Riemannian Geometry

for Dummies

Colin Roberts



Section 1

Introduction

Riemannian geometry is the study of a smooth $manifold\ M$ along with a $Riemannian\ metric\ g.$

The point of Riemmannian geometry is to generalize the
differentiable and metric structure of \mathbb{R}^n .

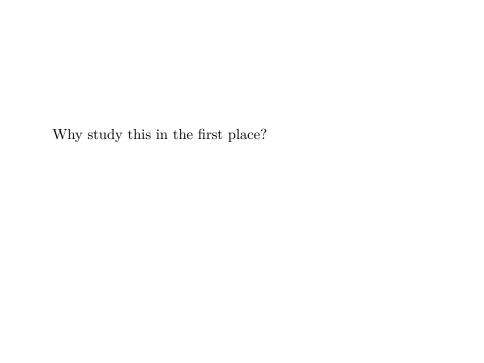
We generalize to space	ces that have i	interesting topolo	gy and
geometry.			
,			

This will require us to rethink some notions we foun	d "easy"
in \mathbb{R}^n .	

But we will gain a very general framework for working with differentiable objects.

Section 2

Motivation



Example: P	artial differentia	al equations (F	PDEs) on spaces
that are not	flat.		

Example: Partial differential equations (PDEs) on spaces that are not flat.

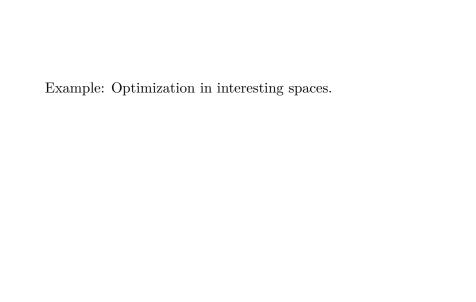
■ Fluid flow on Earth;

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- Fluid flow on Earth;
- Electrical Impedence Tomography (EIT);
- General relativity.



Example: Optimization in interesting spaces.

■ Grassmannians;

Example: Optimization in interesting spaces.

- Grassmannians;
- Flags.

Section 3

Preliminaries

add more math text before/after pics so that people see some notation. More examples.

Subsection 1

Smooth Manifolds

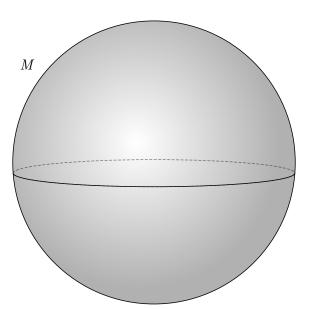
 \blacksquare Start with a topological space M;

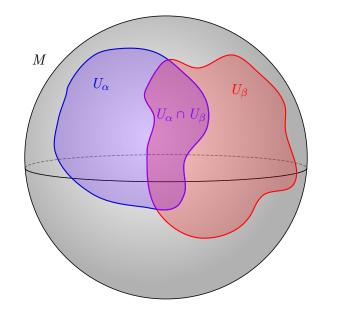
- \blacksquare Start with a topological space M;
- \blacksquare Look at open sets U that cover M;

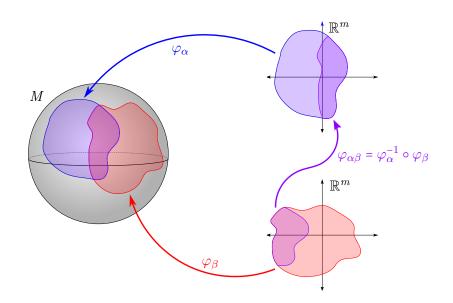
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- Show coordinate transition functions are smooth.

Define the sphere as the set of points in \mathbb{R}^3 ... then say we'll mostly use this as an example so keep it in mind

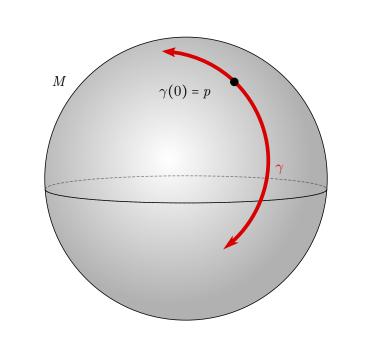


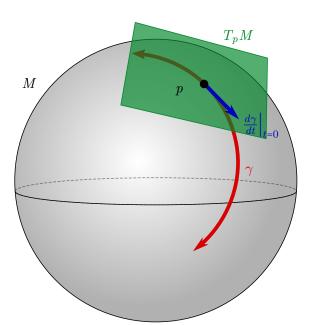


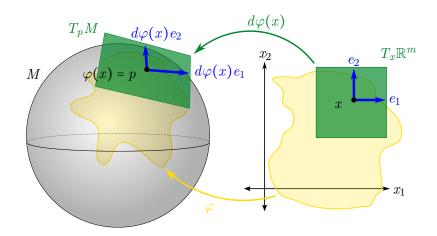


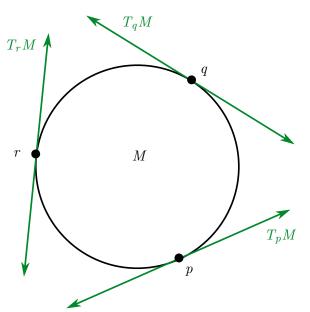
Subsection 2

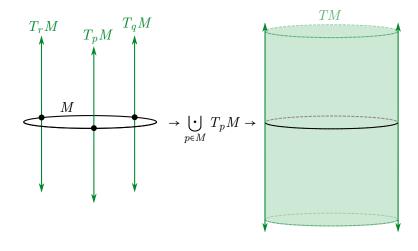
Vector Fields

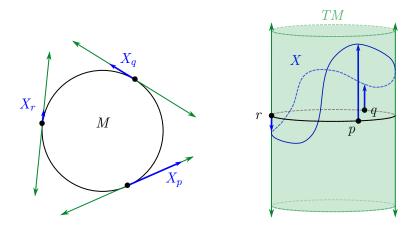








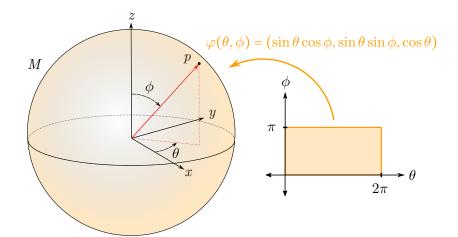


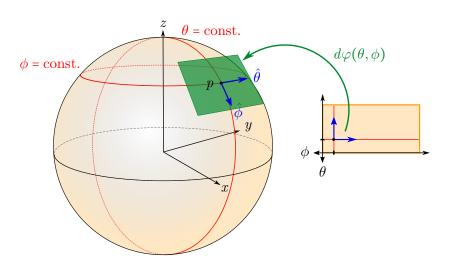


Subsection 3

Example

Example of charts and maps for sphere as well as a vector field. Do spherical coordinates





Section 4

Riemannian Geometry

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- Extract geometrical and analytical qualities of the underlying manifold M.

Subsection 1

Riemannian Metric

 $g_{ij}(x) = \varphi^*(x)e_i \cdot \varphi^*(x)e_k = (\partial_i \varphi) \cdot (\partial_j \varphi) = \sum_{k=1}^m \frac{\partial \varphi_k}{\partial x_i} \frac{\partial \varphi_k}{\partial x_j}$

connection covariant derivative interpretation, covariant derivative in spherical coordinates, second covariant matrix and covariant laplacian from hessian? Compatability with riemannian metric picture. Show a geodesic in the coordinates between cities or something?

From minimization of length/energy. Both are good to mention. Geodesic equation

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0$$

equivalent to

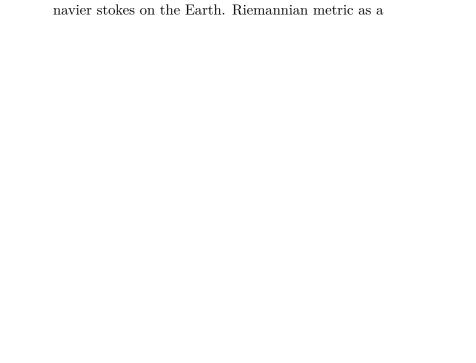
$$\ddot{x}^l + \dot{x}^j \dot{x}^k \Gamma^l_{ik} = 0$$

which is saying that the only "acceleration" of the curve comes from the geometry it lies on. When flat space, $\Gamma^l_{jk} = 0$ and we have $\ddot{x} = 0$.

volume and integration

Section 5

Applications



conductivity?

Section 6

Conclusions