## Clifford Analysis and a Noncommutative Gelfand Representation

Colin Roberts



## Overview

- 1 Introduction
- 2 Clifford analysis
- 3 Gelfand theory
- 4 Future work
- 5 Conclusions

## Section 1

## Introduction

# Motivating problems

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- The *Calderón problem* replaces the medium with a manifold M, conductivity with g, and replaces the voltage-to-current map with the Dirichlet-to-Neumann operator  $\Lambda$ .



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- Do these functions also contain geometric information such as metric data?
- $\blacksquare$  How much can we learn about M if our data is supported only on the boundary?

## Subsection 1

Preliminaries

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- Clifford analysis arrived in the 1980's due to Hestenes, Sobczyk, Sommen, Brackx, and Delenghe in order to enrich Élie Cartan's differential forms. See: [Hestenes, Sobczyk: 1984] and [Doran, Lasenby: 2003].



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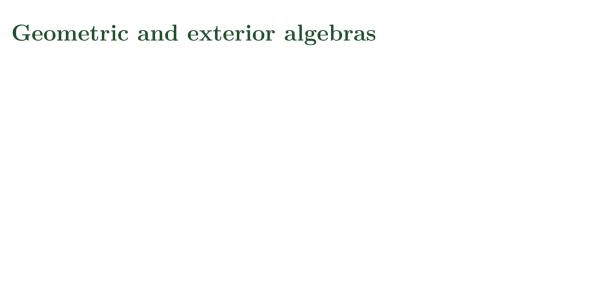
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■ Form the *Clifford algebra* via a quotient

$$C\ell(V,Q) \coloneqq \mathcal{T}(V)/\langle \mathbf{v} \otimes \mathbf{v} - Q(\mathbf{v}) \rangle.$$



## Geometric and exterior algebras

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- The scalar part is symmetric:  $\mathbf{u} \cdot \mathbf{v} = g(\mathbf{u}, \mathbf{v})$ .
- The bivector part is antisymmetric:  $\boldsymbol{u} \wedge \boldsymbol{v} = -\boldsymbol{v} \wedge \boldsymbol{u}$ .



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  - Elements of the even grade subalgebra,  $\mathcal{G}^+$ , are called *spinors*.
- The most general elements are *multivectors* and are given by

$$A = \sum_{r=0}^{n} \langle A \rangle_r,$$

where  $\langle A \rangle_r \in \mathcal{G}^r$  extracts the grade r part of A. So  $\mathcal{G} = \bigoplus_{i=1}^n \mathcal{G}^r$ .

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■ The most important products are

$$A_r \cdot B_s \coloneqq \langle A_r B_s \rangle_{|r-s|} \qquad \qquad A_r \wedge B_s \coloneqq \langle A_r B_s \rangle_{r+s}$$

$$A_r \rfloor B_s \coloneqq \langle A_r B_s \rangle_{s-r} \qquad \qquad A_r \lfloor B_s \coloneqq \langle A_r B_s \rangle_{r-s}$$

# Reciprocals and reverses

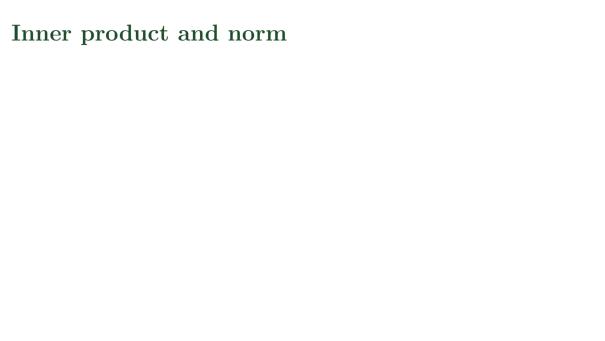
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- The reverse of a multivector is extended linearly from the action on r-blades by

$$\mathbf{A_r}^{\dagger} = (\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r)^{\dagger} = \mathbf{v}_r \wedge \cdots \wedge \mathbf{v}_1.$$



# Inner product and norm

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■ Define the *multivector norm* by

$$|A| \coloneqq \sqrt{(A,A)}.$$

# Adjoint

Note the reverse acts as an adjoint by

$$(CA, B) = (A, C^{\dagger}B)$$
  
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# ${\bf Pseudoscalars}$

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 $\blacksquare$  We define the *unit pseudoscalar* by

$$I \coloneqq \frac{1}{|\boldsymbol{\mu}|} \boldsymbol{\mu}.$$



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- Unit r-blades correspond to r-dimensional subspaces so they correspond to points in Gr(r, n).



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■ Note  $A_r^{\perp} \in \mathcal{G}^{n-r}$ , like the Hodge star  $\star$ .

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$$(u_1 + v_1 \mathbf{B}_{12})(u_2 + v_2 \mathbf{B}_{12}) = u_1 u_2 - v_1 v_2 + (u_1 v_2 + u_2 v_1) \mathbf{B}_{12}.$$

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■ Right multiplication by  $B_{12}$  rotates counter-clockwise by  $\pi/2$ .

# Section 2

# Clifford analysis

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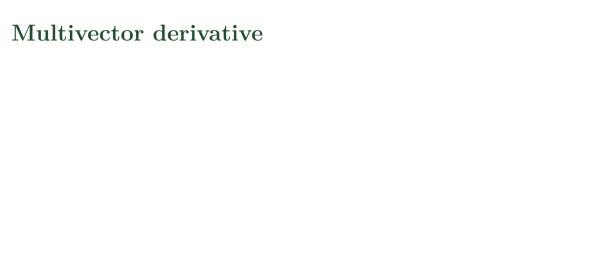
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■ Retain the same naming scheme as before.



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 $\blacksquare$   $\nabla_u$  is compatible with dot and wedge since

$$\nabla_{\mathbf{u}}(A \cdot B) = (\nabla_{\mathbf{u}}A) \cdot B + A \cdot (\nabla_{\mathbf{u}}B)$$
$$\nabla_{\mathbf{u}}(A \wedge B) = (\nabla_{\mathbf{u}}A) \wedge B + A \wedge (\nabla_{\mathbf{u}}B).$$



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■ Note  $\nabla^2 = \Delta$ , the Laplace-Beltrami operator.



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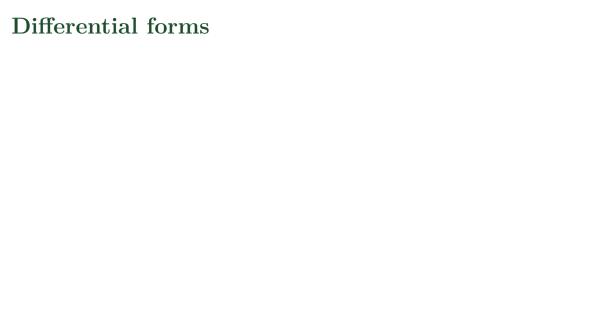
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■ Specifically,

$$\operatorname{curl}(\boldsymbol{v}) = (\nabla \wedge \boldsymbol{v})^{\perp}$$



#### Differential forms

■ Define the r-dimensional directed measure

$$dX_r \coloneqq \mathbf{v}_{j_1} \wedge \dots \wedge \mathbf{v}_{j_r} dx^{j_1} \dots dx^{j_r}$$

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■ Define an r-form  $a_r$  by

$$\alpha_r = A_r \cdot dX_r^{\dagger}$$

where  $A_r = \frac{1}{r!} a_{i_1 \cdots i_r} \mathbf{v}^{i_1} \wedge \cdots \wedge \mathbf{v}^{i_r}$ .

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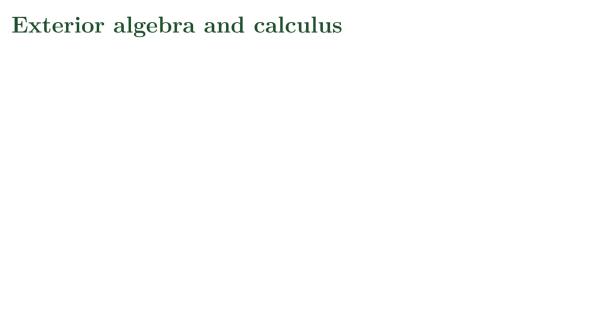
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where  $A_r = \frac{1}{r!} a_{i_1 \cdots i_r} \mathbf{v}^{i_1} \wedge \cdots \wedge \mathbf{v}^{i_r}$ .

■ Refer to  $A_r$  the multivector equivalent of  $\alpha_r$ .



## Exterior algebra and calculus

■ Given r-forms  $a_r$ ,  $b_r$ , and an s-form  $c_s$ , we have

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■ The Hodge star on multivector equivalents is

$$\star a_r = (\boldsymbol{I}^{-1} A_r)^{\dagger} \cdot dX_{n-r}^{\dagger}$$



#### Volume form

 $\blacksquare$  The *volume form* on M is given in local coordinates by

$$\mu = \sqrt{|g|} dx^1 \cdots dx^n = \mathbf{I}^{-1} \cdot dX_n$$

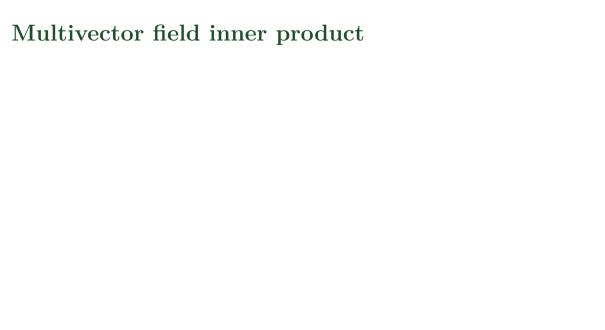
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■ We integrate scalar fields  $A_0$  on M by

$$\int_{M} A_0^{\perp} \cdot dX_n = \int_{M} A_0 \mu.$$



## Multivector field inner product

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■  $\langle\langle A_r, B_s \rangle\rangle$  when  $r \neq s$  so the  $L^2$ -direct sum agrees with the grade based direct sum.



## Boundary

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■ The boundary volume form is

$$\mu_{\partial} \coloneqq \boldsymbol{I}_{\partial}^{-1} \cdot dX_{n-1}$$

and we define

$$\ll A, B \gg_{\partial} \coloneqq \frac{1}{\operatorname{vol}(M)} \int_{\partial M} (A, B) \mu_{\partial}.$$

## Multivector valued integrals

We can define a multivector valued integral on an oriented submanifold R by taking  $A \in \mathcal{G}(M)$  and computing

$$\int_R A \mathbf{I}_R \mu_R.$$

# Theorem (Hestenes, Sobczyk, 1984)

Let 
$$A, B \in \mathcal{G}(M)$$
, then

$$\int_{M} \dot{A} \dot{\nabla} \mathbf{I} \mu = \int_{\partial M} A \mathbf{I}_{\partial} \mu_{\partial}$$

 $\int_{M} \mathbf{I} \nabla B \mu = \int_{\partial M} \mathbf{I}_{\partial} B \mu_{\partial}$ 

 $\int_{\mathcal{M}} \dot{A} \dot{\nabla} \mathbf{I} B \mu = (-1)^n \int_{\mathcal{M}} A \mathbf{I} \nabla B \mu + \int_{\partial \mathcal{M}} A \mathbf{I}_{\partial} B \mu_{\partial}.$ 

#### Theorem

We have the Green's formula for the gradient

 $\ll A, \mathbf{I} \nabla B \gg = (-1)^n \ll \nabla A, \mathbf{I} B \gg + \ll A, \mathbf{I}_{\partial} B \gg_{\partial}.$ 



## Special fields

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■ Let  $f = u + v\mathbf{B} \in \mathcal{G}_2^+(\mathbb{R}^2)$  then  $\nabla f = 0$  yields the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

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■ Define the monogenic fields

$$\mathcal{M}(M) := \{ A \in \mathcal{G}(M) \mid \nabla A = 0 \}.$$

■ Let  $f = u + v\mathbf{B} \in \mathcal{G}_2^+(\mathbb{R}^2)$  then  $\nabla f = 0$  yields the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

■ Define the *gradients* 

$$\nabla \mathcal{G}(M) \coloneqq \{ \nabla A \mid A \in \mathcal{G}(M) \text{ and } A|_{\partial M} = 0 \}.$$



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■ Note,

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■ Define the Cauchy kernel by G(x, x') = E(x' - x).



## Cauchy integral

■ Let  $A \in \mathcal{M}(M)$ , then we have the Cauchy integral formula

$$A(x) = (-1)^n \mathbf{I}^{-1} \int_{\partial M} G(x, x') \mathbf{I}_{\partial}(x') A(x') \mu_{\partial}(x').$$

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■ This uniquely determines a monogenic field from boundary values.

## Lemma

Let  $A \in \mathcal{M}(M)$  be such that  $A|_{\partial M} = 0$ . Then A = 0 on all of M.

Lemma

Fix a multivector field 
$$A \in \mathcal{G}(M)$$
. If

for all  $B \in \mathcal{G}(M)$  with  $B|_{\partial M} = 0$ , then A = 0.

$$\ll A, B \gg = 0$$

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## Theorem (Clifford-Hodge-Morrey Decomposition)

The space of multivector fields  $\mathcal{G}(M)$  has the  $L^2$ -orthogonal decomposition

The space of manifector fields 
$$g(m)$$
 has the  $B$  -orthogonal accomposition

 $\mathcal{G}(M) = \mathcal{M}(M) \oplus \mathbf{I} \nabla \mathcal{G}(M).$ 



## Proof.

• Orthogonality: Let  $A \in \mathcal{M}(M)$  and  $\mathbf{I} \nabla B \in \mathbf{I} \nabla \mathcal{G}(M)$  and note

 $\ll A, \mathbf{I} \nabla B \gg = (-1)^n \ll \nabla A, \mathbf{I} B \gg + \ll A, \mathbf{I}_{\partial} B \gg = 0,$ 

by the multivector Green's formula.

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- lacktriangle Use the Cauchy integral formula, construct a monogenic field  $\tilde{C}$  from  $C|_{\partial M}$  and note  $C = \tilde{C} + C_0$  where  $C_0|_{\partial M} = 0$ .
- Note,  $0 = \langle C, \mathbf{I} \nabla B \rangle = \langle \nabla C_0, \mathbf{I} B \rangle$ .

- By the previous lemmas, it must be that  $C_0 = 0$ . Hence the orthogonal
- complement to  $\mathbf{I}\nabla\mathcal{G}(M)$  is  $\mathcal{M}(M)$ .

# Comparing to Hodge-Morrey

## Comparing to Hodge-Morrey

■ The Hodge-Morrey decomposition reads

$$\Omega^{r}(M) = \underbrace{\mathcal{E}_{D}^{r}(M)}_{\operatorname{Im}(\nabla \wedge)} \oplus \underbrace{\mathcal{C}_{N}^{r}(M)}_{\operatorname{Im}(\nabla \rfloor)} \oplus \underbrace{\mathcal{H}^{r}(M)}_{\operatorname{Ker}(\nabla)}.$$

via [Schwarz: 1995].

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via [Schwarz: 1995].

■ Whereas the Clifford-Hodge-Morrey decomposition ignores specific grades

$$\mathcal{G}(M) = \mathcal{M}(M) \oplus \mathbf{I} \nabla \mathcal{G}(M).$$

## Section 3

## Gelfand theory

■ In [Belishev: 2003], we see an algebraic proof for the 2-dimensional Calderón problem.

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- We prove an analogous result for an arbitrary  $\mathbb{B}$  in  $\mathbb{R}^n$ .
- $\blacksquare$  This approach can hopefully be used to prove the analogous result for any smooth orientable Riemannian manifold M.

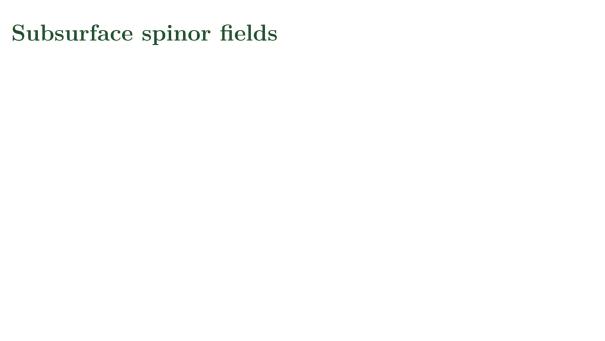
The boundary control (BC) method is implemented in [Belishev: 2003] in the following manner.

■ Determine the algebra  $\mathcal{A}(M)$  of holomorphic functions on M from continuous function algebra on the boundary  $\mathcal{A}(\partial M)$  using  $\Lambda$ .

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- Functions in  $\mathcal{A}(M)$  determine the complex structure on M.
- $\blacksquare$  Thus, we can find a g that is conformal with the complex structure.



## Subsurface spinor fields

■ Let  $\mathbf{B} \in \mathcal{G}(M)$  be a constant unit 2-blade, then  $f_+ \in \mathcal{G}^+(M)$  satisfying

$$f_+ = \mathbf{P}_{\boldsymbol{B}} \circ f_+ \circ \mathbf{P}_{\boldsymbol{B}}$$

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is a subsurface spinor field. Let  $\mathcal{G}_{B}^{+}(M)$  denote the space such fields.

■ The space of monogenic subsurface spinors

$$\mathcal{A}_{\mathbf{B}}(M) = \{ f_+ \in \mathcal{G}_{\mathbf{B}}^+(M) \mid \nabla f_+ = 0 \}$$

is a commutative unital Banach algebra.



## **Functionals**

■ Define the *spinor dual*  $\mathcal{M}^*(M)$  as the continuous right  $\mathcal{G}_n^+$ -module homomorphisms

$$\mathcal{M}^*(M) \coloneqq \{l: \mathcal{M}^+(M) \to \mathcal{G}_n^+ \mid l(fs+g) = l(f)s + l(g), \ \forall f, g \in \mathcal{M}(M), \ s \in \mathcal{G}_n^+ \}$$

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and refer to the elements as *spin functionals*.

■ Assert the weak-\* topology on  $\mathcal{M}^*(M)$  so that every  $x \in M$  corresponds to a continuous map on  $\mathcal{M}^*(M)$ .



■ Define the algebra  $\mathbb{A}_{B}$  to be the algebra generated by 1 and B.

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- The  $spinor\ spectrum\ \mathfrak{M}(M)$  is the set of algebra homomorphisms

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- One example of such characters are point evaluations  $\delta(f) = f(x^{\delta})$ .
- We show these are the only elements in the spectrum.

■ Take the standard basis for  $\mathbb{R}^n$ , and consider  $M = \mathbb{B}_{R,w}$ .

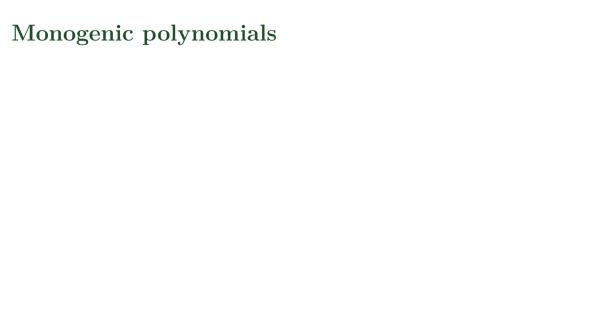
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- Let  $\mathbf{B}_{ij} = \mathbf{e}_i \mathbf{e}_j$ , and define

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■ Note  $z_{ij} \in \mathcal{A}_{\boldsymbol{B}_{ij}}(\mathbb{B}_{R,w})$ .



# Monogenic polynomials

■ Let  $\sigma$  be a permutation of  $\{2, 3, ..., n\}$ , then

$$p_{j_2\cdots j_n}(x) = \frac{1}{j!} \sum_{\text{permutations}} z_{1\sigma(1)}(x)\cdots z_{1\sigma(j)}(x)$$

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■ Collect these into the set of monogenic polynomials

$$\mathcal{M}^{\mathcal{P}}(\mathbb{B}_{R,w}) = \left\{ \sum_{j=0}^{N} \left( \sum_{\substack{j_2 \dots j_n \\ j_2 + \dots + j_n = j}} p_{j_2 \dots j_n} a_{j_2 \dots j_n} \right) \mid j_2 + \dots + j_n = j, N \in \mathbb{N}, \ a_{j_2 \dots j_n} \in \mathcal{G}_n \right\}.$$

# Lemma (Density)

The space  $\mathcal{M}^{\mathcal{P}}(\mathbb{B}_{R,w})$  is dense in  $\mathcal{M}^{+}(\mathbb{B}_{R,w})$ .

Proof sketch.

■ Let  $f \in \mathcal{M}^+(\mathbb{B}_{R,w})$  and use the Cauchy integral formula to define the coefficients  $a_{i_2...i_n} \in \mathcal{G}_n^+$  by

$$a_{j_2\cdots j_n} = \int_{\partial \mathbb{B}_{R,w}} \frac{\partial^j G(w,y)}{\partial y_2^{j_2}\cdots \partial y_n^{j_n}} \boldsymbol{\nu}(y) f(y) \mu_{\partial}(y),$$

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■ Then

$$f(x) = \sum_{j=0}^{\infty} \left( \sum_{\substack{j_2 \dots j_n \\ j_2 + \dots + j_n = j}} p_{j_2 \dots j_n} (x - w) a_{j_2 \dots j_n} \right),$$

converges pointwise for  $x \in \mathbb{B}_{R,w}$  by [Ryan, 2004].



# Idea

■ By linearity, we can note that for  $\delta \in \mathfrak{M}(\mathbb{B}_{R,w})$ 

$$\delta(f(x)) = \sum_{j=0}^{\infty} \left( \sum_{\substack{j_2 \dots j_n \\ j_2 + \dots + j_n = j}} \delta(p_{j_2 \dots j_n}(x-w)) a_{j_2 \dots j_n} \right)$$

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■ On each monogenic polynomial

$$\delta(p_{j_2...j_n}(x)) = \frac{1}{j!} \sum_{\text{permutations}} \delta((z_{1\sigma(1)}(x)) \cdots \delta(z_{1\sigma(j)}(x)))$$

by the multiplicativity of  $\delta$ .

Let  $\delta \in \mathfrak{M}(\mathbb{B}_{R,w})$  and  $z_{ij} \in \mathcal{A}_{\mathbf{B}_{ij}}(\mathbb{B}_{R,w})$ , then  $\delta(z_{ij}) = z_{ij}(x^{\delta})$  for some  $x^{\delta} \in \mathbb{R}^n$ .

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■ The set of constants  $\alpha$  and  $\beta$  are determined by n independent numbers, so we can say  $\delta(z_{ij}) = z_{ij}(x^{\delta})$  for some  $x^{\delta} \in \mathbb{R}^n$ .

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so this sequence not converge due to a singularity at  $x^{\delta}$ .

■ Hence, it must be that  $x^{\delta} \in \mathbb{B}_{R,w}$  by continuity of  $\delta$ .

# Theorem (Noncommutative Gelfand representation)

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 $f \in \mathcal{M}(\mathbb{B}_{R,w})$ . Given the weak-\* topology on  $\mathcal{M}^*(\mathbb{B}_{r,w})$ , the map

 $\gamma: \mathfrak{M}(\mathbb{B}_{R,w}) \to \mathbb{B}_{R,w}, \quad \delta \mapsto x^{\delta}$ 



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- For  $f \in \mathcal{M}^+(\mathbb{B}_{R,w})$  we have

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■ Taking  $n \to \infty$  shows  $\gamma$  and  $\gamma^{-1}$  are continuous so  $\gamma$  is a homeomorphism.

#### Section 4

#### Future work



**Question:** Let (M, g) be an unknown Riemannian manifold with known boundary  $\partial M$ . Consider the forward problem

$$\begin{cases} \Delta \omega = 0 & \text{in } M \\ \iota^* \omega = \phi & \text{on } \partial M \end{cases}$$

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- Define the *Dirichlet-to-Neumann map* on forms by  $\Lambda \phi = \iota^*(\star d\omega)$ .
- Can we determine (M, g) from  $\Lambda$ ?



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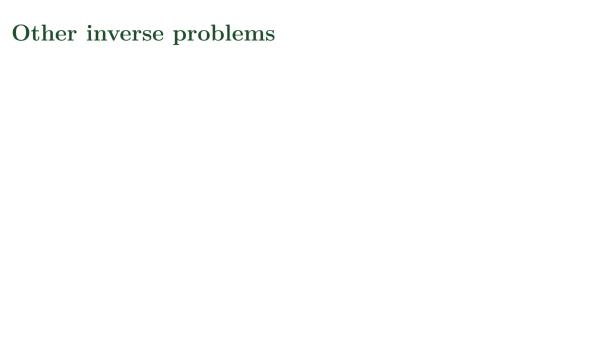
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- Given the algebraic structure of each  $\mathcal{A}_{B}(M) \subset \mathcal{M}^{+}(M)$ , can this be used to determine q?



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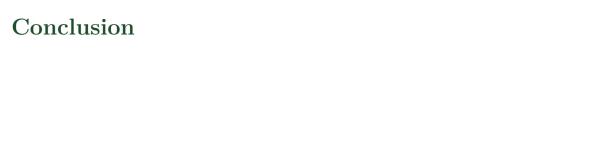
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- Can the Clifford-Hodge-Morrey decomposition can allow us to work on other related boundary inverse problems?
- These problems could include spacetime problems where the metric g is of mixed signature.

#### Section 5

#### Conclusions



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- This provides a new way to decompose fields on domains of  $\mathbb{R}^n$  and this can likely be generalized to arbitrary compact orientable pseudo-Riemannian manifolds.
- Likewise, we have proven that the monogenic fields contain a wealth of topological information and this information is supported on the boundary by the Cauchy integral formula.

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  to a Shallow Water Model.
- We are continuing to work to apply our scheme to new models such as the Modular Arbitrary-Order Ocean-Atmospheric Model (MAOOAM).

