MATH 560, Homework 5

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Solutions

Problem 1. (§4.3 Problem 6) Use Cramer's rule to solve the given system of linear equations.

$$\begin{aligned} x_1 - x_2 + 4x_3 &= -2 \\ -8x_1 + 3x_2 + x_3 &= 0 \\ 2x_1 - x_2 + x_3 &= 6 \end{aligned}$$

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Solution. Cramer's rule states that $x_k = \frac{\det(M_k)}{\det(A)}$. In this case we have

$$\det(A) = \det\left(\begin{bmatrix} 1 & -1 & 4 \\ -8 & 3 & 1 \\ 2 & -1 & 1 \end{bmatrix}\right) = 2$$

So we have

$$\begin{aligned} x_1 &= \frac{\det(M_1)}{\det(A)} = \frac{1}{2}\det\left(\begin{bmatrix} -2 & -1 & 4 \\ 0 & 3 & 1 \\ 6 & -1 & 1 \end{bmatrix}\right) = -43 \\ x_2 &= \frac{\det(M_2)}{\det(A)} = \frac{1}{2}\det\left(\begin{bmatrix} 1 & -2 & 4 \\ -8 & 0 & 1 \\ 2 & 6 & 1 \end{bmatrix}\right) = -109 \\ x_3 &= \frac{\det(M_3)}{\det(A)} = \frac{1}{2}\det\left(\begin{bmatrix} 1 & -1 & 2 \\ -8 & 3 & 0 \\ 2 & -1 & 6 \end{bmatrix}\right) = -17. \end{aligned}$$

So $x_1 = -43$, $x_2 = -109$, and $x_3 = -17$. ■

Problem 2. (§4.3 Problem 18) Complete the proof of Theorem 4.7 by showing that if A is an elementary matrix of type 2 or type 3, then $\det(AB) = \det(A)\cdot\det(B)$.

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Proof. For type 2, we have that A is a diagonal matrix with one in each entry except for the row/column we wish to scale. So

$$A = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \lambda & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

with λ in the kth diagonal entry. It's worth noting that left multiplication of B by A will scale the kth row of B and right multiplication will scale the kth column. Regardless, we have that $\det(A) = \lambda$. So we have that $\det(AB) = \lambda\det(B) = \det(A)\det(B)$ by theorem 4.3.

For A a type 3 elementary matrix, theorem 4.6 tells us that $\det(AB) = \det(B)$. Note that $\det(A) = 1$ and we have that, $\det(AB) = \det(A)\det(B)$. □

Problem 3. (§4.3 Problem 21.) Prove that if $M \in \mathbf{M}_{n\times n}(\mathbb{F})$ can be written in the form

$$M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where A and C are square matrices, then $\det(M) = \det(A)\cdot\det(C)$.

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Proof. Consider first

$$M = \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & I_{q\times q} \end{bmatrix} \begin{bmatrix} I_{p\times p} & 0 \\ 0 & B \end{bmatrix},$$

where $p+q=n$. Then we have that

$$\begin{bmatrix} A & 0 \\ 0 & I_{q\times q} \end{bmatrix} = E_1 E_2 \cdots E_r I_{n\times n}$$

and

$$\begin{bmatrix} I_{p\times p} & 0 \\ 0 & B \end{bmatrix} = E_1' E_2' \cdots E_l' I_{n\times n}$$

where E_i, E_j' are elementary $n\times n$ matrices. The above work shows that $\det(M) = \det(E_1\cdots E_r E_1' \cdots E_l' I_{n\times n}^2) = \det(E_1\cdots E_r)\det(E_1'\cdots E_l') = \det(A)\det(B)$. Finally, note that we can generate

$$M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

by using $n\times n$ type 3 elementary matrices. To show this we have

$$\begin{bmatrix} I_{p\times p} & C \\ 0 & I_{q\times q} \end{bmatrix} = E_1'' E_2'' \cdots E_t'' I_{n\times n}$$

where E_i'' are type 3 matrices. By combining the above work with this, we get that

$$\det(M) = \det(E_1\cdots E_r E_1'\cdots E_l' E_1''\cdots E_t'' I_{n\times n}^3) = \det(A)\det(B)$$

since the determinant is not affected by type 3 matrices.

Note that *Problem 4.* of this assignment is a repeat and I don't have another unique proof to show, so I'd use this one. $\square$

Problem 5. (§5.1 (Problem 3. (a),(c)) For each of the following matrices $A \in \mathbf{M}_{n\times n}(\mathbb{F})$,

(i) Determine all the eigenvalues of A.

(ii) For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ.

(iii) If possible, find a basis for $\mathbb{F}^n$ consisting of eigenvectors of A.

(iv) If successful in finding such a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

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Solution (Part (a)). For matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

with coefficients in $\mathbb{R}$.

i) We get the characteristic polynomial $(1-\lambda)(2-\lambda)-6 = \lambda^2-3\lambda-4 = (\lambda+1)(\lambda-4)$ so we have eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 4$.

ii)

$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}$$

Which yields equations

$$\begin{aligned} x_1 + 2x_2 &= -x_1 \\ 3x_1 + 2x_2 &= -x_2 \end{aligned}$$

Which tells us that $x_2 = -x_1$. So the eigenvector corresponding to λ_1 is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \end{bmatrix}$$

Which yields equations

$$\begin{aligned} x_1 + 2x_2 &= 4x_1 \\ 3x_1 + 2x_2 &= 4x_2 \end{aligned}$$

Which tells us that $x_2 = \frac{3}{2}x_1$. So the eigenvector corresponding to λ_1 is $\begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix}$.

iii) The basis is $\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix}\right\}$.

iv) We want $Q^{-1}AQ = D$ with D diagonal. So we have

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix}.$$

Which tells us that

$$Q = \begin{bmatrix} 1 & 1 \\ -1 & \frac{3}{2} \end{bmatrix} \quad \text{and} \quad Q^{-1} = \frac{1}{5}\begin{bmatrix} 3 & -2 \\ 2 & 2 \end{bmatrix}.$$

So then,

$$Q^{-1}AQ = \frac{1}{5}\begin{bmatrix} 3 & -2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$$ ■

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Solution (Part (c)). For matrix

$$A = \begin{bmatrix} i & 1 \\ 2 & -i \end{bmatrix}$$

with coefficients in $\mathbb{C}$.

i) We get the characteristic polynomial $(i-\lambda)(-i-\lambda)-2 = \lambda^2-1 = (\lambda+1)(\lambda-1)$ so we have eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$.

ii)

$$\begin{bmatrix} i & 1 \\ 2 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Which yields equations

$$\begin{aligned} ix_1 + x_2 &= x_1 \\ 2x_1 - ix_2 &= x_2 \end{aligned}$$

Which tells us that $x_2 = (1-i)x_1$. So the eigenvector corresponding to λ_1 is $\begin{bmatrix} 1 \\ 1-i \end{bmatrix}$.

$$\begin{bmatrix} i & 1 \\ 2 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}$$

Which yields equations

$$\begin{aligned} ix_1 + x_2 &= -x_1 \\ 2x_1 - ix_2 &= -x_2 \end{aligned}$$

Which tells us that $x_2 = (-1-i)x_1$. So the eigenvector corresponding to λ_1 is $\begin{bmatrix} 1 \\ -1-i \end{bmatrix}$.

iii) The basis is $\left\{\begin{bmatrix} 1 \\ 1-i \end{bmatrix}, \begin{bmatrix} 1 \\ -1-i \end{bmatrix}\right\}$.

iv) We want $Q^{-1}AQ = D$ with D diagonal. So we have

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$

and

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1-i \end{bmatrix}.$$

Which tells us that

$$Q = \begin{bmatrix} 1 & 1 \\ 1-i & 1-i \end{bmatrix} \quad \text{and} \quad Q^{-1} = \frac{1}{2}\begin{bmatrix} 1+i & 1 \\ 1-i & -1 \end{bmatrix}.$$

So then,

$$Q^{-1}AQ = \frac{1}{2}\begin{bmatrix} 1+1 & 1 \\ 1-i & -1 \end{bmatrix}\begin{bmatrix} i & 1 \\ 2 & -i \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1-i & -1-i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \blacksquare$$

Problem 6. (§5.1 Problem 4. (e)) For each linear operator T on V, find the eigenvalues of T and an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

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Proof (Part (e)). We have

$$\begin{aligned}T(a_0+a_1x+a_2x^2) &= x(a_1+2a_2x)+x(a_0+2a_1+4a_2)+(a_0+3a_1+9a_2)\\ &= (a_0+3a_1+9a_2)+x(a_0+3a_1+4a_2)+x^2(2a_2).\end{aligned}$$

Let $\alpha = \{1, x, x^2\}$. Then in this basis we have

$$[T]_\alpha x = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_0+3a_1+9a_2 \\ a_0+3a_1+4a_2 \\ 2a_2 \end{bmatrix}.$$

Thus

$$[T]_\alpha = \begin{bmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

The characteristic polynomial is $(1-\lambda)(3-\lambda)(2-\lambda)-3(2-\lambda) = -\lambda(\lambda-4)(\lambda-2)$. Thus we get $\lambda_1 = 0, \lambda_2 = 2$, and $\lambda_3 = 4$. So

$$\begin{bmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This yields equations

$$\begin{aligned} x_1+3x_2+9x_3 &= 0 \\ x_1+3x_2+4x_3 &= 0 \\ 2x_3 &= 0. \end{aligned}$$

Which tells us $x_1 = -3$, $x_2 = 1$, and $x_3 = 0$. Next we get

$$\begin{bmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}.$$

This yields equations

$$\begin{aligned} x_1+3x_2+9x_3 &= 2x_1 \\ x_1+3x_2+4x_3 &= 2x_2 \\ 2x_3 &= 2x_3. \end{aligned}$$

Which tells us that $x_1 = -3$, $x_2 = -13$, and $x_3 = 4$. Finally we get

$$\begin{bmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}.$$

This yields equations

$$\begin{aligned} x_1+3x_2+9x_3 &= 4x_1 \\ x_1+3x_2+4x_3 &= 4x_2 \\ 2x_3 &= 4x_3. \end{aligned}$$

Which tells us that $x_1 = 1$, $x_2 = 1$, and $x_3 = 0$. So our basis of eigenvectors is $\beta = \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -13 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$. So in this basis, T is diagonal. Specifically,

$$[T]_\beta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}. \qquad \square$$

Problem 7. (§5.1 Problem 7.) Let T be a linear operator on a finite-dimensional vector space V. We define the **determinant** of T, denoted $\det(T)$, as follows: Choose any ordered basis β for V, and define $\det(T) = \det([T]_\beta)$.

(a) Prove that the preceding definition is independent of the choice of an ordered basis for V. That is, prove that if β and γ are two ordered bases for V, then $\det([T]_\beta) = \det([T]_\gamma)$.

(b) Prove that T is invertible if and only if $\det(T) \neq 0$.

(c) Prove that if T is invertible, then $\det(T^{-1}) = (\det(T))^{-1}$.

(d) Prove that if U is also a linear operator on V, then $\det(TU) = \det(T) \cdot \det(U)$.

(e) Prove that $\det(T - \lambda I_V) = \det([T]_\beta - \lambda I)$ for any scalar λ and any ordered basis β for V.

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Proof (Part (a)). Since $\det(T) = \det([T]_\beta)$ for any ordered basis β. It is that $\det(T) = \det([T]_\gamma)$ for another ordered basis. Thus $\det([T]_\beta) = \det([T]_\gamma)$. □

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Proof (Part (b)). First suppose that T is invertible. Thus $TT^{-1} = I$. Then, $\det(TT^{-1}) = \det(T)\det(T^{-1}) = \det(I) = 1$. Thus if this is satisfied, we have that $\det(T) \neq 0$. For the converse, suppose for a contradiction that $\det(T) = 0$ but T is invertible. Since $\det(T) = 0$ there is at least one row of $[T]_\beta$ for any basis β is a linear combination of the other rows. This means that for some $x \neq 0$ we have that $[T]_\beta x = 0$. Thus $[T]_\beta$ is not injective and thus not invertible. □

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Proof (Part (c)). We have that $1 = \det(I) = \det(TT^{-1}) = \det(T)\det(T^{-1})$. Thus $\det(T^{-1}) = \det(T)^{-1}$. □

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Proof (Part (d)). We have that U can be created by multiplying elementary matrices of all three distinct types. Since we showed that for all three types elementary matrices E that $\det(ET) = \det(E)\det(T)$ we have that $U = E_1 \cdots E_m$ for E_i an elementary matrix of type 1,2, or 3 and thus $\det(UT) = \det(E_1 \cdots E_m)\det(T) = \det(U)\det(T)$. □

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Proof (Part (e)). We have $\det(T - \lambda I_V) = \det([T - \lambda I_V]_\beta) = \det([T] - \lambda [I]_\beta) = \det([T]_\beta - \lambda I)$ since I is the same no matter which basis. □

Problem 8. (§5.1 Problem 9.) Prove that the eigenvalues of an upper triangular matrix M are the diagonal entries of M.

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Proof. We can do this by cofactor expansion. Let M be upper triangular $n \times n$ matrix, then

$$\begin{aligned}\det(M) &= \sum_{j=1}^{n} (-1)^{1+j} M_{1j} \det(\tilde{M}_{1j}) \\ &= M_{11} \det(\tilde{M}_{11})\end{aligned}$$

Since the only nonzero entry in the first column is M_{11}. It's convenient to rename $\tilde{M}_{11} = M^{(1)}$. The superscript in $M^{(q)}$ tells us that we're looking at a sub-matrix of M with the first q rows and columns removed. Then, we have

$$\begin{aligned}\det(M) &= M_{11} \sum_{j=1}^{n-1} (-1)^{1+j} M^{(1)}_{1j} \det(\tilde{M}^{(1)}_{1j}) \\ &= M_{11} M^{(1)}_{11} \det(\tilde{M}^{(1)}_{11}) \\ &= M_{11} M_{22} \det(\tilde{M}^{(1)}_{11}).\end{aligned}$$

It's worth showing one more iteration before jumping to the final step. Next we have

$$\begin{aligned}\det(M) &= M_{11} M_{22} \sum_{j=1}^{n-2} (-1)^{1+j} M^{(2)}_{1j} \det(\tilde{M}^{(2)}_{1j}) \\ &= M_{11} M_{22} M^{(2)}_{11} \det(\tilde{M}^{(2)}_{11}) \\ &= M_{11} M_{22} M_{33} \det(\tilde{M}^{(2)}_{11}).\end{aligned}$$

Then finally,

$$\begin{aligned}\det(M) &= M_{11} \cdots M_{(n-2)(n-2)} \sum_{j=1}^{2} (-1)^{1+j} M^{(n-2)}_{1j} \det(\tilde{M}^{(n-2)}_{1j}) \\ &= M_{11} \cdots M_{(n-2)(n-2)} M^{(n-1)}_{11} \det(\tilde{M}^{(n-2)}_{11}) \\ &= M_{11} \cdots M_{(n-2)(n-2)} M_{(n-1)(n-1)} M_{nn}.\end{aligned}$$

□

Problem 9. (§5.1 Problem 14.) For any square matrix A, prove that A and A^t have the same characteristic polynomial (and hence the same eigenvalues).

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Proof. By theorem 4.9 we have that $\det(A) = \det(A^t)$. Since we get the characteristic polynomial by subtracting λ from the diagonal entries and the diagonal entries do not change from transposing a matrix, it must be the case that the characteristic polynomial for A and A^t are the same. To show this another way, $\det(A - \lambda I) = \det((A - \lambda I)^t)$ so the characteristic polynomials are equivalent. $\square$

Problem 10. (§5.1 Problem 24.) Use Exercise 21(a) to prove Theorem 5.3.

Theorem 5.3 states: Let $A \in \mathbf{M}_{n\times n}(\mathbb{F})$.

(a) The characteristic polynomial of A is a polynomial of degree n with leading coefficient $(-1)^n$.

(b) A has at most n distinct eigenvalues.

Then note that Exercise 21(a) tells us that $f(t) = (A_{11} - t)(A_{22} - t)\cdots(A_n n - t) + q(t)$ with $f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \ldots + a_1 t + a_0$.

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Proof (Part (a)). If we multiply out $f(t)$ we have that $f(t) = (-t)\cdots(-t) + g(t) = (-1)^n t^n + g(t)$ for some polynomial $g(t)$ which has degree at most $n-1$. So the characteristic polynomial is degree n with leading coefficient $(-1)^n$. □

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Proof (Part (b)). Since the characteristic polynomial is a polynomial of degree n, by the fundamental theorem of algebra there are at most n distinct roots for the polynomial over $\mathbb{C}$. □