Lorentzian Geometry and Topological Electromagnetism

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Thanks and funding

Section 1

Introduction

Outline

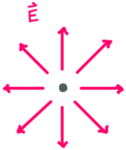
1. Intro Lorentzian geometry
2. Poincaré group $\mathrm{A}(1,3)$ and its Lie algebra $\mathfrak{a}(1,3)$
3. de Rham (Co)homology
4. Topological electromagnetism

Motivation

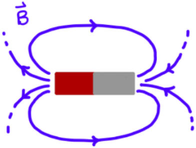
plasmas and what not, robotic motion, computer graphics, PDEs, inverse problems

Maxwell's Equations

Gauss's Laws

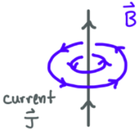


$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho$$



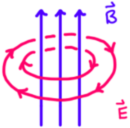
$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

Ampere's Law



$$\boldsymbol{\nabla} \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{J}$$

Faraday's Law



$$\boldsymbol{\nabla} \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0$$

Section 2

Lorentzian Geometry

Geometry through algebra

- Take a vector space V with a quadratic form $Q(-)$
- Create the ***Clifford algebra*** $C\ell(V, Q)$ from the tensor algebra
- Elements of $C\ell(V, Q)$ are ***multivectors*** of grade 0 (scalars) up to grade n (pseudoscalars)

Euclidean space

Take $\mathbb{R}^n$ with Euclidean norm $|-|$ and an orthonormal basis $\boldsymbol{e}_i$.

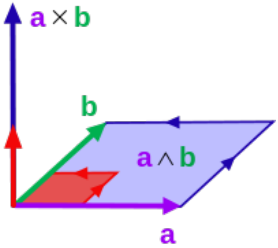
- We have the product in $\mathcal{G}_n := C\ell(\mathbb{R}^n, |-|)$ by

$$\boldsymbol{e}_i \boldsymbol{e}_j = \underbrace{\boldsymbol{e}_i \cdot \boldsymbol{e}_j}_{\text{scalar}} + \underbrace{\boldsymbol{e}_i \wedge \boldsymbol{e}_j}_{\text{bivector}}$$

- $\delta_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$ are the values of the Euclidean inner product on this basis.

$\mathbb{R}^3$

Given $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{G}_3$ bivector $\boldsymbol{a} \wedge \boldsymbol{b}$ represents an oriented plane



and the perpendicular or *dual* $(\boldsymbol{a} \wedge \boldsymbol{b})^{\perp} = \boldsymbol{a} \times \boldsymbol{b}$.

Lorentzian space

Instead, take $\mathbb{R}^4$ with basis $\boldsymbol{e}_0, \cdots, \boldsymbol{e}_3$ so that

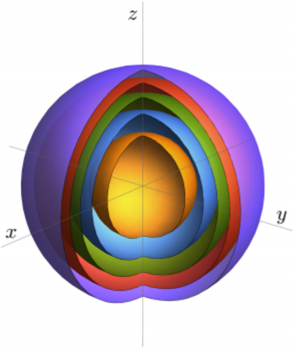
- $\boldsymbol{e}_0^2 = -1$ (temporal)
- $\boldsymbol{e}_i^2 = +1$ for $i = 1, 2, 3$ (spatial)
- $\boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu = 0$ if $\mu \neq \nu$ (orthogonal)
- Build $\mathcal{G}_{1,3}$ from these definitions.

Space Oddity

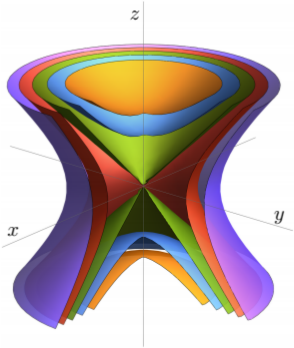
There exist ***null vectors*** $\boldsymbol{c}$ so that $\boldsymbol{c} \cdot \boldsymbol{c} = 0$.

Level sets $\boldsymbol{p} \cdot \boldsymbol{p} =$ constant yield foliations

Euclidean



Lorentzian



Question

What are the symmetries of Euclidean space?

Question

What are the symmetries of ***Lorentzian*** space?

Section 3

Poincaré Group

One way to study geometry is through symmetries of the space.

Want to see what deformations preserve the algebraic structure $\mathcal{G}_n$

- Rotations and reflections via the orthogonal group $\mathrm{O}(n)$ and special orthogonal group $\mathrm{SO}(n)$.
- Translations (if we keep track of base points) via the group $\mathbb{R}^n$.
- These symmetries combine to form ***Euclidean group*** $\mathrm{E}(n) = \mathbb{R}^n \rtimes \mathrm{O}(n)$ and ***special Euclidean group*** $\mathrm{SE}(n) = \mathbb{R}^n \rtimes \mathrm{SO}(n)$

- Given a unit vector $\boldsymbol{n}$ and multivector A, we have

$$\boldsymbol{n} A \boldsymbol{n}^\dagger$$

  reflects A about the hyperplane perpendicular to $\boldsymbol{n}$
- Given another unit vector $\boldsymbol{m}$,

$$\boldsymbol{n}\boldsymbol{m} A (\boldsymbol{n}\boldsymbol{m})^\dagger = \boldsymbol{n}\boldsymbol{m} A \boldsymbol{m}\boldsymbol{n}$$

  yields a rotation in the plane defined by $\boldsymbol{n} \wedge \boldsymbol{m}$.

Pin and Spin

- Unit vectors generate the group $\boldsymbol{n} \in \mathrm{Pin}(n)$ and define a an element $\mathrm{T} \in \mathrm{O}(n)$ by

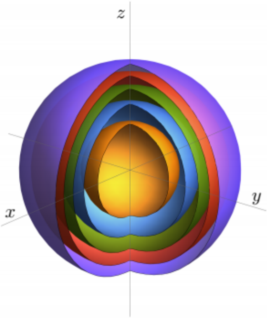
$$\mathrm{T}(\boldsymbol{v}) = \boldsymbol{n}\boldsymbol{v}\boldsymbol{n}^{\dagger}$$

  so the mapping $\boldsymbol{n} \mapsto \mathrm{T}$ is 2-to-1.

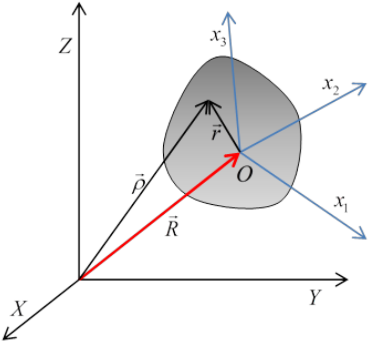
- Likewise, $\boldsymbol{n}\boldsymbol{m} \in \mathrm{Spin}(n)$ defines $\mathrm{R} \in \mathrm{SO}(n)$ by

$$\mathrm{R}(\boldsymbol{v}) = \boldsymbol{n}\boldsymbol{m}\boldsymbol{v}(\boldsymbol{n}\boldsymbol{m})^{\dagger}.$$

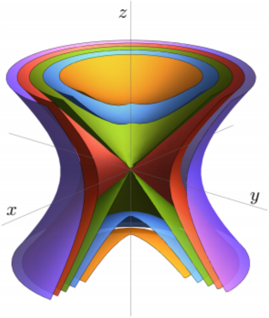
Both Spin(n) and Pin(n) lead to motion on the level sets



Hence, we can take $\mathbb{R}^n \rtimes \mathrm{Spin}(n)$ to be the rigid symmetries of Euclidean space.

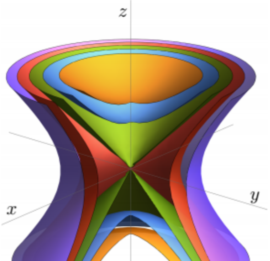


Define $\mathrm{Spin}(1,3)$ and $\mathrm{Pin}(1,3)$ analogously to see motion on the Lorentzian level sets



Relativity

- One sheeted hyperboloids are inaccessible regions of space (often called ***spacelike***)
- Two sheeted hyperboloids represent past and future directions (often called ***timelike***)
- Cone consists of all null vectors $\boldsymbol{c} \cdot \boldsymbol{c} = 0$ which represents ***light***.
- A particle with rest mass m has 4-momentum $m\boldsymbol{v}$ and $\boldsymbol{v} \cdot \boldsymbol{v} = -1$
- The future hyperboloids are foliated by mass.



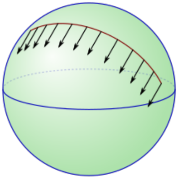
- Motion of a relativistic particle is a curve γ in the ***Poincaré group***

$$\mathrm{A}(1,3) = \mathbb{R}^{1,3} \rtimes \mathrm{Spin}^{+}(1,3)$$

- Since mass is preserved

$$\boldsymbol{\nabla}(\boldsymbol{v} \cdot \boldsymbol{v}) = 0 \quad \Longrightarrow \quad \nabla_{\boldsymbol{v}} \boldsymbol{v} = \boldsymbol{v} \cdot \underbrace{(\boldsymbol{\nabla} \wedge \boldsymbol{v})}_{\text{vorticity } \omega}$$

- Optimal transport of 4-velocity is given by projection onto the vorticity plane



Section 4

de Rham (Co)homology

de Rham (Co)homology

- Given a manifold M, we have the exterior algebra of (compactly supported) forms $\Omega(M)$.
  - de Rham cochain complex is built with the exterior derivative d.
  - de Rham cohomology ring is

$$H^{\bullet}_{dR}(M) = \bigwedge_{k\in\mathbb{N}} H^k_{dR} = \bigwedge_{k\in\mathbb{N}} \ker d_k \ / \ \mathrm{im}\, d_{k-1}.$$

- The dual space $\Omega^*(M)$ is the space of currents $T\colon \Omega(M) \to \mathbb{R}$.
  - de Rham chain complex is built with boundary operator ∂

$$\partial T[\alpha] = T[d\alpha].$$

  - de Rham homology group is

$$H^{dR}_{\bullet} = \bigoplus H^{dR}_k := \bigoplus \ker \partial_k \ / \ \mathrm{im}\partial_{k+1}.$$

Multivector Equivalents of Forms

Given a Riemannian manifold (M, g), take the Clifford algebra bundle $\mathcal{G}(M)$ whose sections are multivector fields.

- A k-form α_k has a multivector equivalent A_k by

$$\alpha_k = A_k \cdot dX_k.$$

    - $d\alpha_k \mapsto \boldsymbol{\nabla} \wedge A_k$.
    - $\delta\alpha_k \mapsto \boldsymbol{\nabla} \cdot A_k$.

- de Rham cohomology: $H^\bullet_{dR}(M) \cong \bigoplus_{n\in\mathbb{N}} \ker \boldsymbol{\nabla} \wedge_k \;\, / \; \mathrm{im}\boldsymbol{\nabla}\wedge_{k+1}$.
- de Rham homology: $H_\bullet^{dR} = \bigoplus_{k\in\mathbb{N}} \ker \boldsymbol{\nabla} \cdot_k \;\, / \; \mathrm{im}\boldsymbol{\nabla}\cdot_{k+1}$.

Multivector Equivalents of Currents

We have the Riemannian volume form μ and the bilinear pairing of multivectors $(-,-)$

- We can take a k-current by

$$T[-] = \int_M (T_k, -)\mu.$$

and for a k-chain K we have the k-current δ_K

$$\delta_K[-] = \int_K \mu_K = \int_M (\boldsymbol{I}_K, -)\mu.$$

- The boundary operator acts accordingly

$$\partial T[\alpha_{k-1}] = T[d\alpha_{k-1}] = \int_M (T_k, \boldsymbol{\nabla} \wedge A_{k-1})\mu = \int_M (\boldsymbol{\nabla} \cdot T_k, A_{k-1})\mu$$

Useful Theorems

Theorem (de Rham's Theorem)

The singular (co)homology over $\mathbb{R}$ is isomorphic to the de Rham (co)homology.

Theorem (Poincaré Duality)

We have $H_k \cong H^{n-k}$ by the dual $\perp$.

Theorem

The cap product is given by $\lrcorner\colon H^\ell(M) \times H_k(M) \to H_{k-\ell}(M)$.

Section 5

Topological Electromagnetism

Axioms

There are four physical postulates for electromagnetism that we write as topological axioms.

- **Axiom 1.** *Conservation of charge:* Current density $\boldsymbol{J}_3$ must flow through boundaries of regions N^4 so

  $$0 = \int_{\partial N^4} \boldsymbol{J}_3 \cdot dX_3 = \int_{N^4} (\boldsymbol{\nabla} \wedge \boldsymbol{J}_3) \cdot dX_4$$

  so $\boldsymbol{J}_3$ is closed. Hence, for a co-closed 3-current δ_{N^3} we have $\delta_{N^3}[j_3] = 0$ and which implies the magnetic excitation H is the potential $\boldsymbol{\nabla} \wedge H = \boldsymbol{J}_3$ or dually $\boldsymbol{J} = \boldsymbol{J}_3^{\perp} = \boldsymbol{\nabla} \cdot H^{\perp}$ defines a homology class.
- **Axiom 2.** *Conservation of flux:* The electromagnetic field F defines a cohomology class in $H^2(M)$ by taking a co-closed

Axioms

- **Axiom 3.** *Constitutive law:* We relate the excitation H with the field F. The simplest case is the linear case given by $F = H^{\perp}$ which yields the Maxwell equations $\boldsymbol{\nabla} F = \boldsymbol{J}$ or, in their more recognizable relativistic form

$$\boldsymbol{\nabla} \wedge F = 0 \qquad \text{(homogeneous)}$$

$$\boldsymbol{\nabla} \cdot F = \boldsymbol{J} \qquad \text{(inhomogeneous)}$$

  The homogeneous equations are Gauss's law for magnetism and Faraday's law whereas the inhomogeneous are Gauss's law for electricity and Ampere's law.

- **Axiom 4.** *Lorentz force:* The motion of a particle with 4-velocity $\boldsymbol{v}$ in a field F is

Faraday Transport

- For a charged particle, we have the Lorentz force law $\nabla_{\boldsymbol{v}}\boldsymbol{v} = \frac{1}{2}\boldsymbol{v} \cdot F$
- In terms of a proper time parameterization $\frac{d\boldsymbol{v}}{d\tau} = \frac{1}{2}\boldsymbol{v} \cdot F(\gamma(\tau))$.