Hodge Theory, Gelfand Theory, and Clifford Analysis Applied to Tomography

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Overview

Section 1

Introduction

Motivating problems

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- The ***Calderón problem*** replaces the medium with a manifold M, conductivity with g, and replaces the voltage-to-current map with the Dirichlet-to-Neumann operator Λ.

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- Can we access these functions from the boundary?

Subsection 1

Preliminaries

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- The associated ***Clifford algebra*** is the quotient

$$C\ell(V, g) := \mathcal{T}(V) / \langle \mathbf{v} \otimes \mathbf{v} - g(\mathbf{v}, \mathbf{v}) \rangle .$$

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- The completely degenerate case is the *exterior algebra*

$$\bigwedge(V) := C\ell(V, 0).$$

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  - The bivector part is antisymmetric: $\boldsymbol{u} \wedge \boldsymbol{v} = -\boldsymbol{v} \wedge \boldsymbol{u}$.

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  - Elements of the even grade subalgebra, $\mathcal{G}^+$, are called ***spinors***.
- Since $\mathcal{G} = \bigoplus_{r=0}^{n} \mathcal{G}^r$ a general ***multivector*** is $A = \sum_{r=0}^{n} \langle A \rangle_r$.

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- The most important products for us are

$$A_r \wedge B_s \coloneqq \langle A_r B_s \rangle_{r+s}$$

$$A_r \,\lrcorner\, B_s \coloneqq \langle A_r B_s \rangle_{s-r}$$

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- The *reverse* $\dagger$ is extended linearly from the action on r-blades

$$\boldsymbol{A_r}^\dagger = (\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_r)^\dagger = \boldsymbol{v}_r \wedge \cdots \wedge \boldsymbol{v}_1.$$

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- The ***projection*** of A into a subspace $\boldsymbol{U_r}$ by

$$\mathrm{P}_{\boldsymbol{U_r}}(A) \coloneqq A \,\lrcorner\, \boldsymbol{U_r} \boldsymbol{U_r}^{-1}.$$

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- We define the ***unit pseudoscalar*** (which corresponds to $V \subset V$) by

$$\boldsymbol{I} := \frac{1}{|\boldsymbol{\mu}|}\boldsymbol{\mu}.$$

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- Dual exchanges products $(A \,\lrcorner\, B)^{\perp} = A \wedge B^{\perp}$.

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- *Complex algebra* $\mathbb{C}$ is isomorphic to $\mathcal{G}_{0,2}^+$.
  - Standard basis $\boldsymbol{e}_1, \boldsymbol{e}_2$, and $\boldsymbol{e}_{12} := \boldsymbol{e}_1\boldsymbol{e}_2$. Then $\boldsymbol{e}_{12}^2 = -1$.
  - Right multiplication of vectors by $\boldsymbol{e}_{12}$ rotates counter-clockwise by $\pi/2$.

Section 2

Clifford analysis

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- Take same naming scheme and notation: $\mathfrak{X}^r(M)$, $\mathfrak{X}^+(M)$, etc.

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- $\boldsymbol{\nabla}^2$ is the Laplace-Beltrami operator.

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- Specifically,

$$\operatorname{curl}(\boldsymbol{v}) = (\boldsymbol{\nabla} \wedge \boldsymbol{v})^{\perp}$$

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- Algebra and calculus descend to multivector equivalent:

$$\alpha_r \wedge \beta_s = (A_r \wedge B_s) \,\lrcorner\, dX_{r+s} \qquad \alpha_r \,\lrcorner\, \beta_s = (A_r \,\lrcorner\, B_s) \,\lrcorner\, dX_{r-s}$$

$$\underbrace{d\alpha_r = (\boldsymbol{\nabla} \wedge A_r) \,\lrcorner\, dX_{r+1}^\dagger}_{\text{exterior derivative}} \qquad \underbrace{\delta\alpha_r = (-\boldsymbol{\nabla} \,\lrcorner\, A_r) \,\lrcorner\, dX_{r-1}^\dagger}_{\text{codifferential}}$$

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- For M this yields $d\mu = \sqrt{|g|}\, dx^1 \cdots dx^n$

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- Define the *multivector field inner product on R* by

$$\langle\!\langle A, B \rangle\!\rangle_R := \int_R A * B d\mu_R$$

Green's formulas

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- From [Hestenes, Sobczyk, 1984] and [Booß- Bavnbek, Wojciechowski, 1993]

$$\langle\!\langle \boldsymbol{\nabla} A, B \rangle\!\rangle = (-1)^n \langle\!\langle A, \boldsymbol{\nabla} B \rangle\!\rangle + \langle\!\langle A, B \rangle\!\rangle_{\partial M}$$

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$$(\!| \boldsymbol{\nabla} A, B |\!) = (-1)^n (\!| A, \boldsymbol{\nabla} B |\!) + (\!| A, B |\!)_{\partial M}$$

- Following from the above

$$\langle\!\langle \boldsymbol{\nabla} A, B \rangle\!\rangle = -\langle\!\langle A, \boldsymbol{\nabla} B \rangle\!\rangle + \langle\!\langle A, \boldsymbol{\nu} B \rangle\!\rangle.$$

Subsection 1

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Cauchy integral

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- There exists a vector-valued ***Cauchy kernel*** G_x where $\boldsymbol{\nabla} G_x = \delta_x$.
- Given $A \in \mathcal{M}(M)$, the ***Cauchy integral*** is

$$A(x) = (-1)^{n-1} \langle\!\langle A, G_x \rangle\!\rangle^{\perp}_{\partial M}.$$

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- Define $\boldsymbol{G}(\boldsymbol{x}) := \frac{1}{S_n} \frac{\mathbf{x}}{|\mathbf{x}|^n}$ then the Cauchy integral is

$$A(\boldsymbol{x}) = \frac{1}{S_n} \int_{\partial M} \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^n} \boldsymbol{\nu}(\boldsymbol{x}') A(\boldsymbol{x}') \, d\mu_{\partial M}(\boldsymbol{x}').$$

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- **Scalar part of the above is the double layer potential.**

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- Cauchy integral is evaluation and an isomorphism from $\mathrm{tr}\mathcal{M}(M)$.

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- In a region $M \subset \mathbb{R}^3$ take a vector field $\boldsymbol{J}$,

$$\mathrm{BS}(\boldsymbol{J})(\boldsymbol{x}) = \left\langle \langle\!| \boldsymbol{J}, G_{\boldsymbol{x}} |\!\rangle^{\perp} \right\rangle_2 = \frac{1}{4\pi} \int_{N^3} \boldsymbol{J}(\boldsymbol{y}) \wedge \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \, d\mu_{N^3}(\mathbf{x}').$$

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- **This is the Biot–Savart formula which recovers magnetic field from current.**

Section 3

Hodge theory

Idea

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- Put some pictures I've already made such as solid torus.

Product on cohomologies

Proposition

The contraction $\lrcorner$ is a product on cohomologies by:

- $\lrcorner : H^r(M) \times H^s(M) \to H^{s-r}(M)$;
- $\lrcorner : H^r(M, \partial M) \times H^s(M, \partial M) \to H^{s-r}(M, \partial M)$;
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- This product is equivalent to the mixed cup product in [Shonkwiler, 2009].

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- But, $\mathcal{M}(M) \neq \bigoplus_{j=1}^n \mathcal{M}^j(M)$.

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- Define the ***Dirac fields*** $\boldsymbol{\nabla}\mathfrak{X}(M)$ as

$$\boldsymbol{\nabla}\mathfrak{X}(M) := \{\boldsymbol{\nabla} A \mid A \in \mathfrak{X}(M) \text{ and } A|_{\partial M} = 0\};$$

Theorem: Clifford–Hodge Decomposition

The space of multivector fields $\mathfrak{X}(M)$ has the orthogonal decomposition

$$\mathfrak{X}(M) = \mathcal{M}(M) \oplus \boldsymbol{\nabla}\mathfrak{X}(M).$$

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- From Hodge–Morrey

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- But the Clifford-Hodge-Morrey is not filtered by grades

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Section 4

Tomography

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- Suppose M free of charges, then the forward problem

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- **Question:** Can we determine (M, σ) from Λ?

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- The smooth cases is still unsolved.