ColoState Spring 2018 MATH 617 Assignment 2

Due Fri. 03/02/2018

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(20 points) *Problem 1.* Define $\mathcal{A} = \{A \subseteq \mathbb{R} : \text{ Either } A \text{ or } A^c \text{ is countable}\}$. For $A \in \mathcal{A}$, define $\mu(A) = 0$ if A is countable and $\mu(A) = 1$ if A^c is countable.

(i) Show that μ is a measure on $\mathcal{A}$.

(ii) Consider the outer measure μ^* on $\mathcal{P}(\mathbb{R})$ induced by μ. Show that μ^* is not finitely additive.

(20 points) *Problem 2.* Assume μ is a measure defined on an algebra $\mathcal{A}$ consisting of subsets of a fixed nonempty set X. Let $(X, \mathcal{S}^*, \mu^*)$ be the measure space obtained through outer measure and the Caratheodory condition. Let $E \subset \mathcal{S}^*$ with $\mu^*(E) < +\infty$. Prove that for any $\epsilon > 0$, there exists $A_\epsilon \in \mathcal{A}$ such that $\mu^*(E \Delta A_\epsilon) < \epsilon$.

(20 points) *Problem 3.* Prove the following regarding the Lebesgue outer measure λ^*:

(i) For $E \in \mathbb{R}$, $\lambda^*(E) = 0$ iff E is a null set.

(ii) For $E \in \mathbb{R}$, if $\lambda^*(E) = 0$, then E has an empty interior.

(20 points) *Problem 4.* Let $E \subset \mathbb{R}$ be bounded. Prove that there exists a Borel set F such that

(i) $E \subseteq F$ and $\lambda^*(E) = \lambda(F)$.

(ii) For any Borel subset $G \subseteq (F \setminus E)$, we have $\lambda(G) = 0$.

Here λ^* is the Lebesgue outer measure and λ is the Lebesgue measure.

(20 points) *Problem 5.* Suppose that $A \subset \mathbb{R}$ is a Lebesgue nonmeasurable set and $0 < \lambda^*(A) < \infty$. Prove that $\lambda(E) < \lambda^*(A)$ for any Lebesgue measurable set $E \subset A$.

Problem 1. Define $\mathcal{A} = \{A \subseteq \mathbb{R} : \text{ Either } A \text{ or } A^c \text{ is countable}\}$. For $A \in \mathcal{A}$, define $\mu(A) = 0$ if A is countable and $\mu(A) = 1$ if A^c is countable.

(i) Show that μ is a measure on $\mathcal{A}$.

(ii) Consider the outer measure μ^* on $\mathcal{P}(\mathbb{R})$ induced by μ. Show that μ^* is not finitely additive.

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Proof.

(i) First we see that $\mu(\emptyset) = 0$ since $\emptyset$ is countable. Next, we must show that μ is countably additive. Let $\{A_n\}_{n\in\mathbb{N}}$ be a countable collection of disjoint sets. Then if $\bigcup_{n\in\mathbb{N}} A_n$ is countable we have that each A_n is countable and hence

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = 0 = \sum_{n\in\mathbb{N}} \mu(A_n).$$

If $\bigcup_{n\in\mathbb{N}} A_n$ is such that $\mathbb{R} \setminus \bigcup_{n\in\mathbb{N}} A_n$ is countable, then in order for each A_n to be disjoint, we must have that only a single A_i is so that $\mathbb{R} \setminus \bigcup_{n\in\mathbb{N}} A_i$ is countable and all other sets are countable by the construction of $\mathcal{A}$. Hence

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = 1 = \sum_{n\in\mathbb{N}} \mu(A_n) = \mu(A_i).$$

So we have that μ is a measure.

(ii) Consider a counter example of a finite collection of two sets $E_1 = [0, 1]$ and $E_2 = [2, 3]$. Then note that

$$\mu^*(E_1 \cup E_2) = 1$$

since there is no countable covering of $E_1 \cup E_2$. But we also have, by the same logic,

$$\mu^*(E_1) + \mu^*(E_2) = 2.$$

Hence, μ^* is not finitely additive. $\square$

Problem 2. Assume μ is a measure defined on an algebra $\mathcal{A}$ consisting of subsets of a fixed nonempty set X. Let $(X, \mathcal{S}^*, \mu^*)$ be the measure space obtained through outer measure and the Caratheodory condition. Let $E \subset \mathcal{S}^*$ with $\mu^*(E) < +\infty$. Prove that for any $\epsilon > 0$, there exists $A_\epsilon \in \mathcal{A}$ such that $\mu^*(E \Delta A_\epsilon) < \epsilon$.

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Proof. Note that the infimum definition of μ^* allows us to find a countable collection of pairwise disjoint sets $A_n \in \mathcal{A}$ so that $E \subseteq \bigcup_{n=1}^\infty A_n$ so that we have

$$\mu^*(E) + \epsilon \geq \sum_{n=1}^\infty \mu^*(A_n) \geq \mu^*(E).$$

Then of course we have $\sum_{n=1}^\infty \mu^*(A_n) < \infty$ which weans that we have some $N \in \mathbb{N}$ so that

$$\sum_{n=N+1}^\infty \mu^*(A_n) < \epsilon/2.$$

We denote $A_\epsilon = \bigcup_{n=1}^N A_n$ and we find that

$$E \setminus A_\epsilon = E \setminus \left(\bigcup_{n=1}^N A_n \right) \subseteq \left(\bigcup_{n=1}^\infty A_n \right) \setminus \left(\bigcup_{n=1}^N A_n \right) = \bigcup_{n=N+1}^\infty A_n.$$

It follows that we have

$$\mu^*(E \setminus A_\epsilon \leq \sum_{n=N+1}^\infty \mu^*(A_n) < \epsilon/2$$

and that

$$A_\epsilon \setminus E \subseteq \left(\bigcup_{n=1}^\infty A_n \right) \setminus E.$$

Then we have

$$\mu^*(A_\epsilon \setminus E) \leq \sum_{n=1}^\infty \mu^*(A_n) - \mu^*(E).$$

Lastly, the countable additivity of μ^* implies that

$$\mu^*(E \Delta A_\epsilon) = \mu^*(E \setminus A_\epsilon) + \mu^*(A_\epsilon \setminus E) \leq \epsilon.$$

Note: I saw a very similar solution in the text. I tried to clean it up and make it as nice as possible. □

Problem 3. Prove the following regarding the Lebesgue outer measure λ^*:

(i) For $E \subseteq \mathbb{R}$, $\lambda^*(E) = 0$ iff E is a null set.

(ii) For $E \subseteq \mathbb{R}$, if $\lambda^*(E) = 0$, then E has an empty interior.

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Proof.

(i) Let $E \subseteq \mathbb{R}$ and that $\lambda^*(E) = 0$. Then

$$\begin{aligned}
&\lambda^*(E) = 0 \\
&\iff \inf\left\{\sum_{i=1}^{\infty} \lambda(I_i) \mid I_i \in \mathcal{I} \forall i, I_i \cap I_j = \emptyset \text{ for } i \neq j \text{ and } E \subseteq \bigcup_{i=1}^{\infty} I_i\right\} = 0 \\
&\iff E \text{ for all } \epsilon > 0, \text{ can be covered by a countable family of intervals with } \sum_{n=1}^{\infty} \lambda(I_n) \leq \epsilon \\
&\iff E \text{ is a null set.}
\end{aligned}$$

(ii) Suppose that $E \subseteq \mathbb{R}$ with $\lambda^*(E) = 0$. Suppose that $\text{int}(E) \neq \emptyset$. Then we have that there exists an open interval $I \subset E$ and we necessarily have that $\lambda^*(I) \geq 0$. Hence, E must have an empty interior. $\square$

Problem 4. Let $E \subset \mathbb{R}$ be bounded. Prove that there exists a Borel set F such that

(i) $E \subseteq F$ and $\lambda^*(E) = \lambda(F)$.

(ii) For any Borel subset $G \subseteq (F \setminus E)$, we have $\lambda(G) = 0$.

Here λ^* is the Lebesgue outer measure and λ is the Lebesgue measure.

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Proof.

(i) By definition of λ^*, we know that $\lambda^*(E)$ is the infimum of the measure of all possible coverings of E by disjoint intervals in $\mathbb{R}$. Hence, for any $m \in \mathbb{N}$ and $\frac{1}{m} > 0$ we have a countable collection of pairwise disjoint intervals $\{I_n^m\}_{n\in\mathbb{N}}$ so that $E \subseteq \bigcup_{n\in\mathbb{N}} I_n^m$. Since E is bounded, $\lambda^*(E) < \infty$ and we have

$$\lambda\left(\bigcup_{n\in\mathbb{N}} I_n^m\right) - \lambda^*(E) < \frac{1}{m}.$$

Since this is true for all $\frac{1}{m} > 0$, we necessarily have $F \subseteq \bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} I_n^m$. Then with this, we see

$$\begin{aligned}\lambda(F) - \lambda^*(E) \leq \lambda\left(\bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} I_n^m\right) - \lambda^*(E) = 0 \\ \implies \lambda^*(E) = \lambda(F).\end{aligned}$$

(ii) Let $G \subseteq (F \setminus E)$ be such that $G = \bigcup_{n\in\mathbb{N}} J_n$ where J_n are Borel sets. Note that since G is a Borel set we have

$$\lambda(G) = \lambda^*(G).$$

We then have

$$\lambda^*(F \setminus E) = 0$$

which, since $G \subseteq F \setminus E$, means

$$\lambda^*(G) = 0$$

□

Problem 5. Suppose that $A \subset \mathbb{R}$ is a Lebesgue nonmeasurable set and $0 < \lambda^*(A) < \infty$. Prove that $\lambda(E) < \lambda^*(A)$ for any Lebesgue measurable set $E \subset A$.

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Proof. First, note that $\lambda^*(E) = \lambda(E)$ since E is measurable. Now, monotonicity of λ^* implies the following:

$$\lambda^*(E) \leq \lambda^*(A).$$

Now, by the Caratheodory condition, we can do the following:

$$\lambda^*(A) = \lambda^*(E \cap A) + \lambda^*(E^c \cap A).$$

Then suppose for a contradiction that we in fact have $\lambda^*(E) = \lambda^*(A)$. This means that

$$\begin{aligned}
& 0 = \lambda^*(E) - \lambda^*(A) \\
\iff & 0 = \lambda^*(E) - \lambda^*(E \cap A) - \lambda^*(E^c \cap A) \\
\iff & 0 = \lambda^*(E^c \cap A), && \text{since } \lambda^*(E) = \lambda^*(E \cap A) \text{ because } E \subset A \\
\iff & A && \text{is measurable.}
\end{aligned}$$

But this is a contradiction since we supposed that A is non-measurable. Hence, we must have that $\lambda(E) < \lambda^*(A)$ since the case for equality provides a contradiction. $\square$