ColoState Spring 2018 MATH 617 Assignment 1

Due Wed. 01/31/2018

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(15 points) *Problem 1.* Let X, Y be fixed nonempty sets, f be a mapping from X to Y, $A, A_n (n \in \mathbb{N})$ subsets of X, and $B, B_n (n \in \mathbb{N})$ subsets of Y.

(i) Prove that $f(\bigcap_{n\in\mathbb{N}} A_n) \subseteq \bigcap_{n\in\mathbb{N}} f(A_n)$.

(ii) Give an example for (i) such that it is indeed a proper subset.

(iii) Prove that $f^{-1}(\bigcap_{n\in\mathbb{N}} B_n) = \bigcap_{n\in\mathbb{N}} f^{-1}(B_n)$.

(iv) Give an example such that $A \subsetneq f^{-1}(f(A))$.

(15 points) *Problem 2.* Study set operations

(i) Let $A_n = \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right]$ for $n \in \mathbb{N}$. Find $\bigcup_{n\in\mathbb{N}} A_n$ and $\bigcap_{n\in\mathbb{N}} A_n$.

(ii) Let $f(x)$ be a real-valued function defined on a subset E of $\mathbb{R}$. Prove that

$$\{x \in E : f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \left\{x \in E : f(x) \geq \alpha + \frac{1}{n}\right\}$$

holds for any $\alpha \in \mathbb{R}$.

(20 points) *Problem 3.* Give an example of an uncountable null set.

(15 points) *Problem 4.* Prove that a countable union of null sets is still a null set.

(15 points) *Problem 5.* Let f be a nonnegative continuous function defined on $[a, b]$. Prove that if the Riemann integral $\int_a^b f(x)dx = 0$, then $f(x) \equiv 0$.

(20 points) *Problem 6.* Prove that any bounded monotone function on a closed finite interval is Riemann integrable.

Problem 1. Let X, Y be fixed nonempty sets, f be a mapping from X to Y, $A, A_n (n \in \mathbb{N})$ subsets of X, and $B, B_n (n \in \mathbb{N})$ subsets of Y.

(i) Prove that $f(\bigcap_{n\in\mathbb{N}} A_n) \subseteq \bigcap_{n\in\mathbb{N}} f(A_n)$.

(ii) Give an example for (i) such that it is indeed a proper subset.

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Proof.

(i) Let $y \in f\left(\bigcap_{n\in\mathbb{N}} A_n\right)$. Thus there exists (possibly many) $x \in \bigcap_{n\in\mathbb{N}} A_n$ so that $f(x) = y$. By definition of the intersection, this implies that $x \in A_n$ for every $n \in \mathbb{N}$ and hence $y \in f(A_n)$ for every n. Thus $y \in \bigcap_{n\in\mathbb{N}} f(A_n)$ and we have $f\left(\bigcap_{n\in\mathbb{N}} A_n\right) \subseteq \bigcap_{n\in\mathbb{N}} f(A_n)$.

(ii) For all $n \in \mathbb{N}$ let $A_n = \left[\frac{-1}{n}, \frac{1}{n}\right]$ so that $\bigcap_{n\in\mathbb{N}} A_n = \{0\}$. Define $f \colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Then we have that $f\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \{0\}$ and $\bigcap_{n\in\mathbb{N}} f(A_n) = \{-1, 0, 1\}$ and hence $f\left(\bigcap_{n\in\mathbb{N}} A_n\right) \subset \bigcap_{n\in\mathbb{N}} f(A_n)$.

(iii) Let $x \in f^{-1}\left(\bigcap_{n\in\mathbb{N}} B_n\right)$. Then there exists $y \in B_n$ for all $n \in \mathbb{N}$ so that $f^{-1}(y) = x$ and hence $x \in f^{-1}(B_n)$ for every n. Thus we have $x \in \bigcap_{n\in\mathbb{N}} f^{-1}(B_n)$ showing that $f^{-1}\left(\bigcap_{n\in\mathbb{N}} B_n\right) \subseteq \bigcap_{n\in\mathbb{N}} f^{-1}(B_n)$. For the other containment we let $x \in \bigcap_{n\in\mathbb{N}} f^{-1}(B_n)$. This means that we have $y \in B_n$ satisfying $f^{-1}(y) = x$ for every $n \in \mathbb{N}$. Hence, $y \in \bigcap_{n\in\mathbb{N}} B_n$ and we have that $x \in f^{-1}\left(\bigcap_{n\in\mathbb{N}} B_n\right)$. Thus $f^{-1}\left(\bigcap_{n\in\mathbb{N}} B_n\right) \subseteq \bigcap_{n\in\mathbb{N}} f^{-1}(B_n)$. Both containments then show $f^{-1}(\bigcap_{n\in\mathbb{N}} B_n) = \bigcap_{n\in\mathbb{N}} f^{-1}(B_n)$.

(iv) Let $f \colon \mathbb{R} \to \mathbb{R}$ and let $A = [0, 1]$. Note that $f(A) = [0, 1]$ and $f^{-1}(f(A)) = f^{-1}([0, 1]) = [-1, 1]$. Thus $A \subsetneq f^{-1}(f(A))$. $\square$

Problem 2. Study set operations

(i) Let $A_n = \left[-1+\frac{1}{n}, 1-\frac{1}{n}\right]$ for $n \in \mathbb{N}$. Find $\bigcup_{n\in\mathbb{N}} A_n$ and $\bigcap_{n\in\mathbb{N}} A_n$.

(ii) Let $f(x)$ be a real-valued function defined on a subset E of $\mathbb{R}$. Prove that

$$\{x \in E : f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \left\{x \in E : f(x) \geq \alpha + \frac{1}{n}\right\}$$

holds for any $\alpha \in \mathbb{R}$.

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Proof.

(i) First we consider $\bigcup_{n\in\mathbb{N}} A_n$. Let $x \in (-1,1)$. Then $\exists N \in \mathbb{N}$ so that $x \in \bigcup_{n=1}^{N} A_n = \left[-1+\frac{1}{N}, 1-\frac{1}{N}\right]$. Suppose x with $|x| \geq 1$ is in $\bigcup_{n\in\mathbb{N}} A_n$. Then $\exists N \in \mathbb{N}$ so that $x \in \left[-1+\frac{1}{N}, 1-\frac{1}{N}\right]$. But, for any N, we have that $\left|-1+\frac{1}{N}\right| \leq 1$ and $\left|1-\frac{1}{N}\right| \leq 1$ which means that $x \notin \left[-1+\frac{1}{n}, 1-\frac{1}{n}\right]$ for any n and hence $\bigcup_{n\in\mathbb{N}} A_n = (-1,1)$.

Next, note that $0 \in A_1$ and since $A_n \supset A_{n+1}$ for every $n \in \mathbb{N}$, we have that $0 \in \bigcap_{n\in\mathbb{N}} A_n$. Suppose some $x \neq 0$ is in $\bigcap_{n\in\mathbb{N}} A_n$, then $x \notin A_1$ since $A_1 = \{0\}$ and hence $\bigcap_{n\in\mathbb{N}} A_n = \{0\}$.

(ii) If we let $p \in \{x \in E \colon f(x) > \alpha\}$ then we have that $f(p) - \alpha > 0$. Hence by the archimedean property, $\exists N \in \mathbb{N}$ so that $f(p) - \alpha > \frac{1}{N}$. Thus we have that

$$\{x \in E \colon f(x) > \alpha\} \subseteq \bigcup_{n=1}^{\infty} \left\{x \in E \colon f(x) \geq \alpha + \frac{1}{n}\right\}.$$

Next, let $p \in \bigcup_{n=1}^{\infty} \left\{x \in E \colon f(x) \geq \alpha + \frac{1}{n}\right\}$. Then we have that $f(p) - \alpha \geq \frac{1}{N}$ for $N \in \mathbb{N}$. This means that $f(p) > \alpha$ and hence

$$\{x \in E \colon f(x) > \alpha\} \supseteq \bigcup_{n=1}^{\infty} \left\{x \in E \colon f(x) \geq \alpha + \frac{1}{n}\right\}.$$

Finally, both containments show equality between sets. $\square$

Problem 3. Give an example of an uncountable null set.

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Proof. Consider the Cantor set K taken by removing the middle $1/3$ open interval from $[0,1]$ and repeat this process for the remaining closed intervals at every step and continuing for infinitely many steps. We will show two major qualities: the Cantor set is uncountable and this set is also a null set.

To see that the Cantor set K is uncountable, note that we represent a point in the Cantor set by using a ternary representation. The ternary representation returns a decimal number that is given by $\sum_{n=1}^{\infty} a_n \cdot 10^{-n}$ where a_n is a sequence of numbers in the set $T = \{0, 1, 2\}$. If x has a representation a_n is either 0 or 2 that means that at the nth iteration, x is in the nth step of creating the Cantor set. However, if $a_n = 1$, then x was in the interval removed at that nth iteration. To restate, if $x \in K$, then each a_n for the ternary representation of x is either 0 or 2. If $x \notin K$, then at some finite $N \in \mathbb{N}$, a_N for the ternary representation is 1 and all subsequent a_n are 1 as well. This means the ternary representation of a member of K is in one to one correspondence with a binary representation of a number in $[0,1]$ and hence K must be uncountable.

Now, to see that K is a null set. Let I_i^n be the ith remaining interval at step n and denote J_i^n be the ith open interval removed at step n. Note that there will be 2^n I_i^n at step n. Then $K = \cap_{n=1}^{\infty} \cup_{i=1}^{2^n} I_i^{(n)}$. Then, the length of all of these intervals at the Nth step is given by

$$\sum_{n=1}^{2^N} \left(\frac{1}{3}\right)^N = \left(\frac{2}{3}\right)^N.$$

Finally, for any $\epsilon > 0$ there exists an $N \in \mathbb{N}$ sufficiently large so that we have $(2/3)^N < \epsilon$. Hence, K is a null set. ☐

Problem 4. Prove that a countable union of null sets is still a null set.

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Proof. Let $\{A_n\}_{n\in\mathbb{N}}$ be a countable collection of null sets and fix $\epsilon > 0$. Then for each A_n we have a countable union of intervals $\{I_{n_i}\}_{i\in\mathbb{N}}$ such that $A_n \subseteq \bigcup_{i\in\mathbb{N}} I_{n_i}$ and that $\sum_{i=1}^{\infty} \lambda\left(I_{n_i}\right) < \frac{\epsilon}{2^i}$ by choosing I_{n_i} sufficiently small for each i. Then we have that

$$\sum_{n=1}^{\infty}\sum_{i=1}^{\infty} \lambda\left(I_{n_i}\right) = \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} \leq \epsilon,$$

which shows that $\{A_n\}_{n\in\mathbb{N}}$ is a null set. $\square$

Problem 5. Let f be a nonnegative continuous function defined on $[a, b]$. Prove that if the Riemann integral $\int_a^b f(x)dx = 0$, then $f(x) \equiv 0$.

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Proof. We will show the contrapositive. Suppose that $f(x) \neq 0$ for some x_0. Then since f is nonnegative, we have that $f(x_0) > 0$ and since f is continuous, there exists a $\delta > 0$ so that for $x \in (x_0 - \delta, x_0 + \delta) \subset [a, b]$ we have $f(x) > 0$. Then note that continuity of f implies that f is Riemann integrable and we have that $\int_{x_0-\delta}^{x_0+\delta} f(x)dx > 0$. It then follows that

$$0 > \int_{x_0-\delta}^{x_0+\delta} f(x)dx \leq \int_a^b f(x)dx.$$

Thus if $\int_a^b f(x)dx = 0$ we must have that $f(x) \equiv 0$. $\square$

Problem 6. Prove that any bounded monotone function on a closed finite interval is Riemann integrable.

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Proof. Let f be a bounded monotone function on the closed finite interval $[a, b]$. Without loss of generality, assume that f is monotone increasing so that $f(x) \geq f(y)$ when $x > y$. We will see where this extra assumption is used and explain why it's safe to do this. Let P_n be a regular partition of $[a, b]$ so that $x_i - x_{i-1} = (b-a)/n$. Then

$$U(P_n, f) = \sum_{k=1}^{n} f\left(a + \frac{k(b-a)}{n}\right)\left(\frac{b-a}{n}\right) L(P_n, f) = \sum_{k=1}^{n} f\left(a + \frac{(k-1)(b-a)}{n}\right)\left(\frac{b-a}{n}\right).$$

Note, if f was monotone decreasing, we just switch the k for the $k-1$ in the two sums previously. Now, we have that

$$\begin{aligned}
\lim_{n\to\infty} U(P_n, f) - L(P_n, f) &= \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{\infty} f\left(a + \frac{k(b-a)}{n}\right) - \left(a + \frac{(k-1)(b-a)}{n}\right) \\
&= \lim_{n\to\infty} \frac{b-a}{n} f(b) - f(a) = 0.
\end{aligned}$$

Thus f is Riemann integrable. □