MATH 560, Homework 7

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Solutions

Problem 1. (§6.1 Problem 6.) Complete the proof of Theorem 6.1. It is as follows:
Let V be an inner product space. Then for $x, y, z \in V$ and $c \in \mathbb{F}$, the following statements are true.

(a) $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$.

(b) $\langle x, cy\rangle = \bar{c}\langle x, y\rangle$.

(c) $\langle x, 0\rangle = \langle 0, x\rangle = 0$.

(d) $\langle x, x\rangle = 0$ if and only if $x = 0$.

(e) If $\langle x, y\rangle = \langle x, z\rangle$ for all $x \in V$, then $y = z$.

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Proof (a). We have

$$\begin{aligned}\langle x, y+z\rangle &= \overline{\langle y+z, x\rangle}\\ &= \overline{\langle y, x\rangle} + \overline{\langle z, x\rangle}\\ &= \langle x, y\rangle + \langle x, z\rangle.\end{aligned}$$ □

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Proof (b). We have

$$\begin{aligned}\langle x, cy\rangle &= \overline{\langle cy, x\rangle}\\ &= \overline{c}\overline{\langle y, x\rangle}\\ &= \overline{c}\langle x, y\rangle.\end{aligned}$$ □

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Proof (c). Let $v \in V$ then

$$\begin{aligned}\langle x, 0v\rangle &= \overline{0}\langle x, v\rangle\\ &= 0.\end{aligned}$$

Similarly

$$\begin{aligned}\langle x, 0v\rangle &= \overline{\langle 0v, x\rangle}\\ \implies &= 0 \quad \text{from above.}\end{aligned}$$

Note that $0v = 0$ and we are done. □

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Proof (d). The converse direction is immediate: Let $x = 0$ then $\langle 0, 0\rangle = 0$. For the forward direction let $\langle x, x\rangle = 0$. Then we have that $\langle x, x\rangle > 0$ if $x \neq 0$ by definition. Thus if $\langle x, x\rangle = 0$ we necessarily have $x = 0$.

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Proof (e). Suppose that $\langle x, y\rangle = \langle x, z\rangle$ for every $x \in V$. Then for any x

$$\begin{aligned}\langle x, y-z\rangle &= \langle x, y\rangle - \langle x, z\rangle\\ &= \langle x, z\rangle - \langle x, z\rangle\\ &= 0.\end{aligned}$$

So $y - z = 0$ which means that $y = z$. □

Problem 2. (§6.1 Problem 7.) Complete the proof of Theorem 6.2. It is as follows:
Let V be an inner product space over $\mathbb{F}$. Then for all $x, y \in V$ and $c \in \mathbb{F}$, the following statements are true.

(a) $\|cx\| = |c| \cdot \|x\|$.

(b) $\|x\| = 0$ if and only if $x = 0$. In any case, $\|x\| \geq 0$.

(c) (Cauchy-Schwarz Inequality) $|\langle x, y\rangle \leq \|x\| \cdot \|y\|$.

(d) (Triangle Inequality) $\|x + y\| \leq \|x\| + \|y\|$.

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Proof (a).

$$\begin{aligned} \langle cx, cx\rangle &= c\bar{c}\langle x, x\rangle && \text{by definition} \\ \implies \|cx\|^2 &= |c|^2\|x\|^2 \\ \implies \|cx\| &= |c|\|x\|. \end{aligned}$$

□

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Proof (b). Suppose that $\|x\| = 0$. Then $\sqrt{\langle x, x\rangle} = 0$. By Theorem 6.2 we have that $x = 0$. If $x = 0$ then $\langle x, x\rangle = 0$. Otherwise, by definition of an inner product space we have that if $x \neq 0$ then $\langle x, x\rangle > 0$ which implies that $\|x\| > 0$ if x is nonzero. So in any case, $\|x\| \geq 0$. □

Note that the proof for (c) and (d) are given in the text.

Problem 3. (§6.1 Problem 9.) Let β be a basis for a finite-dimensional inner product space.

(a) Prove that if $\langle x, z\rangle = 0$ for all $z \in \beta$, then $x = 0$.

(b) Prove that if $\langle x, z\rangle = \langle y, z\rangle$ for all $z \in \beta$, then $x = y$.

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Proof (a). Suppose that $\langle x, z_j\rangle = 0$ for all $z_j \in \beta$. Then since $x \in V$ we can write $x = \sum_{i=1}^{n} \alpha_i z_i$. Then we have for $z_j \in \beta$

$$\begin{aligned} 0 = \langle x, z_j\rangle &= \left\langle \sum_{i=1}^{n} \alpha_i z_i, z_j \right\rangle \\ &= \sum_{i=1}^{n} \alpha_i \langle z_i, z_j\rangle \end{aligned}$$

which means $x = 0$. □

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Proof (b). Consider then $\langle x - y, z\rangle = 0$. This means $x - y = 0$ by (a) and thus $x = y$. □

Problem 4. (§6.1 Problem 10.) Let V be an inner product space, and suppose that x and y are orthogonal vectors in V. Prove that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$. Deduce the Pythagorean theorem in $\mathbb{R}^2$.

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Proof. We have

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y\rangle + 2\langle x,y\rangle\langle y,y\rangle \\ &= \langle x+y, x+y\rangle + \langle y,y\rangle && \text{since } x \text{ and } y \text{ are orthogonal} \\ &= \|x\| + \|y\|. \end{aligned}$$

Then in $\mathbb{R}^2$ we have ae_1 and be_2 in $\mathbb{R}^2$ as the sides of the triangle and $c = ae_1 + be_2$ as the hypotenuse. Then

$$\|c\|^2 = \|ae_1 + be_2\| = \|a\|^2 + \|b\|^2. \qquad \square$$

Problem 5. (§6.1 Problem 12.) Let $\{v_1, v_2, ..., v_k\}$ be an orthogonal set in V, and let $a_1, a_2, ..., a_k$ be scalars. Prove that

$$\left\| \sum_{i=1}^{k} a_i v_i \right\|^2 = \sum_{i=1}^{k} |a_i|^2 \| v_i \|^2.$$

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Proof. We have

$$\begin{aligned} \left\| \sum_{i=1}^{k} a_i v_i \right\|^2 &= \left\langle \sum_{i=1}^{k} a_i v_i, \sum_{i=1}^{k} a_i v_i \right\rangle \\ &= \left\langle a_1 v_1, \sum_{i=1}^{k} a_i v_i \right\rangle \\ &= \langle a_1 v_1, a_1 v_1 \rangle + \cdots + \langle a_k v_k, a_k, a_k v_k \rangle && \text{because of orthonormality} \\ &= \sum_{i=1}^{k} |a_i|^2 \| v_i \|^2 \end{aligned}$$

□

Problem 6. (§6.1 Problem 26.) Let $\|\cdot\|$ be a norm on a vector space V, and define, for each ordered pair of vectors, the scalar $d(x,y) = \|x-y\|$, called the distance between x and y. Prove the following results for all $x, y, z \in V$.

(a) $d(x,y) \geq 0$.

(b) $d(x,y) = d(y,x)$.

(c) $d(x,y) \leq d(x,z) + d(z,y)$.

(d) $d(x,x) = 0$.

(e) $d(x,y) \neq 0$ if $x \neq y$.

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Proof (a). $d(x,y) = \|x-y\| \geq 0$ by properties of the norm. □

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Proof (b). $d(x,y) = \|x-y\| = \|y-x\| = d(y,x)$ again by properties of the norm. □

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Proof (c). We use our favorite analysis trick.

$$\begin{aligned} d(x,y) &= \|x-y\| \\ &= \|x-z+z-y\| \\ &\leq \|x-z\| + \|z-y\| \\ &= d(x,y) + d(z,y) \end{aligned}$$

□

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Proof (d). $d(x,x) = \|x-x\| = 0$. □

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Proof (e). $d(x,y) = \|x-y\| > 0$ means that $x-y$ is nonzero and thus $x \neq y$. □

Problem 7. (§6.2 Problem 2 (a),(j).) Apply the Gram-Schmidt process to the given subset S of the inner product space V to obtain an orthogonal basis for $\mathrm{span}(S)$. Then normalize the vectors in this basis to obtain an orthonormal basis β for $\mathrm{span}(S)$, and compute the Fourier coefficients of the given vector relative to β. Finally, use Theorem 6.5 to verify your result.

(a) $V = \mathbb{R}^3$, $S = \{(1,0,1),(0,1,1),(1,3,3)\}$, and $x = (1,1,2)$.

(j) $V = \mathbb{C}^4$, $S = \{(1,i,2-i,-1),(2+3i,3i,1-i,2i),(-1+7i,6+10i,11-4i,3+4i)\}$, and $x = (-2+yi,6+9i,9-3i,4+4i)$.

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Proof (a). We begin by letting $v_1 = w_1$. Then

$$\begin{aligned} v_2 &= w_2 - \frac{\langle w_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 \\ &= \left(-\frac{1}{2}, 1, \frac{1}{2}\right). \end{aligned}$$

Then

$$\begin{aligned} v_3 &= w_3 - \frac{\langle w_3, v_2\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle w_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2 \\ &= \left(\frac{1}{3}, \frac{1}{3}, \frac{-1}{3}\right). \end{aligned}$$

Then we normalize and get

$$\begin{aligned} \frac{v_1}{\|v_1\|} &= \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \\ \frac{v_2}{\|v_2\|} &= \left(\frac{-1}{\sqrt{6}}, \frac{4}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \\ \frac{v_3}{\|v_3\|} &= \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right). \end{aligned}$$

For the first way of finding coefficients we have

$$f_1 \frac{v_1}{\|v_1\|} + f_2 \frac{v_2}{\|v_2\|} + f_3 \frac{v_1}{\|v_1\|} = x$$

which yields

$$\begin{aligned} f_1 &= \frac{3}{\sqrt{2}} \\ f_2 &= \frac{3}{\sqrt{6}} \\ f_3 &= 0. \end{aligned}$$

This matches up with Theorem 6.5

$$\begin{aligned} f_1 &= \left\rangle x, \frac{v_1}{\|v_1\|} \right\rangle = \frac{3}{\sqrt{2}} \\ f_2 &= \left\rangle x, \frac{v_2}{\|v_2\|} \right\rangle = \frac{3}{\sqrt{6}} \\ f_3 &= \left\rangle x, \frac{v_3}{\|v_3\|} \right\rangle = 0 \end{aligned}$$

□

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Proof (b). We begin by letting $v_1 = w_1$. Then

$$\begin{aligned}\frac{v_2}{\|v_2\|} &= w_2 - \frac{\langle w_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 \\ &= \left(\frac{1}{2^{3/2}}, \frac{i}{2^{3/2}}, \frac{2-i}{2^{3/2}}, \frac{1}{2^{3/2}}\right).\end{aligned}$$

Then

$$\begin{aligned}\frac{v_3}{\|v_3\|} &= w_3 - \frac{\langle w_3, v_2\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle w_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2 \\ &= \left(\frac{3i+1}{2\sqrt{5}}, \frac{i}{\sqrt{5}}, \frac{-1}{2\sqrt{5}}, \frac{2i+1}{2\sqrt{5}}\right).\end{aligned}$$

Then

$$\begin{aligned}\frac{v_4}{\|v_4\|} &= w_4 - \frac{\langle w_4, v_2\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle w_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2 - \frac{\langle w_4, v_3\rangle}{\langle v_3, v_3\rangle} v_3 \\ &= \left(\frac{i-7}{2\sqrt{35}}, \frac{i+3}{\sqrt{35}}, \frac{5}{2\sqrt{35}}, \frac{5}{2\sqrt{35}}\right).\end{aligned}$$

For the first way of finding coefficients we have

$$f_1 \frac{v_1}{\|v_1\|} + f_2 \frac{v_2}{\|v_2\|} + f_3 \frac{v_1}{\|v_1\|} = x$$

which yields

$$\begin{aligned}f_1 &= 6\sqrt{2} \\ f_2 &= 4\sqrt{5} \\ f_3 &= 2 \\ f_4 &= 2\sqrt{35}\end{aligned}$$

This matches up with Theorem 6.5

$$\begin{aligned}f_1 &= \left\langle x, \frac{v_1}{\|v_1\|}\right\rangle = 6\sqrt{2} \\ f_2 &= \left\langle x, \frac{v_2}{\|v_2\|}\right\rangle = 4\sqrt{5} \\ f_3 &= \left\langle x, \frac{v_3}{\|v_3\|}\right\rangle = 2 \\ f_4 &= \left\langle x, \frac{v_4}{\|v_4\|}\right\rangle = 2\sqrt{35}.\end{aligned}$$

□

Problem 8. (§6.2 Problem 11.) Let A be an $n \times n$ matrix with complex entries. Prove that $AA^* = I$ if and only if the rows of A form an orthonormal basis for $\mathbb{C}^n$.

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Proof. First assume that $AA^* = I$. Then have that AA^* is found by taking the inner products of the row vectors of A and the column vectors of A^*. But the column vectors of A^* are exactly the conjugate of the row vectors of A. i.e., we have

$$(AA^*)_{ij} = \langle v_i, v_j \rangle$$

which means that the above must be equal to 1 when $i = j$ and 0 when $i \neq j$. Which means that the rows of A are orthonormal.

If we assume the rows are orthonormal and use the above identity, then we have that $AA^* = I$. □

Problem 9. (§6.2 Problem 16.)

(a) *Bessel's Inequality.* Let V be an inner product space, and let $S = \{v_1, v_2, ..., v_n\}$ be an orthonormal subset of V. Prove that for any $x \in V$ we have

$$\|x\|^2 \geq \sum_{i=1}^{n} |\langle x, v_i \rangle|^2.$$

Hint: Apply Theorem 6.6 to $x \in V$ and $W = \text{span}(S)$. Then use Exercise 10 of Section 6.1.

(b) In the context of (a), prove that Bessel's inequality is an equality if and only if $x \in \text{span}(S)$.

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Proof (a). Let $x = u + w$ with $w \in \text{span}(S)$ and $u \in W^\perp$. Then we have $\|x\|^2 = \|u+w\|^2 \geq \|w\|^2 = \sum_{i=1}^{n} |\langle x, v_i \rangle|^2$.□

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Proof (b). It follows from above that if $\|x\|$ is in $\text{span}(S)$ we have equality. □

Problem 10. (§6.2 Problem 19.) In each of the following parts, find the orthogonal projection of the given vector on the given subspace W of the inner product space V.

(a) $V=\mathbb{R}^2$, $u=(2.6)$, and $W=\{(x,y)\mid y=4x\}$.

(b) $V=\mathbb{R}^3$, $u=(2,1,3)$, and $W=\{(x,y,z)\mid x+3y-2z=0\}$.

(c) $V=P(\mathbb{R})$ with the inner product $\langle f(x),g(x)\rangle=\int_0^1 f(t)g(t)dt$, $h(x)=4+3x-2x^2$, and $W=P_1(\mathbb{R})$.

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Proof (a). Note $(1,4)$ spans W and we normalize to get $\frac{(1,4)}{\sqrt{17}}$. Then the orthogonal projection is

$$\left\langle u,\frac{(1,4)}{\sqrt{17}}\right\rangle\frac{(1,4)}{\sqrt{17}}=\frac{26(1,4)}{17}.$$

□

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Proof (b). Note $\frac{(2,0,1)}{\sqrt{5}}$ and $\frac{(-3,1,0)}{\sqrt{10}}$ span W and are normalized. Then the orthogonal projection is

$$\left\langle u,\frac{(2,0,1)}{\sqrt{5}}\right\rangle\frac{(2,0,1)}{\sqrt{5}}+\left\langle u,\frac{(-3,1,0)}{\sqrt{10}}\right\rangle\frac{(-3,1,0)}{\sqrt{10}}=\frac{7}{5}(2,0,1)+\frac{-1}{2}(-3,1,0)=\left(\frac{43}{10},\frac{-1}{2},\frac{7}{5}\right).$$

□

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Proof (c). Note 1, $\frac{1}{\sqrt{3}}(2x-1)$ spans W and is normalized. Then the orthogonal projection is

$$\langle h,1\rangle+\left\langle\frac{1}{\sqrt{3}}(2x-1)\right\rangle\frac{1}{\sqrt{3}}(2x-1)=\frac{1}{9}(x+1).$$

□