MATH 517, Homework 8

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Solutions

Problem 1. (Rudin 8.1) Define

$$f(x) := \begin{cases} e^{-1/x^2} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Prove that f is infinitely differentiable at $x = 0$ and that $f^{(n)}(0) = 0$ for $n = 1, 2, \ldots$. In particular, this means that the Taylor series for f is the 0 function, even though f is very much nonconstant.

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Proof. We will show this by induction. For the base case, we show that f is differentiable at $x = 0$ and specifically that $f'(0) = 0$. Consider then

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{e^{-1/h^2} - 0}{h} \\ &= \lim_{h \to 0} \frac{\frac{1}{h}}{\frac{1}{e^{-1/h^2}}} \\ &= \lim_{h \to 0} \frac{\frac{-1}{h^2}}{\frac{-2e^{1/h^2}}{h^3}} && \text{via L'Hopitals rule} \\ &= \lim_{h \to 0} \frac{h}{2e^{1/h^2}} \\ &= 0. \end{aligned}$$

Note that we have $\lim_{h \to 0} h^k e^{-1/h^2} = 0$ for all integer values of k. If $k \in \mathbb{Z}$ and $k \geq 0$ then

$$\lim_{h \to 0} h^k e^{-1/h^2} = 0.$$

When $k < 0$ then we show this using L'Hopitals rules as before and put h^k in the denominator with positive power $p = -k$. So

$$\begin{aligned} \lim_{h \to 0} \frac{e^{-1/h^2}}{h^p} &= \lim_{h \to 0} \frac{\frac{1}{h^p}}{\frac{1}{e^{-1/h^2}}} \\ &= \lim_{h \to 0} \frac{\frac{-p}{h^{p+1}}}{\frac{-2e^{-1/h^2}}{h^3}} && \text{via L'Hoptials} \\ &= \lim_{h \to 0} \frac{\frac{p}{2h^{p-2}}}{\frac{1}{e^{-1/h^2}}} \\ &\vdots && \text{repeat L'Hopitals rule} \\ &= \lim_{h \to 0} C h^r e^{-1/h^2} = 0 && \text{by repeated L'Hopitals rule, } r \geq 0 \text{ and } C \text{ is a constant.} \end{aligned}$$

Next, assume that $f^{(n-1)}(0) = 0$. Then we wish to show that $f^{(n)}(0) = 0$. Consider

$$\begin{aligned}
f'(x) &= \frac{2e^{-1/x^2}}{x^3} \\
f''(x) &= \frac{e^{-1/x^2}(4 - 6x^2)}{x^6} \\
f^{(3)}(x) &= \frac{4e^{-1/x^2}(6x^4 - 9x^2 + 2)}{x^9} \\
&\vdots && \text{continue this process} \\
f^{(n)}(x) &= \frac{e^{-1/x^2} P_1(x)}{x^{3n}}.
\end{aligned}$$

Note that $P_1(x)$ is a polynomial in x with degree less than $3n$. Thus

$$\lim_{x \to 0} f^{(n)}(x) = 0, \qquad \square$$

by what we showed above. Thus, by induction, we have that f is infinitely differentiable at $x = 0$.

Problem 2. (Rudin 8.6) Suppose $f(x)f(y) = f(x+y)$ for all $x, y \in \mathbb{R}$.

(a) Assuming f is differentiable and not the zero function, prove that $f(x) = e^{cx}$ for some constant c.

(b) Prove the same thing, but now only assuming that f is continuous and not the zero function. (Of course this implies (a), but you should give the easy proof of (a) first.)

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Proof (a). Suppose f is differentiable and not the zero function and that $f(x)f(y) = f(x+y)$. It then follows that $f(0) = 1$ since we have $f(x) = f(x+0) = f(x)f(0)$. Next, consider

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x)f(h)-f(x)}{h} \\
&= f(x) \lim_{h\to 0} \frac{f(h)-1}{h} \\
&= cf(x) && \text{since } f \text{ is differentiable, } f'(0) \text{ exists and we say } f'(0) = c.
\end{aligned}$$

The last statement shows that f is also analytic on all of $\mathbb{R}$ since the derivative f' at any point x is $cf(x)$. Then to see that this shows $f(x) = e^{cx}$ we just observe that the Taylor series centered at $x = 0$ for f is

$$f(x) = \sum_{n=0}^{\infty} \frac{(cx)^n}{n!} = e^{cx}. \qquad \square$$

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Proof (b). Anything from (a) I'll take as I need without reproving them. Also note that $f(x) > 0$ for every $x \in \mathbb{R}$. To see this, note that f is a real valued function and we have

$$\begin{aligned}
f(x) &= f\left(\frac{x}{2} + \frac{x}{2}\right) \\
&= f\left(\frac{x}{2}\right) f\left(\frac{x}{2}\right) \\
&= f\left(\frac{x}{2}\right)^2 \geq 0.
\end{aligned}$$

To see that $f(x) \neq 0$ for any x, suppose that for some x_0 we have $f(x_0) = 0$. Then for any other $x \in \mathbb{R}$ we have $x = x_0 + y$ for some y and then $f(x) = f(x_0 + y) = f(x_0)f(y) = 0$. Thus f is the zero function, which is a contradiction of our original supposition. So it follows $f(x) > 0$ for every x.

Now define $g(x) = \log(f(x))$ for every x. By assumption $g(x)$ is continuous for every x and is defined since $f(x) > 0$. Note that $\log(f(x+y)) = \log(f(x)) + \log(f(y))$. Now, referencing Homework 6 Question 2(b) here, we have the following:

First, let $g(1) = c$. Then note $g(0) = g(0+0) = 2g(0)$ which implies $g(0) = 0$. Next we have that $g(0) = g(x-x) = g(x) + g(-x) = 0$ which implies $g(x) = -g(-x)$ for any x. Then let $q \neq 0 \in \mathbb{Q}$. Then $q = \frac{m}{n}$ with $m, n \in \mathbb{Z}$. It follows that

$$g(q) = g\left(\frac{m}{n}\right) = g\left(\frac{1}{n} + \ldots + \frac{1}{n}\right) = g\left(\frac{1}{n}\right) + \ldots + g\left(\frac{1}{n}\right) = mg\left(\frac{1}{n}\right),$$

and it follows that if $q = 1$ then $q = \frac{n}{n}$ so

$$\begin{aligned}
g(1) &= ng\left(\frac{1}{n}\right) \\
\implies \frac{1}{n}f(1) &= g\left(\frac{1}{n}\right).
\end{aligned}$$

It follows that for any $q \in Q$ $g(q) = cq$. In other words, g is a linear function if the inputs are rational (including $q = 0$). So now let $\{x_i\}$ be a sequence of rationals converging to a real number x, then by continuity of f we have that $\lim_{i \to \infty} f(x_i)$ converges to $g(x)$. So we have

$$\begin{aligned} g(x) &= \lim_{i \to \infty} g(x_i) \\ &= \lim_{i \to \infty} cx_i \\ &= cx. \end{aligned} \qquad \square$$

So $g(x) = cx$, which is linear for all reals. Thus $g(x) = \log(f(x))$ is differentiable, which specifically means that $f(x)$ is differentiable. It follows from (a) that $f(x) = e^{cx}$.

Problem 3. (Rudin 8.9)

(a) Let $S_N = 1 + \frac{1}{2} + \cdots + \frac{1}{N}$ be the Nth partial sum of the harmonic series. Prove that

$$\lim_{N\to\infty} (S_N - \log N)$$

exists. (*Hint:* $\log(N+1) - \log(N) = \int_N^{N+1} \frac{1}{t} dt$.)

(The limit is defined to be the Euler-Mascheroni constant, denoted γ, which is approximately $0.5772\ldots$. Whether γ is irrational is an open question.

(b) Approximately how large must m be so that $S_{10^m} > 100$?

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Proof (a). We will show that the sequence $S_n - \log_n$ is bounded and monotonic. First, consider 1 as an upper bound and 0 as a lower bound. To show this, just consider an arbitrary $N \in \mathbb{N}$ and then

$$\begin{aligned} S_n - \log n &= 1 + \frac{1}{2} + \cdots + \frac{1}{N} - \int_1^N \frac{1}{t} dt \\ &= 1 + \frac{1}{2} + \cdots + \frac{1}{N} - \int_{N-1}^N \frac{1}{t} dt - \cdots - \int_1^2 \frac{1}{t} dt \\ &\le 1 + \frac{1}{2} + \cdots + \frac{1}{N} - \frac{1}{N} - \frac{1}{N-1} - \cdots - \frac{1}{2} && \text{since } \int_m^{m+1} \frac{1}{t} dt \ge \frac{1}{n+1} \\ &= 1. \end{aligned}$$

Which is achieved when $N = 1$. Then it follows similarly that 0 is a lower bound by

$$\begin{aligned} S_n - \log n &= 1 + \frac{1}{2} + \cdots + \frac{1}{N} - \int_1^N \frac{1}{t} dt \\ &= 1 + \frac{1}{2} + \cdots + \frac{1}{N} - \int_{N-1}^N \frac{1}{t} dt - \cdots - \int_1^2 \frac{1}{t} dt \\ &\ge 1 + \frac{1}{2} + \cdots + \frac{1}{N} - \frac{1}{N-1} - \frac{1}{N-2} - \cdots - 1 && \text{since } \int_m^{m+1} \frac{1}{t} dt \le \frac{1}{m} \\ &= \frac{1}{N} \ge 0. \end{aligned}$$

Finally we need to show that this sequence is monotonic. So consider an arbitrary $m \in \mathbb{N}$ and consider

$$\begin{aligned} (s_{m+1} - \log(m+1)) - (s_m - \log m) &= (s_{m+1} - s_m) - (\log(m+1) - \log m) \\ &= \frac{1}{m+1} - \int_m^{m+1} \frac{1}{t} dt \\ &\le \frac{1}{m+1} - \frac{1}{m+1} && \text{since } \int_m^{m+1} \frac{1}{t} dt \ge \frac{1}{m+1} \\ &= 0. \end{aligned}$$

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Proof (b). It seems that $\log N$ is a good approximation for S_N. Thus if $S_N > 100$ we must satisfy that $\log(N) = 100$ which means $N = e^{100}$ Then we set $N = 10^m$ and $m = \log_{10} N = \log_{10} e^{100} = \frac{100}{\log 10}$. □

Problem 4. (Rudin 8.14) If $f(x) = (\pi - |x|)^2$ on $[-\pi, \pi]$, prove that

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx$$

and conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

(*Hint:* Feel free to use Theorem 8.14, even though we didn't discuss it in class, and standard facts about integration.)

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Proof. I'm going to use the sin and cos version for finding the Fourier series. So we can find the fourier coefficients a_n and b_n for $n \geq 1$ by the following:

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \\
&= 0 && \text{since } f(x) \text{ is even and } \sin x \text{ is odd,} \\
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \\
&= \frac{2}{\pi} \int_{0}^{\pi} (\pi - x)^2 \cos nx dx && \text{since } f(x) \text{ and } \cos nx \text{ are even} \\
&= \frac{4}{n^2}.
\end{aligned}$$

For a_0 we have

$$\begin{aligned}
a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin 0 dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi - |x|)^2 dx \\
&= \frac{1}{\pi} \int_{0}^{\pi} (\pi - x)^2 dx \\
&= \frac{2\pi^2}{3}.
\end{aligned}$$

Fix $x = 0$ and consider any $t \in [-pi, pi]$. Then with $M = 2\pi$ we have

$$\begin{aligned}
|f(t) - f(0)| &= |(\pi - |t|)^2 - (\pi - |0|)^2| \\
&= |\pi^2 - 2\pi|t| + |t|^2 - \pi^2| \\
&= |t| \cdot ||t| - 2\pi| \\
&\leq 2\pi|t| = M|t|
\end{aligned}$$

Thus we satisfy Theorem 8.14. This implies that $f(x) = \sum_{-\infty}^{\infty} c_m e^{imx}$ on $[-\pi, \pi]$. It then follows that

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx.$$

Then we have

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\begin{aligned}
f(0) = \pi^2 = & \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2} \\
\implies \frac{\pi^2}{6} = & \sum_{n=1}^{\infty} \frac{1}{n^2}.
\end{aligned}
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Then from Theorem 8.16 (Perseval's theorem) we have

$$\frac{1}{\pi}\int_{\pi}^{\pi} |f(x)|^2 dx = \frac{2\pi^4}{9} + 16\sum_{n=\infty}^{\infty} \frac{1}{n^4}.$$

Finding that $\int_{\pi}^{\pi} |f(x)|^2 dx = \frac{2\pi^4}{5}$ we have

$$\frac{\frac{2\pi^4}{5} - \frac{2\pi^4}{9}}{16} = \frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4}. \qquad \square$$