MATH 560, Homework 1

Colin Roberts

August 31, 2017

Solutions

§1.2 Problem 11. Let $V = \{0\}$ consist of a single vector 0, and define $0 + 0 = 0$ and $c0 = 0$ for each c in $\mathbb{F}$. Prove that V is a vector space over $\mathbb{F}$.

:

Proof.

(1) To see V is commutative with addition we have $0 + 0 = 0$ and since V is a singleton set, there is just this one value to check.

(2) We have $(0 + 0) + 0 = 0 + 0 = 0 = 0 + (0 + 0)$

(3) V only contains the 0 element and $0 + 0 = 0$.

(4) Let $x, y \in V$, then $x + y = 0 + 0 = 0$.

(5) Let $x \in V$, then $1(x) = 1(0) = 0 = x$.

(6) Let $a, b \in \mathbb{F}$ and $x \in V$, then $(ab)x = (ab)0 = 0 = a(0) = a(b0) = a(bx)$.

(7) Let $a \in \mathbb{F}$ and $x, y \in V$, then $a(x + y) = a(0 + 0) = 0 = a(0) + a(0) = a(x) + a(y)$.

(8) Let $a, b \in \mathbb{F}$ and $x \in V$, then $(a + b)x = (a + b)0 = 0 = a(0) + b(0) = a(x) + b(x)$. $\square$

§1.2 Problem 12. A real-valued function defined on the real line is called an *even function* if $f(-x) = f(x)$ for each real number x. Prove that the set of even functions defined on the real line with the operations of addition and scalar multiplication defined in Example 3 is a vector space.

:

Proof. First note that for any even function $f : \mathbb{R} \to \mathbb{R}$ that $f(t) \in \mathbb{R}\ \forall t \in \mathbb{R}$. This allows us to treat any even function f as a member of $\mathbb{R}$ and use field properties inherited from $\mathbb{R}$

(a) Let $f, g \in V$ and then $f + g = g + f$ by commutivity of addition in $\mathbb{R}$.

(b) Let $f, g, h \in V$, then we have $(f + g) + h = f + (g + h)$ by associativity of $\mathbb{R}$

(c) Let $f, g \in V$ and $g(t) = 0\ \forall t$ (which is even), then $f + g = f + g = f$.

(d) Let $f \in V$ and let $g(t) = -f(t)\ \forall t$ and notice that g is even and thus $g \in V$. Then $f + g = f - f = 0$.

(e) Let $f \in V$ then $1f = f$ since 1 is the multiplicative identity in $\mathbb{R}$.

(f) Let $f \in V$ and $a, b \in \mathbb{R}$, then $(ab)f = a(bf)$ by associativity of multiplication in $\mathbb{R}$.

(g) Let $f, g \in V$ and $a \in \mathbb{R}$, then $a(f + g) = af + ag$ by distribution in $\mathbb{R}$.

(h) Let $f \in V$ and $a, b \in \mathbb{R}$ then $(a + b)f = af + bf$ by distribution in $\mathbb{R}$. $\square$

§1.3 Problem 6. Prove that $\operatorname{tr}(aA+bB) = a\operatorname{tr}(A) + b\operatorname{tr}(B)$ for any $A, B \in M_{n\times n}(\mathbb{F})$

:

Proof.

$$\begin{aligned} \operatorname{tr}(aA+bB) &= (aA_{11} + aA_{22} + \ldots + aA_{nn}) + (bB_{11} + bB_{22} + \ldots + bB_{nn}) \\ &= a(A_{11} + \ldots + A_{nn}) + b(B_{11} + \ldots + B_{nn}) \\ &= a\operatorname{tr}(A) + b\operatorname{tr}(B) \end{aligned}$$

□

§1.3 Problem 12. Verify that the upper triangular matrices form a subspace of $M_{m\times n}(\mathbb{F})$.

:

Proof. First notice that A with $A_{ij} = 0\ \forall i, j$ is upper triangular. And for B upper triangular we have $A + B = 0 + B = B$ shows that the additive identity exists. Next, let A, B be arbitrary upper triangular matrices. Then $(A+B)_{ij} = A_{ij}+B_{ij}$ so that $(A+B)_{ij} = 0\ \forall i > j$. Next, let A be an upper triangular matrix and let $x \in \mathbb{F}$ and then $(aA)_{ij} = a(A_{ij})$ and since $a0 = 0$ we have that A is upper triangular. Finally let A, B be upper triangular matrices. Then if we have $A_{ij} = -B_{ij}$ we have $A_{ij} = 0\ \forall i, j$ thus $A + B = 0$. Thus upper triangular matrices form a subspace. □

§1.3 Problem 23. Let W_1 and W_2 be subspaces of a vector space V.

(a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

(b) Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

:

Proof (a). First note that $W_1 + W_2 = \{w_1 + w_2 | w_1 \in W_1 \text{ and } w_2 \in W_2\}$, and since W_1, W_2 are subspaces we have $0 \in W_1$ and $0 \in W_2$. Thus $0 \in W_1 + W_2$. Second, let $u, v \in W_1 + W_2$ and note that $u = u_1 + u_2$ and $v = v_1 + v_2$ with $u_1, u_2 \in W_1$ and $v_2, v_2 \in W_2$ then consider $u + v = (u_1 + u_2) + (v_1 + v_2) = (u_1 + v_1) + (u_2 + v_2)$ and then we have $u_1 + v_1 \in W_1$ and $u_2 + v_2 \in W_2$ by the fact that W_1 and W_2 are subspaces. So $u + v \in W_1 + W_2$. Next, let $w \in W_1 + W_2$ be given by $w = w_1 + w_2$ with $w_1 \in W_1$ and $w_2 \in W_2$ and let $a \in \mathbb{F}$. Then $aw = a(w_1 + w_2) = aw_1 + aw_2$ thus $aw_1 + aw_2 \in W_1 + W_2$ since $aw_1 \in W_1$ and $aw_2 \in W_2$ by the fact that W_1 and W_2 are subspaces. Finally, let $u, v \in W_1 + W_2$ and note that $u = u_1 + u_2$ and $v = v_1 + v_2$ with $u_1, u_2 \in W_1$ and $v_2, v_2 \in W_2$. Then, since W_1 and W_2 are subspaces, we can let $v_1 = -u_1$ and $v_2 = -u_2$ and then $u + v = (u_1 + v_1) + (u_2 + v_2) = (u_1 - u_2) + (u_2 - u_2) = 0 \in W_1 + W_2$. □

:

Proof (b). Let W be a subspace of V with $W_1 \subseteq W$ and $W_2 \subseteq W$. Thus a vector $w_1 \in W_1$ also satisfies $w_1 \in W$ as well as $w_2 \in W_2$ satisfying $w_2 \in W$. So we can say that $W \supseteq \{w_1 + w_2 | w_1 \in W_1 \text{ and } w_2 \in W_2\}$. So then $W \supseteq W_1 + W_2$. □

§1.3 Problem 28. A matrix M is called **skew-symmetric** if $M^t = -M$. Clearly, a skew-symmetric matrix is square. Let $\mathbb{F}$ be a field. Prove that the set W_1 of all skew-symmetric $n \times n$ matrices with entries from $\mathbb{F}$ is a subspace of $M_{n\times n}(\mathbb{F})$. Now assume that $\mathbb{F}$ is not of characteristic 2, and let W_2 be the subspace of $M_{n\times n}(\mathbb{F})$ consisting of all symmetric $n \times n$ matrices. Prove that $M_{n\times n}(\mathbb{F}) = W_1 \oplus W_2$.

:

Proof (First Part). I will do this by showing the properties hold for all components of a matrix $A = A_{ij}$ to make notation easier. Just note $A^T_{ij} = A_{ji}$ for a general square matrix.
Let A, B be skew-symmetric matrices. Then $(A+B)_{ij} = A_{ij} + B_{ij} = -A_{ji} - B_{ji} = -(A+B)_{ji}$ which shows closure. Second, let A be skew-symmetric and $a \in \mathbb{F}$, then $(aA)_{ij} = a(A_{ij}) = a(-A_{ji}) = -(aA)_{ji}$. Third, let A, B be skew-symmetric with $A_{ij} = 0\ \forall i, j$. Then, $(A+B)_{ij} = A_{ij} + B_{ij} = B_{ij} = -B_{ji}$. Finally, let A, B be skew-symmetric with $A_{ij} = -B_{ij}\ \forall i, j$ then $(A+B)_{ij} = A_{ij} + B_{ij} = A_{ij} - A_{ij} = 0$. So the skew symmetric matrices form a subspace of square matrices. □

:

Proof (Second Part). Again I will do this using indices. Let A be an $n\times n$ matrix, B be a skew-symmetric matrix, and C be a symmetric matrix. Then $A_{ii} = B_{ii}$ and $C_{ii} = 0\ \forall i$ shows the diagonal of an $n \times n$ matrix can be written as a sum of a skew-symmetric and a symmetric matrix. Next, consider the following,

$$\begin{aligned} A_{ij} &= B_{ij} + C_{ij} && \text{and} \\ A_{ji} &= B_{ji} + C_{ji} && \forall j > i \end{aligned}$$

Notice, these equations will allow us to solve for each value in the matrix A. Then with $B_{ji} = B_{ij}$ and $C_{ij} = -C_{ij}$ we have,

$$\begin{aligned} A_{ij} &= B_{ij} + C_{ij} && \text{and} \\ A_{ji} &= B_{ij} - C_{ij} && \forall j > i \end{aligned}$$

Subtracting the two equations yields,

$$\begin{aligned} A_{ij} - A_{ji} &= B_{ij} + C_{ij} - B_{ij} + C_{ij} = 2C_{ij} \\ C_{ij} &= \frac{1}{2}(A_{ij} - A_{ji}) \end{aligned}$$

Finally, we plug C_{ij} into the first equation to yield,

$$B_{ij} = A_{ij} + \frac{1}{2}(A_{ij} - A_{ji})$$

Thus we have B_{ij} and C_{ij} in terms of A_{ij} and A_{ji} for every i and j and so we can write A as a sum of a skew-symmetric matrix B and a symmetric matrix C since char$(\mathbb{F}) \neq 2$. Lastly if $A \in W_1$ and $B \in W_2$ then if $W_1 \cap W_2 \neq 0$ we can find a matrix that is both symmetric and skew-symmetric. First note that if $T \in M_{n\times n}(\mathbb{F})$ is skew symmetric then $T_{ii} = 0\ \forall i$. Then if T is to also be symmetric $T_{ij} = T_{ji}$. However it must also be skew-symmetric so $T_{ij} = -T_{ji}$. So then $T_{ji} = -T_{ji}$ thus $T_{ij} = 0$. So T is the zero matrix. Thus $W_1 \cap W_2 = 0$ and $M_{n\times n}(\mathbb{F}) = W_1 \oplus W_2$. □

§1.3 Problem 30. Let W_1 and W_2 be subspaces of a vector space V. Prove that V is the direct sum of W_1 and W_2 if and only if each vector in V can be uniquely written as $w_1 + w_2$ where $w_1 \in W_1$ and $w_2 \in W_2$.

:

Proof. For the forward direction suppose that $V = W_1 \oplus W_2$. Then we can write $v \in V$ as $w_1 + w_2 = v$ with $w_1 \in W_1$ and $w_2 \in W_2$. Then consider $u_1 \in W_1$ and $u_2 \in W_2$ with $u_1 + u_2 = v$ Then $w_1 + w_2 = u_1 + u_2$ which implies that $u_1 - w_1 = w_2 - u_2$. Since the left hand side $u_1 - w_1 \in W_1$ and the right hand side $w_2 - u_2 \in W_2$ then $v \in W_1 \cap W_2$. Thus $v = 0$ and $u_1 = w_1$ and $u_2 = w_2$ so that w_1 and w_2 are unique elements of W_1 and W_2 respectively.

For the reverse direction, suppose that each vector $v \in V$ can be written uniquely as $w_1 + w_2 = v$ with $w_1 \in W_1$ and $w_2 \in W_2$. Then if $u \in W_1 \cap W_2$ and suppose for a contradiction that $w \neq 0$ and that we have $v = w_1 + w_2 = w_1 + w_2 + u - u = (w_1 - u) + (w_2 + u)$ which means that w_1 and w_2 weren't unique.

§1.3 Problem 31. Let W be a subspace of a vector space V over a field $\mathbb{F}$. For any $v \in V$ the set $\{v\} + W = \{v + w | w \in W\}$ is called the **coset** of W containing v. It is costumary to denote this coset by $v + W$ rather than $\{v\} + W$.

(a) Prove that $v + W$ is a subspace of V if and only if $v \in W$

(b) Prove that $v_1 + W = v_2 + W$ if and only if $v_1 - v_2 \in W$. Addition and scalar multiplication by scalars of $\mathbb{F}$ can be defined in the following collection $S = \{v + W | v \in V\}$ of all cosets of W as follows:

$$(v_1 + W) + (v_1 + W) = (v_1 + v_2) + W$$

for all $v_1, v_2 \in V$ and

$$a(v + W) = av + W$$

(c) Prove that the preceding operations are well defined; that is show that if $v_1 + W = v_1' + W$ and $v_2 + W = v_2' + W$, then

$$(v_1 + W) + (v_2 + W) = (v_1' + W) + (v_2' + W)$$

and

$$a(v_1 + W) = a(v_1' + W)$$

(d) Prove that the set S is a vector space with the operations defined in (c). This vector space is called the **quotient space of** V **modulo** W and is denoted by V/W

:

Proof (a). For the forward direction, suppose that for $v \in V$ we have that $v + W$ is a subspace. Then let $u_1, u_2 \in v + W$. So $u_1 = v + w_1$ and $u_2 = v + w_2$. So $u_1 - u_2 = w_1 - w_2$ and by closure of W we have that $u_1 - u_2 \in W$. So $v + W = W$, which implies $v \in W$. For the reverse direction, suppose that $v \in W$. Then since $v \in W$ we have $v + W = W$ and W is a subspace. □

:

Proof (b). For the forward direction suppose that $v_1 + W = v_2 + W$. Then we have $v_1 + w_1 = v_2 + w_2$ which means that $v_1 - v_2 = w_2 - w_1$. Since W is a subspace, it is closed, and thus $v_1 - v_2 \in W$. For the reverse direction, suppose that $v_1 - v_2 \neq 0 \in W$, then $(v_1 - v_2) + w = 0$. Thus $v_1 + \frac{1}{2}w = v_2 + \frac{1}{2}w$ Thus we have $v_1 + W = v_2 + W$. □

:

Proof (c). Suppose we have $v_1 + W = v_1' + W$ and $v_2 + W = v_2' + W$, then

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W = (v_1' + v_2') + W = (v_1' + W) + (v_2' + W).$$

Next we have $a \in \mathbb{F}$ and then,

$$a(v_1 + W) = av_1 + W = av_1' + W = a(v_1' + W)$$ □

:

Proof (d). First let $u_1, u_2 \in V/W$ be written as $v_1 + W$ and $v_2 + W$ respectively. . Then, $u_1 + u_2 = (v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$ so V/W is closed. Next let $a \in \mathbb{F}$ and consider $a(u_1) = a(v_1 + W) = av_1 + W$ which is also in V/W. Then if we let $v_1 = 0$ so $u_1 = W$ we have $u_1 + u_2 = W + (v_2 + W) = v_2 + W$. So the zero element exists. If we let $u_2 = -u_1$ then $u_1 + u_2 = (v_1 + W) + (v_2 + W) = (v_1 + W) - (v_1 + W) = 0 + W = W$. □

§1.4 Problem 8. Show that $P_n(\mathbb{F})$ is generated by $\{1, x, ..., x^n\}$.

:

Proof. Let $f(x) \in P_n(\mathbb{F})$ be an arbitrary polynomial in x of degree n. It can be written as $a_0 + a_1 x + a_2 x^2 + ... + a_n x^n = a_0(1) + a_1(x) + ... + a_n x^n$. Since we can write $f(x)$ as a linear combination of elements of $\{1, x, ..., x^n\}$ we know that $P_n(\mathbb{F}) = \text{span}(\{1, x, ..., x^n\})$ $\square$

§1.4 Problem 9. Show that the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

generate $M_{2\times 2}(\mathbb{F})$.

:

Proof. Let $A \in M_{2\times 2}(\mathbb{F})$ be arbitrary. So,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{12}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a_{21}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Since A is written as a linear combination of these matrices, these matrices generate $M_{2\times 2}(\mathbb{F})$. □

§1.5 Problem 16. Prove that a set S of vectors is linearly independent if and only if each finite subset of S is linearly independent.

:

Proof. For the forward direction, suppose that S contains linearly independent vectors $\{v_1,...,v_n\}$. Since S is linearly independent we have that $a_1v_1 + ... + a_nv_n = 0$ implies $a_i = 0\ \forall i$. Thus, no finite linear combination containing the same number of terms or less will make it so that $a_i \neq 0$ since they all have to be zero. So any subset of S will contain only linearly independent vectors. For the reverse direction, suppose each finite subset $U \subseteq S$ is linearly independent. Then note $U = S \subseteq S$ so S is linearly independent. □

§1.5 Problem 18. Let S be a set of nonzero polynomials in $P(\mathbb{F})$ such that no two have the same degree. Prove that S is linearly independent.

:

Proof. Suppose for a contradiction that for $p_1, ..., p_n \in S$ we have that $a_1p_1, ..., a_np_n = 0$ and that $a_i \neq 0$. So $(a_1p_1 + ... + a_{n-1}p_{n-1}) = -a_np_n$. Since p_n is arbitrary, let it be of degree n. So $a_np_n \in \text{span}(\{p_1, ..., p_n\}$. But p_n has a different degree than each other p_i, $i \neq n$ and thus $a_1p_n + ... + a_2p_n$ is at most degree $n-1$ and cannot be equal to a_np_n. Thus S is linearly independent. □

§1.6 Problem 29.

(a) Prove that if W_1 and W_2 are finite-dimensional subspaces of a vector space V, then the subspace $W_1 + W_2$ is finite-dimensional, and $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

(b) Let W_1 and W_2 be finite-dimensional subspaces of a vector space V, and let $V = W_1 + W_2$. Deduce that V is the direct sum of W_1 and W_2 if and only if $\dim(V) = \dim(W_1) + \dim(W_2)$.

:

Proof (a). Suppose W_1 and W_2 are finite-dimensional subspaces of a vector space V. Let $\{u_1, ..., u_n\}$ be a basis for $W_1 \cap W_2$. Then we also have $\{u_1, ..., u_n, v_1, ..., v_m\}$ as a basis for W_1 and $\{u_1, ..., u_m, w_1, ..., w_k\}$ as a basis for W_2. Then,

$$\begin{aligned}\dim(W_1 + W_2) &= n + m + k \\ &= (n+m) + (n_k) - n \\ &= \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)\end{aligned}$$

which also shows that $\dim(W_1 + W_2)$ is finite. □

:

Proof (b). For the forward direction suppose that $V = W_1 \oplus W_2$. Then $\dim(W_1 \cap W_2) = 0$ and from the previous exercise we have $\dim(V) = \dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2)$. For the reverse direction suppose that $\dim(V) = \dim(W_1) + \dim(W_2)$. Thus by the previous exercise we have that $\dim(W_1 \cap W_2) = 0$ and so $W_1 \cap W_2 = \{0\}$ and $V = W_1 \oplus W_2$. □

§1.6 Problem 31. Let W_1 and W_2 be subspaces of a vector space V having dimensions m and n, respectively, where $m \geq n$.

(a) Prove that $\dim(W_1 \cap W_2) \leq n$.

(b) Prove that $\dim(W_1 + W_2) \leq m + n$.

:

Proof (a). We have that $\dim(W_1+W_2) = \dim(W_1)+\dim(W_2)-\dim(W_1 \cap W_2) = m+n-\dim(W_1 \cap W_2)$. Suppose $\dim(W_1 \cap W_2) > n$ then $\dim(W_1 + W_2) < m$ which contradicts that W_1 has m basis vectors. □

:

Proof (b). We have that $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) = m + n - \dim(W_1 \cap W_2) = m + n - \dim(W_1 \cap W_2)$. Since $\dim(W_1 \cap W_2) \geq 0$ we have that $\dim(W_1 \cap W_2) \leq m + n$. □