

Hodge Theory, Gelfand Theory, and Clifford Analysis Applied to Tomography

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Overview

- 1 Introduction
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- 3 Hodge theory
- 4 Tomography
- 5 Gelfand theory
- 6 Further results, open questions, conclusion

Section 1

Introduction

Motivating problems

- *Electrical Impedance Tomography (EIT)* asks whether one can determine the conductivity of a medium from the voltage-to-current map.
- The *Calderón problem* replaces the medium with a manifold M , conductivity with g , and replaces the voltage-to-current map with the Dirichlet-to-Neumann operator Λ .

Other questions

- What topological information can we retrieve from functions on a manifold?
- Do these functions also contain metric data?
- Can we access these functions from the boundary?

Subsection 1

Preliminaries

Clifford and geometric algebras

Let V be a vector space over a field K with symmetric bilinear form g .

- Define the *tensor algebra*

$$\mathcal{T}(V) := \bigoplus_{j=0}^{\infty} V^{\otimes j} = K \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots.$$

- The associated *Clifford algebra* is the quotient

$$Cl(V, g) := \mathcal{T}(V) / \langle \mathbf{v} \otimes \mathbf{v} - g(\mathbf{v}, \mathbf{v}) \rangle.$$

Geometric and exterior algebras

- If g is non-degenerate then we have a *geometric algebra*

$$\mathcal{G} := \mathcal{Cl}(V, g).$$

- The completely degenerate case is the *exterior algebra*

$$\bigwedge(V) := \mathcal{Cl}(V, 0).$$

Algebraic structure

\mathcal{G} is generated by scalars and vectors given how \otimes acts in the quotient.

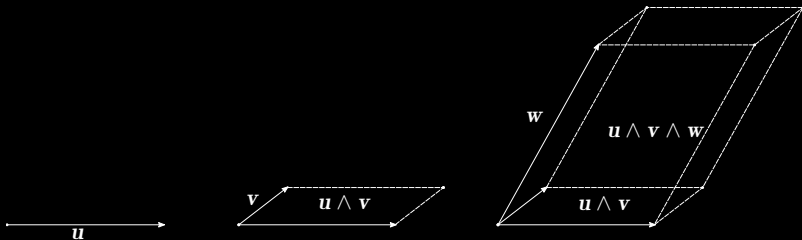
- Given vectors $\mathbf{u}, \mathbf{v} \in \mathcal{G}$ we can take the product

$$\mathbf{u}\mathbf{v} = \underbrace{\mathbf{u} \cdot \mathbf{v}}_{\text{scalar}} + \underbrace{\mathbf{u} \wedge \mathbf{v}}_{\text{bivector}}.$$

- The scalar part is symmetric: $\mathbf{u} \cdot \mathbf{v} = g(\mathbf{u}, \mathbf{v})$.
- The bivector part is antisymmetric: $\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u}$.

Multivectors

- \mathcal{G} is graded and of dimension 2^n .
 - Grade- r elements, \mathcal{G}^r , called *r -vectors*.
 - $\langle A \rangle_r \in \mathcal{G}^r$ extracts the grade- r part of an arbitrary element A .
 - There are $\binom{n}{r}$ independent *r -blades* of the form $\mathbf{A}_r = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r$.
 - Elements of the even grade subalgebra, \mathcal{G}^+ , are called *spinors*.
- Since $\mathcal{G} = \bigoplus_{r=0}^n \mathcal{G}^r$ a general *multivector* is $A = \sum_{r=0}^n \langle A \rangle_r$.



Algebraic Structure

- Extend the multiplication from vectors to multivectors.
- On homogeneous elements,

$$A_r B_s = \langle A_r B_s \rangle_{|r-s|} + \langle A_r B_s \rangle_{|r-s|+2} + \cdots + \langle A_r B_s \rangle_{r+s}$$

- The most important products for us are

$$A_r \wedge B_s := \langle A_r B_s \rangle_{r+s}$$

$$A_r \lrcorner B_s := \langle A_r B_s \rangle_{s-r}$$

Reciprocals and reverses

- Given any vector basis \mathbf{e}_i , define the *reciprocal vectors* by $\mathbf{e}^i \cdot \mathbf{e}_j = \delta_j^i$.
- The *reverse* \dagger is extended linearly from the action on r -blades

$$\mathbf{A}_r^\dagger = (\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r)^\dagger = \mathbf{v}_r \wedge \cdots \wedge \mathbf{v}_1.$$

Inner product and norm

- Define the *multivector inner product* and *multivector norm* by

$$A * B := \langle A^\dagger B \rangle =: |A|^2$$

- Reverse \dagger is the adjoint operator

$$(CA) * B = A * (C^\dagger B)$$

$$(AC) * B = A * (BC^\dagger).$$

- g definite $\implies *$ and $|\blacksquare|$ definite.

Blades and subspaces

- If $|U_r| = \pm 1$, then U_r is a *unit blade*.
- Unit r -blades correspond to subspaces $U \subset V$.
- The *projection* of A into a subspace U_r by

$$P_{U_r}(A) := A \lrcorner U_r U_r^{-1}.$$

Pseudoscalars

- *Pseudoscalars* are the grade- n elements.
- Define the *volume element*

$$\boldsymbol{\mu} = \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n.$$

- Define the *unit pseudoscalar* by

$$I := \frac{1}{|\boldsymbol{\mu}|} \boldsymbol{\mu}.$$

Duality

- The *dual* \perp of a multivector A is

$$A^\perp := A\mathbf{I}^{-1} \in \mathcal{G}^{n-r}.$$

- The *Hodge star* \star_g of a multivector A is

$$\star_g A = (\mathbf{I}^{-1} A)^\dagger.$$

- Dual exchanges products $(A \lrcorner B)^\perp = A \wedge B^\perp$.

Examples

- Define $\mathcal{G}_{p,q}$ by $\mathbf{e}_i^2 = -1$ for $i = 1, \dots, p$ and $\mathbf{e}_i^2 = +1$ otherwise.
- $\mathcal{G}_{1,3}$ is the *spacetime algebra*.
- $\mathcal{G}_{1,3}^2 \cong \mathfrak{spin}(1,3)$ which is the Lie algebra of the Lorentz group.
- *Quaternion algebra* \mathbb{H} is isomorphic to $\mathcal{G}_{0,3}^+$.
- *Complex algebra* \mathbb{C} is isomorphic to $\mathcal{G}_{0,2}^+$.
 - Standard basis $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_{12} := \mathbf{e}_1 \mathbf{e}_2$. Then $\mathbf{e}_{12}^2 = -1$.
 - Right multiplication of vectors by \mathbf{e}_{12} rotates counter-clockwise by $\pi/2$.

Section 2

Clifford analysis

Multivector Fields

- (M, g) is a smooth, compact, connected, oriented n -dimensional Riemannian manifold.
- **Idea: Form the Clifford algebras on tangent spaces.**
 - Form the *geometric algebra bundle*

$$\mathcal{G}M := \bigsqcup_{p \in M} \mathcal{C}\ell(T_p M, g_p).$$

- The *multivector fields* $\mathfrak{X}(M)$ are the C^∞ -sections of $\mathcal{G}M$.
- Take same naming scheme and notation: $\mathfrak{X}^r(M)$, $\mathfrak{X}^+(M)$, etc.

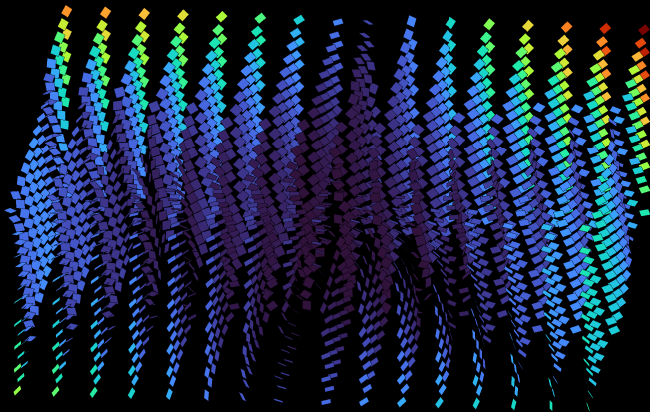
Scalar field

$$\langle p_{(1,0)} + p_{(0,1)} + p_{(1,1)} + p_{(2,0)} + p_{(0,2)} + p_{(3,0)} + p_{(0,3)} + p_{(2,1)}(1 + \mathbf{e}_{31}) + p_{(1,2)}(1 + \mathbf{e}_{31}) \rangle$$



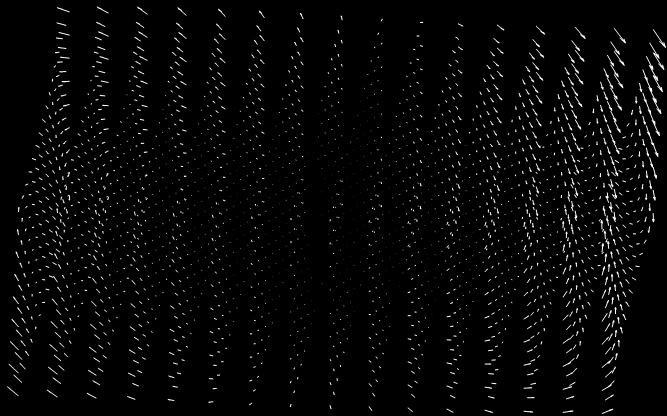
Bivector field

$$\langle p_{(1,0)} + p_{(0,1)} + p_{(1,1)} + p_{(2,0)} + p_{(0,2)} + p_{(3,0)} + p_{(0,3)} + p_{(2,1)}(1 + \mathbf{e}_{31}) + p_{(1,2)}(1 + \mathbf{e}_{31}) \rangle_2$$



Vector field

$$\langle p_{(1,0)} + p_{(0,1)} + p_{(1,1)} + p_{(2,0)} + p_{(0,2)} + p_{(3,0)} + p_{(0,3)} + p_{(2,1)}(1 + \mathbf{e}_{31}) + p_{(1,2)}(1 + \mathbf{e}_{31}) \rangle_2^\perp$$



Hodge–Dirac operator

M has the Levi-Civita connection ∇ and covariant derivative ∇_u which can be extended to act on multivectors [Schindler: 2018].

- Define the *Hodge–Dirac operator* locally by

$$\nabla = \sum_{i=1}^n e^i \nabla_{e_i}$$

- ∇ acts as a vector in $\mathfrak{X}(M)$ with Leibniz rule $\nabla(AB) = \dot{\nabla} \dot{A}B + \dot{\nabla} A \dot{B}$.
- ∇^2 is the Laplace-Beltrami operator.

Examples

- For $A_+ \in \mathfrak{X}^+(\mathbb{R}^2)$ if $\nabla A_+ = 0$ then A_+ is a **holomorphic function**.
- For a vector field $\mathbf{v} \in \mathfrak{X}^1(\mathbb{R}^3)$ we have

$$\nabla \mathbf{v} = \underbrace{\nabla \lrcorner \mathbf{v}}_{\text{divergence}} + \underbrace{\nabla \wedge \mathbf{v}}_{\text{curl}}.$$

where

$$\text{curl}(\mathbf{v}) = (\nabla \wedge \mathbf{v})^\perp$$

Differential forms

- Define the *r-dimensional directed measure* dX_r by

$$dX_r := \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_r} dx^{i_1} \cdots dx^{i_r}.$$

- Any r -form α_r has a *multivector equivalent* A_r so $\alpha_r = A_r \lrcorner dX_r^\dagger$.
- Algebra and calculus descend to multivector equivalent:

$$\alpha_r \wedge \beta_s = (A_r \wedge B_s) \lrcorner dX_{r+s}$$

$$\underbrace{d\alpha_r = (\nabla \wedge A_r) \lrcorner dX_{r+1}^\dagger}_{\text{exterior derivative}}$$

$$\alpha_r \lrcorner \beta_s = (A_r \lrcorner B_s) \lrcorner dX_{r-s}$$

$$\underbrace{\delta\alpha_r = (-\nabla \lrcorner A_r) \lrcorner dX_{r-1}^\dagger}_{\text{codifferential}}$$

Submanifolds

Fix an r -dimensional submanifold R .

- Define the *tangent unit pseudoscalar* \mathbf{I}_R .
- Dual is the *normal blade* $\boldsymbol{\nu}_R = \mathbf{I}_R^\perp$.
- Define the *volume form* on R by

$$d\mu_R := \mathbf{I}_R^{-1} \lrcorner dX_r$$

- For M this yields $d\mu = \sqrt{|g|} dx^1 \cdots dx^n$

Integral products

- Define the *directed integral product on R*

$$\langle A, B \rangle_R := \int_R A^\dagger \mathbf{I}_R B d\mu_R.$$

- Define the *multivector field inner product on R* by

$$\langle\langle A, B \rangle\rangle_R := \int_R A * B d\mu_R$$

Green's formulas

- From [Hestenes, Sobczyk, 1984] and [Booß-Bavnbek, Wojciechowski, 1993]

$$\langle \nabla A, B \rangle = (-1)^n \langle A, \nabla B \rangle + \langle A, B \rangle_{\partial M}$$

- Following from the above

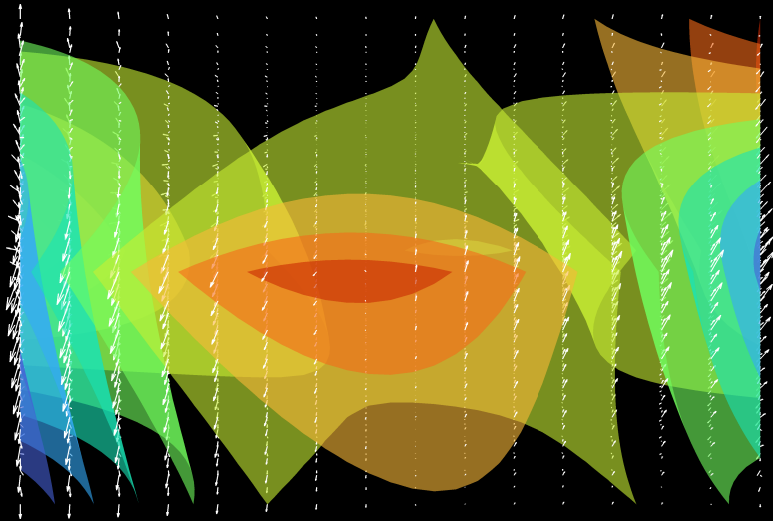
$$\langle \nabla A, B \rangle = -\langle A, \nabla B \rangle + \langle A, \nu B \rangle.$$

Monogenic fields

- The space of *monogenic fields* $\mathcal{M}(M) = \ker \nabla$.
- Ex. $f = u + v\mathbf{e}_{12} \in \mathfrak{X}^+(\mathbb{R}^2)$ then $\nabla f = 0$ is holomorphic

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

- Field $p_{(1,0)} + p_{(0,1)} + \cdots$ is monogenic (or quaternion harmonic).



Properties of monogenic fields

Monogenic fields...

- can be **uniquely continued** [Booß- Bavnbeek, Wojciechowski: 1993];
- have a **Cauchy integral** [Calderbank: 1995];
- have **harmonic components**.

Cauchy integral

- There exists a vector-valued *Cauchy kernel* G_x where $\nabla G_x = \delta_x$.
- Given $A \in \mathcal{M}(M)$, the *Cauchy integral* is

$$A(x) = (-1)^{n-1} \langle A, G_x \rangle_{\partial M}^\perp.$$

- [Calderbank: 1995], this map is an isomorphism.

Example

Consider fields on a region $M \subset \mathbb{R}^n$:

- Define $\mathbf{G}(\mathbf{x}) := \frac{1}{S_n} \frac{\mathbf{x}}{|\mathbf{x}|^n}$ then the Cauchy integral is

$$A(\mathbf{x}) = \frac{1}{S_n} \int_{\partial M} \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^n} \boldsymbol{\nu}(\mathbf{x}') A(\mathbf{x}') d\mu_{\partial M}(\mathbf{x}').$$

- **Scalar part of the above is the double layer potential.**

Inversion

- [Calderbank, 1995] Can solve the equation $\nabla A = B$ by

$$A(x) = (-1)^{n-1} \langle B, G_x \rangle^\perp.$$

- In a region $M \subset \mathbb{R}^3$ take a vector field \mathbf{J} ,

$$\text{BS}(\mathbf{J})(\mathbf{x}) = \left\langle \langle \mathbf{J}, G_{\mathbf{x}} \rangle^\perp \right\rangle_2 = \frac{1}{4\pi} \int_{N^3} \mathbf{J}(\mathbf{y}) \wedge \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^3} d\mu_{N^3}(\mathbf{x}').$$

- **This is the Biot–Savart operator.** [Cantarella, et al.: 2001]

z -polynomials

- Define the functions $z_{ij} = x_j - x_i \mathbf{e}_{ij}$.
- A *homogeneous monogenic polynomial* is

$$p_{\vec{k}} = \frac{1}{k!} \sum_{\sigma} z_{1\sigma(1)} \cdots z_{1\sigma(k)}.$$

- Space of *monogenic polynomials* is $\text{span}_{\mathcal{G}}\{p_{\vec{k}}\}$.

Proposition

Monogenic fields can be written as a power series in z_{ij} and the coefficients are computed with a Cauchy integral.

Section 3

Hodge theory

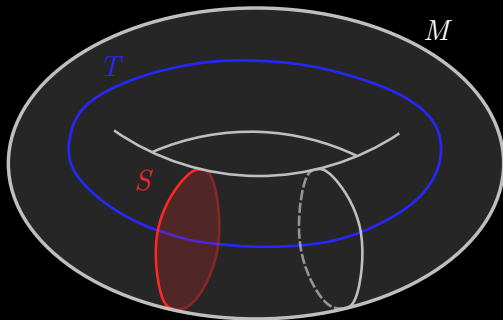
Idea

Hodge theory relates analysis to topology.

■ **Theorem (Hodge Isomorphisms).**

$$H^r(M) \cong \mathcal{M}_N^r(M)$$

$$H^r(M, \partial M) \cong \mathcal{M}^r(M, \partial M).$$



Product on cohomologies

We know $\wedge: H^r(X) \times H^s(X) \rightarrow H^{r+s}(X)$. But...

Proposition

The contraction \lrcorner is a product on cohomologies by:

- $\lrcorner: H^r(M) \times H^s(M) \rightarrow H^{s-r}(M)$;
- $\lrcorner: H^r(M, \partial M) \times H^s(M, \partial M) \rightarrow H^{s-r}(M, \partial M)$;
- $H^r(M) \lrcorner H^s(M, \partial M)$ is trivial;
- $H^r(M, \partial M) \lrcorner H^s(M)$ is trivial;

- This product is equivalent to the mixed cup product in [Shonkwiler, 2009].

Hodge decompositions

Hodge, Morrey, Friedrichs found decompositions of the space of forms.

- **Theorem [Hodge–Morrey].** $\mathfrak{X}^r(M) = \mathcal{E}_D^r(M) \oplus \mathcal{C}_N^r(M) \oplus \mathcal{M}^r(M)$.
- **Theorem [Hodge–Morrey–Friedrichs].**

$$\mathcal{M}^r(M) = \mathcal{M}_D^r(M) \oplus \mathcal{M}_{\mathbf{co}}^r \quad \text{or} \quad \mathcal{M}^r(M) = \mathcal{M}_N^r(M) \oplus \mathcal{M}_{\mathbf{ex}}^r$$

- But, $\mathcal{M}(M) \neq \bigoplus_{j=1}^n \mathcal{M}^j(M)$.

Combining the boundary constraints of exact and coexact fields...

- Define the *Dirac fields* $\nabla\mathfrak{X}(M)$ as

$$\nabla\mathfrak{X}(M) := \{\nabla A \mid A \in \mathfrak{X}(M) \text{ and } A|_{\partial M} = 0\};$$

Theorem: Clifford–Hodge Decomposition

The space of multivector fields $\mathfrak{X}(M)$ has the orthogonal decomposition

$$\mathfrak{X}(M) = \mathcal{M}(M) \oplus \nabla\mathfrak{X}(M).$$

Comparing to Hodge–Morrey

- From Hodge–Morrey

$$\mathfrak{X}(M) = \sum_{r=0}^n \underbrace{\mathcal{E}_D^r(M)}_{\operatorname{im}(\nabla \wedge)} \oplus \underbrace{\mathcal{C}_N^r(M)}_{\operatorname{im}(\nabla \lrcorner)} \oplus \underbrace{\mathcal{H}^r(M)}_{\ker(\nabla)}.$$

- But the Clifford-Hodge-Morrey is not filtered by grades

$$\mathfrak{X}(M) = \underbrace{\mathcal{M}(M)}_{\ker \nabla} \oplus \underbrace{\nabla \mathfrak{X}(M)}_{\operatorname{im} \nabla}.$$

Section 4

Tomography

EIT forward problem

- If M is Ohmic region of \mathbb{R}^3 , represent conductivity with metric g .
- **Ohm's law** $-\nabla \wedge u = \mathbf{J}$ and **conservation** $\nabla \lrcorner \mathbf{J} = 0$.
- Suppose M free of charges, then the forward problem

$$\begin{cases} \nabla^2 u = 0 & \text{in } M \\ u = \phi & \text{on } \partial M \end{cases}$$

EIT inverse problem

We want access to M only from ∂M .

- The *electric Dirichlet-to-Neumann (DN) map*

$$\Lambda_E: \mathfrak{t}\mathfrak{X}^0(M) \rightarrow \mathfrak{t}\mathfrak{X}^0(M) \quad \text{by} \quad \Lambda_E \phi = \boldsymbol{\nu} \lrcorner \boldsymbol{\nabla} \wedge u = \frac{\partial u}{\partial \boldsymbol{\nu}}.$$

- Question: Can we determine (M, σ) from Λ_E ?

Magnetic analog

- Magnetic bivector field B solves the forward problem

$$\begin{cases} \nabla^2 B = 0 & \text{in } M \\ B = \boldsymbol{\nu} \wedge \mathbf{J} & \text{on } \partial M \end{cases}$$

- Define the *magnetic DN operator* $\Lambda_B: \mathfrak{n}\mathfrak{X}^2(M) \rightarrow \mathfrak{n}\mathfrak{X}^2(M)$ by $\Lambda_B(\boldsymbol{\nu} \wedge \mathbf{J}) = \boldsymbol{\nu} \wedge \nabla \lrcorner B$.
- Question: What can we get from Λ_B ?

Calderón problem

Don't forget our goal...

- The problem has been solved in dimension $n = 2$ [Belishev: 2003].
- Solved in dimensions $n \geq 3$ when M is an analytic manifold [Lassas, Taylor, Uhlmann: 2003].
- The smooth cases is still unsolved.

Electromagnetic tomography

- **Can combine to monogenic spinor $A_+ = u + B$.**
- Can we recover the space of monogenic fields from the DN operators?
- If the space of monogenic fields contained geometric data, this would be a step forward in the Calderón problem...

Geometric generalization

In arbitrary dimension:

$$\begin{cases} \nabla^2 A_r = 0 & \text{in } M \\ A_r = \phi_r & \text{on } \partial M \end{cases}$$

■ *Generalized electric DN operator*

$$\Lambda_E: t\mathfrak{X}(M) \rightarrow t\mathfrak{X}(M) \quad \text{by} \quad \Lambda_E \phi_r = \boldsymbol{\nu} \lrcorner \nabla \wedge A_r.$$

■ *Generalized magnetic DN operator*

$$\Lambda_B: n\mathfrak{X}(M) \rightarrow n\mathfrak{X}(M) \quad \text{by} \quad \Lambda_B \phi_r = \boldsymbol{\nu} \wedge \nabla \lrcorner A_r.$$

Comologies from DN operators

- $\ker \Lambda_E = \operatorname{tr} \mathcal{M}_N^r(M)$ and $\ker \Lambda_B = \operatorname{tr} \mathcal{M}_D^r(M)$.
- $\operatorname{tr}: \mathcal{M}(M) \rightarrow \operatorname{tr} \mathcal{M}(M)$ is an isomorphism.
- Applying Hodge isomorphisms...

Theorem

We have $\ker \Lambda_E \cong H^r(M)$ and $\ker \Lambda_B \cong H^r(M, \partial M)$.

- $\Lambda_E \times \Lambda_B$ is equivalent to **complete DN operator** [Shonkwiler, Sharafutdinov: 2013].

Spinor DN operator

- Define the *spinor DN operator* $\mathcal{J}: \text{tr}\mathfrak{X}^\pm(M) \rightarrow \text{tr}\mathfrak{X}^\pm(M)$.
- Specifically: $\mathcal{J}\phi_r = \nu \nabla A_r$.
- Generalized operators are scalar part $\Lambda_E + \Lambda_B = \langle \mathcal{J} \rangle$.

Theorem

We have $\ker \mathcal{J} = \text{tr}\mathcal{M}(M)$.

Section 5

Gelfand theory

Open questions

- In [Belishev: 2003], we see an algebraic proof for the 2-dimensional Calderón problem.
- In [Belishev, Vakulenko: 2017], we see a proof for a noncommutative Gelfand representation using quaternion fields for a ball \mathbb{B} in \mathbb{R}^3 .
- Belishev and Vakulenko ask whether this is true in higher dimensions.
- We will prove this is true for arbitrary regions.

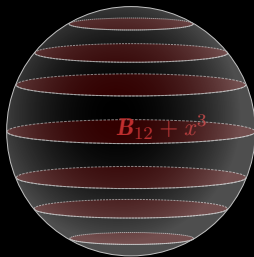
Overview of BC method

The *boundary control (BC) method* in [Belishev: 2003] is as follows:

- Determine the algebra $\mathcal{A}(M)$ of holomorphic functions on M using Λ .
- Gelfand theory implies the spectrum of $\mathcal{A}(M)$ is homeomorphic to M .
- Algebraic structure of $\mathcal{A}(M)$ determines the complex structure on M .
- Find g that conformal with the complex structure.

Subsurface spinor fields

- For O convex, let $B \in \mathfrak{X}(O)$ be parallel translation of a unit 2-blade.
- Refer to $A_+ = P_B \circ A_+$ as a *subsurface spinor*.



- The *algebra of monogenic subsurface spinors* is $\mathcal{A}_B(O)$.
- **Algebra is a commutative Banach algebra.**

Characters

- Define the algebra \mathbb{A}_B to be the algebra generated by 1 and B .
- The *spinor spectrum* $\mathfrak{M}(M)$ consists of *spin characters*:
 - Continuous grade-preserving \mathcal{G}^+ -linear maps $\mathcal{M}^+(M) \rightarrow \mathcal{G}^+$,
 - algebra morphisms $\mathcal{A}_B(O) \rightarrow \mathbb{A}_B$.
- One example of such characters are point evaluations $\delta(A_+) = A_+(x_\delta)$.
- **We show these are the only elements in the spectrum.**

Idea

By linearity, we can note that for $\delta \in \mathfrak{M}(\mathbb{B}_{R,w})$

$$\delta(f(x)) = \sum_{j=0}^{\infty} \left(\sum_{\substack{j_2 \cdots j_n \\ j_2 + \cdots + j_n = j}} \delta(p_{j_2 \cdots j_n}(x - w)) a_{j_2 \cdots j_n} \right)$$

On each monogenic polynomial

$$\delta(p_{j_2 \cdots j_n}(x)) = \frac{1}{j!} \sum_{\text{permutations}} \delta((z_{1\sigma(1)}(x)) \cdots \delta(z_{1\sigma(j)}(x)))$$

by the multiplicativity of δ . Then $z_{ij} \in \mathcal{A}_{e_{ij}}(O)$.

Necessary lemmas

For regions $M \subset \mathbb{R}^n$:

Lemma: Density

The space $\mathcal{M}^{\mathcal{P}}(M)$ is dense in $\mathcal{M}(M)$.

Lemma: Point evaluation

For $\delta \in \mathfrak{M}(M)$ we have $\delta(z_{ij}) = z_{ij}(x_\delta)$ for some $x_\delta \in \mathbb{R}^n$.

Lemma: Identification

Let $A_+ \in \mathcal{M}^+(M)$, then $\delta(A_+) = A_+(x_\delta)$ for some $x_\delta \in M$.

The previous lemmas imply the following:

Theorem: Clifford-algebraic Gelfand theorem

With the weak-* topology on $\mathfrak{M}(M)$, the map

$$\gamma: \mathfrak{M}(M) \rightarrow M, \quad \delta \mapsto x_\delta$$

is a homeomorphism. The Gelfand transform $\widehat{A}_+(\delta) = \delta[A_+]$ is an isometric isomorphism so $\mathcal{M}^+(M) \cong \widehat{\mathcal{M}^+(M)}$.

Section 6

Further results, open questions, conclusion

A Stone–Weierstrass theorem

- Using continuation from a z_{ij} :

Lemma

The space $\overline{\mathcal{M}^+(M)}$ separates points.

- Using [Laville, Ramadanoff: 1996]:

Theorem: Stone–Weierstrass

$\overline{\mathcal{M}^+(M)}$ is dense in $C(M; \mathcal{G}^+)$.

Sheaf

- Using unique continuation:

Theorem

The sheaf \mathcal{M}_M is Hausdorff and the map $\pi: \mathcal{M}_M \rightarrow M$ is a local homeomorphism.

- Can one find a component of \mathcal{M}_M that is homeomorphic to M ?

Future work and open questions

To get a higher dimensional BC method we need:

- The DN operator determines $\text{tr} \mathcal{M}^+(M)$.
- The map $\text{tr}: \vee \mathcal{M}^+(M) \rightarrow \text{tr} \vee \mathcal{M}^+(M)$ is an isometric isomorphism of algebras.
- The space $\mathcal{M}^+(M)$ determines the metric structure of M up to isometry.

Future work and open questions

- Many of these approaches use the Hilbert transform which is also used by Belishev, Sharafutdinov, and Shonkwiler to study the Calderón problem.
- The Hilbert transform is also a classical part of Clifford analysis.
- [Lassas, Uhlmann: 2001] use sheaf theory to solve the analytic Calderón problem.
- Santacesaria proposes another method for solving the Calderón problem using Clifford algebras and complex geometric optics.

Conclusion

- Clifford analysis is a natural way to combine Hodge theory and complex analysis.
- Able to describe DN operators and extract homological information and boundary values of monogenic fields.
- Monogenic fields are uniquely determined by boundary traces via Cauchy integral.
- Monogenic fields are able to tell us the topology of the manifold they are defined on.

Thank you!