MATH 560, Homework 9

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Solutions

Problem 1. (§6.3 Problem 20. (a)) For the following sets of data that follows, use the least squares approximation to find the best fits with both (i) a linear function and (ii) a quadratic function. Compute the error E in both cases.

$$\{(-3,9),(-2,6),(0,2),(1,1)\}$$

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Proof. (i) First, for the linear function, we have

$$A = \begin{bmatrix} -3 & 1 \\ -2 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad \text{and} \qquad y = \begin{bmatrix} 9 \\ 6 \\ 2 \\ 1 \end{bmatrix}.$$

It follows that

$$A^* A = \begin{bmatrix} -3 & -2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ -2 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 14 & -4 \\ -4 & 4 \end{bmatrix}$$

and

$$(A^* A)^{-1} = \frac{1}{20} \begin{bmatrix} 2 & 2 \\ 2 & 7 \end{bmatrix}.$$

Then we get that

$$\begin{bmatrix} c \\ d \end{bmatrix} = x_0 = \frac{1}{20} \begin{bmatrix} 2 & 2 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} -3 & -2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 6 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ \frac{5}{2} \end{bmatrix}.$$

So the line $y = -2t + \frac{5}{2}$ is the line of best fit. The error

$$E = \|Ax_0 - y\|^2 = \left\| \begin{bmatrix} -3 & 1 \\ -2 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ \frac{5}{2} \end{bmatrix} - \begin{bmatrix} 9 \\ 6 \\ 2 \\ 1 \end{bmatrix} \right\|^2 = 1.$$

(ii) For the quadratic function we have,

$$A = \begin{bmatrix} (-3)^2 & -3 & 1 \\ (-2)^2 & -2 & 1 \\ 0^2 & 0 & 1 \\ 1^2 & 0 & 1 \end{bmatrix}$$

□

Problem 2. (§6.4 Problem 17 (a)-(d) Read (e) and (f).) Let T and U be self-adjoint linear operators on an n-dimensional inner product space V, and let $A = [T]_\beta$, where β is an orthonormal basis for V. Prove the following results.

(a) T is positive definite [semidefinite] if and only if all of its eigenvalues are positive [nonnegative].

(b) T is positive definite if and only if

$$\sum_{i,j} A_{ij} a_j \overline{a_i} > 0 \text{ for all nonzero } n\text{-tuples } (a_1, a_2, \ldots, a_n).$$

(c) T is positive semidefinite if and only if $A = B^* B$ for some square matrix B.

(d) If T and U are positive semidefinite operators such that $T^2 = U^2$, then $T = U$.

(e) If T and U are positive definite operators such that $TU = UT$, then TU is positive definite.

(f) T is positive definite [semidefinite] if and only if A is positive definite [semidefinite].

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Proof.

(a) First, suppose that T is positive definite [semidefinite] but that some eigenvalue is negative [nonpositive]. Of course, we can say that the eigenvalues are real since T is self adjoint and that the orthornormal basis β consists of eigenvectors. Now we have $x = a_1 v_1 + \cdots + a_n v_n$ with $v_i \in \beta$ and not all $a_i = 0$. Then with $A = [T]_\beta$

$$\begin{aligned} \langle Ax, x\rangle &= \langle \lambda_1 a_1 v_1 + \cdots \lambda_n a_n v_n, a_1 v_1 + \cdots + a_n v_n\rangle \\ \iff &= \lambda_1 |a_1|^2 + \cdots + \lambda_n |a_n|^2 > 0 && \text{if } T \text{ is positive definite} \\ \iff &= \lambda_1 |a_1|^2 + \cdots + \lambda_n |a_n|^2 \geq 0 && \text{if } T \text{ is positive semidefinite.} \end{aligned}$$

Now the last two statements imply that $\lambda_1, \ldots \lambda_n 0$ if T is positive definite and $\lambda_1, \ldots, \lambda_n \geq 0$ if T is positive semidefinite.

Using the same x, we show the converse by noticing $[T]_\beta x = \lambda_1 a_1 v_1 + \cdots \lambda_n a_n v_n$ with $\lambda_i > 0$ [$\lambda_i \geq 0$]. Now

$$\begin{aligned} \langle Ax, x\rangle &= \lambda_1 a_1^2 + \cdots + \lambda_n a_n^2 > 0 && \text{if } \lambda_i > 0 \\ &= \lambda_1 a_1^2 + \cdots + \lambda_n a_n^2 \geq 0 && \text{if } \lambda_i \geq 0. \end{aligned}$$

It follows that T is positive definite [semidefinite] if all of its eigenvalues are positive [nonnegative].

(b) Let $x = a_1 v_1 + \cdots + a_n v_n$ with $v_i \in \beta$. Then note that $\sum_{i,j} A_{ij} a_j \overline{a_i} = \langle Ax, x\rangle$. Now if $\sum_{i,j} A_{ij} a_j \overline{a_i} > 0$ then $\langle Ax, x\rangle > 0$ and $A = [T]_\beta$ is positive definite. Analogously, if $\langle Ax, x\rangle > 0$ then $\sum_{i,j} A_{ij} a_j \overline{a_i} > 0$.

(c) Suppose T is positive semidefinite and thus since T is also self adjoint we have that $Av_i = \lambda_i v_i$ with $\lambda_i \geq 0$. Now define B so that $Bv_i = \sqrt{\lambda_i} v_i$. It follows that we have for $x = a_1 v_1 + \cdots + a_n v_n$ we have

$$\begin{aligned} \langle B^* Bx, x\rangle &= \langle Bx, Bx\rangle \\ &= \lambda_1 |a_1|^2 + \cdots + \lambda_n |a_n|^2 \\ &= \langle Ax, x\rangle. \end{aligned}$$

Hence, $A = B^* B$.

For the converse, if $A = B^* B$, then for the same x,

$$\begin{aligned}\langle Ax, x\rangle &= \langle B^* Bx, x\rangle \\ &= \langle Bx, Bx\rangle \\ &= \|Bx\|^2 \geq 0.\end{aligned}$$

Hence A is positive semidefinite.

(d) Suppose T and U are positive semidefinite and satisfy $T^2 = U^2$. Then we have that $T = B^* B$ and $U = C^* C$ by (c). It then follows for $x = a_1 v_1 + \cdots + a_n v_n$ with $v_i \in \beta$,

$$\begin{gathered}\langle T^2 x, x\rangle = \langle U^2 x, x\rangle \\ \iff \langle (B^* B)^2 x, x\rangle = \langle (C^* C)^2 x, x\rangle \\ \iff \langle B^* Bx, (B^* B)^* x\rangle = \langle C^* Cx, (C^* C)^* x\rangle \\ \iff \langle B^* Bx, B^* Bx\rangle = \langle C^* Cx, C^* Cx\rangle \\ \iff \|Tx\|^2 = \|Ux\|^2.\end{gathered}$$

Since T and U are positive semidefinite, it must be that $Tx = Ux$ since for the last equality the only other possibility is $Tx = -Ux$, which contradicts T and U being positive semidefinite. Hence, $T = U$. $\square$

Problem 3. (§6.5 Problem 14.) Prove that if A and B are unitarily equivalent matrices, then A is positive definite [semidefinite] if and only if B is positive definite [semidefinite].

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Proof. Suppose A and B are unitarily equivalent matrices, which means we can write $A = U^*BU$ for U a unitary matrix. Then, suppose that A is positive definite [semidefinite]. It follows for $x \in V$ that

$$\begin{aligned} 0 &< \langle Ax, x\rangle && \text{if } A \text{ is positive definite} \\ \text{or } 0 &\leq \langle Ax, x\rangle && \text{if } A \text{ is positive semidefinite} \end{aligned}$$

and

$$\begin{aligned} \langle Ax, x\rangle &= \langle U^*BUx, x\rangle \\ &= \langle BUx, Ux\rangle. \end{aligned}$$

Note that $Ux \neq 0 \in V$ and specifically $\|Ux\| = \|x\|$. Denote $x' = Ux$ and we have that $\langle Bx', x'\rangle > 0$ if A is positive definite and $\langle Bx', x'\rangle \geq 0$ if A is positive semidefinite. Thus we have shown B is positive definite [semidefinite if A is positive definite [semidefinite].

For the converse, we suppose that B is positive definite [semidefinite] and repeat the above proof with $B = U'^*AU'$ with $U' = U^*$. It is exactly analogous. □

Problem 4. (§6.5 Problem 21.) Let A and B be $n \times n$ matrixes that are unitarily equivalent.

(a) Prove that $\operatorname{tr}(A^* A) = \operatorname{tr}(B^* B)$.

(b) Use (a) to prove that

$$\sum_{i,j=1}^{n} |A_{ij}|^2 = \sum_{i,j=1}^{n} |B_{ij}|^2.$$

(c) Use (b) to show that the matrices

$$\begin{bmatrix} 1 & 2 \\ 2 & i \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} i & 4 \\ 1 & 1 \end{bmatrix}$$

are *not* unitarily equivalent.

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Proof.

(a) Suppose that A and B are unitarily equivalent and specifically we have $A = U^* BU$. We have

$$A^* A = (U^* BU)^* (U^* BU) = U^* B^* UU^* BU = U^* B^* B^* U.$$

Finally, by properties of the trace we have

$$\operatorname{tr}(A^* A) = \operatorname{tr}(U^* B^* BU) = \operatorname{tr}(U^* UB^* B) = \operatorname{tr}(B^* B).$$

(b) We have

$$\operatorname{tr}(A^* A) = \sum_{i=1}^{n} (A^* A)_{ii} = \sum_{i,k=1}^{n} (A^*_{ik})(A_{ki}) = \sum_{i,j=1}^{n} |A_{ij}|^2.$$

Then we apply (a) and we find

$$\sum_{i,j=1}^{n} |A_{ij}|^2 = \sum_{i,j=1}^{n} |B_{ij}|^2.$$

(c) We have for the first matrix

$$a = |1|^2 + |2|^2 + |2|^2 + |i|^2 = 10.$$

For the second matrix we have

$$b = |i|^2 + |4|^2 + |1|^2 + |1|^2 = 19. \qquad \square$$

Thus the two matrices are *not* unitarily equivalent.

Problem 5. (§6.6 Problem 2.) Let $V = \mathbb{R}^2$, $W = \text{span}(\{(1,2)\})$, and β be the standard ordered basis for V. Compute $[T]_\beta$, where T is the orthogonal projection of V on W. Do the same for $V = \mathbb{R}^3$ and $W = \text{span}(\{(1,0,1)\})$.

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Proof. With β the standard basis we find $[T]_\beta$ by projecting the standard vectors onto $W = \text{span}(\{(1,2)\})$. So we have

$$\frac{(1,0)\cdot(1,2)}{\|(1,2)\|^2}(1,2) = \left(\frac{1}{5},\frac{2}{5}\right)$$
$$\frac{(0,1)\cdot(1,2)}{\|(1,2)\|^2}(1,2) = \left(\frac{2}{5},\frac{4}{5}\right)$$
$$.$$

This tells us that

$$[T]_\beta = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix},$$

where the left column is the first projection and the right column is the second.

Now with $W = \text{span}(\{(1,0,1)\})$, we have

$$\frac{(1,0,0)\cdot(1,0,1)}{\|(1,0,1)\|^2}(1,0,1) = \left(\frac{1}{2},0,\frac{1}{2}\right)$$
$$\frac{(0,1,0)\cdot(1,0,1)}{\|(1,0,1)\|^2}(1,0,1) = (0,0,0)$$
$$\frac{(0,0,1)\cdot(1,0,1)}{\|(1,0,1)\|^2}(1,0,1) = \left(\frac{1}{2},0,\frac{1}{2}\right).$$

And we have that

$$[T]_\beta = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$ □

Problem 6. (§6.6 Problem 3 (b).) For the following matrix A

(1) Verify that L_A possesses a spectral decomposition.

(2) For each eigenvalue of L_A, explicitly define the orthogonal projection on the corresponding eigenspace.

(3) Verify your results using the spectral theorem.

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

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Proof.

(i) First we find the eigenvalues which are solutions to $\lambda^2 + 1 = 0$. Namely, $\lambda_1 = i$ and $\lambda_2 = -i$. Then we have that the eigenvectors are $v_1 = (i, 1)$ and $v_2 = (-i, 1)$. This yields

$$\frac{(1,0)\cdot(i,1)}{\|(i,1)\|^2}(i,1) = \frac{i}{2}(i,1)$$
$$\frac{(0,1)\cdot(i,1)}{\|(i,1)\|^2}(i,1) = \frac{1}{2}(i,1),$$

implying that

$$T_1 = \frac{1}{2}\begin{bmatrix} -1 & i \\ i & 1 \end{bmatrix}.$$

Similarly,

$$\frac{(1,0)\cdot(-i,1)}{\|(i,1)\|^2}(-i,1) = \frac{-i}{2}(-i,1)$$
$$\frac{(0,1)\cdot(-i,1)}{\|(i,1)\|^2}(-i,1) = \frac{1}{2}(-i,1),$$

giving

$$T_2 = \frac{1}{2}\begin{bmatrix} -1 & -i \\ i & 1 \end{bmatrix}.$$

Now we verify that

$$A = \lambda_1 T_1 + \lambda_2 T_2 = \frac{1}{2}\left(i \begin{bmatrix} -1 & i \\ -i & 1 \end{bmatrix} - i \begin{bmatrix} -1 & -i \\ i & 1 \end{bmatrix} \right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \qquad \square$$

Hence we have shown that L_A has a spectral decomposition.

(ii) We have T_1 and T_2 above for λ_1 and λ_2 respectively.

(iii) To verify we notice that $E_{\lambda_1} \oplus E_{\lambda_2} = V$, and that $E_{\lambda_1} = E_{\lambda_2}^{\perp}$ (showing (a) and (b)). Now $T_1 T_2 = 0$, $T_1^2 = T_1$, and $T_2^2 = T_2$ since T_i is a projection matrix, which shows (c). Finally, for (d) we have $I = T_1 + T_2$ and we showed (e) in part (i) of this proof. So we have verified the spectral theorem.

Problem 7. (§6.7 Problem 2 (a),(b).) Let $T: V \to W$ be a linear transformation of rank r, where V and W are finite-dimensional inner product spaces. In each of the following, find orthonormal bases $\{v_1, v_2, \dots, v_n\}$ for V and $\{u_1, u_2, \dots, u_m\}$ for W, and the nonzero singular values $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_r$ of T such that $T(v_i) = \sigma_i u_i$ for $1 \le i \le r$.

(a) $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x_1, x_2) = (x_1, x_1 + x_2, x_1 - x_2)$.

(b) $T: P_2(R) \to P_1(R)$, where $T(f(x)) = f''(x)$, and the inner products are defined as in Example 1.

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Proof. (a) We can let the matrix $A = [T]_\beta$ with β the standard basis. Then

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

We know that the eigenvectors of A^*A will generate a basis for W. We have

$$A^*A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

This implies that the eigenvectors corresponding to A^*A are $v_1 = (1,0)$ and $v_2 = (0,1)$. So we then let

$$V^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now

$$AA^* = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix},$$

which has eigenvalues $\lambda_1 = 3$, $\lambda_2 = 2$, and $\lambda_3 = 0$ with corresponding normalized eigenvectors $u_1 = \frac{1}{\sqrt{3}}(1,1,1)$, $u_2 = \frac{1}{\sqrt{2}}(0,-1,1)$, and $u_3 = \frac{1}{\sqrt{6}}(-2,1,1)$. It follows that

$$U = \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}.$$

Now we note that $\sigma_1 = \sqrt{2}$ and $\sigma_2 = \sqrt{3}$ and check this by confirming $Av_1 = (1,1,1) = \sigma_1 u_1$ and $Av_2 = (0,1,-1) = \sigma_2 u_2$.

(b) Again, let $A = [T]_\beta$ with β the standard basis. Now we have

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

So we get that

$$A^*A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix},$$

which has eigenvalues $\lambda_1 = 0$, $\lambda_2 = 0$, and $\lambda_3 = 4$ corresponding to eigenvectors $v_1' = (1,0,0)$, $v_2' = (0,1,0)$, and $v_3' = (0,0,1)$. So we write

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Next we have

$$AA^* = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix},$$

which has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 4$ corresponding to eigenvectors $u_1' = (1,0)$ and $u_2' = (0,1)$. Then we write

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now, translating these vectors to the corresponding vectors for $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$ we find

$$\begin{aligned} v_1 &= \sqrt{\frac{5}{8}}(3x^2 - 1) \\ v_2 &= \sqrt{\frac{3}{2}}x \\ v_3 &= \frac{1}{\sqrt{2}} && \text{and} \\ u_1 &= \sqrt{\frac{3}{2}}x \\ u_2 &= \frac{1}{\sqrt{2}}. \end{aligned}$$

We verify by noting that $\sigma_1 = 4$ and $T(v_1) = \sigma_1 u_1$ and else $T(v_2) = T(v_3) = 0$. $\square$

Problem 8. (§6.7 Problem 4 (a).) Find a polar decomposition for the following matrices.

(a) $\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$.

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Proof. First we find the eigenvectors of A^*A. We have

$$A^*A = \begin{bmatrix} 5 & -3 \\ 3 & 5 \end{bmatrix},$$

which has eigenvalues $\lambda_1 = 8$ and $\lambda_2 = 2$ with corresponding eigenvectors $v_1 = \frac{1}{\sqrt{2}}(-1,1)$ and $v_2 = \frac{1}{\sqrt{2}}(1,1)$. So $\sigma_1 = \sqrt{8}$ and $\sigma_2 = \sqrt{2}$ and

$$V = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$
$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and

$$U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

So we have

$$W = UV^* = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix},$$
$$P = V\Sigma V^* = \frac{1}{2} \begin{bmatrix} \frac{3}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{3}{2\sqrt{2}} \end{bmatrix}.$$

Of course, this is for $A = WP$. □