Riemannian Geometry

for Dummies

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Section 1

Introduction

Riemannian geometry is the study of a ***smooth manifold*** M along with a ***Riemannian metric*** g.

The point of Riemmannian geometry is to generalize the differentiable and metric structure of $\mathbb{R}^n$.

We generalize to spaces that have interesting topology and geometry.

This will require us to rethink some notions we found “easy" in $\mathbb{R}^n$.

But we will gain a very general framework for working with differentiable objects.

Section 2

Motivation

Why study this in the first place?

Example: Partial differential equations (PDEs) on spaces that are not flat.

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- Fluid flow on Earth;
- Electrical Impedence Tomography (EIT);
- General relativity.

Example: Optimization in interesting spaces.

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- Matrix (symmetry) groups;

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- Matrix (symmetry) groups;
- Grassmannians, Flags;
- Curved spacetime.

Section 3

Preliminaries

add more math text before/after pics so that people see some notation. More examples.

Subsection 1

Smooth Manifolds

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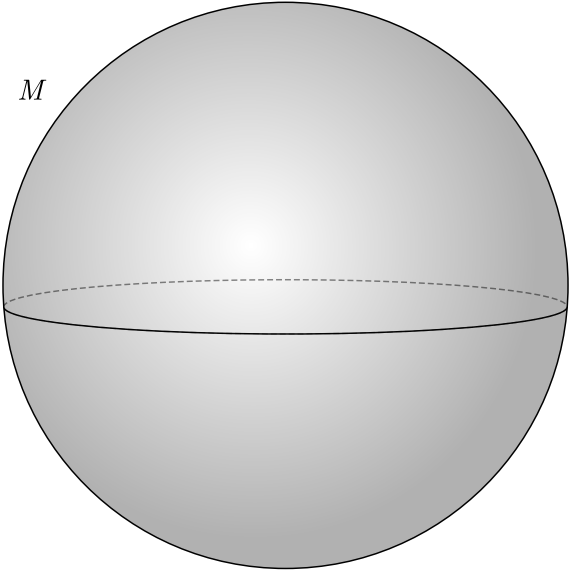
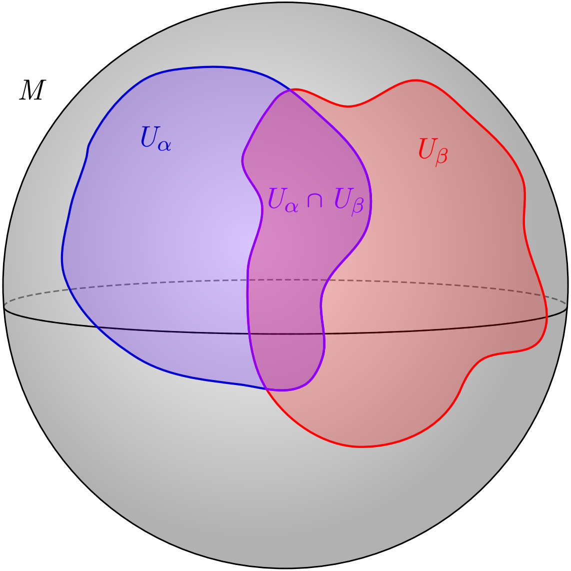
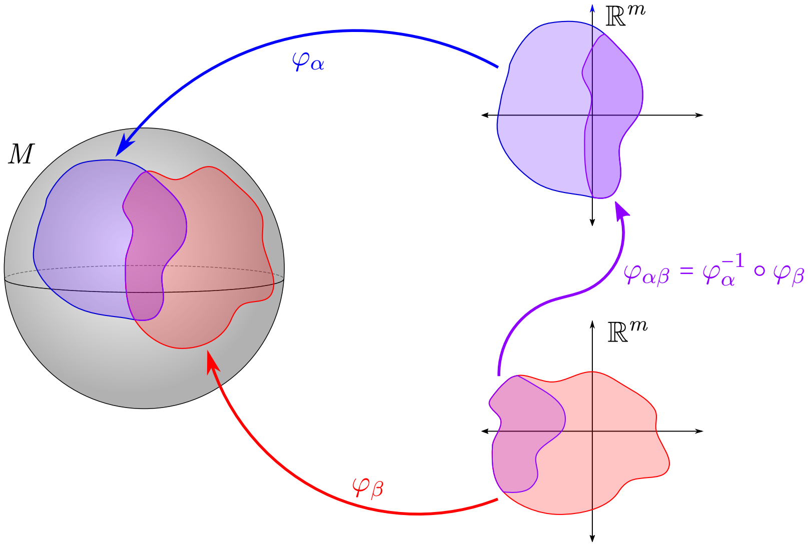
- Start with a topological space M;
- Look at open sets U that cover M;
- Construct local coordinates φ;
- Show coordinate transition functions are smooth.

Define the sphere as the set of points in $\mathbb{R}^3$... then say we'll mostly use this as an example so keep it in mind


M

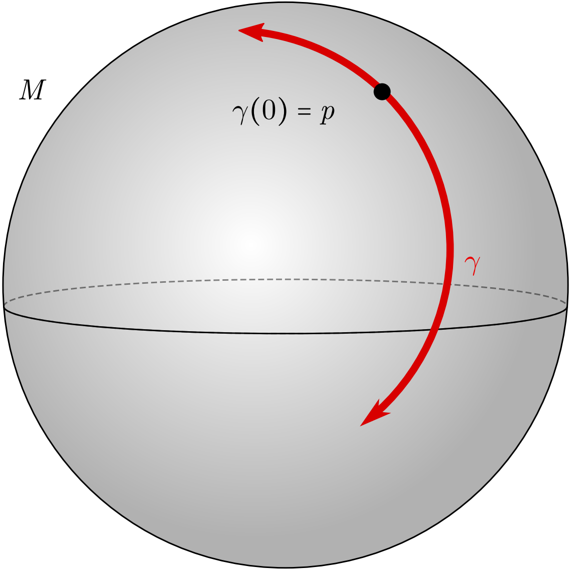

M
U_α
U_β
$U_\alpha \cap U_\beta$

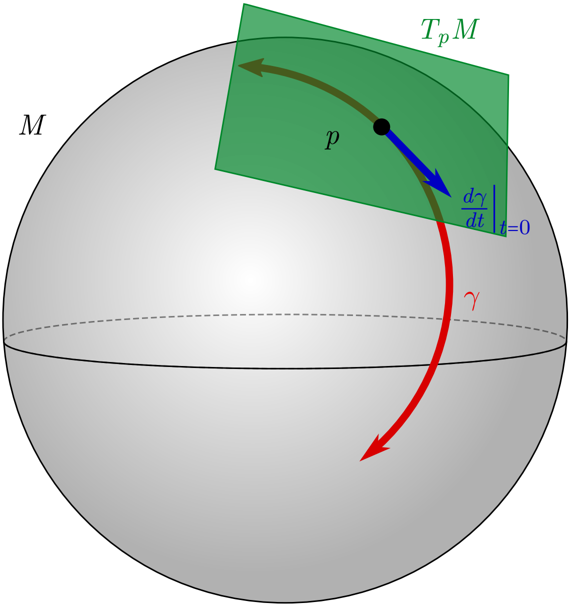


M
φ_α
$\mathbb{R}^m$
$\varphi_{\alpha\beta} = \varphi_\alpha^{-1} \circ \varphi_\beta$
$\mathbb{R}^m$
φ_β

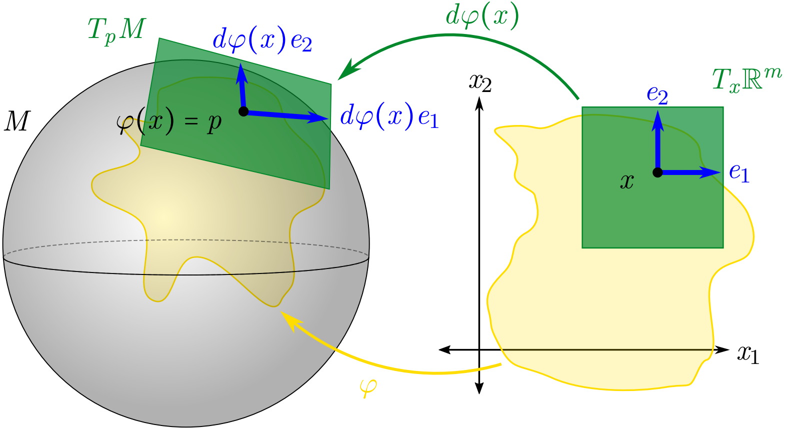
Subsection 2

Vector Fields

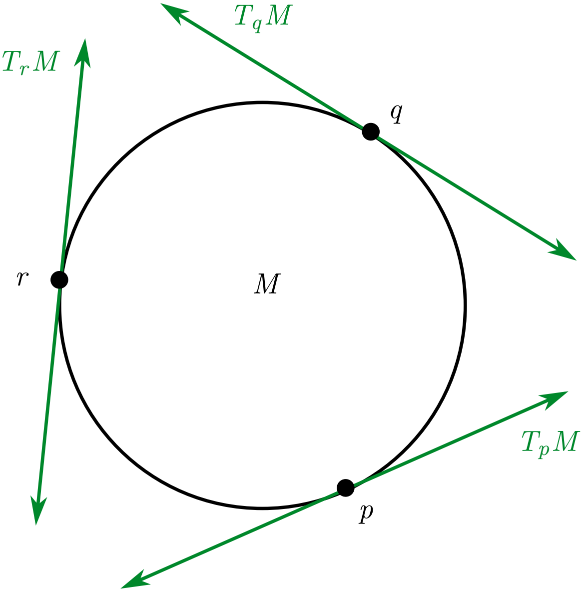

M
$\gamma(0) = p$
γ



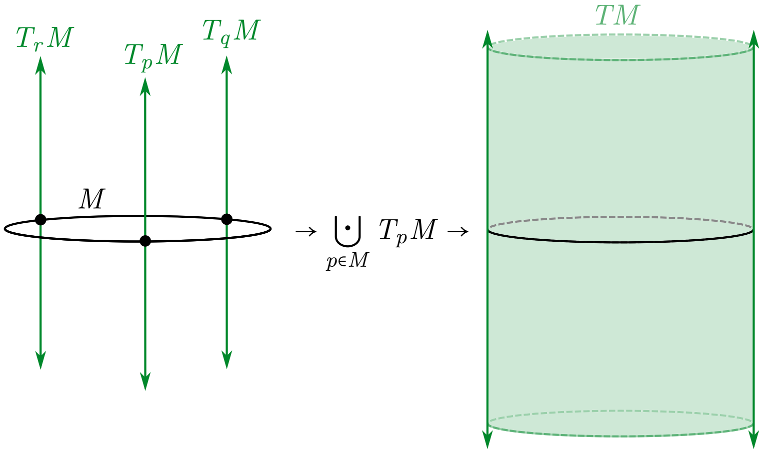
T_pM
M
p
$\frac{d\gamma}{dt}\Big|_{t=0}$
γ

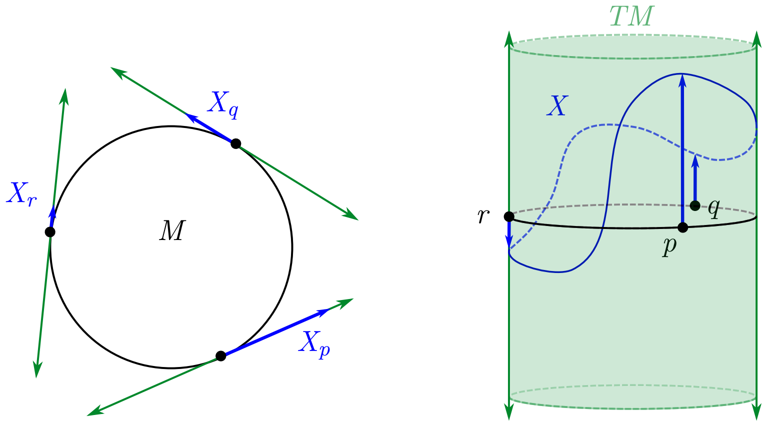


T_pM
$d\varphi(x)e_2$
$d\varphi(x)$
M
$\varphi(x) = p$
$d\varphi(x)e_1$
x_2
$T_x\mathbb{R}^m$
e_2
e_1
x
x_1
φ

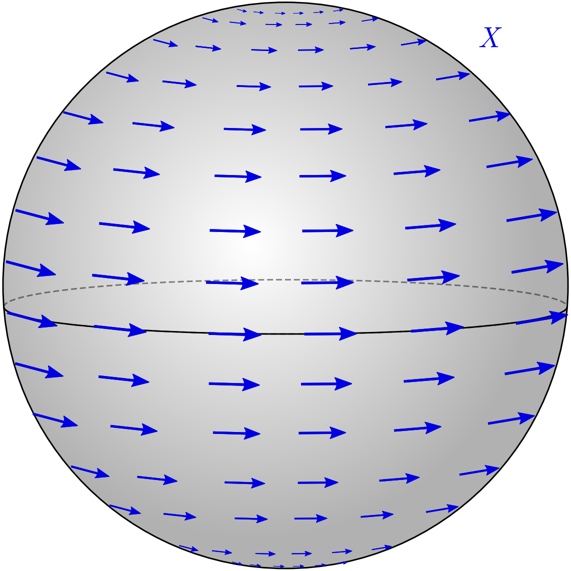


T_qM
T_rM
q
r
M
T_pM
p


T_rM
T_pM
T_qM
M
$\rightarrow \dot\bigcup_{p\in M} T_pM \rightarrow$
TM



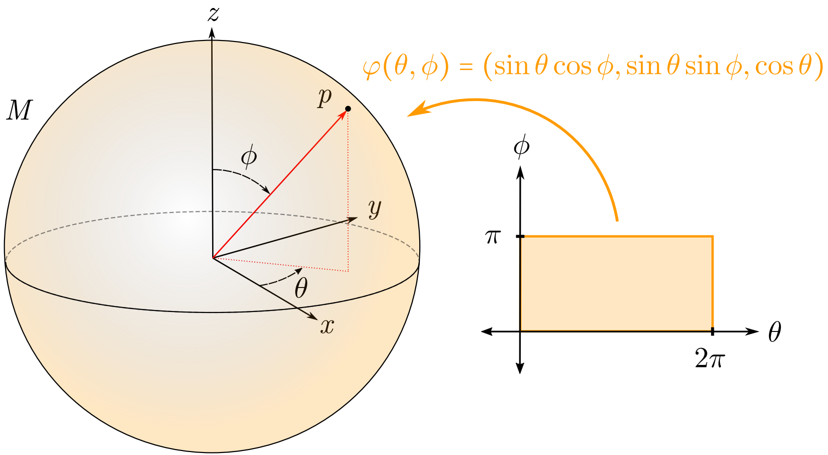
X_q
X_r
M
X_p
TM
X
r
q
p

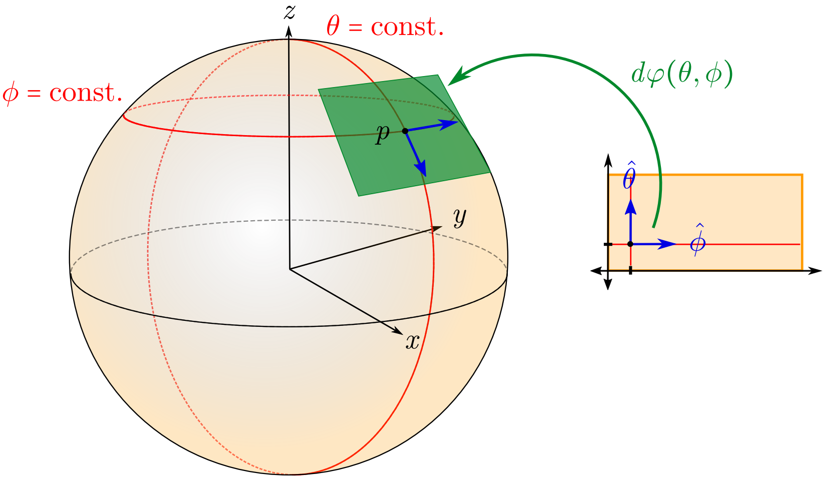

X

Subsection 3

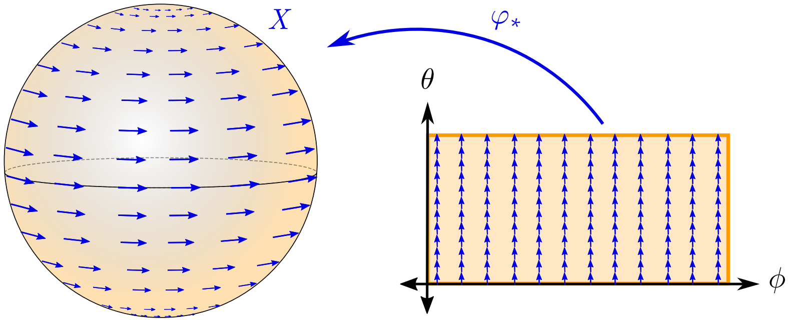
Example

Example of charts and maps for sphere as well as a vector field. Do spherical coordinates


z
M
p
ϕ
y
θ
x
$\varphi(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$
ϕ
π
θ
2π


z
θ = const.
ϕ = const.
$d\varphi(\theta,\phi)$
p
y
x
$\hat{\theta}$
$\hat{\phi}$

h



X
φ_*
θ
ϕ

Section 4

Riemannian Geometry

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- Build an inner product on the tangent space T_pM;
- Have the inner product vary smoothly as we vary the point p;
- Define this as our Riemannian metric tensor field g;
- Extract geometrical and analytical qualities of the underlying manifold M.

Subsection 1

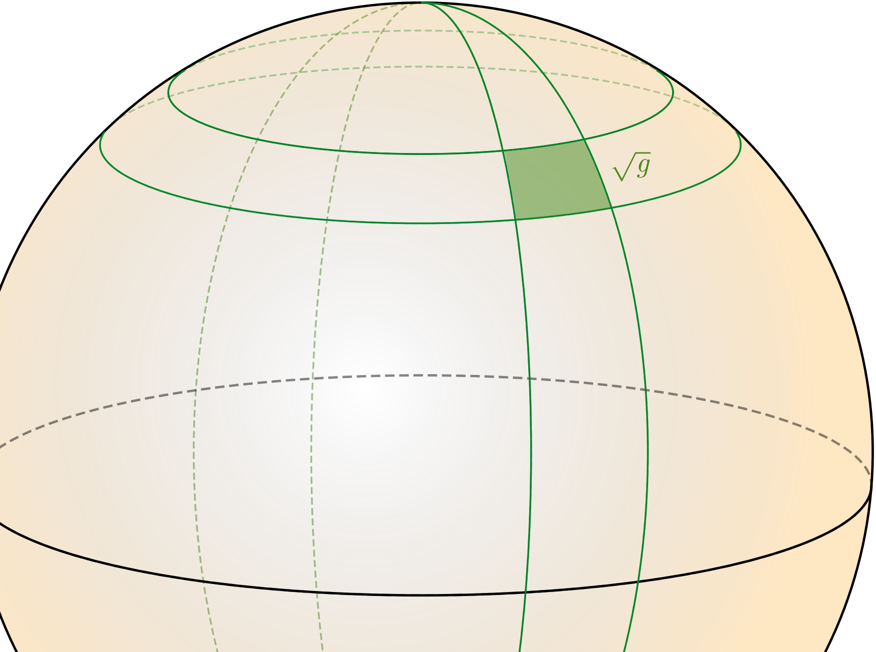
Riemannian Metric

$$g_{ij}(x) = \varphi^*(x)e_i \cdot \varphi^*(x)e_k = (\partial_i\varphi)\cdot(\partial_j\varphi) = \sum_{k=1}^{m} \frac{\partial\varphi_k}{\partial x_i}\frac{\partial\varphi_k}{\partial x_j}$$

Riemannian metric tells you how your vectors in your coordinates really look on the manifold. Thus, we also know

- How lengths are distorted.
- How volume is distorted.

But this allows us to integrate or differentiate (covariant der?) in our coordinates but think of it as happening on the manifold.


$\sqrt{g}$

Can see how $d\phi$ doesn't change as θ changes in previous pic.

can show how the area and lengths change on the sphere from the determinant (or alternating product) on all tangent vectors.

Distance on the manifold computed using minimal length curves.

$$S = \{\gamma : [0,1] \to M \mid \gamma(0) = p,\ \gamma(1) = q\}$$

$$d(p,q) = \inf_{\gamma \in S} \operatorname{length}(\gamma) \int_0^1 \sqrt{g(\dot{\gamma},\dot{\gamma})}\, dt$$

connection covariant derivative interpretation, covariant derivative in spherical coordinates,second covariant matrix and covariant laplacian from hessian? Compatability with riemannian metric picture. Show a geodesic in the coordinates between cities or something?

From minimization of length/energy. Both are good to mention. Geodesic equation

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0$$

equivalent to

$$\ddot{x}^l + \dot{x}^j \dot{x}^k \Gamma^l_{jk} = 0$$

which is saying that the only "acceleration" of the curve comes from the geometry it lies on. When flat space, $\Gamma^l_{jk} = 0$ and we have $\ddot{x} = 0$.

volume and integration

Section 5

Applications

navier stokes on the Earth. Riemannian metric as a conductivity?

Section 6

Conclusions