Riemannian Geometry

for Dummies

Colin Roberts

Colorado State University

Section 1

Introduction

Riemannian geometry is the study of a ***smooth manifold*** M along with a ***metric tensor field*** g.

The point of Riemmannian geometry is to generalize the differentiable and metric structure of $\mathbb{R}^n$.

We generalize to spaces that have interesting topology and geometry.

This will require us to rethink some notions we found “easy" in $\mathbb{R}^n$.

But we will gain a very general framework for working with differentiable objects.

Section 2

Motivation

Why study this in the first place?

Example: Partial differential equations (PDEs) on spaces that are not flat.

Example: Partial differential equations (PDEs) on spaces that are not flat.

- Fluid flow on Earth;

Example: Partial differential equations (PDEs) on spaces that are not flat.

- Fluid flow on Earth;
- Electrical Impedence Tomography (EIT);

Example: Partial differential equations (PDEs) on spaces that are not flat.

- Fluid flow on Earth;
- Electrical Impedence Tomography (EIT);
- General relativity.

Example: Optimization in interesting spaces.

Example: Optimization in interesting spaces.

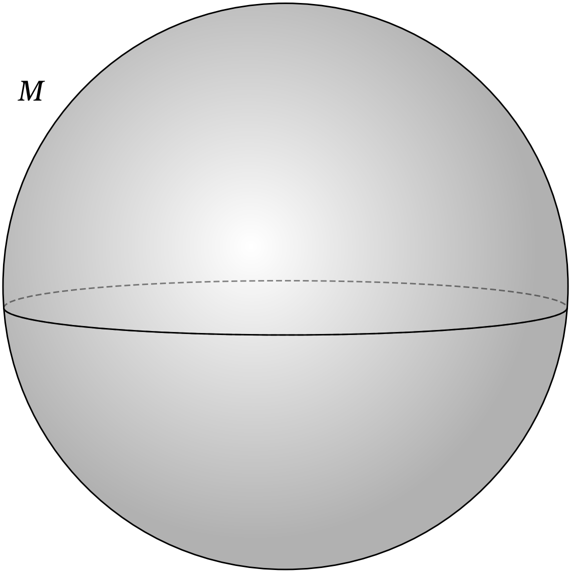
- Grassmannians;

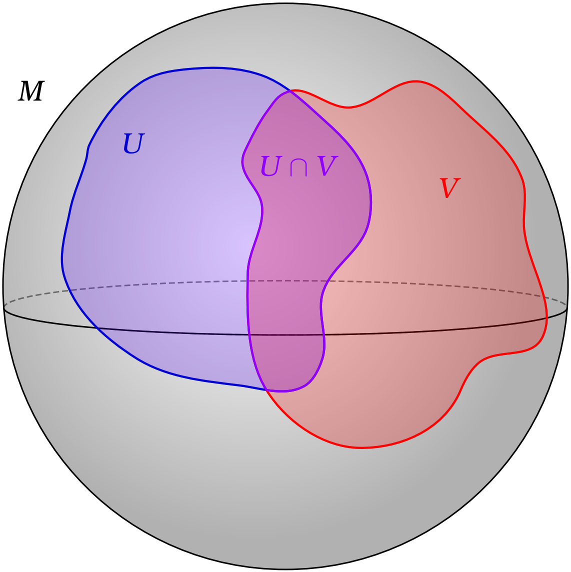
Example: Optimization in interesting spaces.

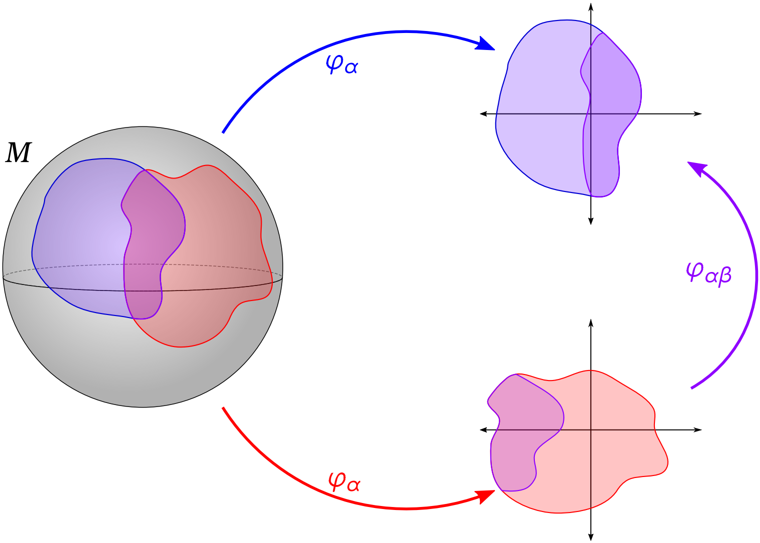
- Grassmannians;
- Flags.

Section 3

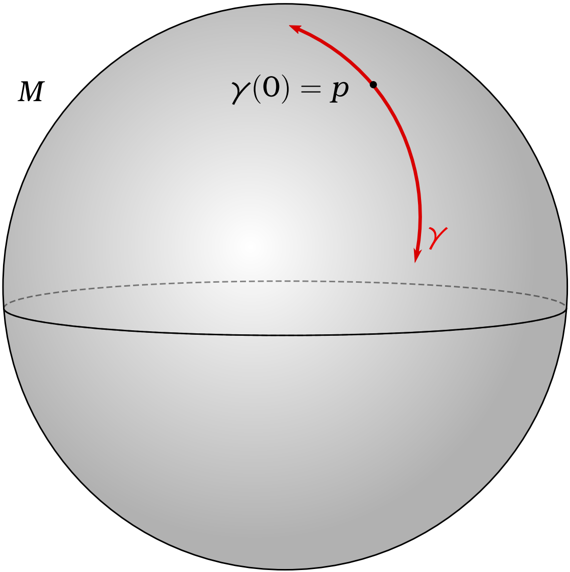
Preliminaries


M

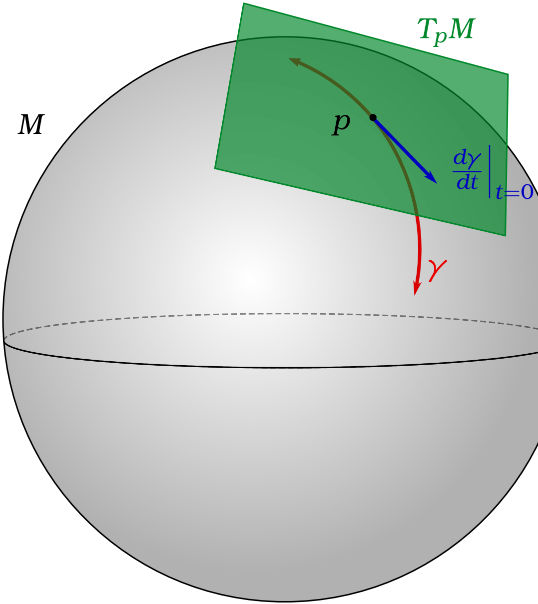

M
U
$U \cap V$
V



M
φ_α
$\varphi_{\alpha\beta}$
φ_α



M
$\gamma(0) = p$
γ


T_pM
M
p
$\left.\frac{d\gamma}{dt}\right|_{t=0}$
γ

A ***topological space*** M is a set of points with a collection of subsets $\mathcal{O}$ that we define to be open. These sets satisfy

A ***topological space*** M is a set of points with a collection of subsets $\mathcal{O}$ that we define to be open. These sets satisfy

- $\emptyset, M \in \mathcal{O}$;

A ***topological space*** M is a set of points with a collection of subsets $\mathcal{O}$ that we define to be open. These sets satisfy

- $\emptyset, M \in \mathcal{O}$;
- Arbitrary unions of open sets are open;

A ***topological space*** M is a set of points with a collection of subsets $\mathcal{O}$ that we define to be open. These sets satisfy

- $\emptyset, M \in \mathcal{O}$;
- Arbitrary unions of open sets are open;
- Finite intersections of open sets are open.

Just do diffeomorphisms and forget this

- A ***homeomorphism*** is a continuous bijection $f\colon M \to N$ with continuous inverse f^{-1}.

Just do diffeomorphisms and forget this

- A ***homeomorphism*** is a continuous bijection $f\colon M \to N$ with continuous inverse f^{-1}.
- We say M and N are homeomorphic.

ignore boundary

A ***manifold*** (with boundary) is a space that locally looks like $\mathbb{R}^n$ (or $\mathbb{R}^{n^+}$).

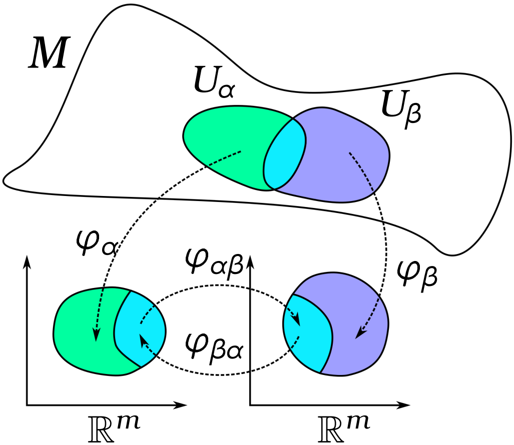
ignore boundary

A ***manifold*** (with boundary) is a space that locally looks like $\mathbb{R}^n$ (or $\mathbb{R}^{n^+}$).

$$\downarrow$$

A ***manifold*** M (with boundary) is a topological space such that each open set in $\mathcal{O}$ is homeomorphic to $\mathbb{R}^n$ (or $\mathbb{R}^{n^+}$).

Manifolds



M
U_α
U_β
φ_α
$\varphi_{\alpha\beta}$
φ_β
$\varphi_{\beta\alpha}$
$\mathbb{R}^m$
$\mathbb{R}^m$

Section 4

Applications

Section 5

Conclusions