, for example, allows one to show that Λ determines the Betti numbers of M [Belishev, Sharafutdinov: 2008].
- Can the Clifford-Hodge-Morrey decomposition can allow us to work on other related boundary inverse problems?

Section 5

Conclusions



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- This provides a new way to decompose fields on domains of \mathbb{R}^n and this can likely be generalized to arbitrary compact orientable Riemannian manifolds.
- Likewise, we have proven that the monogenic spinors contain a wealth of topological information.

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- We are continuing to work to apply our scheme to new models such as the Modular Arbitrary-Order Ocean-Atmospheric Model (MAOOAM).

