Clifford Analysis and a Noncommutative Gelfand Representation

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Overview

Section 1

Introduction

Motivating problems

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- ***Electrical Impedance Tomography (EIT)*** asks whether one can determine the conductivity of a medium from the voltage-to-current map.

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- The ***Calderón problem*** replaces the medium with a manifold M, conductivity with g, and replaces the voltage-to-current map with the Dirichlet-to-Neumann operator Λ.

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- Do these functions also contain geometric information such as metric data?
- How much can we learn about M if our data is supported only on the boundary?

Subsection 1

Preliminaries

- ***Clifford algebra*** originated in 1878 with William Kingdon Clifford's work that extends Hermann Grassmann's ***exterior algebra***.

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- ***Clifford analysis*** arrived in the 1980's due to Hestenes, Sobczyk, Sommen, Brackx, and Delenghe in order to enrich Élie Cartan's ***differential forms***. See: **[Hestenes, Sobczyk: 1984]** and **[Doran, Lasenby: 2003]**.

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$$\mathcal{T}(V) \coloneqq \bigoplus_{j=0}^{\infty} V^{\otimes j} = \mathbb{F} \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots.$$

- Form the ***Clifford algebra*** via a quotient

$$C\ell(V, Q) \coloneqq \mathcal{T}(V) / \langle \boldsymbol{v} \otimes \boldsymbol{v} - Q(\boldsymbol{v}) \rangle.$$

Geometric and exterior algebras

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- The ***exterior algebra*** is given by

$$\bigwedge(V) \coloneqq C\ell(V, 0).$$

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  - The bivector part is antisymmetric: $\boldsymbol{u} \wedge \boldsymbol{v} = -\boldsymbol{v} \wedge \boldsymbol{u}$.

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    - Elements of the even grade subalgebra, $\mathcal{G}^+$, are called ***spinors***.
- The most general elements are ***multivectors*** and are given by

$$A = \sum_{r=0}^{n} \langle A \rangle_r,$$

where $\langle A \rangle_r \in \mathcal{G}^r$ extracts the grade r part of A. So $\mathcal{G} = \bigoplus_{r=0}^{n} \mathcal{G}^r$.

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- The most important products are

$$A_r \cdot B_s \coloneqq \langle A_r B_s \rangle_{|r-s|} \qquad A_r \wedge B_s \coloneqq \langle A_r B_s \rangle_{r+s}$$

$$A_r \rfloor B_s \coloneqq \langle A_r B_s \rangle_{s-r} \qquad A_r \lfloor B_s \coloneqq \langle A_r B_s \rangle_{r-s}$$

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- The ***reverse*** of a multivector is extended linearly from the action on r-blades by

$$\boldsymbol{A_r}^\dagger = (\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_r)^\dagger = \boldsymbol{v}_r \wedge \cdots \wedge \boldsymbol{v}_1.$$

Inner product and norm

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- Define the ***multivector norm*** by

$$|A| \coloneqq \sqrt{(A, A)}.$$

Adjoint

Note the reverse acts as an adjoint by

$$(CA, B) = (A, C^\dagger B)$$

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- We define the ***unit pseudoscalar*** by

$$\boldsymbol{I} \coloneqq \frac{1}{|\boldsymbol{\mu}|}\boldsymbol{\mu}.$$

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- If $|\boldsymbol{A_r}| = 1$, then $\boldsymbol{A_r}$ is a ***unit blade***.
- Unit r-blades correspond to r-dimensional subspaces so they correspond to points in $\mathrm{Gr}(r, n)$.

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- Note $A_r^{\perp} \in \mathcal{G}^{n-r}$, like the Hodge star $\star$.

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- Both are grade preserving.

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  - Take the standard basis $\boldsymbol{e}_1, \boldsymbol{e}_2$, and define $\boldsymbol{B}_{12} = \boldsymbol{e}_1 \boldsymbol{e}_2$ and note $\boldsymbol{B}_{12}^2 = -1$. Thus,

$$(u_1 + v_1 \boldsymbol{B}_{12})(u_2 + v_2 \boldsymbol{B}_{12}) = u_1 u_2 - v_1 v_2 + (u_1 v_2 + u_2 v_1)\boldsymbol{B}_{12}.$$

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  - Right multiplication by $\boldsymbol{B}_{12}$ rotates counter-clockwise by $\pi/2$.

Section 2

Clifford analysis

Multivector Fields

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- Retain the same naming scheme as before.

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$$\nabla_{\boldsymbol{u}} A_r = \langle \nabla_{\boldsymbol{u}} A_r \rangle_r.$$

- $\nabla_{\boldsymbol{u}}$ is compatible with dot and wedge since

$$\nabla_{\boldsymbol{u}}(A \cdot B) = (\nabla_{\boldsymbol{u}} A) \cdot B + A \cdot (\nabla_{\boldsymbol{u}} B)$$

$$\nabla_{\boldsymbol{u}}(A \wedge B) = (\nabla_{\boldsymbol{u}} A) \wedge B + A \wedge (\nabla_{\boldsymbol{u}} B).$$

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$$\boldsymbol{\nabla}(AB) = \dot{\boldsymbol{\nabla}} \dot{A} B + \dot{\boldsymbol{\nabla}} A \dot{B}.$$

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- Note $\boldsymbol{\nabla}^2 = \Delta$, the Laplace-Beltrami operator.

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- Specifically,

$$\operatorname{curl}(\boldsymbol{v}) = (\boldsymbol{\nabla}\wedge\boldsymbol{v})^{\perp}$$

Differential forms

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- Define the ***r-dimensional directed measure***

$$dX_r \coloneqq \mathbf{v}_{j_1} \wedge \cdots \wedge \mathbf{v}_{j_r} \, dx^{j_1} \cdots dx^{j_r}$$

where $1 \leq j_1 < \cdots < j_r \leq n$ and summation is implied.

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- Define an r-form a_r by

$$\alpha_r = A_r \cdot dX_r^{\dagger}$$

  where $A_r = \frac{1}{r!} a_{i_1 \cdots i_r} \boldsymbol{v}^{i_1} \wedge \cdots \wedge \boldsymbol{v}^{i_r}$.

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- Refer to A_r the *multivector equivalent* of α_r.

Exterior algebra and calculus

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- Given r-forms a_r, b_r, and an s-form c_s, we have

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- The Hodge star on multivector equivalents is

$$\star\, a_r = (\boldsymbol{I}^{-1} A_r)^\dagger \cdot dX_{n-r}^\dagger$$

Volume form

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- The ***volume form*** on M is given in local coordinates by

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- We integrate scalar fields A_0 on M by

$$\int_M A_0^{\perp} \cdot dX_n = \int_M A_0 \mu.$$

Multivector field inner product

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- $\ll A_r, B_s \gg = 0$ when $r \neq s$ so the L^2-direct sum agrees with the grade based direct sum.

Boundary

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- The boundary volume form is

$$\mu_\partial \coloneqq \boldsymbol{I}_\partial^{-1} \cdot dX_{n-1}$$

and we define

$$\ll A, B \gg_\partial \coloneqq \frac{1}{\mathrm{vol}(M)} \int_{\partial M} (A, B) \mu_\partial.$$

Multivector valued integrals

We can define a multivector valued integral on an oriented submanifold R by taking $A \in \mathcal{G}(M)$ and computing

$$\int_R A\boldsymbol{I}_R \mu_R.$$

Theorem (Hestenes, Sobczyk, 1984)

Let $A, B \in \mathcal{G}(M)$, *then*

$$\begin{aligned}
\int_M \dot{A}\dot{\boldsymbol{\nabla}}\boldsymbol{I}\mu &= \int_{\partial M} A\boldsymbol{I}_\partial \mu_\partial \\
\int_M \boldsymbol{I}\boldsymbol{\nabla} B\mu &= \int_{\partial M} \boldsymbol{I}_\partial B\mu_\partial \\
\int_M \dot{A}\dot{\boldsymbol{\nabla}}\boldsymbol{I} B\mu &= (-1)^n \int_M A\boldsymbol{I}\boldsymbol{\nabla} B\mu + \int_{\partial M} A\boldsymbol{I}_\partial B\mu_\partial .
\end{aligned}$$

Theorem

We have the Green's formula for the gradient

$$\ll A, \boldsymbol{I}\nabla B \gg = (-1)^n \ll \nabla A, \boldsymbol{I} B \gg + \ll A, \boldsymbol{I}_\partial B \gg_\partial .$$

Special fields

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- Let $f = u + v\boldsymbol{B} \in \mathcal{G}_2^+(\mathbb{R}^2)$ then $\nabla f = 0$ yields the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

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$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

- Define the ***gradients***

$$\nabla\mathcal{G}(M) \coloneqq \{\nabla A \mid A \in \mathcal{G}(M) \text{ and } A|_{\partial M} = 0\}.$$

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  where S_n is the surface area of the unit ball.

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$$E(x) \coloneqq \frac{1}{S_n} \frac{x}{|x|^n}$$
where S_n is the surface area of the unit ball.
- Note,
$$\nabla E(x) = -\dot{E}(x)\dot{\nabla} = \delta(x).$$

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- For the remainder, take M imbedded in $\mathbb{R}^n$ with $n \geq 2$.
- Define the vector field
$$E(x) \coloneqq \frac{1}{S_n} \frac{x}{|x|^n}$$
where S_n is the surface area of the unit ball.
- Note,
$$\nabla E(x) = -\dot{E}(x)\dot{\nabla} = \delta(x).$$
- Define the ***Cauchy kernel*** by $G(x, x') \coloneqq E(x' - x)$.

Cauchy integral

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- Let $A \in \mathcal{M}(M)$, then we have the ***Cauchy integral formula***

$$A(x) = (-1)^n \boldsymbol{I}^{-1} \int_{\partial M} G(x, x') \boldsymbol{I}_\partial(x') A(x') \mu_\partial(x').$$

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- This uniquely determines a monogenic field from boundary values.

Lemma

Let $A \in \mathcal{M}(M)$ *be such that* $A|_{\partial M} = 0$. *Then* $A = 0$ *on all of* M.

Lemma

Fix a multivector field $A \in \mathcal{G}(M)$. *If*

$$\ll A, B \gg = 0$$

for all $B \in \mathcal{G}(M)$ *with* $B|_{\partial M} = 0$, *then* $A = 0$.

Theorem (Clifford-Hodge-Morrey Decomposition)

The space of multivector fields $\mathcal{G}(M)$ has the L^2-orthogonal decomposition

$$\mathcal{G}(M) = \mathcal{M}(M) \oplus \boldsymbol{I}\nabla\mathcal{G}(M).$$

Proof.

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- *Orthogonality:* Let $A \in \mathcal{M}(M)$ and $\boldsymbol{I}\nabla B \in \boldsymbol{I}\nabla\mathcal{G}(M)$ and note

$$\ll A, \boldsymbol{I}\nabla B \gg = (-1)^n \ll \nabla A, \boldsymbol{I}B \gg + \ll A, \boldsymbol{I}_\partial B \gg = 0,$$

by the multivector Green's formula.

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- Note,

$$0 = \ll C, \boldsymbol{I}\nabla B \gg = \ll \nabla C_0, \boldsymbol{I}B \gg .$$

- By the previous lemmas, it must be that $C_0 = 0$. Hence the orthogonal complement to $\boldsymbol{I}\nabla\mathcal{G}(M)$ is $\mathcal{M}(M)$.

Comparing to Hodge-Morrey

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- The Hodge-Morrey decomposition reads

$$\Omega^r(M) = \underbrace{\mathcal{E}_D^r(M)}_{\mathrm{Im}(\nabla\wedge)} \oplus \underbrace{\mathcal{C}_N^r(M)}_{\mathrm{Im}(\nabla\rfloor)} \oplus \underbrace{\mathcal{H}^r(M)}_{\mathrm{Ker}(\nabla)}.$$

via [Schwarz: 1995].

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via **[Schwarz: 1995]**.

- Whereas the Clifford-Hodge-Morrey decomposition ignores specific grades

$$\mathcal{G}(M) = \mathcal{M}(M) \oplus \boldsymbol{I}\nabla\mathcal{G}(M).$$

Section 3

Gelfand theory

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- Belishev and Vakulenko as whether this is true in higher dimensions.
- We prove an analogous result for an arbitrary $\mathbb{B}$ in $\mathbb{R}^n$.
- This approach can hopefully be used to prove the analogous result for any smooth orientable Riemannian manifold M.

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- Thus, we can find a g that is conformal with the complex structure.

Subsurface spinor fields

Subsurface spinor fields

- Let $\boldsymbol{B} \in \mathcal{G}(M)$ be a constant unit 2-blade, then $f_+ \in \mathcal{G}^+(M)$ satisfying

$$f_+ = \mathrm{P}_{\boldsymbol{B}} \circ f_+ \circ \mathrm{P}_{\boldsymbol{B}}$$

  is a ***subsurface spinor field***. Let $\mathcal{G}^+_{\boldsymbol{B}}(M)$ denote the space such fields.

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- The space of monogenic subsurface spinors

$$\mathcal{A}_{\boldsymbol{B}}(M) = \{f_+ \in \mathcal{G}^+_{\boldsymbol{B}}(M) \mid \nabla f_+ = 0\}$$

  is a commutative unital Banach algebra.

Functionals

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- Define the ***spinor dual*** $\mathcal{M}^*(M)$ as the continuous right $\mathcal{G}_n^+$-module homomorphisms

  $$\mathcal{M}^*(M) \coloneqq \{l\colon \mathcal{M}^+(M) \to \mathcal{G}_n^+ \mid l(fs+g) = l(f)s + l(g),\ \forall f, g \in \mathcal{M}(M),\ s \in \mathcal{G}_n^+\}$$

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  and refer to the elements as ***spin functionals***.
- Assert the weak-$\star$ topology on $\mathcal{M}^*(M)$ so that every $x \in M$ corresponds to a continuous map on $\mathcal{M}^*(M)$.

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  and refer to the elements as ***spin characters***.
- One example of such characters are point evaluations $\delta(f) = f(x^\delta)$.
- We show these are the only elements in the spectrum.

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- Note $z_{ij} \in \mathcal{A}_{\boldsymbol{B}_{ij}}(\mathbb{B}_{R,w})$.

Monogenic polynomials

Monogenic polynomials

- Let σ be a permutation of $\{2, 3, \ldots, n\}$, then

$$p_{j_2 \cdots j_n}(x) = \frac{1}{j!} \sum_{\text{permutations}} z_{1\sigma(1)}(x) \cdots z_{1\sigma(j)}(x)$$

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- Collect these into the set of ***monogenic polynomials***

$$\mathcal{M}^{\mathcal{P}}(\mathbb{B}_{R,w}) = \left\{ \sum_{j=0}^{N} \left(\sum_{\substack{j_2 \cdots j_n \\ j_2 + \cdots + j_n = j}} p_{j_2 \cdots j_n} a_{j_2 \cdots j_n} \right) \mid j_2 + \cdots + j_n = j, N \in \mathbb{N},\ a_{j_2 \cdots j_n} \in \mathcal{G}_n \right\}.$$

Lemma (Density)

The space $\mathcal{M}^{\mathcal{P}}(\mathbb{B}_{R,w})$ is dense in $\mathcal{M}^{+}(\mathbb{B}_{R,w})$.

Proof sketch.

- Let $f \in \mathcal{M}^{+}(\mathbb{B}_{R,w})$ and use the Cauchy integral formula to define the coefficients $a_{j_2 \cdots j_n} \in \mathcal{G}_n^{+}$ by

$$a_{j_2 \cdots j_n} = \int_{\partial \mathbb{B}_{R,w}} \frac{\partial^{j} G(w,y)}{\partial y_2^{j_2} \cdots \partial y_n^{j_n}} \boldsymbol{\nu}(y) f(y) \mu_{\partial}(y),$$

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- Then

$$f(x) = \sum_{j=0}^{\infty} \left(\sum_{\substack{j_2 \cdots j_n \\ j_2 + \cdots + j_n = j}} p_{j_2 \cdots j_n}(x-w) a_{j_2 \cdots j_n} \right),$$

converges pointwise for $x \in \mathbb{B}_{R,w}$ by [Ryan, 2004].

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- By linearity, we can note that for $\delta \in \mathfrak{M}(\mathbb{B}_{R,w})$

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- On each monogenic polynomial

$$\delta(p_{j_2 \ldots j_n}(x)) = \frac{1}{j!} \sum_{\text{permutations}} \delta\Big((z_{1\sigma(1)}(x)\Big) \cdots \delta\Big(z_{1\sigma(j)}(x)\Big)$$

by the multiplicativity of δ.

Lemma (Point evaluation)

Let $\delta \in \mathfrak{M}(\mathbb{B}_{R,w})$ *and* $z_{ij} \in \mathcal{A}_{\boldsymbol{B}_{ij}}(\mathbb{B}_{R,w})$, *then* $\delta(z_{ij}) = z_{ij}(x^\delta)$ *for some* $x^\delta \in \mathbb{R}^n$.

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- The set of constants α and β are determined by n independent numbers, so we can say $\delta(z_{ij}) = z_{ij}(x^\delta)$ for some $x^\delta \in \mathbb{R}^n$.

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  $$\lim_{n\to\infty} \delta(G_n) = \lim_{n\to\infty} G_n(x^\delta)$$

  so this sequence not converge due to a singularity at x^δ.
- Hence, it must be that $x^\delta \in \mathbb{B}_{R,w}$ by continuity of δ.

Theorem (Noncommutative Gelfand representation)

For any $\delta \in \mathfrak{M}(\mathbb{B}_{R,w})$, *there is a point* $x^\delta \in \mathbb{B}_{R,w}$ *such that* $\delta(f) = f(x^\delta)$ *for any* $f \in \mathcal{M}(\mathbb{B}_{R,w})$. *Given the weak-⋆ topology on* $\mathcal{M}^*(\mathbb{B}_{r,w})$, *the map*

$$\gamma : \mathfrak{M}(\mathbb{B}_{R,w}) \to \mathbb{B}_{R,w}, \quad \delta \mapsto x^\delta$$

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- To see that γ is a homeomorphism, take a sequence $\delta_n \to \delta$ in $\mathfrak{M}(\mathbb{B}_{R,w})$.
- For $f \in \mathcal{M}^+(\mathbb{B}_{R,w})$ we have

$$f(\gamma(\delta_n)) = f(x^{\delta_n}) = \delta_n(f) = \gamma^{-1}(x^{\delta_n})(f).$$

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- Taking $n \to \infty$ shows γ and γ^{-1} are continuous so γ is a homeomorphism.

Section 4

Future work

Calderón problem

Calderón problem

- **Question:** Let (M, g) be an unknown Riemannian manifold with known boundary ∂M. Consider the forward problem

$$\begin{cases} \Delta\omega = 0 & \text{in } M \\ \iota^*\omega = \phi & \text{on } \partial M \end{cases}$$

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- Define the ***Dirichlet-to-Neumann map*** on forms by $\Lambda\phi = \iota^*(\star d\omega)$.
- Can we determine (M,g) from Λ?

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- The problem has been solved in dimension $n = 2$ **[Belishev: 2003]**.
- Solved in dimensions $n \geq 3$ when M is an analytic manifold **[Lassas, Taylor, Uhlmann: 2003]**.
- The smooth cases is still unsolved.

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- If Λ can provide us $b|_{\partial M}$, then we can possibly reconstruct $\mathcal{M}^{+}(M)$.

Thoughts

- Even monogenic fields have harmonic components. Given a harmonic r-vector A_r, can we reconstruct a monogenic multivector containing A_r?
- For $n = 3$, the scalar potential u and magnetic bivector field b are two parts of a monogenic field $f = u + b$ due to Ohm's and Ampere's laws

$$-\nabla \wedge u = \boldsymbol{j} = \nabla \rfloor b.$$

- If Λ can provide us $b|_{\partial M}$, then we can possibly reconstruct $\mathcal{M}^+(M)$.
- Given the algebraic structure of each $\mathcal{A}_{\boldsymbol{B}}(M) \subset \mathcal{M}^+(M)$, can this be used to determine g?

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- These problems could include spacetime problems where the metric g is of mixed signature.

Section 5

Conclusions

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Conclusion

- We have utilized multivector fields to serve as a meaningful generalization of both the complex numbers and differential forms.
- This provides a new way to decompose fields on domains of $\mathbb{R}^n$ and this can likely be generalized to arbitrary compact orientable pseudo-Riemannian manifolds.
- Likewise, we have proven that the monogenic fields contain a wealth of topological information and this information is supported on the boundary by the Cauchy integral formula.

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- We have submitted *Model and Data Reduction for Data Assimilation: Particle Filters Employing Projected Forecasts and Data with Application to a Shallow Water Model*.
- We are continuing to work to apply our scheme to new models such as the Modular Arbitrary-Order Ocean-Atmospheric Model (MAOOAM).

Thank you!