ColoState Spring 2018 MATH 617 Final.PartII

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Work independently. Please write down all **necessary** *steps, partial credit will be given if deserved.*

(20 points) *Problem 3.* Let E be a subset of $[0,1]$. Prove that E is Lebesgue measurable if and only if

$$\lambda^*(E) = \sup\{\lambda(F) : F \text{ is closed and } F \subseteq E\}.$$

(20 points) *Problem 4.* Let λ be the Lebesgue measure on the real line. Consider a Lebesgue measurable subset E of $[0,1]$ with the following property:

$$\lambda(E \cap [a,b]) \geq c(b-a) \qquad \forall [a,b] \subseteq [0,1],$$

where $c > 0$ is a constant. Show that $\lambda(E) = 1$.

(20 points) *Problem 5.* Prove or provide a counterexample for the following statement: If f is absolutely continuous on $[a,b]$, g is continuous on $[a,b]$, and $f' = g$ almost everywhere on $[a,b]$, then $f' = g$ everywhere on $[a,b]$.

Problem 3. Let E be a subset of $[0,1]$. Prove that E is Lebesgue measurable if and only if

$$\lambda^*(E) = \sup\{\lambda(F) \;:\; F \text{ is closed and } F \subseteq E\}.$$

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Proof. This is an equivalent statement for the outer regularity of the Lebesgue measure. For the forward direction, we suppose that $E \subseteq [0,1]$ is Lebesgue measurable. Now, if E is empty, then the statement is vacuously true since the $\emptyset$ contains no subsets. Specifically,

$$0 = \lambda^*(\emptyset) = \lambda^*(E) = \sup\{\lambda(F) \;:\; F \text{ is closed and } F \subseteq E\}.$$

For E nonempty, note that by Theorem 4.2.2. there exists $F_n \subseteq E$ such that F_n is closed and

$$\lambda^*(E \setminus F_n) < \frac{1}{n}.$$

Note that $E \setminus F_n$ is Lebesgue measurable as well (since F_n is a closed subset of $\mathbb{R}$) which means that

$$\lambda^*(E \setminus F_n) = \lambda(E \setminus F_n),$$

since the Lebesgue measurable sets are a σ-algebra. Then,

$$\begin{aligned} \lambda^*(E \setminus F_n) = \lambda^*(E \setminus F_n) &< \frac{1}{n} \\ \implies \lambda^*(E) - \frac{1}{n} &< \lambda^*(F_n). \end{aligned}$$

Taking $n \to \infty$ is equivalent to taking $\sup\{\lambda(F) \;:\; F \text{ closed and } F \subseteq E\}$, and we find that

$$\lambda^*(E) \leq \sup\{\lambda(F) \;:\; F \text{ closed and } F \subseteq E\}.$$

To see that $\lambda^*(E) \geq \sup\{\lambda(F) \;:\; F \text{ closed and } F \subseteq E\}$ just note that since for any closed (and hence Lebesgue measurable) $F \subseteq E$ we have that

$$\begin{aligned} &\lambda^*(E) = \lambda(E) \geq \lambda(F) \\ &\quad \implies \lambda^*(E) \geq \sup\{\lambda(F) \;:\; F \text{ closed and } F \subseteq E\}. \end{aligned}$$

Thus we have that

$$\lambda^*(E) = \sup\{\lambda(F) \;:\; F \text{ closed and } F \subseteq E\}.$$

Now, for the converse, we suppose that

$$\lambda^*(E) = \sup\{\lambda(F) \;:\; F \text{ closed and } F \subseteq E\}$$

and show that E is Lebesgue measurable. The above statement implies that for any $\epsilon > 0$ we have some closed $F_\epsilon \subseteq E$ such that

$$\lambda^*(E) - \lambda(F_\epsilon) < \epsilon.$$

Then, since F_ϵ is Lebesgue measureable and E is Lebesgue outer measurable we have that

$$\lambda^*(E) - \lambda(F_\epsilon) = \lambda^*(E) - \lambda^*(F_\epsilon) = \lambda^*(E \setminus F_\epsilon) < \epsilon. \qquad \square$$

Thus, by Theorem 4.2.2., we have that E must be Lebesgue measurable.

Problem 4. Let λ be the Lebesgue measure on the real line. Consider a Lebesgue measurable subset E of $[0,1]$ with the following property:

$$\lambda(E \cap [a,b]) \geq c(b-a) \quad \forall [a,b] \subseteq [0,1],$$

where $c > 0$ is a constant. Show that $\lambda(E) = 1$.

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Proof. Define

$$f(x) = \int_a^x \chi_E d\lambda(t)$$

and note that f is differentiable almost everywhere. By our supposition, we then have that

$$f'(x) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} \chi_E d\lambda(t) \geq \lim_{h \to 0} \frac{ch}{h} = c > 0$$

almost everywhere. Then, since we defined f as the integral of χ_E, $f'(x) = \chi_E(x)$ almost everywhere. Since we also showed $f'(x) > 0$ almost everywhere, it must be that $f'(x) = 1$ almost everywhere. In particular, this means that $\chi_E(x) = 1$ almost everywhere and so over $[a,b]$ the set in which $\chi_E(x) = 0$ must be a null set. Ultimately, this means that $\lambda(E) = 1$.$\square$

Problem 5. Prove or provide a counterexample for the following statement: If f is absolutely continuous on $[a, b]$, g is continuous on $[a, b]$, and $f' = g$ almost everywhere on $[a, b]$, then $f' = g$ everywhere on $[a, b]$.

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Proof. Define

$$f(x) = \int_a^x g(t) d\lambda(t)$$

and note that by this definition, $f'(x) = g(x)$ almost everywhere. Now, since g is continuous, for any $x_0 \in (a, b)$ we have that for $h > 0$ and some $z \in [x_0, x_0 + h] \subseteq (a, b)$ (equivalently a $\tilde{z} \in [x_0 - h, x_0]$) that

$$g(z) = \frac{1}{h} \int_{x_0}^{x_0+h} g(t) d\lambda(t),$$

and equivalently

$$g(\tilde{z}) = \frac{1}{h} \int_{x_0-h}^{x_0} g(t) d\lambda(t).$$

(Note that the above fact is critical for showing the remaining calculations.) Then, taking the limit as $h \to 0$ we have

$$\begin{aligned} f'(x_0) &= \frac{f(x_0 + h) - f(x_0)}{h} \\ &= \lim \frac{1}{h} \int_{x_0}^{x_0+h} g(t) d\lambda(t) \\ &= g(x_0), \end{aligned}$$

and

$$\begin{aligned} f'(x_0) &= \frac{f(x_0 - h) - f(x_0)}{h} \\ &= \lim \frac{1}{h} \int_{x_0}^{x_0-h} g(t) d\lambda(t) \\ &= g(x_0), \end{aligned}$$

which shows the limits from both sides agree and, since $x_0 \in (a, b)$ was arbitrary, that $f'(x) = g(x)$ everywhere in (a, b). To see that $f'(x) = g(x)$ for $x = a$, just take $x_0 = a$ above and $h > 0$ that

$$\begin{aligned} f'(a) &= \frac{f(a + h) - f(a)}{h} \\ &= \lim \frac{1}{h} \int_a^{a+h} g(t) d\lambda(t) \\ &= g(a). \end{aligned}$$

and for $[b - h, b]$ with $h > 0$, letting $h \to 0$ we have

$$f'(b) = g(b).$$

Thus, $f'(x) = g(x)$ for all $x \in [a, b]$. □