MATH 317, Homework 4

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Solutions

Problem 1. Let $f : \mathbb{R} \setminus \{5\} \to \mathbb{R}$ by $f(x) = x \cos \frac{1}{x-5} - 5 \cos \frac{1}{x-5}$. Show that $\lim_{x \to 5} f(x) = 0$.

Proof. Fix $\epsilon > 0$ and let $\delta = \epsilon$. Then for $x \in \mathbb{R} \setminus \{5\}$ and $0 < |x-5| < \delta$ we have,

$$\begin{aligned}
|f(x) - 0| &= \left| x \cos\left(\frac{1}{x-5}\right) - 5 \cos\left(\frac{1}{x-5}\right) \right| \\
&= \left| (x-5) \cos\left(\frac{1}{x-5}\right) \right| \\
&\leq |x-5||1| \\
&\leq |x-5| \\
&< \delta = \epsilon
\end{aligned}$$

Thus f has a limit 0 at $x = 5$. $\square$

Problem 2. Let $f\colon (a,\infty) \to \mathbb{R}$ for some $a > 0$, and let $g\colon (0, \frac{1}{a}) \to \mathbb{R}$ be defined by $g(x) = f(\frac{1}{x})$. Prove that f has a limit point at ∞ if and only if g has a limit at 0.

Proof. For the forward direction, suppose that f has a limit L at ∞. Fix $\epsilon > 0$, then $\exists P > 0$ such that if $x > \max\{P, a\}$ then $\left|f(x) - L\right| < \epsilon$. With the same ϵ, fix $\delta > \frac{1}{P}$ and for $x \in (a,\infty)$ and $0 < |x - 0| < \delta$ we have,

$$|g(x) - L| = \left|f\left(\tfrac{1}{x}\right) - L\right|$$

But if we have $0 < |x - 0| < \frac{1}{P}$, then

$$\begin{aligned} &\leq \left|f(P) - L\right| \\ &< \epsilon \end{aligned}$$

Thus we have that $g(x)$ has a limit at $x = 0$. □

Next, suppose that $g(x)$ has a limit L at $x = 0$. Fix $\epsilon > 0$, then $\exists\delta > 0$ such that if $0 < |x - 0| < \delta$ we have $|g(x) - L| < \epsilon$. Keep the same ϵ, if $f(x)$ has a limit at ∞ then $\exists P > 0$ such that if $x > \max\{P, a\}$ we have $|f(x) - L| < \epsilon$. Let $P > \frac{1}{\delta}$, then we have,

$$\begin{aligned} |f(x) - L| &= \left|g\left(\tfrac{1}{x}\right) - L\right| \\ &\leq \left|g\left(\frac{1}{P}\right) - L\right| \\ &< \epsilon \end{aligned}$$

Since $\frac{1}{P} < \delta$, we know $f(x)$ has a limit at ∞. Thus we know that $f(x)$ has a limit at ∞ iff $f\left(\frac{1}{x}\right)$ has a limit at 0.

Problem 3. Give an example of a function $f\colon (0,1) \to \mathbb{R}$ which has a limit at every point of $(0,1)$ except at $x = \frac{1}{2}$.

Proof. First let's show that f does not have a limit $L \in \mathbb{R}$ at $x = \frac{1}{2}$. Fix $\epsilon = \frac{1}{4} + |L|$. Then $\forall \delta > 0$ and for $x \in D$, $|x - \frac{1}{2}| < \delta$ we have,

$$\begin{aligned} |f(x) - L| &= \left| \frac{1}{x - \frac{1}{2}} - L \right| \\ &\leq \left| \frac{1}{x - \frac{1}{2}} \right| + |L| \end{aligned}$$

Notice, $\left| \frac{1}{x-\frac{1}{2}} \right|$ is minimized if $\left| x - \frac{1}{2} \right|$ is maximized. Thus if we let $x = 1, 0$ we have $\left|1 - \frac{1}{2}\right| = \left|0 - \frac{1}{2}\right| = 1/2$. Since $x \in (0,1)$, we have,

$$\begin{aligned} \left| \frac{1}{x - \frac{1}{2}} \right| + |L| &< \frac{1}{2} + |L| \\ &> \epsilon \end{aligned}$$

Now we must show that all other points $x \in (0,1) \setminus \left\{\frac{1}{2}\right\}$ have defined limits. In fact, the limit at each point other than $x = \frac{1}{2}$ is the function evaluated at that point. More specifically, $\lim_{x \to x_0} f(x) = f(x_0)$ $\forall x_0 \in (0,1) \setminus \left\{\frac{1}{2}\right\}$. Fix $\epsilon > 0$. Then let $\delta < \frac{\epsilon |x - x_0 + 2xx_0|}{2}$ and let $x, x_0 \in (0,1) \setminus \left\{\frac{1}{2}\right\}$ be such that $0 < |x - x_0| < \delta$. Then,

$$\begin{aligned} |f(x) - f(x_0)| &= \left| \frac{1}{x - \frac{1}{2}} - \frac{1}{x_0 - \frac{1}{2}} \right| \\ &= \left| \frac{\left(x_0 - \frac{1}{2}\right) - \left(x - \frac{1}{2}\right)}{\left(x - \frac{1}{2}\right)\left(x_0 - \frac{1}{2}\right)} \right| \\ &= \left| \frac{2(x - x_0)}{x - x_0 + xx_0} \right| \\ &< \frac{2\delta}{|x - x_0 + 2xx_0|} \\ &< \epsilon \end{aligned}$$

Thus we know a limit exists $\forall x \in (0,1) \setminus \left\{\frac{1}{2}\right\}$. $\square$

Problem 4. Let $f: D \to \mathbb{R}$ with x_0 an accumulation point of D, and suppose that f has a limit at x_0. Prove *from the definition of the limit* that this limit is unique.

Proof. Suppose that $\lim_{x\to x_0} f(x) = L_1$ and $\lim_{x\to x_0} f(x) = L_2$ where $L_1 \neq L_2$. Since we have the first limit, $\forall \epsilon > 0$, $\exists \delta_1 > 0$ such that if $0 < |x - x_0| < \delta_1$ and $x \in D$ we have $|f(x) - L_1| < \epsilon$. Fix $\epsilon = |L_1 - L_2|$, then $\exists \delta_2 > 0$ such that if $0 < |x - x_0| < \delta_2$ we have, $|f(x) - L_2| < \epsilon$. Pick $\delta = \min\{\delta_1, \delta_2\}$ and we have,

$$\begin{aligned}
|f(x) - L_2| &= |f(x) - L_1 + L_1 - L_2| \\
&\leq |f(x) - L_1| + |L_1 - L_2| \\
&< \epsilon + \epsilon = 2\epsilon
\end{aligned}$$

Which is a contradiction since. Thus $L_1 = L_2$. $\square$

Problem 5. Define $f\colon (0,1) \to \mathbb{R}$ by $f(x) = \frac{x^3+6x^2+x}{x^2-6x}$. Determine whether or not f has a limit at 0 and prove your claim.

Proof. First, let's do some algebra and reduce the fraction (all joking aside, I used *FullSimplify* in *Mathematica*).

$$\begin{aligned} f(x) &= \frac{x^3-6x^2+x}{x^2-6x} \\ &= \frac{1+6x^4}{x-6} \end{aligned}$$

From here, it is fairly easy to see that plugging in 0 is possible, and gives us the result $\frac{-1}{6}$. Thus, I guess the limit must be that. Fix ϵ and let $\delta = \frac{9}{\epsilon}$ and for $x \in (0,1)$ and $0 < |x-0| < \delta$ we have,

$$\begin{aligned} \left| f(x) - \left(\frac{1}{6}\right) \right| &\leq \left| \frac{1+6}{x-6} \right| + \left| \frac{1}{6} \right| \\ &= \left| \frac{7}{x-6} \right| + \frac{1}{6} \\ &\leq \frac{7}{|x|+|6|} + \frac{1}{6} \\ &= \frac{7}{x+6} + \frac{\frac{x}{6}+1}{x+6} \\ &= \frac{8+\frac{x}{6}}{x+6} \\ &\leq \frac{9}{x} \\ &< \epsilon \end{aligned}$$

So the limit at 0 exists and is equal to $\frac{-1}{6}$. $\square$

Problem 6. Suppose $f, g, h\colon D \to \mathbb{R}$ with x_0 an accumulation point of D. Suppose further that $f(x) \leq g(x) \leq h(x)$ for all $x \in D$ and that f and h both have limits at x_0 with $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x)$.

(i) Prove that g has a limit at x_0.

(ii) Prove that $lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = \lim_{x\to x_0} h(x)$.

Proof. We will do (i) and (ii) in just one cohesive proof. Suppose that we have $f(x) \leq g(x) \leq h(x)$ $\forall x \in D$. Also we have $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = L$. Fix $\epsilon > 0$, then $\exists \delta_1 > 0$ such that $\forall x \in D$ where $0 < |x - x_0| < \delta_1$ we have $|f(x) - L| < \epsilon$. With the same ϵ, $\exists \delta_2 > 0$ such that $\forall x \in D$ where $0 < |x - x_0| < \delta_2$ we have $|h(x) - L| < \epsilon$. Let $\delta = \min\{\delta_1, \delta_2\}$, then we have,

$$\begin{aligned} |g(x) - L| < \epsilon &\iff -\epsilon < g(x) - L < \epsilon \\ &\iff -\epsilon + L < g(x) < \epsilon + L \end{aligned}$$

But since we have chosen $0 < |x - x_0| < \delta$ and since $\forall x \in D$ we have $f(x) \leq g(x) \leq h(x)$. Thus we have,

$$-\epsilon + L < f(x) \leq g(x) \leq h(x) < \epsilon + L$$

Thus $\lim_{x\to x_0} g(x) = \lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = L$. $\square$

Problem 7. Assume that $f : \mathbb{R} \to \mathbb{R}$ is such that $f(x+y) = f(x)f(y)$ for every $x, y \in \mathbb{R}$, and suppose that f has a limit as 0.

(i) Prove that f has a limit at every point in $\mathbb{R}$.

(ii) Prove that $f(x) = 0$ for all $x \in \mathbb{R}$ or $\lim_{x\to 0} f(x) = 0$.

Proof. Since we know that $f(x+y) = f(x)f(y) \quad \forall x, y \in \mathbb{R}$ we know that $f(0) = f(x-x) = f(x)f(-x)$. Since x is arbitrary, f must be defined for all $x \in mathbbR$ and thus the limit at any point is $f(x)$. Thus we have (i). Next, suppose that $f(0) = L \neq 1$ and $L \neq 0$, then fix $\epsilon > 0$ then $\exists \delta > 0$ such that $\forall x \in \mathbb{R}$ where $|x| < \delta$ we have $|f(x) - 1| < \epsilon$ since the limit is defined as the function evaluated at the point. But this means,

$$\begin{aligned}
|f(x) - L| &= |f(x+0) - L| \\
&= |f(x)f(0) - L| \\
&= \left| f(x) - \frac{L}{f(0)} \right| \\
\implies \frac{L}{f(0)} &= L
\end{aligned}$$

But since $L \neq 1$ and $L \neq 0$ this is a contradiction. Thus either $L = 1$ or $L = 0$.

If $f(0) \neq 1$ then we have $L = \frac{1}{f(0)} = \frac{1}{0}$ which is not possible. However, if we allow $f(0) = 0$ and thus $\lim_{x\to 0} f(x) = 0$, we have,

$$\begin{aligned}
|f(x) - 0| &= |f(x)| \\
&= |f(x+0)| \\
&= |f(x)f(0)| \\
&= 0 < \epsilon
\end{aligned}$$

Thus if $f(0) \neq 1$ then $f(x) = 0$ for all $x \in \mathbb{R}$.

If we have $f(0) \neq 0$ then we must have $f(0) = 1$ or we contradict the statement that $L = 1$ or $L = 0$. Thus we know that $f(x) = f(x)f(0) = 0$ for all $x \in \mathbb{R}$ or $\lim_{x\to 0} f(x) = 1$. $\square$