MATH 317, Homework 2

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Solutions

Problem 1. Let A and B be sets. Suppose that A is a finite set and that there exists a bijection $f: A \to B$. Prove that the set B is finite.

Proof. Suppose that B is infinite. Since $f: A \to B$ is surjective, $\forall b \in B, \exists a \in A$ such that $f(a) = b$. Since A is finite, we can say for some $n \in \mathbb{N}$ that the members of A are $a_1, a_2, ..., a_n$. Since f is also injective we have $f(a_i) = b_i$ where each $b_i \in B$. But this is a contradiction! The list $b_1, b_2, ..., b_n$ is the same length as $a_1, a_2, ..., a_n$ which we stated was finite. Thus B must also be finite. $\square$

Problem 2. Find the supremum and infimum of the set $S = \left\{1 + \frac{(-1)^n}{n} \mid n \in \mathbb{N}\right\}$. Prove your claims.

Here we can see that the contribution from $\frac{(-1)^n}{n}$ gets smaller and smaller in magnitude as n gets larger. So we expect that the infimum and supremum occur with small n and can check some values.

$$n = 1 \implies 1 + \frac{-1}{1} = 0$$

$$n = 2 \implies 1 + \frac{(-1)^2}{2} = \frac{3}{2}$$

$$n = 3 \implies 1 + \frac{(-1)^3}{3} = \frac{2}{3}$$

If we go on, the no greater or lesser values are achieved. $\sup S = \frac{3}{2}$ and $\inf S = 0$.

Proof (Infimum). Suppose $\exists n_0 \in \mathbb{N}$ such that $1 + \frac{(-1)^{n_0}}{n_0} < 0$ Then,

$$(-1)^{n_0} < -n_0$$

Since $n_0 \in \mathbb{N}$, we have $-n_0 \leq -1$.

$$(-1)^{n_0} \leq -n_0 < -1$$

This is not possible. The left hand side can only equal ± 1 and will never be less than -1. This is a contradiction and so $\inf S = 0$. $\square$

Proof (Supremum). Suppose $\exists n_0 \in \mathbb{N}$ such that,

$$1 + \frac{(-1)^{n_0}}{n_0} > \frac{3}{2}$$

$$\implies (-1)^{n_0} > \frac{1}{2} n_0$$

This is untrue $\forall n \in \mathbb{N}$ as if $n_0 = 1$ we have $-1 > \frac{1}{2}$. If $n_0 = 2$ we have $1 > 1$ which is also untrue. Past this, as n_0 were to increase, the right hand side grows monotonically and is unbounded, and the left oscillates between ± 1 and thus the left hand side will never be greater than the left. This is a contradiction and thus $\sup S = \frac{3}{2}$. $\square$

Problem 3. Find the supremum and infimum of the set $S = \left\{\frac{1}{n} - (-1)^n \mid n \in \mathbb{N}\right\}$. Prove your claims.

$$\begin{aligned} n = 1 &\implies 1 - (-1) = 2 \\ n = 2 &\implies \frac{1}{2} - (-1)^2 = \frac{-1}{2} \\ n = 3 &\implies \frac{1}{3} - (-1)^3 = \frac{4}{3} \\ n = 4 &\implies \frac{1}{4} - (-1)^4 = \frac{-3}{4} \end{aligned}$$

Since the fraction $\frac{1}{n}$ is decreasing, it is largest when $n = 1$ and $\sup S = 2$. As n gets larger, $\frac{1}{n}$ will decrease to 0 while the contribution from $(-1)^n$ will oscillate between ± 1 giving us $\inf S = -1$.

Proof (Infimum). Suppose that $\inf S < -1$, then $\exists n_0 \in \mathbb{N}$ such that,

$$\begin{aligned} \frac{1}{n_0} - (-1)^{n_0} &< -1 \\ \frac{1}{n_0} &< -1 + (-1)^{n_0} \end{aligned}$$

Here on the right hand side we have a number that oscillates only between the values $-2, 0$. On the left we have $\frac{1}{n_0}$ which is positive and nonzero $\forall n \in \mathbb{N}$. Thus this is a contradiction and $\inf S = -1$. $\square$

Proof (Supremum). Suppose that $\sup S > 2$ for some $n_0 \in \mathbb{N}$. Then,

$$\begin{aligned} \frac{1}{n_0} - (-1)^{n_0} &> 2 \\ \frac{1}{n_0} &> 2 + (-1)^{n_0} \end{aligned}$$ $\square$

Here the right hand side oscillates between 1 and 3. The left hand side decreases from 1 to arbitrarily close to 0 as n increases from 1 to n. Because of this, the right hand side will never be less than the left which contradicts our statement about n_0. Thus $\sup S = 2$.

Problem 4.

(a) Show $|b| \leq a$ if and only if $-a \leq b \leq a$.

(b) Prove $||a| - |b|| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$

(a)We want to show that $|b| \leq a \iff -a \leq b \leq a$. Based on these definitions, we know that $a \geq 0$.

Proof (Part (a)). This is a biconditional statement, so I will prove the forward direction first.

Suppose that $|b| \leq a$. Using the fact that $b = 0$, $b > 0$, or $b < 0$,
(1) If $b = 0$ and $a \geq 0$ then we have that $-a \leq 0$. Which is what we wanted to satisfy.
(2) If $b > 0$ and $|b| \geq a$, then $a \geq b$ which implies $-a \leq -b \leq b$. Thus $-a \leq -b < b$.
(3) If $b < 0$ then $|b| = -b \leq a$. This implies $b \geq -a$ and $-b > b$ so $-a \leq b \leq a$.

Next, suppose that $-a \leq b \leq a$.
(1) If $b = 0$ and $a \geq b$ then $a \geq 0 = |b|$.
(2) If $b > 0$ and $b \leq a$ then $|b| = b \leq a = |a|$.
(3) If $b < 0$ and $-a \leq b$, then $-b \leq a$. Since $b < 0$, $|b| = -b \leq a$. □

Proof (Part (b)). We have a few different conditions here to prove:
(1) If $a = 0$ and $b = 0$ then we have

$$\begin{aligned} ||0| - |0|| &= |0 - 0| \\ &= |0| \\ &\leq |0| + |0| \end{aligned}$$

(2) If $a < 0$ and $b = 0$ then we have

$$\begin{aligned} ||a| - |b|| &= ||a| - 0| \\ &= ||a|| \\ &= |(a)| \\ &= |a| \\ &\leq |a| + |b| \end{aligned}$$

(3) If $a > 0$ and $b = 0$ then we have

$$\begin{aligned} ||a| - |b|| &= ||a| - 0| \\ &= ||a|| \\ &= |(a)| \\ &\leq |a| + |b| \end{aligned}$$

(4) If $a = 0$ and $b < 0$ then we have

$$\begin{aligned} ||a| - |b|| &= |0 - |b|| \\ &= | - (-b)| \\ &= |b| \\ &\leq |a| + |b| \end{aligned}$$

(5) If $a = 0$ and $b > 0$ then we have

$$\begin{aligned} ||a| - |b|| &= |0 - |b|| \\ &= | - |b|| \\ &= | - (-b)| \\ &= |b| \\ &\leq |a| + |b| \end{aligned}$$

(6) If $a < 0$ and $b < 0$ then we have

$$\begin{aligned} ||a| - |b|| &= ||a| - (-b)| \\ &= |(a) + b| \\ &\leq |a| + |b| \end{aligned}$$

(7) If $a > 0$ and $b > 0$ then we have

$$\begin{aligned} ||a| - |b|| &= |(a) - (b)|| \\ &= |a - b| \end{aligned}$$

Since both $a > 0$ and $b > 0$

$$\begin{aligned} &< |a + b| \\ &\leq |a| + |b| \end{aligned}$$

(8) If $a < 0$ and $b > 0$ then we have

$$||a| - |b|| = | - a - (b)|$$

Since $-a > 0$ and $b > 0$

$$< |a + b| \qquad \leq |a| + |b|$$

(9) If $a > 0$ and $b < 0$ then we have

$$\begin{aligned} ||a| - |b|| &= |a - (-b)| \\ &= |a + b| \qquad \leq |a| + |b| \end{aligned}$$

Given all of the possible combinations of conditions are satisfied, we know that $\forall a, b \in \mathbb{R}$, $||a| - |b|| \leq |a| + |b|$ □

Note: I think there is probably a more elegant way to argue this. I thought of the brute force way, and it was fairly easy and algebraic (plus LaTeX lets me copy and paste).

Problem 5.

(a) Prove $|a+b+c| \leq |a|+|b|+|c|$ for all $a, b, c \in \mathbb{R}$.

(b) Use induction to prove

$$|a_1 + a_2 + \ldots + a_n| \leq |a_1| + |a_2| + \ldots + |a_n|$$

for n numbers $a_1, a_2, \ldots, a_n$.

Proof (Part (a)).

$$\begin{aligned} |a+b+c| &= |(a+b)+c| \\ &\leq |(a+b)|+|c| \\ &= |a+b|+|c| \\ &\leq |a|+|b|+|c| \end{aligned}$$

Thus, $|a+b+c| \leq |a|+|b|+|c|$. □

Proof (Part (b)). Depending on your point of view, the base case is given by the original triangle inequality $|a_1 + a_2| \leq |a_1| + |a_2|$. Or we can assume the case in (a). Either way, it has been shown. Let's assume $|a_1 + a_2 + \ldots + a_n| \leq |a_1| + |a_2| + \ldots + |a_n|$, and prove for $(n+1)$.

$$\begin{aligned} |a_1 + a_2 + \ldots + a_n + a_{n+1}| &= |(a_1 + a_2 + \ldots + a_n) + a_{n+1}| \\ &\leq |(a_1 + a_2 + \ldots + a_n)| + |a_{n+1}| \\ &= |a_1| + |a_2| + \ldots + |a_n| + |a_{n+1}| \end{aligned}$$

Thus, $|a_1 + a_2 + \ldots + a_n + a_{n+1}| \leq |a_1| + |a_2| + \ldots + |a_n| + |a_{n+1}|$. Since it also holds true for $(n+1)$ we have shown $\forall n \in \mathbb{N}$. □

Problem 6. Let $f\colon X \to Y$ be a function, and let $A, B \subseteq Y$. Prove the following:

(i) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

(ii) $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$

Proof. Let $x \in f^{-1}(A \cup B) = \{x \in X \mid f(x) \in A \cup B\}$. Suppose, for a contradiction, that $x \notin f^{-1}(A) \cup f^{-1}(B)$. This is equivalent to saying $x \in \{x \in X \mid f(x) \notin A \text{ and } f(x) \notin B\}$. Since $f(x) \notin A$ and $f(x) \notin B$, we contradict the original statement $f(x) \in A \cup B$. Thus, $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$.

Next, let $x \in f^{-1}(A) \cup f^{-1}(B) = \{x \in X \mid f(x) \in A \text{ or } f(x) \in B\}$. Suppose, for a contradiction, $x \notin f^{-1}(A \cup B)$. This is equivalent to saying $x \in \{x \in X \mid f(x) \notin A \cup B\}$. Since $f(x) \notin A$ and $f(x) \notin B$ we contradict the original statement $f(x) \in A$ or $x \in B$. Thus $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$

Since $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$, $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ $\square$

Problem 7. Let $A, B \subseteq \mathbb{R}$ be bounded (compact?) sets. Define $A+B := \{a+b \mid a \in A \text{ and } b \in B\}$. Prove or disprove the following statement:

$$\sup(A+B) = \sup A + \sup B$$

Proof. Suppose that $\sup(A+B) > \sup A + \sup B$. This means $\exists a_0 \in A$ and $\exists b_0 \in B$ such that $a_0 + b_0 > \sup A + \sup B$. Thus,

$$\begin{aligned} a_0 + b_0 &> \sup A + \sup B \\ 0 &> (\sup A - a_0) + (\sup B - b_0) \end{aligned}$$

Since $a_0 \leq \sup A$ and $b_0 \leq \sup B$ we have a contradiction. It is not possible for the side on the right to be less than zero since that would require at least one of a_0 or b_0 to be greater than one of the supremums which contradicts the fact that they are the least upper bound of the sets. Thus we have that $\sup(A+B) \leq \sup A + \sup B$.

Next, suppose that $\sup(A+B) < \sup A + \sup B$. Then $\forall a_0 \in A$ and $\forall b_0 \in B$, $a_0 + b_0 < \sup A + \sup B$. We can write this in a similar way,

$$\begin{aligned} a_0 + b_0 &< \sup A + \sup B \\ 0 &< (\sup A - a_0) + (\sup B - b_0) \end{aligned}$$

Since a_0 is at most $\sup A$ and b_0 is at most $\sup B$ by definition since those are the least upper bounds of the set. However, if $a_0 = \sup A$ and $b_0 = \sup B$ we have $0 < 0$ which is false. Thus it must be that $\sup(A+B) \geq \sup A + \sup B$.

Since we have shown that $\sup(A+B) \leq \sup A + \sup B$ and $\sup(A+B) \geq \sup A + \sup B$, we know $\sup(A+B) = \sup A + \sup B$. $\square$