

MATH 272, HOMEWORK 3  
DUE FEBRUARY 17<sup>TH</sup>

**Problem 1.** Compute the Fourier series for the following functions on the interval  $[0, L]$ . Then plot your result (for  $N = 1, 50, 100, 500$ ) compared to the original function. What do you notice if you plot the Fourier series outside the range of  $[0, L]$ ?

(a)  $f(x) = \frac{1}{2} \cos\left(\frac{2\pi x}{L}\right) + \sin\left(\frac{-4\pi x}{L}\right).$

(b)  $\sin\left(\frac{3\pi x}{L}\right).$

(c)  $\delta(x - L/2).$

**Problem 2.** Consider a function  $f(x)$  that describes the height of a rubber string with rest length  $L$ . We can attach the ends of the string at  $x = 0$  and  $x = L$  by requiring that  $f(0) = f(L) = 0$ . Then, one can subject the string to an external force  $g(x)$  and find the profile of the string by solving

$$-\frac{d^2}{dx^2}f(x) = g(x).$$

(a) Let  $g(x) = \delta(x - L/2)$  and let  $f(x)$  be given by some Fourier series. Using the equation above, solve for the coefficients of the Fourier series for  $f(x)$ .

(b) Plot the Fourier series for  $f(x)$  for  $N = 1, 5, 50$ .

This is an extremely important to solve. The fact that we can determine a solution  $f(x)$  where the external force is the Dirac delta function means that we have the ability to determine a the deformation of a string from a point force.

**Problem 3.** Compute the following Fourier transforms (using a table or WolframAlpha if need be).

(a)  $\sin(3\pi x).$

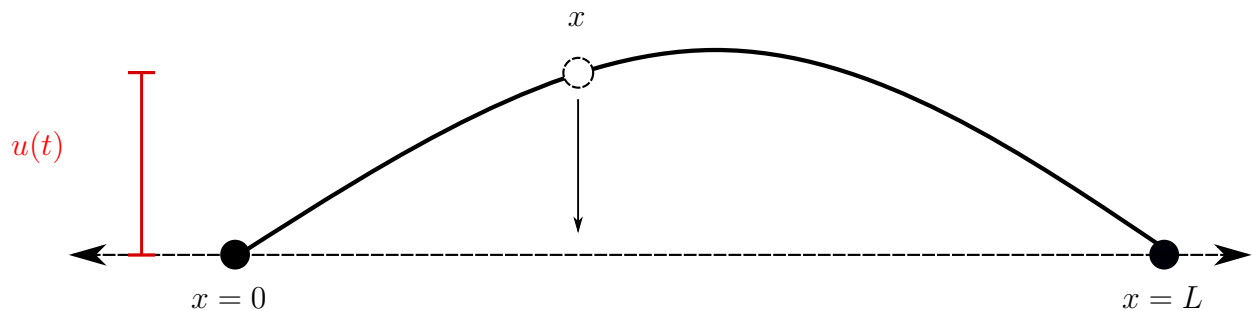
(b)  $e^{-x^2}2.$

(c)  $\delta(x).$

**Problem 4.** A common application for the Fourier transform is to solve differential equations whose domain is time  $t \in [0, \infty)$ . We can model how a point  $x$  on a rubber string oscillates over time consider the differential equation

$$u''(t) + v^2 u(t) = 0,$$

with initial conditions  $u(0) = L$  and  $u'(0) = 0$ . Here  $u(t)$  is the displacement of the string at position  $x$  with the initial conditions describing the string being pulled tight at time  $t = 0$ .



To solve this equation, we could use methods we learned previously, or apply the Fourier transform to the whole equation by

$$\mathcal{F}(u''(t) + v^2 u(t)) = \mathcal{F}(0).$$

- (a) Compute the Fourier transform above.
- (b) One should then have a new equation

$$-4\pi^2 k^2 \hat{u}(k) + v^2 \hat{u}(k) = 0.$$

Solve for  $\hat{u}(k)$  in terms of  $k$ .

- (c) Use the inverse Fourier transform to compute  $u(t)$  from  $\hat{u}(k)$ . This is the solution to the differential equation.