Riemannian Geometry

for Dummies

Colin Roberts

Colorado State University

Section 1

Introduction

Riemannian geometry is the study of a ***smooth manifold*** M along with a ***metric tensor field*** g.

The point of Riemmannian geometry is to generalize the differentiable and metric structure of $\mathbb{R}^n$.

We generalize to spaces that have interesting topology and geometry.

This will require us to rethink some notions we found “easy" in $\mathbb{R}^n$.

But we will gain a very general framework for working with differentiable objects.

Section 2

Motivation

Why study this in the first place?

Example: Partial differential equations (PDEs) on spaces that are not flat.

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- Fluid flow on Earth;
- Electrical Impedence Tomography (EIT);
- General relativity.

Example: Optimization in interesting spaces.

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- Grassmannians;

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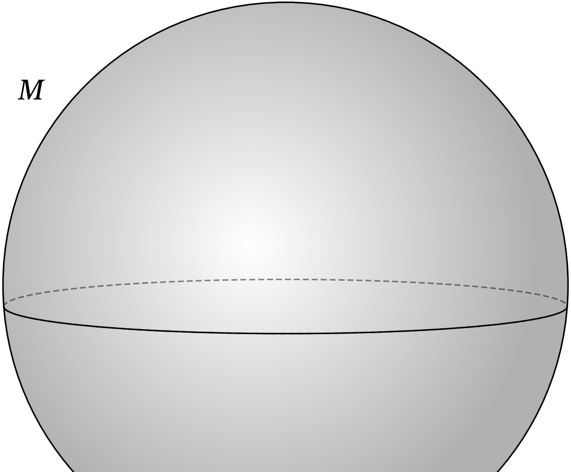
- Grassmannians;
- Flags.

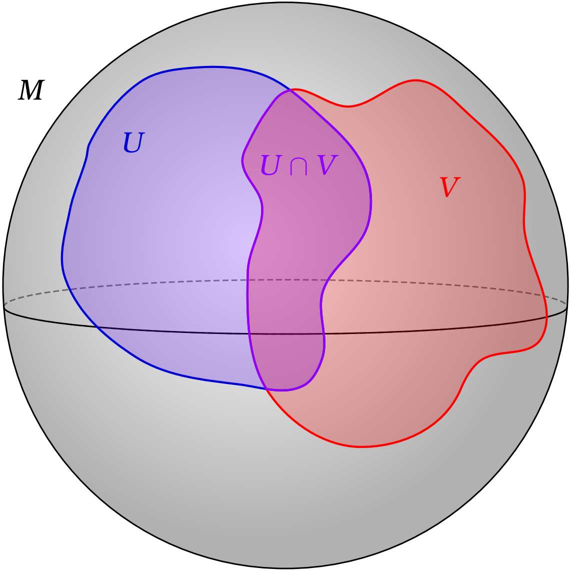
Section 3

Preliminaries

Manifold



Open sets



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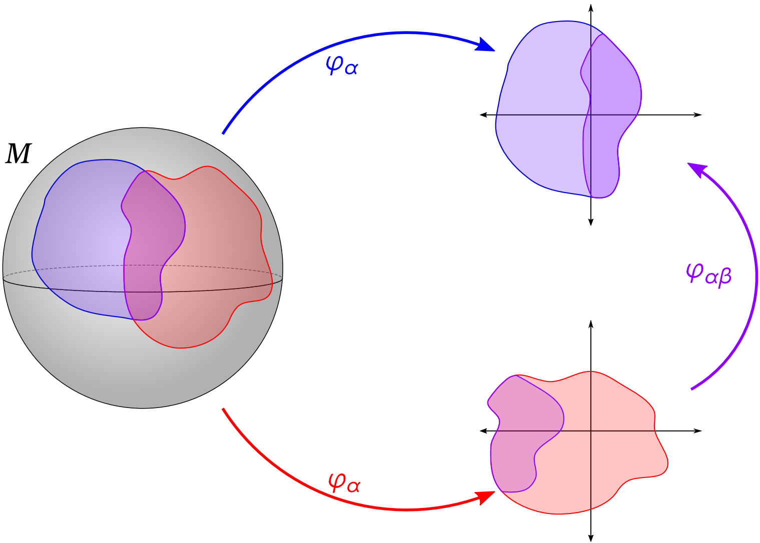
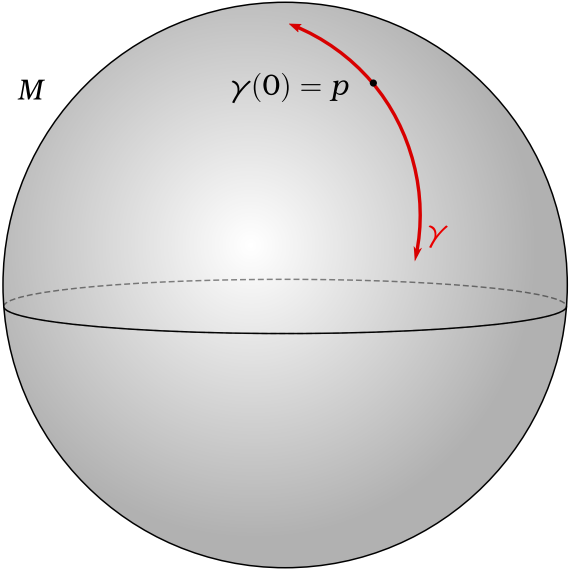
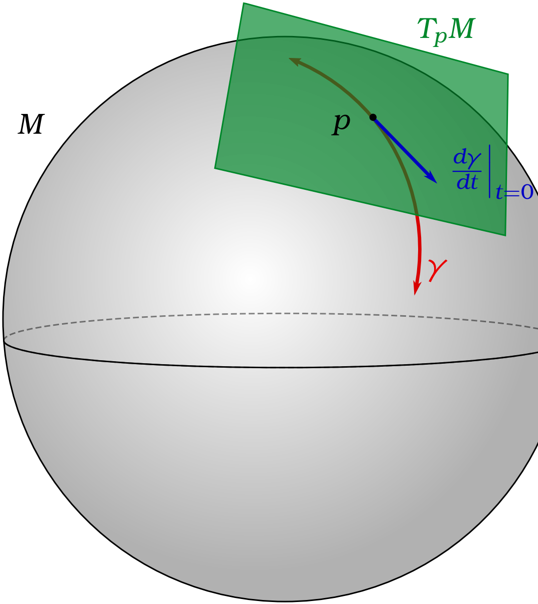


Curves



Tangent spaces



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- Arbitrary unions of open sets are open;
- Finite intersections of open sets are open.

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- A ***homeomorphism*** is a continuous bijection $f\colon M \to N$ with continuous inverse f^{-1}.
- We say M and N are homeomorphic.

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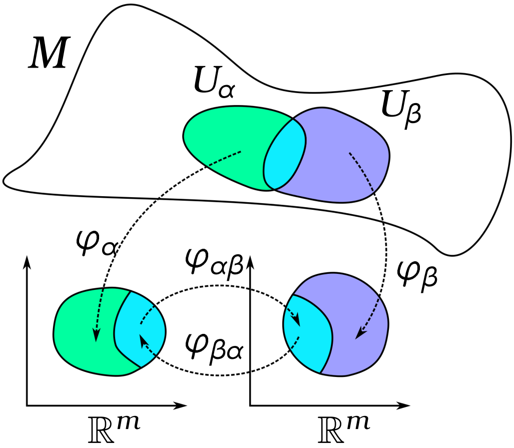
$$\downarrow$$

A ***manifold*** M (with boundary) is a topological space such that each open set in $\mathcal{O}$ is homeomorphic to $\mathbb{R}^n$ (or $\mathbb{R}^{n^+}$).

Manifolds



M
U_α
U_β
φ_α
$\varphi_{\alpha\beta}$
φ_β
$\varphi_{\beta\alpha}$
$\mathbb{R}^m$
$\mathbb{R}^m$

Section 4

Applications

Section 5

Conclusions