MATH 570, Homework 11

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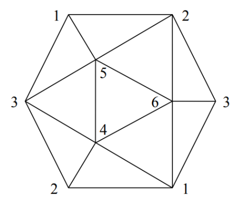
Solutions

Problem 1. Let X be the 2-dimensional simplicial complex with ten 2-simplices drawn below; recall that X is homeomorphic to the projective plane $\mathbb{RP}^2$.



Show that the 2-dimensional simplicial homology group of X is $H_2(X) \cong 0$.

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Proof. Henry has a hint online. ☐

Problem 2. Prove Proposition 13.5 in our book: If X is a space, $\{X_\alpha\}_{\alpha\in X}$ is the set of path components of X, and $\iota_\alpha\colon X_\alpha \hookrightarrow X$ is the corresponding inclusion, then for each $p \geq 0$ the map $\oplus_{\alpha\in A} H_p(X_\alpha) \to H_p(X)$ whose restriction to singular homology group $H_p(X_\alpha)$ is $(\iota_\alpha)_*\colon H_p(X_\alpha) \to H_p(X)$ is an isomorphism. Proceed by the following steps.

(a) Show the maps $(\iota_\alpha)_\#\colon C_p(X_\alpha) \to C_p(X)$ give an isomorphism $g_C\colon \oplus_{\alpha\in A} C_p(X_\alpha) \to C_p(X)$, defined by $g_C((c_\alpha)_{\alpha\in A}) = \sum_\alpha (\iota_\alpha)_\#(c_\alpha)$, where $c_\alpha \in C_p(X_\alpha)$. Injectivity is clear, and the first sentence of the proof in our book implies surjectivity.

(b) Show that restricting g_C gives an isomorphism $g_Z\colon \oplus_{\alpha\in A} Z_p(X_\alpha) \to Z_p(X)$. The injectivity of g_Z follows from that of g_C; you need to show that g_Z is well-defined and surjective.

(c) Show that restricting g_C gives an isomorphism $g_B\colon \oplus_{\alpha\in A} B_p(X_\alpha) \to B_p(X)$. The injectivity of g_B follows from that of g_C; you need to show that g_B is well-defined and surjective.

(d) Deduce that g_C induces an isomorphism $\oplus_{\alpha\in A} H_p(X_\alpha) \to H_p(X)$.

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Proof. (a)

Problem 3. Prove Proposition 13.6: For any topological space X, the singular homology group $H_0(X)$ is a free abelian group with basis consisting of an arbitrary point in each path component.

Remark: Our book contains a detailed proof; you can learn and use this proof!

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Proof. First, assume X is path-connected. Let an element in C_0 be given by $c = \sum_{i=1}^m$ and define the map $\epsilon\colon C_0(X) \to \mathbb{Z}$ by

$$\epsilon\left(\sum_{i=1}^m n_i x_i\right) = \sum_{i=1}^m n_i.$$

Clearly, ϵ is a surjective group homomorphism.

Now choose a point $x_0 \in X$ and for every $x \in X$ let $\alpha(x)$ be a path from x_0 to x. This path α is a singular 1-simplex whose boundary is the 0-chain $x - x_0$. So for an arbitrary 0-chain c, we have

$$\partial\left(\sum_{i=1}^m \alpha(x_i)\right) = \sum_{i=1}^m - \sum_{i=1}^m n_i x_0 = c - \epsilon(c)x_0.$$

If we then let $c \in \ker\epsilon$ so that $\epsilon(c) = 0$ we find that $c \in B_0(X)$. This means that $\ker\epsilon \subseteq B_0(X)$.

Next, note that $B_0(X) \in \ker\epsilon$ since for any singular 1-simplex σ we have $\partial\sigma = \sigma(1) - \sigma(0)$ and in particular, $\epsilon(\partial\sigma) = 1 - 1 = 0$. This means $B_0(X) \subseteq \ker\epsilon$.

Then we have that $\ker\epsilon = B_0(X)$, and by the first isomorphism theorem, that ϵ induces and isormorphism $H_0 \to \mathbb{Z}$. Finally, by Proposition 13.5, we have that $H_0(X)$ is a direct sum of infinite cyclic groups, one for each path component. Hence, we have $H_0(X)$ is a free abelian group with basis consisting of an arbitrary point in each path component. $\square$

Problem 4. Use the Mayer-Vietoris Theorem to prove Theorem 13.23: For $n \geq 1$, the singular homology groups of the sphere S^n are

$$H_p(S^n) \cong \begin{cases} \mathbb{Z} & \text{if } p = 0 \text{ or } n \\ 0 & \text{if } 0 < p < n \text{ or } p > n. \end{cases}$$

Remark: Our book contains a detailed proof; you can learn and use this proof!

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Proof. We let N and S represent the north and south poles of S^n. Now $U = S^n \backslash \{N\}$ and $V = S^n \backslash \{S\}$. Since U and V are contractible, the Mayer Vietoris sequence is

$$0 \to H_p(S^n) \xrightarrow{\partial_*} H_{p-1}(U \cap V) \to 0, \qquad \square$$

which gives us that ∂_* is an isomorphism. Since $U \cap V$ is homotopy equivalent to S^{n-1},

$$H_p(S^n) \cong H_{p-1}(U \cap V) \cong H_{p-1}(S^{n-1}) \;\text{ for } p > 1, n \geq 1.$$

Now we finish by induction. For the case $n = 1$, $H_0(S^1) \cong H_1(S^1) \cong \mathbb{Z}$ by Proposition 13.6 and Corollary 13.15. For $p > 1$, The Mayer Vietoris theorem tells us that $H_p(S^1) \cong H_{p-1}(S^0)$. Since S^0 is the disjoint union of two points, $H_{p-1}(S^0)$ is the trivial group by Propositions 13.7 and 13.5.

Next, suppose the result is true for S^{n-1} for $n > 1$, then for $p = 0$ and $p = 1$ we have the result by Proposition 13.6 and Corollary 13.15. For $p > 1$, the Mayer Vietoris theorem and the inductive hypothesis imply that

$$H_p(S^n) \cong H_{p-1}(S^{n-1}) \cong \begin{cases} 0 & \text{if } p < n, \text{or} p > n, \\ \mathbb{Z} & \text{if } p = n. \end{cases}$$

Hence, we are finished. $\square$