MATH 519, Homework 2

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Solutions

Problem 1. (S & S 2.2.) Show that $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$.
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Proof. First, consider breaking this into four contours: $\gamma_1 = [-R, -\epsilon]$, $\gamma_2 = \{z \in \mathbb{C} : z = \epsilon e^{-it}, \leq t \leq 1\}$, $\gamma_3 = [\epsilon, R]$, and $\gamma_4 = \{z \in \mathbb{C} : z = Re^{it}, 0 \leq 1 \leq \pi\}$. Then we denote $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$. Then

$$\int_\gamma \frac{\sin(z)}{z} dz = \int_{\gamma_1} \frac{\sin(z)}{z} dz + \int_{\gamma_2} \frac{\sin(z)}{z} dz + \int_{\gamma_3} \frac{\sin(z)}{z} dz + \int_{\gamma_4} \frac{\sin(z)}{z} dz.$$

Note that by Cauchy's theorem, we have that for all $\epsilon > 0$ that

$$\int_\gamma \frac{\sin(z)}{z} dz = 0,$$

since $\frac{\sin(z)}{z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$. We can then rearrange the above integrals slightly to receive the following:

$$\begin{aligned} 0 &= \int_{\gamma_1} \frac{\sin(z)}{z} dz + \int_{\gamma_2} \frac{\sin(z)}{z} dz + \int_{\gamma_3} \frac{\sin(z)}{z} dz + \int_{\gamma_4} \frac{\sin(z)}{z} dz \\ \implies -\int_{\gamma_1} \frac{\sin(z)}{z} dz - \int_{\gamma_3} \frac{\sin(z)}{z} dz &= \int_{\gamma_2} \frac{\sin(z)}{z} dz + \int_{\gamma_4} \frac{\sin(z)}{z} dz. \end{aligned}$$

Notice that in the limit as $R \to \infty$ and $\epsilon \to 0$ we have

$$-2 \int_0^\infty \frac{\sin(x)}{x} dx = \int_{\gamma_2} \frac{\sin(z)}{z} dz + \int_{\gamma_4} \frac{\sin(z)}{z} dz,$$

since $\frac{\sin(x)}{x}$ is even and where we of course let the limits be taken for γ_2 and γ_4 as well. This leaves us to evaluate the contour integrals for γ_2 and γ_4 and we are done.

Note that we can take $\Im\left(\int_{\gamma_2} \frac{e^{iz}-1}{z} dz\right) = \int_{\gamma_2} \frac{\sin(z)}{z}$ and so we have for the γ_2 contour integral

$$\begin{aligned} \int_{\gamma_2} \frac{\sin(z)}{z} dz &= \Im\left(\int_{\gamma_2} \frac{e^{iz}}{z} dz\right) \\ &= \Im\left(\int_{\gamma_2} \frac{\left(1 + iz - \frac{z^2}{2} - \frac{iz^3}{3!} + \cdots\right) - 1}{z} dz\right). \end{aligned}$$

Then we want to apply the ML theorem,

$$\left| \Im\left(\int_{\gamma_2} \left(i - \frac{z}{2} - \frac{iz^2}{3!} + \cdots\right) dz\right)\right| \leq \pi\epsilon M.$$

Note that the integrand is bounded and we denote this bound by M. Then as $\epsilon \to 0$ we find that $\pi\epsilon M \to 0$ and hence

$$\int_{\gamma_2} \frac{\sin(z)}{z} dz = 0.$$

Next, consider the γ_4 contour integral. We have

$$\int_{\gamma_2} \frac{\sin(z)}{z} dz = \Im\left(\int_{\gamma_4} \frac{e^{iz}-1}{z} dz\right).$$

Concentrating on just the integral portion, we find

$$\begin{aligned}
\int_{\gamma_4} \frac{e^{iz}-1}{z} dz &= \int_{\gamma_4} \frac{e^{iz}}{z} dz - \int_{\gamma_4} \frac{1}{z} dz \\
&= -\int_{\gamma_4} \frac{1}{z} dz && \text{since we showed the other integral goes to zero in class} \\
&= \int_0^{\pi} \frac{1}{Re^{it}} iRe^{it} dt \\
&= -i \int_0^{\pi} dt \qquad = -i\pi.
\end{aligned}$$

Hence the imaginary part of this integral is just $-\pi$.

Thus we have

$$\begin{aligned}
-2\int_0^{\infty} \frac{\sin(x)}{x} dx &= \int_{\gamma_4} \frac{\sin(z)}{z} dz = -\pi \\
\implies \int_0^{\infty} \frac{\sin(x)}{x} dx &= \frac{\pi}{2}.
\end{aligned}$$

Note: I got help from Jeremy on this problem. □

Problem 2.(S & S 2.12.a) Let u be a real valued function defined on the unit disk $\mathbb{D}$. Suppose that u is twice continuously differentiable and harmonic, that is,

$$\Delta u(x,y) = 0$$

for all $(x,y) \in \mathbb{D}$.

Prove that there exists a holomorphic function f on the unit disk such that $\Re(f) = u$. Also show that the imaginary part of f is uniquely defined up to an additive (real) constant.

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Proof. We will use the hint. First, let $g(z) = 2\frac{\partial u}{\partial z}$ and note that

$$\begin{aligned}
2\frac{\partial}{\partial z} u(x,y) &= \left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) u(x,y) \\
&= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}.
\end{aligned}$$

Now we let $\frac{\partial u}{\partial x} = w(x,y)$ and $-\frac{\partial u}{\partial y} = v(x,y)$ and hence $g(z) = w + iv$. Then taking the partial derivatives, we find

$$\begin{aligned} w_x &= \frac{\partial^2 u}{\partial x^2} \\ w_y &= \frac{\partial}{\partial y}\frac{\partial u}{\partial x} \\ v_x &= -\frac{\partial}{\partial x}\frac{\partial v}{\partial y} \\ v_y &= -\frac{\partial^2 v}{\partial y^2}. \end{aligned}$$

Note that $w_x = v_y$ since u is harmonic and that $w_y = -v_x$ by commuting partial derivatives (u is twice continuously differentiable). Hence this shows that $g(z)$ is holomorphic.

Since $g(z)$ is holomorphic we have that there exists a primitive F such that $F' = g$. Now we want that $f(x,y) = u(x,y) + i(\nu(x,y) + K)$ where k is some real constant. Let $F = \mu + i\nu$ and we have

$$F' = 2\frac{d\mu}{dz} = 2\frac{du}{dz}$$

so that $\Re(F) = u + c$ with $c \in \mathbb{C}$. We can choose F so that $c = 0$ and hence $\Re(F) = u$. So we take Now since F is holomorphic we have that the CREs hold and this means that

$$\nu = \int \frac{\partial u}{\partial x} dy + \phi(y) = -\int \frac{\partial u}{\partial y} dx + \psi(x)$$

by just integrating the CREs. Note that this was done via real integration and hence the potential functions $\phi(y)$ and $\psi(x)$ are real. Now we differentiate with respect to x to find

$$\begin{aligned} \int \frac{\partial^2 u}{\partial x^2} dy &== \int \frac{\partial^2 u}{\partial y \partial x} dx + \psi'(x) \\ \implies -\frac{\partial u}{\partial y} &= -\int \frac{\partial^2 u}{\partial y^2} dy = \frac{-\partial u}{\partial y} + \psi'(x) \\ \implies \psi'(x) = 0. \end{aligned}$$

Letting $f = u + iv$ we have that $v = \nu + K$ where $K = \psi(x)$ is a constant.
Note: I got help from Emily and Jeremy on this problem. $\square$

Problem 3. (S & S 2.13.) Suppose that f is an analytic function defined everywhere in $\mathbb{C}$ and such that for each $z_0 \in \mathbb{C}$ at least one coefficient in the expansion

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

is equal to 0. Prove that f is a polynomial.

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Proof. First note that $c_n = \frac{f^(n)(z_0)}{n!}$. Then, suppose for a contradiction that no $f^(n)(z)$ is indentically zero. This means that

$$\bigcup_{n=0}^{\infty} (f^(n))^{-1}(0)$$

is a countable set. However, this is a contradiction since the original supposition is that for each $z_0 \in \mathbb{C}$ at least one coefficient is zero and since $\mathbb{C}$ is uncountable. Hence, the set of points where some derivative $f^(n)$ vanishes must be uncountable (since a countable union of countable sets is still countable). This then implies that f must be a polynomial.
Note: I found some help online for this problem. □

Problem 4. S & S. Complete the proof of Theorem 4.4, stated on page 49 (the bit about a_n).
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Proof. We wish to show that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where coefficients

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Up to this point, we have that

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) \cdot (z - z_0)^n.$$

Note that the Cauchy integral formula for derivatives is as follows:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta,$$

and this holds by the assumptions of the theorem. Hence a slight rearrangement and letting $z = z_0$ gives that

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta.$$

Hence we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$ □

Problem 5. Let C be the boundary of the triangle with vertices at the points 0, $3i$, and -4, with positive orientation. Show that $\left|\int_C (e^z - \overline{z}) dz\right| \leq 60$.
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Proof. First note that the length of the contour C is $L(C) = 12$. Then consider

$$\begin{aligned} \sup_{z \in C} |e^z - \overline{z}| &\leq \sup_{z \in C} |e^z| + \sup_{z \in C} |\overline{z}| \\ &= 1 + 4 = 5 = M. \end{aligned}$$

Then by the ML theorem we have that

$$\left| \int_C (e^z - \overline{z}) dz \right| \leq ML(C) = 60.$$ □

Problem 6. Evaluate these integrals using any paths between the limits of integration:

(a) $\int_i^{i/2} e^{\pi z} dz$.

(b) $\int_1^3 (z-2)^3 dz$.

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Proof.

(a) Let $\gamma\colon [0,1] \to \mathbb{C}$ be given by $\gamma(t) = i - \frac{i}{2}t$. Then we have

$$\begin{aligned}
\int_i^{i/2} \exp(\pi z) dz &= \int_\gamma \exp(\pi\gamma(t))\gamma'(t) dt \\
&= \int_0^1 \exp\left(i - \frac{i}{2}t\right)\left(\frac{-i}{2}\right) dt \\
&= \frac{i}{2}\int_0^1 \exp\left(-\frac{i\pi}{2}t\right) dt \\
&= \frac{i}{2}\left[\frac{-2}{i\pi}\exp\left(-\frac{i\pi}{2}t\right)\right]_0^1 \\
&= \frac{-1}{\pi}(-i-1) = \frac{1+i}{\pi}.
\end{aligned}$$

(b) Let $\gamma\colon [0,1] \to \mathbb{C}$ by $\gamma(t) = 1 + 2t$. Then we have

$$\begin{aligned}
\int_1^3 (z-2)^3 dz &= \int_\gamma (\gamma(t)-2)^3\gamma'(t) dt \\
&= \int_0^1 (1+2t-2)^3 2dt \\
&= 2\int_0^1 (2t-1)^3 dt \\
&= \frac{1}{4}\left[(2t-1)^4\right]_0^1 \\
&= \frac{1}{4}(1^4 - (-1)^4) \\
&= 0.
\end{aligned}$$

□

Problem 7. Let Cauchy help you evaluate these integrals, where C_1 is the square with sides along $x = \pm 2$ and $y = \pm 2$ (positively oriented) and C_2 is the positively-oriented circle $|z - i| = 2$:

(a) $\int_{C_1} \frac{e^{-z}}{z - \frac{\pi i}{2}} dz$.

(b) $\int_{C_1} \frac{z}{2z+1} dz$.

(c) $\int_{C_2} \frac{1}{z^2+4} dz$.

(d) $\int_{C_2} \frac{1}{(z^2+4)^2} dz$.

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Proof.

Cauchy's integral formula extends to rectangles as well, so we find

$$\begin{aligned}
\int_{C_1} \frac{\exp(-z)}{z - \frac{\pi i}{2}} dz &= 2\pi i \left(\exp\left(-\frac{\pi i}{2} \right) \right) \\
&= 2\pi.
\end{aligned}$$

Again, we have

$$\begin{aligned}
\int_{C_1} \frac{z}{2z+1} dz &= \frac{1}{2} \int_{C_1} \frac{z}{z + \frac{1}{2}} dz \\
&= \frac{1}{2} (2\pi i) \left(-\frac{1}{2} \right) \\
&= -\frac{\pi i}{2}.
\end{aligned}$$

We have

$$\int_{C_2} \frac{1}{z^2+4} dz = \int_{C_2} \frac{1}{(z-2i)(z+2i)} dz.$$

Now, note that $\dfrac{1}{z+2i}$ is holomorphic on the interior of C_2. Hence we find

$$\begin{aligned}
\int_{C_2} \frac{1}{z^2+4} dz &= \int_{C_2} \frac{\left(\frac{1}{z+2i} \right)}{z-2i} dz \\
&= 2\pi i \left(\frac{1}{2i+2i} \right) \\
&= \frac{\pi}{2}.
\end{aligned}$$

We have

$$\begin{aligned}
\int_{C_2} \frac{1}{(z^2+4)^2} dz &= \int_{C_2} \frac{1}{(z-2i)^2(z+2i)^2} dz \\
&= \int_{C_2} \frac{1}{(z+2i)^2} \cdot \frac{1}{(z-2i)^2} dz.
\end{aligned}$$

Letting $f(z) = \frac{1}{(z+2i)^2}$ and noting that f is holomorphic on the interior of C_2, we find

$$\begin{aligned}
\int_{C_2} \frac{1}{(z^2+4)^2} dz &= 2\pi i f'(z)|_{z=2i} \\
&= 2\pi i \left(-\frac{2}{(2i+2i)^3} \right) \\
&= \frac{\pi}{16}.
\end{aligned}$$

□