Hodge Theory, Gelfand Theory, and Clifford Analysis Applied to Tomography

Colin Roberts

Colorado State University

Overview

Section 1

Introduction

Motivating problems

- ***Electrical Impedance Tomography (EIT)*** asks whether one can determine the conductivity of a medium from the voltage-to-current map.
- The ***Calderón problem*** replaces the medium with a manifold M, conductivity with g, and replaces the voltage-to-current map with the Dirichlet-to-Neumann operator Λ.

Other questions

- What topological information can we retrieve from functions on a manifold?
- Do these functions also contain metric data?
- Can we access these functions from the boundary?

Subsection 1

Preliminaries

Clifford and geometric algebras

Let V be a vector space over a field K with symmetric bilinear form g.

- Define the ***tensor algebra***

$$\mathcal{T}(V) := \bigoplus_{j=0}^{\infty} V^{\otimes j} = K \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots .$$

- The associated ***Clifford algebra*** is the quotient

$$C\ell(V, g) := \mathcal{T}(V) / \langle \mathbf{v} \otimes \mathbf{v} - g(\mathbf{v}, \mathbf{v}) \rangle.$$

Geometric and exterior algebras

- If g is non-degenerate then we have a *geometric algebra*

$$\mathcal{G} := C\ell(V, g).$$

- The completely degenerate case is the *exterior algebra*

$$\bigwedge(V) := C\ell(V, 0).$$

Algebraic structure

$\mathcal{G}$ is generated by scalars and vectors given how $\otimes$ acts in the quotient.

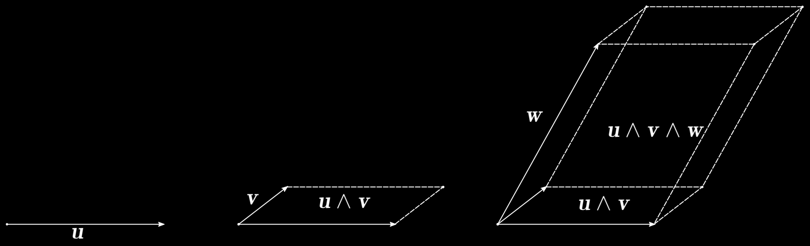
- Given vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{G}$ we can take the product

$$\boldsymbol{u}\boldsymbol{v} = \underbrace{\boldsymbol{u} \cdot \boldsymbol{v}}_{\text{scalar}} + \underbrace{\boldsymbol{u} \wedge \boldsymbol{v}}_{\text{bivector}} .$$

  - The scalar part is symmetric: $\boldsymbol{u} \cdot \boldsymbol{v} = g(\boldsymbol{u}, \boldsymbol{v})$.
  - The bivector part is antisymmetric: $\boldsymbol{u} \wedge \boldsymbol{v} = -\boldsymbol{v} \wedge \boldsymbol{u}$.

Multivectors

- $\mathcal{G}$ is graded and of dimension 2^n.
  - Grade-r elements, $\mathcal{G}^r$, called ***r-vectors***.
  - $\langle A \rangle_r \in \mathcal{G}^r$ extracts the grade-r part of an arbitrary element A.
  - There are $\binom{n}{r}$ independent ***r-blades*** of the form $\boldsymbol{A_r} = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r$.
  - Elements of the even grade subalgebra, $\mathcal{G}^+$, are called ***spinors***.
- Since $\mathcal{G} = \bigoplus_{r=0}^{n} \mathcal{G}^r$ a general ***multivector*** is $A = \sum_{r=0}^{n} \langle A \rangle_r$.



Algebraic Structure

- Extend the multiplication from vectors to multivectors.
- On homogeneous elements,

$$A_r B_s = \langle A_r B_s \rangle_{|r-s|} + \langle A_r B_s \rangle_{|r-s|+2} + \cdots + \langle A_r B_s \rangle_{r+s}$$

- The most important products for us are

$$A_r \wedge B_s \coloneqq \langle A_r B_s \rangle_{r+s}$$

$$A_r \lrcorner B_s \coloneqq \langle A_r B_s \rangle_{s-r}$$

Reciprocals and reverses

- Given any vector basis $\boldsymbol{e}_i$, define the ***reciprocal vectors*** by $\boldsymbol{e}^i \cdot \boldsymbol{e}_j = \delta^i_j$.
- The ***reverse*** $\dagger$ is extended linearly from the action on r-blades

$$\boldsymbol{A_r}^\dagger = (\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_r)^\dagger = \boldsymbol{v}_r \wedge \cdots \wedge \boldsymbol{v}_1.$$

Inner product and norm

- Define the *multivector inner product* and *multivector norm* by

$$A * B := \left\langle A^\dagger B \right\rangle =: |A|^2$$

- Reverse $\dagger$ is the adjoint operator

$$(CA) * B = A * (C^\dagger B)$$
$$(AC) * B = A * (BC^\dagger).$$

- **g definite $\implies$ $*$ and $|\blacksquare|$ definite.**

Blades and subspaces

- If $|\boldsymbol{U_r}| = \pm 1$, then $\boldsymbol{U_r}$ is a *unit blade*.
- Unit r-blades correspond to subspaces $U \subset V$.
- The *projection* of A into a subspace $\boldsymbol{U_r}$ by

$$\mathrm{P}_{\boldsymbol{U_r}}(A) \coloneqq A \,\lrcorner\, \boldsymbol{U_r} \boldsymbol{U_r}^{-1}.$$

Pseudoscalars

- *Pseudoscalars* are the grade-n elements.
- Define the *volume element*

$$\boldsymbol{\mu} = \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_n.$$

- Define the *unit pseudoscalar* by

$$\boldsymbol{I} := \frac{1}{|\boldsymbol{\mu}|}\boldsymbol{\mu}.$$

Duality

- The ***dual*** $\perp$ of a multivector A is

$$A^{\perp} := A\boldsymbol{I}^{-1} \in \mathcal{G}^{n-r}.$$

- The ***Hodge star*** $\star_g$ of a multivector A is

$$\star_g A = (\boldsymbol{I}^{-1}A)^{\dagger}.$$

- Dual exchanges products $(A \lrcorner B)^{\perp} = A \wedge B^{\perp}$.

Examples

- Define $\mathcal{G}_{p,q}$ by $\boldsymbol{e}_i^2 = -1$ for $i = 1, \ldots, p$ and $\boldsymbol{e}_i^2 = +1$ otherwise.
- $\mathcal{G}_{1,3}$ is the ***spacetime algebra***.
- $\mathcal{G}_{1,3}^2 \cong \mathfrak{spin}(1,3)$ which is the Lie algebra of the Lorentz group.
- ***Quaternion algebra*** $\mathbb{H}$ is isomorphic to $\mathcal{G}_{0,3}^+$.
- ***Complex algebra*** $\mathbb{C}$ is isomorphic to $\mathcal{G}_{0,2}^+$.
  - Standard basis $\boldsymbol{e}_1, \boldsymbol{e}_2$, and $\boldsymbol{e}_{12} \coloneqq \boldsymbol{e}_1 \boldsymbol{e}_2$. Then $\boldsymbol{e}_{12}^2 = -1$.
  - Right multiplication of vectors by $\boldsymbol{e}_{12}$ rotates counter-clockwise by $\pi/2$.

Section 2

Clifford analysis

Multivector Fields

- (M, g) is a smooth, compact, connected, oriented n-dimensional Riemannian manifold.
- **Idea: Form the Clifford algebras on tangent spaces.**
  - Form the ***geometric algebra bundle***

$$\mathcal{G}M := \bigsqcup_{p \in M} C\ell(T_pM, g_p).$$

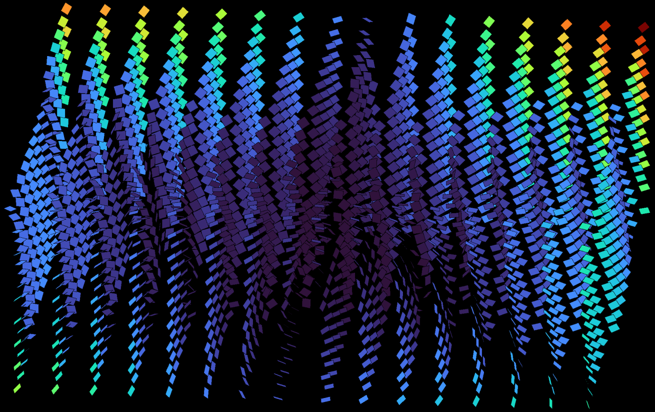
  - The ***multivector fields*** $\mathfrak{X}(M)$ are the C^∞-sections of $\mathcal{G}M$.
- Take same naming scheme and notation: $\mathfrak{X}^r(M)$, $\mathfrak{X}^+(M)$, etc.

Scalar field

$$\langle p_{(1,0)} + p_{(0,1)} + p_{(1,1)} + p_{(2,0)} + p_{(0,2)} + p_{(3,0)} + p_{(0,3)} + p_{(2,1)}(1 + \mathbf{e}_{31}) + p_{(1,2)}(1 + \mathbf{e}_{31}) \rangle$$

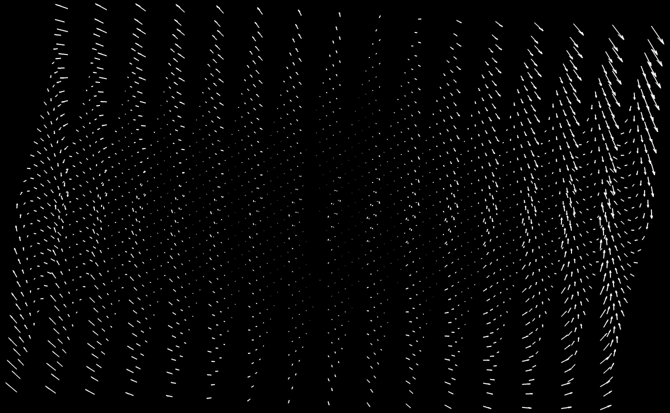
Bivector field

$$\left\langle p_{(1,0)} + p_{(0,1)} + p_{(1,1)} + p_{(2,0)} + p_{(0,2)} + p_{(3,0)} + p_{(0,3)} + p_{(2,1)}(1 + \mathbf{e}_{31}) + p_{(1,2)}(1 + \mathbf{e}_{31}) \right\rangle_2$$



Vector field

$$\left\langle p_{(1,0)} + p_{(0,1)} + p_{(1,1)} + p_{(2,0)} + p_{(0,2)} + p_{(3,0)} + p_{(0,3)} + p_{(2,1)}(1 + \boldsymbol{e}_{31}) + p_{(1,2)}(1 + \boldsymbol{e}_{31}) \right\rangle_2^{\perp}$$



Hodge–Dirac operator

M has the Levi-Civita connection ∇ and covariant derivative $\nabla_{\boldsymbol{u}}$ which can be extended to act on multivectors [Schindler: 2018].

- Define the *Hodge–Dirac operator* locally by

$$\boldsymbol{\nabla} = \sum_{i=1}^{n} \boldsymbol{e}^i \nabla_{\boldsymbol{e}_i}$$

- $\boldsymbol{\nabla}$ acts as a vector in $\mathfrak{X}(M)$ with Leibniz rule $\boldsymbol{\nabla}(AB) = \dot{\boldsymbol{\nabla}}\dot{A}B + \dot{\boldsymbol{\nabla}}A\dot{B}$.
- $\boldsymbol{\nabla}^2$ is the Laplace-Beltrami operator.

Examples

- For $A_+ \in \mathfrak{X}^+(\mathbb{R}^2)$ if $\boldsymbol{\nabla} A_+ = 0$ then A_+ is a **holomorphic function**.
- For a vector field $\boldsymbol{v} \in \mathfrak{X}^1(\mathbb{R}^3)$ we have

$$\boldsymbol{\nabla}\boldsymbol{v} = \underbrace{\boldsymbol{\nabla} \lrcorner\, \boldsymbol{v}}_{\textbf{divergence}} + \underbrace{\boldsymbol{\nabla} \wedge \boldsymbol{v}}_{\textbf{curl}}.$$

where

$$\operatorname{curl}(\boldsymbol{v}) = (\boldsymbol{\nabla} \wedge \boldsymbol{v})^{\perp}$$

Differential forms

- Define the *r-dimensional directed measure dX_r* by

$$dX_r := \boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_r}\, dx^{i_1} \cdots dx^{i_r}\,.$$

- Any r-form α_r has a ***multivector equivalent*** A_r so $\alpha_r = A_r \,\lrcorner\, dX_r^\dagger$.
- Algebra and calculus descend to multivector equivalent:

$$\alpha_r \wedge \beta_s = (A_r \wedge B_s) \,\lrcorner\, dX_{r+s} \qquad \alpha_r \,\lrcorner\, \beta_s = (A_r \,\lrcorner\, B_s) \,\lrcorner\, dX_{r-s}$$

$$\underbrace{d\alpha_r = (\boldsymbol{\nabla} \wedge A_r) \,\lrcorner\, dX^\dagger_{r+1}}_{\textbf{exterior derivative}} \qquad \underbrace{\delta\alpha_r = (-\boldsymbol{\nabla} \,\lrcorner\, A_r) \,\lrcorner\, dX^\dagger_{r-1}}_{\textbf{codifferential}}$$

Submanifolds

Fix an r-dimensional submanifold R.

- Define the ***tangent unit pseudoscalar*** $\boldsymbol{I}_R$.
- Dual is the ***normal blade*** $\boldsymbol{\nu}_R = \boldsymbol{I}_R^{\perp}$.
- Define the ***volume form*** on R by

$$d\mu_R := \boldsymbol{I}_R^{-1} \lrcorner \, dX_r$$

- For M this yields $d\mu = \sqrt{|g|}\, dx^1 \cdots dx^n$

Integral products

- Define the *directed integral product on R*

$$\langle\!| A, B |\!\rangle_R := A^\dagger \boldsymbol{I}_R B d\mu_R.$$

- Define the *multivector field inner product on R* by

$$\langle\!\langle A, B \rangle\!\rangle_R := \int_R A * B d\mu_R$$

Green's formulas

- From [Hestenes, Sobczyk, 1984] and [Booß- Bavnbek, Wojciechowski, 1993]

$$\llparenthesis \boldsymbol{\nabla} A, B \rrparenthesis = (-1)^n \llparenthesis A, \boldsymbol{\nabla} B \rrparenthesis + \llparenthesis A, B \rrparenthesis_{\partial M}$$

- Following from the above

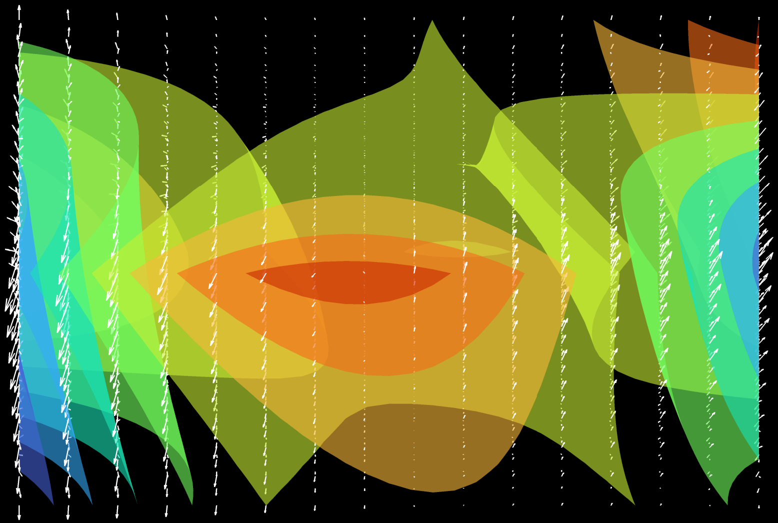
$$\langle\!\langle \boldsymbol{\nabla} A, B \rangle\!\rangle = -\langle\!\langle A, \boldsymbol{\nabla} B \rangle\!\rangle + \langle\!\langle A, \boldsymbol{\nu} B \rangle\!\rangle.$$

Monogenic fields

- The space of ***monogenic fields*** $\mathcal{M}(M)$ is the kernel of $\boldsymbol{\nabla}$.
- Ex. $f = u + v\mathbf{e}_{12} \in \mathfrak{X}^+(\mathbb{R}^2)$ then $\boldsymbol{\nabla} f = 0$ is holomorphic

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

- Field $p_{(1,0)} + p_{(0,1)} + \cdots$ is monogenic (or quaternion harmonic).



Properties of monogenic fields

Monogenic fields...

- can be **uniquely continued** [Booß- Bavnbek, Wojciechowski: 1993];
- have a **Cauchy integral** [Calderbank: 1995];
- have **harmonic components**.

Cauchy integral

- There exists a vector-valued ***Cauchy kernel*** G_x where $\nabla G_x = \delta_x$.
- Given $A \in \mathcal{M}(M)$, the ***Cauchy integral*** is

$$A(x) = (-1)^{n-1} \llparenthesis A, G_x \rrparenthesis^{\perp}_{\partial M}.$$

- [Calderbank: 1995], this map is an isomorphism.

Example

Consider fields on a region $M \subset \mathbb{R}^n$:

- Define $\boldsymbol{G}(\boldsymbol{x}) := \frac{1}{S_n} \frac{\mathbf{x}}{|\mathbf{x}|^n}$ then the Cauchy integral is

$$A(\boldsymbol{x}) = \frac{1}{S_n} \int_{\partial M} \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^n} \boldsymbol{\nu}(\boldsymbol{x}') A(\boldsymbol{x}') \, d\mu_{\partial M}(\boldsymbol{x}').$$

- **Scalar part of the above is the double layer potential.**

Inversion

- [Calderbank, 1995] Can solve the equation $\boldsymbol{\nabla} A = B$ by

$$A(x) = (-1)^{n-1} \llparenthesis B, G_x \rrparenthesis^{\perp}.$$

- In a region $M \subset \mathbb{R}^3$ take a vector field $\boldsymbol{J}$,

$$\mathrm{BS}(\boldsymbol{J})(\boldsymbol{x}) = \left\langle \llparenthesis \boldsymbol{J}, G_{\mathbf{x}} \rrparenthesis^{\perp} \right\rangle_2 = \frac{1}{4\pi} \int_{N^3} \boldsymbol{J}(\boldsymbol{y}) \wedge \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} \, d\mu_{N^3}(\boldsymbol{x}').$$

- **This is the Biot–Savart operator.** [Cantarella, et al.: 2001]

z-polynomials

- Define the functions $z_{ij} = x_j - x_i \mathbf{e}_{ij}$.
- A ***homogeneous monogenic polynomial*** is

$$p_{\vec{k}} = \frac{1}{k!} \sum_{\sigma} z_{1\sigma(1)} \cdots z_{1\sigma(k)}.$$

- Space of ***monogenic polynomials*** is $\mathrm{span}_{\mathcal{G}}\{p_{\vec{k}}\}$.

Proposition

Monogenic fields can be written as a power series in z_{ij} and the coefficients are computed with a Cauchy integral.

Section 3

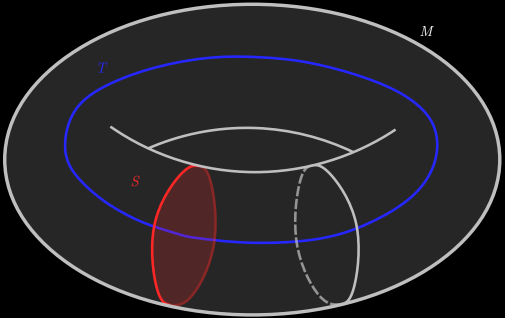
Hodge theory

Idea

Hodge theory relates analysis to topology.

- **Theorem (Hodge Isomorphisms).**

$$H^r(M) \cong \mathcal{M}^r_N(M) \qquad\qquad H^r(M, \partial M) \cong \mathcal{M}^r(M, \partial M).$$



Product on cohomologies

We know $\wedge\colon H^r(X) \times H^s(X) \to H^{r+s}(X)$. But...

Proposition

The contraction $\lrcorner$ is a product on cohomologies by:

- $\lrcorner\colon H^r(M) \times H^s(M) \to H^{s-r}(M)$;
- $\lrcorner\colon H^r(M, \partial M) \times H^s(M, \partial M) \to H^{s-r}(M, \partial M)$;
- $H^r(M) \lrcorner H^s(M, \partial M)$ is trivial;
- $H^r(M, \partial M) \lrcorner H^s(M)$ is trivial;

- This product is equivalent to the mixed cup product in [Shonkwiler, 2009].

Hodge decompositions

Hodge, Morrey, Friedrichs found decompositions of the space of forms.

- **Theorem [Hodge–Morrey].** $\mathfrak{X}^r(M) = \mathcal{E}^r_D(M) \oplus \mathcal{C}^r_N(M) \oplus \mathcal{M}^r(M)$.
- **Theorem [Hodge–Morrey–Friedrichs].**

$$\mathcal{M}^r(M) = \mathcal{M}^r_D(M) \oplus \mathcal{M}^r_{\mathbf{co}} \qquad \textbf{or} \qquad \mathcal{M}^r(M) = \mathcal{M}^r_N(M) \oplus \mathcal{M}^r_{\mathbf{ex}}$$

- But, $\mathcal{M}(M) \neq \bigoplus_{j=1}^{n} \mathcal{M}^j(M)$.

Combining the boundary constraints of exact and coexact fields...

- Define the ***Dirac fields*** $\boldsymbol{\nabla}\mathfrak{X}(M)$ as

$$\boldsymbol{\nabla}\mathfrak{X}(M) := \{\boldsymbol{\nabla}A \mid A \in \mathfrak{X}(M) \text{ and } A|_{\partial M} = 0\};$$

Theorem: Clifford–Hodge Decomposition

The space of multivector fields $\mathfrak{X}(M)$ has the orthogonal decomposition

$$\mathfrak{X}(M) = \mathcal{M}(M) \oplus \boldsymbol{\nabla}\mathfrak{X}(M).$$

Comparing to Hodge–Morrey

- From Hodge–Morrey

$$\mathfrak{X}(M) = \sum_{r=0}^{n} \underbrace{\mathcal{E}_D^r(M)}_{\mathrm{im}(\boldsymbol{\nabla}\wedge)} \oplus \underbrace{\mathcal{C}_N^r(M)}_{\mathrm{im}(\boldsymbol{\nabla}\lrcorner)} \oplus \underbrace{\mathcal{H}^r(M)}_{\ker(\boldsymbol{\nabla})}.$$

- But the Clifford-Hodge-Morrey is not filtered by grades

$$\mathfrak{X}(M) = \underbrace{\mathcal{M}(M)}_{\ker \boldsymbol{\nabla}} \oplus \underbrace{\boldsymbol{\nabla}\mathfrak{X}(M)}_{\mathrm{im}\, \boldsymbol{\nabla}}.$$

Section 4

Tomography

EIT forward problem

- If M is Ohmic region of $\mathbb{R}^3$, represent conductivity with metric g.
- **Ohm's law** $-\boldsymbol{\nabla} \wedge u = \boldsymbol{J}$ and **conservation** $\boldsymbol{\nabla} \lrcorner \boldsymbol{J} = 0$.
- Suppose M free of charges, then the forward problem

$$\begin{cases} \boldsymbol{\nabla}^2 u = 0 & \text{in } M \\ u = \phi & \text{on } \partial M \end{cases}$$

EIT inverse problem

We want access to M only from ∂M.

- The *electric Dirichlet-to-Neumann (DN) map*

$$\Lambda_E \colon \boldsymbol{t}\mathfrak{X}^0(M) \to \boldsymbol{t}\mathfrak{X}^0(M) \quad \text{by} \quad \Lambda_E \phi = \boldsymbol{\nu} \lrcorner \boldsymbol{\nabla} \wedge u = \frac{\partial u}{\partial \boldsymbol{\nu}}.$$

- <u>**Question:**</u> Can we determine (M, σ) from Λ_E?

Magnetic analog

- Magnetic bivector field B solves the forward problem

$$\begin{cases} \boldsymbol{\nabla}^2 B = 0 & \text{in } M \\ B = \boldsymbol{\nu} \wedge \boldsymbol{J} & \text{on } \partial M \end{cases}$$

- Define the ***magnetic DN operator*** $\Lambda_B \colon \boldsymbol{n}\mathfrak{X}^2(M) \to \boldsymbol{n}\mathfrak{X}^2(M)$ by $\Lambda_B(\boldsymbol{\nu} \wedge \boldsymbol{J}) = \boldsymbol{\nu} \wedge \boldsymbol{\nabla} \lrcorner B$.
- **Question:** What can we get from Λ_B?

Calderón problem

Don't forget our goal...

- The problem has been solved in dimension $n = 2$ [Belishev: 2003].
- Solved in dimensions $n \geq 3$ when M is an analytic manifold [Lassas, Taylor, Uhlmann: 2003].
- The smooth cases is still unsolved.

Electromagnetic tomography

- **Can combine to monogenic spinor $A_+ = u + B$.**
- Can we recover the space of monogenic fields from the DN operators?
- If the space of monogenic fields contained geometric data, this would be a step forward in the Calderón problem...

Geometric generalization

In arbitrary dimension:

$$\begin{cases} \nabla^2 A_r = 0 & \text{in } M \\ A_r = \phi_r & \text{on } \partial M \end{cases}$$

- *Generalized electric DN operator*

$$\Lambda_E \colon \boldsymbol{t}\mathfrak{X}(M) \to \boldsymbol{t}\mathfrak{X}(M) \quad \text{by} \quad \Lambda_E \phi_r = \boldsymbol{\nu} \lrcorner \boldsymbol{\nabla} \wedge A_r.$$

- *Generalized magnetic DN operator*

$$\Lambda_B \colon \boldsymbol{n}\mathfrak{X}(M) \to \boldsymbol{n}\mathfrak{X}(M) \quad \text{by} \quad \Lambda_B \phi_r = \boldsymbol{\nu} \wedge \boldsymbol{\nabla} \lrcorner A_r.$$

Comologies from DN operators

- $\ker \Lambda_E = \mathrm{tr}\mathcal{M}^r_N(M)$ and $\ker \Lambda_B = \mathrm{tr}\mathcal{M}^r_D(M)$.
- $\mathrm{tr}\colon \mathcal{M}(M) \to \mathrm{tr}\mathcal{M}(M)$ is an isomorphism.
- Applying Hodge isomorphisms...

Theorem

We have $\ker \Lambda_E \cong H^r(M)$ and $\ker \Lambda_B \cong H^r(M, \partial M)$.

- $\Lambda_E \times \Lambda_B$ is equivalent to **complete DN operator** [Shonkwiler, Sharafutdinov: 2013].

Spinor DN operator

- Define the ***spinor DN operator*** $\mathcal{J} : \mathrm{tr}\mathfrak{X}^{\pm}(M) \to \mathrm{tr}\mathfrak{X}^{\pm}(M)$.
- Specifically: $\mathcal{J}\phi_r = \boldsymbol{\nu}\boldsymbol{\nabla} A_r$.
- Generalized operators are scalar part $\Lambda_E + \Lambda_B = \langle \mathcal{J} \rangle$.

Theorem

We have $\ker \mathcal{J} = \mathrm{tr}\mathcal{M}(M)$.

Section 5

Gelfand theory

Open questions

- In [Belishev: 2003], we see an algebraic proof for the 2-dimensional Calderón problem.
- In [Belishev, Vakulenko: 2017], we see a proof for a noncommutative Gelfand representation using quaternion fields for a ball $\mathbb{B}$ in $\mathbb{R}^3$.
- Belishev and Vakulenko as whether this is true in higher dimensions.
- We will prove this is true for arbitrary regions.

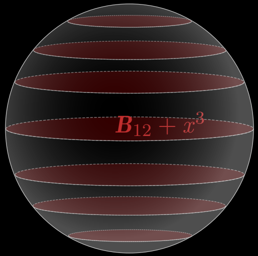
Overview of BC method

The ***boundary control (BC) method*** in [Belishev: 2003] is as follows:

- Determine the algebra $\mathcal{A}(M)$ of holomorphic functions on M using Λ.
- Gelfand theory implies the spectrum of $\mathcal{A}(M)$ is homeomorphic to M.
- Algebraic structure of $\mathcal{A}(M)$ determines the complex structure on M.
- Find g that conformal with the complex structure.

Subsurface spinor fields

- For O convex, let $\boldsymbol{B} \in \mathfrak{X}(O)$ be parallel translation of a unit 2-blade.
- Refer to $A_+ = \mathrm{P}_{\boldsymbol{B}} \circ A_+$ as a ***subsurface spinor***.



- The ***algebra of monogenic subsurface spinors*** is $\mathcal{A}_{\boldsymbol{B}}(O)$.
- **Algebra is a commutative Banach algebra.**

Characters

- Define the algebra $\mathbb{A}_{\boldsymbol{B}}$ to be the algebra generated by 1 and $\boldsymbol{B}$.
- The ***spinor spectrum*** $\mathfrak{M}(M)$ consists of ***spin characters***:
  - Continuous grade-preserving $\mathcal{G}^+$-linear maps $\mathcal{M}^+(M) \to \mathcal{G}^+$,
  - algebra morphisms $\mathcal{A}_{\boldsymbol{B}}(O) \to \mathbb{A}_{\boldsymbol{B}}$.
- One example of such characters are point evaluations $\delta(A_+) = A_+(x_\delta)$.
- **We show these are the only elements in the spectrum.**

Idea

By linearity, we can note that for $\delta \in \mathfrak{M}(\mathbb{B}_{R,w})$

$$\delta(f(x)) = \sum_{j=0}^{\infty} \left(\sum_{\substack{j_2 \cdots j_n \\ j_2 + \cdots + j_n = j}} \delta(p_{j_2 \cdots j_n}(x - w)) a_{j_2 \cdots j_n} \right)$$

On each monogenic polynomial

$$\delta(p_{j_2 \ldots j_n}(x)) = \frac{1}{j!} \sum_{\text{permutations}} \delta\left((z_{1\sigma(1)}(x)\right) \cdots \delta\left(z_{1\sigma(j)}(x)\right)$$

by the multiplicativity of δ. Then $z_{ij} \in \mathcal{A}_{\boldsymbol{e}_{ij}}(O)$.

Necessary lemmas

For regions $M \subset \mathbb{R}^n$:

Lemma: Density

The space $\mathcal{M}^{\mathcal{P}}(M)$ is dense in $\mathcal{M}(M)$.

Lemma: Point evaluation

For $\delta \in \mathfrak{M}(M)$ we have $\delta(z_{ij}) = z_{ij}(x_\delta)$ for some $x_\delta \in \mathbb{R}^n$.

Lemma: Identification

Let $A_+ \in \mathcal{M}^+(M)$, then $\delta(A_+) = A_+(x_\delta)$ for some $x_\delta \in M$.

The previous lemmas imply the following:

Theorem: Clifford-algebraic Gelfand theorem

With the weak-$*$ topology on $\mathfrak{M}(M)$, the map

$$\gamma\colon \mathfrak{M}(M) \to M, \quad \delta \mapsto x_\delta$$

is a homeomorphism. The Gelfand transform $\widehat{A_+}(\delta) = \delta[A_+]$ is an isometric isomorphism so $\mathcal{M}^+(M) \cong \widehat{\mathcal{M}^+(M)}$.

Section 6

Further results, open questions, conclusion

A Stone–Weierstrass theorem

- Using continuation from a z_{ij}:

Lemma

The space $\overline{\mathcal{M}^+(M)}$ separates points.

- Using [Laville, Ramadanoff: 1996]:

Theorem: Stone–Weierstrass

$\vee\overline{\mathcal{M}^+(M)}$ is dense in $C(M;\mathcal{G}^+)$.

Sheaf

- Using unique continuation:

Theorem

The sheaf $\mathcal{M}_M$ is Hausdorff and the map $\pi\colon \mathcal{M}_M \to M$ is a local homeomorphism.

- Can one find a component of $\mathcal{M}_M$ that is homeomorphic to M?

Future work and open questions

To get a higher dimensional BC method we need:

- The DN operator determines $\mathrm{tr}\mathcal{M}^+(M)$.
- The map $\mathrm{tr}\colon \vee\,\mathcal{M}^+(M) \to \mathrm{tr}\vee\mathcal{M}^+(M)$ is an isometric isomorphism of algebras.
- The space $\mathcal{M}^+(M)$ determines the metric structure of M up to isometry.

Future work and open questions

- Many of these approaches use the Hilbert transform which is also used by Belishev, Sharafutdinov, and Shonkwiler to study the Calderón problem.
- The Hilbert transform is also a classical part of Clifford analysis.
- [Lassas, Uhlmann: 2001] use sheaf theory to solve the analytic Calderón problem.
- Santacesaria proposes another method for solving the Calderón problem using Clifford algebras and complex geometric optics.

Section 7

Conclusions

Conclusion

- Clifford analysis is a natural setting for studying PDEs and Hodge theory on manifolds.
- Able to describe DN operators and extract homological information and boundary values of special functions.
- Special functions are able to tell us the topology of the manifold they are defined on.

Thank you!