Clifford Analysis and a Noncommutative Gelfand Representation

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Overview

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Section 1

Introduction

Subsection 1

Motivation

Electrical Impedance Tomography

Electrical Impedance Tomography (EIT) asks whether one can determine the conductivity of a medium based on measurements along the boundary.

Calderón problem

- Let M be a smooth, connected, oriented Riemannian manifold with boundary ∂M with metric g.
- Conductivity is represented by g.
- Forward problem: Let $\Delta u = 0$ in M and $u = \phi$ on ∂M.
- Inverse problem: Given the ***Dirichlet-to-Neumann map*** $\Lambda\phi = \frac{\partial u}{\partial \nu}$, can we recover (M, g)?

Subsection 2

Preliminaries

- ***Clifford algebra*** originated in 1878 with William Kingdon Clifford's work that extends Hermann Grassmann's ***exterior algebra***.
- ***Clifford analysis*** arrived in the 1980's due to Hestenes, Sobczyk, Sommen, Brackx, and Delenghe in order to enrich Éllie Cartan's ***differential forms***.
  - Atiyah-Singer Dirac operator and spin manifolds.

Clifford algebras

Let V be a vector space over a field $\mathbb{F}$ with quadratic form Q.

- Define the tensor algebra

$$\mathcal{T}(V) \coloneqq \bigoplus_{j=0}^{\infty} V^{\otimes j} = \mathbb{F} \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots.$$

- Form the ***Clifford algebra*** via a quotient

$$C\ell(V, Q) \coloneqq \mathcal{T}(V) / \langle \mathbf{v} \otimes \mathbf{v} - Q(V) \rangle.$$

Geometric and exterior algebras

Let V be a vector space over a field $\mathbb{F}$ with quadratic form Q.

- Given a (pseudo) inner product g, we set $Q(\cdot) = g(\cdot, \cdot)$ and define a ***geometric algebra***

$$\mathcal{G} := C\ell(V, g).$$

- The ***exterior algebra*** is given by

$$\bigwedge(V) := C\ell(V, 0).$$

Algebra structure

We define a multiplication in $\mathcal{G}$ by noting how the product $\otimes$ acts in the quotient.

- Given $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{G}$ we can take the product

$$\boldsymbol{u}\boldsymbol{v} = \underbrace{\boldsymbol{u} \cdot \boldsymbol{v}}_{\text{scalar}} + \underbrace{\boldsymbol{u} \wedge \boldsymbol{v}}_{\text{bivector}} .$$

  - The scalar part is symmetric: $\boldsymbol{u} \cdot \boldsymbol{v} = g(\boldsymbol{u}, \boldsymbol{v})$.
  - The bivector part is antisymmetric: $\boldsymbol{u} \wedge \boldsymbol{v} = -\boldsymbol{v} \wedge \boldsymbol{u}$.

Multivectors

- $\mathcal{G}$ is graded and of dimension 2^n.
  - There are $\binom{n}{r}$ elements of grade r called ***r-vectors***.
  - Those that are exterior products of r independent vectors are ***r-blades***.
    E.g., $\boldsymbol{A_r} = \boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_r$.
- The most general elements are ***multivectors*** and are given by

$$A = \sum_{r=0}^{n} \langle A \rangle_r,$$

where $\langle A \rangle_r \in \mathcal{G}^r$ extracts the grade r part of A.

Algebra Structure

Extend the multiplication from vectors to multivectors by

$$A_r B_s = \langle A_r B_s \rangle_{|r-s|} + \langle A_r B_s \rangle_{|r-s|+2} + \cdots + \langle A_r B_s \rangle_{r+s}$$

with

$$A_r \cdot B_s \coloneqq \langle A_r B_s \rangle_{|r-s|} \qquad A_r \wedge B_s \coloneqq \langle A_r B_s \rangle_{r+s}$$

$$A_r \rfloor B_s \coloneqq \langle A_r B_s \rangle_{s-r} \qquad A_r \lfloor B_s \coloneqq \langle A_r B_s \rangle_{r-s}$$

Reciprocals

Given any vector basis $\boldsymbol{v}_i$ we define the ***reciprocal vectors*** by $\boldsymbol{v}^i \cdot \boldsymbol{v}_j = \delta^i_j$.

Reverse

The ***reverse*** of a multivector is extended linearly from the action on r-blades by

$$\boldsymbol{A_r}^{\dagger} = (\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_r)^{\dagger} = \boldsymbol{v}_r \wedge \cdots \wedge \boldsymbol{v}_1.$$

Inner product and norm

We define the ***multivector inner product*** by

$$(A, B) \coloneqq \left\langle A^\dagger B \right\rangle$$

which is bilinear, symmetric, and positive definite if g is positive definite. Then we define the ***multivector norm*** by

$$|A| \coloneqq \sqrt{(A, A)}.$$

Adjoint

Note the reverse acts as an adjoint by

$$(CA, B) = (A, C^\dagger B) \tag{1}$$

$$(AC, B) = (A, BC^\dagger). \tag{2}$$

Pseudoscalars

Pseudoscalars are the grade-n elements. For example, $\boldsymbol{\mu} = \boldsymbol{v}_1 \wedge \boldsymbol{v}_n$. We define the ***unit pseudoscalar*** by

$$\boldsymbol{I} \coloneqq \frac{1}{|\boldsymbol{\mu}|}\boldsymbol{\mu}.$$

Blades and subspaces

If $|\boldsymbol{A_r}| = 1$, then $\boldsymbol{A_r}$ is a ***unit blade***.

All unit r-blades correspond to an r-dimensional subspace and can be identified with points in $\mathrm{Gr}(r, n)$.

Duality

Given any multivector A, we can take its ***dual***

$$A^{\perp} \coloneqq A\boldsymbol{I}^{-1}.$$

Note $A_r^{\perp} \in \mathcal{G}^{n-r}$.

Projection and rejection

We can define the *projection* of B into a subspace $\boldsymbol{A_r}$ by

$$\mathrm{P}_{\boldsymbol{A_r}}(B) \coloneqq B \rfloor \boldsymbol{A_r} \boldsymbol{A_r}^{-1}$$

and the *rejection* by

$$\mathrm{R}_{\boldsymbol{A_r}}(B) \coloneqq B \wedge \boldsymbol{A_r} \boldsymbol{A_r}^{-1}.$$

Both are grade preserving.

Complex Numbers

Do a more thorough example like in my thesis to wrap everything up Maybe it is worth including the hermitian inner product example in both?

Claim: $\mathbb{C}$ arises naturally as the even subalgebra $\mathcal{G}_2^+$.

Take the standard basis $\boldsymbol{e}_1, \boldsymbol{e}_2$, and define $\boldsymbol{B}_{12} = \boldsymbol{e}_1 \boldsymbol{e}_2$ and note $\boldsymbol{B}_{12}^2 = -1$. Thus

$$(u_1 + v_1 \boldsymbol{B}_{12})(u_2 + v_2 \boldsymbol{B}_{12}) = u_1 u_2 - v_1 v_2 + (u_1 v_2 + u_2 v_1)\boldsymbol{B}_{12}.$$

Moreover, right multiplication of vectors by $\boldsymbol{B}_{12}$ rotates counter-clockwise by $\pi/2$.

Examples

Claim: The quaternion algebra arises naturally inside the even subalgebra $\mathcal{G}_3^+$.

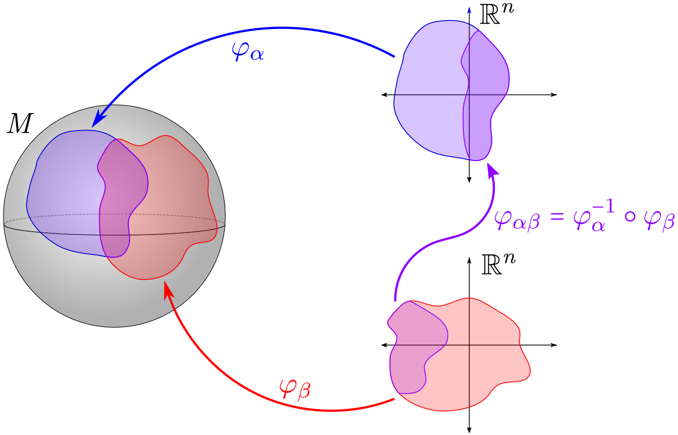
Claim: The spacetime algebra is $\mathcal{G}_{3,1}$.

Subsection 3

Manifolds and fields

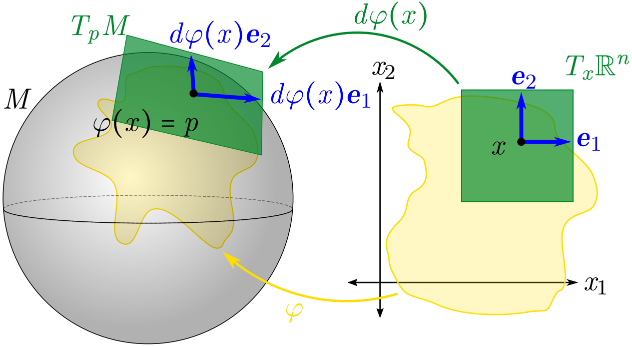
The playing field

We let M be a smooth, compact, connected, and oriented n-dimensional Riemannian manifold with metric g (unless otherwise stated).



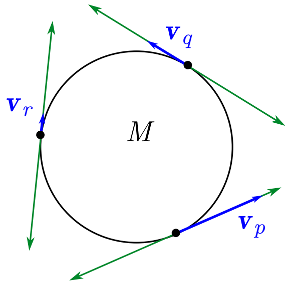
The playing field

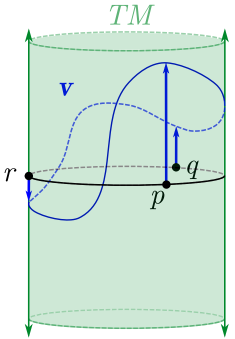
At each point on M, we have the tangent space T_pM.



The playing field

From M, we create the tangent bundle TM whose sections are vector fields.





Idea: Form the Clifford algebras on tangent spaces.

- Each $C\ell(T_pM, g_p)$ is a ***geometric tangent space*** which we glue together to form
$$C\ell(TM, g) \coloneqq \bigsqcup_{p \in M} C\ell(T_pM, g_p).$$
- The space of ***(smooth) multivector fields*** is
$$\mathcal{G}(M) \coloneqq \{C^\infty\text{-smooth sections of } C\ell(TM, g)\}.$$

Section 2

Clifford analysis

Subsection 1

Differentiation

Covariant derivative

On M we have the unique torsion free Levi-Civita connection ∇ and covariant derivative $\nabla_{\boldsymbol{u}}$.

- $\nabla_{\boldsymbol{u}}$ can be extended to multivectors Explain briefly how and it is grade preserving
$$\nabla_{\boldsymbol{u}} A_r = \langle \nabla_{\boldsymbol{u}} A_r \rangle_r.$$
- $\nabla_{\boldsymbol{u}}$ is compatible with dot and wedge since
$$\nabla_{\boldsymbol{u}}(A \cdot B) = (\nabla_{\boldsymbol{u}} A) \cdot B + A \cdot (\nabla_{\boldsymbol{u}} B)$$
$$\nabla_{\boldsymbol{u}}(A \wedge B) = (\nabla_{\boldsymbol{u}} A) \wedge B + A \wedge (\nabla_{\boldsymbol{u}} B).$$

Gradient

We define the ***gradient*** (or ***Dirac operator***) in some local basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ by

$$\boldsymbol{\nabla} = \sum_{i=1}^{n} \boldsymbol{v}^i \nabla_{\boldsymbol{v}_i}$$

Note that this has the algebraic properties of a vector in $\mathcal{G}(M)$.

Gradient

We define the ***gradient*** (or ***Dirac operator***) in some local basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ by

$$\nabla = \sum_{i=1}^{n} \boldsymbol{v}^i \nabla_{\boldsymbol{v}_i}$$

Note that this has the algebraic properties of a vector in $\mathcal{G}(M)$ and has the Leibniz rule

$$\nabla(AB) = \dot{\nabla}\dot{A}B + \dot{\nabla}A\dot{B}.$$

Subsection 2

Integration

Differential forms

We define the *r-dimensional directed measure*

$$dX_r \coloneqq \boldsymbol{v}_{j_1} \wedge \cdots \boldsymbol{v}_{j_r} dx^1 \cdots dx^n$$

where $1 \leq j_1 < \cdots < j_r \leq n$ is an increasing set of indices. This allows us to define an r-form α_r by

$$\alpha_r = A_r \cdot dX_k^\dagger$$

where $A_r = \frac{1}{r!} \alpha_{i_1 \dots i_r} \boldsymbol{v}^{i_1} \wedge \cdots \wedge \boldsymbol{v}^{i_r}$. We call A_r the *multivector equivalent* of α_r.

Volume form

The ***volume form*** on M is given in local coordinates by

$$\mu = \sqrt{|g|}dx^1 \cdots dx^n = \boldsymbol{I}^{-1} \cdot dX_n$$

Hence, we can integrate scalar fields A_0 on M by

$$\int_M A_0^{\perp} \cdot dX_n = \int_M A_0 \mu.$$

Exterior algebra

Given an r- and s-form α_r and β_s we have

$$\alpha_r + \beta_s = (A_r + B_s) \cdot dX_r^\dagger$$

$$\alpha_r \wedge \beta_s = (A_r \wedge B_s) \cdot dX_{r+s}^\dagger.$$

Exterior derivative

The exterior derivative on multivector equivalents is

$$d\alpha_r = (\nabla \wedge A_r) \cdot dX_{r+1}^{\dagger}$$

Hodge star

The Hodge star on multivector equivalents is

$$\star\alpha_r = (\boldsymbol{I}^{-1} A_r)^\dagger \cdot dX_{n-r}^\dagger$$

Multivector field inner product

We define an inner product on multivector fields by

$$\ll A, B \gg \coloneqq \frac{1}{\mathrm{vol}(M)} \int_M (A, B)\mu$$

which realizes the r-form inner product

$$\int_M \alpha_r \wedge \star \beta_r = \int_M \langle A_r \dagger B_r \rangle \mu = \mathrm{vol}(M) \ll A, B \gg .$$

Remark: By definition of the multivector inner product, A_r and B_s are orthogonal when $r \neq s$ so this agrees with the grade direct sum $\oplus$ – we use the same notation for both.

Boundary

On the boundary ∂M, we have the boundary pseudoscalar $\boldsymbol{I}_\partial$ and the boundary normal $\boldsymbol{\nu} = \boldsymbol{I}_\partial^\perp$. Then

$$\mu_\partial \coloneqq \boldsymbol{I}_\partial^{-1} \cdot dX_{n-1}$$

and we define

$$\ll A, B \gg_\partial \coloneqq \frac{1}{\mathrm{vol}(M)} \int_{\partial M} (A, B) \mu_\partial.$$

Multivector valued integrals

We can define a multivector valued integral on an oriented submanifold R by taking $A \in \mathcal{G}(M)$ and computing

$$\int_R A\boldsymbol{I}_R \mu_R.$$

Subsection 3

Clifford-Hodge-Morrey decomposition

Fundamental theorems of geometric calculus

Let $A, B \in \mathcal{G}(M)$, then

$$
\begin{aligned}
\int_M \dot{A} \dot{\nabla} \boldsymbol{I} \mu &= \int_{\partial M} A \boldsymbol{I}_\partial \mu_\partial \\
\int_M \boldsymbol{I} \nabla B \mu &= \int_{\partial M} \boldsymbol{I}_\partial B \mu_\partial \\
\int_M \dot{A} \dot{\nabla} \boldsymbol{I} B \mu &= (-1)^n \int_M A \boldsymbol{I} \nabla B \mu + \int_{\partial M} A \boldsymbol{I}_\partial B \mu_\partial .
\end{aligned}
$$

Theorem: (Multivector Green's formula)

We have the Green's formula for the gradient

$$\ll A, \boldsymbol{I} \nabla B \gg = (-1)^n \ll \nabla A, \boldsymbol{I} B \gg + \ll A, \boldsymbol{I}_\partial B \gg_\partial .$$

Proof

Fix $A^\dagger, B \in \mathcal{G}(M)$ and note that

$$\begin{aligned}\int_M A^\dagger \boldsymbol{I} \nabla B \mu &= (-1)^n \int_M \dot{A}^\dagger \dot{\nabla} \boldsymbol{I} B \mu + \int_{\partial M} A^\dagger \boldsymbol{I}_\partial B \mu_\partial \\ &= (-1)^n \int_M (\nabla A)^\dagger \boldsymbol{I} B \mu + \int_{\partial M} A^\dagger \boldsymbol{I}_\partial B \mu_\partial .\end{aligned}$$

Then, take the scalar part and divide by $\mathrm{vol}(M)$ to find

$$\ll A, \boldsymbol{I} \nabla B \gg = (-1)^n \ll \nabla A, \boldsymbol{I} B \gg + \ll A, \boldsymbol{I}_\partial B \gg_\partial .$$

Monogenic fields

Let $A \in \mathcal{G}(M)$. Then we say that A is ***monogenic*** if $\nabla A = 0$. We denote the space of monogenic fields by $\mathcal{M}(M)$.

Holomorphic functions

Take the coordinates x and y and let $f = u + v\boldsymbol{B} \in \mathcal{G}_2(\mathbb{R}^2)$ then $\nabla f = 0$ yields the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Cauchy integral

For M a domain in $\mathbb{R}^n$ with $n \geq 2$, we have the vector valued field

$$E(x) \coloneqq \frac{1}{S_n} \frac{x}{|x|^n}$$

where S_n is the surface area of the unit ball. Note

$$\nabla E(x) = -\dot{E}(x)\dot{\nabla} = \delta(x).$$

We then define the ***Cauchy kernel*** by $G(x, x') \coloneqq E(x' - x)$.

Cauchy integral

If $A \in \mathcal{M}(M)$, then we have the ***Cauchy integral formula***

$$A(x) = (-1)^n \boldsymbol{I}^{-1} \int_{\partial M} G(x, x') \boldsymbol{I}_\partial(x') A(x') \mu_\partial(x').$$

This allows us to uniquely determine a monogenic field from boundary values $A|_{\partial M}$.

Lemma

Let $A \in \mathcal{M}(M)$ be such that $A|_{\partial M} = 0$. Then $A = 0$ on all of M. *Proof sketch:* Utilize the Cauchy integral formula for A to deduce that $A = 0$ on M.

Lemma

Fix a multivector field $A \in \mathcal{G}(M)$. If

$$\ll A, B \gg = 0$$

for all $B \in \mathcal{G}(M)$ with $B|_{\partial M} = 0$, then $A = 0$.

Proof sketch:

- Use mollifiers to smooth indicator functions χ_U on open subsets U to be supported only on closed ϵ neighborhood $\overline{U^\epsilon}$. Call these functions χ_U^ϵ.
- Write $A = \sum_J A_J \boldsymbol{V}^J$ with $\boldsymbol{V}^J = \boldsymbol{v}^{j_1} \wedge \cdots \wedge \boldsymbol{v}^{j_r}$. Then note

$$\ll A, A_J \boldsymbol{V}_J \chi_U^\epsilon \gg = 0$$

  implies $A_J = 0$ on U^ϵ for all J since $(\boldsymbol{V}^J, \boldsymbol{V}_K) = \delta_K^J$. Hence $A = 0$ on U^ϵ.
- Cover M in such U^ϵ and repeat the argument leaving the $A|_{\partial M}$ undetermined. But, by smoothness of A, $A = 0$ on M.

Section 3

Gelfand theory

Section 4

Conclusions

Other projects

Data assimilation.