MATH 570, Homework 10

Colin Roberts

November 9, 2017

Solutions

Problem 1. Let X be the abstract simplicial complex

$$\{\{0\},\{1\},\{2\},\{3\},\{4\},\{5\},\{0,1\},\{0,2\},\{0,3\},\{0,4\},\{1,2\},\{3,4\}\}.$$

(a) Draw the geometric realization of X.

(b) Compute the simplicial homology group $H_0(X)$.

(c) Compute the simplicial homology group $H_1(X)$.

(d) Compute the simplicial homology group $H_2(X)$.

:

Proof. For now I'm noting that $Z_p(X) = \ker(\partial_p)$, $B_p(X) = \operatorname{im}(\partial_{p+1})$, and $H_p(X) = Z_p(X)/B_p(X)$. Also we define $\partial_p \colon C_p(X) \to C_{p-1}(X)$ by

$$\partial_p([x_0,\ldots,x_p]) = \sum_{i=0}^{p}(-1)^i[x_0,\ldots,\hat{x_i},\ldots,x_p].$$

(a) We have X:

(b) Now $H_0(X) = Z_0(X)/B_0(X) = \ker(\partial_0)/\operatorname{im}(\partial_1)$, so we find $Z_0(X)$ and $B_0(X)$. First we have $C_1(X) = \{a[0,1]+b[0,2]+c[0,3]+d[0,4]+e[1,2]+f[3,4] \mid a,b,c,d,e,f \in \mathbb{Z}\}$ and $C_0(X) = \{a[0]+b[1]+c[2]+d[3]+e[4]+g[5] \mid a,b,c,d,e,f \in \mathbb{Z}\}$. Then

$$\begin{aligned}
B_0(X) &= \operatorname{im}(\partial_1)\\
&= \{a([1]-[0])+b([2]-[0])+c([3]-[0])+d([4]-[0])+e([2]-[1])+f([4]-[3])\\
&\quad \mid a,b,c,d,e,f \in \mathbb{Z}\}\\
&= \{a([1]-[0])+b([2]-[0])+c([3]-[0])+d([4]-[0]) \mid a,b,c,d \in \mathbb{Z}\} \cong \mathbb{Z}^4.
\end{aligned}$$

Notice that $e([2]-[1])$ and $f([4]-[3])$ are $\mathbb{Z}$ linear combinations of the other three.

Now $\partial_0 = 0$ so we have that $Z_0(X) = \ker(\partial_0) = C_0(X) \cong \mathbb{Z}^6$. Then we have $H_0(X) = \mathbb{Z}^6/\mathbb{Z}^4 = \mathbb{Z}^2$. This tells us that there are two connected components.

(c) Now $H_1(X) = Z_1(X)/B_1(X) = \ker(\partial_1)/\operatorname{im}(\partial_2)$ and we have $C_2(X) = 0$ and $C_1(X) = \{a[0,1]+b[0,2]+c[0,3]+d[0,4]+e[1,2]+f[3,4] \mid a,b,c,d,e,f \in \mathbb{Z}\}$. Note that $B_1(X) = \operatorname{im}(\partial_2) = \langle e \rangle$, as the trivial group. Now from above we have $\operatorname{im}(\partial_1) \cong \mathbb{Z}^3$ and since $C_1(X) \cong \mathbb{Z}^5$ we have $Z_1(X) = \ker(\partial_1) \cong \mathbb{Z}^2$. So it follows $H_1(X) \cong \mathbb{Z}^2$.

(d) We have no 2-simplices, so $H_2(X) = \langle e \rangle$. $\square$

Problem 2. Let X be the simplicial complex which is the boundary of a tetrahedron. That is, X has 4 vertices (say labeled 0,1,2,3), all $\binom{4}{2} = 6$ possible edges, all $\binom{4}{3} = 4$ possible 2-simplices, and no tetrahedra.

(a) Draw the geometric realization of X.

(b) Compute the simplicial homology group $H_1(X)$. What group is $Z_1(X)$ isomorphic to?

(c) Compute the simplicial homology group $H_2(X)$.

(a) We have X:

(b) Now $H_1(X) = Z_1(X)/B_1(X) = \ker(\partial_1)/\mathrm{im}(\partial_2)$ and we have $C_2(X) = \{a[0,1,2] + b[0,1,3] + c[0,2,3] + d[1,2,3] \mid a,b,c,d \in \mathbb{Z}\}$ and $C_1(X) = \{a[0,1] + b[0,2] + c[0,3] + d[1,2] + e[1,3] + f[2,3] \mid a,b,c,d,e,f \in \mathbb{Z}\}$. Now

$$\begin{aligned} B_1(X) &= \mathrm{im}(\partial_2) \\ &= \{a([1,2]-[0,2]+[0,1]) + b([1,3]-[0,3]+[0,1]) + c([2,3]-[0,3]+[0,2])+ \\ &\quad d([2,3]-[1,3]+[1,2]) \mid a,b,c,d \in \mathbb{Z}\} \end{aligned}$$

From the extra work below, we have $\mathrm{im}(\partial_1) \cong \mathbb{Z}^3$ and since $C_1(X) \cong \mathbb{Z}^6$ we have $Z_1(X) = \ker(\partial_1) \cong \mathbb{Z}^3$. This can be seen by letting the ordered basis vectors be $\{[0,1],[0,2],[0,3],[1,2],[1,3],[2,3]\}$ and augmenting a matrix (really the matrix for ∂_2) as follows:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{which reduces to} \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This above row reduction shows that we have 3 linearly independent vectors, which shows that $Z_1(X) \cong \mathbb{Z}^3$. So it follows $H_1(X) \cong \mathbb{Z}^3/\mathbb{Z}^3 \cong \langle 3 \rangle$, the trivial group.

(c) We have $C_2(X) = \{a[0,1,2]+b[0,1,3]+c[0,2,3]+d[1,2,3] \mid a,b,c,d \in \mathbb{Z}\}$. $H_2(X) = Z_2(X)/B_2(X) = \ker(\partial_2)/\mathrm{im}(\partial_3)$, and we have that $\mathrm{im}(\partial_3) = 0$ since there are no 3-simplices. Then $\ker(\partial_2) = \mathbb{Z}$ since $\mathrm{im}(\partial_2) = \mathbb{Z}^3$. Thus we have $H_2(X) \cong \mathbb{Z}$.

(d) *Extra work:* We have $C_1(X) = \{a[0,1]+b[0,2]+c[0,3]+d[1,2]+e[1,3]+f[2,3] \mid a,b,c,d,e,f \in \mathbb{Z}\}$ and $C_0(X) = \{a[0]+b[1]+c[2]+d[3] \mid a,b,c,d,e,f \in \mathbb{Z}\}$. Then

$$\begin{aligned}
B_0(X) &= \mathrm{im}(\partial_1) \\
&= \{a([1]-[0]) + b([2]-[0]) + c([3]-[0]) + d([2]-[1]) + e([3]-[1]) + f([3]-[2]) \\
&\mid a,b,c,d,e,f \in \mathbb{Z}\} \\
&= \{a([1]-[0]) + b([2]-[0]) + c([3]-[0]) \mid a,b,c \in \mathbb{Z}\} \cong \mathbb{Z}^3.
\end{aligned}$$

Notice that $d([2]-[1])$ and $e([3]-[1])$ are $\mathbb{Z}$ linear combinations of the other three.

Now $\partial_0 = 0$ so we have that $Z_0(X) = \ker(\partial_0) = C_0(X) \cong \mathbb{Z}^4$. Then we have $H_0(X) = \mathbb{Z}^4/\mathbb{Z}^3 = \mathbb{Z}$. This registers the one connected component.

Problem 3. Choose any old homework or exam problem, or a portion thereof. Clearly state both the problem and the homework/exam number. Write out a solution that is as clear as possible, with no extraneous steps.

Problem 2. Homework 8: Let S^1 be the unit circle and let $C = S^1 \times [-1, 1]$ be a cylinder. Prove that $S^1 \cong C$.

:

Proof. Define the maps $f\colon S^1 \to C$ and $g\colon C \to S^1$ with $f(x) = (x, 0)$ and $g(x, s) = x$. Then we show that $f \circ g \simeq \mathrm{Id}_C$ and $g \circ f \simeq \mathrm{Id}_{S^1}$. Clearly we have $g \circ f = \mathrm{Id}_{S^1}$ which shows $g \circ f \simeq \mathrm{Id}_{S^1}$. Now we have $H\colon C \times I \to C$ defined by $H((x, s), t) = (x, st)$ is continuous and satisfies $H((x, s), 0) = f(x)$ and $H((x, s), 1) = \mathrm{Id}_C(x, s)$ which shows that $f \circ g \simeq \mathrm{Id}_C$. Hence, $S^1 \simeq C$. $\square$