MATH 519, Homework 3

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Solutions

Problem 1. (S & S 3.1.) Using Euler's formula

$$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i},$$

show that the complex zeros of $\sin \pi z$ are exactly at the integers, and that they are each of order 1.
Calculate the residue of $1/\sin \pi z$ at $z = n \in \mathbb{Z}$.
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Proof. First, we have that

$$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i} = 0$$
$$\iff \frac{e^{-i\pi z}(e^{2i\pi z} - 1)}{2i} = 0,$$

and so we must have that $e^{2i\pi z} = 1$ which means that z must be an integer. Clearly these are also zeros of order 1.

To calculate the residue, we take

$$\lim_{z \to n} \frac{z-n}{\sin \pi z} = \frac{1}{\pi} \qquad \text{via L'Hopital's rule.}$$ □

Problem 2.(S & S 3.2) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}.$$

Where are the poles of $1/(1+z^4)$?
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Proof.

Problem 3. (S & S 3.6.) Show that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \pi.$$

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Proof.

Problem 4. (S & S 3.14.) Prove that all entire functions that are also injective take the form $f(z) = az + b$ with $a, b \in \mathbb{C}$ and $a \neq 0$.
Hint: Apply the Casorati-Weierstrass theorem to $f(1/z)$.]

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Proof.

Problem 5. (S & S 3.21ab) Certain sets have geometric properties that guarantee they are simply connected

(a) An open set $\Omega \in \mathbb{C}$ is **convex** if for any two points in Ω, the straight line segment between them is contained in Ω. Prove that a convex open set is simply connected.

(b) More generally, an open set $\Omega \in \mathbb{C}$ is **star-shaped** if there exists a point $z_0 \in \Omega$ such that for any $z \in \Omega$, the straight line segment between z and z_0 is contained in Ω. Prove that a star shaped open set is simply connected. Conclude that the slit plane $\mathbb{C} \setminus \{(-\infty, 0]\}$ (and more generally any sector, convex or not) is simply connected.

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Proof.

Problem 6. Find the residues at the (obvious) singularities:

(a) $\frac{1}{z+z^2}$.

(b) $z \cos\left(\frac{1}{z}\right)$.

(c) $\frac{z - \sin z}{z}$.

(d) $z^2 e^{1/z}$.

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Proof.