MATH 272, HOMEWORK 8, *Solutions*

DUE APRIL 6^{TH}

Problem 1. Plot each of the following vector fields.

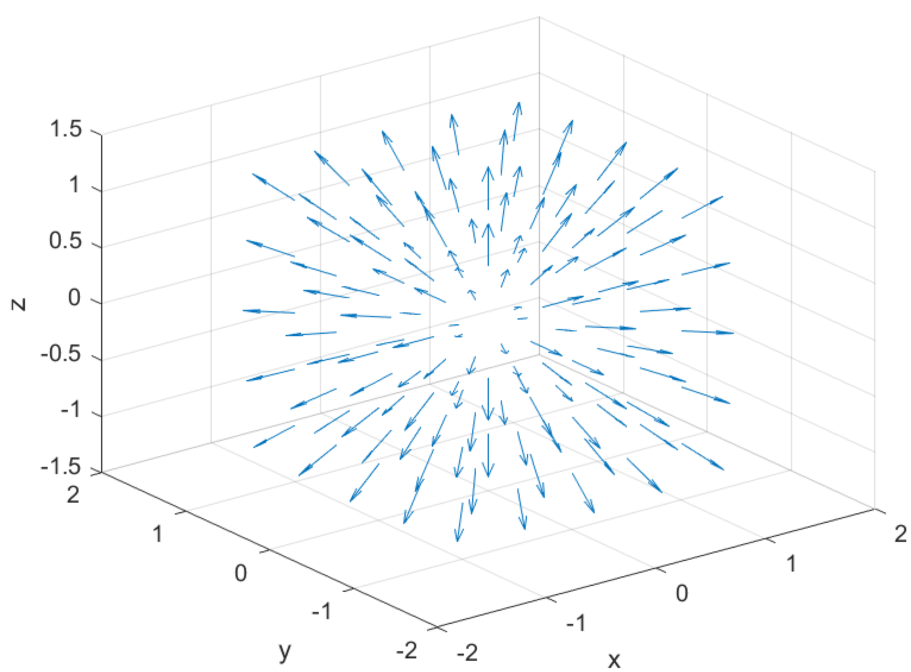
(a) $\hat{\boldsymbol{r}} = \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{x}} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{y}} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{z}}$.

(b) $\hat{\boldsymbol{\theta}} = \frac{-y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{x}} + \frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{y}}$.

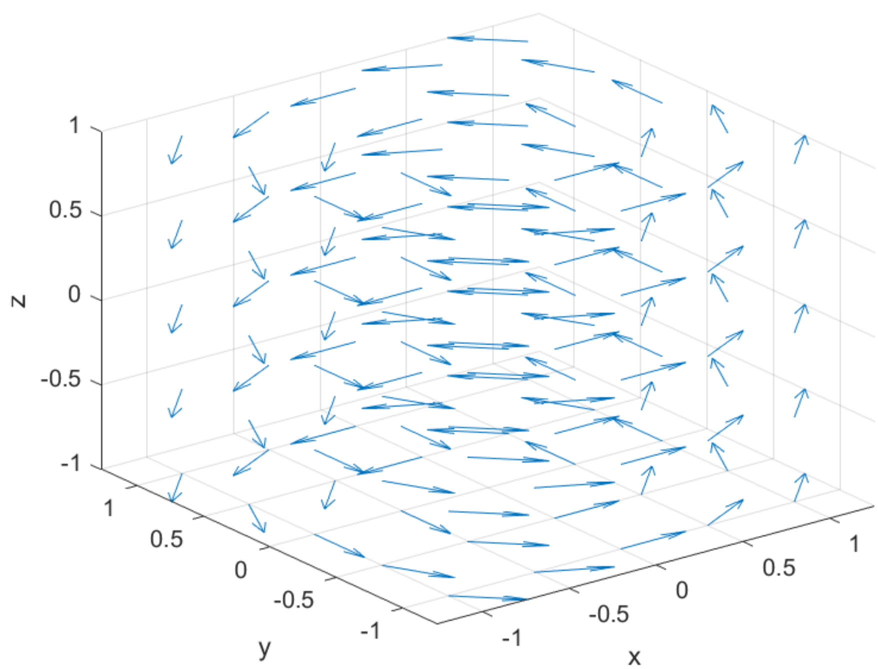
(c) $\hat{\boldsymbol{\phi}} = \frac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{x}} + \frac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{y}} + \frac{-\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{z}}$.

Solution 1.

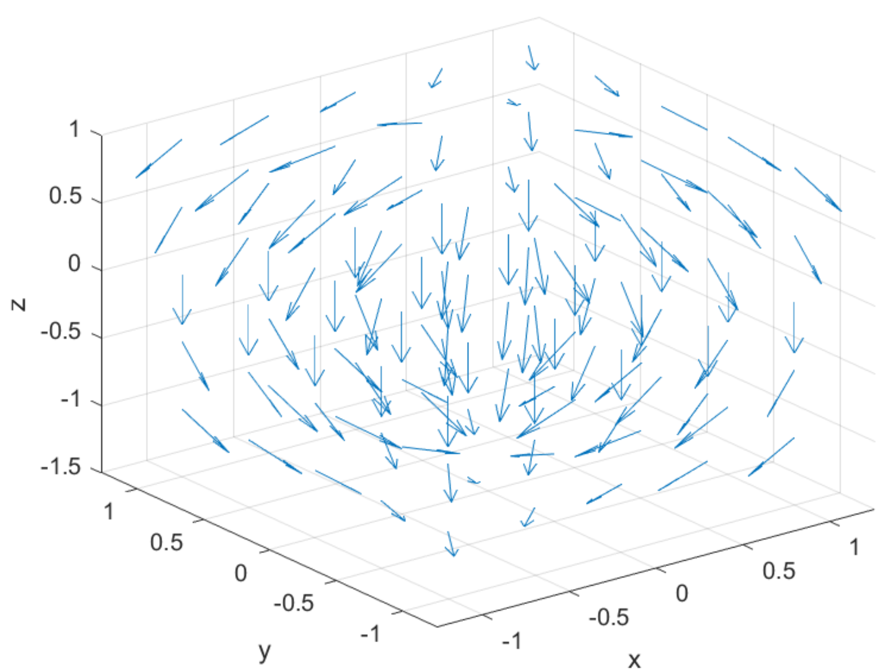
(a) Here is the plot for $\hat{\boldsymbol{r}}$.



(b) Here is the plot for $\hat{\boldsymbol{\theta}}$.



(c) Here is the plot for $\hat{\boldsymbol{\phi}}$.



Problem 2. Consider the following vector field

$$\vec{\boldsymbol{E}} = \frac{x}{(x^2+y^2+z^2)^{3/2}}\hat{\boldsymbol{x}} + \frac{y}{(x^2+y^2+z^2)^{3/2}}\hat{\boldsymbol{y}} + \frac{z}{(x^2+y^2+z^2)^{3/2}}\hat{\boldsymbol{z}},$$

which you can think of as the electric field of a positive point charge. We argued that this field $\vec{\boldsymbol{E}}$ is conservative in a previous homework problem. Specifically, $\vec{\boldsymbol{E}} = \vec{\boldsymbol{\nabla}}\phi$, for some scalar field ϕ. This follows from Faraday's law for static charges.

(a) Compute the integral

$$T = \int_{\vec{\gamma}} \vec{\boldsymbol{E}} \cdot d\vec{\gamma} \qquad \text{where} \qquad \vec{\gamma}(t) = \begin{pmatrix} t \\ t \\ t \end{pmatrix},$$

and $a \leq t \leq b$ with a and b both greater than 0. Note that this integral T describes the gain in kinetic energy of a charged particle that moved along the path $\vec{\gamma}$.

(b) Equivalently, since $\vec{\boldsymbol{E}}$ is conservative, we have

$$T = \int_{\vec{\gamma}} \vec{\boldsymbol{E}} \cdot d\vec{\gamma} = \phi(\vec{\gamma}(b)) - \phi(\vec{\gamma}(a)).$$

Show that this is true for the given vector field and potential. This shows that the choice of path does not matter; only the endpoints $\vec{\gamma}(a)$ and $\vec{\gamma}(b)$ matter.

(c) Argue why the integral around any closed curve must be zero.

Solution 2. (a) First, note that

$$d\vec{\gamma} = \dot{\vec{\gamma}}dt = (\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}})dt.$$

Thus, we have that

$$\vec{\boldsymbol{E}} \cdot d\vec{\gamma} = \frac{x+y+z}{(x^2+y^2+z^2)^{3/2}}dt.$$

Now, we have

$$\begin{aligned}
\int_{\vec{\gamma}} \vec{\boldsymbol{E}} \cdot d\gamma &= \int_a^b \frac{t+t+t}{(t^2+t^2+t^2)^{3/2}}dt \\
&= \int_a^b \frac{3t}{(3t^2)^{3/2}}dt \\
&= \int_a^b \frac{1}{\sqrt{3}}\frac{1}{t^2}dt \\
&= \frac{-1}{\sqrt{3}}\left(\frac{1}{b} - \frac{1}{a}\right).
\end{aligned}$$

(b) Note that we have

$$\phi(x, y, z) = \frac{-1}{\sqrt{x^2 + y^2 + z^2}},$$

from previous homeworks. Then, we have

$$\begin{aligned}
\phi(\vec{\gamma}(b)) - \phi(\vec{\gamma}(a)) &= \frac{-1}{\sqrt{b^2 + b^2 + b^2}} - \frac{1}{\sqrt{a^2 + a^2 + a^2}} \\
&= \frac{-1}{\sqrt{3}} \left(\frac{1}{b} - \frac{1}{a} \right)
\end{aligned}$$

(c) For a closed curve, we have $\vec{\gamma}(b) = \vec{\gamma}(a)$ and thus

$$\int_{\vec{\gamma}} \vec{\boldsymbol{E}} \cdot d\gamma = \phi(\vec{\gamma}(b)) - \phi(\vec{\gamma}(a)) = \phi(\vec{\gamma}(a)) - \phi(\vec{\gamma}(a)) = 0.$$

Problem 3. Let us see some of the benefit of using spherical coordinates.

(a) Using the fact that

$$\hat{\boldsymbol{r}} = \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{x}} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{y}} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{z}},$$

convert the vector field $\vec{\boldsymbol{E}}$ into spherical coordinates (i.e., only a function of r, θ, ϕ, and $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$).

(b) Parameterize the surface of a sphere of radius R (which we'll call Σ) as well as the outward normal vector $\hat{\boldsymbol{n}}$ and in spherical coordinates.

(c) Compute the following integral using spherical coordinates that we have found:

$$\iint_{\Sigma} \vec{\boldsymbol{E}} \cdot \hat{\boldsymbol{n}} d\Sigma,$$

where $d\Sigma$ will be the area form in spherical coordinates.

Solution 3.

(a) We have

$$\begin{aligned}
\vec{\boldsymbol{E}} &= \frac{x}{(x^2+y^2+z^2)^{3/2}}\hat{\boldsymbol{x}} + \frac{y}{(x^2+y^2+z^2)^{3/2}}\hat{\boldsymbol{y}} + \frac{z}{(x^2+y^2+z^2)^{3/2}}\hat{\boldsymbol{z}} \\
&= \frac{1}{x^2+y^2+z^2}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{x}} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{y}} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{z}}\right) \\
&= \frac{1}{x^2+y^2+z^2}\hat{\boldsymbol{r}} \\
&= \frac{1}{r^2}\hat{\boldsymbol{r}}.
\end{aligned}$$

(b) We have the parameterization of the surface of a unit sphere given by letting $r = R$ and $\theta \in [0, 2\pi)$ and $\phi \in [0, \pi]$. If we then attempt to compute the unit vector, we can use the implicit equation

$$f(x, y, z) = x^2 + y^2 + z^2 = R^2.$$

From this, we have

$$\begin{aligned}
\hat{\boldsymbol{n}} &= \frac{\vec{\boldsymbol{\nabla}} f}{\left|\vec{\boldsymbol{\nabla}} f\right|} \\
&= \frac{2x}{\sqrt{4x^2+4y^2+4z^2}}\hat{\boldsymbol{x}} + \frac{2y}{\sqrt{4x^2+4y^2+4z^2}}\hat{\boldsymbol{y}} + \frac{2z}{\sqrt{4x^2+4y^2+4z^2}}\hat{\boldsymbol{z}} \\
&= \hat{\boldsymbol{r}}.
\end{aligned}$$

(c) From the previous work, we have that

$$\vec{\boldsymbol{E}} \cdot \hat{\boldsymbol{n}} = \frac{1}{r^2}.$$

Then, this gives us

$$\begin{aligned}
\iint\limits_{\Sigma} \vec{\boldsymbol{E}} \cdot \hat{\boldsymbol{n}} d\Sigma &= \int_0^{\pi} \int_0^{2\pi} \frac{1}{R^2} R^2 \sin\phi \, d\theta \, d\phi \\
&= 2\pi \int_0^{\pi} \sin\phi \, d\phi \\
&= 4\pi.
\end{aligned}$$

One can notice that the radius of the sphere does not come into play here.

Problem 4. Note that the Laplacian Δ in cylindrical coordinates is given by

$$\Delta f(\rho, \theta, z) = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}.$$

Compute the Laplacian of

$$f(\rho, \theta, z) = \sqrt{\rho^2 + z^2}\, z\cos(\theta).$$

Solution 4. Let's compute each term and then add them together. We have

$$\begin{aligned} A = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) &= \frac{z\cos\theta}{\rho}\frac{\partial}{\partial \rho}\left(\frac{\rho^2}{\sqrt{\rho^2+z^2}}\right) \\ &= \frac{z\cos\theta}{\rho}\left(\frac{2\rho}{\sqrt{\rho^2+z^2}} - \frac{\rho}{(\rho^2+z^2)}\right) \\ &= z\cos\theta\left(\frac{2}{\sqrt{\rho^2+z^2}} - \frac{1}{(\rho^2+z^2)}\right). \end{aligned}$$

Likewise, we have

$$B = \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2} = -\frac{\sqrt{\rho^2+z^2}\, z\cos\theta}{\rho^2}.$$

Lastly, we have

$$\begin{aligned} C = \frac{\partial^2 f}{\partial z^2} &= \cos\theta \frac{\partial}{\partial z}\frac{\partial}{\partial z}\left(z\sqrt{\rho^2+z^2}\right) \\ &= \cos\theta \frac{\partial}{\partial z}\left(\sqrt{\rho^2+z^2} + \frac{z^2}{\sqrt{\rho^2+z^2}}\right) \\ &= \cos\theta\left(\frac{z}{\sqrt{\rho^2+z^2}} + \frac{2z}{\sqrt{\rho^2+z^2}} - \frac{z^3}{(\rho^2+z^2)^{3/2}}\right). \end{aligned}$$

Then we have that

$$\Delta f = A + B + C.$$

Problem 5. Note that the Laplacian Δ in spherical coordinates is given by

$$\Delta f(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 f}{\partial \phi^2}.$$

Compute the Laplacian of

$$f(r,\theta,\phi) = r^2\cos(\theta)\cos(\phi).$$

Solution 5. Again, we will do this piece by piece. First, we have

$$\begin{aligned} A = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) &= \frac{\cos\theta\cos\phi}{r^2}\frac{\partial}{\partial r}\left(2r^3\right) \\ &= \frac{\cos\theta\cos\phi}{r^2}6r^2 \\ &= 6\cos\theta\cos\phi. \end{aligned}$$

Next, we have

$$\begin{aligned} B = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) &= \frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial \theta}\left(-\sin^2\theta\right) \\ &= -2\cos\theta\cos\phi. \end{aligned}$$

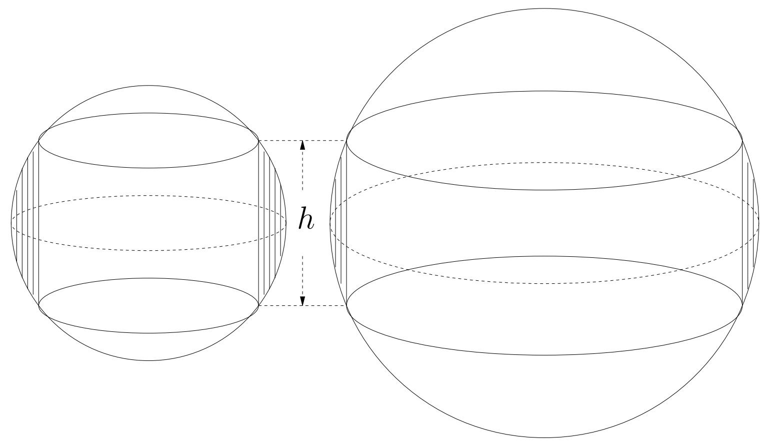
Lastly, we have

$$\begin{aligned} C = \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} &= \frac{\cos\theta}{\sin^2\theta}\left(-\cos\phi\right) \\ &= \frac{-\cos\theta\cos\phi}{\sin^2\theta}. \end{aligned}$$

Then,

$$\Delta f = A + B + C.$$

Problem 6. (BONUS) The following problem is a somewhat pop-culture math paradox known as the *napkin ring problem* (see Vsauce for more). Consider the following problem. We want to compute the volume inside a ball of radius R after drilling out an inscribed cylinder of height h. See the following picture.



The question is, does the left over volume (of the napkin ring) depend on the radius R of the sphere. You have your choice of working in spherical or cylindrical coordinates. Use whichever helps you most.

Solution 6.