MATH 519, Homework 1

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Solutions

Problem 1. Use the CREs to show that $f(z) = e^{-y}\sin x - ie^{-y}\cos x$ is entire.
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Proof. We must show that $f(z) = f(x,y) = u(x,y) + iv(x,y)$ from above satisfies $u_x = v_y$ and $u_y = -v_x$ for all $z \in \mathbb{C}$. We have

$$\begin{aligned} u_x &= e^{-y}\cos x \\ u_y &= -e^{-y}\sin x \\ v_x &= e^{-y}\sin x \\ v_y &= e^{-y}\cos x. \end{aligned}$$

This shows that $u_x = v_y$ and $u_y = -v_x$, and thus f is entire. □

Problem 2. Where are the Cauchy-Riemann equations satisfied for $g(z) = z\Im(z)$?
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Proof. First we write this in terms of x and y so we get

$$\begin{aligned} & g(z) = z\Im(z) \\ \implies & g(x,y) = (x+iy)(y) \\ & \qquad\quad = x^2 + iy^2. \end{aligned}$$

This gives us that

$$\begin{aligned} u(x,y) &= x^2 \\ v(x,y) &= y^2. \end{aligned}$$

Then, taking the partial derivatives,

$$\begin{aligned} u_x &= 2x \\ u_y &= 0 \\ v_x &= 0 \\ v_y &= 2y. \end{aligned}$$

Then, asserting that $u_x = v_y$ and $u_y = -v_x$, we find that we must have $x = y$. Thus the CREs are satisfied only when $x = y$. □

Problem 3. S&S 1.1. Describe geometrically the sets of points z in the complex plane defined by the following relations:

(a) $|z - z_1| = |z - z_2|$ where $z_1, z_2 \in \mathbb{C}$.

(b) $1/z = \overline{z}$.

(c) $\Re(z) = 3$.

(d) $\Re(z) > c$, (resp., $\geq c$) where $c \in \mathbb{R}$.

(e) $\Re(az + b) > 0$ where $a, b \in \mathbb{C}$.

(f) $|z| = \Re(z) + 1$.

(g) $\Im(z) = c$ with $c \in \mathbb{R}$.

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Proof.

(a) This is the set of points that are equidistant from z_1 and z_2. In fact, the set is a line of points that passes through the midpoint of the two points z_1 and z_2. The line will have slope that is orthogonal to the line between z_1 and z_2.

(b) This is the unit circle.

(c) This is the vertical line that passes through 3 on the real axis.

(d) This is all complex numbers that are to the right of the vertical line passing through c on the real line, but not including the line passing through c itself (except when we allow for $\geq c$).

(e) Note that $az + b$ is an affine translation of the complex plane. b moves the origin, and a scales and rotates the plane. Now, since we just want the real part of this to be positive, we just need the real part of az and b to both be positive. Then the way a affects the set of points z that satisfy $\Re(az + b)$ is a bit more complicated. But what will happen is we will end up with an open half plane that is rotated by the argument of a (i.e., $a = re^{i\theta}$ and $\arg(a) = \theta$) and translated by b.

(f) Here we have that $z = x + iy$ and that

$$\begin{aligned} |z|^2 &= (x+1)^2 \\ \implies x^2 + y^2 &= x^2 + 2x + 1 \\ \implies y^2 &= 2x + 1. \end{aligned}$$

This is a parabola.

(g) This is a horizontal line that is c units above the real axis. $\square$

Problem 4. S&S 1.3. With $\omega = se^{i\varphi}$, where $s \geq 0$ and $\varphi \in \mathbb{R}$, solve the equation $z^n = \omega$ in $\mathbb{C}$ where n is a natural number. How many solutions are there?

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Proof. There are n unique solutions. We note that if $z = \sqrt[n]{s}e^{i\varphi/n}$ then $z^n = \omega$. However, we also have that $z = \sqrt[n]{s}e^{i\left(\frac{\varphi}{n}+\frac{2\pi ik}{n}\right)}$ for $k = 0, 1, \ldots, n-1$ are solutions (with the $k = 0$ being the first case I mentioned). □

Problem 5. S&S 1.10. Show that

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} = 4\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} = \Delta,$$

where Δ is the **Laplacian**

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

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Proof. We have that

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right).$$

Then we have

$$\begin{aligned}4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} &= 4\frac{1}{4}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)\\ &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\\ &= \Delta\\ &= 4\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z}.\end{aligned}$$

Note the last equality is due to commutivity. □

Problem 6. S&S 1.11. Use Exercise 10 to prove that if f is holomorphic in the open set Ω, then the real and imaginary parts of f are **harmonic**; that is, their Laplacian is zero.

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Proof. If f is holomorphic on Ω, then for $z_0 \in \Omega$, $\frac{\partial f}{\partial \overline{z}}(z_0) = 0$. Thus $\Delta = 0$ since $\frac{\partial}{\partial \overline{z}} = 0$. □

Problem 7. S&S 1.13ab. Suppose that f is holomorphic in an open set Ω. Prove that in any one of the following cases:

(a) $\Re(f)$ is constant;

(b) $\Im(f)$ is constant;

one can conclude that f is constant.

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Proof. Use the CREs. We have $f(x, y) = u(x, y) + iv(x, y)$. If $\Re(f)$ is constant, then $v_x = v_y = 0$ and hence $u_x = u_y = 0$ and thus f is constant. The proof for (b) is analogous. $\square$

Problem 8. S&S 1.24. Let γ be a smooth curve in $\mathbb{C}$ parametrized by $z(t)\colon [a, b] \to \mathbb{C}$. Let γ^- denote the curve with the same image as γ but with the reverse orientation. Prove that for any continuous function f on γ

$$\int_\gamma f(z)dz = -\int_\gamma f(z)dz.$$

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Proof. We have

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_a^b f(z(t))z'(t)dt \\
&= \int_a^b f(z(b+a-t))(-z'(b+a-t))dt \\
&= -\int_a^b f(z(b+a-t))z'(b+a-t)dt \\
&= -\int_{\gamma^-} f(z)dz.
\end{aligned}$$

$\square$