MATH 517, Homework 3

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Solutions

Problem 1. **(Rudin 3.8)** If $\sum a_n$ converges and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges.

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Proof. Since $\sum a_n$ converges we know that the partial sums A_N form a bounded sequence. Since $\{b_n\}$ is bounded and monotonic we have that $\{b_n\} \to L$. Then if $\{b_n\}$ is nondecreasing we have that $\lim_{n\to\infty} L - b_n = 0$. Call the sequence $\{c_n\} = \{L - b_n\}$, and note that $\sum a_n c_n = \sum a_n b_n - b \sum a_n$ so we have that $\sum a_n b_n$ converges. Otherwise $\{b_n\}$ is nonincreasing so $\lim_{n\to\infty} b_n - L = 0$ and we can call $\{d_n\} = \{b_n - L\}$. Now we have $\sum a_n d_n = L \sum a_n - \sum a_n b_n$. So $\sum a_n b_n$ also converges. $\square$

Problem 2. (Rudin 3.16) Fix $\alpha > 0$. Choose $x_1 > \sqrt{\alpha}$ and recursively define the sequence $\{x_n\}$ by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right)$$

(a) Prove that $\{x_n\}$ decreases monotonically and that $\lim x_n = \sqrt{\alpha}$.

(b) Let $\epsilon_n = x_n - \sqrt{\alpha}$ be the error in approximating $\sqrt{\alpha}$ by x_n, and show that

$$\epsilon_{n+1} = \frac{e_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}}$$

Conclude that, with $\beta = 2\sqrt{\alpha}$

$$\epsilon_{n+1} < \beta\left(\frac{\epsilon_1}{\beta}\right)^{2n}$$

(c) Part (b) shows that this is an excellent method of approximating square roots. As an example, show that for $\alpha = 3$ and $x_1 = 2$, then $\frac{\epsilon_1}{\beta} < \frac{1}{10}$, and hence

$$\epsilon_5 < 4 \times 10^{-16} \text{ and } \epsilon_6 < 4 \times 10^{-32}$$

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Proof (Part (a)). We will prove monotonicity by induction. So for the base case, consider

$$\begin{aligned}
x_2 &= \frac{1}{2}\left(x_1 + \frac{\alpha}{x_1}\right) \\
&= \frac{x_1}{2}\left(1 + \frac{\alpha}{x_1^2}\right) \\
&< \frac{x_1}{2}(1+1) && \text{since } x_1^2 > \alpha \\
&= x_1
\end{aligned}$$

Now assume this is true for $i = 1, \ldots, n-1$. Then we have to show $x_n < x_{n-1}$. So, assume that $x_n \geq x_{n-1}$ so thus we have that

$$\begin{aligned}
& x_{i+1} \leq x_n \leq x_i \\
\iff & \frac{x_i + \alpha}{2} \leq \frac{x_{n-1} + \alpha}{2} \leq \frac{x_{i-1} + \alpha}{2}
\end{aligned}$$

Which contradicts $x_{n-1} < x_{n-2}$, since $x_{n-1} < x_{n-2} < \ldots < x_{i+1} < x_i$. Thus we have x_n is decreasing.

To find $\lim_{n\to\infty} x_n$ show that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = L$ So we have

$$\begin{aligned}
L &= \frac{1}{2}\left(L + \frac{\alpha}{L}\right) \\
\frac{L}{2} &= \frac{\alpha}{2L} \\
L^2 &= \alpha \\
L &= \pm\sqrt{\alpha}
\end{aligned}$$

We can show that $+\sqrt{\alpha}$ is a lower bound for our sequence $\{x_n\}$ by supposing that for some $x_n \leq \sqrt{\alpha}$ and we have that $x_{n+1} \leq x_n$. But, instead, we have that

$$\begin{aligned}
x_{n+1} &= \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) \\
&> \frac{x_n}{2}(1+1) \\
&= x_n
\end{aligned}$$

Which contradicts that x_n is decreasing. Thus we have that $\sqrt{\alpha}$ is a lower bound, thus we can choose the positive root from above, and we are done. □

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Solution (Part (b)). We have

$$\begin{aligned}
\epsilon_{n+1} &= x_{n+1} - \sqrt{\alpha} \\
&= \frac{x_n^2 + \alpha}{2x_n} - \sqrt{\alpha} \\
&= \frac{(\epsilon_n + \sqrt{\alpha})^2}{2x_n} - \sqrt{\alpha} \\
&= \frac{\epsilon_n + 2\epsilon\sqrt{\alpha} + 2\alpha}{2x_n} - \sqrt{\alpha} \\
&= \frac{\epsilon_n^2}{2x_n} + \frac{\epsilon_n\sqrt{\alpha} + \alpha}{\epsilon_n + \sqrt{\alpha}} - \sqrt{\alpha} \\
&= \frac{\epsilon_n^2}{2x_n} + \sqrt{\alpha} - \sqrt{\alpha} \\
&= \frac{\epsilon_n^2}{2x_n} \\
&< \frac{\epsilon_n^2}{2\sqrt{\alpha}} && \text{since } x_n > \sqrt{\alpha}
\end{aligned}$$

Then let $\beta = 2\sqrt{\alpha}$ and we have that

$$\begin{aligned}
\frac{\epsilon_{n+1}}{\beta} &< \epsilon_n^2 \\
&< \frac{\epsilon_{n-1}^2}{\beta} \\
&< \left(\frac{\epsilon_{n-2}^2}{\beta}\right)^2 \\
&< \left(\left(\frac{\epsilon_{n-3}^2}{\beta^2}\right)^2\right)^2 \\
&\vdots \\
&< \left(\frac{\epsilon_1}{\beta}\right)^{2n} \\
\implies \epsilon_{n+1} &< \beta\left(\frac{\epsilon_1}{\beta}\right)^{2n}
\end{aligned}$$

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Problem 3. (Rudin 3.20) Suppose that $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_n\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p. (*Note:* We are not assuming X is compact or $\mathbb{R}^k$, so you can't immediately say that $\{p_n\}$ is Cauchy and therefore it converges.)

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Proof. Since $\{p_n\}$ converges we have that for $N_1 \in \mathbb{N}$ that $|p_{n_k} - p| < \frac{\epsilon}{2}$ for $n_k > N_1$. Then since $\{p_n\}$ is Cauchy we have that $|p_n - p_m| < \frac{\epsilon}{2}$ for $n > N_2 \in \mathbb{N}$. Then take $N = \max(\{N_1, N-2\})$ and we have that for $n, n_k > N$

$$\begin{aligned} |p_{n_k} - p| &= |p_{n_k} - p_n + p_n - p| \\ &\leq |p_{n_k} - p_n| + |p_n - p| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Which shows that $\{p_n\} \to p$. $\square$

Problem 4. Let $\{a_n\}$ be a sequence of real numbers satisfying $\liminf |a_n| = 0$. Prove that there exists a subsequence $\{a_{n_k}\}$ so that $\sum_{k=1}^{\infty} a_{n_k}$ converges.

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Proof. Since $\liminf |a_n| = 0$ we have that some subsequence $\{a_{n_k}\}$ converges to 0. Fix $\epsilon > 0$ and, specifically, choose $a_{n_k} < \frac{\epsilon}{2^k}$ so that

$$\begin{aligned} \left| \sum_{k=n}^{m} a_k \right| &= \left| \sum_{k=n}^{m} \frac{\epsilon}{2^k} \right| \\ &< \epsilon \end{aligned}$$

So we have that $\sum_{k=1}^{\infty} a_{n_k}$ converges since we have shown that the series is Cauchy. $\square$