ColoState Spring 2018 MATH 617 Assignment 2

Due Fri. 03/23/2018

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(20 points) *Problem 1.* Let $(X, \mathcal{S})$ be a measurable space and $\langle \mu_n \rangle_{n\in\mathbb{N}}$ a sequence of measures such that for any $E \in \mathcal{S}$, we have $\mu_n(E) \le \mu_{n+1}(E)$. For any $E \in \mathcal{S}$, define $\mu(E) := \lim_{n\to\infty} \mu_n(E)$. Prove that μ is a measure on $\mathcal{S}$.

(20 points) *Problem 2.* Let $(\mathbb{R}, \mathcal{L}, \lambda)$ be the Lebesgue measure space and $A \in \mathcal{L}$ be a measurable bounded set with $\lambda(A) > 0$. Prove that for any $0 < b < \lambda(A)$, there exists a $B \in \mathcal{L}$ such that $B \subset A$ and $\lambda(B) = b$. *Hint: Assume $A \subseteq [-a, a]$. Apply the Intermediate Value Theorem.*

(20 points) *Problem 3.* Let $f(x)$ be a continuous real-valued function defined on a closed finite interval $[a, b]$. Prove that

(i) f is a bounded measurable function;

(ii) $f \in L_1[a, b]$.

(20 points) *Problem 4.* Textbook p.141 Problem 5.3.23.

(20 points) *Problem 5.* Assume $(X, \mathcal{S}, \mu)$ is a complete measure space, $f \in L_1(X, \mathcal{S}, \mu)$. Prove that for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $E \in \mathcal{S}$ with $\mu(E) \le \delta$, we have $\int_E |f| d\mu < \epsilon$. (*Hint: First consider f is bounded. For the case that f is unbounded, construct a bounded monotone sequence that converges to f.*)

Problem 1. Let $(X, \mathcal{S})$ be a measurable space and $\langle \mu_n \rangle_{n \in \mathbb{N}}$ a sequence of measures such that for any $E \in \mathcal{S}$, we have $\mu_n(E) \leq \mu_{n+1}(E)$. For any $E \in \mathcal{S}$, define $\mu(E) := \lim_{n \to \infty} \mu_n(E)$. Prove that μ is a measure on $\mathcal{S}$.

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Proof. Note that since each μ_n is a measure and since for any $E \in \mathcal{S}$, $\mu_n(E) \leq \mu_{n+1}(E)$ that necessarily $\mu \colon \mathcal{S} \to [0, \infty)$. Now, to see that $\mu(\emptyset) = 0$, we show

$$\mu(\emptyset) = \lim_{n \to \infty} \mu_n(\emptyset) = \lim_{n \to \infty} 0 = 0.$$

Hence $\mu(\emptyset) = 0$. Now, we need to show that μ is countably additive so we let $A = \bigcup_{i=1}^{\infty} A_i$ be a countable union of disjoint sets $A_i \in \mathcal{S}$. Then note that

$$\mu(A) - \mu_1(A) = \sum_{n=1}^{\infty} (\mu_{n+1}(A) - \mu_n(A)).$$

Working with this, we see that

$$\begin{aligned}
\mu(A) - \mu_1(A) &= \sum_{n=1}^{\infty} (\mu_{n+1}(A) - \mu_n(A)) \\
&= \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} \mu_{n+1}(A_i) - \sum_{i=1}^{\infty} \mu_n(A_i) \right) \\
&= \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} (\mu_{n+1}(A_i) - \mu_n(A_i)) \\
&= \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} (\mu_{n+1}(A_i) - \mu_n(A_i)) && \text{since each term here is positive by } \mu_n(E) \leq \mu_{n+1}(E) \\
&= \sum_{i=1}^{\infty} (\mu(A_i) - \mu_1(A_i)) \\
\implies \mu(A) &= \sum_{i=1}^{\infty} \mu(A_i) && \text{since } \mu_1 \text{ is a measure.}
\end{aligned}$$

Hence, μ is countably additive and thus is a measure. $\square$

Problem 2. Let $(\mathbb{R}, \mathcal{L}, \lambda)$ be the Lebesgue measure space and $A \in \mathcal{L}$ be a measurable bounded set with $\lambda(A) > 0$. Prove that for any $0 < b < \lambda(A)$, there exists a $B \in \mathcal{L}$ such that $B \subset A$ and $\lambda(B) = b$.
Hint: Assume $A \subseteq [-a, a]$. Apply the Intermediate Value Theorem.

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Proof. Without loss of generality, we can assume $A \subseteq [-a, a]$ since translation does not affect measure and since A is bounded. Now consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto \lambda(A \cap (A + x))$. Note that since $\lambda(A) < \infty$, theorem 4.3.4 implies that f is a continuous function. We have

$$f(-2a) = \lambda(A \cap (A - 2a)) = \lambda(\emptyset) = 0$$

by construction as well as

$$f(0) = \lambda(A \cap (A + 0)) = \lambda(A).$$

By continuity of f, there exists $c \in (-2a, 0)$ such that $f(c) = b$. Then we have that $B = A \cap (A + c)$ is Lebesgue measurable and that $B \subset A$. $\square$

Problem 3. Let $f(x)$ be a continuous real-valued function defined on a closed finite interval $[a, b]$. Prove that

(i) f is a bounded measurable function;

(ii) $f \in L_1[a, b]$.

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Proof.

(i) To see that f is bounded note that the continuous image of a compact set is compact and that compact subsets of $\mathbb{R}$ are bounded.

To see that f is measurable, let $E \subseteq f([a, b])$. By outer regularity of λ, we know

$$\lambda(E) = \inf\{\lambda(U) : U \supseteq E \text{ with } U \text{ open}\}.$$

It's important to note that $[a, b]$ is open as a subset of $[a, b]$ in order for the case where $E = [a, b]$ to be understood. Now, note that

$$\lambda(f^{-1}(E)) = \inf\{\lambda(f^{-1}(U)) : U \supseteq E \text{ with } U \text{ open}\}.$$

Since the preimage of open sets is open under a continuous function, we have that $f^{-1}(U)$ is open for each open U and hence we have that $f^{-1}(E)$ must be measurable. Thus f is a measurable function. □

Problem 4. Let $f \in \mathbb{L}$. For $x \in X$ and $n \geq 1$, define

$$f_n(x) := \begin{cases} f(x) & \text{if } |f(x)| \leq n, \\ n & \text{if } f(x) > n, \\ -n & \text{if } f(x) < -n. \end{cases}$$

Prove the following:

(i) $f_n \in \mathbb{L}$ and $|f_n(x)| \leq n \ \forall n$ and $\forall x \in X$.

(ii) $\lim_{n\to\infty} f_n(x) = f(x) \ \forall x \in X$.

(iii) $\|f_n(x)\| := \min\{|f_n(x)|, n\} := (|f| \wedge n)(x)$ is an element of $\mathbb{L}^+$ and

$$\lim_{n\to\infty} \int \|f_n\| d\mu = \int \|f\| d\mu.$$

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Proof.

(i) Fix an arbitrary n_0 and an arbitrary x_0. Note that if $|f(x_0)| \leq n_0$ then we have $|f_{n_0}(x_0)| \leq n_0$. Now if $|f(x)| > n_0$ we have that $|f_{n_0}(x_0)| = n_0$ hence $|f_{n_0}(x_0)| \leq n_0$ for arbitrary n_0 and arbitrary x_0. Now, consider a measurable subset $E \subseteq \text{Image}(f_n(x)) \subseteq [-n, n]$. Note that we then have $f_n^{-1}(E) = f^{-1}(E)$ is measurable and hence $f_n(x) \in \mathbb{L}$.

(ii) Fix x. Then note that $\lim_{n\to\infty} f_n(x) = f_\infty(x)$ is defined so that $f_\infty(x) = f(x)$ if $|f(x)| \leq \infty$. Hence we have that $\lim_{n\to\infty} = f_\infty(x) = f(x)$.

(iii) First we show that $\|f_n(x)\|$ is in $\mathbb{L}^+$. To see this, note that if f_n is measurable, then $|f_n|$ is measurable. Then we have that $\|f_n(x)\|$ is a piecewise function where each piece is positive and measurable. So we have that $\|f_n(x)\| \in \mathbb{L}^+$.

Then note that $\{\|f_n(x)\|\}_{n\in\mathbb{N}}$ is clearly an increasing sequence of functions since $n < n+1$ and $|f_n(x)| < |f_{n+1}(x)|$ by definition (just note Image($f_n(x)$) from (i)). Now, we have that $\lim_{n\to\infty} f_n(x) = f(x)$ and so $\lim_{n\to\infty} \|f_n(x)\| \to \|f\|$ as well (see that $\min\{|f_\infty(x), \infty\} = \min\{|f_\infty(x)|\} = |f_\infty(x)| = \|f(x)\|$). So by the monotone convergence theorem (5.2.7) we have that

$$\int \|f\| d\mu = \lim_{n\to\infty} \int \|f_n\| d\mu. \qquad \square$$

Problem 5. Assume $(X, \mathcal{S}, \mu)$ is a complete measure space, $f \in L_1(X, \mathcal{S}, \mu)$. Prove that for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $E \in \mathcal{S}$ with $\mu(E) \leq \delta$, we have $\displaystyle\int_E |f| d\mu < \epsilon$. (*Hint: First consider f is bounded. For the case that f is unbounded, construct a bounded monotone sequence that converges to f.*)

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Proof. Since f is bounded we have that $|f(x)| \leq M$ for all $x \in X$. Fix $\epsilon > 0$ and let $\delta < \frac{\epsilon}{M}$. Consider any $E \in \mathcal{S}$ such that $\mu(E) \leq \delta$. Then

$$\begin{aligned}\int_E |f| d\mu &\leq \int_E M d\mu \\ &= M \int_E d\mu \\ &\leq M\delta \\ &< M\frac{\epsilon}{M} = \epsilon.\end{aligned}$$

Using the fact that f is integrable iff $|f|$ is integrable, we consider the case where $|f|$ is unbounded. Let $\{|f_n|\}_{n\in\mathbb{N}}$ be defined by $|f_n|(x) = (|f| \wedge n)(x)$ and note that $\{|f_n|\}_{n\in\mathbb{N}}$ is a bounded monotone sequence that converges to $|f|$ by Problem 4. Then note that $|f_n|$ is bounded by M so $|f_n| \leq M$. Fix $\epsilon > 0$, then we also have the ability to choose $E \in \mathcal{S}$ such that $\mu(E) \leq \delta$ with $\delta = \frac{\epsilon}{2M}$.

$$\begin{aligned}\int_E |f| d\mu &= \int_E |f| - |f_n| + |f_n| d\mu \\ &= \int_E |f| - |f_n| d\mu + \int_E |f_n| d\mu.\end{aligned}$$

Note that $\exists N \in \mathbb{N}$ such that for $n \geq N$ we have $\int_E |f| - |f_n| d\mu < \frac{\epsilon}{2}$ since $|f_n|$ converges monotonically to $|f|$. Hence

$$\begin{aligned}\int_E |f| d\mu &< \frac{\epsilon}{2} + \delta M \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

□