

Clifford Analysis and a Noncommutative Gelfand Representation

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Section 1

Introduction

Subsection 1

Motivation

Electrical Impedance Tomography

Electrical Impedance Tomography (EIT) asks whether one can determine the conductivity of a medium based on measurements along the boundary.

Calderón problem

- Let M be a smooth, connected, oriented Riemannian manifold with boundary ∂M with metric g .
- Conductivity is represented by g .
- Forward problem: Let $\Delta u = 0$ in M and $u = \phi$ on ∂M .
- Inverse problem: Given the *Dirichlet-to-Neumann map* $\Lambda\phi = \frac{\partial u}{\partial \nu}$, can we recover (M, g) ?

Subsection 2

Preliminaries

- *Clifford algebra* originated in 1878 with William Kingdon Clifford's work that extends Hermann Grassmann's *exterior algebra*.
- *Clifford analysis* arrived in the 1980's due to Hestenes, Sobczyk, Sommen, Brackx, and Delenghe in order to enrich Élie Cartan's *differential forms*.
 - Atiyah-Singer Dirac operator and spin manifolds.

Clifford algebras

Let V be a vector space over a field \mathbb{F} with quadratic form Q .

- Define the tensor algebra

$$\mathcal{T}(V) := \bigoplus_{j=0}^{\infty} V^{\otimes j} = \mathbb{F} \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots.$$

- Form the *Clifford algebra* via a quotient

$$Cl(V, Q) := \mathcal{T}(V) / \langle \mathbf{v} \otimes \mathbf{v} - Q(\mathbf{v}) \rangle.$$

Geometric and exterior algebras

Let V be a vector space over a field \mathbb{F} with quadratic form Q .

- Given a (pseudo) inner product g , we set $Q(\cdot) = g(\cdot, \cdot)$ and define a *geometric algebra*

$$\mathcal{G} := Cl(V, g).$$

- The *exterior algebra* is given by

$$\bigwedge(V) := Cl(V, 0).$$

Algebra structure

We define a multiplication in \mathcal{G} by noting how the product \otimes acts in the quotient.

- Given $\mathbf{u}, \mathbf{v} \in \mathcal{G}$ we can take the product

$$\mathbf{u}\mathbf{v} = \underbrace{\mathbf{u} \cdot \mathbf{v}}_{\text{scalar}} + \underbrace{\mathbf{u} \wedge \mathbf{v}}_{\text{bivector}}.$$

- The scalar part is symmetric: $\mathbf{u} \cdot \mathbf{v} = g(\mathbf{u}, \mathbf{v})$.
- The bivector part is antisymmetric: $\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u}$.

Multivectors

- \mathcal{G} is graded and of dimension 2^n .
 - There are $\binom{n}{r}$ elements of grade r called *r -vectors*.
 - Those that are exterior products of r independent vectors are *r -blades*.
E.g., $\mathbf{A}_r = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r$.
- The most general elements are *multivectors* and are given by

$$A = \sum_{r=0}^n \langle A \rangle_r,$$

where $\langle A \rangle_r \in \mathcal{G}^r$ extracts the grade r part of A .

Algebra Structure

Extend the multiplication from vectors to multivectors by

$$A_r B_s = \langle A_r B_s \rangle_{|r-s|} + \langle A_r B_s \rangle_{|r-s|+2} + \cdots + \langle A_r B_s \rangle_{r+s}$$

with

$$A_r \cdot B_s := \langle A_r B_s \rangle_{|r-s|}$$

$$A_r \wedge B_s := \langle A_r B_s \rangle_{r+s}$$

$$A_r \rfloor B_s := \langle A_r B_s \rangle_{s-r}$$

$$A_r \lrcorner B_s := \langle A_r B_s \rangle_{r-s}$$

Reciprocals

Given any vector basis \mathbf{v}_i we define the *reciprocal vectors* by $\mathbf{v}^i \cdot \mathbf{v}_j = \delta_j^i$.

Reverse

The *reverse* of a multivector is extended linearly from the action on r -blades by

$$\mathbf{A}_r^\dagger = (\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r)^\dagger = \mathbf{v}_r \wedge \cdots \wedge \mathbf{v}_1.$$

Inner product and norm

We define the *multivector inner product* by

$$(A, B) := \langle A^\dagger B \rangle$$

which is bilinear, symmetric, and positive definite if g is positive definite. Then we define the *multivector norm* by

$$|A| := \sqrt{(A, A)}.$$

Adjoint

Note the reverse acts as an adjoint by

$$(CA, B) = (A, C^\dagger B) \tag{1}$$

$$(AC, B) = (A, BC^\dagger). \tag{2}$$

Pseudoscalars

Pseudoscalars are the grade- n elements. For example, $\boldsymbol{\mu} = \mathbf{v}_1 \wedge \mathbf{v}_n$. We define the *unit pseudoscalar* by

$$\boldsymbol{I} := \frac{1}{|\boldsymbol{\mu}|} \boldsymbol{\mu}.$$

Blades and subspaces

If $|\mathbf{A}_r| = 1$, then \mathbf{A}_r is a *unit blade*.

All unit r -blades correspond to an r -dimensional subspace and can be identified with points in $\text{Gr}(r, n)$.

Duality

Given any multivector A , we can take its *dual*

$$A^\perp := A\mathbf{I}^{-1}.$$

Note $A_r^\perp \in \mathcal{G}^{n-r}$.

Projection and rejection

We can define the *projection* of B into a subspace \mathbf{A}_r by

$$P_{\mathbf{A}_r}(B) := B \rfloor \mathbf{A}_r \mathbf{A}_r^{-1}$$

and the *rejection* by

$$R_{\mathbf{A}_r}(B) := B \wedge \mathbf{A}_r \mathbf{A}_r^{-1}.$$

Both are grade preserving.

Complex Numbers

Do a more thorough example like in my thesis to wrap everything up Maybe it is worth including the hermitian inner product example in both?

Claim: \mathbb{C} arises naturally as the even subalgebra \mathcal{G}_2^+ .

Take the standard basis $\mathbf{e}_1, \mathbf{e}_2$, and define $\mathbf{B}_{12} = \mathbf{e}_1 \mathbf{e}_2$ and note $\mathbf{B}_{12}^2 = -1$. Thus

$$(u_1 + v_1 \mathbf{B}_{12})(u_2 + v_2 \mathbf{B}_{12}) = u_1 u_2 - v_1 v_2 + (u_1 v_2 + u_2 v_1) \mathbf{B}_{12}.$$

Moreover, right multiplication of vectors by \mathbf{B}_{12} rotates counter-clockwise by $\pi/2$.

Examples

Claim: The quaternion algebra arises naturally inside the even subalgebra \mathcal{G}_3^+ .

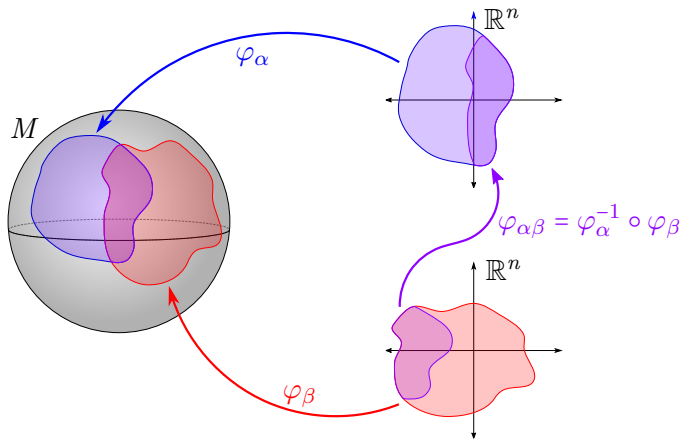
Claim: The spacetime algebra is $\mathcal{G}_{3,1}$.

Subsection 3

Manifolds and fields

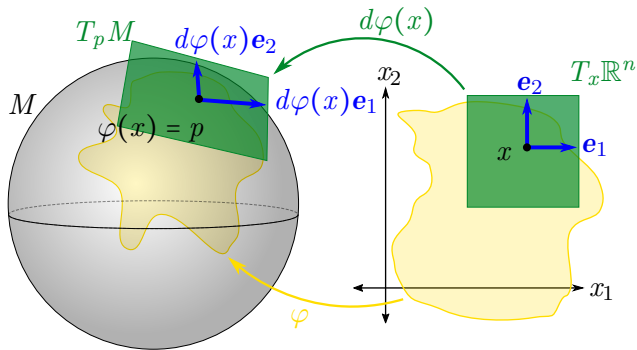
The playing field

We let M be a smooth, compact, connected, and oriented n -dimensional Riemannian manifold with metric g (unless otherwise stated).



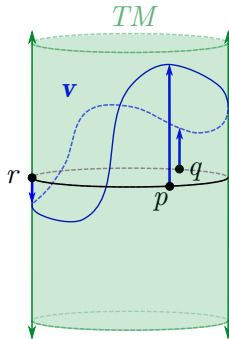
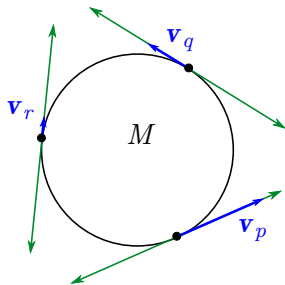
The playing field

At each point on M , we have the tangent space $T_p M$.



The playing field

From M , we create the tangent bundle TM whose sections are vector fields.



Idea: Form the Clifford algebras on tangent spaces.

- Each $Cl(T_p M, g_p)$ is a *geometric tangent space* which we glue together to form

$$Cl(TM, g) := \bigsqcup_{p \in M} Cl(T_p M, g_p).$$

- The space of *(smooth) multivector fields* is

$$\mathcal{G}(M) := \{C^\infty\text{-smooth sections of } Cl(TM, g)\}.$$

Section 2

Clifford analysis

Subsection 1

Differentiation

Covariant derivative

On M we have the unique torsion free Levi-Civita connection ∇ and covariant derivative $\nabla_{\mathbf{u}}$.

- $\nabla_{\mathbf{u}}$ can be extended to multivectors **Explain briefly how** and it is grade preserving

$$\nabla_{\mathbf{u}} A_r = \langle \nabla_{\mathbf{u}} A_r \rangle_r.$$

- $\nabla_{\mathbf{u}}$ is compatible with dot and wedge since

$$\nabla_{\mathbf{u}}(A \cdot B) = (\nabla_{\mathbf{u}} A) \cdot B + A \cdot (\nabla_{\mathbf{u}} B)$$

$$\nabla_{\mathbf{u}}(A \wedge B) = (\nabla_{\mathbf{u}} A) \wedge B + A \wedge (\nabla_{\mathbf{u}} B).$$

Gradient

We define the *gradient* (or *Dirac operator*) in some local basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ by

$$\nabla = \sum_{i=1}^n \mathbf{v}^i \nabla_{\mathbf{v}_i}$$

Note that this has the algebraic properties of a vector in $\mathcal{G}(M)$.

Gradient

We define the *gradient* (or *Dirac operator*) in some local basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ by

$$\nabla = \sum_{i=1}^n \mathbf{v}^i \nabla_{\mathbf{v}_i}$$

Note that this has the algebraic properties of a vector in $\mathcal{G}(M)$ and has the Leibniz rule

$$\nabla(AB) = \dot{\nabla} \dot{A} B + \dot{\nabla} A \dot{B}.$$

Subsection 2

Integration

Differential forms

We define the *r-dimensional directed measure*

$$dX_r := \mathbf{v}_{j_1} \wedge \cdots \mathbf{v}_{j_r} dx^1 \cdots dx^n$$

where $1 \leq j_1 < \cdots < j_r \leq n$ is an increasing set of indices. This allows us to define an r -form α_r by

$$\alpha_r = A_r \cdot dX_k^\dagger$$

where $A_r = \frac{1}{r!} \alpha_{i_1 \cdots i_r} \mathbf{v}^{i_1} \wedge \cdots \wedge \mathbf{v}^{i_r}$. We call A_r the *multivector equivalent* of α_r .

Volume form

The *volume form* on M is given in local coordinates by

$$\mu = \sqrt{|g|} dx^1 \cdots dx^n = \mathbf{I}^{-1} \cdot dX_n$$

Hence, we can integrate scalar fields A_0 on M by

$$\int_M A_0^\perp \cdot dX_n = \int_M A_0 \mu.$$

Exterior algebra

Given an r - and s -form α_r and β_s we have

$$\alpha_r + \beta_s = (A_r + B_s) \cdot dX_r^\dagger$$

$$\alpha_r \wedge \beta_s = (A_r \wedge B_s) \cdot dX_{r+s}^\dagger.$$

Exterior derivative

The exterior derivative on multivector equivalents is

$$d\alpha_r = (\nabla \wedge A_r) \cdot dX_{r+1}^\dagger$$

Hodge star

The Hodge star on multivector equivalents is

$$\star \alpha_r = (\mathbf{I}^{-1} A_r)^\dagger \cdot dX_{n-r}^\dagger$$

Multivector field inner product

We define an inner product on multivector fields by

$$\ll A, B \gg := \frac{1}{\text{vol}(M)} \int_M (A, B) \mu$$

which realizes the r -form inner product

$$\int_M \alpha_r \wedge \star \beta_r = \int_M \langle A_r \dagger B_r \rangle \mu = \text{vol}(M) \ll A, B \gg .$$

Remark: By definition of the multivector inner product, A_r and B_s are orthogonal when $r \neq s$ so this agrees with the grade direct sum \oplus – we use the same notation for both.

Boundary

On the boundary ∂M , we have the boundary pseudoscalar \mathbf{I}_∂ and the boundary normal $\boldsymbol{\nu} = \mathbf{I}_\partial^\perp$. Then

$$\mu_\partial := \mathbf{I}_\partial^{-1} \cdot dX_{n-1}$$

and we define

$$\ll A, B \gg_\partial := \frac{1}{\text{vol}(M)} \int_{\partial M} (A, B) \mu_\partial.$$

Multivector valued integrals

We can define a multivector valued integral on an oriented submanifold R by taking $A \in \mathcal{G}(M)$ and computing

$$\int_R A \mathbf{I}_R \mu_R.$$

Subsection 3

Clifford-Hodge-Morrey decomposition

Fundamental theorems of geometric calculus

Let $A, B \in \mathcal{G}(M)$, then

$$\begin{aligned}\int_M \dot{A} \dot{\nabla} \mathbf{I} \mu &= \int_{\partial M} A \mathbf{I} \partial \mu \partial \\ \int_M \mathbf{I} \nabla B \mu &= \int_{\partial M} \mathbf{I} \partial B \mu \partial \\ \int_M \dot{A} \dot{\nabla} \mathbf{I} B \mu &= (-1)^n \int_M A \mathbf{I} \nabla B \mu + \int_{\partial M} A \mathbf{I} \partial B \mu \partial.\end{aligned}$$

Theorem: (Multivector Green's formula)

We have the Green's formula for the gradient

$$\langle\langle A, \mathbf{I} \nabla B \rangle\rangle = (-1)^n \langle\langle \nabla A, \mathbf{I} B \rangle\rangle + \langle\langle A, \mathbf{I} \partial B \rangle\rangle_{\partial} .$$

Proof

Fix $A^\dagger, B \in \mathcal{G}(M)$ and note that

$$\begin{aligned}\int_M A^\dagger \mathbf{I} \nabla B \mu &= (-1)^n \int_M \dot{A}^\dagger \dot{\nabla} \mathbf{I} B \mu + \int_{\partial M} A^\dagger \mathbf{I}_\partial B \mu_\partial \\ &= (-1)^n \int_M (\nabla A)^\dagger \mathbf{I} B \mu + \int_{\partial M} A^\dagger \mathbf{I}_\partial B \mu_\partial.\end{aligned}$$

Then, take the scalar part and divide by $\text{vol}(M)$ to find

$$\ll A, \mathbf{I} \nabla B \gg = (-1)^n \ll \nabla A, \mathbf{I} B \gg + \ll A, \mathbf{I}_\partial B \gg_\partial .$$

Monogenic fields

Let $A \in \mathcal{G}(M)$. Then we say that A is *monogenic* if $\nabla A = 0$.

Holomorphic functions

Take the coordinates x and y and let $f = u + v\mathbf{B} \in \mathcal{G}_2(\mathbb{R}^2)$ then $\nabla f = 0$ yields the Cauchy-Riemann equations

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}.\end{aligned}$$

Section 3

Gelfand theory

Section 4

Conclusions

Other projects

Data assimilation.