Lorentzian Geometry and Topological Electromagnetism

Colin Roberts

Colorado State University

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- Collaborators: Cameron Krulewski, Michael Robinson, Clayton Shonkwiler

Section 1

Introduction

Outline

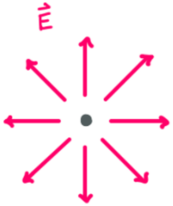
1. Intro Lorentzian geometry
2. Poincaré group $\mathrm{A}(1,3)$ and its Lie algebra $\mathfrak{a}(1,3)$
3. de Rham (Co)homology
4. Topological electromagnetism
5. Other thoughts

Motivation

- Study plasmas in a topological way
- Conceptualize robotic motion or computer graphics
- Nice playing field for PDEs and inverse problems

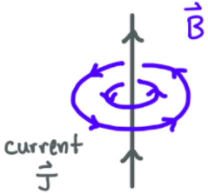
Maxwell's inhomogeneous equations

Gauss's law for electricity



$$\nabla \cdot \boldsymbol{E} = \rho$$

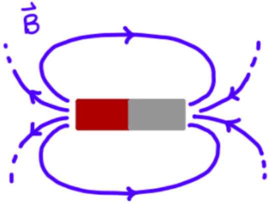
Ampere's Law



$$\nabla \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{J}$$

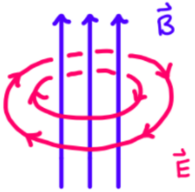
Maxwell's homogeneous equations

Gauss's law for magnetism



$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

Faraday's Law



$$\boldsymbol{\nabla} \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0$$

Section 2

Lorentzian Geometry

Geometry through algebra

- Take a vector space V with a quadratic form $Q(-)$
- Create the ***Clifford algebra*** $C\ell(V, Q)$ from the tensor algebra
- Elements of $C\ell(V, Q)$ are ***multivectors*** of grade 0 (scalars) up to grade n (pseudoscalars)

Euclidean space

Take $\mathbb{R}^n$ with Euclidean norm $|-|$ and an orthonormal basis $\boldsymbol{e}_i$.

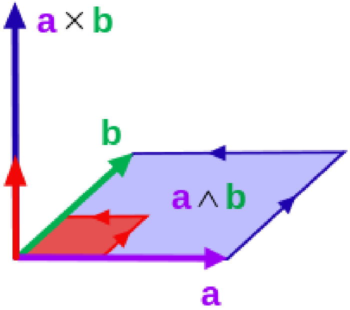
- We have the product in $\mathcal{G}_n := C\ell(\mathbb{R}^n, |-|)$ by

$$\boldsymbol{e}_i \boldsymbol{e}_j = \underbrace{\boldsymbol{e}_i \cdot \boldsymbol{e}_j}_{\text{scalar}} + \underbrace{\boldsymbol{e}_i \wedge \boldsymbol{e}_j}_{\text{bivector}}$$

- $\delta_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$ are the values of the Euclidean inner product on this basis.

$\mathbb{R}^3$

Given $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{G}_3$ bivector $\boldsymbol{a} \wedge \boldsymbol{b}$ represents an oriented plane



and the perpendicular or *dual* $(\boldsymbol{a} \wedge \boldsymbol{b})^{\perp} = \boldsymbol{a} \times \boldsymbol{b}$.

Lorentzian space

Instead, take $\mathbb{R}^4$ with basis $\boldsymbol{e}_0, \cdots, \boldsymbol{e}_3$ so that

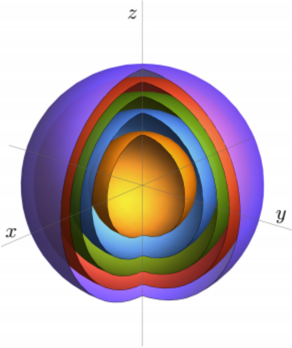
- $\boldsymbol{e}_0^2 = -1$ (temporal)
- $\boldsymbol{e}_i^2 = +1$ for $i = 1, 2, 3$ (spatial)
- $\boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu = 0$ if $\mu \neq \nu$, $\mu, \nu = 0, 1, 2, 3$ (orthogonal)
- Build $\mathcal{G}_{1,3}$ from this basis

Space Oddity

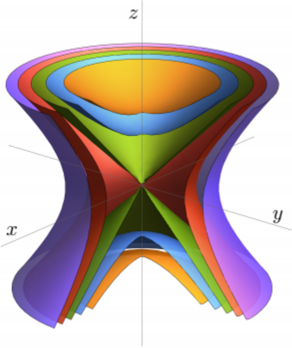
There exist ***null vectors*** $\boldsymbol{c}$ so that $\boldsymbol{c} \cdot \boldsymbol{c} = 0$.

Level sets $\boldsymbol{p} \cdot \boldsymbol{p} =$ constant yield foliations

Euclidean



Lorentzian



Question

What are the symmetries of Euclidean space?

Question

What are the symmetries of ***Lorentzian*** space?

Section 3

Poincaré Group

We can study geometry and topology through symmetry.

Symmetries of $\mathcal{G}_n$

- Rotations and reflections via the orthogonal group $\mathrm{O}(n)$ and special orthogonal group $\mathrm{SO}(n)$.
- Translations via the group $\mathbb{R}^n$.
- Combining yields the ***Euclidean group*** $\mathrm{E}(n) = \mathbb{R}^n \rtimes \mathrm{O}(n)$
- Removing reflections yields the ***special Euclidean group*** $\mathrm{SE}(n) = \mathbb{R}^n \rtimes \mathrm{SO}(n)$
- One could replace $\mathrm{O}(n)$ with the conformal group $\mathrm{CO}(n)$ and preserve $\mathcal{G}_n$

Rotations and reflections in $\mathcal{G}_n$

- Given a unit vector $\boldsymbol{n}$ and multivector A, we have

$$\boldsymbol{n}A\boldsymbol{n}^\dagger$$

  reflects A about the hyperplane perpendicular to $\boldsymbol{n}$

- Given another unit vector $\boldsymbol{m}$,

$$\boldsymbol{nm}A(\boldsymbol{nm})^\dagger = \boldsymbol{nm}A\boldsymbol{mn}$$

  yields a rotation in the plane defined by $\boldsymbol{n} \wedge \boldsymbol{m}$.

Pin and Spin

- Unit vectors generate the group $\boldsymbol{n} \in \mathrm{Pin}(n)$ and define a an element $\underline{T} \in \mathrm{O}(n)$ by

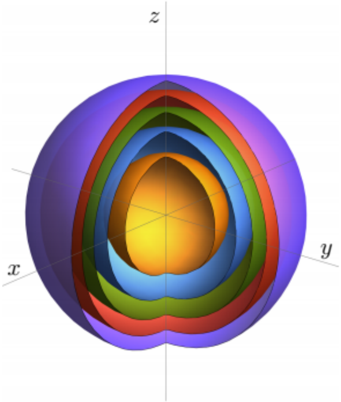
$$\underline{T}(\boldsymbol{v}) = \boldsymbol{n}\boldsymbol{v}\boldsymbol{n}^{\dagger}$$

  so the mapping $\boldsymbol{n} \mapsto \underline{T}$ is 2-to-1.
- Likewise, $R = \boldsymbol{n}\boldsymbol{m} \in \mathrm{Spin}(n)$ defines $\underline{R} \in \mathrm{SO}(n)$ by

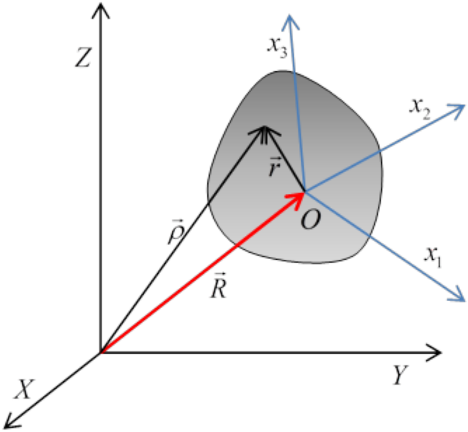
$$\underline{R}(\boldsymbol{v}) = R\boldsymbol{v}R^{\dagger}.$$

- We refer to the $R \in \mathrm{Spin}(n)$ as ***unit spinors***

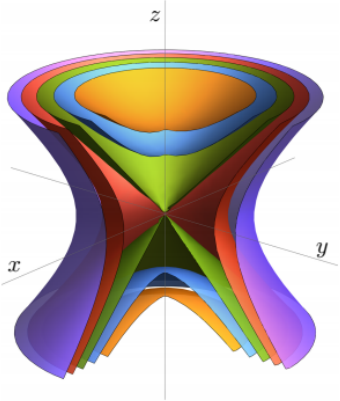
Orbits of vectors under action of $\mathrm{Spin}(n)$ and $\mathrm{Pin}(n)$ yield the level sets



Hence, we can take $\mathbb{R}^n \rtimes \mathrm{Spin}(n)$ to be the rigid symmetries of Euclidean space.

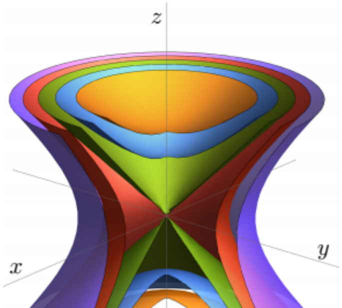


Defining $\mathrm{Spin}(1,3)$ and $\mathrm{Pin}(1,3)$ analogously lead to level sets via orbits



Relativity

- One sheeted hyperboloids are inaccessible regions of space (often called ***spacelike***)
- Two sheeted hyperboloids represent past and future directions (often called ***timelike***)
- Cone consists of all null vectors $\boldsymbol{c} \cdot \boldsymbol{c} = 0$ which represents ***light***.
- A particle with rest mass m has 4-momentum $\boldsymbol{p} = m\boldsymbol{v}$ and $\boldsymbol{p} \cdot \boldsymbol{p} = -m^2$
- $\implies$ future/past hyperboloids are foliated by mass.



Clifford analysis

- Reciprocal basis $\boldsymbol{e}^i$ defined by $\boldsymbol{e}^i \cdot \boldsymbol{e}_j = \delta^i_j$
- The gradient operator is defined to be $\boldsymbol{\nabla} = \boldsymbol{e}^i \frac{\partial}{\partial x^i}$
- Gradient splits like a vector

$$\boldsymbol{\nabla} A = \underbrace{\boldsymbol{\nabla} \cdot A}_{\text{grade lowering}} + \underbrace{\boldsymbol{\nabla} \wedge A}_{\text{grade raising}}$$

- Directional (or covariant) derivative

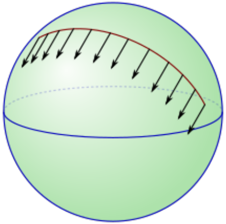
$$\nabla_{\boldsymbol{v}} A \coloneqq (\boldsymbol{v} \cdot \boldsymbol{\nabla}) A.$$

- Dynamics of a relativistic particle is a curve γ in the ***Poincaré group***

$$\mathrm{A}(1,3) \coloneqq \mathbb{R}^{1,3} \rtimes \mathrm{Spin}^{+}(1,3)$$

- Since mass is preserved $\boldsymbol{\nabla}(\boldsymbol{v}\cdot\boldsymbol{v}) = 0 \quad \Longrightarrow \quad \nabla_{\boldsymbol{v}}\boldsymbol{v} = \boldsymbol{v}\cdot\underbrace{(\boldsymbol{\nabla}\wedge\boldsymbol{v})}_{\text{vorticity } \omega}$

- Optimal transport of 4-velocity is given by projection onto the vorticity plane



Infinitesimal dynamics

Dynamics on a Lie group come from infinitesimals which are elements of the Lie algebra.

- Lie algebra of $\mathrm{A}(1,3)$ is the extension $\mathfrak{a}(1,3) = \mathbb{R}^{1,3} \rtimes \mathfrak{spin}^{+}(1,3)$
- $\mathfrak{spin}(1,3) = \mathcal{T} \oplus \mathcal{S}$ where

$$\mathcal{T} = \{\boldsymbol{e}_0 \boldsymbol{e}_i \mid i = 1, 2, 3\}$$

$$\mathcal{S} = \{\boldsymbol{e}_i \boldsymbol{e}_j \mid i \neq j,\ i, j = 1, 2, 3\} = \mathfrak{spin}(3)$$

- $\mathcal{T}$ corresponds to accelerations and the representations correspond mass.
- $\mathcal{S}$ corresponds to non-physical motions and the representations correspond to spin

Section 4

de Rham (Co)homology

- On a manifold M, we have the multivector fields $\mathcal{G}(M)$
- ***de Rham cohomology*** ring is

$$H^{\bullet}_{dR}(M) := \bigwedge_{k\in\mathbb{N}} \ker \boldsymbol{\nabla} \wedge_k \ / \ \mathrm{im}\boldsymbol{\nabla}\wedge_{k-1}$$

- Dual are the currents $T\colon \mathcal{G}(M) \to \mathbb{R}$ with boundary operator ∂

$$\partial T[A] = T[\boldsymbol{\nabla} \wedge A]$$

- ***de Rham homology*** group is

$$H^{dR}_{\bullet} := \bigoplus_{k\in\mathbb{N}} \ker \partial_k \ / \ \mathrm{im}\partial_{k+1}.$$

Multivector Equivalents of Currents

- Riemannian volume form μ and the bilinear pairing $(-,-)$
- Define k-current by distributional multivector

$$T[-] = \int_M (T_k, -)\mu.$$

- The boundary operator acts (on compactly supported fields)

$$\partial T[A] = T[\boldsymbol{\nabla} \wedge A] = \int_M (T_k, \boldsymbol{\nabla} \wedge A_{k-1})\mu = \int_M (\boldsymbol{\nabla} \cdot T_k, A_{k-1})\mu$$

- So $H_k^{dR} = \mathsf{ker}\, \boldsymbol{\nabla} \cdot_k / \mathrm{im} \boldsymbol{\nabla} \cdot_{k+1}$

Useful Theorems

Theorem (de Rham's Theorem)

The singular (co)homology over $\mathbb{R}$ is isomorphic to the de Rham (co)homology.

Theorem (Poincaré Duality)

We have $H_k \cong H^{n-k}$ by the dual $\perp$.

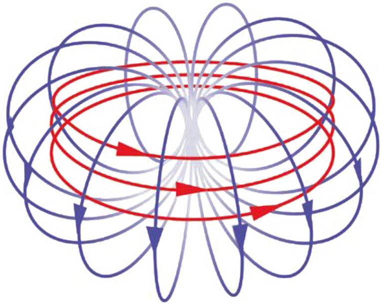
Theorem (Potentials)

Let A be a k-vector, then if $T[A] = 0$ for all $T \in H_k(M)$ we have A is potential, $A = \boldsymbol{\nabla} \wedge H$.

Section 5

Topological Electromagnetism

There are four physical postulates for electromagnetism that we write as topological axioms.



Axiom 1: Conservation of charge

- Current density $\boldsymbol{J}_3$ must flow through boundaries of regions N^4 so

$$0 = \int_{\partial N^4} \boldsymbol{J}_3 \cdot dX_3 \underbrace{=}_{\text{de Rham}} \partial N^4[\boldsymbol{J}_3] = N^4[\boldsymbol{\nabla} \wedge \boldsymbol{J}_3]$$

  so $\boldsymbol{J}_3$ is closed since N^4 is arbitrary
- Hence, for co-closed 3-current N^3 we have $N^3[\boldsymbol{J}_3] = 0$
- Thus magnetic excitation H is the potential $\boldsymbol{\nabla} \wedge H = \boldsymbol{J}_3$ by potentials theorem
- By Poincaré, $\boldsymbol{J} = \boldsymbol{J}_3^{\perp} = (\boldsymbol{\nabla} \wedge H)^{\perp} = \boldsymbol{\nabla} \cdot H^{\perp}$ defines a homology class in $H_1(M)$

Axiom 2: Conservation of flux

- The electromagnetic field F defines a cohomology class in $H^2(M)$ by taking a co-closed 2-current N^2 and noting
$$0 = \int_{N^2} F \cdot dX_2 = N^2[F]$$
implies $\boldsymbol{\nabla} \wedge F = 0$.
- Note F is not necessarily potential!

Axiom 3: Constitutive law

- We relate the excitation H with the field F.
- Simplest case is given by $F = H^{\perp}$ which yields the Maxwell equations $\boldsymbol{\nabla} F = \boldsymbol{J}$ or, in their more recognizable relativistic form

$$\boldsymbol{\nabla} \wedge F = 0 \qquad \text{(homogeneous)}$$

$$\boldsymbol{\nabla} \cdot F = \boldsymbol{J} \qquad \text{(inhomogeneous)}$$

- Homogeneous equations are Gauss's law for magnetism and Faraday's law
- Inhomogeneous are Gauss's law for electricity and Ampere's law

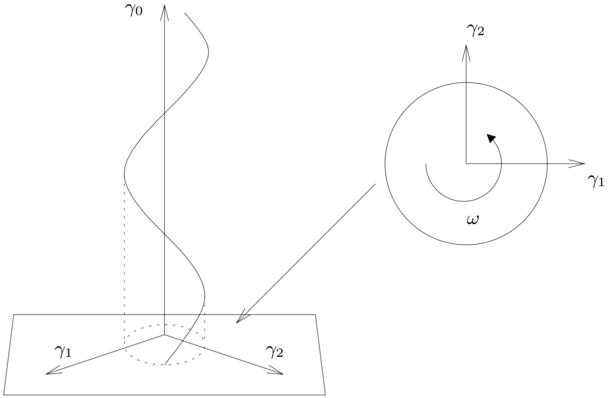
Axiom 4: Lorentz force

- Transport of charged particle with charge q and 4-velocity $\boldsymbol{v}$ in a field F is

$$\underbrace{\nabla_{\boldsymbol{v}} \boldsymbol{v} = \frac{q}{m} \boldsymbol{v} \cdot F}_{\text{Faraday transport}}$$

- F plays the role of vorticity ω in Fermi transport equation

- For a charged particle, we have the Lorentz force law $\nabla_{\boldsymbol{v}} \boldsymbol{v} = \frac{1}{2} \boldsymbol{v} \cdot F$
- In terms of a proper time parameterization $\frac{d\boldsymbol{v}}{d\tau} = \frac{1}{2} \boldsymbol{v} \cdot F(\gamma(\tau))$.



Spinor Equations

- Since $\boldsymbol{v} \cdot \boldsymbol{v}$ is constant, velocity at any τ is given by time varying isometries

$$\boldsymbol{v}(\tau) = \underline{R}_\tau(\boldsymbol{v}_0)$$

- $R \in \mathrm{Spin}(1,3)$ induces $\underline{R}_\tau(\boldsymbol{v}_0) = R(\tau)\boldsymbol{v}_0 R(\tau)^\dagger$
- Hence particle motion tracked as curve in the Poincaré group
- Fermi-Faraday transport of a spinor is given by $\frac{dR}{d\tau} = FR$

Section 6

Conclusions and questions

This can likely be generalized to charged fluids (plasmas)

- Take $m, q, \boldsymbol{v}$ in $\mathcal{G}(M)$ so $\boldsymbol{p} = m\boldsymbol{v}$ and $\boldsymbol{J} = q\boldsymbol{v}$
- If we allow mass to flow separately from the velocity

$$\boxed{m\nabla_{\boldsymbol{v}}\boldsymbol{v} + (\nabla_{\boldsymbol{v}} m)\boldsymbol{v} = q\boldsymbol{v} \cdot \boldsymbol{F}}$$

- Given EM axioms and the constraint that *charge to mass is constant* yields

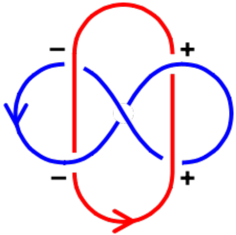
$$\boxed{\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0}$$

- Maxwell's equations hold

$$\boxed{\boldsymbol{\nabla} F = \boldsymbol{J}}$$

- What kind of topology can be extracted from these equations?
- E.g.,

$$\boldsymbol{v}\boldsymbol{\omega} = \underbrace{\boldsymbol{v} \cdot \boldsymbol{\omega}}_{\text{transport}} + \underbrace{\boldsymbol{v} \wedge \boldsymbol{\omega}}_{\text{helicity}}$$



- Can we make any predictions with this formulation?

Thank you!