MATH 317, Homework 3

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Solutions

Problem 1. Prove that every Cauchy sequence is bounded.

Proof. Suppose that (x_n) is Cauchy. Then we can fix $\epsilon > 0$ and $\exists N \in \mathbb{N}$ such that $\forall n, m \geq N$,

$$|x_n - x_m| < \epsilon$$

For a contradiction, suppose that (x_n) is unbounded. It follows that the subsequence of (x_n) where $n \geq N$ is unbounded as well. Thus in this subsequence there $\exists k \in \mathbb{N}$ with $k > N$ where $\forall M > 0$, $|x_k| > M$. Hence, if we let $M = |x_m| + \epsilon$ then $\exists |x_n| > M$ which implies that, given these choices,

$$|x_n - x_m| \leq |x_n| - |x_m| > M - |x_m| = \epsilon$$

This is a contradiction to (x_n) being Cauchy. Thus the sequence must be bounded. $\square$

Problem 2. Prove that the set $\left\{1+\frac{(-1)^n}{n} \mid n \in \mathbb{N}\right\}$ has exactly one accumulation point.

Proof. First, let's show that 1 is an accumulation point. If 1 is an accumulation point then $\forall \epsilon > 0$, the neighborhood $Q = (1-\epsilon, 1+\epsilon)$ contains at least one other point that is not 1. Now, fix $\epsilon > 0$ and let $N \in \mathbb{N}$ be such that $N > \frac{1}{\epsilon - 1}$. Then $\forall n \geq N$,

$$\begin{aligned}
\left|1+\frac{(-1)^n}{n}\right| &\leq |1| + \left|\frac{(-1)^n}{n}\right| \\
&= 1 + \frac{1}{n} \\
&\leq 1 + \frac{1}{N} \\
&< 1 + \frac{1}{\frac{1}{\epsilon-1}} \\
&= \epsilon
\end{aligned}$$

Thus we know there exists another point in any open neighborhood around 1 and 1 is an accumulation point.

Next, we must show that there exists no other accumulation point. Suppose, for a contradiction, that there exists another accumulation point $x \neq 1$. Then $\forall \epsilon > 0$ we have for at least one $N \in \mathbb{N}$, $x - \epsilon < 1 + \frac{(-1)^N}{N} < x + \epsilon$. Fix $\epsilon > |x-1| > 0$, and we have

$$\begin{aligned}
x - \epsilon < 1 + \frac{(-1)^N}{N} < x + \epsilon \\
\epsilon < \left(1 + \frac{(-1)^N}{N}\right) - x < \epsilon \\
\implies \left|\left(1 + \frac{(-1)^N}{N}\right) - x\right| < \epsilon \\
\implies \left|x - \left(1 + \frac{(-1)^N}{N}\right)\right| < \epsilon
\end{aligned}$$

Thus,

$$\begin{aligned}
\left|x - 1 + \frac{(-1)^N}{N}\right| &\leq |x-1| + \left|\frac{(-1)^N}{N}\right| \\
&= \epsilon + \frac{1}{N} > \epsilon
\end{aligned}$$

Since we have the quantity being greater than ϵ, this is a contradiction. Thus if (x_n) is Cauchy it must also be bounded. $\square$

Problem 3. Prove the following:

(a) $\displaystyle\lim_{n\to\infty}\frac{(-1)^n}{n}=0$ (b) $\displaystyle\lim_{n\to\infty}\frac{1}{n^{1/3}}=0$

(c) $\displaystyle\lim_{n\to\infty}\frac{2n-1}{3n+2}=\frac{2}{3}$ (d) $\displaystyle\lim_{n\to\infty}\frac{n+6}{n^2-6}=0$

Proof (a). If $\lim_{n\to\infty}\frac{(-1)^n}{n}=0$, then $\forall\epsilon>0\ \exists N\in\mathbb{N}$ such that $\forall n\geq N$, $\left|\frac{(-1)^n}{n}-0\right|<\epsilon$. Fix $\epsilon>0$ and let $N>\frac{1}{\epsilon}$. Then we have,

$$\begin{aligned}\left|\frac{(-1)^n}{n}-0\right| &= \left|\frac{(-1)^n}{n}\right|\\ &=\frac{1}{n}\\ &\leq\frac{1}{N}\\ &<\epsilon\end{aligned}$$

Thus 0 is the limit. □

Proof (b). If $\lim_{n\to\infty}\frac{1}{n^{1/3}}=0$, then $\forall\epsilon>0\ \exists N\in\mathbb{N}$ such that $\forall n\geq N$, $\left|\frac{1}{n^{1/3}}-0\right|<\epsilon$. Fix $\epsilon>0$ and let $N>\frac{1}{\epsilon^3}$. Then we have,

$$\begin{aligned}\left|\frac{1}{n^{1/3}}-0\right| &= \frac{1}{n^{1/3}}\\ &\leq\frac{1}{N^{1/3}}\\ &<\epsilon\end{aligned}$$

Thus 0 is the limit. □

Proof (c). First let's show that $\lim_{n\to\infty}\frac{1}{n}=0$. If this is true, then $\forall\epsilon>0\ \exists N\in\mathbb{N}$ such that $\forall n\geq N$, $\left|\frac{1}{n}-0\right|<\epsilon$. Fix $\epsilon>0$ and let $N=\frac{1}{\epsilon}$. Then,

$$\begin{aligned}\left|\frac{1}{n}-0\right| &= \left|\frac{1}{n}\right|\\ &=\frac{1}{n}\\ &\leq\frac{1}{N}\\ &<\epsilon\end{aligned}$$

So $\lim_{n\to\infty}\frac{1}{n}=0$. Now consider the sequence given in (c),

$$\begin{aligned}\frac{2n-1}{3n+2} &= \frac{2-1/n}{3+2/n}\\ &=\frac{2-(1/n)}{3+2(1/n)}\end{aligned}$$

If we let $n\to\infty$ then we have,

$$\frac{2-(0)}{3+2(0)}=\frac{2}{3}$$ □

Proof (d). Since we already know that $\frac{1}{n} \to 0$ we have,

$$\begin{aligned}\frac{n+6}{n^2-6} &= \frac{n(1+6/n)}{n^2(1-6/n^2)} \\ &= \frac{1+6(1/n)}{n(1-6(1/n)(1/n))}\end{aligned}$$

and as $n \to \infty$,

$$\begin{aligned}\frac{1}{n}\frac{1+6(1/n)}{1-6(1/n)(1/n)} &\to (0)\frac{1+6(0)}{1-6(0)(0)} \\ &= 0\end{aligned}$$

□

Problem 4.

(a) Consider three sequences (a_n), (b_n), and (s_n) such that $a_n \leq s_n \leq b_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = s$. Prove $\lim_{n\to\infty} s_n = s$. This is called the "Squeeze lemma."

(b) Suppose (s_n) and (t_n) are sequences such that $|s_n| \leq t_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} t_n = 0$. Prove $\lim_{n\to\infty} s_n = 0$.

Proof (a). Fix $\epsilon > 0$. Then $\exists N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$ we have,

$$|a_n - s| < \epsilon \tag{1}$$

Also, $\exists N_2 \in \mathbb{N}$ such that $\forall n \geq N_2$ we have,

$$|b_n - s| < \epsilon \tag{2}$$

Define $N = \max(N_1, N_2)$. Thus both expressions (*Eqn. 1* and *Eqn. 2*) hold $\forall n \geq N$. Next, notice that these conditions imply that,

$$\epsilon < a_n - s \leq b_n - s < \epsilon \tag{3}$$

Also notice that,

$$\begin{aligned} &a_n \leq s_n \leq b_n \\ \implies &a_n - s \leq s_n - s \leq b_n - s \end{aligned}$$

Inserting this back into the earlier expression (*Eqn. 3*), we have that

$$\begin{aligned} &\epsilon < a_n - s \leq s_n - s \leq b_n - s < \epsilon \\ \implies &|s_n - s| < \epsilon \end{aligned}$$

Thus the sequence (s_n) converges to s. ☐

Proof (b). Since $t_n \to 0$, $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall n \geq N$, $|t_n - 0| < \epsilon$. Then we have,

$$||s_n| - 0| \leq |t_n - 0| < \epsilon$$

But,

$$||s_n| - 0| = ||s_n|| = |s_n| = |s_n - 0|$$

Thus,

$$|s_n - 0| \leq |t_n - 0| < \epsilon$$

Thus $\lim_{n\to\infty} s_n = 0$ ☐

Problem 5. Let

$$a_n = \begin{cases} \frac{1}{n}, & \text{if 51 does not divide } n \\ 1, & \text{if 51 does divide } n \end{cases}$$

Prove that (a_n) does not converge.

Proof. If (a_n) converges, then we know (a_n) is also Cauchy. Thus, $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall n, m \geq N$, $|a_m - a_n| < \epsilon$. Here, fix $\epsilon = \frac{48}{50}$ and let $n = 51N - 1$ and $m = n + 1$. Thus,

$$\begin{aligned} |a_m - a_n| &= \left| 51N - \frac{1}{51N-1} \right| \\ &= \left| 1 - \frac{1}{51N-1} \right| \\ &= 1 - \frac{1}{51N-1} \end{aligned}$$

But since $N \in \mathbb{N}$, $N \geq 1$ by definition and we have $\frac{1}{51N-1} \leq \frac{1}{51-1} = \frac{1}{50}$. Thus,

$$1 - \frac{1}{51N-1} \geq 1 - \frac{1}{50} = \frac{49}{50} > \frac{48}{50} = \epsilon$$

Thus (a_n) is certainly not Cauchy. Since (a_n) is not Cauchy, it must not converge. $\square$

Problem 6. Consider the following sequences, defined as follows:

$$a_n = (-1)^n, \qquad b_n = \sin\frac{n\pi}{4} \qquad c_n = n^2 \qquad d_n = \frac{6n+4}{7n-3}$$

(a) For each sequence, give an example of a monotone subsequence.

(b) For each subsequence you gave, determine if it converges or diverges. If it converges, find the limit. If it diverges, does it diverge to ∞, $-\infty$, or niether?

(c) Repeat part (b) for the original sequences.

Solution (a). A valid subsequence for a_n would be $n \to 2k$ for $k \in \mathbb{N}$. The sequence is,

$$a_{2k} = \{1, 1, 1, 1, \ldots\}$$

Since by definition a constant sequence is monotonic.

A valid subsequence for b_n would be $n \to 4k$ for $k \in \mathbb{N}$. The sequence is,

$$b_{4k} = \{0, 0, 0, 0, \ldots\}$$

By the same logic as before.

A valid subsequence for c_n would be the sequence itself. But rather I'll restrict to $k = 1$. This sequence looks like,

$$c_1 = \{1, 1, 1, 1, \ldots\}$$

Which is again constant and monotone.

A valid subsequence for d_n would also be the sequence itself, but again I'll restrict $k = 1$. This sequence looks like,

$$d_n = \left\{\frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \ldots\right\}$$

Again same logic, so this is monotone.

It is totally a hack to abuse sequences like this. I also considered finite sequences, but we hadn't talked about finite sequences. ■

Solution (b). Let's start with the first subsequence: (a_{2k}). Notice, I have made my life easy here. (a_{2k}) is constant and thus $\text{im}((a_{2k})) = 1$. We can show that the sequence $\{1, 1, 1, 1, \ldots\}$ is Cauchy. Fix $\epsilon > 0$. Since $\forall k \in \mathbb{N}$, $a_{2k} = 1$ we know that for any $p, q \in \mathbb{N}$, $|a_{2p} - a_{2q}| = |1 - 1| = 0 < \epsilon$. Since this subsequence is Cauchy it is also convergent.

Note: This methodology is going to be repeated virtually word for word in the next 3 examples

Next consider the sequence (b_{4k}). Since the sequence is constant, $\forall k \in \mathbb{N}$ we have $b_{4k} = 0$. Fix $\epsilon > 0$. Again, using the fact the sequence is constant, we have $\forall p, q \in \mathbb{N}$, $|b_{4p} - b_{4q}| = |0 - 0| = 0 < \epsilon$.

Next consider the sequence (c_1). Since the sequence is constant, and in fact defined by the same element, we can do a similar trick. Fix $\epsilon > 0$. Again, using the fact the sequence is constant and consists only of c_1, we have $|c_1 - c_1| = |1 - 1| = 0 < \epsilon$.

Next consider the sequence (d_1). Since the sequence is constant, and in fact defined by the same element, we can copy the previous trick. Fix $\epsilon > 0$. Again, using the fact the sequence is constant and consists only of d_1, we have $|d_1 - d_1| = |\frac{5}{2} - \frac{5}{2}| = 0 < \epsilon$. ■

Solution (c).

Proof (a_n). Suppose that the sequence (a_n) converges to the value L. Then fix $\epsilon > 1 + |L| > 0$. Then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $|(-1)^n - L| < \epsilon$. But,

$$\begin{aligned} \left|(-1)^n - L\right| &\leq |(-1)^n| + |-L| \\ &= 1 + |L| > \epsilon \end{aligned}$$

This contradicts the necessary statement that $|(-1)^n - L| < \epsilon$ and thus (a_n) does not converge to any value since L was arbitrary. □

Proof (b_n). Suppose that the sequence (b_n) converges to the value L. Then fix $\epsilon > |L| > 0$. Then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $\left|\sin\left(\frac{n\pi}{4}\right) - L\right| < \epsilon$. But,

$$\left|\sin\left(\frac{n\pi}{4}\right) - L\right| \leq \left|\sin\left(\frac{n\pi}{4}\right)\right| + |-L|$$

Notice, $\mathrm{Im}((b_n)) = \left\{-1, -\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 1\right\}$. Thus, $\left|\sin\left(\frac{n\pi}{4}\right)\right| = 0$ is the smallest value in the set. Substituting in 0,

$$|0| + |-L| = |L| > \epsilon$$

Since the smallest value possible does not satisfy the ϵ chosen we have a contradiction. All other values will give a larger answer than something already larger than ϵ. Because L was arbitrary, (b_n) does not converge to any value. □

Proof (c_n). The sequence (c_n) diverges to $+\infty$. We can show this by fixing $M > 0$. Then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $a_n > M$. If we choose $N^2 > M$ then we have,

$$a_n = n^2 \geq N^2 > M$$

So (c_n) diverges to ∞. □

Proof (d_n). The sequence (d_n) converges to $\frac{6}{7}$. Here I will use the fact that $\lim_{n\to\infty} \frac{1}{n} = 0$ and the arithmetic operations of sequences.

$$\begin{aligned} d_n &= \frac{6n+4}{7n-3} \\ &= \frac{6 + 4(1/n)}{7 - 3(1/n)} \end{aligned}$$

Now,

$$\begin{aligned} \lim_{n\to\infty} d_n &= \lim_{n\to\infty} \frac{6+4(1/n)}{7-3(1/n)} \\ &= \frac{6+4(0)}{7-3(0)} \\ &= \frac{6}{7} \end{aligned}$$ □

Showing that $(d_n) \to \frac{6}{7}$.

Problem 7.

(a) Let (s_n) be a sequence such that

$$|s_{n+1} - s_n| < 2^{-n} \quad \forall n \in \mathbb{N}$$

Prove (s_n) is a Cauchy sequence and hence a convergent sequence.

(b) Is the results in (a) true if we only assume $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$

Proof (a). To make this a bit nicer, let me first show a *Lemma* involving a summation of inverse powers of two.

Lemma. Consider, for $n, i \in \mathbb{N}$,

$$\sum_{i=1}^{n} \frac{1}{2^i}$$

After a close look at the first few terms, a pattern begins to form. I took the guess that the sum evaluates to $\frac{2^p-1}{2^p}$. Let's prove it using induction.

Base: For $n = 1$ we have,

$$\sum_{i=1}^{1} \frac{1}{2^i} = \frac{1}{2} = \frac{2-1}{2}$$

which is true.

Next, assume the statement is true for n.

Induction: We want to show that $\sum_{i=1}^{n+1} \frac{1}{2^i} = \frac{2^{n+1}-1}{2^{n+1}}$. Start with the sum,

$$\begin{aligned}
\sum_{i=1}^{n+1} \frac{1}{2^i} &= \frac{1}{2^{n+1}} + \sum_{i=1}^{n} \frac{1}{2^i} \\
&= \frac{1}{2^{n+1}} + \frac{2^{n-1}}{2^n} \\
&= \frac{1}{2^{n+1}} + \frac{2(2^n-1)}{2^{n+1}} \\
&= \frac{2^{n+1}-1}{2^{n+1}}
\end{aligned}$$

Thus we have proven the statement about the summation. $\square$

We have (s_n) defined to be such that,

$$|s_{n+1} - s_n| < 2^{-n}$$

Fix $\epsilon > 2^{-(N-1)} \frac{2^p-1}{2^p} > 0$. Then $\exists N \in \mathbb{N}$ such that $\forall n, m \geq N$, $|s_m - s_m| < \epsilon$. Without loss of generality, let

m be arbitrarily larger than n by letting $m = n + p$ for any $p \in \mathbb{N}$. Thus,

$$\begin{aligned}
|s_m - s_n| &= |s_{n+p} - s_n| \\
&= |s_{n+p} - s_{n+p-1} + s_{n+p-1} - s_n| \\
&\leq |s_{n+p} - s_{n+p-1}| + |s_{n+p-1} - s_n| \\
&< 2^{-n+p-1} + |s_{n+p-1} - s_n| \\
&= 2^{-n+p-1} + |s_{n+p-1} - s_{n+p-2} + s_{n+p-1} - s_n| \\
&\leq 2^{-(n+p-1)} + |s_{n+p-1} - s_{n+p-2}| + |s_{n+p-1} - s_n| \\
&< 2^{-(n+p-1)} + 2^{-(n+p-2)} + |s_{n+p-1} - s_n|
\end{aligned}$$

We can continue in this fashion, and ultimately,

$$\begin{aligned}
|s_m - s_n| &< 2^{-(n+p-1)} + 2^{-(n+p-2)} + \ldots + 2^{-(n+1)} + 2^{-n} \\
&= 2^{-(n-1)} \left(2^{-p} + 2^{-(p-1)} + \ldots + 2^{-2} + 2^{-1}\right)
\end{aligned}$$

But by the *Lemma* above, $0 < 2^{-p} + 2^{-(p-1)} + \ldots + 2^{-2} + 2^{-1} < 1$. In fact, it is equal to $\frac{2^p - 1}{2^p}$. Thus,

$$\begin{aligned}
|s_m - s_n| &< 2^{-(n-1)} \left(2^{-p} + 2^{-(p-1)} + \ldots + 2^{-2} + 2^{-1}\right) \\
&= 2^{-(n-1)} \frac{2^p - 1}{2^p} \\
&\leq 2^{-(N-1)} \frac{2^p - 1}{2^p} \\
&< \epsilon
\end{aligned}$$

□

Thus we know the sequence is Cauchy.

Notice: The value of N needed also depends on how much larger m is than n. This is why p shows up in the definition. In my mind this just dictates how we choose N. I believe I could rid of p entirely by letting epsilon be defined in terms of N differently.

Proof (b). Here we again want to show that this sequence is Cauchy. Thus, $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall n, m \geq N$, $|s_m - s_n| < \epsilon$. Without loss of generality, let $m = n + p$ where $p \in \mathbb{N}$ and fix $\epsilon > 1 + \frac{p}{N}$. Thus we have,

$$\begin{aligned}
|s_m - s_n| &= |s_{n+p} - s_n| \\
&= |s_{n+p} - s_{n+p-1} + s_{n+p-1} - s_n| \\
&\leq |s_{n+p} - s_{n+p-1}| + |s_{n+p-1} - s_n| \\
&< \frac{1}{n+p-1} + |s_{n+p-1} - s_{n+p-2} + s_{n+p-2} - s_n| \\
&\leq \frac{1}{n+p-1} + |s_{n+p-1} - s_{n+p-2}| + |s_{n+p-2} - s_n| \\
&< \frac{1}{n+p-1} + \frac{1}{n+p-2} + |s_{n+p-2} - s_{n+p-3} + s_{n+p-3} - s_n|
\end{aligned}$$

If we continue in this fashion,

$$
\begin{aligned}
&< \frac{1}{n+p-1} + \frac{1}{n+p-2} + \ldots + \frac{1}{n+1} + \frac{1}{n} \\
&\leq \frac{n+p}{n} \\
&= 1 + \frac{p}{n} \\
&< 1 + \frac{p}{N} \\
&< \epsilon
\end{aligned}
$$

Thus the sequence is in fact Cauchy. □