Riemannian Geometry

for Dummies

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Section 1

Introduction

Riemannian geometry is the study of a ***smooth manifold*** M along with a ***metric tensor field*** g.

The point of Riemmannian geometry is to generalize the differentiable and metric structure of $\mathbb{R}^n$.

We generalize to spaces that have interesting topology and geometry.

This will require us to rethink some notions we found “easy" in $\mathbb{R}^n$.

But we will gain a very general framework for working with differentiable objects.

Section 2

Motivation

Why study this in the first place?

Example: Partial differential equations (PDEs) on spaces that are not flat.

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- Fluid flow on Earth;
- Electrical Impedence Tomography (EIT);
- General relativity.

Example: Optimization in interesting spaces.

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- Grassmannians;

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- Grassmannians;
- Flags.

Section 3

Preliminaries

add more math text before/after pics so that people see some notation. More examples.

Subsection 1

Smooth Manifolds

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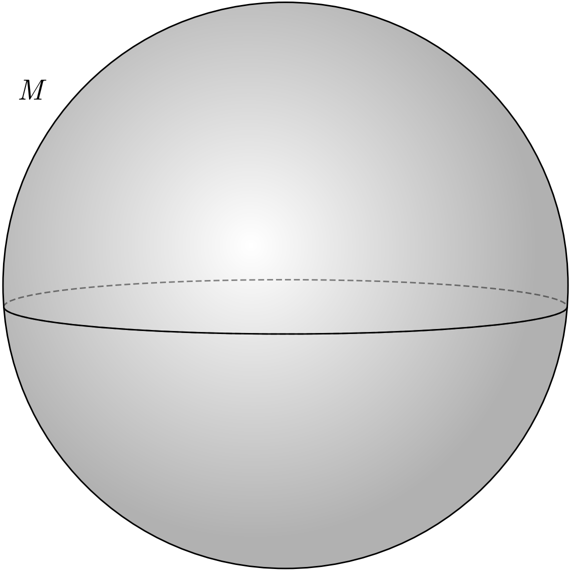
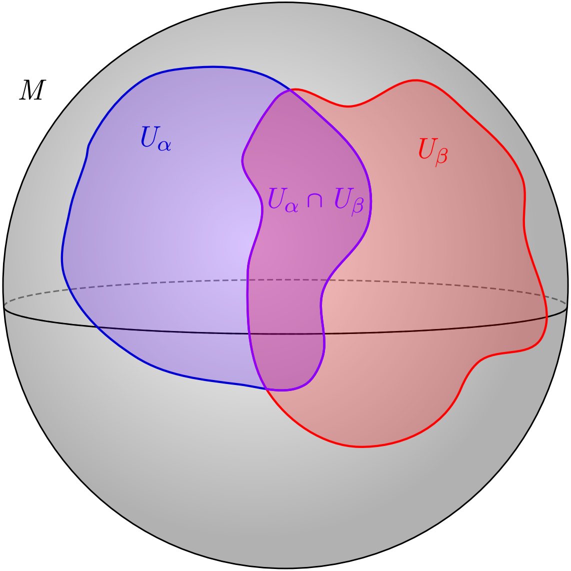
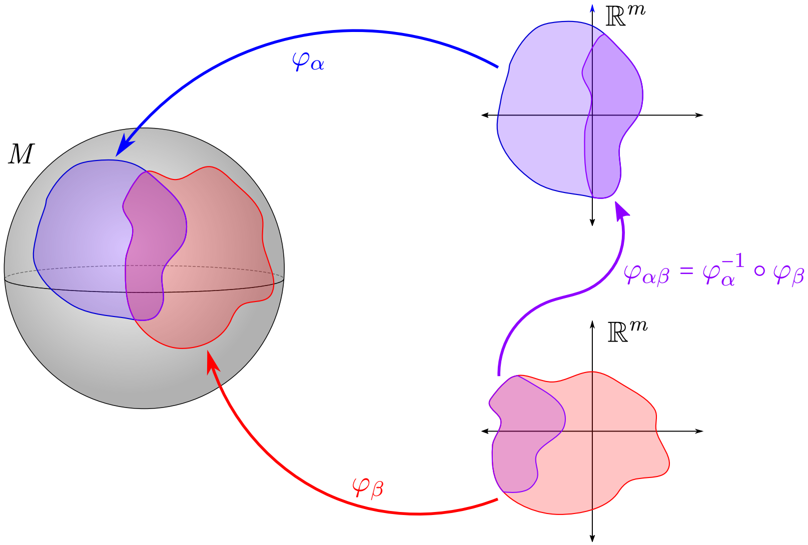
- Look at open sets U that cover M;
- Construct local coordinates φ;
- Show coordinate transition functions are smooth.

Define the sphere as the set of points in $\mathbb{R}^3$... then say we'll mostly use this as an example so keep it in mind


M

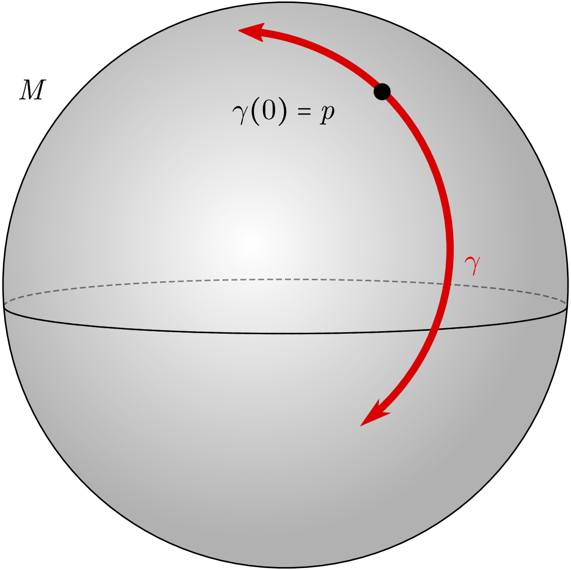

M
U_α
U_β
$U_\alpha \cap U_\beta$

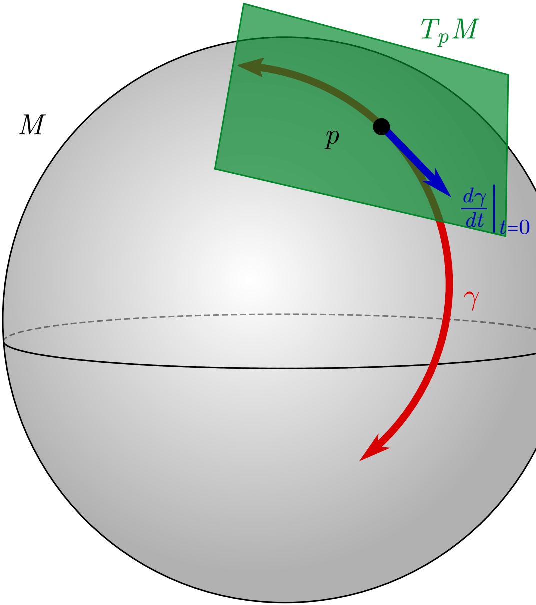


M
φ_α
$\mathbb{R}^m$
$\varphi_{\alpha\beta} = \varphi_\alpha^{-1} \circ \varphi_\beta$
$\mathbb{R}^m$
φ_β

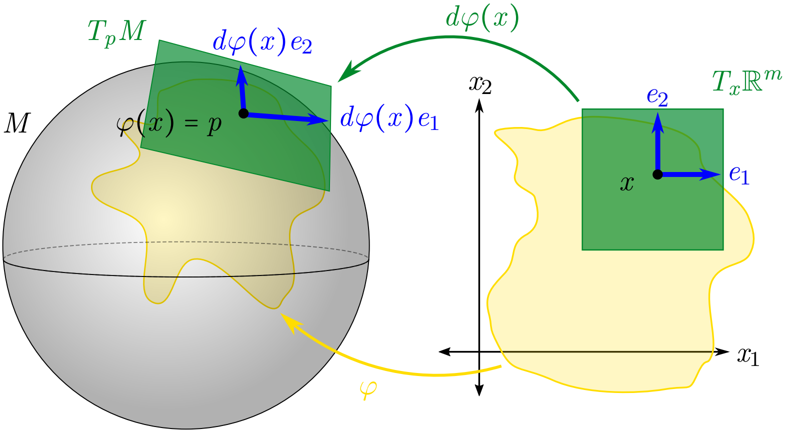
Subsection 2

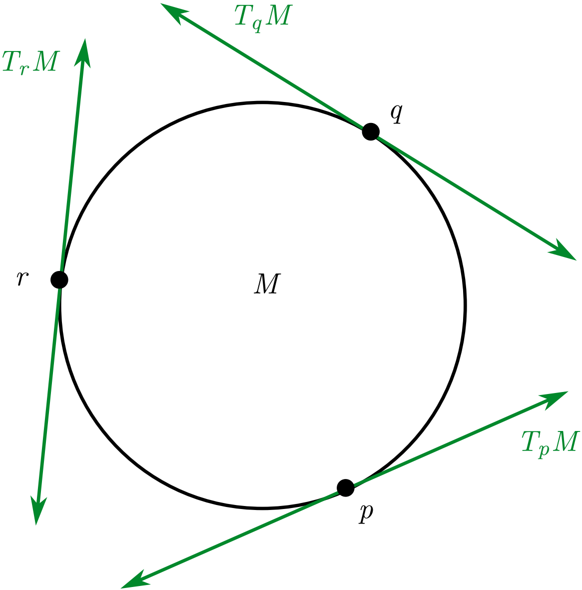
Vector Fields

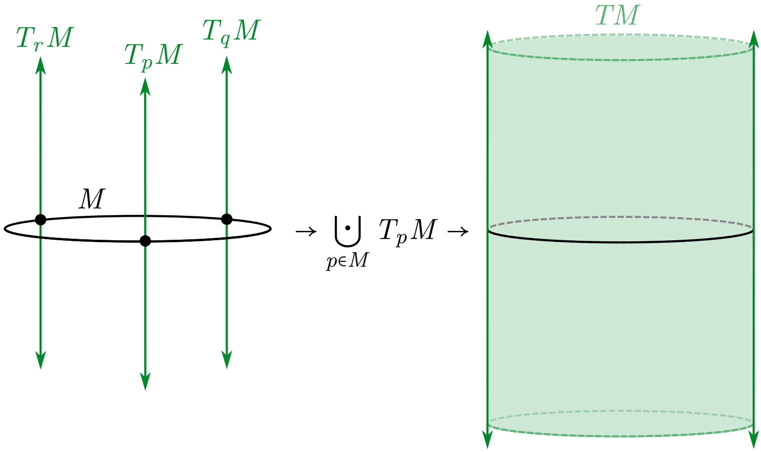

M
$\gamma(0) = p$
γ



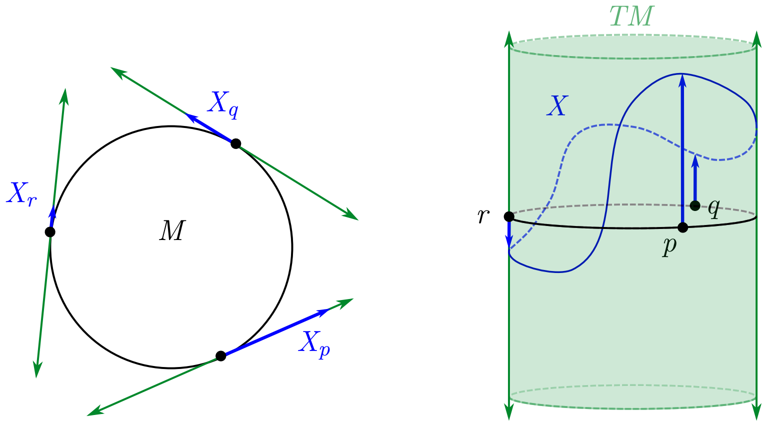
T_pM
M
p
$\frac{d\gamma}{dt}\Big|_{t=0}$
γ


$d\varphi(x)$
T_pM
$d\varphi(x)e_2$
M
$\varphi(x) = p$
$d\varphi(x)e_1$
$T_x\mathbb{R}^m$
x_2
e_2
x
e_1
x_1
φ


T_qM
T_rM
q
r
M
T_pM
p



$T_r M$
$T_p M$
$T_q M$
M
$\rightarrow \bigcupdot_{p \in M} T_p M \rightarrow$
TM

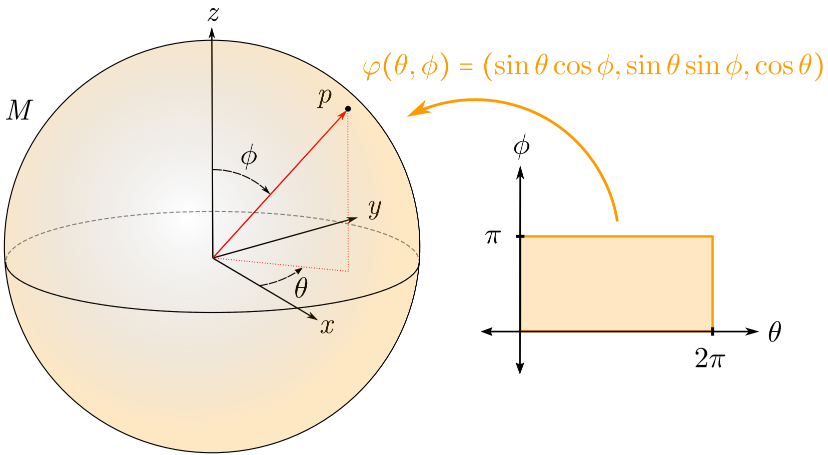


X_q
X_r
M
X_p
TM
X
r
q
p

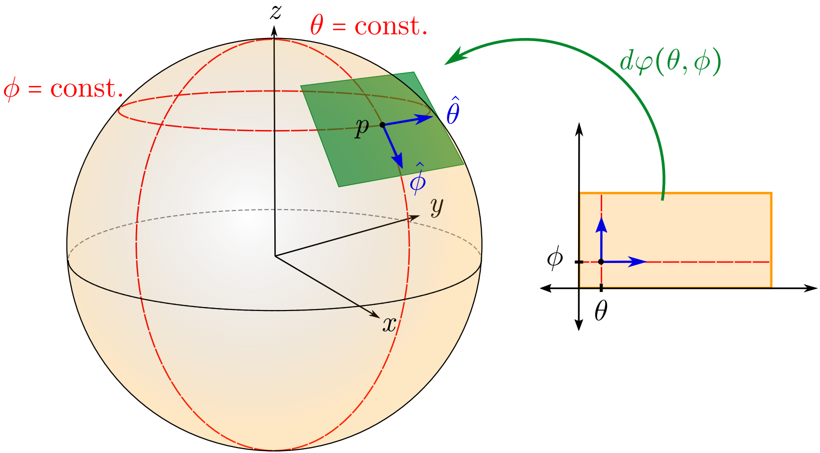
Subsection 3

Example

Example of charts and maps for sphere as well as a vector field. Do spherical coordinates



$\varphi(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$
z
M
p
ϕ
y
θ
x
ϕ
π
θ
2π



z
θ = const.
ϕ = const.
$d\varphi(\theta, \phi)$
p
$\hat{\theta}$
$\hat{\phi}$
y
x
ϕ
θ

Section 4

Riemannian Geometry

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- Have the inner product vary smoothly as we vary the point p;
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- Extract geometrical and analytical qualities of the underlying manifold M.

Subsection 1

Riemannian Metric

$$g_{ij}(x) = \varphi^*(x)e_i \cdot \varphi^*(x)e_k = (\partial_i\varphi) \cdot (\partial_j\varphi) = \sum_{k=1}^{m} \frac{\partial\varphi_k}{\partial x_i}\frac{\partial\varphi_k}{\partial x_j}$$

From minimization of length/energy. Both are good to mention. Geodesic equation

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0$$

equivalent to

$$\ddot{x}^l + \dot{x}^j\dot{x}^k\Gamma^l_{jk} = 0$$

which is saying that the only "acceleration" of the curve comes from the geometry it lies on. When flat space, $\Gamma^l_{jk} = 0$ and we have $\ddot{x} = 0$.

Section 5

Applications

Section 6

Conclusions