MATH 570, Homework 6

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October 5, 2017

Solutions

Problem 1. Two paths $f, g\colon I \to X$ in space X are *path-homotopic*, denoted $f \sim g$, if they are homotopy equivalent relative $\{0,1\} \subseteq I$. A *reparametrization* of a path $f\colon I \to X$ is a path of the form $f \circ \varphi$ for some continuous map $\varphi\colon I \to I$ fixing 0 and 1. Prove Lemma 7.9 in our book: any reparametrization of a path f is path-homotopic to f.

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Proof. Since φ fixes $\{0,1\}$ then we have that $f \circ \varphi(0) = f(0)$ and $f \circ \varphi(1) = f(1)$. Consider the straight line homotopy $H\colon I \times I \to I$ given by which deforms φ to Id_I. Then we have that $f \circ H(x,t)$ is a homotopy between f and $f \circ \varphi$ relative $\{0,1\}$ since $f \circ H(x,0) = f(x)$ and $f \circ H(x,1) = f \circ \varphi(x)$. $\square$

Problem 2. Let $f, g\colon I \to X$ be two paths from p to q in space X. Draw a picture to show that if $f \sim g$, then $f \cdot \bar{g} \sim c_p$, where c_p is the constant path at p.
Remark: If you draw a nice picture, you don't need to write down any sentences or math equations. I recommend trying to draw a homotopy $H\colon I \times I \to X$ in the domain $I \times I$ instead of trying to draw it in the codomain X.
Remark: Instead of drawing a picture you could alternatively give an explicit definition of such a homotopy using math equations.

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Proof. Since $f \sim g$ relative $\{p, q\}$ (endpoints) we have that $\bar{f} \sim \bar{g}$ relative $f\{p, q\}$ via a homotopy F. This means we could perform this homotopy F twice as fast, and then use the fact that we have a homotopy H for $f \cdot \bar{f} \sim c_p$. This homotopy is in our text and given by the proof for Theorem 7.11 (b). It is as follows:

$$H(s,t) = \begin{cases} f(2s), & 0 \le s \le t/2; \\ f(t), & t/2 \le s \le 1 - t/2; \\ f(2-2s), & 1 - t/2 \le s \le 1. \end{cases} \qquad \square$$

A picture is as follows: In the domain,

and in the codomain,

Problem 3. If topological space X is path-connected and $\pi_1(X)$ is trivial, then we say that X is *simply connected*. Prove that if $X \subseteq \mathbb{R}^n$ is convex, then X is simply connected.

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Proof. If $X \subseteq \mathbb{R}^n$ is convex. Thus for any two points $x, y \in X$, we have that there exists a straight line path between x and y. So a convex subset X of $\mathbb{R}^n$ is path-connected. Let $f \colon I \to X$ be a loop in X based at p. Then, since each point is connected by a straight line, we have a straight line homotopy from f to Id. Since f was an arbitrary loop, we have that $\pi_1(X)$ is trivial. Since X is path-connected and $\pi_1(X)$ is trivial, X is simply connected. $\square$

Problem 4. Choose any old homework or exam problem, or a portion thereof. Clearly state both the problem and the homework/exam number. Write out a solution that is as clear as possible, with no extraneous steps.

I will redo Problem 2 off of Homework 4. The category stuff is the most new to me, and I want to make sure I get it nailed down.

Homework 4 Problem 2: Let $(X_\alpha)_{\alpha\in A}$ be a family of topological spaces, and equip $\coprod_{\alpha\in A} X_\alpha$ with the disjoint union topology. Prove that $\coprod_{\alpha\in A} X_\alpha$ is the coproduct of $(X_\alpha)_{\alpha\in A}$ in the category of topological spaces as follows.

(a) Define the maps $\iota_\alpha\colon X_\alpha \to \coprod_{\alpha\in A} X_\alpha$.

(b) Prove that $(\coprod_{\alpha\in A} X_\alpha, (\iota_\alpha))$ satisfies the necessary universal property.

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Solution (a). Note that $\coprod_{\alpha\in A} X_\alpha = \{(x,\alpha) \mid \alpha \in A \text{ and } x \in X_\alpha\}$. We define $\iota X_\alpha \to \coprod_{\alpha\in A} X_\alpha$ by $\iota_\alpha(x) = (x,\alpha)$.■

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Proof (b). This diagram will be useful:

$$\begin{tikzcd}
\coprod_{\alpha\in A} X_\alpha & \\
W \arrow[u, dashed, "f"] & X_\alpha \arrow[ul, "\iota_\alpha"'] \arrow[l, "f_\alpha"]
\end{tikzcd}$$

Let W be any topological space with morphisms $f_\alpha\colon X_\alpha \to W$ for each $\alpha \in A$. Then define $f\colon W \to \coprod_{\alpha\in A} X_\alpha$ by $f(x,\alpha) = f_\alpha(x)$. Note that f is a morphism since f^{-1} restricted to any X_α is just f_α. So f pulls open sets back to open sets in the disjoint union topology. It then follows that we have that the diagram above commutes, i.e., $f(\iota_\alpha(x)) = f_\alpha(x) = (x,\alpha)$. Finally, we have that f was unique since if we had another distinct map $g\colon W \to \coprod_{\alpha\in A} X_\alpha$ with $g(x,\alpha) \neq f(x,\alpha)$ then $g(\iota_\alpha(x)) \neq f_\alpha(x)$ and the diagram would not commute. □