ColoState Spring 2018 MATH 617 Assignment 2

Due Wed. 02/14/2018

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(20 points) *Problem 1.* Let $f_n(x)$ be a sequence of Riemann integrable functions on $[a, b]$ and $f_n(x)$ converges uniformly on $[a, b]$ to $f(x)$. Prove that $f(x)$ is also Riemann integrable and $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$.

(15 points) *Problem 2.* Let $A = \mathbb{Q} \cap [0, 1]$. If $\{I_n\}$ is a finite collection of open intervals covering A, then $\sum_n \lambda(I_n) \geq 1$.

(15 points) *Problem 3.* Check whether this "easier proof" for $\mu(A) \leq \mu^*(A)$ (Textbook Prop.3.7.4(iv)) is correct. Provide a correct proof if this one is incorrect.

Since $A \subseteq X$, the definition of μ^* as an infimum implies that there exist $A_n \in \mathcal{A}(n \in \mathbb{N})$ such that $A \subseteq \bigcup_{n=1}^\infty A_n$ and $\sum_{n=1}^\infty \mu(A_n) < \mu^*(A) + \varepsilon$.

By the monotonicity and countable subadditivity of μ as a measure, we have

$$\mu(A) \leq \mu(\bigcup_{n=1}^\infty A_n) \leq \sum_{n=1}^\infty \mu(A_n) < \mu^*(A) + \varepsilon.$$

So for any $\varepsilon > 0$, we have

$$\mu(A) < \mu^*(A) + \varepsilon.$$

Letting $\varepsilon \to 0$ yields

$$\mu(A) \leq \mu^*(A).$$

(20 points) *Problem 4.* Textbook (p.69) Exercise 3.6.9.

(15 points) *Problem 5.* Let $\mathcal{A}$ be an algebra of subsets of a nonempty set X and $\langle\mu_n\rangle_{n\in\mathbb{N}}$ be a sequence of measures on $\mathcal{A}$. Assume $\mu_n(X) < +\infty, \forall n \in \mathbb{N}$. For any $A \in \mathcal{A}$, define

$$\mu(A) = \sum_{n=1}^\infty \frac{1}{3^n}\mu_n(A).$$

Prove that μ is a measure on $\mathcal{A}$.

(15 points) *Problem 6.* Assume μ is a measure defined on an algebra $\mathcal{A}$ consisting of subsets of a fixed nonempty set X. Let μ^* be the outer measure induced by μ and $\mathcal{S}^*$ be obtained through the Caratheodory condition. Prove that μ^* is countably additive on $\mathcal{S}^*$.

Problem 1. Let $f_n(x)$ be a sequence of Riemann integrable functions on $[a,b]$ and $f_n(x)$ converges uniformly on $[a,b]$ to $f(x)$. Prove that $f(x)$ is also Riemann integrable and $\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b f(x)dx$.

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Proof. To see that f is Riemann integrable, fix $\epsilon > 0$. Let $0 < \eta$, $0 < \delta$, and $0 < \eta + \delta < \epsilon$. Note that $f_n \to f$ uniformly implies that $\exists N \in \mathbb{N} : \ n \geq N \implies |f_n(x) - f(x)| < \frac{\delta}{2(b-a)}$ $\forall x$. Let P_m be a regular partition of $[a,b]$ into m segments. Letting $M_{i,n} = \sup_{x\in[x_i,x_{i+1}]}(f_n(x))$ and $m_{i,n} = \inf_{x\in[x_i,x_{i+1}]}(f_n(x))$, we then have $\forall n\ \exists m \in \mathbb{N}$ such that

$$\begin{aligned} & U(P_m, f_n) - L(P_m, f_n) < \eta \\ \iff & \sum_{i=1}^{m}(M_{i,n} - m_{i,n})(x_{i+1} - x_i) < \eta. \end{aligned}$$

Note that uniform convergence implies that $\frac{\delta}{b-a} + M_{i,n} - m_{i,n} > M_i - m_i$ where $M_i = \sup_{x\in[x_i,x_{i+1}]}(f(x))$ and $m_i = \inf_{x\in[x_i,x_{i+1}]}(f(x))$. Hence we have

$$\begin{aligned} U(P_m, f) - L(P_m, f) &= \sum_{i=1}^{m}(M_i - m_i)(x_{i+1} - x_i) \\ &< \left(\frac{\delta}{b-a} + M_{i,n} - m_{i,n}\right)(x_{i+1} - x_i) \\ &= \delta + \eta < \epsilon. \end{aligned}$$

Hence f is Riemann integrable. To see that this shows $\int_a^b f_n dx \to \int_a^b f dx$, note that the previous work shows that

$$\lim_{m\to\infty, n\to\infty} |(U(P_m, f_n) - L(P, f_n)) - (U(P_m, f) - L(P_m, f))| = 0. \qquad \square$$

Problem 2. Let $A = \mathbb{Q} \cap [0,1]$. If $\{I_n\}$ is a finite collection of open intervals covering A, then $\sum_n \lambda(I_n) \geq 1$.

Note: I will let $\overline{I}$ denote the closure of the open interval I.

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Proof. First suppose that we have a finite covering of A with a single interval I. Then to contain all points in A, we must have that $I \supseteq [0,1]$ and hence

$$\lambda(I) \geq 1.$$

Assume this is true up to a covering with $n-1$ intervals and suppose there exists a covering with n intervals so that

$$\sum_{k=1}^{n} \lambda(I_k) < 1.$$

By density of the rationals and by the fact that $\lambda(\{x\}) = 0$, we have that for some $i, j \in \{1, \ldots, n\}$ that $\overline{I_i} \cap \overline{I_j} = \{x\}$. It must be that $I_i \cup \{x\} \cup I_j = I_0$ is an open interval and hence we can create a new covering by removing I_i and I_j from the original covering $\{I_k\}_{k=1,\ldots,n}$ and replacing with the interval I_0. However, this new collection is then a covering of A using $n-1$ sets, which contradicts our supposition. Hence, by induction, we must have that $\sum_{i=1}^{n} \lambda(I_n) \geq 1$ for any collection of open intervals covering A. $\square$

Problem 3. Check whether this “easier proof" for $\mu(A) \leq \mu^*(A)$ (Textbook Prop.3.7.4(iv)) is correct. Provide a correct proof if this one is incorrect.

Since $A \subseteq X$, the definition of μ^* as an infimum implies that there exist $A_n \in \mathcal{A}(n \in \mathbb{N})$ such that $A \subseteq \bigcup_{n=1}^{\infty} A_n$ and $\sum_{n=1}^{\infty} \mu(A_n) < \mu^*(A) + \varepsilon$.

By the monotonicity and countable subadditivity of μ as a measure, we have

$$\mu(A) \leq \mu(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n) < \mu^*(A) + \varepsilon.$$

So for any $\varepsilon > 0$, we have

$$\mu(A) < \mu^*(A) + \varepsilon.$$

Letting $\varepsilon \to 0$ yields

$$\mu(A) \leq \mu^*(A).$$

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Solution. One mistake is that we must require that each A_i is mutally disjoint from any other A_j for $i \neq j$. We must also check the case for when $\mu^*(A) = +\infty$. In this case we surely have that $\mu^*(A) \geq \mu(A)$. Also, in order to apply monotonicity we must instead consider

$$\bigcup_{n=1}^{\infty} (A_n \cap A)$$

as opposed to $\bigcup_{n=1}^{\infty}(A_n)$ since $\bigcup_{n=1}^{\infty}(A_n) \supseteq A$ and thus could contain more than just A.

Also it should be that we let $\epsilon > 0$ be arbitrary in the beginning and then noting this fact instead of where the proof lets $\epsilon \to 0$ would work. I think letting $\epsilon \to 0$ is a fine way of saying it though! ∎

Problem 4. Let $X = \mathbb{N}$, the set of natural numbers. For every finite set $A \subseteq X$, let $\#A$ denote the number of elements in A. Define for $A \subseteq X$,

$$\mu_n(A) := \frac{\#\{m : 1 < m < n, m \in A\}}{n}.$$

Show that μ_n is countably additive for every n on $P(X)$. In a sense, μ_n is the proportion of integers between 1 to n which are in A. Let $\mathcal{C} = \{A \subseteq X : \lim_{n\to\infty} \mu_n(A) \quad \text{exists}\}$. Show that $\mathcal{C}$ is closed under taking complements, finite disjoint unions and proper differences. Is it an algebra?

Note: I will let $|\cdot|$ denote the cardinality of a set and I will use $\coprod$ as the notation for disjoint union.
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Proof. To see that μ_n is countably additive, let $\{A_m\}_{m\in\mathbb{N}}$ be a collection of disjoint sets from $P(X)$. Then we want to show that

$$\mu_n\left(\coprod_{m\in\mathbb{N}} A_m\right) = \sum_{m\in\mathbb{N}} \mu_n(A_m).$$

Now we have

$$\mu_n\left(\coprod_{m\in\mathbb{N}} A_m\right) = \mu_n\left(\coprod_{m\in\mathbb{N}} (A_m \cap \{1,\dots,n\})\right).$$

Note that since the A_m are disjoint there are sets A_{m_i} with $i = 1, dots, n$ so that $A_{m_i} \cap \{1,\dots,n\}$ is possibly nonempty (there may be no sets that interesect $\{1,\dots,n\}$ or at most n). This means that we have

$$\begin{aligned}
\mu_n\left(\coprod_{m\in\mathbb{N}} (A_m \cap \{1,\dots,n\})\right) &= \mu_n\left(\coprod_{i=1}^{n} (A_{m_i} \cap \{1,\dots,n\})\right) \\
&= \sum_{i=1}^{n} \frac{1}{n} \left|A_{m_i} \cap \{1,\dots,n\}\right|,
\end{aligned}$$

which holds since the cardinality of the finite union of disjoint sets is additive. Then

$$\sum_{i=1}^{n} \frac{1}{n} \left|A_{m_i} \cap \{1,\dots,n\}\right| = \sum_{m\in\mathbb{N}} \frac{1}{n} \left|A_m \cap \{1,\dots,n\}\right|,$$

which holds since all the other sets than the A_{m_i} have an empty intersection and hence the intersections of these sets has a cardinality of 0. Finally,

$$\sum_{m\in\mathbb{N}} \frac{1}{n} \left|A_m \cap \{1,\dots,n\}\right| = \sum_{m\in\mathbb{N}} \mu_n(A_m),$$

which shows the countable additivity.

Now, let $A \in \mathcal{C}$. Let $\lim_{n\to\infty} \mu_n(A) = L$ and note that $L \in [0,1]$. Then we have

$$\begin{aligned}
1 &= \lim_{n\to\infty} \mu_n(X) \\
&= \lim_{n\to\infty} (\mu_n(A) + \mu_n(A^c)) \qquad \text{by the countable (and hence finite) additivity of } \mu_n \\
&= L + \lim_{n\to\infty} \mu_n(A^c) \\
\implies \lim_{n\to\infty} \mu_n(A^c) &= 1 - L.
\end{aligned}$$

The limit existing shows $A^c \in \mathcal{C}$.

To see that finite disjoint unions are in $\mathcal{C}$ it suffices to show that the union of two disjoint sets are in $\mathcal{C}$. Let $A, B \in \mathcal{C}$ so that $A \cap B = \emptyset$. Then letting $\lim_{n\to\infty} \mu_n(A) = L_A$ and $\lim_{n\to\infty} \mu_n(B) = L_B$ we have

$$\begin{aligned}
\lim_{n\to\infty} \mu_n(A \coprod B) &= \lim_{n\to\infty} (\mu_n(A) + \mu_n(B)) && \text{by the countable (and hence finite) additivity of } \mu_n \\
&= \lim_{n\to\infty} \mu_n(A) + \lim_{n\to\infty} \mu_n(B) \\
&= L_A + L_B.
\end{aligned}$$

Thus the finite disjoint union of two sets is in $\mathcal{C}$.

Finally, let $A, B \in \mathcal{C}$ be such that $B \subset A$ (proper subset). We wish to show that $A \setminus B \in \mathcal{C}$. To see this, we let $\lim_{n\to\infty} \mu_n(A) = L_A$ and $\lim_{n\to\infty} \mu_n(B) = L_B$ and we have

$$\begin{aligned}
\lim_{n\to\infty} \mu_n(A) &= \lim_{n\to\infty} \mu_n((A \setminus B) \cup B) \\
&= \lim_{n\to\infty} \mu_n(A \setminus B) + \lim_{n\to\infty} \mu_n(B) && \text{by countable additivity} \\
\iff L_A - L_B &= \lim_{n\to\infty} \mu_n(A \setminus B).
\end{aligned}$$

Thus the proper differences are in $\mathcal{C}$.

Lastly, I do think that $\mathcal{C}$ is an algebra. But I've found proving this or finding a counter example is extremely hard!!! $\square$

Problem 5. Let $\mathcal{A}$ be an algebra of subsets of a nonempty set X and $\langle \mu_n \rangle_{n\in\mathbb{N}}$ be a sequence of measures on $\mathcal{A}$. Assume $\mu_n(X) < +\infty, \forall n \in \mathbb{N}$. For any $A \in \mathcal{A}$, define

$$\mu(A) = \sum_{n=1}^{\infty} \frac{1}{3^n} \mu_n(A).$$

Prove that μ is a measure on $\mathcal{A}$.

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Proof. In order to show that μ is a measure, we must show that $\mu(\emptyset) = 0$ and that μ is countably additive. Note that $\mathcal{A}$ is an algebra, and hence $\emptyset \in \mathcal{A}$ and since each μ_n is a measure we have that $\mu_n(\emptyset) = 0$ for every $n \in \mathbb{N}$. Thus

$$\begin{aligned}\mu(\emptyset) &= \sum_{i=1}^{\infty} \frac{1}{3^n} \mu_n(\emptyset) \\ &= 0,\end{aligned}$$

since each term in the series is identically 0. To see that μ is countably additive, let $\{A_m\}_{m\in\mathbb{N}}$ be a countable and disjoint collection of subsets of $\mathcal{A}$ which exists due to the fact $\mathcal{A}$ is an algebra. Note, if $\mathcal{A}$ is not infinite, then μ is vacuously countably additive. We want to show that

$$\mu\left(\coprod_{m\in\mathbb{N}} A_m\right) = \sum_{m\in\mathbb{N}}^{\infty} \mu(A_m).$$

Note that each μ_n is a measure and is countably additive which allows us to do the following:

$$\begin{aligned}\mu\left(\coprod_{m\in\mathbb{N}} A_m\right) &= \sum_{n\in\mathbb{N}} \frac{1}{3^n} \mu_n \left(\coprod_{m\in\mathbb{N}} A_m\right) \\ &= \sum_{m\in\mathbb{N}} \sum_{n\in\mathbb{N}} \frac{1}{3^n} \mu_n(A_m) \\ &= \sum_{m\in\mathbb{N}} \mu(A_m).\end{aligned}$$

$\square$

Note the last equality and the ability to swap the summations is due to the fact that $\mu_n(A_m) < \infty$ for all n, m.

Problem 6. Assume μ is a measure defined on an algebra $\mathcal{A}$ consisting of subsets of a fixed nonempty set X. Let μ^* be the outer measure induced by μ and $\mathcal{S}^*$ be obtained through the Caratheodory condition. Prove that μ^* is countably additive on $\mathcal{S}^*$.

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Proof. First note that we have μ^* is countably subadditive. Hence, it suffices to show for an arbitrary countable collection of disjoint disjoint subsets $\{A_n\}_{n\in\mathbb{N}}$ of $\mathcal{S}^*$ that

$$\mu^*\left(\coprod_{n\in\mathbb{N}} A_n\right) \geq \sum_{n\in\mathbb{N}} \mu^*(A_n).$$

Then we have

$$\begin{aligned}
\mu^*\left(\coprod_{n\in\mathbb{N}} A_n\right) &= \mu^*(A_1) + \mu^*(A_1^c) \\
&= \mu^*(A_1) + \mu^*(A_1^c \cap A_2) + \mu^*(A_1^c \cap A_2^c) \\
&\vdots \\
&= \sum_{i=1}^{n} \mu^*(A_i) + \mu^*\left(\bigcap_{i=1}^{n} A_i^c\right) \\
&= \sum_{i=1}^{n} \mu^*(A_i) + \mu^*\left(\left(\coprod_{i=1}^{n} A_i\right)^c\right).
\end{aligned}$$

Now we let $n \to \infty$ we we find

$$\begin{aligned}
\mu^*\left(\coprod_{n\in\mathbb{N}} A_n\right) &\geq \sum_{i=1}^{\infty} \mu^*(A_i) + \mu^*\left(\left(\coprod i = 1^{\infty} A_i\right)^c\right) \\
&= \sum_{i=1}^{\infty} \mu^*(A_i).
\end{aligned}$$

Thus we have that μ^* is countably additive on $\mathcal{S}^*$.

Note: I did see this solution in the text. But I digested it and tried to simplify it some. ☐