MATH 271, Homework 9

Due November 13th

Problem 1. Compute the following:

(a)

$$[A] = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.$$

(b)

$$[B] = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 3 & 2 \\ 2 & 3 \end{pmatrix}$$

(c) Take

$$[M] = \begin{pmatrix} 10 & 15 \\ 20 & 10 \end{pmatrix}$$

and

$$[N] = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Compute $[M][N]$ and $[N][M]$ to see that matrices do not commute in general.

Problem 2. A linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is given by the matrix

$$[T] = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}.$$

(a) Compute how T transforms the standard basis elements for $\mathbb{R}^3$. That is, find

$$T(\hat{\boldsymbol{x}}), \qquad T(\hat{\boldsymbol{y}}), \qquad T(\hat{\boldsymbol{z}})$$

and relate these values to the columns of $[T]$.

(b) Are the vectors $T(\hat{\boldsymbol{x}})$, $T(\hat{\boldsymbol{y}})$, and $T(\hat{\boldsymbol{z}})$ linearly independent? Do these vectors form a basis for $\mathbb{R}^3$?

(c) If we apply this linear transformation to the unit cube (that is, all points who have (x, y, z) coordinates with $0 \le x \le 1$, $0 \le y \le 1$, and $0 \le z \le 1$), what will the volume of the transformed cube be? (*Hint: the determinant of this matrix* $[T]$ *provides us this information.*)

Problem 3. Consider some linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$. Let $\vec{\boldsymbol{v}}_1, \ldots, \vec{\boldsymbol{v}}_k$ be vectors in $\mathrm{Null}(T)$.

(a) Show that the span of these vectors is also in the nullspace of T.

(b) How many linearly independent vectors can be in the nullspace?

Problem 4.

(a) Show that for any 2×2-matrix that the sign of the determinant changes if either a row or column is swapped. *Note: this is true for square matrices of any size.*

(b) Show that for any 2×2-matrix that multiplying a column by a constant scales the determinant by that constant as well. *Note: this is true for square matrices of any size.*

(c) Show that for any 2×2-matrix that adding a scalar multiple one column to the other will not change the determinant. *Note: this is true in broader generality. In fact, adding linear combinations of columns to another column will not change the determinant.*

(d) Using these facts, argue why a square matrix with columns that are linearly dependent must have a determinant of zero.

Problem 5. Consider the equation

$$[A]\vec{\boldsymbol{v}} = \vec{\mathbf{0}},$$

where

$$[A] = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(a) Are the columns of $[A]$ linearly independent or dependent? Explain.

(b) What vector(s) $\vec{\boldsymbol{v}}$ satisfy this equation? In other words, what is $\mathrm{Null}([A])$?

(c) Using what you found above, what must $\det([A])$ be equal to? *Hint: you do not need to compute the determinant!*

Problem 6. Compute the following determinants:

(a)

$$\det([A]) = \begin{vmatrix} -3 & 1 & 5 \\ -3 & 4 & 2 \\ -3 & 2 & 1 \end{vmatrix}$$

(b)

$$\det([B]) = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

(c) Compute $\det([A][B])$ using properties of the determinant. *Hint: this should be very quick to do. Do not compute the product of the matrices $[A]$ and $[B]$!*

Problem 7.

(a) What does a zero determinant indicate about the solutions of a non-homogeneous system of linear equations? (Think geometrically!)

(b) What does a zero determinant indicate about the solutions of a homogeneous system of linear equations? (Think geometrically!)

Problem 8. Given the matrices

$$[A] = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ -2 & -2 & 0 \end{pmatrix} \qquad \text{and} \qquad [B] = \begin{pmatrix} -3 & 1 & 1 \\ 2 & -2 & 4 \\ -1 & -1 & -1 \end{pmatrix}.$$

(a) Compute $\mathrm{tr}([A])$ and $\mathrm{tr}([B])$.

(b) Compute $\mathrm{tr}([A][B])$ and compare it to $\mathrm{tr}([B][A])$.