

Geometric Algebra and Calculus on Riemannian Manifolds

Colin Roberts

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1 Clifford Bundles

What are the eigenspinors of laplace beltrami on a manifold? Can one show that with some Q the Clifford sections on M form a Banach algebra?

https://en.wikipedia.org/wiki/Clifford_algebra Look at "basis and dimension" to show that we will work with orthonormal frames. Let (M, Q) be an m -dimensional smooth manifold with a quadratic form Q on the tangent bundle. By polarisation, this induces

$$g(u, v) = \frac{1}{2}(Q(u + v) - Q(u) - Q(v)).$$

Thus any quadratic manifold is a Riemannian manifold (M, g) . Then we can consider an induced Clifford bundle $Cl(TM, g)$ on M with the quadratic form $Q(v) = g(v, v)$. Other sources make the choice of $Q(v) = -g(v, v)$ in order to mimic the complex and quaternionic fields.

Specifically, we define $Cl(T_p M, g_p)$ to be the Clifford algebra on the tangent space $T_p M$ and let

$$Cl(TM, g) := \bigcup_{p \in M} Cl(T_p M, g_p)$$

be the Clifford bundle.

We can then define the space of Clifford sections by noting we have a natural projection

$$\pi: Cl(TM, g) \rightarrow M$$

that maps a Clifford element to the point at which it is based and putting

$$\Gamma Cl(TM, g) := \{\sigma: M \rightarrow Cl(TM, g) \mid \pi \circ \sigma = \text{Id}_M\}.$$

How do we define smoothness here? Maybe just consider a section of the tensor algebra. Then we can show the smoothness and algebra properties from a quotient map from there.

Definition 1.1. Define a k -blade field as a smooth section of

$$\odot^k(TM)$$

Proposition 1.1. *The space $\Gamma Cl(TM, g)$ forms an algebra.*

Proposition 1.2. *There exists a norm on $\Gamma Cl(TM, g)$ via involution (multiplication by the pseudoscalar).*

Proposition 1.3. *With this norm, $\Gamma Cl(TM, g)$ is a Banach algebra.*

Proposition 1.4. *With (blah) we have that $\Gamma Cl(TM, g)$ is a Banach $*$ -algebra.*

We need that the algebra of clifford sections is a Banach $*$ -algebra. https://en.wikipedia.org/wiki/Gelfand%E2%80%93Naimark_theorem

https://en.wikipedia.org/wiki/Banach_algebra#Spectral_theory This ties up the knot for the relationship of different notions of spectra.

Can we define an inner product on the clifford algebra in a natural way? <https://math.stackexchange.com/questions/2606319/is-the-natural-norm-on-the-exterior-algebra-submult>. That is, multiply the components of rank n in a meaningful way like in the link.

Relationship to frame bundles and flags?

Two versions of a Clifford norm and mention of $Cl(p, q, r)$ <https://math.stackexchange.com/questions/1128844/about-the-definition-of-norm-in-clifford-algebra>

Also this <https://mathoverflow.net/questions/176140/norms-on-clifford-algebra-c-norm>

I feel like a good Ph.D. project would be to fucking make a single source that has information on clifford algebras on manifolds with all the relations to function algebras and such.

This is cool <https://www.jstor.org/stable/pdf/1970397.pdf>

Clifford homology/cohomology/elliptic complexes with dirac operator (derivative)

How do we define this dirac operator in a coordinate free way? Will it be able to give us $\text{Im} \subseteq \text{Ker}$? Maybe there is some way to say "yes, up to non-harmonic functions" as D^2 is a k-blade laplacian.

This definition is essentially defined throughout [1]

Definition 1.2. Define

$$D_L = \sum_{i=1}^n e_i \frac{\partial}{\partial x^i}$$

and

$$D_R = \sum_{i=1}^n \frac{\partial}{\partial x^i} e_i$$

Definition 1.3. Define the *exterior derivative* $D \wedge$ by

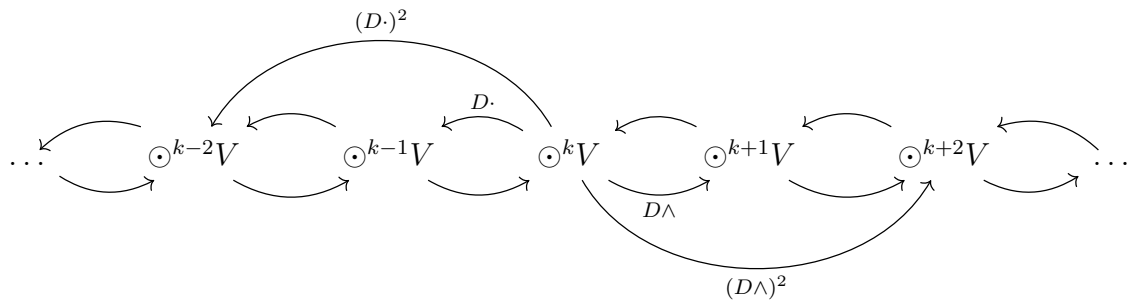
$$\frac{1}{2} (D_L - D_R)$$

and the *interior derivative* $D \cdot$ by

$$\frac{1}{2} (D_L + D_R).$$

And again, in [1], it's shown that

$$(D\cdot)^2 = 0 = (D\wedge)^2.$$



probabilistic (norm one) clifford sections?

left/right dirac operators

in lectures on clifford algebras - clifford analysis lecture, there is a generalization of cauchy integral formula

2 C^* -Algebras

References

- [1] Chris Doran. *Geometric Algebra for Physicists*. Cambridge University Press, 2003.