Riemannian Geometry

for Dummies

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Section 1

Introduction

Riemannian geometry is the study of a ***smooth manifold*** M along with a ***Riemannian metric*** g.

The point of Riemmannian geometry is to generalize the differentiable and metric structure of $\mathbb{R}^n$.

We think of living on the manifold. We refer to this as ***intrinsic***.

We generalize to spaces that have interesting topology and geometry.

This will require us to rethink some notions we found “easy" in $\mathbb{R}^n$.

But we will gain a very general framework for working with differentiable objects.

Section 2

Motivation

Why study this in the first place?

Example: Partial differential equations (PDEs) on spaces that are not flat.

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- General relativity

Example: Optimization in interesting spaces.

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- Matrix (symmetry) groups

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- Curved spacetime

Example: Rephrasing classical problems.

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- EIT
- Polymer growth
- Electrodynamics

Section 3

Preliminaries

Subsection 1

Smooth Manifolds

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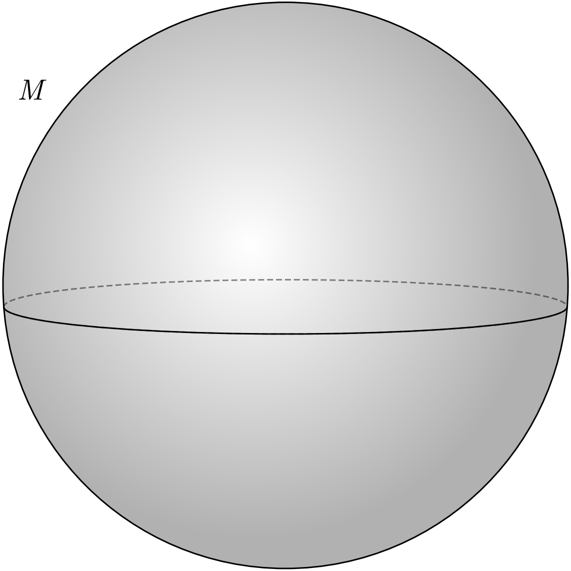
- Show coordinate transition functions are smooth

Working example: 2-sphere

$$S^2 \coloneqq \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$


M

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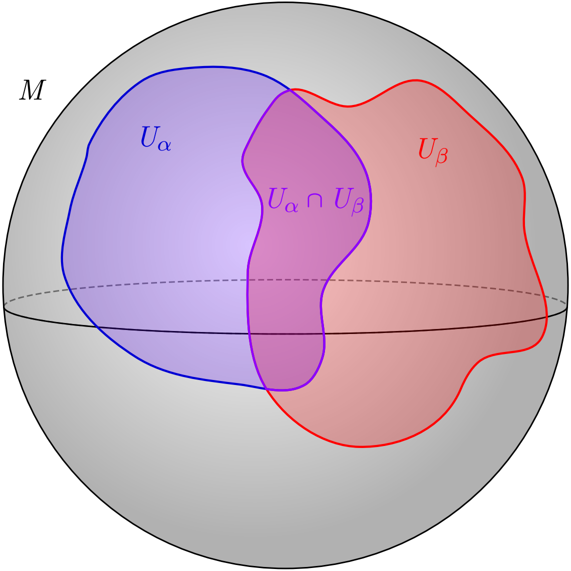
These are our ***local coordinates***.


M
U_α
U_β
$U_\alpha \cap U_\beta$

Our local coordinates must work together on overlaps

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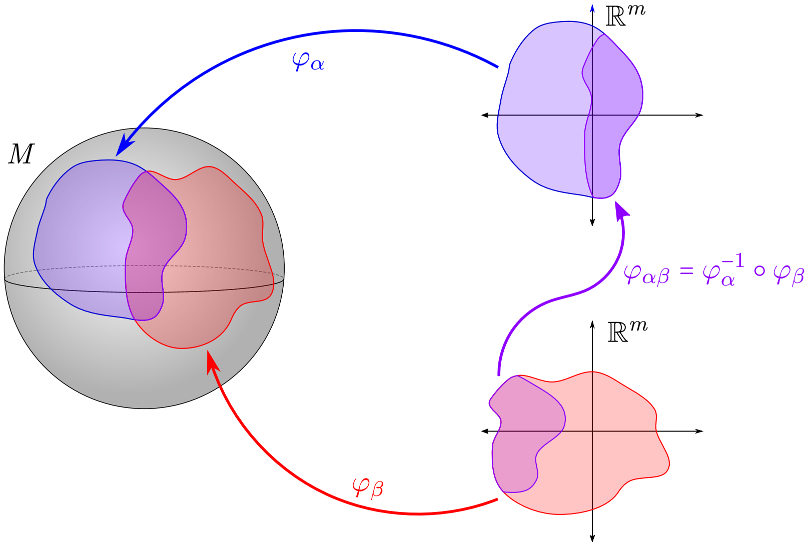
We check the ***transition function***

$$\phi_{\alpha\beta} = \varphi_\alpha^{-1} \circ \varphi_\beta$$

is smooth and invertible as a function on $\mathbb{R}^m$.

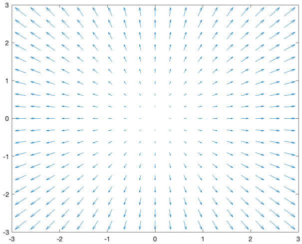


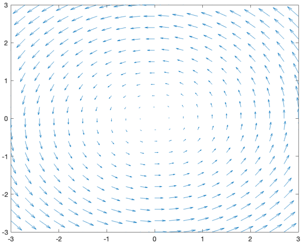
M
φ_α
$\mathbb{R}^m$
$\varphi_{\alpha\beta} = \varphi_\alpha^{-1} \circ \varphi_\beta$
$\mathbb{R}^m$
φ_β

Subsection 2

Vector Fields

Vector fields on $\mathbb{R}^m$ are functions $\vec{f}: \mathbb{R}^m \to \mathbb{R}^m$.





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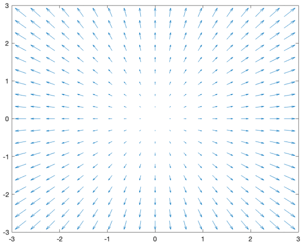

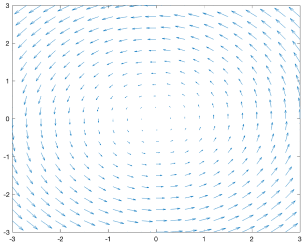

Intrinsic vector fields on manifolds carry geometric information.

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- Properly define vector fields X as ***sections*** of the tangent bundle

- Start with a curve $\gamma(-1,1) \to M$ with $\gamma(0) = p$

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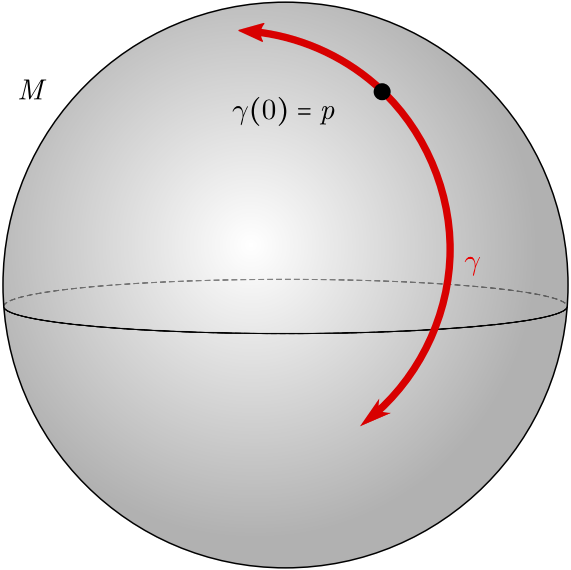
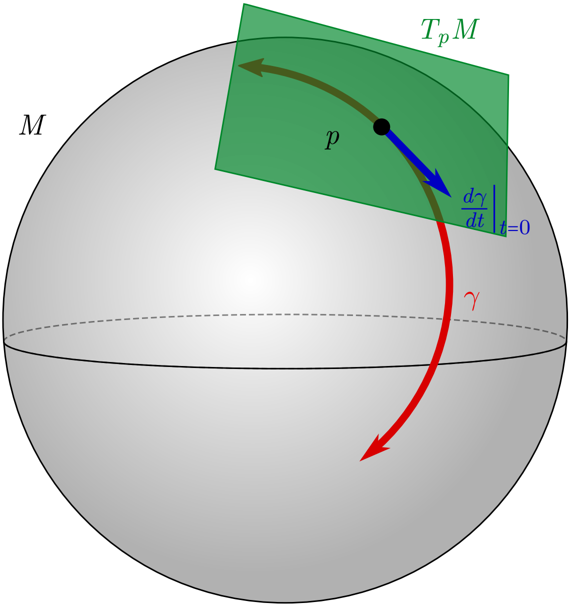
- All possible tangent vectors form the tangent space T_pM.


M
$\gamma(0) = p$
γ


T_pM
M
p
$\frac{d\gamma}{dt}\Big|_{t=0}$
γ

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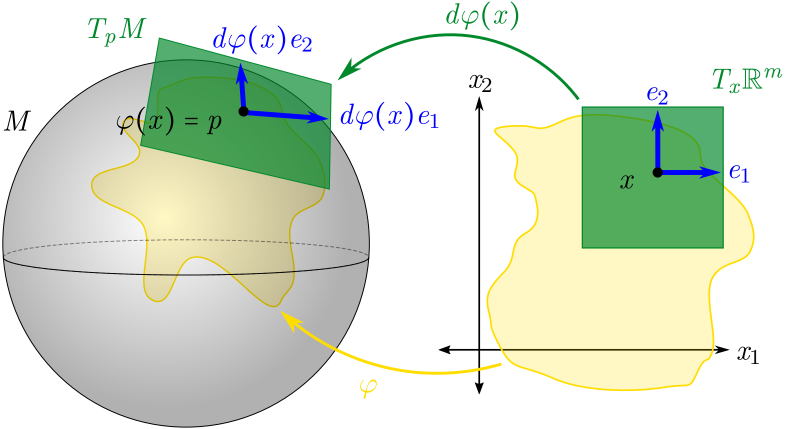
- If $\varphi(x) = p$, then $d\varphi(x)\colon T_x\mathbb{R}^m \to T_pM$


T_pM
$d\varphi(x)e_2$
$d\varphi(x)$
$T_x\mathbb{R}^m$
x_2
e_2
M
$\varphi(x) = p$
$d\varphi(x)e_1$
e_1
x
x_1
φ

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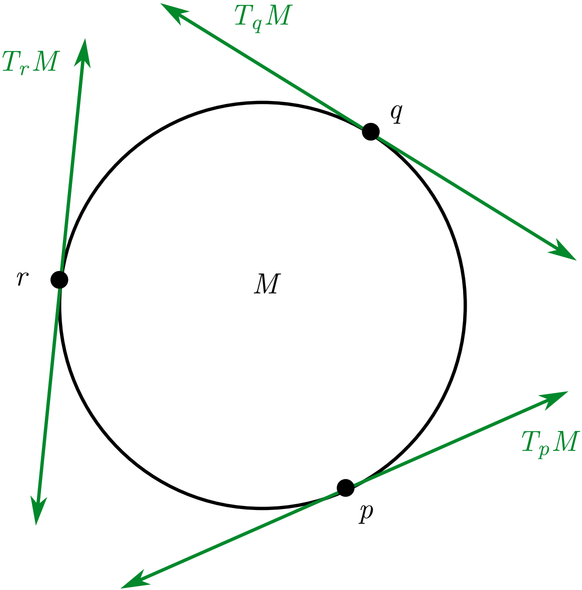
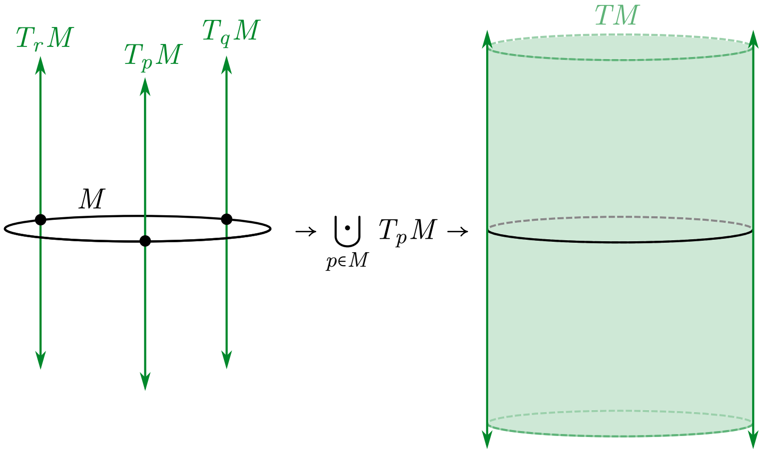
- Properly gluing tangent spaces T_pM to the manifold M allows us to build a larger manifold TM.
- This allows us to see how tangent vectors move around the whole manifold.

We briefly drop a dimension to the 1-sphere

$$S^1 \coloneqq \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$


T_qM
T_rM
q
r
M
T_pM
p


T_rM
T_pM
T_qM
M
$\rightarrow \bigcupdot_{p\in M} T_pM \rightarrow$
TM

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- For example, $X_p = v \in T_pM$
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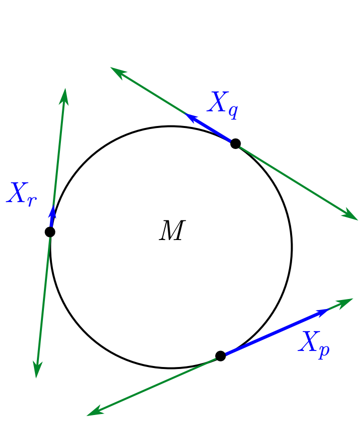
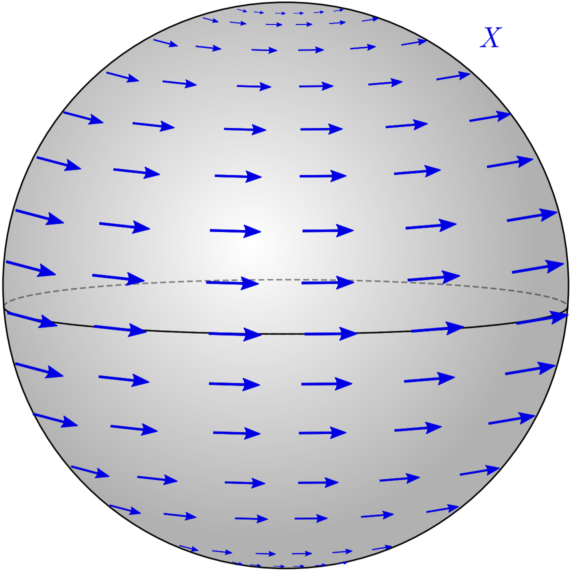
- X is a ***section*** if $\pi \circ X = \mathrm{Id_M}$ (vertical line test)



X_q
X_r
M
X_p
TM
X
r
q
p

Back to the 2-sphere.

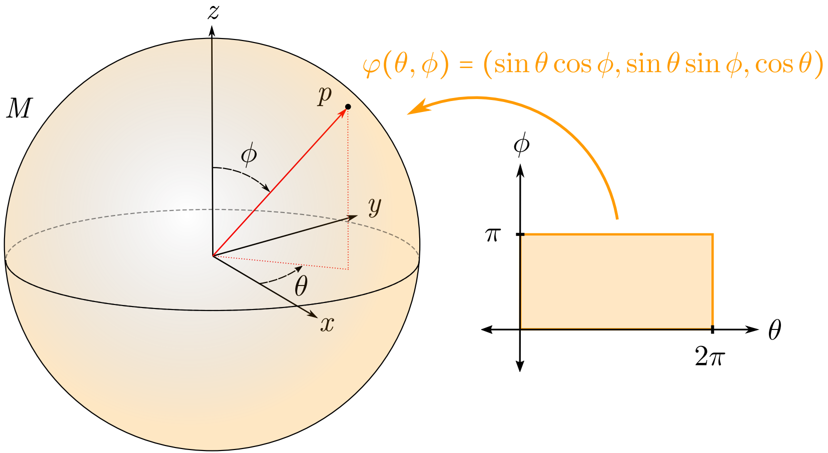


X

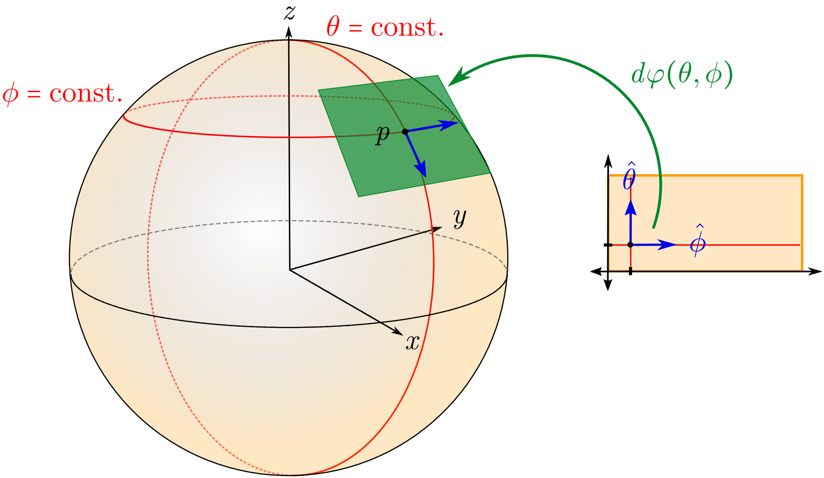
Subsection 3

Specific Coordinates

We should work with specific coordinates on S^2.



z
M
p
ϕ
y
θ
x
$\varphi(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$
ϕ
π
θ
2π



z
θ = const.
ϕ = const.
p
y
x
$d\varphi(\theta, \phi)$
$\hat{\theta}$
$\hat{\phi}$

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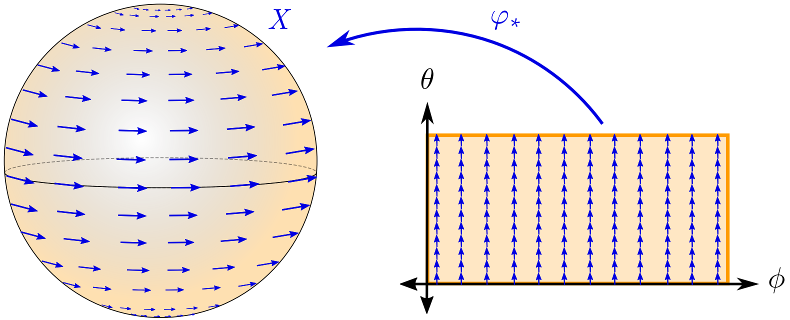
- This bundle map $\varphi_*\colon T\mathbb{R}^m \to TM$ is the ***pushforward***



X
φ_*
θ
ϕ

Section 4

Riemannian Geometry

Our To-Do List:

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- Have the inner product vary smoothly as we vary the point p;
- Define this as our Riemannian metric tensor field g;
- Extract geometrical and analytical qualities of the underlying manifold M.

Subsection 1

Riemannian Metric

We use the differential and dot product to form a matrix at each point

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This matrix is the ***Riemannian metric***.

Riemannian metric provides an inner product for tangent vectors on M. Thus, we know

- how lengths are distorted;
- how volume is distorted.

This allows us to integrate or differentiate in our coordinates but think of it as intrinsic to the manifold.

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$$\gamma\colon [0,1] \to M \qquad \gamma(0) = p,\ \gamma(1) = q$$

We need to solve

$$\inf_{\gamma} \ell(\gamma) \coloneqq \int_0^1 \sqrt{g(\dot{\gamma}, \dot{\gamma})}\, dt$$

- Reminder: in $\mathbb{R}^m$, the speed of a curve is $\sqrt{\dot{\gamma}, \dot{\gamma}}$

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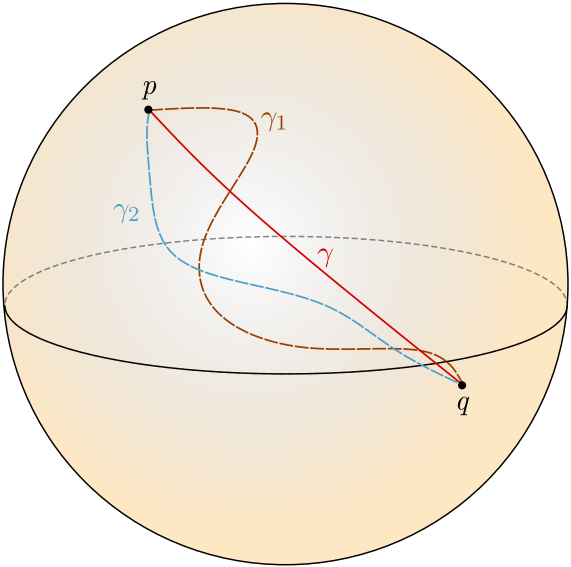
- $g(\dot{\gamma}, \dot{\gamma})$ is the speed on M
- We put $g(\dot{\gamma}, \dot{\gamma})$ to mean $\sum_{i,j=1}^{m} g_{ij}\dot{\gamma}_i\dot{\gamma}_j$.



p
γ_1
γ_2
γ
q

Solving this optimization problem yields the ***geodesic equation***

$$\ddot{x}^l + \dot{x}^j \dot{x}^k \Gamma^l_{jk} = 0$$

where Γ^l_{jk} are the ***Christoffel symbols*** which are formed by derivatives of the metric.

This defines an intrinsic derivative ∇ called the

Levi-Civita connection

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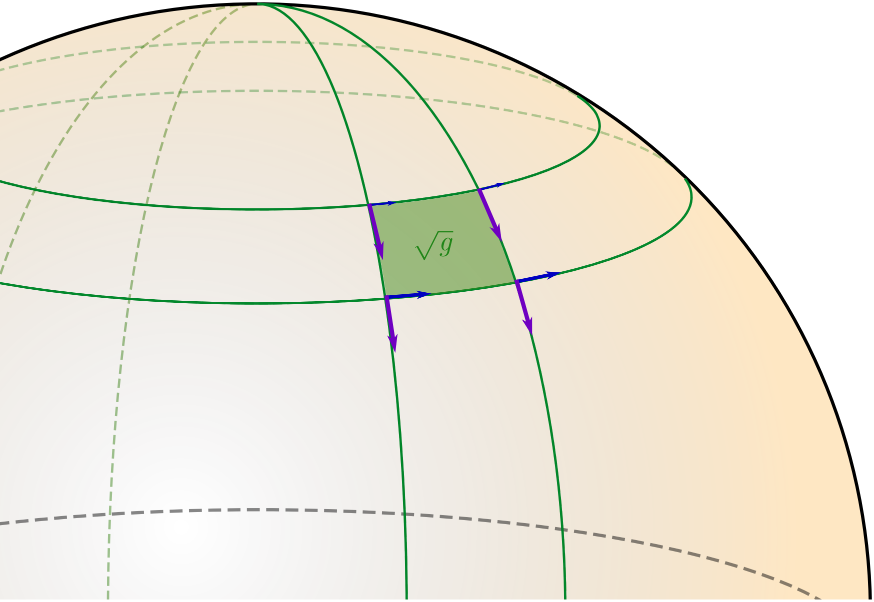
- The determinant gives us area information.
- Then $\sqrt{|\det(g(x))|}$ gives us the volume on M

In spherical coordinates, $\sqrt{|\det(g)|} = \sin\varphi$ which gives us the integrand

$$\sin\varphi\, d\varphi\, d\theta.$$


$\sqrt{g}$

Section 5

Conclusions

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- We generalized vector fields X to M
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- No measurement depends on the choice of coordinates

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- Hence, we can define lengths and volumes
- Thus, we can integrate

- g induces a derivative ∇

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- g provides an intrinsic length function on M

This is just the beginning!