MATH 517, Homework 7

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Solutions

Problem 1. (Rudin 7.3) Give an example of sequences $\{f_n\}$, $\{g_n\}$ of uniformly converging functions on some set E so that $\{f_n g_n\}$ does not converge uniformly on E.

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Proof. Let $f_n, g_n \colon \mathbb{R} \to \mathbb{R}$ with each defined by $f_n(x) = x$ and $g_n(x) = \frac{1}{n}$. Then $f_n(x) \to f(x) = x$ and $g_n(x) \to g(x) = 0$ both converge uniformly yet $f_n g_n = \frac{x}{n}$ does not.

First, fix $\epsilon > 0$ then we have that $\forall n \in \mathbb{N}$ and any $x \in \mathbb{R}$

$$|f_n(x) - f(x)| = |x - x| = 0 < \epsilon.$$

So we've shown f_n converges uniformly.

Next, fix $\epsilon > 0$ and $\forall n > N \geq \frac{1}{\epsilon}$ with $n, N \in \mathbb{N}$ we have for every x,

$$\begin{aligned} |g_n(x) - g(x)| &= \left|\frac{1}{n}\right| \\ &< \epsilon. \end{aligned}$$

So we've shown that g_n also converges uniformly.

Note that $f_n g_n$ converges to the 0 function pointwise. To see this, fix x and $\epsilon > 0$ then let $N \in \mathbb{N}$ be such that $N \geq \frac{|x|}{\epsilon}$. Then for $n > N$ we have

$$\begin{aligned} |(f_n g_n)(x) - 0| &= \left|\frac{x}{n}\right| \\ &< \epsilon. \end{aligned}$$

Finally, suppose that $f_n g_n$ converges uniformly to the 0 function. So for any x, we have for $\exists N \in \mathbb{N}$ such that $\forall n > N$ we have $|(f_n g_n)(x) - 0| < \epsilon$. However

$$|(f_n g_n)(x) - 0| = \left|\frac{x}{n}\right|$$

and we can choose $x \in \mathbb{R}$ so that $\frac{x}{n} > M$ for any positive real M. Which means that $|(f_n g_n)(x) - 0| > \epsilon$, which contradicts the supposition that $f_n g_n$ converges uniformly. $\square$

Problem 2. (Rudin 7.7) Define $f_n(x) = \frac{x}{1+nx^2}$ for each $n = 1, 2, ...$

(a) Show that $\{f_n\}$ converges uniformly to a function f on $\mathbb{R}$.

(b) Show that $f'(x) = \lim_{n\to\infty} f_n'(x)$ for all $x \neq 0$, but that this fails when $x = 0$.

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Proof (a). We will show that this function converges to $f(x) = 0$. Fix $\epsilon > 0$ and let $N \in \mathbb{N}$ be such that $N > \frac{1}{\epsilon}$. Then we have two cases. First if $|x| < 1$, then for $n > N$ we have

$$\begin{aligned} |f_n(x) - 0| &= \left|\frac{x}{1+nx^2}\right| \\ &< \left|\frac{1}{1+nx^2}\right| \\ &= \left|\frac{1}{x^2}\frac{1}{\frac{1}{x^2}+n}\right| \\ &< \left|\frac{1}{\frac{1}{x^2}+n}\right| \\ &< \left|\frac{1}{1+n}\right| < \epsilon. \end{aligned}$$

If $|x| \geq 1$, then for $n > N$ we have

$$\begin{aligned} |f_n(x) - 0| &= \left|\frac{x}{1+nx^2}\right| \\ &\leq \left|\frac{x}{1+nx}\right| \\ &< \left|\frac{x}{nx}\right| \\ &= \left|\frac{1}{n}\right| < \epsilon. \end{aligned}$$

Thus $\{f_n\}$ converges uniformly to $f(x) = 0$ on $\mathbb{R}$. It's worth noting that N did not depend on the value of x. The two cases were just easiest to show by breaking them up. $\square$

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Proof (b). We have that $f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2}$. We showed above that $f_n(x) \to f(x) = 0$ by part (a), and thus $f' = 0$ since f is a constant function. Then for $x \neq 0$ we have

$$\begin{aligned} \lim_{n\to\infty} |f_n'(x) - 0| &= \lim_{n\to\infty}\left|\frac{1-nx^2}{(1+nx^2)^2}\right| \\ &< \lim_{n\to\infty}\left|\frac{1-nx^2}{1+2nx^2+n^2x^4}\right| \\ &< \lim_{n\to\infty}\left|\frac{1-nx^2}{n^2x^4}\right| \\ &= \lim_{n\to\infty}\left|\frac{1}{n^2}\frac{\frac{1}{n^2}-\frac{x^2}{n}}{x^4}\right| \\ &= \lim_{n\to\infty}\frac{1}{n^2}\left|\frac{\frac{1}{n^2}-\frac{x^2}{n}}{x^4}\right| = 0 \end{aligned}$$

For $x = 0$ we have $f_n'(0) = 1$ for every n and thus this does not converge to zero. $\square$

Problem 3. Prove that every uniformly convergent sequence of bounded real-valued functions is uniformly bounded (i.e., there exists $M > 0$ so that $|f_n(x)| < M$ for all n and all x.)

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Proof. Let f_n be a sequence of bounded real-valued functions on a domain X and this sequence of functions converges uniformly. Thus we have that $|f_n(x)| < M_n$ for every n and for any $x \in X$. Also for any $\epsilon > 0$ we have that $\exists N \in \mathbb{N}$ such that for $n > m \geq N$ and any x we have

$$\begin{aligned}
&|f_n(x) - f_m(x)| < \epsilon \\
&\iff |f_n(x)| < \epsilon + |f_m(x)| && \text{since } |f_n(x)| - |f_m(x)| < |f_n(x) - f_m(x)| \\
&\iff |f_n(x)| < \epsilon + M_m \\
&\iff M_n \leq M_m && \text{since } \epsilon > 0 \text{ was arbitrary.}
\end{aligned}$$

This means that for any $n > m$ we have $M_n \leq M_m$. Then consider the finite set of M_i for $i = 1, ..., m$ and note that $\max(\{M_1, ..., M_m\}) = M$ exists and is finite. Then we have that for any n and all x, $|f_n(x)| < M$ by how we constructed this M. $\square$

Problem 4. A family $\mathcal{F}$ of (real- or complex-valued) functions on a set E in a metric space X is *equicontinuous* on E if for every $\epsilon > 0$ there exists $\delta > 0$ so that

$$|f(x) - f(y)| < \epsilon$$

whenever $d(x, y) < \delta$, $x, y \in E$, and $f \in \mathcal{F}$.
Give an example of an equicontinuous sequence $\{f_n\}$ of functions on some metric space that converges pointwise but not uniformly.

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Proof. Consider the sequence of functions $f_n = \frac{x}{n}$ defined on all of $\mathbb{R}$. Then note that this is an equicontinuous sequence of functions. To see this, fix $\epsilon > 0$ and let $0 < \delta < \epsilon$. Then we have for any n and $|x - y| < \delta$

$$\begin{aligned} |f_n(x) - f_n(y)| &= \left|\frac{x}{n} - \frac{y}{n}\right| \\ &= \left|\frac{x - y}{n}\right| \\ &< |x - y| < \epsilon. \end{aligned}$$

Thus we have that this sequence is in fact equicontinuous. Note that in Problem 1 I showed that this sequence converges pointwise but not uniformly. The proof would be the same, so I'll omit that here.□