MATH 517, Homework 10

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Solutions

Problem 1. (Rudin 9.9) If $E \subseteq \mathbb{R}^n$ is a connected open set and $F\colon E \to \mathbb{R}^m$ is differentiable such that $F'(\vec{x}) = \vec{0}$ for all $\vec{x} \in E$, prove that F is constant on E.

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Proof. Since F is differentiable, we have that $F'(\vec{x})_{ij} = \frac{\partial_j F}{\partial x_i}(\vec{x}) = 0$. Consider then the mean value theorem on the components of F, F_j. Denote $E_i \subseteq \mathbb{R}$ as the set containing the ith components of the vectors in E. Now choose $\vec{p} = (p_1, \ldots, p_n)$ and let $F(\vec{x}) = q$ and consider $\tilde{F_j}\colon \mathbb{R} \to \mathbb{R}$ defined by $F_{j_i}(\tilde{p_i}) = F(p_1, p_2, \ldots, \tilde{p_i}, \ldots, p_n)$ where each p_l are fixed except $l = i$, where $\tilde{p_i} \in (a_i, b_i) \subseteq E_i$. Then we have by Theorem 5.11 that F_{j_i} is constant on this interval. This is true for all F_{j_i}, and so we consider the set $X = \{\vec{x} \in E \mid F(\vec{x}) = q\}$. We have by construction that X is open since it is the finite product of unions of open sets. Finally, consider a limit point $\vec{r} \in X$ and consider the sequence $\{r_n\}_{n\in\mathbb{N}} \in F$ converging to $\vec{r}$. We have that $F(\vec{r_n}) = p$ for all n since $\vec{r} \in X$, and since F is continuous, $\lim_{n\to\infty} F(q_n) = F(q) = p$. This implies that X is also closed, and thus since E is connected, the only open and closed subsets of E are E itself and $\emptyset$. Certainly X is nonempty, and thus $X = E$ and we have F is constant on E. $\square$

Problem 2. **(Rudin 9.12)** Fix two real numbers $0 < a < b$. Define $F\colon \mathbb{R}^2 \to \mathbb{R}^3$ where $F(s,t) = (f_1(s,t), f_2(s,t), f_3(s,t))$ with

$$\begin{aligned} f_1(s,t) &= (b + a\cos s)\cos t \\ f_2(s,t) &= (b + a\cos s)\sin t \\ f_3(s,t) &= a\sin s. \end{aligned}$$

(a) Describe the range T of F (it is a compact subset of $\mathbb{R}^3$).

(b) Show that there are exactly 4 points $\vec{p} \in T$ such that

$$(\nabla f_1)(F^{-1}(\vec{p})) = \vec{0}.$$

(c) Determine the set of all $\vec{q} \in T$ such that

$$(\nabla f_3)(F^{-1}(\vec{q})) = \vec{0}.$$

(d) Show that one of the points $\vec{p}$ found in part (b) corresponds to a local maximum of f_1, one corresponds to a local minimum, and that the other two are neither (they are saddle points).

Which of the points $\vec{q}$ found in part (c) correspond to maxima or minima?

(e) Let $\lambda \in \mathbb{R}$ be irrational, and define $G(t) = F(t, \lambda t)$. Prove that G is an injective mapping of $\mathbb{R}$ onto a dense subset of T, and show that

$$|G'(t)|^2 = a^2 + \lambda^2(b + a\cos t)^2.$$

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Proof. (a) The range of F is a hollow torus with b as the distance from the center of the "donut hole" to the center of the tube portion and a is the radius of the tube.

(b) For $(\nabla f_1)(s,t) = 0$, we see that $(\nabla f_1)(s,t) = (-a\sin s\cos t, -(b + a\cos s)\sin t)$. Note that these functions are 2π periodic, and we can restrict $s, t \in [0, 2\pi)$. Then we have,

$$\begin{aligned} -a\sin s\cos t &= 0 \\ -(b + a\cos s)\sin t) &= 0. \end{aligned}$$

The first equation is 0 when $s \in \{0, \pi\}$ and $t \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$. The second equation is 0 when $t \in \{0, \pi\}$, and is otherwise nonzero. So we find that the solutions for this are $\{(0,0), (0,\pi), (\pi,0), (\pi,\pi)\}$. By plugging these into F, we find that these points correspond to $\vec{q} \in \{(b+a), 0, 0), (b-a, 0, 0), (-b+a, 0, 0), (-b-a, 0, 0)\}$.

(c) We have that $(\nabla f_3)(s,t) = (a\cos s, 0)$ which means that $s = \frac{\pi}{2}$ or $s = \frac{3\pi}{2}$ and $t \in [0, 2\pi)$. The image of these points gives us two circles of radius b in the planes $z = \pm a$.

(d) Notice that $(a + b, 0, 0)$ was the largest possible value of $f_1(s,t)$ and that $(-a - b, 0, 0)$ was the smallest. These two points correspond to the local maxima and minima and for $(s,t) = (0,0)$ and $(s,t) = (0,\pi)$ respectively. The other two points, $(s,t) = (\pi, 0)$ and $(s,t) = \pi, \pi)$ are saddle points. We show this by looking at $f_1(\pi, t) = (b-a)\cos t$ and for $t = \pi$ we have that this is a minimum yet for $f_1(s,\pi) = -(b + a\cos s)$ we have that $s = \pi$ is a maximum. Likewise for the point $(s,t) = (\pi, 0)$ we have $f_1(\pi,t) = (b-a)\cos t$ is maximal when $t = 0$ and $f_1(s,0) = (b + a\cos s)$ is minimal when $s = 0$. Hence (π,π) and $(\pi,0)$ are saddle points.

(e) To see that $G(t) = ((b + a\cos t)\cos\lambda t, (b + a\cos t)\sin\lambda t, \sin t)$ is injective, consider distinct $t_1, t_2 \in \mathbb{R}$, then suppose we have $G(t_1) = G(t_2)$. Now

$$\begin{aligned} &G(t_1) = G(t_2) \\ \implies &\sin t_1 = \sin t_2. \end{aligned}$$

This shows that $t_1 - t_2 = 2n\pi$ for any nonzero integer n. We also have that

$$\sin\lambda t_1 = \sin\lambda t_2,$$

which implies $\lambda(t_1 - t_2) = 2m\pi$ for any nonzero integer m and $(t_1 - t_2) = \frac{2m}{\lambda}\pi$. However, since λ is irrational, that means n and m both had to be 0, else both conditions cannot be simultaneously true. Thus $t_1 = t_2$.

(*Note: I found some help from StackExchange (Question 449756) for this portion. This was really tough, so my aim was to just figure it out.* To show that the range of $G(t)$ is dense in T, we can use Kronecker's Estimation Theorem which states: Given any $\alpha \in [0, 1]$, any irrational λ, and any $\epsilon > 0$, there exist an integer $k > 0$ such that

$$|k\lambda - \lfloor k\lambda \rfloor - \alpha| < \epsilon.$$

This can be extended to showing that this is true for any $\alpha \in [0, 2\pi]$ by replacing the floor function with $[x]$ symbolizing taking "modulo 2π." This shows that for $k \in \mathbb{N}$ we have $k\lambda$ is dense modulo 2π.

Now let $f(s_0, t_0)$ be any point on T and consider $g(s_0 + 2\pi n)$. By Kronecker's Estimation Theorem we have

$$|(t_0 - \lambda s_0) - 2n\pi\lambda + 4\pi m| < \epsilon.$$

This implies that

$$\begin{aligned} |\sin t_0 - \sin\lambda(s_0 + 2\pi n)| &\leq 2\left|\sin\frac{t_0 - \lambda s_0 - 2\pi n\lambda}{2}\right| && \text{by trigonometric identities} \\ &= 2\left|\sin 2\pi\left(\frac{t_0 - \lambda s_0}{4\pi} - n\frac{\lambda}{2} + m\right)\right| \\ &\leq 4\pi\left|\frac{t_0 - \lambda s_0}{4\pi} - n\frac{\pi}{2} + m\right| && \text{by } \sin x \leq x \\ &< \epsilon. \end{aligned}$$

There is an analogous result for $|\cos t_0 - \cos\lambda(s_0 + 2\pi n)| < \epsilon$. Thus it has been shown that an arbitrary point, $f(s_0, t_0)$, is a limit point of $g(s_0 + 2\pi n)$, meaning the image of g is dense in T.

We calculate

$$G'(t) = \begin{bmatrix} -a\sin t\cos\lambda t - \lambda(b + a)\sin\lambda t) \\ -a\sin t\cos\lambda t + \lambda(b + a)\cos\lambda t) \\ a\cos t \end{bmatrix}.$$

Then $|G'(t)|^2$ is found by

$$\begin{aligned}
(G'(t))(G'(t))^T &= \begin{bmatrix} -a\sin t\cos\lambda t - \lambda(b+a\cos t)\sin\lambda t) \\ -a\sin t\sin\lambda t + \lambda(b+a\cos t)\cos\lambda t) \\ a\cos t \end{bmatrix} \begin{bmatrix} -a\sin t\cos\lambda t - \lambda(b+a\cos t)\sin\lambda t) \\ -a\sin t\sin\lambda t + \lambda(b+a\cos t)\cos\lambda t) \\ a\cos t \end{bmatrix}^T \\
&= (-a\cos\lambda t\sin t - \lambda(b+a\cos t)\sin\lambda t)^2 + (-a\sin\lambda t\sin t + \lambda(b+a\cos t)\cos\lambda t) \\
&\quad + a^2\cos^2\lambda t \\
&= a^2\cos^2\lambda t\sin^2 t + 2\lambda a(b+a\cos t)\cos\lambda t\sin t\sin\lambda t + \lambda^2(b+a\cos t)^2\sin^2\lambda t \\
&\quad + a^2\sin^2\lambda t\sin^2 t - 2\lambda a(b+a\cos t)\cos\lambda t\sin t\sin\lambda t + \lambda^2(b+a\cos t)^2\cos^2\lambda t \\
&\quad + a^2\cos^2\lambda t \\
&= a^2\sin^2\lambda t + \lambda^2(b+a\cos t)^2 + a^2\cos^2 t \\
&= a^2 + \lambda^2(b+a\cos t)^2. \qquad \square
\end{aligned}$$

Problem 3. Let $F = (f_1, f_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f_1(x, y) = e^x \cos y, \quad f_2(x, y) = e^x \sin y.$$

(a) What is the range of F?

(b) Show that the Jacobian of F is not zero at any point of $\mathbb{R}^2$, so every point of $\mathbb{R}^2$ has a neighborhood on which F is injective. However, F is not injective globally.

(c) Put $\vec{a} = (0, \pi/3)$, $\vec{b} = F(\vec{a})$, and let G be the continuous inverse of F defined in a neighborhood of $\vec{b}$ so that $G(\vec{b}) = \vec{a}$. Find an explicit formula for G, compute $F'(\vec{a})$ and $G'(\vec{b})$, and verify that they satisfy the equation

$$G'(\vec{b}) = \left[F'(G(\vec{b}))\right]^{-1}$$

that came up in the proof of the Inverse Function Theorem.

(d) What are the images under F of lines parallel to the coordinate axes of $\mathbb{R}^2$?

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Proof.

(a) The range of F is $\mathbb{R}^2 \setminus \{0\}$. Note that any point on a circle is given by $(\cos y, \sin y)$ and e^x will scale the radius of that circle, but e^x is always nonzero.

(b) We have

$$F'(x, y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}.$$

The Jacobian is then

$$J = \det(F'(x, y)) = e^{2x} \cos^2 y + e^{2x} \sin^2 y = e^{2x},$$

which is nonzero everywhere. To see F is not injective everywhere, just consider a fixed x_0 and note that $F(x_0, y) = F(x_0, y + 2\pi)$.

(c) We have $\vec{b} = (\cos \pi/3, \sin \pi/3) = (1/2, \sqrt{3}/2)$, so we define $G(x, y) = (\sqrt{x^2 + y^2}, \tan^{-1} y/x)$ so that $G(\vec{b}) = \vec{a}$. Note I found this by realizing that $\sqrt{x^2 + y^2}$ provides the length of the vector and $\tan^{-1} y/x$ proves the angle. Now we have

$$G'(x, y) = \begin{bmatrix} \frac{1}{x} & \frac{1}{y} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix}.$$

$$F'(\vec{a}) = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$
$$G'(\vec{b}) = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}.$$

Now we just compute

$$F'(G(\vec{b})) = F'(\vec{a}) = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix},$$

which has an inverse

$$\left[F'(G(\vec{b}))\right]^{-1} = F'(\vec{a}) = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}.$$

Hence we have shown that they are equivalent.

(d) If we fix x_0 and let y vary, we find that the image of $F(x_0, y) = (e^{x_0} \cos y, e^{x_0} \sin y)$ is a circle with radius e^{x_0}. This is the image of lines parallel to the y axis. Now if we fix y_0 and let x vary we have $F(x, y_0) = (e^x \cos y_0, e^x \sin y_0)$ which are lines with a slope of $\frac{\sin y_0}{\cos y_0}$. □