# MATH 560, Homework 10

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November 16, 2017

Solutions

**Problem 1. (§6.7 Problem 5. (c))** Find an explicit formula for the following.

$T^\dagger(a+bx+cx^2)$, where $T$ is the linear transformation given by $T\colon P_2(\mathbb{R}) \to P_1(\mathbb{R})$, where $T(f(x)) = f''(x)$, and the inner product is $\langle g, h\rangle = \int_{-1}^{1} g(t)h(t)\,dt$.

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*Proof.* From the previous homework we have for the matrix $A = [T]_\beta$ with the basis $\beta = \left\{\sqrt{\frac{1}{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{5}{8}}(3x^2-1)\right\}$,

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\Sigma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

From this we have that

$$\Sigma^\dagger = \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \\ 0 & 0 \end{bmatrix}.$$

Then $A^\dagger$ is given by

$$\begin{aligned} A^\dagger &= V\Sigma^\dagger U^* \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & = \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \\ 0 & 1/2 \end{bmatrix}. \end{aligned}$$

□

**Problem 2.** **(§6.7 Problem 6. (c))** Use the results of Exercise 3 to find the pseudoinverse of the following matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

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*Proof.* We find

$$A^* A = \begin{bmatrix} 2 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 2 \end{bmatrix}.$$

This matrix has eigenvalues $\lambda_1 = 5$, $\lambda_2 = 1$, $\lambda_3 = \lambda_4 = 0$, with corresponding eigenvectors $v_1 = (2,1,1,2)$, $v_2 = (0,-1,1,0)$, $v_3 = (-1,0,0,1)$, $v_4 = (-1,1,1,0)$. This gives us

$$V = \begin{bmatrix} 2 & 0 & -1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \end{bmatrix}.$$

Next we find

$$A^* A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix},$$

which has eigenvalues $\lambda_1 = 5$ and $\lambda_2 = 1$ with corresponding eigenvectors $u_1 = (1,1)$ and $u_2 = (-1,1)$. This gives

$$U = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

It follows that

$$\Sigma = \begin{bmatrix} \sqrt{5} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\Sigma^\dagger = \begin{bmatrix} 1/\sqrt{5} & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Now

$$\begin{aligned} A^\dagger &= V\Sigma^\dagger U^* \\ &= \begin{bmatrix} -1+2/\sqrt{5} & -1-2/\sqrt{5} \\ 1/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & -1/\sqrt{5} \\ 1+2/\sqrt{5} & 1-2/sqrt5 \end{bmatrix}. \end{aligned}$$

□

**Problem 3. (§6.7 Problem 9. (a))** Let $V$ and $W$ be finite-dimensional inner product spaces over $\mathbb{F}$, and suppose that $\{v_1, v_2, \ldots, v_n\}$ and $\{u_1, u_2, \ldots, u_m\}$ are orthonormal bases for $V$ and $W$, respectively. Let $T\colon V \to W$ be a linear transformation of rank $r$, and suppose that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ are such that

$$T(v_i) = \begin{cases} \sigma_i u_i & 1 \leq i \leq r \\ 0 & r < i. \end{cases}$$

Prove that $\{u_1, u_2, \ldots, u_m\}$ is a set of eigenvectors of $TT^*$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$, where

$$\lambda_i = \begin{cases} \sigma_i^2 & 1 \leq i \leq r \\ 0 & r < i. \end{cases}$$

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*Proof.* Choose any basis $\beta$ and denote $A = [T]_\beta$. Then we have $A = [U\Sigma V^*]_\beta$ via the SVD. It follows that

$$\begin{aligned} AA^* &= [(U\Sigma V^*)(V\Sigma^* U^*)]_\beta \\ &= [U\Sigma\Sigma^* U^*]_\beta. \end{aligned}$$

Let $[\Sigma\Sigma^*]_\beta = D$ with $D$ diagonal with entries $\sigma_i^2 = \lambda_i$ for the $i$th diagonal element. Now notice that we have

$$\begin{aligned} & (AA^*)[U]_\beta = D[U]_\beta && \text{since } U \text{ is unitary, and } D \text{ is diagonal} \\ \implies & (AA^*)[u_i]_\beta = \lambda_i [u_i]_\beta && \text{by taking the } i\text{th columns of } [U]_\beta \text{ to be } [u_i]_\beta. \end{aligned}$$

Hence we have that $TT^*(u_i) = \lambda u_i$. $\square$

**Problem 4. (§6.7 Problem 13.)** Prove that if $A$ is a positive semidefinite matrix, then the singular values of $A$ are the same as the eigenvalues of $A$.

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*Proof.* Since $A$ is positive definite we can write $A = BB^*$. Now $B = U\Sigma V^*$ via the SVD. By the previous problem, we then have

$$BB^* = U\Sigma\Sigma^* U^*.$$

We then have that $A = U\Sigma^2 U^{-1}$ by above as a singular value decomposition for $A$ with singular values $\sigma_i^2$. This is also an eigenvalue decomposition with eigenvalues $\sigma_i^2$. □

**Problem 5. (§6.7 Problem 14.)** Prove that if $A$ is a positive definite matrix and $A = U\Sigma V^*$ is a singular value decomposition of $A$, then $U = V$.

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*Proof.* If $A$ is positive definite then $A$ is also positive semidefinite. By the previous problem we found the singular values of $A$ are the same as the eigenvalues of $A$. It follows that the singular value decomposition $A = U\Sigma V^*$ is equivalent to $A = PDP^{-1}$ which means that $\Sigma = D$ and hence $V^* = P^{-1} = U^{-1}$. Specifically we have that $V = U$. □

**Problem 6. (§6.7 Problem 15.)** Let $A$ be a square matrix with a polar decomposition $A = WP$.

(a) Prove that $A$ is normal if and only if $WP^2 = P^2W$.

(b) Use (a) to prove that $A$ is normal if and only if $WP = PW$.

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*Proof.*

(a) First note that $P^2 = V\Sigma^2V^*$, $P^*P = V\Sigma^*\Sigma V^*$ and $PP^* = V\Sigma\Sigma^*V^*$. But we have that $\Sigma^2 = \Sigma^*\Sigma = \Sigma\Sigma^*$. Thus $P^2 = P^*P = PP^*$. Then

$$\begin{aligned} AA^* &= A^*A \\ \iff WPP^*W^* &= P^*P \\ \iff WP^2 &= P^2W. \end{aligned}$$

Hence we have that $A$ is normal if and only if $WP^2 = P^2W$.

(b) Using (a) we have

$$\begin{aligned} P^2 &= WP^2W* = (WPW^*)^2 \\ \iff P &= WPW^* \\ \iff PW &= WP. \end{aligned}$$

□

**Problem 7. (§6.7 Problem 21.)** Let $V$ and $W$ be finite-dimensional inner product spaces, and let $T\colon V \to W$ be linear. Prove the following results.

(a) $TT^\dagger T = T$.

(b) $T^\dagger T T^\dagger = T^\dagger$.

(c) Both $T^\dagger T$ and $TT^\dagger$ are self-adjoint.

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*Proof.* For each part we will let $A = [T]_\beta$ for some basis $\beta$ and let $A = U\Sigma V^*$ by SVD.

(a) We have

$$\begin{aligned} AA^\dagger A &= (U\Sigma V^*)(V\Sigma^\dagger U^*)(U\Sigma V^*) \\ &= U\Sigma\Sigma^\dagger\Sigma V^* \\ &= U\Sigma V^* = A. \end{aligned}$$

(b) We have

$$\begin{aligned} A^\dagger AA^\dagger &= (V\Sigma^\dagger U^*)(U\Sigma V^*)(V\Sigma^\dagger U^*) \\ &= V\Sigma^\dagger\Sigma\Sigma^\dagger U^* \\ &= V\Sigma^\dagger U^* = A^\dagger \end{aligned}$$

(c) We have

$$\begin{aligned} (A^\dagger A)^* &= (V\Sigma^\dagger\Sigma V^*)^* = (VV^*)^* = A^\dagger A \qquad \text{and} \\ (AA^\dagger)^* &= (U\Sigma\Sigma U^*)^* = (UU^*) = AA^\dagger. \end{aligned}$$

So both are self adjoint. $\square$