Clifford Analysis and a Noncommutative Gelfand Representation

Colin Roberts

Overview

Section 1

Introduction

Motivating problems

- ***Electrical Impedance Tomography (EIT)*** asks whether one can determine the conductivity of a medium from the voltage-to-current map.
- The ***Calderón problem*** replaces the medium with a manifold M, conductivity with g, and replaces the voltage-to-current map with the Dirichlet-to-Neumann operator Λ.

Other questions

- What topological information can we retrieve from functions on a manifold M?
- Do these functions also contain geometric information such as metric data?
- How much can we learn about M if our data is supported only on the boundary?

Subsection 1

Preliminaries

- ***Clifford algebra*** originated in 1878 with William Kingdon Clifford's work that extends Hermann Grassmann's ***exterior algebra***.
- ***Clifford analysis*** arrived in the 1980's due to Hestenes, Sobczyk, Sommen, Brackx, and Delenghe in order to enrich Élie Cartan's ***differential forms***. See: **[Hestenes, Sobczyk: 1984]** and **[Doran, Lasenby: 2003]**.

Clifford algebras

Let V be a vector space over a field $\mathbb{F}$ with quadratic form Q.

- Define the tensor algebra

$$\mathcal{T}(V) \coloneqq \bigoplus_{j=0}^{\infty} V^{\otimes j} = \mathbb{F} \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots.$$

- Form the ***Clifford algebra*** via a quotient

$$C\ell(V, Q) \coloneqq \mathcal{T}(V) / \langle \mathbf{v} \otimes \mathbf{v} - Q(\mathbf{v}) \rangle.$$

Geometric and exterior algebras

- Given a (pseudo) inner product g, we set $Q(\cdot) = g(\cdot,\cdot)$ and define a ***geometric algebra***

$$\mathcal{G} \coloneqq C\ell(V, g).$$

- The ***exterior algebra*** is given by

$$\bigwedge(V) \coloneqq C\ell(V, 0).$$

Algebra structure

Multiplication in $\mathcal{G}$ is seen by looking at how $\otimes$ acts in the quotient.

- Given $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{G}$ we can take the product

$$\boldsymbol{u}\boldsymbol{v} = \underbrace{\boldsymbol{u} \cdot \boldsymbol{v}}_{\text{scalar}} + \underbrace{\boldsymbol{u} \wedge \boldsymbol{v}}_{\text{bivector}} .$$

  - The scalar part is symmetric: $\boldsymbol{u} \cdot \boldsymbol{v} = g(\boldsymbol{u}, \boldsymbol{v})$.
  - The bivector part is antisymmetric: $\boldsymbol{u} \wedge \boldsymbol{v} = -\boldsymbol{v} \wedge \boldsymbol{u}$.

Multivectors

- $\mathcal{G}$ is graded and of dimension 2^n.
  - There are $\binom{n}{r}$ elements in the space of grade-r elements, $\mathcal{G}^r$, called ***r-vectors***.
  - Those that are exterior products of r independent vectors are ***r-blades***. E.g., $\boldsymbol{A_r} = \boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_r$.
  - Elements of the even grade subalgebra, $\mathcal{G}^+$, are called ***spinors***.
- The most general elements are ***multivectors*** and are given by

$$A = \sum_{r=0}^{n} \langle A \rangle_r,$$

where $\langle A \rangle_r \in \mathcal{G}^r$ extracts the grade r part of A. So $\mathcal{G} = \bigoplus_{r=0}^{n} \mathcal{G}^r$.

Algebraic Structure

- Extend the multiplication from vectors to multivectors.
- On homogeneous elements,

$$A_r B_s = \langle A_r B_s \rangle_{|r-s|} + \langle A_r B_s \rangle_{|r-s|+2} + \cdots + \langle A_r B_s \rangle_{r+s}$$

- The most important products are

$$A_r \cdot B_s \coloneqq \langle A_r B_s \rangle_{|r-s|} \qquad A_r \wedge B_s \coloneqq \langle A_r B_s \rangle_{r+s}$$

$$A_r \rfloor B_s \coloneqq \langle A_r B_s \rangle_{s-r} \qquad A_r \lfloor B_s \coloneqq \langle A_r B_s \rangle_{r-s}$$

Reciprocals and reverses

- Given any vector basis $\boldsymbol{v}_i$, define the ***reciprocal vectors*** by $\boldsymbol{v}^i \cdot \boldsymbol{v}_j = \delta^i_j$.
- The ***reverse*** of a multivector is extended linearly from the action on r-blades by

$$\boldsymbol{A_r}^\dagger = (\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_r)^\dagger = \boldsymbol{v}_r \wedge \cdots \wedge \boldsymbol{v}_1.$$

Inner product and norm

- Define the ***multivector inner product*** by

$$(A, B) \coloneqq \left\langle A^\dagger B \right\rangle$$

  which is bilinear, symmetric, and positive definite if g is positive definite.

- Define the ***multivector norm*** by

$$|A| \coloneqq \sqrt{(A, A)}.$$

Adjoint

Note the reverse acts as an adjoint by

$$(CA, B) = (A, C^\dagger B)$$
$$(AC, B) = (A, BC^\dagger).$$

Pseudoscalars

- ***Pseudoscalars*** are the grade-n elements. For example, the volume element

$$\boldsymbol{\mu} = \boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_n.$$

- We define the ***unit pseudoscalar*** by

$$\boldsymbol{I} \coloneqq \frac{1}{|\boldsymbol{\mu}|}\boldsymbol{\mu}.$$

Blades and subspaces

- If g is positive definite all blades are invertible [Chisholm: 2012].
- If $|\boldsymbol{A_r}| = 1$, then $\boldsymbol{A_r}$ is a ***unit blade***.
- Unit r-blades correspond to r-dimensional subspaces so they correspond to points in $\mathrm{Gr}(r, n)$.

Duality

- Given any multivector A, we can take its ***dual***

$$A^{\perp} \coloneqq A\boldsymbol{I}^{-1}.$$

- Note $A_r^{\perp} \in \mathcal{G}^{n-r}$, like the Hodge star $\star$.

Projection and rejection

- The ***projection*** of B into a subspace $\boldsymbol{A_r}$ by

$$\mathrm{P}_{\boldsymbol{A_r}}(B) \coloneqq B \rfloor \boldsymbol{A_r} \boldsymbol{A_r}^{-1}$$

- The ***rejection*** by

$$\mathrm{R}_{\boldsymbol{A_r}}(B) \coloneqq B \wedge \boldsymbol{A_r} \boldsymbol{A_r}^{-1}.$$

- Both are grade preserving.

Examples

- Define $\mathcal{G}_{p,q}$ by letting $\boldsymbol{e}_i^2 = -1$ for $i = 1, \ldots, p$ and $\boldsymbol{e}_i^2 = +1$ otherwise.
- **Claim:** $\mathbb{H}$ arises naturally as the even subalgebra $\mathcal{G}_3^+ \coloneqq \mathcal{G}_{0,3}^+$.
- **Claim:** $\mathbb{C}$ arises naturally as the even subalgebra $\mathcal{G}_2^+ \coloneqq \mathcal{G}_{0,2}^+$.
  - Take the standard basis $\boldsymbol{e}_1, \boldsymbol{e}_2$, and define $\boldsymbol{B}_{12} = \boldsymbol{e}_1\boldsymbol{e}_2$ and note $\boldsymbol{B}_{12}^2 = -1$. Thus,

$$(u_1 + v_1\boldsymbol{B}_{12})(u_2 + v_2\boldsymbol{B}_{12}) = u_1u_2 - v_1v_2 + (u_1v_2 + u_2v_1)\boldsymbol{B}_{12}.$$

  - Right multiplication by $\boldsymbol{B}_{12}$ rotates counter-clockwise by $\pi/2$.

Section 2

Clifford analysis

Multivector Fields

- Let M be a smooth, compact, connected, and oriented n-dimensional Riemannian manifold with metric g.
- <u>**Idea:**</u> Form the Clifford algebras on tangent spaces.
  - Each $C\ell(T_pM, g_p)$ is a ***geometric tangent space*** which we glue together to form
$$C\ell(TM, g) \coloneqq \bigsqcup_{p \in M} C\ell(T_pM, g_p).$$
  - The space of ***(smooth) multivector fields*** is
$$\mathcal{G}(M) \coloneqq \{C^\infty\text{-smooth sections of } C\ell(TM, g)\}.$$
- Retain the same naming scheme as before.

Multivector derivative

On M, take the unique Levi-Civita connection ∇ and covariant derivative $\nabla_{\boldsymbol{u}}$.

- $\nabla_{\boldsymbol{u}}$ is extended to multivectors and is grade preserving [Schindler: 2018],

$$\nabla_{\boldsymbol{u}} A_r = \langle \nabla_{\boldsymbol{u}} A_r \rangle_r.$$

- $\nabla_{\boldsymbol{u}}$ is compatible with dot and wedge since

$$\nabla_{\boldsymbol{u}}(A \cdot B) = (\nabla_{\boldsymbol{u}} A) \cdot B + A \cdot (\nabla_{\boldsymbol{u}} B)$$
$$\nabla_{\boldsymbol{u}}(A \wedge B) = (\nabla_{\boldsymbol{u}} A) \wedge B + A \wedge (\nabla_{\boldsymbol{u}} B).$$

Gradient

- Define the ***gradient*** (or ***Dirac operator***) locally by

$$\boldsymbol{\nabla} = \sum_{i=1}^{n} \boldsymbol{v}^i \nabla_{\boldsymbol{v}_i}$$

- $\boldsymbol{\nabla}$ acts as a vector in $\mathcal{G}(M)$ and obeys the Leibniz rule

$$\boldsymbol{\nabla}(AB) = \dot{\boldsymbol{\nabla}}\dot{A}B + \dot{\boldsymbol{\nabla}}A\dot{B}.$$

- Note $\boldsymbol{\nabla}^2 = \Delta$, the Laplace-Beltrami operator.

Example

- In $\mathcal{G}_3(\mathbb{R}^3)$, $\nabla_{\boldsymbol{e}_i}$ is the partial derivative.
- Take a vector field $\boldsymbol{v}$, then

$$\boldsymbol{\nabla}\boldsymbol{v} = \underbrace{\boldsymbol{\nabla}\cdot\boldsymbol{v}}_{\text{divergence}} + \underbrace{\boldsymbol{\nabla}\wedge\boldsymbol{v}}_{\text{curl}}.$$

- Specifically,

$$\operatorname{curl}(\boldsymbol{v}) = (\boldsymbol{\nabla}\wedge\boldsymbol{v})^{\perp}$$

Differential forms

- Define the *r-dimensional directed measure*

$$dX_r \coloneqq \mathbf{v}_{j_1} \wedge \cdots \wedge \mathbf{v}_{j_r} dx^{j_1} \cdots dx^{j_r}$$

  where $1 \le j_1 < \cdots < j_r \le n$ and summation is implied.
- Define an r-form α_r by

$$\alpha_r = A_r \cdot dX_r^\dagger$$

  where $A_r = \frac{1}{r!}\alpha_{i_1 \cdots i_r} \mathbf{v}^{i_1} \wedge \cdots \wedge \mathbf{v}^{i_r}$.
- Refer to A_r the *multivector equivalent* of α_r.

Exterior algebra and calculus

- Given an r- and s-form α_r and β_s we have

$$\alpha_r + \beta_s = (A_r + B_s) \cdot dX_r^\dagger, \qquad \alpha_r \wedge \beta_s = (A_r \wedge B_s) \cdot dX_{r+s}^\dagger.$$

- The exterior derivative on multivector equivalents is

$$d\alpha_r = (\nabla \wedge A_r) \cdot dX_{r+1}^\dagger$$

- The Hodge star on multivector equivalents is

$$\star\alpha_r = (\boldsymbol{I}^{-1} A_r)^\dagger \cdot dX_{n-r}^\dagger$$

Volume form

- The ***volume form*** on M is given in local coordinates by

$$\mu = \sqrt{|g|}dx^1 \cdots dx^n = \boldsymbol{I}^{-1} \cdot dX_n$$

- We integrate scalar fields A_0 on M by

$$\int_M A_0^\perp \cdot dX_n = \int_M A_0 \mu.$$

Multivector field inner product

- We define the L^2-inner product on multivector fields by

$$\ll A, B \gg \coloneqq \frac{1}{\mathrm{vol}(M)} \int_M (A, B)\mu$$

- This realizes the r-form inner product

$$\int_M \alpha_r \wedge \star\beta_r = \int_M \left\langle A_r^\dagger B_r \right\rangle \mu = \mathrm{vol}(M) \ll A, B \gg$$

- $\ll A_r, B_s \gg = 0$ when $r \neq s$ so the L^2-direct sum agrees with the grade based direct sum.

Boundary

- On ∂M, the boundary pseudoscalar $\boldsymbol{I}_\partial$ induces the boundary normal

$$\boldsymbol{\nu} = \boldsymbol{I}_\partial^\perp.$$

- The boundary volume form is

$$\mu_\partial \coloneqq \boldsymbol{I}_\partial^{-1} \cdot dX_{n-1}$$

and we define the ***multivector field inner product***

$$\ll A, B \gg_\partial \coloneqq \frac{1}{\mathrm{vol}(M)} \int_{\partial M} (A, B) \mu_\partial .$$

Multivector valued integrals

We can define a multivector valued integral on an oriented submanifold R by taking $A \in \mathcal{G}(M)$ and computing

$$\int_R A \boldsymbol{I}_R \mu_R.$$

Theorem (Hestenes, Sobczyk, 1984)

Let $A, B \in \mathcal{G}(M)$, *then*

$$\int_M \dot{A}\dot{\boldsymbol{\nabla}}\boldsymbol{I}\mu = \int_{\partial M} A\boldsymbol{I}_\partial \mu_\partial$$

$$\int_M \boldsymbol{I}\boldsymbol{\nabla} B\mu = \int_{\partial M} \boldsymbol{I}_\partial B\mu_\partial$$

$$\int_M \dot{A}\dot{\boldsymbol{\nabla}}\boldsymbol{I}B\mu = (-1)^n \int_M A\boldsymbol{I}\boldsymbol{\nabla} B\mu + \int_{\partial M} A\boldsymbol{I}_\partial B\mu_\partial.$$

Theorem

We have the Green's formula for the gradient

$$\ll A, \boldsymbol{I}\nabla B \gg = (-1)^n \ll \nabla A, \boldsymbol{I}B \gg + \ll A, \boldsymbol{I}_\partial B \gg_\partial .$$

Proof. Fix $A^\dagger, B \in \mathcal{G}(M)$ and note that

$$\begin{aligned}\int_M A^\dagger \boldsymbol{I}\nabla B\mu &= (-1)^n \int_M \dot{A}^\dagger \dot{\nabla} \boldsymbol{I} B\mu + \int_{\partial M} A^\dagger \boldsymbol{I}_\partial B \mu_\partial \\ &= (-1)^n \int_M (\nabla A)^\dagger \boldsymbol{I} B\mu + \int_{\partial M} A^\dagger \boldsymbol{I}_\partial B\mu_\partial .\end{aligned}$$

Take the scalar part and divide by $\mathrm{vol}(M)$ to find

$$\ll A, \boldsymbol{I}\nabla B \gg = (-1)^n \ll \nabla A, \boldsymbol{I}B \gg + \ll A, \boldsymbol{I}_\partial B \gg_\partial .$$

Special fields

- Define the ***monogenic fields***

$$\mathcal{M}(M) \coloneqq \{A \in \mathcal{G}(M) \mid \nabla A = 0\}.$$

- Let $f = u + v\boldsymbol{B} \in \mathcal{G}_2^+(\mathbb{R}^2)$ then $\nabla f = 0$ yields the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

- Define the ***gradients***

$$\nabla\mathcal{G}(M) \coloneqq \{\nabla A \mid A \in \mathcal{G}(M) \text{ and } A|_{\partial M} = 0\}.$$

Cauchy kernel

- For the remainder, take M imbedded in $\mathbb{R}^n$ with $n \geq 2$.
- Define the vector field
$$E(x) \coloneqq \frac{1}{S_n} \frac{x}{|x|^n}$$
where S_n is the surface area of the unit ball.
- Note,
$$\nabla E(x) = -\dot{E}(x)\dot{\nabla} = \delta(x).$$
- Define the ***Cauchy kernel*** by $G(x, x') \coloneqq E(x' - x)$.

Cauchy integral

- Let $A \in \mathcal{M}(M)$, then we have the ***Cauchy integral formula***

$$A(x) = (-1)^n \boldsymbol{I}^{-1} \int_{\partial M} G(x, x') \boldsymbol{I}_\partial(x') A(x') \mu_\partial(x').$$

- This uniquely determines a monogenic field from boundary values.

Lemma

Let $A \in \mathcal{M}(M)$ *be such that* $A|_{\partial M} = 0$. *Then* $A = 0$ *on all of* M.

Lemma

Fix a multivector field $A \in \mathcal{G}(M)$. *If*

$$\ll A, B \gg = 0$$

for all $B \in \mathcal{G}(M)$ *with* $B|_{\partial M} = 0$, *then* $A = 0$.

Theorem (Clifford-Hodge-Morrey Decomposition)

The space of multivector fields $\mathcal{G}(M)$ has the L^2-orthogonal decomposition

$$\mathcal{G}(M) = \mathcal{M}(M) \oplus \boldsymbol{I}\nabla\mathcal{G}(M).$$

Proof.

- *Orthogonality:* Let $A \in \mathcal{M}(M)$ and $\boldsymbol{I}\nabla B \in \boldsymbol{I}\nabla\mathcal{G}(M)$ and note

$$\ll A, \boldsymbol{I}\nabla B \gg = (-1)^n \ll \nabla A, \boldsymbol{I}B \gg + \ll A, \boldsymbol{I}_\partial B \gg = 0,$$

by the multivector Green's formula.

- Let $C \in \mathcal{G}(M)$ be in the orthogonal complement to $\boldsymbol{I}\nabla\mathcal{G}(M)$.
- Use the Cauchy integral formula, construct a monogenic field $\tilde{C}$ from $C|_{\partial M}$ and note $C = \tilde{C} + C_0$ where $C_0|_{\partial M} = 0$.
- Note,

$$0 = \ll C, \boldsymbol{I}\nabla B \gg = \ll \nabla C_0, \boldsymbol{I}B \gg .$$

- By the previous lemmas, it must be that $C_0 = 0$. Hence the orthogonal complement to $\boldsymbol{I}\nabla\mathcal{G}(M)$ is $\mathcal{M}(M)$.

Comparing to Hodge-Morrey

- The Hodge-Morrey decomposition reads

$$\Omega^r(M) = \underbrace{\mathcal{E}^r_D(M)}_{\mathrm{Im}(\nabla\wedge)} \oplus \underbrace{\mathcal{C}^r_N(M)}_{\mathrm{Im}(\nabla\rfloor)} \oplus \underbrace{\mathcal{H}^r(M)}_{\mathrm{Ker}(\nabla)}.$$

  via **[Schwarz: 1995]**.

- Whereas the Clifford-Hodge-Morrey decomposition ignores specific grades

$$\mathcal{G}(M) = \mathcal{M}(M) \oplus \boldsymbol{I}\nabla\mathcal{G}(M).$$

Section 3

Gelfand theory

Open questions

- In [Belishev: 2003], we see an algebraic proof for the 2-dimensional Calderón problem.
- In [Belishev, Vakulenko: 2017], we see a proof for a noncommutative Gelfand representation using quaternion fields for a ball $\mathbb{B}$ in $\mathbb{R}^3$.
- Belishev and Vakulenko as whether this is true in higher dimensions.
- We prove an analogous result for an arbitrary $\mathbb{B}$ in $\mathbb{R}^n$.
- This approach can hopefully be used to prove the analogous result for any smooth orientable Riemannian manifold M.

2-dimensional BC method

The boundary control (BC) method is implemented in [Belishev: 2003] in the following manner.

- Determine the algebra $\mathcal{A}(M)$ of holomorphic functions on M from continuous function algebra on the boundary $\mathcal{A}(\partial M)$ using Λ.
- The classical Gelfand representation shows $\mathcal{A}(M)$ is homeomorphic to M via the weak-$*$ topology.
- Functions in $\mathcal{A}(M)$ determine the complex structure on M.
- Thus, we can find a g that is conformal with the complex structure.

Subsurface spinor fields

- Let $\boldsymbol{B} \in \mathcal{G}(M)$ be a constant unit 2-blade, then $f_+ \in \mathcal{G}^+(M)$ satisfying

$$f_+ = \mathrm{P}_{\boldsymbol{B}} \circ f_+ \circ \mathrm{P}_{\boldsymbol{B}}$$

  is a ***subsurface spinor field***. Let $\mathcal{G}^+_{\boldsymbol{B}}(M)$ denote the space such fields.
- The space of monogenic subsurface spinors

$$\mathcal{A}_{\boldsymbol{B}}(M) = \{f_+ \in \mathcal{G}^+_{\boldsymbol{B}}(M) \mid \nabla f_+ = 0\}$$

  is a commutative unital Banach algebra.

Functionals

- Define the ***spinor dual*** $\mathcal{M}^*(M)$ as the continuous right $\mathcal{G}_n$-module homomorphisms

  $$\mathcal{M}^*(M) \coloneqq \{l\colon \mathcal{M}^+(M) \to \mathcal{G}_n^+ \mid l(fs+g) = l(f)s + l(g),\ \forall f,g \in \mathcal{M}(M),\ s \in \mathcal{G}_n^+\}$$

  and refer to the elements as ***spin functionals***.
- Assert the weak-$\ast$ topology on $\mathcal{M}^*(M)$ so that every $x \in M$ corresponds to a continuous map on $\mathcal{M}^*(M)$.

Characters

- Define the algebra $\mathbb{A}_{\boldsymbol{B}}$ to be the algebra generated by 1 and $\boldsymbol{B}$.
- The ***spinor spectrum*** $\mathfrak{M}(M)$ is the set of algebra homomorphisms

  $$\mathfrak{M}(M) \coloneqq \{\delta \in \mathcal{M}^*(M) \mid \delta(f) \in \mathbb{A}_{\boldsymbol{B}},\ \delta(fg) = \delta(f)\delta(g),\ \forall f, g \in \mathcal{A}_{\boldsymbol{B}}(M),\ \boldsymbol{B} \in \mathrm{Gr}(2, n)\}$$

  and refer to the elements as ***spin characters***.
- One example of such characters are point evaluations $\delta(f) = f(x^\delta)$.
- We show these are the only elements in the spectrum.

z analogs

- Take the standard basis for $\mathbb{R}^n$, and consider $M = \mathbb{B}_{R,w}$.
- Let $\boldsymbol{B}_{ij} = \boldsymbol{e}_i \boldsymbol{e}_j$, and define

$$z_{ij} = x_j - x_i \boldsymbol{B}_{ij}$$

- Note $z_{ij} \in \mathcal{A}_{\boldsymbol{B}_{ij}}(\mathbb{B}_{R,w})$.

Monogenic polynomials

- Let σ be a permutation of $\{2, 3, \dots, n\}$, then

$$p_{j_2\cdots j_n}(x) = \frac{1}{j!} \sum_{\text{permutations}} z_{1\sigma(1)}(x) \cdots z_{1\sigma(j)}(x)$$

  is a monogenic homogeneous polynomial of degree j.

- Collect these into the set of ***monogenic polynomials***

$$\mathcal{M}^{\mathcal{P}}(\mathbb{B}_{R,w}) = \left\{ \sum_{j=0}^{N} \left(\sum_{\substack{j_2\cdots j_n \\ j_2+\cdots+j_n=j}} p_{j_2\cdots j_n} a_{j_2\cdots j_n} \right) \mid j_2 + \cdots + j_n = j,\, N \in \mathbb{N},\ a_{j_2\cdots j_n} \in \mathcal{G}_n \right\}.$$

Lemma (Density)

The space $\mathcal{M}^{\mathcal{P}}(\mathbb{B}_{R,w})$ is dense in $\mathcal{M}(\mathbb{B}_{R,w})$.

Proof sketch.

- Let $f \in \mathcal{M}(\mathbb{B}_{R,w})$ and use the Cauchy integral formula to define the coefficients $a_{j_2\cdots j_n} \in \mathcal{G}_n^+$ by

$$a_{j_2\cdots j_n} = \int_{\partial\mathbb{B}_{R,w}} \frac{\partial^j G(w,y)}{\partial y_2^{j_2}\cdots\partial y_n^{j_n}} \boldsymbol{\nu}(y) f(y) \mu_\partial(y),$$

- Then

$$f(x) = \sum_{j=0}^{\infty} \left(\sum_{\substack{j_2\cdots j_n \\ j_2+\cdots+j_n=j}} p_{j_2\cdots j_n}(x-w) a_{j_2\cdots j_n} \right),$$

converges pointwise for $x \in \mathbb{B}_{R,w}$ by **[Ryan, 2004]**.

Idea

- By linearity, we can note that for $\delta \in \mathfrak{M}(\mathbb{B}_{R,w})$

$$\delta(f(x)) = \sum_{j=0}^{\infty} \left(\sum_{\substack{j_2 \cdots j_n \\ j_2 + \cdots + j_n = j}} \delta(p_{j_2 \cdots j_n}(x-w)) a_{j_2 \cdots j_n} \right)$$

- On each monogenic polynomial

$$\delta(p_{j_2 \dots j_n}(x)) = \frac{1}{j!} \sum_{\text{permutations}} \delta\left((z_{1\sigma(1)}(x)\right) \cdots \delta\left(z_{1\sigma(j)}(x)\right)$$

by the multiplicativity of δ.

Lemma (Point evaluation)

Let $\delta \in \mathfrak{M}(\mathbb{B}_{R,w})$ and $z_{ij} \in \mathcal{A}_{\boldsymbol{B}_{ij}}(\mathbb{B}_{R,w})$, then $\delta(z_{ij}) = z_{ij}(x^\delta)$ for some $x^\delta \in \mathbb{R}^n$.

Proof sketch.

- We have $\delta(z_{ij}) = \alpha_{ij} + \beta_{ij}\boldsymbol{B}_{ij}$.
- Note $z_{ij}\boldsymbol{B}_{ji} = -z_{ji}$ and $z_{ij} = z_{kj} + z_{ik}\boldsymbol{B}_{kj}$ yield the relationships

$$\alpha_{ji} = -\beta_{ij} \qquad \alpha_{ij} = \alpha_{kj} \qquad \beta_{ij} = \beta_{ik} \qquad \alpha_{ik} = -\beta_{kj}.$$

- The set of constants α and β are determined by n independent numbers, so we can say $\delta(z_{ij}) = z_{ij}(x^\delta)$ for some $x^\delta \in \mathbb{R}^n$.

Lemma (Identification)

Let $f \in \mathcal{M}(\mathbb{B}_{R,w})$, then $\delta(f) = f(x^\delta)$ for some $x^\delta \in \mathbb{B}_{R,w}$.

Proof.

- Fix $\delta \in \mathfrak{M}(\mathbb{B}_{R,w})$ and suppose $x^\delta \notin \mathbb{B}_{R,w}$.
- Take a sequence $x_n \to x^\delta$ with $x_n \notin \mathbb{B}_{R,w}$.
- Define $G_n(x) \coloneqq G(x, x_n)\boldsymbol{e_1} \in \mathcal{M}^+(\mathbb{B}_{R,w})$.
- Note,

$$\lim_{n\to\infty} \delta(G_n) = \lim_{n\to\infty} G_n(x^\delta)$$

  so this sequence not converge due to a singularity at x^δ.
- Hence, it must be that $x^\delta \in \mathbb{B}_{R,w}$ by continuity of δ.

Theorem (Noncommutative Gelfand representation)

For any $\delta \in \mathfrak{M}(\mathbb{B}_{R,w})$, *there is a point* $x^{\delta} \in \mathbb{B}_{R,w}$ *such that* $\delta(f) = f(x^{\delta})$ *for any* $f \in \mathcal{M}(\mathbb{B}_{R,w})$. *Given the weak-$\star$ topology on* $\mathcal{M}^{*}(\mathbb{B}_{r,w})$, *the map*

$$\gamma : \mathfrak{M}(\mathbb{B}_{R,w}) \to \mathbb{B}_{R,w}, \quad \delta \mapsto x^{\delta}$$

is a homeomorphism.

Proof.

- The lemmas show that $\gamma\colon \mathfrak{M}(\mathbb{B}_{R,w}) \to \mathbb{B}_{R,w}$ is bijective.
- To see that γ is a homeomorphism, take a sequence $\delta_n \to \delta$ in $\mathfrak{M}(\mathbb{B}_{R,w})$.
- For $f \in \mathcal{M}^+(\mathbb{B}_{R,w})$ we have

$$f(\gamma(\delta_n)) = f(x^{\delta_n}) = \delta_n(f) = \gamma^{-1}(x^{\delta_n})(f).$$

- Taking $n \to \infty$ shows γ and γ^{-1} are continuous so γ is a homeomorphism.

Section 4

Future work

Calderón problem

- **Question:** Let (M,g) be an unknown Riemannian manifold with known boundary ∂M. Consider the forward problem

$$\begin{cases} \Delta\omega = 0 & \text{in } M \\ \iota^*\omega = \phi & \text{on } \partial M \end{cases}$$

- Define the ***Dirichlet-to-Neumann map*** on forms by $\Lambda\phi = \iota^*(\star d\omega)$.
- Can we determine (M,g) from Λ?

Calderón problem

- This problem is equivalent to the electical impedance tomography problem in dimension 3.
- The problem has been solved in dimension $n = 2$ **[Belishev: 2003]**.
- Solved in dimensions $n \geq 3$ when M is an analytic manifold **[Lassas, Taylor, Uhlmann: 2003]**.
- The smooth cases is still unsolved.

Thoughts

- Even monogenic fields have harmonic components. Given a harmonic r-vector A_r, can we reconstruct a monogenic multivector containing A_r?
- For $n = 3$, the scalar potential u and magnetic bivector field b are two parts of a monogenic field $f = u + b$ due to Ohm's and Ampere's laws

$$-\nabla \wedge u = \boldsymbol{j} = \nabla \rfloor b.$$

- If Λ can provide us $b|_{\partial M}$, then we can possibly reconstruct $\mathcal{M}^{+}(M)$.
- Given the algebraic structure of each $\mathcal{A}_{\boldsymbol{B}}(M) \subset \mathcal{M}^{+}(M)$, can this be used to determine g?

Other inverse problems

- Can the magnetic impedance tomography problem can provide some extra insight on the EIT problem?
- The Hodge-Morrey decomposition is an instrumental tool for boundary value problems that, for example, allows one to show that Λ determines the Betti numbers of M **[Belishev, Sharafutdinov: 2008]**.
- Can the Clifford-Hodge-Morrey decomposition can allow us to work on other related boundary inverse problems?
- These problems could include spacetime problems where the metric g is of mixed signature.

Section 5

Conclusions

Conclusion

- We have utilized multivector fields to serve as a meaningful generalization of both the complex numbers and differential forms.
- This provides a new way to decompose fields on domains of $\mathbb{R}^n$ and this can likely be generalized to arbitrary compact orientable pseudo-Riemannian manifolds.
- Likewise, we have proven that the monogenic fields contain a wealth of topological information and this information is supported on the boundary by the Cauchy integral formula.

Data Assimilation

- Over the past two years I have also worked with a team on developing new techniques for data assimilation.
- We have submitted *Model and Data Reduction for Data Assimilation: Particle Filters Employing Projected Forecasts and Data with Application to a Shallow Water Model*.
- We are continuing to work to apply our scheme to new models such as the Modular Arbitrary-Order Ocean-Atmospheric Model (MAOOAM).