1 Introduction

The Calderón problem for requires reconstructing the conformal structure of a manifold Ω given the Dirchlet-to-Neumann map on the boundary. It has been demonstrated in dimensions up to two in various ways. The method in which we emply is the Boundary Control (BC) method [4].

2 Outline

2.1 Belishev's Complex Idea

The following facts are in use:

(i) (Ω, g) has the complex structure of a Riemann surface; this structure determines the class of metrics conformally equivalent to g;

(ii) the algebra $\mathcal{A}(\Omega)$ of functions continuous in Ω and holomorphic in $\mathrm{int}(\Omega)$ is nontrivial; functions $w \in \mathcal{A}(\Omega)$ (as local homeomorphisms $\Omega'' \to \mathbb{C}$) determine the complex structure;

(iii) algebra $\mathcal{A}(\Omega)$ is generic: its (topologized) spectrum is homeomorphic to the manifold, $\mathrm{sp}\mathcal{A}(\Omega) \asymp \Omega$, whereas the algebra itself is identical to its Gelfand transform, $\hat{\mathcal{A}}(\Omega) \equiv \mathcal{A}(\Omega)$;

(iv) the algebra of traces $\mathcal{A}(\Gamma) \coloneqq \{w|_\Gamma \mid w \in \mathcal{A}(\Omega)\}$ is isometrically isomorphic to $\mathcal{A}(\Omega)$ (through the map $\mathrm{tr}\colon w'' \to w|_\Gamma$); the isometry yields $\mathrm{sp}\mathcal{A}(\Gamma) \asymp \mathrm{sp}\mathcal{A}(\Omega)$, $\hat{\mathcal{A}}(\Gamma) \equiv \hat{\mathcal{A}}(\Omega)$ that leads to $\mathrm{sp}\mathcal{A}(\Gamma) \asymp \Omega$ and $\hat{\mathcal{A}}(\Gamma) \equiv \mathcal{A}(\Omega)$;

(v) the algebra $\mathcal{A}(\Gamma)$ is determined by the DN-map Λ_g.

To solve the Calderon problem we use these facts in reverse order:

(a) from the operator Λ_g, one recovers the trace algebra $\mathcal{A}(\Gamma)$;

(b) Finding its spectrum and Gelfand transform, we get $\Omega \asymp \mathrm{sp}\mathcal{A}(\Gamma)$ and $\mathcal{A}(\Omega) \equiv \hat{\mathcal{A}}(\Gamma)$;

(c) Using functions of the algebra $\mathcal{A}(\Omega)$ we endow Ω with the complex structure;

(d) introducing a metric g on Ω conformal to the complex structure we get the manifold (Ω, g) whose DN-map coincides with Λ_g by construction.

This procedure (a)-(d) gives a canonical representative of the class of conformally equivalent manifolds which has the given DN-map. The assertion of Theorem 1 is a simple corollary of determinacy of this procedure.

2.2 Rephrasing with the Geometric Algebra

3 Harmonic Functions and Fields

Let (Ω, g) be a oriented 2-dimensional compact Riemannian manifold with boundary Γ. Then the set of ***Harmonic functions*** on Ω is

$$\mathrm{Harm}_g\Omega := \{u \mid \Delta_g = 0 \text{ in } \mathrm{int}\Omega\}$$

4 Notes

The key seems to be that for a complex function $\Psi = u(z) + iv(z)$ that $\frac{\partial}{\partial z}\Psi = 0$ implies that both $u(z)$ and $v(z)$ are harmonic functions. This is true for even grade multivectors f as well. So, if we take $f = f_0 + f_2$ then $\boldsymbol{\nabla} f = 0$ (i.e., f is ***monogenic***) means that f_0 and f_2 are harmonic. Hence, by knowing f is monogenic and knowing either f_0 or f_2, we know f all together. So the goal is then to recover the algebra of monogenic functions on Ω via this connection.

In section 6.3 of Geometric Algebra for Physicists, the eigenvalue equation arises. This might connect nicely to the spectral theory somehow.

We can talk about the notation for taking derivatives which is:

- In the absence of brackets, $\boldsymbol{\nabla}$ acts on the object to its immediate right.
- When $\boldsymbol{\nabla}$ is followed by brackets, the derivative acts on all of the terms in the brackets.
- When the $\boldsymbol{\nabla}$ acts on a multivector to which it is not adjacent, we use overdots to describe the scope.

To construct the ***algebra of monogenic multivectors*** we can take two monogenic multivectors A and B and then we have

$$\begin{aligned}\boldsymbol{\nabla}(AB) &= \boldsymbol{\nabla} AB + \dot{\boldsymbol{\nabla}} A\dot{B} \\ &= 0\end{aligned}$$

, since $\boldsymbol{\nabla} A = \boldsymbol{\nabla} B = 0$. Then let $\mathcal{M}(\Omega)$ be the algebra of monogenic multivectors on Ω.

We also want to show that if a even field f is monogenic, then its components are harmonic. So take $f = f_0 + f_2$ to be monogenic, then

$$\boldsymbol{\nabla} f = \boldsymbol{\nabla} f_0 + \boldsymbol{\nabla} f_2 = 0.$$

Thus we have

$$(\delta + d)f_0 + (\delta + d)f_2 = df_0 + \delta f_2 + df_2 = 0.$$

Hence it must be that

$$df_0 = -\delta f_2 \qquad \text{and} \qquad df_2 = 0,$$

which are like the CREs (shown below also). Then we want to show that

$$(d\delta + \delta d) f_0 = 0 \qquad \text{and} \qquad (d\delta + \delta d) f_2 = 0.$$

For the first we have $\delta f_0 = 0$ and this means that we just want

$$\delta d f_0 = 0$$

which is true since we know that

$$df_0 = -\delta f_2 \implies \delta d f_0 = -\delta^2 f_2 = 0.$$

So f_0 is harmonic. Then for f_2 we have that $df_2 = 0$, and hence we need only show that

$$d\delta f_2 = 0$$

which is true since

$$\delta f_2 = df_0 \implies d\delta f_2 = d^2 f_0 = 0.$$

Thus, if f is an even monogenic field, its components are harmonic.

5 Set Up

Let (M, g) be an n-dimensional smooth Riemannian manifold. Then we can consider an induced Clifford bundle $C\ell(TM, g)$ on M with the quadratic form $Q(v) = g(v, v)$. This is a natural choice, but not necessarily the "correct" one. Specifically, we define $C\ell(T_pM, g_p)$ to be the Clifford algebra on the tangent space T_pM and let

$$C\ell(TM, g) := \dot{\bigcup}_{p \in M} C\ell(T_pM, g_p)$$

be the Clifford bundle.

We can then define the space of Clifford sections by noting we have a natural projection

$$\pi \colon C\ell(TM, g) \to M$$

that maps a Clifford element to the point at which it is based and putting

$$\Gamma C\ell(TM, g) := \{\sigma \colon M \to C\ell(TM, g) \mid \pi \circ \sigma = \mathrm{Id}_M\}.$$

Question 5.1. There's probably some notation for this somewhere. How do we assure that this is a *smooth* section?

Question 5.2. When is this product operation smooth? Is it always smooth? What does smoothness really mean here?

Belishev is looking at quaternionic functions that satisfy both $\Delta\alpha = 0$ and $\vec{\Delta} u = 0$ in the weak sense. This should be easier to state with the below set up.

Consider a 3-dimensional Riemannian manifold (M, g) with the associated ***Clifford bundle*** $\mathrm{C}\ell(TM)$ generated from the metric g. We call a $f \in \Gamma(\mathrm{C}\ell(TM))$ a ***Clifford field*** or just field.

Problem 5.1. Problem statement to recover g from Λ and algebra of "holomorphic" functions.

Let $i\colon \partial M \to M$ be inclusion of the boundary in M.

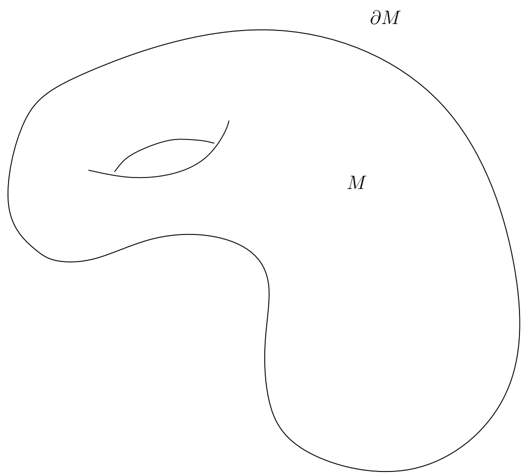

Figure 1: I fixed the caption

Think of even grade (spinor?) fields $f \in \Gamma(C\ell(TM))$ as (maybe?) the voltage and current flux multivector. That is

$$\langle f \rangle_0 = f_0 \qquad \text{and} \qquad \langle f \rangle_2 = f_2$$

where the other grades are zero. f_0 could represent the voltage scalar and f_2 the current flux plane field.

Specifically the problem reads:

$$\begin{cases} df = \delta f = 0 & & \text{in } M \\ i^* f = \varphi & i^*(\delta f) = 0 & \text{on } \partial M \end{cases}$$

Do we want f_0 above and not f since that is the scalar part of this problem? The f_2 part comes in when we're talking about conjugate stuff. Belishev defines ***harmonic quaternion fields*** $\{\alpha, u\}$ to be the ones satisfying

$$d\alpha = \star du \qquad \delta u = 0$$

where we're thinking of u as a 1-form (really a purely imaginary quaternion field).

Then the ***Dirichlet to Neumann operator*** for forms gives

$$\Lambda\varphi = i^*(\star d\varphi) = (-1)^{k+1} i^*(\delta \star \varphi).$$

It seems like the idea is we know f_0 solves the Dirchlet problem and we can relate f_2 to f_0 by the CREs. Then f_2 is related to the Neumann data in the problem.

6 Cauchy Riemann Equations

Letting ∇ represent the ***Dirac operator*** or ***geometric derivative*** (which depends on the metric, as we'll see) we have

$$\nabla = e^i \partial_i = d + \delta,$$

where the first equality is induced from a basis on the tangent space e_i and e^i represents the dual basis satisfying $e^i \cdot e_j = \delta^i_j$. The second equality is more coordinate free where d and δ are the ***exterior derivative*** and ***interior derivative*** respectively. One may define these via

$$\nabla$$

$$\begin{aligned} df &= \frac{1}{2}\left(\nabla f - f\nabla\right) \\ \delta f &= \frac{1}{2}\left(\nabla f + f\nabla\right), \end{aligned}$$

If we consider this field to be ***harmonic*** in the interior of M, then

$$\nabla^2 f = 0$$

or more "weakly" (I believe)

$$df = 0 \qquad \text{and} \qquad \delta f = 0.$$

If we have

$$\begin{aligned} 0 = \nabla^2 f &= (d\delta + \delta d) f \\ 0 &= d\delta f_0 + (d\delta + \delta d) f_2 \\ \implies d\delta f_0 &= -(d\delta + \delta d) f_2 \end{aligned}$$

For any test form φ we should have the weak equations

$$\begin{aligned} 0 &= \langle \nabla^2 f, \varphi \rangle & &= \langle (d\delta + \delta d) f, \varphi \rangle \\ &= \langle d\delta f, \varphi \rangle + \langle \delta d f, \varphi \rangle \\ &= \langle \delta f, \delta \varphi \rangle + \langle df, d\varphi \rangle \end{aligned}$$

which means we have that $\delta f = df = 0$.

Using the second requirement for harmonic, we find the "***Cauchy Riemann equations***" are

$$\begin{aligned} 0 &= df = \delta f \\ df_0 + df_2 &= \delta f_0 + \delta f_2 \\ df_0 + df_2 &= \delta f_2, \end{aligned}$$

since $\delta f_0 = 0$. Now, if we match grades we arrive at four equations

$$\begin{array}{cccc} \langle df \rangle_0 = \langle \delta f \rangle_0 & \langle df \rangle_1 = \langle \delta f \rangle_1 & \langle df \rangle_2 = \langle \delta f \rangle_2 & \langle df \rangle_3 = \langle \delta f \rangle_3 \\ 0 = 0 & df_0 = \delta f_2 & 0 = 0 & df_2 = 0. \end{array}$$

The "Cauchy Riemann equations" are then

$$df_0 = \delta f_2$$

and $df_2 = 0$ states that the current flux plane field is closed. (what does this mean physically? Does it make sense? Does it mean that all "currents" form loops?)

7 Dirichlet to Neumann Operator

The ***Dirichlet to Neumann operator*** for differential forms is defined by

$$\Lambda\colon \Omega^k(\partial M) \to \Omega^{n-k-1}\partial M$$

given by

$$\Lambda\varphi = i^* \star d\omega = (-1)^{k+1} i^*(\delta \star \varphi),$$

where φ is the ***Dirichlet data*** on the boundary and $d\omega$ is the ***Neumann data*** on the boundary.

If we define $\star$ for $\Gamma(C\ell(TM))$ we could use the pseudoscalar I at each point (which is global since it is a volume form). The inclusion map is defined in the usual way and d is the exterior derivative.

8 Algebras

Even (spinor?) fields form a subalgebra of $C\ell(TM)$. This is similar to the result for Section 2 in Belishev [3].

Question 8.1. Can we place a norm on it to form it into a Banach algebra?

Maybe we can just take the l^2 norm of each part individually or something (similar to how Belishev does in the same algebra section.

Question 8.2. Is the subspace of harmonic fields an algebra? Or is it at least dense in some way? See Theorem 1 in [3]

Question 8.3. Does this algebra of Clifford sections form a Banach algebra?

Definition 8.1. A *Banach algebra* is an associative algebra $\mathcal{A}$ over $\mathbb{R}$ or $\mathbb{C}$ that at the same time is also a Banach space, i.e. a normed space and complete in the metric induced by the norm. The norm is required to satisfy

$$\forall x, y \in \mathcal{A} : \ \|xy\| \leq \|x\|\|y\|.$$

Question 8.4. If we can make the algebra of Clifford sections a (unital) Banach algebra, can we compute the spectrum? (If it forms a Banach algebra, the scalar element 1 is likely the unit.)

Definition 8.2. The *spectrum* $\sigma(x)$ of $x \in \mathcal{A}$ is the set

$$\sigma(x) \coloneqq \{\lambda \mid x - \lambda 1 \text{ is not invertible.}\}$$

Of course this has all been said before, see `https://en.wikipedia.org/wiki/Clifford_bundle`. It seems that the signs are flipped from mine.

Question 8.5. Does $|Q(v)|$ form a norm on the Clifford algebra fiber? If so, can we extend this to a norm on the manifold?

Definition 8.3. A *norm* $\|\cdot\|$ on a vector space V over a field $\mathbb{F}$ is a map

$$\|\cdot\|\colon V \to \mathbb{F}$$

that satisfies

(i) (Triangle inequality) $\|u+v\| \leq \|u\| + \|v\| \; \forall u, v \in V$,

(ii) (Scalar behavior) $\|\alpha v\| = |\alpha| \|v\| \; \forall v \in V$ and $\alpha \in \mathbb{F}$,

(iii) (Unique zero) $\|v\| = 0 \iff v \equiv 0$.

If $\|\cdot\|$ only satisfies (i) and (ii) then we call $\|\cdot\|$ a *seminorm*.

Proposition 8.1. *Let V be a n-dimensional vector space over a field $\mathbb{F}$, Q be a nondegenerate quadratic form, and $C\ell(V,Q)$ be the Clifford algebra over V with Q, then the function*

$$|Q(\cdot)|\colon V \to \mathbb{F}$$

is a norm on $C\ell(V,Q)$.

Proof. We show the three properties listed above. Let $\{e_1, \dots, e_{2^n}\}$ be an arbitrary basis for $C\ell(V,Q)$. We note that given $v \in C\ell(V,Q)$ we can write

$$v = \sum_{i=1}^{2^n} \alpha_i e_i.$$

We then take A to be a symmetric $2^n \times 2^n$-matrix (with rank 2^n by the nondegeneracy) and define

$$Q(v) \coloneqq \sum_{i=1}^{2^n} \sum_{j=1}^{2^n} A_{ij} \alpha_i \alpha_j.$$

(i) Let $v, u \in C\ell(V,Q)$. Then put

$$v = \sum_{i=1}^{2^n} \alpha_i e_i,$$
$$u = \sum_{i=1}^{2^n} \beta e_i.$$

$$|Q(v+u)| = \left|\sum_{i=1}^{2^n}\sum_{j=1}^{2^n} A_{ij}(\alpha_i+\beta_i)(\alpha_j+\beta_j)\right|$$
$$= \left|\sum_{i=1}^{2^n}\sum_{j=1}^{2^n} A_{ij}(\alpha_i\alpha_j+\beta_i\alpha_j+\alpha_i\beta_j+\beta_i\beta_j)\right|$$

$$|Q(v)|+|Q(u)| = \left|\sum_{i=1}^{2^n}\sum_{j=1}^{2^n} A_{ij}\alpha_i\alpha_j\right| + \left|\sum_{i=1}^{2^n}\sum_{j=1}^{2^n} A_{ij}\beta_i\beta_j\right|$$

□

9 Questions and Thoughts

- Do some searching for "clifford algebras, eit, voltage to current, dirichlet to neumann, ..."

Question 9.1. What is the Dirichlet to Neumann operator?

Question 9.2. If we define the Dirichlet to Neumann operator as in [2], then this only uses the exterior derivative. What if we instead consider the Dirac operator?

Question 9.3. What is the Hilbert transform?

Question 9.4. Can we relate the Hilbert transform to the D to N map?

Question 9.5. Can we recover the algebra of "holomorphic" functions from the above information?

10 Random Things

- `http://math.uchicago.edu/~amathew/dirac.pdf` These seem like useful notes

Question 10.1. Is there some kind of Hodge-Clifford decomposition we can do for clifford sections?

Question 10.2. Is there some kind of Clifford homology/cohomology?

Outside of the realm of the Clifford stuff, what about the "tensor" dirac derivative operator? Consider the operator

$$D = e_i \otimes \nabla_{e_i}$$

so that, for example, on a vector field we achieve

$$DF = e_i \otimes \nabla_{e_i} f^j e_j = e_i \otimes e_j \nabla_{e_i} f^j$$

which we can write as a matrix that we call the Jacobian. This should transform nicely under coordinates more naturally.

Then if we consider this object not in the tensor algebra but in the quotient (the Clifford algebra) we achieve

$$DF = \operatorname{div}(F) + [DF]_2$$

the superalgebra splitting into the div and curl like components.

Then

$$D^2F = (e_1 \otimes \nabla_{e_i})(e_j \otimes \nabla_{e_j})f^k e_k = e_i \otimes e_j \otimes e_k \nabla_{e_i} \nabla_{e_j} f^k,$$

is like the Hessian tensor. In other words, if instead f was a scalar (rank 0), then

$$D^2 f = (e_i \otimes e_j)\nabla_{e_i}\nabla_{e_j} f$$

is the hessian.

References

[1] Doran, C. & Lasenby A. (2003). *Geometric Algebra for Physicists*. Cambridge University Press.

[2] The Complete Dirichlet-to-Neumann Map.

[3] *On Algebraic and uniqueness properties of 3d harmonic quaternion fields*

[4] *The Calderon Problem for Two-Dimensional Manifolds by the BC-Method*