OPERATOR ALGEBRAS: AN INFORMAL OVERVIEW

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CONTENTS

ABSTRACT. In this article we give a short and informal overview of some aspects of the theory of C*- and von Neumann algebras. We also mention some classical results and applications of these families of operator algebras.

1. INTRODUCTION

Any introduction to the theory of operator algebras, a subject that has deep interrelations with many mathematical and physical disciplines, will miss out important elements of the theory, and this introduction is no exception. The purpose of this article is to give a brief and informal overview on C*- and von Neumann algebras which are the main actors of this summer school. We will also mention some of the classical results in the theory of operator algebras that have been crucial for the development of several areas in mathematics and mathematical physics. Being an overview we can not provide details. Precise definitions, statements and examples can be found in [1] and references cited therein. The main aim of this article is to illustrate in a few pages the richness and diversity of possible applications of this

topic. We have also included a few exercises to motivate further thoughts on the subjects treated.

This article is an extended and modified version of the author's introduction to his *Habilitation Thesis: Operatoralgebraic Methods in Mathematical Physics - Duality of Compact Groups and Gauge Quantum Field Theory* at the RWTH-Aachen University, Germany (cf. [28]).

2. Operator algebras

We begin with a preliminary definition of the two classes of operator algebras that will be mainly considered in this summer school: C*-algebras and von Neumann algebras. This definition is concrete in the sense that the elements of the algebra are given as operators on some complex Hilbert space. One can also introduce these algebras in an abstract setting, i.e. independent of any concrete Hilbert space realization (see e.g. [1, 47]).

2.1. **What are operator algebras?** By operator algebras we mean subalgebras of bounded linear operators on a complex Hilbert space $\mathcal{H}$ which are closed under the adjoint operation $A \to A^*$. We will consider here two main classes according to their completeness properties:

- *C*-algebras* are operator algebras closed with respect to the uniform topology, i.e. the topology defined by the operator norm.
- *Von Neumann algebras* are operator algebras closed with respect to the weak operator topology.

From this preliminary definition we can think of these classes of operator algebras as a rich algebraic structure on which we impose analytic conditions. This characteristic union of algebra and analysis reappears in several fundamental theorems of the theory. For example, one can interpret the following statements as a way of having an algebraic characterization of certain analytical properties or vice versa:

- The norm of *any* element A of a C*-algebra equals the square root of the spectral radius of the self-adjoint element A^*A, i.e.

$$\|A\| = (\operatorname{spr}(A^*A))^{\frac{1}{2}} .$$

  This fact already implies that there is at most one norm on a *-algebra making it a C*-algebra. (The crucial property of the operator norm to show this result is the equation $\|A^*A\| = \|A\|^2$. It is called C*-property.)
- Any *-homomorphism between unital C*-algebras, $\pi\colon \mathcal{A}_1 \to \mathcal{A}_2$, is automatically continuous and

$$\|\pi(A_1)\| \leq \|A_1\| , \quad A_1 \in \mathcal{A}_1 .$$

  If π is injective, then it must be necessarily isometric.

○ Von Neumann's double commutant theorem says that a nondegenerate *-subalgebra $\mathcal{M}$ of bounded linear operators on a Hilbert space $\mathcal{H}$ is weakly closed iff

$$\mathcal{M} = \mathcal{M}'',$$

where $\mathcal{M}'' = (\mathcal{M}')'$. (Recall that the commutant $\mathcal{M}'$ of $\mathcal{M}$ is the set of all bounded linear operators on $\mathcal{H}$ commuting with every operator in $\mathcal{M}$.)

Example 2.1. The set of compact operators on a Hilbert space $\mathcal{H}$ or the set $\mathcal{L}(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$ are examples of C*-algebras realized concretely on a Hilbert space (see also [1]). A source of examples of von Neumann algebras is provided by the fact that the commutant of *any* self-adjoint set $\mathcal{S}$ in $\mathcal{L}(\mathcal{H})$ (i.e. $T \in \mathcal{S}$ implies $T^* \in \mathcal{S}$) makes up a von Neumann algebra (see also Subsection 3.3). The reason for this fact lies in the the double commutant theorem mentioned before and the inclusion $\mathcal{S} \subset \mathcal{S}''$. In fact, from the preceding inclusion it follows immediately that $(\mathcal{S}') = (\mathcal{S}')''$.

F. Riesz was apparently the first mathematician to work with the algebra $\mathcal{L}(\mathcal{H})$ together with its strong operator topology (cf. [45, Chapitre V] and [7, Chapter VII, §2 and §5]).

Exercise 2.2. Show that any idempotent P (i.e. $P^2 = P$) on the Hilbert space $\mathcal{H}$ with $\|P\| = 1$ is self-adjoint, i.e. $P^* = P$. [Hint: Use $\sup_{x \in \mathcal{H}} \|Px\|^2 = \sup_{x \in \mathrm{ima}P^*} \|Px\|^2$ to show that $\mathrm{ima}\, P^* \subseteq \mathrm{ima}\, P$.] (The reverse implication $0 \neq P = P^2 = P^* \Rightarrow \|P\| = 1$ is easy to show using the C*-property.)

2.2. **Differences and analogies between C*- and von Neumann algebras.** Although, strictly speaking, any von Neumann algebra is a C*-algebra (since any von Neumann algebra is automatically closed with respect to the finer operator norm topology) it is useful to separate clearly between these two classes of operator algebras. Von Neumann algebras where introduced (as *rings of operators*) in 1929 by von Neumann in his second paper on spectral theory [35]. This was twelve years before the first elementary properties of normed algebras were considered in [15] (see also [12]). The following commutative prototypes also illustrate the different nature of both families of operator algebras:

• The space $C_0(X)$ of the continuous functions over a locally compact Hausdorff space X which vanish at infinity is an Abelian C*-algebra with complex conjugation as involution and norm given by

$$\|f\| := \sup_{x \in X} |f(x)|.$$

• The space $L^\infty(Z, d\mu)$ of essentially bounded and measurable functions for some σ-finite measure space $(Z, d\mu)$ may be identified with an Abelian von Neumann algebra. The elements of $L^\infty(Z, d\mu)$ are understood as multiplication operators on the complex Hilbert space

$\mathcal{H} = L^2(Z, d\mu)$. The measure space $(Z, d\mu)$ is essentially $[0, 1]$ with the Lebesgue measure $d\mu$ or some countable discrete space.

The fact that any von Neumann algebra is, in particular, a C*-algebra translates in the commutative context to the following result: any commutative von Neumann algebra $\mathcal{A}$ is isomorphic to the algebra $C(X)$ of continuous functions over an extremely disconnected compact Hausdorff space X. (Recall, that extremely disconnected means that the closure of each open set in X is open (as well as closed). This implies, in particular, that X is totally disconnected, i.e. each pair of points can be separated by sets which are both open and closed.)

Exercise 2.3. Show that the Abelian algebra $\mathcal{A} = L^\infty(Z, d\mu) \subset \mathcal{L}(\mathcal{H})$ with $(Z, d\mu)$ a σ-finite measure space is maximal Abelian, i.e. $\mathcal{A} = \mathcal{A}'$. Recall that the elements in $L^\infty(Z, d\mu)$ are understood as multiplication operators on the complex Hilbert space $\mathcal{H} := L^2(Z, d\mu)$. (Hint: Show maximal abelianess first in the case where Z is a finite measure space, i.e. $\mu(Z) < \infty$ and extend then the argument to the σ-finite situation; recall that the measure space is σ-finite if Z can be decomposed into a countable, disjoint union of subsets with finite measure.) In [29] we will give a very short proof of the equation $\mathcal{A} = \mathcal{A}'$ using Modular Theory.

From the preceding Abelian prototypes one can also recognize the following useful general properties of von Neumann algebras which are not necessarily true in the context of C*-algebras.

- ○ Von Neumann algebras have many projections (even more, they can be generated out of the set of projections) and always have an identity. In the Abelian case mentioned above the projections are given by multiplication with characteristic functions of measurable sets. In contrast to these facts, if X is a Hausdorff locally compact but not compact space, then the identity function is not contained in $C_0(X)$. If, in addition, X is connected then $C_0(X)$ has no nontrivial projections.
- ○ Von Neumann algebras can be more easily classified. In fact, von Neumann and his collaborator Murray already described a reduction theory for von Neumann algebras to factors (i.e. von Neumann algebras $\mathcal{M}$ having a trivial center: $\mathcal{M}' \cap \mathcal{M} = \mathbb{C}\mathbb{1}$) and gave a (rough) classification of factors into types I, II and III. With the help of an essentially unique dimension function on (equivalence classes of) projections, one can refine this classification of factors into type I_n, $n \in \mathbb{N} \cup \{\infty\}$, II_1 and II_∞. The factors of type I_n, $n \in \mathbb{N}$ and type II_1 are called finite (cf. [1]). The finer classification of type III factors into type III_0, III_λ, $0 < \lambda < 1$, and III_1 came much later and used deep results in Modular Theory (see e.g. [50, Chapter XII], [29]).
- ○ The set of continuous functions $C([0, 1])$ is separable w.r.t. the supremum norm, while $L^\infty(0, 1)$ is not. This fact suggests that in the

theory of von Neumann algebras other topologies than the uniform topology defined by the operator norm are needed.

Exercise 2.4. Consider the complex Hilbert space $\mathcal{H} = L^2(0,1)$. Show that the space of all bounded linear operators $\mathcal{L}(\mathcal{H})$ is not separable w.r.t. the topology defined by the operator norm. [Hint: Consider the set of projections given by multiplication with characteristic functions associated with the intervals $[0, \lambda]$, with $0 \leq \lambda \leq 1$.]

2.3. **Relevance of operator algebras.** Von Neumann algebras were born in the middle of three fundamental developments in mathematics: the *theory of group representations*, *Hilbert space theory* including the study of continuous linear operators, as well as *quantum mechanics* and the attempts of several mathematicians of that time to put the emerging theory on a firm mathematical footing. Some years later von Neumann and Murray laid the foundation of this field in a series of papers on *rings of operators* (renamed von Neumann algebras by J. Dixmier and J. Dieudonné) (see [32, 33, 38, 34, 39, 37] or [51]). We will recall here some qualified opinions on this classic series of papers:

> "By the wealth and novelty of their techniques and their results, these wonderful papers are certainly the most profound and most difficult which von Neumann ever wrote...; they revealed a large number of completely unsuspected phenomena..." (J. Dieudonné, 1981)
>
> "...perhaps the most original major work in mathematics in the twentieth century." (I.E. Segal, 1996).

It is also worth remembering the original motivations of the authors to start a systematic analysis of von Neumann algebras:

> "In his earliest work with operators..., von Neumann recognized the need for a detailed study of families of operators. Many of the subtler properties of an operator are to be found only in the internal algebraic structure of the algebra of polynomials in the operator (and its closures relative to various operator topologies) or in the action of this algebra on the underlying Hilbert space. His interest in ergodic theory, group representations, and quantum mechanics contributed significantly to von Neumann's realization that a theory of operator algebras was the next important stage in the development of this area of mathematics. The dictates of the subject itself had called for this development." [20, p. 61]
>
> "The problems discussed in this paper arose naturally in continuation of the work begun in a paper of one of us ... Their solution seems to be essential for the further advance of abstract operator theory in Hilbert space under several aspects. First, the formal calculus with operator-rings leads to them. Second, our attempts to generalize the theory of unitary group representations essentially

beyond their classical frame have always been blocked by unsolved questions connected with these problems. Third, various aspects of quantum mechanical formalism suggest strongly the elucidation of this subject. Fourth, the knowledge obtained in these investigations gives an approach to abstract algebras without a finite basis, which seems to differ essentially from all types hitherto investigated." [32, Introduction]

These motivations seem to be fully verified and, even more, they still provide inspiration for present day investigations in functional analysis, harmonic analysis and quantum physics. Independently of applications, operator algebras are of great intrinsic interest. They show various aspects of *infinity* and present fascinating new phenomena like continuous dimensions.

3. Different ways to think about operator algebras

In the present section we will present three different ways one may look at operator algebras.

3.1. **Operator algebras as non-commutative spaces.** There are structure theorems stated in [1] saying that, essentially, the prototypes mentioned in Subsection 2.2

$$\Big(C_0(X), \|\cdot\|\Big) \quad \text{and} \quad L^\infty(Z, d\mu)$$

are the only possible commutative examples of C*- and von Neumann algebras, respectively. In the context of commutative C*-algebras it is also possible to recapture the topological space X from the *algebraic structure* of the set of continuous functions on X decaying at infinity. It is therefore reasonable to think of non-commutative C*-algebras as the non-commutative counterpart of topological spaces. In the same way non-commutative von Neumann algebras can be associated with non-commutative measure spaces. The correspondence

$$\text{space} \quad \leftrightarrow \quad \text{algebraic structure}$$

opens, in the non-commutative setting, a wide and difficult field of current research that includes advanced topics like non-commutative geometry, non-commutative L^p-spaces or quantum groups (see, for example, [6, 16, 42, 24]).

3.2. **Operator algebras as a natural universe for spectral theory.** In the present subsection we will motivate that operator algebras are a natural universe for studying properties of a single operator. In fact the following proposition shows that the fundamental constituents in which one may decompose a single operator are contained in the corresponding von Neumann algebra. In other words, von Neumann algebras are stable under natural operations performed with its elements.

Proposition 3.1. *Let $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ be a von Neumann algebra and $M \in \mathcal{M}$.*

(i) *If $M = V|M|$ is the polar decomposition, then $V \in \mathcal{M} \ni |M|$. (Recall that $|M| := (M^*M)^{\frac{1}{2}}$ is a positive operator and that V is a partial isometry satisfying $\ker V = \ker M$).*

(ii) *If $M = M^*$ and $M = \int \lambda dE_M(\lambda)$ is the corresponding spectral decomposition of the self-adjoint operator, then for the set of spectral projections we have*

$$\{E_M(B) \mid B \subset \mathbb{R}, \text{ Borel}\} \subset \mathcal{M}.$$

(iii) *If $M = M^*$ and $f \in C([-\|M\|, \|M\|])$, then $f(M)$ is in any C*-algebra containing M. In particular, $f(M) \in \mathcal{M}$.*

Proof. We sketch only a few ideas of the proof: to show that any operator is contained in the von Neumann algebra $\mathcal{M}$, it is enough to verify that it commutes with all unitaries $U' \in \mathcal{M}'$. To prove (i) note that for any $U' \in \mathcal{M}'$ we have

$$V|M| = M = U' M (U')^* = (U' V (U')^*)(U' |M| (U')^*).$$

From the uniqueness of the polar decomposition we conclude that

$$(U' V (U')^*) = V \quad \text{and} \quad U' |M| (U')^* = |M| \quad \text{for all} \quad U' \in \mathcal{M}',$$

hence $V, |M| \in \mathcal{M}'' = \mathcal{M}$. Item (ii) is shown similarly using the uniqueness of the spectral decomposition of self-adjoint operators. For (iii) take a sequence p_n of polynomials approximating f in the sup-norm. Then it follows that $p_n(M) \in \mathcal{M}$ approximates in the operator-norm the operator $f(M)$. Hence $f(M)$ is in any C*-algebra containing M. Since any von Neumann algebra is also closed with respect to the operator norm we conclude that $f(M) \in \mathcal{M}$. $\qquad\square$

The precedent proposition implies that any von Neumann algebra is generated as a norm closed subspace by the set of the spectral projections corresponding to its self-adjoint elements.

Exercise 3.2. Let $f \in C([0,1]) \subset L^\infty(0,1)$. What is the polar decomposition of f? Note that, in general, the corresponding partial isometry is contained in $L^\infty(0,1)$ but not in $C([0,1])$.

3.3. **Von Neumann algebras as symmetry algebras.** Kadison suggests in [21, § 2] that von Neumann algebras grew initially out of the early period of group representations. In particular, Schur's characterization of irreducible representations in terms of commutants, Peter-Weyl's theory of compact groups as well as Wedderburn's structural results for matrix algebras were a motivational background in the systematic study of von Neumann algebras.

As already stated before, commutants of an arbitrary self-adjoint set of bounded operators in a Hilbert space, provide a rich source of examples of von Neumann algebras. In particular, if U is a unitary representation of a

group $\mathcal{G}$ on a complex Hilbert space $\mathcal{H}$, then the intertwiner space of the representation

$$(U, U) := \{U_g \mid g \in \mathcal{G}\}' \subset \mathcal{L}(\mathcal{H})$$

is a von Neumann algebra. Even more, *any* von Neumann algebra $\mathcal{N}$ arises in this way. (Take the group of all unitaries in $\mathcal{N}'$.) Therefore, von Neumann algebras may be seen as *symmetry algebras* of unitary group representations on a Hilbert space. This fact partially explains why these structures have been so successfully applied in many branches of mathematics and theoretical physics. The previous observation also shows that the unitary representation theory of groups is deeply related to the theory of operator algebras. For example, one says that a unitary representation U of a group $\mathcal{G}$ is primary if the von Neumann algebra (U, U) is a factor, i.e. if

$$\{U_g \mid g \in \mathcal{G}\}' \cap \{U_g \mid g \in \mathcal{G}\}'' = \mathbb{C}\mathbb{1}\,.$$

Moreover, the classification theory of factors mentioned above can be directly applied to the classification of primary representations (see [30, Chapter 1] and [9, Part II] for further details).

Some miscellaneous statements that show the close relation between group theory and von Neumann algebras are:

- A group $\mathcal{G}$ is of type I iff for every continuous unitary representation U of $\mathcal{G}$ the von Neumann algebra generated by this representation, i.e. $\{U_g \mid g \in \mathcal{G}\}''$, is of type I.
- There exists a bijective correspondence between the continuous unitary representations of a locally compact group $\mathcal{G}$ and the non-degenerate representation of the group C*-algebra $C^*(\mathcal{G})$. (Recall that the group C*-algebra of $\mathcal{G}$ is the enveloping C*-algebra of the convolution algebra $L^1(G)$.)

4. Some classical results

In the present section we recall some classical applications of operator algebras in mathematics and mathematical physics.

4.1. Operator algebras in functional analysis.

At the heart of the following results lies the structure theorem for commutative C*- and von Neumann algebras.

4.1.1. *Spectral theorem.* An immediate success of operatoralgebraic methods in functional analysis was the proof of the spectral theorem for bounded as well as *unbounded* normal operators on a Hilbert space (cf. [35]). The spectral theorem is a generalization of the elementary result that any normal linear operator on $\mathbb{C}^n$ is unitary equivalent to a diagonal matrix. It can be stated in many ways (see e.g. [44] or [40, §17.4])). One of them says that any normal operator is equivalent to a multiplication operator. In applications the spectral theorem is often stated in terms of the spectral resolution $E(\cdot)$ of a self-adjoint operator. (Recall that the orthogonal projections $\{E(\lambda)\}_\lambda$

satisfy the usual properties of monotonicity, right continuity and completeness.) For additional comments and results concerning the spectral theorem see [13, §9] and references therein.

Theorem 4.1. *For any self-adjoint operator T on a complex Hilbert space $\mathcal{H}$ there is a unique spectral resolution $E_T(\cdot)$ such that*

$$T = \int_{\mathrm{sp}\,(T)} \lambda \, dE_T(\lambda) \ .$$

Here, $\mathrm{sp}\,(T)$ *denotes the spectrum of the operator T and the right-hand integral is a Riemann-Stieltjes integral.*

4.1.2. *Decomposable operators.* In the analysis of finite operators (e.g. finite dimensional representations of a group) their decomposition into a direct sum of more fundamental pieces is an important step. As was seen in the preceding subsection the notion of a direct sum is too narrow to deal with more general operators on infinite dimensional spaces. In this situation it is still possible to give a "continuous" decomposition using so-called direct integrals, a technique that uses the theory of von Neumann algebras. A direct integral is a generalization of the concept of direct sum and may be applied to spaces as well as to operators. For measure-theoretic details of direct integrals of Hilbert spaces and operators see e.g. [22, Chapter 14],[49, 9, 8].

Let $(Z, d\mu)$ be a suitable measure space. Denote by $\mathfrak{H} := \int_Z^{\oplus} \mathcal{H}(z) \, dz$ the direct integral of the family of separable Hilbert spaces $\{\mathcal{H}(z)\}_{z \in Z}$ indexed by points in Z and with the corresponding measurability and convergence restrictions. It can be shown that $\mathfrak{H}$ is again a separable Hilbert space. An operator $T \in \mathcal{L}(\mathfrak{H})$ is decomposable with respect to $\int_Z^{\oplus} \mathcal{H}(z) \, dz$ if there is a function $Z \ni z \to T(z) \in \mathcal{L}(\mathcal{H}(z))$ such that for each $x \in \mathfrak{H}$ we have $T(z)x(z) = (Tx)(z)$ for almost every $z \in Z$. In particular, if $T(z) = f(z)\mathbb{1}_{\mathcal{H}(z)}$ for some measurable scalar function f we say that T is diagonalizable. We denote the set of decomposable (resp. diagonalizable) operators by $\mathcal{R}$ (resp. $\mathcal{D}$). The following result characterizes the set of decomposable operators in terms of commutants.

Theorem 4.2. *The set of decomposable operators $\mathcal{R}$ coincides with the commutant of the set of diagonalizable operators, i.e.*

$$\mathcal{R} = \mathcal{D}' \ .$$

Direct integrals allowed von Neumann to reduce the classification of von Neumann algebras on separable Hilbert spaces to the classification of so-called factors (i.e. von Neumann algebras whose centers consist of scalar multiples of the identity operator). In fact, any von Neumann $\mathcal{M}$ can be decomposed with respect to its center as direct integral

$$\mathcal{M} = \int_Z^{\oplus} \mathcal{M}(z) \, dz \ ,$$

where $\mathcal{M}(z)$, $z \in Z$, are factors a.e.

4.1.3. *Unbounded operators.* Many interesting operators in applications like, e.g., Schrödinger operators, are unbounded. Even if operator algebras involve only bounded operators, many families of unbounded operators are also closely related to operator algebras. Let T be a closed unbounded operator. We say that T on $\mathcal{H}$ is *affiliated* to a von Neumann algebra $\mathcal{M} \subset \mathcal{L}(\mathcal{H})$ if $UTU^{-1} = T$ for every unitary $U \in \mathcal{M}'$. In this context we have the following natural characterization: if $T = V \cdot |T|$ is the corresponding polar decomposition of the closed operator, then T is affiliated to $\mathcal{M}$ iff $V \in \mathcal{M} \supset \{E_{|T|}(B) \mid B \subset \mathbb{R}^+ \,,\ \text{Borel}\,\}$. Moreover, it can be shown that an (unbounded) operator is normal on a Hilbert space $\mathcal{H}$ iff it is affiliated to an Abelian von Neumann algebra $\mathcal{A}$ (cf. [23, Theorem 5.6.18]). A symmetric operator affiliated with a finite factor is automatically self-adjoint.

Remark 4.3. In particular Type II$_1$ von Neumann algebras were privileged by von Neumann, because the unbounded operators affiliated with them allow elementary algebraic manipulations. In fact, quoting his 1954 address to the International Congress of Mathematicians: "...one can show that any finite number of them, in fact any countable number of them, are simultaneously defined on an everywhere dense set; one can prove that one can indulge in operations like adding and multiplying operators and one never gets into any difficulty whatever. The whole symbolic calculus goes through." (see e.g. [43]).

Exercise 4.4. Let $T\colon \operatorname{dom} T \subset \ell_2 \to \ell_2$ be the linear operator defined on the domain
$$\operatorname{dom} T := \{a = (a_1, a_2, ...) \in \ell_2 \mid Ta \in \ell_2\}$$
by means of $(Ta)_n := n a_n$, $n \in \mathbb{N}$. Show that T is an unbounded and closed operator.

4.2. **Operator algebras in harmonic analysis.** We apply here the techniques of direct integral decomposition to the theory of unitary group representations. Let $\mathcal{G}$ be a separable locally compact group and let U be a continuous unitary representation of $\mathcal{G}$ on a Hilbert space $\mathcal{H}$. Denote by
$$\mathcal{M} := \{U_g \mid g \in G\}''$$
the von Neumann algebra generated by the representation U and let
$$\mathcal{M}' = (U, U) := \{M' \in \mathcal{B}(\mathcal{H}) \mid M' U_g = U_g M' \,,\ g \in \mathcal{G}\}$$
be the von Neumann algebra of intertwining operators for the representation V. If $\mathcal{A}$ is an Abelian von Neumann subalgebra of $\mathcal{M}'$, then there exists a compact, separable Hausdorff space Z, a regular Borel measure $d\mu$ on Z and a unitary transformation onto a direct integral Hilbert space
$$F\colon \mathcal{H} \longrightarrow \int_Z^{\oplus} \mathcal{H}(z)\, dz\,,$$

such that

$$F \, \mathcal{A} \, F^{-1} = \{M_f \mid f \in L^\infty(Z, d\mu)\}$$

(M_f being the multiplication operator with f) and

$$F \, U_g \, F^{-1} = \int_Z^{\oplus} U_g(z) \, dz$$

(see [54, Section 14.8 ff.]). There are several natural choices for the Abelian von Neumann algebra $\mathcal{A}$:

(i) If $\mathcal{A} = \mathcal{M} \cap \mathcal{M}'$ is the center of $\mathcal{M}$, then, for a.e. $z \in Z$, the von Neumann algebra generated by the representations $V(z)$ are factors, i.e.

$$\mathcal{M}(z) \cap \mathcal{M}(z)' := \{U_g(z) \mid g \in G\}'' \cap \{U_g(z) \mid g \in \mathcal{G}\}' = \mathbb{C}\mathbb{1}_{\mathcal{H}(z)}.$$

This choice is due to von Neumann.

(ii) If $\mathcal{A}$ is maximal Abelian in $\mathcal{M}'$, i.e. $\mathcal{A} = \mathcal{A}' \cap \mathcal{M}'$, then the components $U(z)$ of the direct integral decomposition of U are irreducible a.e. This choice is due to Mautner.

Finally, we mention a class of groups, where the previous decomposition results become particularly simple. A group $\mathcal{G}$ is of type I if all its unitary continuous representations U are of type I, i.e. each U is quasi-equivalent to some multiplicity free representation. Compact or Abelian groups are examples of type I groups. If $\mathcal{G}$ is of type I, then the dual $\widehat{\mathcal{G}}$ (i.e. the set of all equivalence classes of continuous unitary irreducible representations of $\mathcal{G}$) becomes a nice measure space ("smooth" in the terminology of [30, Chapter 2]). In this case one can take $\widehat{\mathcal{G}}$ as the measure space Z in the Mautner decomposition mentioned in the preceding item (ii).

4.3. **Operator algebras in quantum physics.** The publication of the seminal books of Weyl, Wigner and van der Waerden (cf. [55, 56, 53]) in the late twenties show that quantum mechanics was using group theoretical methods almost from its birth. A nice summary of this circle of ideas can be found in [2]. Moreover, it is suggested by Ulam in [52, pp. 22-23] that the spectral theorem and functional calculus are as fundamental to quantum mechanics, as infinitesimal calculus is for classical mechanics. Therefore, operatoralgebraic methods are indirectly present in quantum physics through the representation theory of groups and functional analysis. A direct application of operatoralgebraic methods in the first years of quantum theory was von Neumann's rigorous proof of the mathematical equivalence of the two main competing formalisms at that time: the wave mechanics of Schrödinger and the matrix mechanics of Born, Heisenberg and Jordan (see [36] or the review article [48]; for a thorough historical account on the equivalence problem see [31]).

Remark 4.5. A brief historical introduction to the relation between the representation theory of groups and quantum mechanics is given in [27, Section 1]. In this paper the author also proposes K-theory for operator algebras as a new synthesis of these topics.

4.3.1. *The C*-algebras of the canonical commutation/anticommutation relations.* There are two useful examples of C*-algebras that one can associate with systems of point particles in quantum mechanics. For fermions resp. bosons the generators of these algebras are labeled by points of the even-dimensional Hilbert space $\mathcal{H}$ with scalar product $\langle \cdot, \cdot \rangle$. In certain cases the reference space $\mathcal{H}$ may be interpreted as the phase space of the quantum system.

- The *CAR-algebra* is the C*-algebra that is associated to the canonical anticommutation relations. It is generated by operators $a(\varphi)$, $\varphi \in \mathcal{H}$, satisfying

$$a(\varphi_1)a(\varphi_2)^* + a(\varphi_2)^*a(\varphi_1) = \langle \varphi_1, \varphi_2 \rangle \, \mathbb{1} \,.$$

  (More details on this algebra are given in [29, Appendix] and references cited therein.)
- The *CCR-algebra* is the C*-algebra that is associated to the canonical commutation relations. It is generated by Weyl elements W_φ, $\varphi \in \mathcal{H}$, satisfying the Weyl form of the canonical commutation relations

$$W_\varphi \cdot W_\psi = e^{-\frac{i}{2}\mathrm{Im}\,\langle \varphi, \psi \rangle} \, W_{\varphi + \psi} \,, \quad \varphi, \psi \in \mathcal{H} \qquad (\text{Weyl relation}) \,.$$

Exercise 4.6. Position and momentum operators in quantum mechanics:
Let P and Q be linear operators in a Hilbert space $\mathcal{H}$ which satisfy the following commutation relations:

$$QP - PQ = i\,\mathbb{1} \qquad\qquad (\text{with the convention } \hbar = 1) \,.$$

  (i) Show that the dimension of $\mathcal{H}$ can not be finite. (Hint: Use the following identity of the trace $\mathrm{Tr}\,(AB) = \mathrm{Tr}\,(BA)$.)
  (ii) Show that P and Q can not be both bounded operators (Hint: Show the following relation by induction: $Q^n P - P Q^n = i\,n\,Q^{n-1}$, $n \in \mathbb{N}$.)

Remark 4.7. The previous exercise suggests that the canonical commutation relations must be modified in order to express them in the context of C*-algebras. Typically one uses bounded functions of the operators P and Q. In fact, the Weyl relations of are an "exponentiated" version of the canonical commutation relations (see e.g. [5, 41]). Another possibility is to use resolvents in order to encode the canonical commutation relations (cf. [4]).

The preceding CAR- respectively CCR-algebras are used to model fermionic respectively bosonic quantum systems, in particular to describe free quantum fields. In this case the reference space $\mathcal{H}$ becomes infinite dimensional and this introduces important differences in the representation theory of these algebras.

4.3.2. *Local quantum physics.* In quantum mechanics there are two fundamental notions: *observables* and *states* (see e.g. [26, Part I]). One of the conceptual advantages of C*-algebras in the description of the quantum world is the neat distinction between the abstract algebra and its state space or the corresponding representations on a concrete Hilbert space. This point of view particularly pays off in quantum field theory, where there is an abundance of inequivalent representations (cf. [14]). In fact, Haag and Kastler proposed in the sixties an approach to quantum field theory using the language of operator algebras. In this context the observables become the primary objects of the theory and are described by elements in an abstract C*-algebra. This approach is called nowadays algebraic quantum field theory or local quantum physics. More precisely, the central notion here is a net of local C*-algebras indexed by open and bounded regions in Minkowski space, i.e. an assignment

$$\mathbb{R}^4 \supset \Theta \mapsto \mathcal{A}(\Theta) \,,$$

that satisfies certain natural properties called Haag-Kastler axioms. The elements of $\mathcal{A}(\Theta)$ are interpreted as physical operations performable within the spacetime region Θ. This approach puts the concept of locality in the middle of synthesis of quantum mechanics and special relativity. In particular, causality is expressed in this context in the following natural way: if Θ_1 and Θ_2 are space-like separated regions in Minkowski space, then $\mathcal{A}(\Theta_1)$ commutes elementwise with $\mathcal{A}(\Theta_2)$ (see [19, 18, 3, 25] for further details). Non-local aspects in quantum field theory like the notion of the vacuum, S-matrix etc. are related to the states. Local quantum physics complements other modern developments in relativistic quantum field theory and is particularly powerful in the analysis of structural questions as well as for the rigorous treatment of models. Algebraic quantum field theory has been very successfully applied in superselection theory, the theory that studies three characteristic aspects of elementary particle physics: composition of charges, classification of statistics and charge conjugation (cf. [10, 11, 46]). For applications of Modular Theory to quantum field theory see also [17, 29] and references therein. Further details, developments and references related to this approach to quantum field theory can be found in `http://www.lpq.uni-goettingen.de`.

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