

MATH 272, HOMEWORK 3, *Solutions*  
DUE FEBRUARY 17<sup>TH</sup>

**Problem 1.** Compute the Fourier series for the following functions on the interval  $[0, L]$ . Then plot your result (for  $N = 1, 50, 100, 500$ ) compared to the original function. What do you notice if you plot the Fourier series outside the range of  $[0, L]$ ?

(a)  $f(x) = \frac{1}{2} \cos\left(\frac{2\pi x}{L}\right) + \sin\left(\frac{-4\pi x}{L}\right).$

(b)  $\sin\left(\frac{3\pi x}{L}\right).$

(c)  $\delta(x - L/2).$

**Solution 1.**

(a) Here, we can use orthogonality to greatly reduce the amount of work we must do. Notice as well that our function is essentially written as a Fourier series. We have

$$\begin{aligned} a_n &= \left\langle f, \sqrt{2} \cos\left(\frac{2n\pi x}{L}\right) \right\rangle \\ &= \left\langle \frac{1}{2} \cos\left(\frac{2\pi x}{L}\right) + \sin\left(\frac{-4\pi x}{L}\right), \sqrt{2} \cos\left(\frac{2n\pi x}{L}\right) \right\rangle \\ &= \underbrace{\left\langle \frac{1}{2} \cos\left(\frac{2\pi x}{L}\right), \sqrt{2} \cos\left(\frac{2n\pi x}{L}\right) \right\rangle}_{= 0 \text{ unless } n=1} + \underbrace{\left\langle \sin\left(\frac{-4\pi x}{L}\right), \sqrt{2} \cos\left(\frac{2n\pi x}{L}\right) \right\rangle}_{= 0 \text{ always}}. \end{aligned}$$

Hence, we have that the only nonzero  $a_n$  term is  $a_1$  and we have

$$\begin{aligned} a_1 &= \left\langle \frac{1}{2} \cos\left(\frac{2\pi x}{L}\right), \sqrt{2} \cos\left(\frac{2\pi x}{L}\right) \right\rangle \\ &= \frac{1}{2\sqrt{2}} \left\langle \sqrt{2} \cos\left(\frac{2\pi x}{L}\right), \sqrt{2} \cos\left(\frac{2\pi x}{L}\right) \right\rangle \\ &= \frac{1}{2\sqrt{2}}. \end{aligned}$$

An analogous argument shows that  $f(x)$  is orthogonal to 1 which shows that  $a_0 = 0$  as well. Similarly, another analogous argument shows that the only nonzero  $b_n$  term is  $b_2$  since  $\sin(-x) = -\sin(x)$ . So we have that

$$b_2 = \frac{1}{\sqrt{2}}.$$

Hence, we have found all of our coefficients for the Fourier series.

(b) First, let's compute

$$\begin{aligned} a_0 &= \left\langle \sin\left(\frac{3\pi x}{L}\right), 1 \right\rangle \\ &= \frac{1}{L} \int_0^L \sin\left(\frac{3\pi x}{L}\right) dx \\ &= \frac{2}{3\pi}. \end{aligned}$$

Next, we can compute

$$\begin{aligned} a_n &= \left\langle \sin\left(\frac{3\pi x}{L}\right), \sqrt{2} \cos\left(\frac{2n\pi x}{L}\right) \right\rangle \\ &= \frac{1}{L} \int_0^L \sin\left(\frac{3\pi x}{L}\right) \sqrt{2} \cos\left(\frac{2n\pi x}{L}\right) dx \\ &= \frac{6\sqrt{2}}{9\pi - 4\pi n^2}. \end{aligned}$$

Finally, we compute

$$\begin{aligned} b_n &= \left\langle \sin\left(\frac{3\pi x}{L}\right), \sqrt{2} \sin\left(\frac{2n\pi x}{L}\right) \right\rangle \\ &= 0 \end{aligned} \quad \text{by orthogonality.}$$

Thus, our Fourier series is given by

$$\boxed{\sin\left(\frac{3\pi x}{L}\right) = \frac{2}{3\pi} + \sum_{n=1}^{\infty} \frac{12}{9\pi - 4\pi n^2} \cos\left(\frac{2n\pi x}{L}\right)}.$$

First, we compute

$$a_0 = \langle \delta(x - L/2), 1 \rangle = \frac{1}{L} \int_0^L \delta(x - L/2) dx = \frac{1}{L}.$$

Then, we have

$$\begin{aligned} a_n &= \left\langle \delta(x - L/2), \sqrt{2} \cos\left(\frac{2n\pi x}{L}\right) \right\rangle \\ &= \frac{1}{L} \int_0^L \delta(x - L/2) \sqrt{2} \cos\left(\frac{2n\pi x}{L}\right) dx \\ &= \sqrt{2} \cos(n\pi) \\ &= (-1)^n \frac{\sqrt{2}}{L}. \end{aligned}$$

Finally, we compute

$$\begin{aligned}
 b_n &= \left\langle \delta(x - L/2), \sqrt{2} \sin\left(\frac{2n\pi x}{L}\right) \right\rangle \\
 &= \frac{1}{L} \int_0^L \delta(x - L/2) \sqrt{2} \sin\left(\frac{2n\pi x}{L}\right) dx \\
 &= \frac{\sqrt{2}}{L} \sin(n\pi) \\
 &= 0.
 \end{aligned}$$

Hence, our Fourier series is given by

$$\delta(x - L/2) = \frac{1}{L} + \sum_{n=1}^{\infty} (-1)^n \frac{2}{L} \cos\left(\frac{2n\pi x}{L}\right).$$

**Problem 2.** Consider a function  $f(x)$  that describes the height of a rubber string with rest length  $L$ . We can attach the ends of the string at  $x = 0$  and  $x = L$  by requiring that  $f(0) = f(L) = 0$ . Then, one can subject the string to an external force  $g(x)$  and find the profile of the string by solving

$$-\frac{d^2}{dx^2}f(x) = g(x).$$

- (a) Let  $g(x) = \delta(x - L/2)$  and let  $f(x)$  be given by some Fourier series. Using the equation above, solve for the coefficients of the Fourier series for  $f(x)$ .
- (b) Plot the Fourier series for  $f(x)$  for  $N = 1, 5, 50$ .

This is an extremely important to solve. The fact that we can determine a solution  $f(x)$  where the external force is the Dirac delta function means that we have the ability to determine a the deformation of a string from a point force.

**Solution 2.**

- (a) Here we let  $f(x)$  be written as an arbitrary Fourier series by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \sqrt{2} \cos\left(\frac{2n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sqrt{2} \sin\left(\frac{2n\pi x}{L}\right).$$

Then we can put

$$-\frac{d^2}{dx^2}f(x) = \frac{4n^2\pi^2}{L^2} \left( \sum_{n=1}^{\infty} a_n \sqrt{2} \cos\left(\frac{2n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sqrt{2} \sin\left(\frac{2n\pi x}{L}\right) \right).$$

Setting the left hand side of the ODE equal to the right, we have

$$-\frac{d^2}{dx^2}f(x) = g(x)$$

$$\frac{4n^2\pi^2}{L^2} \left( \sum_{n=1}^{\infty} a_n \sqrt{2} \cos\left(\frac{2n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sqrt{2} \sin\left(\frac{2n\pi x}{L}\right) \right) = \frac{1}{L} + \sum_{n=1}^{\infty} (-1)^n \frac{2}{L} \cos\left(\frac{2n\pi x}{L}\right).$$

Hence, we just need to determine the coefficients  $a_0$ ,  $a_n$ , and  $b_n$  for our function  $f(x)$ . It's clear to see that the  $b_n$  terms for  $f(x)$  must be zero. Solving for the  $a_n$  terms, we find

$$\frac{4n^2\pi^2}{L^2} a_n = (-1)^n \frac{2}{\sqrt{2}L},$$

which yields

$$a_n = (-1)^n \frac{L}{2\sqrt{2}n^2\pi^2}.$$

At this point, we have that

$$f(x) = a_0 + \sum_{n=1}^{\infty} (-1)^n \frac{L}{2n^2\pi^2} \cos\left(\frac{2n\pi x}{L}\right).$$

We can determine  $a_0$  by enforcing our boundary conditions. Namely, we take

$$\begin{aligned} 0 = f(0) &= a_0 + \sum_{n=1}^{\infty} (-1)^n \frac{L}{2n^2\pi^2} \\ &= a_0 - \frac{L}{24}, \end{aligned}$$

where I used WolframAlpha to evaluate the infinite series above. This means that  $a_0 = \frac{L}{24}$ . Thus, we arrive at

$$\boxed{f(x) = \frac{L}{24} + \sum_{n=1}^{\infty} (-1)^n \frac{L}{2n^2\pi^2} \cos\left(\frac{2n\pi x}{L}\right)}.$$

**Problem 3.** Compute the following Fourier transforms (using a table or WolframAlpha if need be).

(a)  $\sin(3\pi x)$ .

(b)  $e^{\frac{-x^2}{2}}$ .

(c)  $\delta(x)$ .

**Solution 3.**

(a) To compute the Fourier transform, one could use a table. However, we can take

$$\mathcal{F}[\sin(3\pi x)] = \int_{-\infty}^{\infty} \sin(3\pi x) e^{-i2\pi kx} dx.$$

The techniques used to compute this integral by hand are found in complex analysis. Specifically, one uses the fact that  $\sin(3\pi x)$  is an analytic function on  $\mathbb{C}$  as well as the method of contour integration (seen here: [https://en.wikipedia.org/wiki/Contour\\_integration](https://en.wikipedia.org/wiki/Contour_integration)). This is not a topic for our class (though we will learn how to integrate along curves).

Anyways, computing this Fourier transform yields

$$\mathcal{F}[\sin(3\pi x)] = \frac{\delta(k - \frac{3}{2}) - \delta(k + \frac{3}{2})}{2i}.$$

Here, I used a table to compute this (as WolframAlpha claims the integral does not converge).

(b) Again, we want to compute

$$\mathcal{F}\left[e^{\frac{-x^2}{2}}\right] = \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} e^{-i2\pi kx} dx.$$

I was able to compute this using WolframAlpha by entering

`integrate[e^(-x^2/2)e^{-i2*pi*k*x},{x,-infty,infty}]`

This yields

$$\mathcal{F}\left[e^{\frac{-x^2}{2}}\right] = \sqrt{2\pi} e^{-2\pi^2 k^2}.$$

One thing to note here is that the Fourier transform of a Gaussian is a Gaussian!

(c) Since, in some sense, the Dirac delta is a Gaussian function (with a variance of zero), we expect to get another type of Gaussian out as well. Let's see what happens. We take

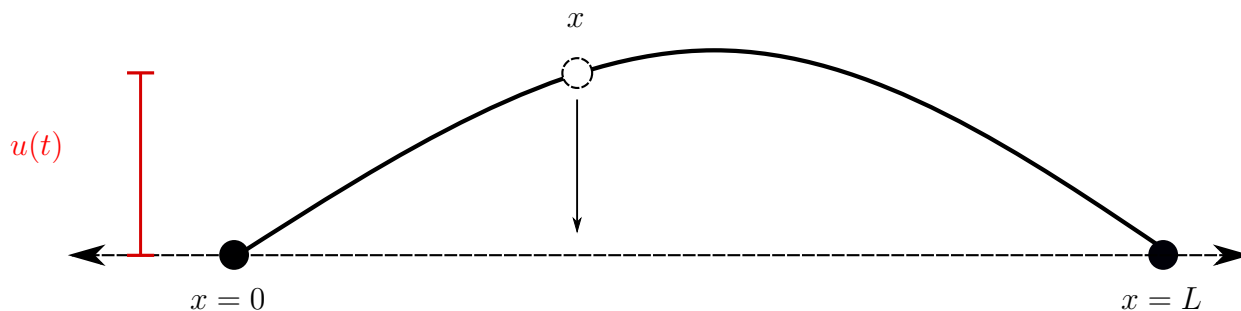
$$\mathcal{F}[\delta(x)] = \int_{-\infty}^{\infty} \delta(x) e^{-i2\pi kx} dx = 1.$$

The constant 1 function is much like a Gaussian with an infinite variance! This is maybe a “handwavy” way to look at this.

**Problem 4.** A common application for the Fourier transform is to solve differential equations whose domain is time  $t \in [0, \infty)$ . We can model how a point  $x$  on a rubber string oscillates over time consider the differential equation

$$u''(t) + v^2 u(t) = 0,$$

with initial conditions  $u(0) = L$  and  $u'(0) = 0$ . Here  $u(t)$  is the displacement of the string at position  $x$  with the initial conditions describing the string being pulled tight at time  $t = 0$ .



To solve this equation, we could use methods we learned previously, or apply the Fourier transform to the whole equation by

$$\mathcal{F} [u''(t) + v^2 u(t)] = \mathcal{F} [0].$$

- (a) Compute the Fourier transform above.
- (b) One should then have a new equation

$$-4\pi^2 k^2 \hat{u}(k) + v^2 \hat{u}(k) = 0.$$

Solve this new equation for  $k$ .

- (c) One should have two values  $k_1$  and  $k_2$  from the work in (b). This corresponds to the solution

$$\hat{u}(k) = \delta(k - k_1) \quad \text{and} \quad \hat{u}(k) = \delta(k - k_2).$$

Compute the inverse Fourier transform of the two delta functions. A linear combination of these correspond to your solution  $u(t)$ .

**Solution 4.**

- (a) We have that the Fourier transformed equation is given by

$$-4\pi^2 k^2 \hat{u}(k) + v^2 \hat{u}(k) = 0,$$

since the Fourier transform converts differentiation into multiplication.

(b) Indeed, we have this equation. If we solve this equation for  $k$ , we have

$$(-4\pi^2 k^2 + v^2)\hat{u}(k) = 0,$$

which yields

$$k = \pm \frac{|v|}{2\pi}.$$

(c) Put  $k_1 = -\frac{|v|}{2\pi}$  and  $k_2 = \frac{|v|}{2\pi}$ . Thus, we get

$$\hat{u}(k) = \delta\left(k + \frac{|v|}{2\pi}\right) \quad \text{and} \quad \hat{u}(k) = \delta\left(k - \frac{|v|}{2\pi}\right).$$

Then we have that

$$\mathcal{F}\left[\delta\left(k + \frac{|v|}{2\pi}\right)\right] = e^{-ivt}$$

and

$$\mathcal{F}\left[\delta\left(k - \frac{|v|}{2\pi}\right)\right] = e^{ivt}.$$

If we take a linear combination of these solutions we arrive at

$$u(t) = C_1 e^{ivt} + C_2 e^{-ivt},$$

which is the solution we have found for the harmonic oscillator equation before!