MATH 571, Homework 4

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Solutions

Problem 1. The classification of closed surfaces says that a connected closed surface (where "closed" here means compact with no boundary) is homeomorphic to exactly one of the following:

- the sphere $M_0 := S^2$,
- the connected sum of g tori for $g \geq 1$, denoted M_g, and also called the torus of genus g, or
- the connected sum of $g \geq 1$ projective planes for $g \geq 1$, denoted N_g.

The connected sum of two surfaces is obtained by deleting a disk from each, and then gluing the two surfaces together along their two boundary circles. It turns out that the M_g surfaces are orientable, whereas the N_g surfaces are not.

1. Show that M_g has a CW complex structure with one 0-cell, $2g$ 1-cells, and one 2-cell, and deduce from this CW structure that the fundamental group of M_g is

$$\pi_1(M_g) \cong \langle a_1, b_1, \ldots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \rangle$$

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2. Show that N_g has a CW complex structure with one 0-cell, g 1-cells, and one 2-cell, and deduce from this CW structure that the fundamental group of N_g is

$$\pi_1(N_g) \cong \langle a_1, \ldots, a_g \mid a_1^2 \cdots a_g^2 \rangle$$

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3. "Abelianization" is a way to turn any group into an ableian group. Indeed, there is a functor $\mathrm{Ab} \colon \mathrm{Grp} \to \mathrm{AbGrp}$ (called abelianization) from the category of groups to the category of abelian groups. Compute the abelianizations of $\pi_1(M_g)$ and $\pi_1(N_g)$.

4. Conclude that none of the connected closed surfaces M_g for $g \geq 0$ or N_g for $g \geq 1$ are homeomorphic (or even homotopy equivalent) to each other.

Remark: Hatcher talks about this on pages 51-52; the point here is to learn all of the details.
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Proof.

(a) For $g = 1$ we have the fundamental group of the torus which is given by $\langle a_1, b_1 \mid a_1 b_1 a_1^{-1} b_1^{-1} \rangle$, which confirms our base case. Then, we assume this is true up to $g-1$, and we show it is true for

g. We then construct M_g from M_{g-1} by taking $M_{g-1}\#M_1$. See the diagram below.

Hence, $\pi_1(M_g) \cong \langle a_1, b_1, \ldots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \rangle$.

(b) For $g = 1$ we have the fundamental group of the projective plane which is given by $\langle a_1 \mid a_1^2 \rangle$. We then assume this is true up to $g-1$ and show it is true for g as well. We construct N_g by taking $N_{g-1}\#N_1$. See the diagram below.

Hence, $\pi_1(N_g) \cong \langle a_1, \ldots, a_g \mid a_1^2 a_2^2 \cdots a_g^2 \rangle$.

(c) The abelianization of a group G is the group $G/[G, G]$, where $[G, G]$ is the commutator subgroup. For $\pi_1(M_g)$, we find that $[\pi_1(M_g), \pi_1(M_g)]$ is generated by $aba^{-1}b^{-1}$ for $a, b \in \pi_1(M_g)$. Then we have

$$\begin{aligned}
\pi_1(M_g)/[\pi_1(M_g), \pi_1(M_g)] &\cong \langle a_1, b_1, \ldots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \rangle / [\pi_1(M_g), \pi_1(M_g)] \\
&\cong \langle a_1, b_1, \ldots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}, a_1 b_1 a_1^{-1} b_1^{-1}, a_1 b_2 a_1^{-1} b_2^{-1}, \\
&\ldots, a_1 b_g a_1^{-1} b_g^{-1}, a_2 b_1 a_2^{-1} b_1^{-1}, \ldots, a_2 b_g a_2^{-1} b_g^{-1}, \ldots, a_g b_g a_g^{-1} b_g^{-1} \rangle \\
&\cong \mathbb{Z}^{2g}.
\end{aligned}$$

Now we do the same process for $\pi_1(N_g)$ and we find that

$$\begin{aligned}
\pi_1(N_g)/[\pi_1(N_g), \pi_1(N_g)] &\cong \langle a_1, \ldots, a_g \mid a_1^2 \cdots a_g^2 \rangle / [\pi_1(N_g), \pi_1(N_g)] \\
&\cong \langle a_1, dots, a_g \mid a_1^2 \cdots a_g^2, a_1 a_2 a_1^{-1} a_2^{-1}, \ldots, a_1 a_g a_1^{-1} a_g^{-1} \rangle \\
&\cong \mathbb{Z}_2 \oplus \mathbb{Z}^{g-1}.
\end{aligned}$$

(d) To see that no M_g is homotopy equivalent to any other $M_{g'}$ with $g \neq g'$, note that the abelianization of $\pi_1(M_g)$ is the direct sum of $2g$ copies of $\mathbb{Z}$ and hence $2g' = 2g$. But this contradicts $g \neq g'$ and thus no M_g is homotopy equivalent to any other $M_{g'}$.

To see that no M_g is homotopy equivalent to any $N_{g'}$, just note that the abelianization of $\pi_1(N_{g'}$ is $\mathbb{Z}_2$ direct sum with $g' - 1$ copies of $\mathbb{Z}$. This is not isomorphic to any abelianization of $\pi_1(M_g)$ and thus no M_g is homotopy equivalent to any $N_{g'}$ even if $g = g'$. □

Problem 2. Hatcher Exercise 9 on page 79: Show that if a path-connected, locally path-connected space X has $\pi_1(X)$ finite, then every map $X \to S^1$ is nullhomotopic.

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Proof. Suppose X is path-connected and locally path-connected and then let $f\colon X \to S^1$ be an arbitrary map. These suppositions allow us to consider Proposition 1.33 and use this to prove our propoistion in this problem. Since $\pi_1(X)$ is finite, and $f_*\colon \pi_1(X) \to \pi_1(S^1)$ is an induced homomorphism, we know that $\ker f_* = \pi_1(X)$ since there can only be a trivial homomorphism between a group of finite order and a group with infinite order. Now, letting $\mathbb{R} = \tilde{X}$ be the universal cover of S^1, we get that $f_*(\pi_1(X)) \subset p_*(\pi_1(\mathbb{R}))$. This implies that $\exists \tilde{f}$ that lifts f to $\tilde{f}$ given by the following commutative diagram:

$$\begin{tikzcd}
 & \mathbb{R} \arrow[d, "p"] \\
X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & S^1
\end{tikzcd}$$

Then we know $f = p \circ \tilde{f}$. Note that any path γ in X lifts to a path in $\mathbb{R}$, and since $\mathbb{R}$ is contractible, we have that $\tilde{f}(\gamma)$ is contractible. Then $p(\tilde{f}(\gamma))$ is then contractible in S^1, which means that our arbitrary f is a nullhomotopic map, and hence we are done. □

Problem 3. Hatcher Exercise 10 on page 79: Find all the connected 2-sheeted and 3-sheeted covering spaces of $S^1 \vee S^1$ up to isomorphism of covering spaces without basepoints (defined on page 67). You do not need to prove your answer is correct.

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Proof. Here are the drawings of the different possibilities with the respective group presentation. □