MATH 560, Homework 2

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August 31, 2017

Solutions

Problem 1. Prove that a map between topological spaces is continuous if and only if the preimage of every closed subset is closed.

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Proof. For the forward direction suppose that we have a continuous map $f\colon X \to Y$. Let $C \subseteq X$ be closed in X. Thus we have that $C = Y \setminus O$ for some open set $O \subseteq Y$. Then $f^{-1}(O)$ is open and $f^{-1}(0) \setminus f^{-1}(C) = X \setminus f^{-1}(C)$. Thus we have that $f^{-1}(C)$ is closed.

Suppose that every preimage of a closed set is closed. Then consider $C \subseteq X$ closed in X and note that we can write $C = X \setminus 0$ for 0 open in Y. Then $f^{-1}(C) = f^{-1}(Y \setminus 0) = X \setminus f^{-1}(O)$. Then we have that $f^{-1}(C)$ is closed which implies that $f^{-1}(O)$ must be open. So f is continuous. $\square$

Problem 2. Let D be a discrete topological space, let T be a space with the trivial (indiscrete) topology, let H be a Hausdorff space, and let A be an arbitrary topological space.

(a) Show that every function $f: D \to A$ is continuous.

(b) Show that every function $f: A \to T$ is continuous.

(c) Show that $f: T \to H$ is continuous if and only if it is a constant map.

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Proof (Part (a)). Let $f: D \to A$. Then $O \subseteq A$ be an open set. Then consider $f^{-1}(O) \subseteq D$. Since any subset of D is open, we have that $f^{-1}(O)$ is open and thus f is continuous. ☐

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Proof (Part (b)). Let $f: A \to T$. Then let $O \subseteq T$ thus $O = \emptyset$ or $O = T$. If $O = \emptyset$ then $f^{-1}(O) = \emptyset$ which is open. Then if $O = T$ we have $f^{-1}(O) = T$ which is also open. Thus f is continuous. ☐

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Proof (Part (c)). For the forward direction, suppose that $f: T \to H$ is continuous. Let $x_1, x_2 \in T$ be unique. Then $f(x_1), f(x_2) \in H$. Suppose that $f(x_1) \neq f(x_2)$, then $\exists O_1 \ni f(x_1)$ and $O_2 \ni f(x_2)$ with O_1 and O_2 open and $O_1 \cap O_2 = \emptyset$. Then f being continuous implies that $f^{-1}(O_1) = T = f^{-1}(O_2)$ Since we said we had two unique elements x_1, x_2 we have that $f^{-1}(O_1) \neq \emptyset \neq f^{-1}(O_2)$. Then note that $f(f^{-1}(O_1)) \subseteq O_1$ and $f(f^{-1}(O_2)) \subseteq O_2$. Thus we have that $O_1 \cap O_2 \neq \emptyset$ and we contradict H being Hausdorff. Thus f is a constant map.

Suppose $f: T \to H$ is a constant map. Let $O \subseteq H$ be open. But $f(x) = h \in H$ $\forall x$ thus we have $f^{-1}(O) = T$ $\forall O \subseteq H$ that are open. So f is continuous.

Note: I worked on this problem with Zach and Tarun. ☐

Problem 3. True or false:

(a) The intervals $[0,1)$ and $(0,\infty)$ in the real line, equipped with the Euclidean topology.

(b) The subsets $\{1,2,3,4,...\}$ and $\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},...\}$, equipped with the Euclidean topology.

(c) The rationals $\mathbb{Q}$ with the discrete topology and the rationals $\mathbb{Q}$ with the Euclidean topology.

(d) $S^2 \setminus \{(0,0,1)\}$ and $\mathbb{R}^2$.

(e) $S^2 \setminus \{(0,0,1),(0,0,-1)\}$ and $\{x \in \mathbb{R}^2 | 1 < \|\|x\|\| < 3\}$.

(f) S^1 and $S^1 \cup \{(x,0) \in \mathbb{R}^2 | 1 \leq x \leq 2\}$.

(g) $\mathbb{R}^n$ and $\mathbb{R}^m$ for $n \neq m$.

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Solution. (a) false

(b) true

(c) false

(d) true

(e) true

(f) false

(g) false ■

Problem 4. Let X be a topological space and let $A \subseteq X$.

(a) Prove that a point $x \in X$ is in $\bar{A}$ if and only if every neighborhood of x contains a point of A.

(b) Suppose $\{x_i\}$ is a sequence of points in A that converges to a point $x \in X$. Prove that $x \in A$.

(c) Show that there exists a sequence for Figure 2.1(c) that converges to more than one limit point.

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Proof (a). For the forward direction, let $x \in \bar{A}$ and let $N_x \cap A = \emptyset$. Then for N_x a neighborhood of x we have $X \setminus N_x$ is closed and since $N_x \cap A = \emptyset$, $X \setminus N_x \supset \bar{A} \supset A$. This implies that $x \notin \bar{A}$, so that implies that $N_x \cap A \neq \emptyset$. Thus every neighborhood of x contains a point of A.

Suppose that every neighborhood of x contains a point of A. Then, for a contradiction, suppose that every open neighborhood $N_x \ni x$, $N_x \cap A \neq \emptyset$. Suppose that $x \notin \bar{A}$. Then we have $x \in X \setminus \bar{A}$ is open. This implies that $\exists N_x \subseteq X \setminus \bar{A}$ so then $N_x \cap A = \emptyset$. $\square$

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Proof (b). Since $x_i \in A \; \forall i$, then by definition of convergence $\forall N_x$, $N_x \cap A \neq \emptyset$. So by (a), $x \in \bar{A}$. $\square$

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Proof (c). Let $x_i = 1 \; \forall i$. Then $\{x_i\} \to 1$. since $x_i \in N_1 \forall i$. Then let $N_2 = \{1,2\}$. Note that $\forall i$, $1 \in N_2$. Thus $\{x_2\} \to 2$ as well. $\square$

Problem 5. Prove that a second countable space X contains a countable dense subset.

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Proof. Let X be a second countable space. Thus we have a basis for the topology on X given by open sets U_i $\forall i \in \mathbb{N}$. Then let $A = \{x_i | i \in \mathbb{N}\}$ so that each $x_i \in U_i$. Notice that $X \setminus \bar{A}$ is open since $\bar{A}$ is closed. Then suppose that $\exists x \in X \setminus \bar{A}$ and that $\exists N_x$ with $N_x \subseteq X \setminus \bar{A}$. But since U_i form a basis, we have that for $\alpha \subseteq \mathbb{N}$, $N_x = \cup_{i \in \alpha} U_i$. Thus $N_x \cap A \neq \emptyset$ since N_x must contain at least $x_i \in U_i$ for some i. This contradicts $N_x \subseteq X \setminus \bar{A}$. So no $x \in X \setminus \bar{A}$ so $X \setminus \bar{A} = \emptyset$. $\square$