Commutative Banach Algebras of Multivectors from the Scalar Dirichlet-to-Neumann Operator

Colin Roberts

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Abstract

The problem of determining an unknown Riemannian manifold given the Dirichlet-to-Neumann (DN) operator is known as the Calderón problem. One method of solving this problem in the two dimensional case is through the Boundary Control method. There, one uses the DN operator to construct a Banach algebra of holomorphic functions on the manifold. The Gelfand transform of this algebra is then homeomorphic to the manifold. In higher dimensions, we replace the complex field with a Clifford algebra and use the DN operator to determine a $\mathrm{Spin}(n)$ invariant space of monogenic multivector fields. Using a power series representation for monogenic fields, one decomposes the space of monogenics into products of commutative algebras of $(0+2)$-vector fields constant on translations of planes and monogenic in $\mathbb{R}^n$. Using this decomposition, we define spinor characters on the space of monogenic fields that correspond to Dirac measures on the manifold. The set of these Dirac measures is then homeomorphic to the underlying manifold with the Gelfand topology.

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1 Introduction

In 1980, Alberto Calderón proposed an inverse problem in his paper *On an inverse boundary value problem* calderon_inverse_2006 [7] where he asks if one can determine the conductivity matrix of a medium from Cauchy data supplied on the boundary. In dimensions $n > 2$, this is equivalent to determining a Riemannian manfiold up to isometry from the scalar Dirichlet-to-Neumann (DN) operator feldman_calderproblem_nodate, salo_calderon_nodate, uhlmann_inverse_2014 [10, 14, 18]. The DN operator takes any given Dirichlet boundary values and outputs the corresponding Neumann data of a solution to Laplace's equation in order to generate the relevant Cauchy data.

One approach to reconstructing the Riemannian metric in dimension $n = 2$ appears in belishev_calderon_2003 [2], where the author uses the Boundary–Control (BC) method to determine the manifold up to conformal class. Add in a bunch of other citations to the BC method. The BC method takes an algebraic approach. Specifically, the DN operator determines the algebra of holomorphic functions on M and realizes M as homeomorphic to the Gelfand spectrum of this commutative algebra. The metric g is then recovered after providing M with a complex structure. In dimension $n = 2$, the Laplace-Beltrami operator is conformally invariant, and this result cannot be improved. An attempt to generalize this approach to dimension $n = 3$ can be found in by replacing the complex structure with a quaternionic structure but this has not lead to a complete solution belishev_algebras_2017, belishev_algebraic_2019 [3, 4]. It has been shown that when M is the 3-dimensional ball in $\mathbb{R}^3$, there is an associated space of harmonic quaternion fields that has a quaternion spectrum homeomorphic to the ball. But, a connection to the DN operator has not been made, and this method has also not been generalized to higher dimensions.

In this paper, I show that there exists a space of spin characters $\mathfrak{M}$ acting on a $\mathrm{Spin}(n)$ invariant space of monogenic multivector fields on the n-dimensional ball that is homeomorphic to the ball. We then observe that this space of monogenics is determined from the DN map, and thus recover the ball up to homeomorphism from the boundary data. This is summarized in two main theorems.

Theorem 1. *The set of multiplicative $\mathfrak{spin}(n)$-linear functionals on the $\mathrm{Spin}(n)$ invariant space of monogenic fields $\mathcal{M}$ on the n-dimensional ball $\mathbb{B}$ is homeomorphic to $\mathbb{B}$ with the Gelfand topology.*

Theorem 2. *The scalar DN operator determines the $\mathrm{Spin}(n)$ invariant space of monogenic fields on regions in $\mathbb{R}^n$.*

The second theorem can be extended to Riemannian manifolds quite readily.

We first introduce the Clifford algebra setting. Given a vector space with an inner product, we can create the graded Clifford algebra. In particular, we extend these Clifford algebras to Clifford algebra valued functions (or multivector fields) on regions $M \subset \mathbb{R}^n$. Inside the multivector fields sit the even graded multivectors consisting of scalars, bivectors, and other $2k$-vectors. In $\mathbb{R}^2$ with the Euclidean inner product, this space is isomorphic to the $\mathbb{C}$-algebra and so the functions valued in this even sub-Clifford algebra can be thought of as complex valued functions. Clifford analysis generalizes the notion of holomorphicity to monogenicity and we find that monogenic functions lie in the kernel of the Dirac operator $\boldsymbol{\nabla}$ just as $\mathbb{C}$-holomorphic functions lie in the kernel of the Wirtinger derivative $\frac{\partial}{\partial \bar{z}}$. Moreover, one has that $\boldsymbol{\nabla}$ is the square root Laplace-Beltrami operator $\Delta = \boldsymbol{\nabla}^2$. Even monogenic multivector fields are $\mathrm{Spin}(n)$ invariant and each grade is harmonic (in the kernel of Δ).

When M is the n-ball, we have that space of even monogenics $\mathcal{M}$ which can be generated by the algebras of even graded B-planar monogenic biparavector fields (each field constant on translations of the B-plane in $\mathbb{R}^n$). Those generating subalgebras are individually isomorphic to the algebra of holomorphic functions on the complex unit disk $\mathbb{D}$. On these spaces, one can define $\mathfrak{spin}(n)$-linear multiplicative functionals $\mathfrak{M}$, referred to as spin characters. Each spin character is equivalent to a Dirac measure on the n-ball which, with the Gelfand topology, provide a homeomorphic copy of the n-ball.

The space of $(0+2)$-vector monogenics is found from the DN operator in the following sense. The DN operator determines a Hilbert transform on multivector fields that allows one to determine the monogenic conjugate bivector field b corresponding to a scalar solution u to the Laplace equation $\Delta u = 0$ so that $f = u + b$ is monogenic. Haven't actually done this yet Considering all smooth boundary conditions generates the relevant space of monogenics, from which we determine the space of spin characters. Thus, the DN operator provides a means of constructing a homeomorphic of the n-ball.

2 Preliminaries

2.1 Clifford algebras

The complex algebra $\mathbb{C}$ can be generalized in a handful of ways. Some of which can be found through the use of Clifford algebras and, more specifically, in geometric algebras. We define the more general Clifford algebras first and realize geometric algebras as particularly nice Clifford algebras with a quadratic form arising from an inner product. Elements of a geometric algebra are known as multivectors and these multivectors carry a wealth of geometric information in their algebraic structure. $\mathbb{C}$ itself can be realized as a special subalgebra of biparavectors in the geometric algebra on $\mathbb{R}^2$ with the Euclidean inner product and the quaternions $\mathbb{H}$ are realized as an analogous algebra on $\mathbb{R}^3$. In particular, both $\mathbb{C}$ and $\mathbb{H}$ arise as the 2- and 3-dimensional even Clifford groups Γ^+ respectively.

Formally, we let (V, Q) be an n-dimensional vector space V over some field K with an arbitrary quadratic form Q. The tensor algebra is given by

$$\mathcal{T}(V) \coloneqq \bigoplus_{j=0}^{\infty} V^{\otimes j} = K \bigoplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots,$$

where the elements (tensors) inherit a multiplication $\otimes$ (the tensor product). From the tensor algebra $\mathcal{T}(V)$, we can quotient by the ideal generated by $v \otimes v - Q(v)$ to define *Clifford algebra* $C\ell(V, Q)$. That is,

$$C\ell(V, Q) = \mathcal{T}(V) \ / \ \langle v \otimes v - Q(v) \rangle.$$

To see how the tensor product descends to the quotient, we let $e_1,\dots,e_n$ be an arbitrary basis for V, then we can consider the tensor product of basis elements $e_i \otimes e_j$ which induces a product in the quotient $C\ell(V,Q)$ which we refer to as the *Clifford multiplication*. In this basis, we write this product as concatenation e_ie_j and define the multiplication by

$$e_ie_j = \begin{cases} Q(e_i) & \text{if } i=j, \\ e_i \wedge e_j & \text{if } i \neq j, \end{cases}$$

where $\wedge$ is the typical exterior product satisfying $v \wedge w = -w \wedge v$ for all $v, w \in V$. As a consequence, the exterior algebra $\bigwedge(V)$ can be realized as a subalgebra of any Clifford algebra over V or as a Clifford algebra with a trivial quadratic form $Q = 0$.

Note that $C\ell(V,Q)$ is a $\mathbb{Z}$-graded algebra with elements of grade-0 up to elements of grade-n. We refer to grade-0 elements as scalars, grade-1 elements as vectors, grade-2 elements as *bivectors*, grade-k elements as k-*vectors*, and grade-n elements as *pseudoscalars*. We denote the space of k-vectors by $C\ell(V,Q)^k$. For each grade there is a basis of $\binom{n}{k}$ k-*blades* which are k-vectors of the form

$$A_k = \prod_{j=1}^{k} v_j, \text{ for } v_j \in V.$$

For example, if $\dim(V) = 3$, then there are $\binom{3}{2} = 3$ 2-blades that form a basis for the bivectors. One particular choice given our vector basis of V would be the following list of 2-blades

$$B_{12} = e_1 \wedge e_2, \quad B_{13} = e_1 \wedge e_3, \quad B_{23} = e_e \wedge e_3.$$

We refer to an $(n-1)$-blade as a *pseudovector* and it should be noted that every $(n-1)$-vector is a pseudovector. In other literature, some will refer to a k-blade as a *simple* or a *decomposable* k-vector.

In general, an element $A \in C\ell(V,Q)$ is written as a linear combination of basis elements of all possible grades and we refer to A as a *multivector*. To extract the grade-k components of A, we use the notation

$$\langle A \rangle_k$$

to denote the grade-k components of the multivector A. Any multivector A can then be given by

$$A = \sum_{k=0}^{n} \langle A \rangle_k$$

which shows the decomposition

$$C\ell(V,Q) = \bigoplus_{j=0}^{n} C\ell(V,Q)^j.$$

For example, $A \in C\ell(\mathbb{R}^3, \|\cdot\|)$ is given by

$$A = a + \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \beta_{12}B_{12} + \beta_{13}B_{13} + \beta_{23}B_{23} + re_1 \wedge e_2 \wedge e_3$$

in general, and we have

$$\langle A\rangle_0 = a, \quad \langle A\rangle_1 = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3, \quad \langle A\rangle_2 = \beta_{12}B_{12} + \beta_{13}B_{13} + \beta_{23}B_{23}, \quad \langle A\rangle_3 = re_1 \wedge e_2 \wedge e_3.$$

If A contains only grade-k components, then we say that A is *homogeneous*. For example, when we refer to vectors we realize them as homogeneous grade-1 multivectors and likewise we realize bivectors as homogeneous grade-2 multivectors. We also refer elements in

$$C\ell(V,Q)^{0+2} = C\ell(V,Q) \oplus C\ell(V,Q)^2$$

as *biparavectors*.

The Clifford multiplication of vectors can be extended to multiplication of vectors with homogeneous grade-k multivectors. In particular, given a vector $v \in C\ell(V,Q)$ and a homogeneous grade-k multivector $A_k \in C\ell(V,Q)$, we have

$$aA_k = \langle aA_k\rangle_{k-1} + \langle aA_k\rangle_{k+1}, \tag{1}$$

eq:vector_m

which decomposes the multiplication into a grade lowering *interior product* and a grade raising *exterior product*. This allows us to extend the Clifford multiplication further. Given a homogeneous grade-s multivector B_s, we have

$$A_kB_s = \langle A_rB_s\rangle_{|r-s|} + \langle A_rB_s\rangle_{|r-s|+2} + \cdots + \langle A_rB_s\rangle_{r+s}. \tag{2}$$

eq:general_

This rule for multiplication then allows for the multiplication of two general multivectors in $C\ell(V,Q)$.

Some specific graded elements of the above product are worth noting here,

$$A_k \cdot B_s := \langle A_kB_s\rangle_{|k-s|} \tag{3}$$

$$A_k \wedge B_s := \langle A_kB_s\rangle_{k+s} \tag{4}$$

$$A_k \rfloor B_s := \langle A_kB_s\rangle_{s-k} \tag{5}$$

eq:left_co

$$A_k \lfloor B_s := \langle A_kB_s\rangle_{k-s}. \tag{6}$$

eq:right_c

These products are particularly emphasized as many helpful identities used in this paper are phrased using these notions. Another key reason behind the additional multiplication symbols $\rfloor$ and $\lfloor$ is to avoid needing to pay special attention to the specific grade of each multivector in a product. The product $\cdot$ on A_k and B_s depends on k and s and as such given by either $\rfloor$ or $\lfloor$ but one must know k and s in order to define this product exactly.

We also have the identities

$$A_r \cdot B_s = A_r \rfloor B_s \qquad \text{if } k \leq s \tag{7}$$

eq:left_co

$$A_r \cdot B_s = A_r \lfloor B_s \qquad \text{if } k \geq s. \tag{8}$$

eq:right_c

For homogeneous k-vectors A_k and B_k, the products above simplify to

$$A_k \lfloor B_k = A_k \rfloor B_k = A_k \cdot B_s. \tag{9}$$

dot_equiva

Using this notation, for a vector α we have

$$\alpha A_k = \alpha \rfloor A_k + \alpha \wedge A_k, \tag{10}$$

so the $\cdot$ and $\lfloor$ notation coincide for left multiplication by vectors. If we are given two k-blades $A_k = \alpha_1 \wedge \cdots \wedge \alpha_k$ and $B_k = \beta_1 \wedge \cdots \wedge \beta_k$ we have

$$A_k \cdot B_k^\dagger = \det(\alpha_i \cdot \beta_j)_{i,j=1}^k, \tag{11}$$ eq:dot_pro

which is equivalent to $A_k \rfloor B_k$ and $A_k \lfloor B_k$ through dot_equivalent_contraction 9 and this is extended to all k-vectors as is typically seen when constructing the inner product of k-vectors (see hestenes_clifford_1986 [11]. If we are given two bivectors B and B', then we have another special multiplication

$$B \times B' := \langle BB' \rangle_2 = \frac{1}{2}(BB' - B'B), \tag{12}$$ eq:bivecto

which is the grade preserving anti-symmetric portion of the product BB'.

As discussed, $C\ell(V,Q)$ is naturally a $\mathbb{Z}$-graded algebra but we also find that it carries a $\mathbb{Z}/2\mathbb{Z}$-grading as well. This additional grading can be realized by sorting k-vectors in $C\ell(V,Q)$ into the sets where k is even or odd. We say a k-vector is *even* (resp. *odd*) k is even (resp. odd) and in general if a multivector A is a sum of only even (resp. odd) grade elements we also refer to A as even (resp. odd). Taking note of the multiplication defined in eq:general_clifford_multiplication 2, one can see that the multiplication of even multivectors with another even multivectors outputs an even multivector. Thus, the even multivectors form closed subalgebra of $C\ell(V,Q)$ which we denote by $C\ell(V,Q)^+$.

resentation **Example 2.1.**

- Let $V = \mathbb{R}^2$ and let the quadratic form Q be given by the Euclidean norm $Q(\cdot) = \|\cdot\|$. Let e_1 and e_2 be the standard unit vectors and note that we have 1 as the basis scalar, and $B_{12} = e_1 \wedge e_2 = e_1 e_2$ as the basis pseudoscalar. Thus, a general multivector m and r can be written as

$$m = m_0 + m_1 e_1 + m_2 e_2 + m_{12} B_{12}, \qquad r = r_0 + r_1 e_1 + r_2 e_2 + r_{12} B_{12}.$$

We can then multiply mr and find

$$\langle mr \rangle_0 = m_0 r_0 + m_1 r_1 + m_2 r_2 - m_{12} r_{12},$$

$$\langle mr \rangle_1 = (m_0 r_1 + m_1 r_0 - m_2 r_{12} + m_{12} r_2) e_1 + (m_0 r_2 + m_2 r_0 + m_1 r_{12} - m_{12} r_1) e_2,$$

and

$$\langle mr \rangle_2 = (m_1 r_2 - m_2 r_1) B_{12}.$$

Most notably, we see that $B_{12}^2 = -1$ and this allows us to consider a biparavector

$$z = x + y B_{12}$$

as a representation of the complex number $\zeta = x + iy$ in $\mathcal{G}_n^{0+2}$. Thus, the even subalgebra of this Clifford algebra is indeed isomorphic to the complex numbers $\mathbb{C}$.

- If $V = \mathbb{R}^n$, with $n \geq 2$, and with the analogous Q, then there are natural copies of $\mathbb{C}$ contained inside of $C\ell(V,Q)$. In particular, we have the isomorphism

$$\mathbb{C} \cong \{\lambda + \beta B \mid \lambda, \beta \in C\ell(V,Q)^0,\ B \in C\ell(V,Q)^2,\ B^2 = -1.\},$$

which shows that complex numbers arise as biparavectors. Given the standard basis $e_1, \dots, e_n$ we have copies of $\mathbb{C}$ for each of the $\binom{n}{2}$ unit bivectors B_{jk} with $k = 2, \dots, n$ and $j < k$. Note that $B_{jk}B_{jk} = -1$ and we have the representation of $\mathbb{C}$ since

$$\zeta = x + yB,$$

behaves as a complex number $z = x + iy$.

uaternions **Example 2.2.** Let $V = \mathbb{R}^3$ and $Q(\cdot) = \|\cdot\|$. Then, let

$$B_{23} = e_2e_3, \quad B_{31} = e_3e_1, \quad B_{12} = e_1e_2,$$

and note that we can write a even multivector as

$$q = a + \beta_{23}B_{23} + \beta_{31}B_{31} + \beta_{12}B_{12}.$$

Note as well that

$$B_{23}^2 = B_{31}^2 = B_{12}^2 = -1,$$

and

$$B_{23}B_{31}B_{12} = +1.$$

In this case, this even subalgebra is extremely close to being a copy of the quaternion algebra $\mathbb{H}$. Indeed, one can arrive at a representation of the quaternions by taking

$$\boldsymbol{i} \leftrightarrow B_{23}, \quad \boldsymbol{j} \leftrightarrow -B_{31} = B_{13}, \quad \boldsymbol{k} \leftrightarrow B_{12},$$

and noting that we then have $ijk = -1$ as well as $i^2 = j^2 = k^2 = -1$. A more in depth explanation is provided in doran_geometric_2003 [9].

Once again, quaternions arise naturally as parabivectors since we can put

$$q = \alpha + u_1B_{23} - u_2B_{13} + u_3B_{12},$$

and recover the necessary arithmetic seen in $\mathbb{H}$.

In the case where V has a (pseudo) inner product g, we can induce a quadratic form Q by $Q(v) = g(v,v)$ and give rise to a Clifford algebra $C\ell(V,Q)$. This is a special case and we refer to this type of Clifford algebra as a *geometric algebra*. We generally put $\mathcal{G}(V)$ and assume the inner product will be given alongside or will be clear from context. For example, when $V = \mathbb{R}^n$ and we define Q from the Euclidean inner product, we have $C\ell(V,Q) = \mathcal{G}(\mathbb{R}^n)$ and moreover we put $\mathcal{G}(\mathbb{R}^n) = \mathcal{G}_n$. For more information on the topic of geometric algebras see the classical text hestenes_clifford_1986 [11] or the text doran_geometric_2003 [9] which also provides a wide range of applications to physics problems. Both these sources include much of the other necessary preliminaries I cover in the remainder of this section. Finally, the paper chisolm_geometric_2012 [8] proves many of the useful identities I claimed above.

me_algebra **Example 2.3.** If instead we take $V = \mathbb{R}^4$ we take the vector basis e_t, e_1, e_2, e_3 with the pseudo inner products

$$e_t \cdot e_t = -1 \qquad e_t \cdot \gamma_i = 0 \; i = 1,2,3, \quad e_i \cdot e_j = \delta_{ij}.$$

For this basis, we can denote the matrix for this inner product $\eta = \operatorname{diag}(-+++)$ and define Q from η. Then, we have for a vector $A = A_t e_t + A_1 e_1 + A_2 e_2 + A_3 e_3$ we have

$$A \cdot A = -A_t^2 + \sum_{i=1}^{3} A_i^2.$$

For the cases with pseudo inner products with p vectors satisfying $e_i^2 = -1$ for $i = 1, \dots, p$ and q vectors satisfying $e_j^2 = 1$ for $q = p+1, \dots, p+q$, we will denote the algebras by $\mathcal{G}_{p,q}$.

2.1.1 Duality and pseudoscalars

udoscalars

For the remainder of this paper we will be working with geometric algebras with a positive definite inner product g. Given access to an inner product we have a natural isomorphism between V and V^* by the Riesz representation. Namely, given an arbitrary basis e_i for V there exists the dual basis f_i for V^* such that $f_i(e_j) = \delta_{ij}$. This dual basis resides inside V itself in the following manner. There is then a unique map $\sharp \colon V^* \to V$ with $f \mapsto f^\sharp$ such that

$$f_i^\sharp \cdot e_j = \delta_{ij},$$

where δ_{ij} is the Kronecker delta symbol. In terms of the geometric algebra, we put $e^i := f_i^\sharp$ and can note that e^i is simply a vector in the geometric algebra. For an arbitrary basis $e_1, \dots, e_n$ for V, the coefficients for the inner product g are given by $g_{ij} = e_i \cdot e_j$ and we can put $e^i = g^{ij} e_j$ where g^{ij} is the coefficients to matrix inverse of g_{ij}. There is inverse isomorphism $\flat \colon V \to V^*$ given by $e \mapsto e^\flat$ satisfying

$$e_i^\flat(e_j) = \delta_{ij}.$$

Given these identifications, there is no need to distinguish between the vector space V and its dual V^* as it suffices to consider V itself with reciprocal basis elements e^i with the application of the scalar product.

A volume element can be defined by $\mu = e_1 \wedge e_2 \wedge \cdots \wedge e_n = \sqrt{|g|} I$ where $\sqrt{|g|}$ is the square root of the determinant of the matrix g_{ij} and I is the unit pseudoscalar. It follows that the unit pseudoscalar is given by $I = \frac{1}{\sqrt{|g|}} e_1 \wedge e_2 \wedge \cdots e_n$. We can define μ^{-1} such that $\mu^{-1}\mu = 1 = \mu\mu^{-1}$ and analogously I^{-1}. One can equivalently put $e^j = (-1)^{j-1} e_1 \wedge e_2 \wedge \cdots \wedge \breve{e}_j \wedge \cdots \wedge e_n \mu^{-1}$ and note that this gives $\mu^{-1} = e^n \wedge \cdots \wedge e^1$. Conveniently, the unit pseudoscalar satisfies the relation

$$I A_k = (-1)^{k(n-1)} A_k I.$$

Thus, I commutes with the even subalgebra, and anticommutes with the odd subalgebra. Moreso, the pseudoscalar allows one to exchange the interior and exterior products as

$$(A_k \wedge B_s) I = A_k \cdot (B_s I) \tag{13}$$ eq:wedge_t

for homogeneous k and s-vectors A_k and B_s. The above holds true if we replace I with I^{-1} when working in spaces where g is positive definite due to the fact that I^{-1} differs only by a sign. If $B_s = C_{n-s}I$ then,

$$(A_k \cdot B_s)I^{-1} = A_k \cdot (C_{n-s}I) = (A_k \wedge C_{n-s})I = (A_k \wedge (B_sI))I,$$

and in particular

$$(A_k \cdot B_s)I^{-1} = A_k \wedge (B_sI). \tag{14}$$

eq:dot_to_

This shows the duality between the inner and exterior products. The duality extends further to provide an isomorphism between the spaces of k-vectors and $(n-k)$-vectors. For any k-vector A_k, we can take $A_kI^{-1} = B_{n-k}$ to get the corresponding $(n-k)$-vector B_{n-k}. It is under this isomorphism one can see that all pseudovectors are $(n-1)$-blades.

ss_product

Example 2.4. Consider $\mathcal{G}_3$ with the standard orthonormal vector basis $e_1, \ldots, e_n$ and Euclidean inner product. Then, we can define the *cross product* of two vectors u and v by

$$u \times v = (u \wedge v)I^{-1}.$$

The special fact of $\mathcal{G}_3$ is that vectors and bivectors (pseudoscalars in 3-dimensions) are dual to one another. One can also note that the vector $w = u \times v$ is sometimes refered to as axial and in other cases the pseudovector $u \wedge v$ is referred to as axial.

The $\times$ symbol is now overloaded from the bivector definition we saw prior to this example. But, referring back to Example ex:quaternions 2.2, we can realize the cross product of vectors as the antisymmetric product of bivectors

$$(uI^{-1}) \times (vI^{-1}).$$

The necessary relationships for the cross product are seen clearly on the products of the basis blades B_{23}, B_{31}, and B_{12}. In particular, $e_1 = B_{23}I^{-1}$, $e_2 = B_{31}I^{-1}$, and $e_3 = B_{12}I^{-1}$.

2.1.2 Reverse, inverses, and the Clifford and spin groups

We had used the notation $^{-1}$ to denote the inverse for the pseudoscalar, but there are other invertible elements in a geometrical algebra. In particular, all blades are invertible. From this, we can construct a group of all invertible elements referred to as the *Clifford group* Γ for a geometric algebra $\mathcal{G}$ by

$$\Gamma := \left\{ \prod_{j=1}^{k} v_i \mid k \in \mathbb{Z}^+,\ \forall j\ : 1 \leq j \leq k\ :\ \ v_i \in \mathbb{R}^n \text{ such that } |v_i| \neq 0 \right\}.$$

We refer to elements of the Clifford group as *Clifford multivectors.* For any Clifford multivectors $A = v_1 \cdots v_k$ in the group Γ, we have that multiplicative inverse A^{-1} is given by $A^{-1} = v^k \ldots v^1$ as we can see that $A^{-1}A = AA^{-1} = 1$ by construction. Of note is the fact that all scalars, vectors, pseudovectors, and pseudoscalars are always in the Clifford group and have multiplicative inverses. The inverse of a vector v is given by $\frac{v}{v \cdot v}$. It becomes useful to define the *reverse* † such that $A^\dagger = v_k \cdots v_1$. For a k-blade A_k, the reverse also satisfies the relationship

$$A_k^\dagger = (-1)^{k(k-1)/2} A_k. \tag{15}$$

eq:reverse

One can then see that the inverse for the unit pseudoscalar is $I^{-1} = I^\dagger$ which is an identification I will often use. One can see that the multiplicative inverse of an element of the Clifford group A is the reverse of the corresponding product of reciprocal vectors since $A_k^{-1} = (v^1 \cdots v^k)^\dagger$. Note as well that elements $s \in \Gamma^+$ act as rotations on $A \in \mathcal{G}_n$ given the conjugate action

$$A \mapsto sAs^{-1}.$$

In fact, all nonzero vectors $v \in \Gamma$ define a reflection in the hyperplane perpendicular to v via the same conjugation action above.

Following these realizations, one can see that the Clifford group contains important subgroups such as the orthogonal and special orthogonal groups as quotients

$$\mathrm{O}(n) \cong \Gamma/\mathbb{R} \qquad \text{and} \qquad \mathrm{SO}(n) \cong \Gamma^+/\mathbb{R}.$$

This motivatives the following definition.

Definition 2.1. The *Clifford norm* $\|\cdot\|$ for $s \in \Gamma$ is given by

$$\|s\|^2 := ss^\dagger.$$

Note that for vectors the Clifford norm corresponds with the norm induced from the inner product in that with a vector v we have $\|v\| = vv^\dagger = v \cdot v$. We also give the name *unit* to k-blades A_k with unit spinor norm $1 = \|A_k\|$. We can also see that

$$\|\mu\| = \sqrt{|g|}, \tag{16}$$

eq:pseudos

and so

$$\|I\| = 1.$$

With this, we have the *pin* and *spin groups*

$$\mathrm{Pin}(n) := \{s \in \Gamma \mid \|s\| = 1\}.$$
$$\mathrm{Spin}(n) := \{s \in \Gamma^+ \mid \|s\| = 1\}.$$

Moreover,

$$\mathrm{Pin}(n) \cong \Gamma/\mathbb{R}^+ \qquad \text{and} \qquad \mathrm{Spin}(n) \cong \Gamma^+/\mathbb{R}^+.$$

The spin group $\mathrm{Spin}(n)$ is a Lie group and its associated Lie algebra is denoted by $\mathfrak{spin}(n)$. In particular, the $\mathfrak{spin}(n)$ is isomorphic to the algebra of bivectors with the antisymmetric product $\times$

provide a citation.

. Then, for any bivector B, we have an element in the spin group given by

$$e^B = \sum_{j=0}^{\infty} \frac{B^n}{n!}.$$

Fundamentally, Spin(n) acts on the even subalgebra $\mathcal{G}_n^+$. A *spinor* ψ is an element that transforms under a left action of an element of Spin(n) to produce another spinor. In terms of geometric algebra, a spinor is simply an even multivector (i.e., an element of $\mathcal{G}_n^+$). Of note are the two cases we have had as examples before.

f_bivector **Example 2.5.** Consider $\mathcal{G}_2$ and note that we have shown the algebra of spinors $\mathcal{G}_2^+$ is isomorphic to the complex numbers $\mathbb{C}$. Indeed, there is one unit 2-blade B_{12} in $\mathcal{G}_2$ to form the spin algebra $\mathfrak{spin}(2) \cong \mathbb{R}$ and as a consequence all unit norm elements in $\mathcal{G}_2^+$ can be written as

$$e^{\theta B_{12}} = \sum_{n=0}^{\infty} \frac{\theta B_{12}}{n!} = \cos(\theta) + B_{12}\sin(\theta),$$

where θB_{12} is a general bivector in $\mathcal{G}_2$. Hence, we arrive at $\mathrm{Spin}(2) \cong \mathrm{U}(1)$. Any element in $\mathbb{C}$ is also a scaled version of an element of the spin group Spin(2). Hence, we can use a spin representation for an element in $\mathbb{C}$ via $z = re^{\theta B_{12}} \in \mathbb{R} \times \mathrm{Spin}(2)$. This special case shows that parabivectors in $\mathcal{G}_2$ have a unique spin representation and they are spinors as well.

Example 2.6. Consider $\mathcal{G}_3$ and note that we have shown the spinors $\mathcal{G}_3^+$ are isomorphic to the quaternion $\mathbb{H}$ algebra. We also realize $\mathbb{H}$ as scalar copies of elements of $\mathrm{Spin}(3) \cong \mathrm{Sp}(1)$. That is to say that $\mathbb{H} \cong \mathbb{R} \times \mathrm{Spin}(3)$. Indeed, since elements of $\mathcal{G}_3^+$ are simply biparavectors, the biparavectors once again admit a natural spin representation. Likewise,

finish this.

2.1.3 Projection and rejection

There is a direct relationship between unit k-blades and k-dimensional subspaces. Indeed, each unit k-blade B_k ($\|B_k\| = 1$) corresponds to a k-dimensional subspace. That is, each point in $\mathrm{Gr}(k, n)$ corresponds to a unit k-blade. Since blades represent subspaces, they also give us a compact way of projecting multivectors into subspaces. In general, given an multivector A the *projection* onto the subspace spanned by B_k is

$$\mathrm{P}_{B_k}(A) \coloneqq (A \rfloor B_k) B_k^{-1}. \tag{17}$$

eq:project

By definition, we have

$$\mathrm{P}_{B_k}(A) \in \bigoplus_{j=0}^{k} \mathcal{G}_n^j = \mathcal{G}_n^{0+\cdots+k}$$

Specifically,

$$\mathrm{P}_{B_k}(\langle A \rangle_j) \in G_n^j,$$

shows the projection preserves grades.

G a vector v, the projection onto the subspace spanned by the k-blade A_k is given by the identity

$$(v \rfloor A_k) A_k^{-1} = (v \rfloor A_k) \rfloor A_k^{-1} = (v \cdot B_k) \cdot B_k^{-1}. \tag{18}$$

eq:vector_

and more enlightening is to take a projection of a vector v onto another vector u

$$(v \rfloor u) u^{-1} = (v \cdot u) \frac{u}{\|u\|^2},$$

which is the expected result.

A dual notion also exists. We define the *rejection* of a multivector from the subspace spanned by B_k as

$$\mathrm{R}_{B_k}(A) = (A \wedge B_k)B_k^{-1}. \tag{19}$$

eq:rejecti

In the case we have a vector v, we can note

$$\mathrm{P}_{B_k}(v) + \mathrm{R}_{B_k}(v) = v. \tag{20}$$

eq:project

To see this in action, we let $v = v^1e_1 + v^2e_2 + v^3e_3$ and let $B_{12} = e_1e_2$ and note

$$\begin{aligned}\mathrm{R}_{B_{12}}(v) &= [(v^1e_1 + v^2e_2 + v^3e_3) \wedge (e_1e_2)]B_{12}^{-1} \\ &= v^3e_3e_1e_2e^2e^1 \\ &= v^3e_3.\end{aligned}$$

Both the notion of projection and rejection prove to be useful. For vectors u and v, we can find

$$\mathrm{P}_{uI^{-1}}(v) = \mathrm{R}_u(v). \tag{21}$$

eq:project

Change the above equation slightly for the use in the Calderon section. Can we use rejection to simplify the B-planar proofs

2.2 Multivector fields

We want to generalize the setting of geometric algebra to include a smooth structure. One can take the work above for $\mathcal{G}_n$ and consider a C^∞-module structure as opposed to the $\mathbb{R}$-algebra structure in the proceeding section. For brevity, we put $\mathcal{G}_n(\mathbb{R}^n)$ for the C^∞-module and $\mathcal{G}_n$ for the $\mathbb{R}$-algebra. The multivectors themselves can be realized as constant multivector fields so that $\mathcal{G}_n \subset \mathcal{G}_n(\mathbb{R}^n)$. This smooth setting simply makes the coefficients of the global basis blades given by C^∞ functions as opposed to $\mathbb{R}$ scalars. In this case, we refer to a generic element in the C^∞-module $\mathcal{G}_n$ as a *multivector field*. We take $\Omega \subset \mathbb{R}^n$ as a connected region in $\mathbb{R}^n$ for the entirety of this paper and we put

$$\mathcal{G}_n(\Omega) \coloneqq \{f \colon \Omega \to \mathcal{G}_n \mid f \text{ is } C^\infty\text{-smooth}\},$$

where smoothness is meant in terms of the C^∞-module structure.

Perhaps the C^∞-module structure obfuscates the point slightly. Instead, one should think of the fields in $\mathcal{G}_n(\Omega)$ as multivector valued functions on $\Omega \subset \mathbb{R}^n$. Taking this identification allows for an extended toolbox at our disposal. In particular, points in Ω are uniquely identified with constant vector fields in $\mathcal{G}_n^1$ and one can consider endomorphisms living in $\mathcal{G}_n$ (acting on $\mathcal{G}_n^1$) as acting on the input of fields in $\mathcal{G}_n(\Omega)$ as well. Thus, there is not only an algebraic structure on the fields themselves, but on the point in which the field is evaluated. This is perhaps the key insight on why authors developed the so-called vector manifolds widely used in the geometric algebra landscape.

Example 2.7. Consider a multivector field f valued in $\mathcal{G}_n(\mathbb{R}^n)$. With $x \in \mathbb{R}^n$ being identified with the vector in $\mathcal{G}_n^1$, we output a multivector $f(x) \in \mathcal{G}_n$ at each point x. One may be interested in the restriction of f to a vector subspace of $\mathbb{R}^n$ which amounts to using projection on the input. For example, perhaps we wish to know how f behaves on the subspace generated by some k-blade A_k. As such, it suffices to then study $f(\mathrm{P}_{A_k}(x))$.

We refer to smooth fields valued in $\mathcal{G}_n^+$ as *spinor fields* and put $\mathcal{G}_n^+(\Omega)$ to refer to the C^∞-module counterpart. These fields will be shown to carry a Banach algebra structure.

2.2.1 Directional derivative and gradient

Note that $\mathbb{R}^n$ has global coordinates and thus we can choose a global constant vector field basis $e_1, \ldots, e_n$ and we generate $\mathcal{G}_n$ from this basis. Note that we will adopt the Einstein summation convention when needed. With respect to these fields, we have the *directional derivative* ∇_ω with $\omega = \omega^i e_i$. The *gradient* (or *Dirac operator*) is defined as $\boldsymbol{\nabla} = \sum_i e^i \nabla_{e_i}$ and it acts a grade-1 element in the algebra. Note then that $\omega \cdot \boldsymbol{\nabla} = \nabla_\omega$ defines the directional derivative via the gradient. The directional derivative is also grade preserving in that for a multivector A

$$\nabla_\omega \langle A \rangle_k = \langle \nabla_\omega A \rangle_k.$$

This structure defined above is typically referred to as *geometric calculus*. The setting for geometric calculus extends the setting of differential forms and reduces some of the complexity with tensor computations. Since $\boldsymbol{\nabla}$ is a grade-1 object, it acts on a homogeneous k-vector A_k by

$$\boldsymbol{\nabla} A_k = \langle \boldsymbol{\nabla} A_k \rangle_{k-1} + \langle \boldsymbol{\nabla} A_k \rangle_{k+1} \coloneqq \boldsymbol{\nabla} \cdot A_k + \boldsymbol{\nabla} \wedge A_k.$$

Thus, the gradient splits into two operators $(\boldsymbol{\nabla} \cdot)\colon \mathcal{G}_n^k(\Omega) \to \mathcal{G}_n^{k-1}(\Omega)$ (or $\boldsymbol{\nabla} \lfloor$) and $(\boldsymbol{\nabla} \wedge)\colon \mathcal{G}_n^k \to \mathcal{G}_n^{k+1}$. Here, $\boldsymbol{\nabla} \wedge$ can be identified with the exterior derivative d and $\boldsymbol{\nabla} \cdot$ can be identified with the codifferential δ on differential forms up to a sign (see schindler_geometric_2020 [15] There are more citations to use). This of course means the standard properties that apply to d and δ apply to $\boldsymbol{\nabla} \wedge$ and $\boldsymbol{\nabla} \cdot$. Namely, we have

$$(\boldsymbol{\nabla} \wedge)^2 = 0 \qquad (\boldsymbol{\nabla} \cdot)^2 = 0, \tag{22}$$

eq:differer

when acting on a homogeneous k-vector. Since eq:differential_properties 22 holds, the gradient operator gives rise to the grade preserving *Laplace-Beltrami operator*

$$\Delta = \boldsymbol{\nabla}\boldsymbol{\nabla} = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \wedge + \boldsymbol{\nabla} \wedge \boldsymbol{\nabla} \cdot,$$

which is manifestly coordinate invariant by definition. It also motivates the use of the physicist notation $\boldsymbol{\nabla}^2 = \Delta$, but we do not adopt this here. We refer to multivector fields f in the kernel of the Laplace-Beltrami operator *harmonic*.

2.2.2 Monogenic fields

Geometric calculus includes another definition for multivectors that is a big motivation for those who study Clifford analysis.

Definition 2.2. Let $f \in \mathcal{G}_n(\Omega)$. Then we say that f is *monogenic* if $f \in \ker(\boldsymbol{\nabla})$.

What monogenics are we really caring about here? Just the ones we can recover which are...

Monogenic fields are of utmost importance as they have many beautiful properties. One should find them as a suitable generalization of the notion of complex holomorphicity. For example, in regions of Euclidean spaces, a monogenic field f can be completely determined by its Dirichlet boundary values through a generalized Cauchy integral formula. For any even monogenic field, the each of the graded components of f are harmonic.

We put

$$\mathcal{M}(\Omega) \coloneqq \{f \in \mathcal{G}_n(\Omega) \mid \boldsymbol{\nabla} f = 0\}$$

to refer to elements of this set as *monogenic fields* on Ω. As a subset, we also have the *monogenic spinors* $\mathcal{M}^+(\Omega)$, which are simply the even monogenic fields and the *monogenic parabivectors* $\mathcal{M}^{0+2}(\Omega)$. Though these spaces do not form algebras in their own right, they do indeed form a vector space as sums of monogenic functions are monogenic due to the linearity of the gradient. Moreover, the monogenic spinors are invariant under multiplication from the Clifford group Γ^+.

_invariant **Lemma 2.1.** *Let* $s \in \mathrm{Spin}(n)$ *then* $\boldsymbol{\nabla} \circ s = s \circ \boldsymbol{\nabla}$. *In particular, the space of monogenic spinors* $\mathcal{M}^+(\Omega)$ *is* $\mathrm{Spin}(n)$ *invariant.*

This lemma is classical in the theory of the Dirac operator, Clifford analysis, and harmonic analysis so we omit a proof. One can see janssens_special_nodate [12] for example.

2.3 Differential forms and integration

diff_forms

2.3.1 Directed measures and k-forms

Naturally, we would also like to be able to integrate multivectors. In order to do so, we appeal to the language of differential forms and build a relationship between multivectors and forms. Forms have their appeal in global understanding via their properties through integration (e.g., Stokes' and Green's theorems). What we build here provides us a way to carry out a full multivector treatment of boundary value problems.

Given the coordinate system x^i on $\mathbb{R}^n$, we form the basis of tangent vector fields $\partial_i = \frac{\partial}{\partial x^i}$ with the reciprocal 1-forms dx^i (a section of $T^*\Omega$) which are the differentials of the coordinate functions. Thinking of 1-forms as linear functions on tangent vectors, we have $dx^i(\partial_j) = \delta^i_j$. The benefit of this definition is that the 1-forms dx^i carry a natural measure and we can form product measures via the exterior product. For example, we have the surface *directed measure* $d\Sigma = e_i \wedge e_j dx^i dx^j$ and we can note that $(e^j \wedge e^i) \cdot d\Sigma = dx^i dx^j - dx^j dx^i$ is antisymmetric and provides us a surface measure we can integrate; this is a differential form.

In an n-dimensional space with a position dependent inner product g, we have the n-dimensional volume directed measure $d\Omega = \sqrt{|g|}dx^1 \dots dx^n$. If we then define $dX_n = e^n \wedge \cdots \wedge e^1 dx^1 \dots dx^n$ we then find that

$$d\Omega = I^\dagger \cdot dX_n.$$

Here I is the pseudoscalar field defined on Ω with respect to g. Similarly, for $k < n$, we can define the k-dimensional volume measure as

$$dX_k = \frac{1}{k!}(e^{i_k} \wedge \cdots \wedge e^{i_1}) dx^{i_1} \cdots dx^{i_k}.$$

We can now write a k-form α_k as $\alpha_k = A_k \cdot dX_k$. In this sense, a differential form is made up of two essential components namely the multivector field and the k-dimensional volume directed measure. For example, if we wish to write a 2-form α_2 we take $dX_2 = \frac{1}{2!}e^j \wedge e^i dx^i dx^j$ and $A_2 = a_{ij}e_i \wedge e_j$ to yield

$$\alpha_2 = A_2 \cdot dX_2 = \frac{a_{ij}}{2!}(e_i \wedge e_j) \cdot (e^j \wedge e^i)dx^i dx^j = \frac{a_{ij}}{2!}(dx^i dx^j - dx^j dx^i)$$

Thus, we arrive at an isomorphism between k-forms and k-vectors as a contraction with the k-dimensional volume directed measure dX_k since

$$\alpha_k = A_k \cdot dX_k.$$

Hence, we can see now how a differential form simply appends the measure attached to the underlying space. We can also see how this generalizes the musical isomorphisms $\sharp$ and $\flat$ by taking a vector field a and noting

$$a \cdot dX_1 = a^i e_i \cdot e^j dx^j = a^i dx^i, \tag{23}$$

eq:line_el

corresponds to the usual $\flat$ map on vector fields.

2.3.2 Exterior algebra

The exterior algebra of differential forms comes with an addition $+$ and exterior multiplication $\wedge$. We note that the sum of two k-forms α_k and β_k that $\alpha_k + \beta_k$ is also a k-form which we can see by letting $\alpha_k = A_k \cdot dX_k$ and $\beta_k = B_k \cdot dX_k$ and putting

$$\alpha_k + \beta_k = (A_k \cdot dX_k) + (B_k \cdot dX_k) = (A_k + B_k) \cdot dX_k,$$

due to the linearity of $\cdot$. If instead had an s form β_s then we have the exterior product

$$\alpha_k \wedge \beta_s = (A_k \wedge B_k) \cdot dX_{k+s},$$

where $dX_{k+s} = 0$ if $k + s > n$.

2.3.3 Exterior derivative

With differential forms one also has the exterior derivative d giving rise to the calculus of forms. Given we can write a differential k-form as $\alpha_k = A_k \wedge dX_k$, In particular, we have

$$d\alpha_k = (\boldsymbol{\nabla} \wedge A_k) \cdot dX_{k+1},$$

which realizes the exterior derivative as the grade raising component of the gradient $\boldsymbol{\nabla}$. Of course, for scalar fields, this returns the gradient as desired.

2.3.4 Integration on submanifolds

Given a k-dimensional submanifold of $K \subset \Omega$ with a k-form α_k defined on K, we can integrate the k-form. Using the multivector equivalents leads us to the k-dimensional directed measure dK for the submanifold K. Given K is a submanifold of Ω, for any $x \in K$ we have tangent space T_xK which corresponds to a tangent k-blade $I_K(x)$. We put I_K as the smooth k-blade field everywhere tangent to K. Then we have the directed volume measure on K given by

$$dK = I_K^\dagger \cdot dX_k.$$

For a tangent k-vector field A_k on K, we must have for any $x \in K$ that $f = \mathrm{P}_{I_K} \circ f$ so that these fields lie purely tangent to K. In particular, we can always put $A_k = AI_k^\dagger$ for a scalar field A. These fields can contract with the directed measure dX_k to create a k-form on K by $\alpha_k = A_k \cdot dX_k = AdK$ which can be integrated as

$$\int_K \alpha = \int_K AdK.$$

Hence, on Ω itself, we can integrate top forms $\omega = WI^\dagger$ for a scalar field W by

$$\int_\Omega \omega = \int_\Omega W d\Omega.$$

There is also the normal space N_xK that is everywhere orthogonal (with respect to g on Ω) to T_xK. In particular, we have the normal $(n-k)$-blade field $\nu = I_K^\dagger I$. Note that for a unit k-blade I_K we have $I_K^{-1} = I_K^\dagger$ and we see $I_K \nu = I$. Since K is a submanifold of Ω we have the inclusion $\iota \colon K \to \Omega$ and the induced pullback on forms ι^* which is equivalent to the tangent projection operator $\boldsymbol{t}_K$ seen in schwarz_hodge_1995 [16]. Given a p-form ω defined on Ω, we have that $\boldsymbol{t}_K \omega = \omega \circ \mathrm{P}_{I_K}$. Specifically, $\omega = W \cdot dX_p$ we have

$$\iota^* \omega = W \cdot (dX_p \circ \mathrm{P}_{I_K}) = \mathrm{P}_{I_K}(W) \cdot dX_p.$$

The normal projection $\boldsymbol{n}_K$ is then $\boldsymbol{n}_K \omega = \omega - \boldsymbol{t}_K \omega$.

This is pertinent when we take the submanifold $\Sigma = \partial\Omega$. There, I_Σ yields the directed measure

$$d\Sigma \coloneqq I_\Sigma^\dagger \cdot dX_{n-1}.$$

The normal space is 1-dimensional and ν is the unit normal vector to the boundary. The pullback coincides with projection into the tangent space given by I_Σ. Then, for 1-forms $\alpha = \cdot dX_1$ it is apparent that $\boldsymbol{t}_\Sigma \alpha = \mathrm{P}_{I_\Sigma}(a) \cdot dX_1$ and $\boldsymbol{n}_\Sigma \alpha = \mathrm{R}_{I_\Sigma}(a) \cdot dX_1$ by Equations eq:projection+re 20 and eq:projection_rejection_vector 21. One can then find the flux of a vector field through Σ arises as an $(n-1)$-form $\mathrm{P}_\nu(a) I^{-1} \cdot dX_{n-1}$. Once again we see that the flux is determined both by the vector field a and the local geometry of Σ captured by $d\Sigma$ in the following way. Note that $\nu^{-1} = \nu$ since $\|\nu\| = 1$ everywhere on Σ and so $\mathrm{P}_\nu(a) I^{-1} = a \cdot \nu\nu I^{-1} = a \cdot \nu I_\Sigma^\dagger$ which gives us the corresponding form $a \cdot \nu d\Sigma$ and the total flux of a through Σ is then

$$\int_\Sigma (\mathrm{P}_\nu(a) I^{-1}) \cdot dX_{n-1} = \int_\Sigma a \cdot \nu d\Sigma.$$

2.3.5 k-form inner product

Show that this relates back to the spinor norm.

For smooth k-forms $\alpha_k = A_k \cdot dX_k$ and $\beta_k = B_k \cdot dX_k$, we have an inner product

$$\langle \alpha_k, \beta_k \rangle = \int_\Omega \alpha_k \wedge \star \beta_k$$

where $\star$ is the Hodge star. The Hodge star on k-forms inputs a k-form and outputs a a specific dual $(n-k)$-form so that we always have $\alpha_k \wedge \star \beta_k = (A_k \cdot B_k^\dagger) d\Omega$ as we note Equation ??. eq:dot_product Thus, we can realize how $\star$ acts on multivector representative. We let $\star \beta_k = B_k^\star \cdot dX_{n-k}$ by $B_k^\star = (I^{-1} B_k)^\dagger$. Indeed, we have

$$\begin{aligned} \alpha_k \wedge \star \beta_k &= (A_k \wedge B_k^\star) \cdot dX_n \\ &= A_k \cdot B_k^\dagger d\Omega. \end{aligned}$$

2.3.6 Stokes' and Green's theorem

For regions Ω with boundary Σ, we have a compact form of Stokes' theorem

$$\int_\Omega d\omega = \int_\Sigma \boldsymbol{t}\omega,$$

for sufficiently smooth $(n-1)$-forms ω.

More on integration and do Ohms law as an example of some of this stuff. Do all of hodge decomposition and stuff?

3 Algebras of multivector fields

3.1 Banach algebras of Clifford fields

Finish this section. I'm saying this here but it should go later on, but this should lead to the weak formulation for the laplace equation??? Does there exist an inner product instead of just a norm?

Letting Ω be a region in $\mathbb{R}^n$, recall that the space of monogenic fields $\mathcal{M}(\Omega)$ is not an algebra. However, $\mathcal{M}(\Omega)$ does contain algebras that are commutative Banach algebras. To start, consider the algebra of spinor fields $\mathcal{G}_n^+(\Omega)$ which has a norm induced from the spinor norm in the L_2 sense by

$$\|s\|_{L_2} = \int_\Omega s s^\dagger d\Omega$$

This gives us a normed algebra of spinor fields. One can see that we also have the unit 1 in this algebra. Spinors $s, r \in \mathcal{G}_n^+$ satisfy

$$\|sr\| = \|s\| \|r\|$$

since

$$\|sr\|^2 = (sr)(sr)^\dagger = srr^\dagger s^\dagger = s\|rr^\dagger\|^2 s^\dagger = \|s\|^2 \|r\|^2.$$

It follows immediately that for the non-constant C^∞-fields s and r

$$\|sr\|_{L_2} \leq \|s\|_{L_2} \|r\|_{L_2}.$$

Identifying the constant fields in the algebra $\mathcal{G}_n^+(\Omega)$ with points in $\mathbb{R}^{2^{n-1}}$ (since $\dim(\mathcal{G}_n^+) = 2^{n-1}$) we see that the algebra is also complete. Thus we have shown that the space $\mathcal{G}_n^+(\Omega)$ is a (noncommutative) Banach algebra. There are more algebras to discover.

3.1.1 Planar monogenic fields

Generically, if I take some multivector A times a monogenic field f, Af need not be monogenic which is a reason why $\mathcal{M}(\Omega)$ fails to be an algebra. But, there are certain types of monogenic fields in which this property is true. We describe a set of parabivectors that operate entirely on a plane given by a unit bivector B. These specific fields will be of great utility for the remainder of this paper.

Definition 3.1. Let f be a parabivector and B a unit 2-blade. Then f is a *B-planar field* if $f = \mathrm{P}_B \circ f \circ \mathrm{P}_B$.

We then refer to the *B-planar monogenic fields f* when f is both B-planar and monogenic. Planar monogenic fields will serve as a realization of complex valued functions since they carry over some additional nice properties and admit a nice representation.

Lemma 3.1. *Let f be a B-planar monogenic field, then:*

- *The directional derivatives in all directions other than in the B plane are zero;*
- *We have the representation $f = u + \beta B$ for a $u, \beta \in G_n^0(\Omega)$ and B the given unit bivector.*

Proof.

- Let v be a unit vector not in the B plane so that $\mathrm{P}_B(v) = 0$. Since f is B-planar, we know $f = f \circ \mathrm{P}_B$ which shows that $f(x + \epsilon v) = f(x)$. It follows that $\nabla_v f = 0$.
- Let $f = u + b$ for $u \in \mathcal{G}_n^0$ and $b \in G_n^2$. Then $f = \mathrm{P}_B(v) \circ f$ and so $\mathrm{P}_B(u + b) = u + b$. In particular, $\mathrm{P}_B = b$ and thus $b = \beta B$ for a scalar $\beta \in \mathcal{G}_n^0$.

□

To get a geometric interpretation of B-planar fields we can note that they are constant on translations of the B-plane. It follows that

$$(\boldsymbol{\nabla} \wedge B) f = 0. \tag{24}$$

eq:exterio

In $\mathbb{R}^3$, for example, this amounts to fields constant along an axis $\omega = IB^{-1}$ perpendicular to B as

$$\boldsymbol{\nabla} \wedge B = \boldsymbol{\nabla} \wedge \omega I = \boldsymbol{\nabla} \cdot \omega = \nabla_\omega. \tag{25}$$

eq:omega_a

Rephrase this with rejection?

ex:complex_representation

Recall from Example 2.1 that multivectors in the form $\zeta = x + yB$ mimic the complex number ζ when B is a unit 2-blade since $B^2 = -1$. Planar monogenic fields are thus a direct analog of $\mathbb{C}$-holomorphic functions. Indeed, for simplicity take the orthonormal basis e_i and the blade $B = B_{12}$ and for scalar fields u and β put

$$f = u + \beta B_{12}$$

and note

$$\boldsymbol{\nabla} f = 0$$

yields the Cauchy-Riemann equations

$$\nabla_{e_1} u = \nabla_{e_2} \beta \qquad \text{and} \qquad \nabla_{e_2} u = -\nabla_{e_1} \beta.$$

Holomorphic functions form an algebra and we shall show the B-planar monogenic fields do as well.

We let

$$\mathcal{A}_B(\Omega) = \{f \mid f \text{ is } B\text{-planar and monogenic}\}$$

be the space of B-planar monogenic fields. For any 2-blade B in $\mathrm{Gr}(2,n)$, we have a space $\mathcal{A}_B(\Omega)$. Multiplication of two B-planar fields $f = u_f + \beta_f B$ and $g = u_g + \beta_g B$ is given by

$$fg = u_f u_g - \beta_f \beta_g + B(u_f b_g + u_g b_f) = gf. \tag{26}$$

eq:axial_mu

Another property mimics $\mathbb{C}$-holomorphicity. Namely, scaling a holomorphic function by constant complex numbers remains holomorphic. We realize this for B-planar fields as Spin(2) invariance (really $\mathbb{R}\times\mathrm{Spin}(2)$ invariant). The following corollary follows from Lemma lem:clifford_invariant 2.1 since $\mathbb{R}\times\mathrm{Spin}(2)$ is a subgroup of Γ^+

_monogenic

Corollary 3.1. *Let $f = u + \beta B$ be an B-planar monogenic field and let $\zeta = x + yB$ for constant scalars x and y. Then ζf is a B-planar monogenic.*

Proof. Note that ζ admits the representation $\zeta = re^{\theta B}$ as seen in Example ex:exponential_of_bivector 2.5 for some $r, \theta \in \mathbb{R}$ with $r = \|\zeta\|$. If $\|\zeta\| = 1$, then this corollary follows immediately from Lemma lem:clifford_inva 2.1 as $\zeta \in \mathrm{Spin}(n)$. If $r \neq 1$, we note that the the corollary remains true given the $\mathbb{R}$-linearity of $\boldsymbol{\nabla}$. ☐

The point here is that we have now effectively found functions that can be scaled by B-planar constants ζ and remain monogenic.

With the above, we show the space $\mathcal{A}_B(\Omega)$ is closed under multiplication and is in fact abelian.

monogenics

Lemma 3.2. *Let f and g be monogenic and B-planar. Then $fg = gf$, and fg is a B-planar monogenic.*

Proof.

- First, it is clear that $fg = gf$ by Equation eq:axial_multiplication 26.
- The product fg is B-planar since u_f, u_g, β_f, and β_g are all constant on translations of the B-plane, i.e. that $fg = fg \circ \mathrm{P}_B$. Due again to Equation eq:axial_multiplication 26 we have $fg = \mathrm{P}_B \circ fg$ as well.
- To see that the product is monogenic, we have

$$\boldsymbol{\nabla}(fg) = \boldsymbol{\nabla}(u_f u_g - b_f b_g + B(u_f b_g + u_g b_f)).$$

Then the grade-1 components are

$$\langle \boldsymbol{\nabla}(fg) \rangle_1 = \boldsymbol{\nabla} \wedge (u_f u_g - b_f b_g) + \boldsymbol{\nabla} \cdot B(u_f b_g + u_g b_f),$$

and note that we have

$$\begin{aligned}\boldsymbol{\nabla}(u_f u_g - b_f b_g) &= (\boldsymbol{\nabla} u_f)u_g + u_f(\boldsymbol{\nabla} u_g) - (\boldsymbol{\nabla} b_f)b_g - b_f(\boldsymbol{\nabla} b_g)\\ \boldsymbol{\nabla}\cdot B(u_f b_g + u_g b_f) &= (\boldsymbol{\nabla}\cdot B u_f)b_g + u_f(\boldsymbol{\nabla}\cdot B b_g) + b_f(\boldsymbol{\nabla}\cdot B u_g) + (\boldsymbol{\nabla}\cdot B b_f)u_g,\end{aligned}$$

and since f and g are both monogenic we have

$$\langle \boldsymbol{\nabla}(fg)\rangle_1 = (\boldsymbol{\nabla}\cdot B u_f - \boldsymbol{\nabla} b_f)b_g + (\boldsymbol{\nabla}\cdot B u_g - \boldsymbol{\nabla} b_g)b_f.$$

$$0 = \langle \boldsymbol{\nabla} B f\rangle_1 = \boldsymbol{\nabla}\cdot B u_f - \boldsymbol{\nabla} b_f$$

by Corollary cor:mult_by_i_monogenic 3.1 and likewise for $\langle \boldsymbol{\nabla} B g\rangle_1$. Thus,

$$\langle \boldsymbol{\nabla}(fg)\rangle_1 = 0.$$

The grade-3 components for the gradient are

$$\langle \boldsymbol{\nabla}(fg)\rangle_3 = \boldsymbol{\nabla}\wedge B(u_f b_g + u_g b_f),$$

and we can note that $\boldsymbol{\nabla}\wedge B = 0$ since u_f, b_g, u_g, and b_f are all B-planar.

□

From the above work, we realize that for each $\mathcal{A}_B(\Omega)$ we have a well defined multiplicative structure. This realizes that $\mathcal{A}_B(\Omega)$ sits inside of the space of monogenic spinors $\mathcal{M}^+(\Omega)$. We arrive at the following corollary.

Corollary 3.2. *The space $\mathcal{A}_B(\Omega)$ is a commutative unital Banach algebra.*

Proof. Let f and g be B-planar monogenic fields. It is clear that the sum $f+g$ is a B-planar monogenic by the linearity of $\boldsymbol{\nabla}$ and the projection. Since $fg = gf$ is B-planar and monogenic we find that each $\mathcal{A}_B(\Omega)$ is an algebra. Since $\mathcal{A}_B(\Omega)$ is a commutative subalgebra of $\mathcal{G}_n^+(\Omega)$, it is also a commutative Banach algebra. Shorten a lot □

3.1.2 ω-axial fields

The authors in belishev_algebraic_2019,belishev_algebras_2019 [4, 5] give a thorough treatment of an analogous story but with quaternion fields. We show the relationship between the two stories in this section and we find them to be entirely equivalent. As in Example ex:quaternions 2.2, we can see these quaternion fields as parabivector fields. The authors work exclusively in 3-dimensions and quickly specialize to the fields which are ω-axial due to their rich algebraic structure. There, ω is a purely imaginary unit quaternion. Their harmonic ω-axial fields are equivalent to monogenic B-planar fields if we take the axis $\omega = BI^{-1}$. First, note we define ω-axial in the same way.

Definition 3.2. Let $A \in \mathcal{G}_3(\Omega)$ be a multivector field then A is *ω-axial* if $A(x+t\omega) = A(x+t\omega)$.

This definition allows us to perfectly coincide the notions of B-planar monogenic fields with ω-axial harmonic quaternion fields.

Proposition 3.1. *In $\mathbb{R}^3$, every B-planar monogenic field is in correspondence with an ω-axial harmonic quaternion field $h = \varphi + \psi\omega$.*

Proof. Let f be a B-planar monogenic field with $\tilde{\omega} = BI^{-1}$ and note that $f(x + t\tilde{\omega}) = f(x)$ since $\mathrm{P}_B(t\omega) = 0$. Thus, f is $\tilde{\omega}$-axial.

Given the quaternion multiplication is a left handed bivector multiplication (see Example ex:quaternions 2.2, we can replace the purely imaginary quaternion ω and get a vector in $\mathcal{G}_3^1$ by using the correspondence $\boldsymbol{i} \leftrightarrow e_1$, $\boldsymbol{j} \leftrightarrow e_2$, and $\boldsymbol{k} \leftrightarrow e_3$ we generate $\tilde{\omega} \in \mathcal{G}_3^1$. We then have the 2-blade $B = \tilde{\omega} I$ such that

$$\tilde{h} = \varphi + \psi B,$$

is the corresponding parabivector in $\mathcal{G}_3$. It's clear that $\mathrm{P}_B \circ \tilde{h} = \tilde{h}$. Likewise, since φ and ψ were constant on the axis given by ω, then by the previous work $\varphi \circ \mathrm{P}_B$ and $\psi \circ \mathrm{P}_B$ implies that $\tilde{h} \circ \mathrm{P}_B$ and so $\tilde{h}$ is a B-planar. Hence, setting $\varphi = u$ and $\psi = \beta$, we recover a unique f from a given h.

Then, if $h = \varphi + \psi\omega$ is harmonic, we know

$$\boldsymbol{\nabla}\psi = \omega \times \boldsymbol{\nabla}\varphi,$$

where we take the vector cross product $\times$. Based on Example ex:cross_product 2.4, we can see that corresponding B-planar field $f = u + \beta B$ yields the analogous equation

$$\boldsymbol{\nabla} u = \boldsymbol{\nabla} \cdot \beta B = (\boldsymbol{\nabla} \wedge \tilde{\omega}) I = \tilde{\omega} \times \boldsymbol{\nabla}\beta.$$

Thus, the notions of an ω-axial harmonic quaternion field coincides with B-planar monogenic fields in $\mathbb{R}^3$ so long as $B = \tilde{\omega} I$. $\square$

The ω-axial fields do not generalize properly and this definition is solely a happy circumstance seen in $\mathbb{R}^3$ given the duality between vectors and bivectors. In higher dimensions, the notion of B-planar retains all the desired properties that let us define a notion of a Gelfand spectrum.

3.1.3 Spinor spectrum

This story no longer continues in higher dimensions and one can find the two and three dimensional cases to be happy accidents. Instead, now we must deal fully with the situation at hand to dissect the relevant algebras. At our disposal are the algebras $\mathcal{A}_B(\Omega)$ of B-planar monogenic fields. Take the case where the domain $\mathbb{B} \subset \mathbb{R}^n$ is the unit n-ball and moreover let $\mathbb{D}$ be the unit disk in $\mathbb{C} \cong \mathbb{R}^2$. By Gelfand, the maximal ideal space of the commutative Banach algebra $\mathcal{A}_B(\mathbb{B})$ is homeomorphic to the disk given the isomorphism mapping the blade $B \leftrightarrow i$ in the complex plane. Since the space $\mathcal{M}$ is no longer commutative let alone an algebra, we are at a loss to determine maximal ideals. Instead, one can note that maximal ideals of a commutative Banach algebra correspond to the multiplicative linear functionals. Using this identification, we carry on and describe functionals on the monogenic fields.

It's probably worth phrasing this as some kind of algebra morphism

Definition 3.3. Define the *spinor dual* $\mathcal{M}^\times(\Omega)$ as

$$\mathcal{M}^\times(\Omega) \coloneqq \{ l \in \mathcal{L}(\mathcal{M}^+(\Omega); \mathcal{G}_n^+) \mid l(sf) = sl(f),\ \forall f \in \mathcal{M}(\Omega),\ s \in \mathcal{G}_n^+ \},$$

and refer to elements of $\mathcal{M}^\times(\Omega)$ are *spinor functionals*. Maybe have $s \in \mathfrak{spin}(n)$ instead?

Similarly, we will now define the spinor functionals that are multiplicative on the B-planar monogenics. In other words, spin characters are simply algebra homomorphisms from $\mathcal{A}_B(\Omega)$ to $\mathcal{G}_n^+$.

Definition 3.4. The *spinor spectrum* $\mathfrak{M}(\Omega)$ is the set

$$\mathfrak{M}(\Omega) \coloneqq \{\mu \in \mathcal{M}^\times(\Omega) \mid \mu(fg) = \mu(f)\mu(g),\ \forall f \in \mathcal{A}_B(\Omega) \text{ and } \forall g \in \mathcal{A}_{B'},\ B, B' \in \mathrm{Gr}(2,n)\},$$

and we refer to the elements as *spinor characters*.

Maybe we don't need multiplicative on different algebras somehow?

Example 3.1. Edit this In the case where Ω itself is 2-dimensional and compact, we realize $\mathcal{G}_n^+$ is isomorphic to $\mathbb{C}$ and we find that these match the typical definition for characters $\mu \in \mathfrak{M}(\Omega)$. These spin characters each amount to function evalation. Take $f \in \mathcal{M}(\Omega)$ and note that $f \in \mathcal{A}_B(\Omega)$ as well. f is then a holomorphic function when we identify $B \leftrightarrow i$ and as such the spin character μ acts by $\mu(f) = f(x_\mu)$ for some point $x_\mu \in \Omega$ showing the correspondence of points in Ω with spin characters in $\mathfrak{M}(\Omega)$. Hence, with the weak-$*$ topology, the space $\mathfrak{M}(\Omega)$ is homeomorphic to Ω.

There is the question now on what is the homeomorphism type of $\mathcal{A}_B(\Omega)$ for an arbitrary Ω and for a given B. Use 2d Belishev somehow? Describe the weak-$*$ topology here to use later.

4 Gelfand theory

4.1 Topology from monogenics

We seek to determine that the space $\mathfrak{M}(\Omega)$ is homeomorphic to Ω. Thinking of the Calderón problem, we may only have access to functions defined on Ω and not the whole of Ω itself. If one can recover the spinor characters $\mathfrak{M}(\Omega)$, we can utilize the following result.

Theorem 4.1. *For any $\mu \in \mathfrak{M}(\Omega)$, there is a point $x^\mu \in \Omega$ such that $\mu(f) = f(x_\mu)$ for any $f \in \mathcal{M}(\Omega)$ a monogenic spinor field. Given the weak-$*$ topology on $\mathfrak{M}(\Omega)$, the map*

$$\gamma\colon \mathfrak{M}(\Omega) \to \Omega, \quad \mu \mapsto x^\mu$$

is a homeomorphism. The Gelfand transform

$$\widehat{\ }\colon \mathcal{M}(\Omega) \to C(\mathfrak{M}(\Omega); \mathcal{G}_n), \quad \widehat{f}(\mu) \coloneqq \mu(f), \quad \mu \in \mathfrak{M}(\Omega),$$

is an isometry onto its image, so that $\mathfrak{M}(\Omega)$ is isomorphic to $\widehat{\mathcal{M}(\Omega)}$ as algebras.

We prove this theorem in two main parts and discuss the result in this section. First, we can realize a power series representation for elements in a ball $\mathbb{B}$ and denote this sit as $\mathcal{M}(\mathbb{B})$. This power series is constructed using specific B-planar monogenic fields. Finally, we constructively show a correspondence between $\mu \in \mathfrak{M}(\mathbb{B})$ with $x^\mu \in \mathbb{B}$. Then we can use these to cover Ω or something?

4.1.1 Power series

This really is a honest to god Taylor series so I should call it that.

One beautiful result in Clifford analysis is the celebrated generalization of the Cauchy integral formula for $\mathbb{C}$-holomorphic functions. Details of the Cauchy integral formula and Hilbert transform for multivector fields can be found in brackx_hilbert_2008 [6]. We have the fundamental solution to $\boldsymbol{\nabla}$ is a vector field given by

$$E(x) = \frac{1}{a_m}\frac{x}{\|x\|^m},$$

for $x \in \mathbb{R}^n$. That is to say that $\boldsymbol{\nabla} E(x) = \delta(x)$. For any region $\Omega \subset \mathbb{R}^n$ with boundary Σ, we define the *Cauchy kernel* for $x \in \mathbb{R}^n$ and $y \in \Sigma$ using the fundamental solution E as

$$C(y,x) = -\frac{1}{a_n}\nu(x_0)E(x-y),$$

where a_n is the surface area of the n-ball and $\nu(x_0)$ is the outward normal at x_0. The *Cauchy integral* for $\phi \in L_2(\Sigma)$ is then

$$\mathcal{C}[\phi](x) = \frac{1}{a_n}\int_\Sigma \frac{y-x}{\|x-y\|^n}\nu(y)\phi(y)d\Sigma(y).$$

The Cauchy integral is indeed a monogenic function and note that for a scalar ϕ we have $\mathcal{C}[\phi] \in \mathcal{M}(\Omega)$ since it must be a parabivector as well.

Fix a basis $e_1, \dots, e_n$ in $\mathbb{R}^n$ and we can define the functions $z^i_j = x^j - x^i e^i e_j$. Recall that for an orthonormal basis the reciprocal basis elements $e^i = e_i$ satisfy $e^i \cdot e_j = 1$. Ryan uses e_i^{-1} actually. Are the reciprocal basis elements the inverses? Yes see `https://math.stackexchange.com/questions/811248/wedge-product-between-nonorthogonal-basis-and-its-rec`
To further condense notation, we let $B_{ij} = e_i e_j$ be the 2-blade acting as the pseudoscalar for the $e_i e_j$-plane and likewise put $B^i_j = e^i e_j$ and $B^{ij} = e^i e^j$ as necessary. In the same vein, the functions z^i_j are very analogous to z in $\mathbb{C}$ but rather in the B^i_j plane. We then note

$$z^i_j = x^j - x^i B^i_j = e_j \operatorname{P}_{B^i_j}(x).$$

One can quickly confirm that the z^i_j are monogenic and are indeed B^i_j-planar by construction. These functions find their use in a power series representation for monogenic fields f.

- Consider the function $z^1_{\sigma(j)}(x) = x^{\sigma(j)} - x^1 B^i_{\sigma(j)}$ for $\sigma \in \{2, \dots, n\}$ a permutation.
- Let $f \in \mathcal{M}^+(\Omega)$. Then by Theorem 4 in ryan_clifford_2004 [13], we can center a ball of radius R at w to get the monogenic polynomials

$$P_{j_2\cdots j_n}(x) = \frac{1}{j!}\sum_{\text{permutations}} z^1_{\sigma(1)}(x-w)\cdots z^1_{\sigma(j)}(x-w).$$

Each polynomial in the collection

$$\mathcal{P}(\Omega) = \{P_{j_2\cdots j_n} \mid j_2 + \cdots + j_n = j,\ 0 \le j < \infty\}$$

is monogenic and linearly independent. These polynomials generate f as a power (Taylor) series as

$$f(x) = \sum_{j=0}^{\infty} \left(\sum_{j_2 \cdots j_n j_2 + \cdots j_n = j} P_{j_2 \cdots j_n}(x-w) a_{j_2 \cdots j_n}(w) \right),$$

where the coefficients are found using the Cauchy integral

$$a_{j_2 \cdots j_n} = \frac{1}{a_n} \int_{\partial B(w,R)} \frac{\partial^j G(x-w)}{\partial x_2^{j_2} \cdots \partial x_n^{j_n}} \nu(x) f(x) d\Sigma(x).$$

Each coefficient $a_{j_2 \cdots j_n} \in \mathcal{G}_n^+$. Yes but these are coming in as a right module multiplication. So this should be noted and checked

- This series converges uniformly to f for points $x \in \mathbb{B}$.

We have now found that all monogenic fields are generated as power series of homogeneous polynomials in the variables z_j^i. Thus, we have a direct route between the algebras $\mathcal{A}_{B_j^i}(\mathbb{B})$ and the monogenic spinor fields $\mathcal{M}(\mathbb{B})$. In each algebra $\mathcal{A}_{B_j^i}(\mathbb{B})$ the z_j^i act much like a realization of $z \in \mathbb{C}$. We will find that the action of the spin characters on z_j^i can be understood and extended through the power series to all monogenic spinors. The power series representation seen here is one of the strong reasons to utilize geometric calculus and study the results of Clifford analysis.

4.1.2 Correspondence

The functions z_j^i play a crucial role in the above power series representation but they also play a key part in determining the behavior of the spin characters $\mu \in \mathfrak{M}$. If we are able to deduce the action $\mu(z_j^i)$, then we can extend this to any monogenic f via the power series representation. Note that $\mu(1) = 1$ since it is an algebra homomorphish and so for any 2-blade B and $\mu \in \mathfrak{M}(\mathbb{B})$ that the image of the axial algebras $\mathbb{A}_B = \mu(\mathcal{A}_B(\mathbb{B}))$ are all commutative subalgebras of $\mathcal{G}_n^+$. In particular, for a constant $\alpha + \beta B \in \mathcal{A}_B(\mathbb{B})$, $\mu(\alpha + \beta B) = \alpha + \beta B$ by definition and so we retrieve $\mathbb{A}_B$ must be generated by linear combinations of the scalar 1 and the bivector B. Thus, $\mathbb{A}_B$ is an isomorphic copy of $\mathcal{G}_2^+ \cong \mathbb{C}$ as the even subalgebra of the B-plane.

Working in terms of an arbitrary basis and applying μ yields

$$\mu(z_j^i) = \alpha_j^i + \beta_j^i B_j^i,$$

for some constants α_j^i and α_j^i. The z_j^i are not independent from one another. In fact, we have two key relationships in that

$$z_j^i B_i^j = -z_i^j. \tag{27}$$

eq:z_recip

Similarly, we have

$$z_j^i = z_j^k + z_k^i B_j^k. \tag{28}$$

eq:z_relat

Thus, we can take μ of Equations eq:z_reciprocal_relationship 27 and eq:z_relationship 28 and determine a relationship on the constants α^i_j and β^i_j. First, using Equation eq:z_reciprocal_relationship 27

$$\mu(z^i_j B^j_i) = \mu(z^i_j)B^j_i = -\mu(z^j_i)$$

yields

$$(\alpha^i_j + \beta^i_j B^i_j)B^j_i = \beta^i_j + \alpha^i_j B^j_i = -\alpha^j_i - \beta^j_i B^j_i$$

and so $\alpha^j_i = -\beta^i_j$ for all $i \neq j$. Next, using Equation eq:z_relationship 28

$$\mu(z^i_j) = \mu(z^k_j + z^i_k B^k_j) = \mu(z^k_j) + \mu(z^i_k)B^k_j$$

and so

$$a^i_j + b^i_j B^i_j = \alpha^k_j + \beta^k_j B^k_j + (\alpha^i_k + \beta^i_k B^i_k)B^k_j = \alpha^k_j + \beta^i_k B^i_j + (\alpha^i_k + \beta^k_j)B^k_j$$

yields the relationships $\alpha^i_j = \alpha^k_j$, $\beta^i_j = \beta^i_k$, and $\alpha^i_k = -\beta^k_j$.

Briefly, picture α^i_j and β^i_j as components of the $n \times n$ matrices α and β. We can index rows by the superscript and columns by the subscript and see that α and β both have zero diagonal (since we do not have functions z^i_i). The relationship $\alpha^j_i = -\beta^i_j$ for $i \neq j$ then shows that $\alpha = -\beta^\top$. Then we have $\alpha^i_j = \alpha^k_j$ for $i \neq j \neq k$ shows that α is constant along rows and hence β is constant along columns (which shows $\alpha = -\beta^\top$ is consistent with the additional relationship $\beta^i_j = \beta^i_k$). The final relationship $\alpha^i_k = -\beta^k_j$ is consistent as well. The matrices α and β are thus uniquely determined by n numbers. Moreover, treating $\mu(z^i_j) = z^i_j(x_\mu)$ for some $x_\mu \in \mathbb{R}^n$ satisfies the relationships granted above. Thus, we simply find the x_μ such that we retrieve the desired components for α and β.

Using the power series representation for a monogenic spinor f we can extend μ to act on $\mathcal{M}^+(\mathbb{B})$ by the multiplicative and $\mathcal{G}^+_n$ linear nature of μ since we also note again that the coefficients $a_{j_2\cdots j_n} \in \mathcal{G}^+_n$. Using the correspondence, we then realize $\mu(f) = f(x_\mu)$ for the corresponding $x_\mu \in \mathbb{R}^n$. To see that this point $x_\mu \in \mathbb{B}$, we take a field defined on $\mathcal{G}_n(\mathbb{R}^n)$ and monogenic in $\mathcal{G}_n(\Omega)$. For any $x_0 \in \mathbb{R}^n \setminus \mathbb{B}$ we have the field $E(x_0 - x)$ is monogenic for $x \in \mathbb{B}$. Then for a spin character μ we have a sequence of functions $E_n \to E(x_\mu - x)$ such that $\mu(E_n)$ is bounded for all n but diverges in the limit. Can we actually just argue that we can determine all x_0 such that $E(x_0 - x)$ is monogenic on Ω therefore we can determine $\mathbb{R}^n \setminus \Omega$?

Make thie more explicit and do an example or something in 3D. Show that $x\mu$ is in the ball.Finish this and note that this proves the theorem.

I'm not even sure we need to do this with $\Omega = \mathbb{B}$ other than for part of the proof with the power series. But if Ω is compact, it fits inside a ball of some radius r and so we should still be able to represent all the monogenics on Ω with this. The trick is we have a function that is monogenic except at a point.

If work with weak monogenic functions then we can probably use mollifiers and stitch together monogenics on Ω from various open balls in Ω that are monogenic except at some set of measure zero. Then this should allow us to probably speak more accurately about the delta function and E and probably suup this all up to determine the homeomorphism type of any embedded manifold.

4.2 Discussion

Perhaps the above result should not be so surprising. One could venture to the Atiyah-Singer index theorem which relates the topological information of a manifold with the elliptic operators. In particular, the Dirac operator (the gradient $\boldsymbol{\nabla}$) is indeed elliptic. Indeed, this seemingly sparks the motivation for the Calderón problem. There, the elliptic operator is the Laplace-Beltrami operator Δ. However, this is an inverse problem in which we do not know the space (or the metric) and are asked to, in a sense, determine the Laplace-Beltrami operator from information on the boundary of a Riemannian manifold. With this boundary data, one would hopefully be able to decipher Δ and as such, construct a copy of the desired Riemannian manifold.

5 Calderón problem

Okay, we can surely recover $\mathcal{M}^{0+2}(\Omega)$ which is $\mathcal{M}^{+}(\Omega)$ when Ω is dimension 3 or less. Is this all we really need? Otherwise, we may be at a loss here.

Go over Ohm's law (or do it in the forms and integration section) but relate it back to the stuff here so that the conjugate field gets some interpretation.

Explain this using the variational approach and explain that Ω is ohmic where Ohm's law is a linearization of conductivity and such (just like linear elasticity). The electromagnetic potential (or something) is a monogenic spinor?

5.1 Electromagnetism

Consider the spacetime algebra $\mathcal{G}_{1,3}$ seen in Example ex:spacetime_algebra 2.3. Then the spacetime multivector fields on Ω are $\mathcal{G}_{1,3}(\Omega)$ with the basis vector fields e_t, e_1, e_2, e_3. A vector field F on Ω is then given by

$$A = A_t e_t + A_1 e_1 + A_2 e_2 + A_3 e_3,$$

where each coefficient is a smooth scalar field. We denote now by $\boldsymbol{\nabla}_{st} = e^t \nabla_{e_t} + \sum_{j=1}^3 e^j \nabla_{e_j}$ the spacetime gradient and take $\boldsymbol{\nabla} = \sum_{j=1}^3 e^j \nabla_{e_j}$ as the spatial gradient. Let $\mathbf{A} = A_1 e_1 + A_2 e_2 + A_3 e_3$ be the spacelike part of the spacetime vector A and let $\phi = A_t$ be the timelike part. If the vector field does not depend on the temporal variable t we have $\nabla_{e_t} A = 0$ and thus

$$\boldsymbol{\nabla}_{st} A = \boldsymbol{\nabla} u e_t + \boldsymbol{\nabla} \wedge \mathbf{A} + \boldsymbol{\nabla} \cdot \mathbf{A}.$$

If we then take the Lorenz gauge condition $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, we have

$$\|\boldsymbol{\nabla}_{st} A\|^2 = \|\boldsymbol{\nabla} u\|^2 + \|\boldsymbol{\nabla} \wedge \mathbf{A}\|^2.$$

The Lagrangian for this field is then

$$\mathcal{L}(A) = \|\boldsymbol{\nabla}_{st} A\|^2 - A \cdot J,$$

where $J = \rho e_t + J_1 e_1 + J_2 e_2 + J_3 e_3$ is a spacetime vector field A lagrange multiplier?. The Euler-Lagrange equations with the gauge condition yields

$$\boldsymbol{\nabla}_{st}^2 A = J.$$

Let $\mathbf{J} = J_1 e_1 + J_2 e_2 + J_3 e_3$ and if we take a static four current $\nabla_{e_t} J = 0$ we must have $\nabla_{e_t} A = 0$ and we arrive at two equations

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \wedge u e_t = \rho e_t \qquad \text{and} \qquad \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \wedge \mathbf{A} = \mathbf{J},$$

of course one can take $\Delta u = \rho$, but we this equation arises from the spacetime formulation itself. Note that we did not force an inner product on the spatial vectors e_1, e_2, e_3 other than they are orthogonal with the temporal vector e_t. These equations we have are the invariant forms of the equations with respect to any (positive definite) spatial inner product. This will be important momentarily.

In this, we have realized the electric and magnetic fields

$$\boldsymbol{\nabla} \wedge u e_t = e \qquad \text{and} \qquad \boldsymbol{\nabla} \wedge \mathbf{A} = b,$$

and note the electric field e is a spacetime bivector and the magnetic field b is a purely spatial bivector that $\boldsymbol{\nabla} \wedge e = 0$ and $\boldsymbol{\nabla} \wedge b = 0$ are satisfied. The fact that e is a spacetime bivector means it behaves like a spacelike vector when acted on by spatial gradient $\boldsymbol{\nabla}$ owing to the static Faraday's law $\boldsymbol{\nabla} \times E = 0$. Since b is purely spatial, we see $\boldsymbol{\nabla} \wedge b = 0$ mimics the Gauss's law for magnetism if we take the unit spacelike trivector I and let $B = bI$ be the magnetic vector field we have $\boldsymbol{\nabla} \cdot B = 0$.

In the EIT problem, we begin with a region Ω with unknown symmetric positive definite conductivity matrix γ. We apply a static scalar potential ϕ on Σ which produces the potential u^ϕ in the interior. We assume Ω is an ohmic material in that Ohm's law $-\gamma \boldsymbol{\nabla} \wedge u^\phi = \mathbf{J}$ is satisfied. It follows that

$$-\gamma \boldsymbol{\nabla} \wedge u^\phi = \boldsymbol{\nabla} \cdot b.$$

The conservation law

$$\int_\Sigma J \cdot \nu d\Sigma = 0,$$

implies $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$ and we arrive at $\boldsymbol{\nabla} \cdot (\gamma \boldsymbol{\nabla} \wedge u) = 0$. See, for example, feldman_calderproblem_nodate [10].

The conductivity matrix was given in an terms of an orthonormal spatial basis under the Euclidean inner product and we can write the components γ^{ij} for $i, j = 1, 2, 3$. In uhlmann_inverse_2014 [18] we find a relationship between the intrinsic Riemannian metric on a space and the conductivity by

$$g_{ij} = (\det \gamma^{k\ell})^{\frac{1}{n-2}} (\gamma^{ij})^{-1}, \quad \gamma^{ij} = (\det g_{k\ell})^{\frac{1}{2}} (g_{ij})^{-1}. \tag{29}$$ eq:conducti

If we impose the inner product on the spatial components come from g with coefficients in the basis given by the above equations, we can note we have Ohm's law by

$$-\boldsymbol{\nabla} \wedge u^\phi = \mathbf{J},$$

In the static case when there are no free charges inside Ω, we have

$$\Delta_{st} A = \mathbf{J} \quad \text{in } \Omega,$$

and we arrive at $\Delta u = 0$ for the scalar potential and $\Delta \mathbf{A} = \mathbf{J}$ for the magnetic vector potential. In terms of the magnetic field bivector, we have $\boldsymbol{\nabla} \cdot b = \mathbf{J}$ and once again by Ohm's law we have $-\boldsymbol{\nabla} \wedge u^\phi = \boldsymbol{\nabla} \cdot b$. This leads us to consider the parabivector field $f = u + b$. We can note that f is (spatially) monogenic since

$$\boldsymbol{\nabla} f = 0 \quad \Longleftrightarrow \quad -\boldsymbol{\nabla} \wedge u^\phi = \boldsymbol{\nabla} \cdot b \text{ and } \boldsymbol{\nabla} \wedge b = 0,$$

is satisfied. We see now that the fact that the body Ω is ohmic gives us a necessary coupling between the scalar potential and the magnetic field.

5.2 Biot Savart Law

Recall the Biot Savart law from electromagnetic theory

$$\vec{B}(y) = \frac{1}{4\pi}\int_\Omega \mathbf{J} \times \frac{y-x}{|y-x|^3} d\Omega(x),$$

which satisfies

$$\boldsymbol{\nabla} \times \vec{B} = \mathbf{J} + \frac{1}{4\pi}\boldsymbol{\nabla} \wedge \int_\Omega \frac{\boldsymbol{\nabla}\cdot\mathbf{J}}{|y-x|}d\Omega(x) - \frac{1}{4\pi}\boldsymbol{\nabla}\wedge\int_\Sigma \frac{\mathbf{J}\cdot\nu}{|y-x|}d\Sigma(x)$$

In the EIT problem we do not allow charges to accumulate in the interior and so we must have

$$\boldsymbol{\nabla}\cdot\mathbf{J} = 0,$$

so long as $\boldsymbol{\nabla}\cdot\mathbf{J}$ is continuous feldman_calderproblem_nodate [10]. Hence we are left with

$$\boldsymbol{\nabla} \times \vec{B} = \mathbf{J} - \frac{1}{4\pi}\boldsymbol{\nabla}\wedge\int_\Sigma \frac{\Lambda(\phi)}{|y-x|}d\Sigma(x),$$

where Λ is the DN map.

Remark 5.1. It seems like this now says that u has a conjugate field B if and only if

$$\frac{1}{4\pi}\boldsymbol{\nabla}\wedge\int_\Sigma \frac{\Lambda(\phi)}{|y-x|}d\Sigma(x) = 0.$$

Assuming we can swap differentiation and integration we have

$$\boldsymbol{\nabla}\wedge\int_\Sigma \frac{\Lambda(\phi)}{|y-x|}d\Sigma(x) = \int_\Sigma \frac{\Lambda(\phi)(y-x)}{|y-x|^3}d\Sigma(x),$$

since $\boldsymbol{\nabla}\wedge\Lambda = 0$ In B.V. DN-Forms.

Remark 5.2. Perhaps we can just rearrange to see:

$$\mathcal{C}[\mathbf{J}] = \frac{1}{4\pi}\int_\Sigma \frac{y-x}{|y-x|^3}(\nu\cdot\mathbf{J} + \nu\wedge\mathbf{J})d\Sigma(x)$$

and we note $\mathbf{J}\cdot\nu = \Lambda(\phi)$ for which we have found

$$\frac{1}{4\pi}\int_\Sigma \frac{\Lambda(\phi)(y-x)}{|y-x|^3} = \boldsymbol{\nabla}\times\vec{B} - \mathbf{J},$$

Hence

$$\mathcal{C}[\mathbf{J}] = \boldsymbol{\nabla}\times\vec{B} - \mathbf{J} + \frac{1}{4\pi}\int_\Sigma \frac{y-x}{|y-x|^3}\nu\wedge\mathbf{J}d\Sigma(x)$$

In terms of geometric algebra, we wish to show the analogous statement for the magnetic bivector field

$$B(y) = \frac{1}{4\pi}\int_\Omega \mathbf{J}\times\frac{y-x}{|y-x|^3}d\Omega(x),$$

in that

$$\boldsymbol{\nabla}\cdot\frac{1}{4\pi}\int_\Omega \mathbf{J}\wedge\frac{y-x}{|y-x|^3}d\Omega(x) = \mathbf{J}.$$

5.2.1 Discussion

The scalar potential in the EIT problem arises inside of a four vector potential for the electromagnetic field. The electromagnetic potential satisfies Maxwell's equations which can be succinctly stated as $\boldsymbol{\nabla}^2_{st} A = J$, for the four current J. When the four current J does not depend on time, we arrive at the static equations where the electrostatic potential u and magnetic spatial vector potential $\mathbf{A}$ are split into separate equations. Removal of time dependence decouples these potentials. We realize the magnetic field as the bivector $b = \boldsymbol{\nabla} \wedge \mathbf{A}$ and the electric vector field $E = \boldsymbol{\nabla} \wedge u$.

These fields interact with materials which carry an intrinsic inner product related to the conductivity by eq:conductivity_metric 29. If the material is ohmic, we have Ohm's law given by $\boldsymbol{\nabla} \wedge u = \boldsymbol{\nabla} \cdot b$ which leads to the parabivector field $f = u + b$ to be monogenic. This relationship is important and is not fully realized without the proper treatment of the electromagnetic potential.

In an electrostatic boundary value problem, one can supply the scalar potential ϕ on the boundary of a region. This forced scalar potential induces the scalar potential inside of the region and the scalar potential is harmonic when the interior is free of charges. This scalar potential drives a current $\mathbf{J}$ via Ohm's law, and this current is related to the magnetic bivector field b. One may only have access to the boundary of the region and can make measurements of the resulting current flux $\mathrm{P}_\nu(\mathbf{J})$ that corresponds to a given input scalar potential ϕ. Is this enough to determine the underlying inner product of the region?

5.3 Generalization

Explain how we can put γ as a spatial metric and incorporate this into the geometric algebra for $\mathcal{G}_{1,n}(\Omega)$ stuff. Contract away time part again and we get the same equations.

What we have seen for the electromagnetic field is there is a coupling between the electric bivector field and the magnetic bivector field via the four vector potential. This can be generalized to fields in $\mathcal{G}_{1,n}(\Omega)$ to produce analogous equations.

5.4 Inverse problem

A particular application for the work we have done thus far is with the Calderón inverse problem. One can work with differential forms, but we have found forms to be rooted in multivectors contracted with a directed measure. We also note the previous portion on electromagnetism provides a convenient understanding for this problem. The forward problem in terms of geometric calculus is given by the following scenario. We have an ohmic Ω and we find the electrostatic potential u satisfying the Dirichlet problem

$$\begin{cases} \Delta u^\phi = 0 & \text{in } \Omega \\ u^\phi|_\Sigma = \phi & \text{on } \Sigma. \end{cases} \tag{30}$$

eq:dirichl

In the realm of Electrical Impedence Tomography (EIT), the Dirichlet data ϕ amounts to an input voltage along the boundary and by Ohm's law $\mathbf{J} = \boldsymbol{\nabla} \wedge u^\phi$ provides us the current. For any given solution to the boundary value problem, there is the corresponding Neumann data $\mathbf{J}^\perp = \mathrm{P}_\nu(\boldsymbol{\nabla} u^\phi)$ where ν is the normal to the boundary Σ defined by $\nu = I_\Sigma I$ for the oriented boundary pseudoscalar I_Σ. This motivates the so called Voltage-to-Current (VC) operator $\phi \mapsto \mathbf{J}^\perp$. In general, we refer to set of both boundary conditions $(\phi, \mathbf{J}^\perp)$ $\forall \phi$ as the *Cauchy data* and define the *Dirichlet-to-Neumann (DN) operator* Λ such that $\Lambda\phi = \mathbf{J}^\perp$. This mimics the VC operator in EIT. With our notation from before we have

$$\Lambda\phi = \mathrm{P}_\nu(\boldsymbol{\nabla} u^\phi) = \mathbf{J}^\perp.$$

Note that this operator Λ is often referred to as the *scalar* DN operator since the input is the scalar field ϕ whereas a more general operator on differential k-forms has been described in belishev_dirichlet_2008,sharafutdinov_complete_2013 [1, 17]. The inverse problem follows.

Calderón problem. Let Ω be an unknown Riemannian manifold with unknown metric g and with known boundary Σ and known DN operator Λ. Can one recover Ω and the spatial inner product g from knowledge of Σ and Λ?

5.5 Recovering monogenic fields from Λ

With the DN operator, we can reconstruct the boundary four current J. On Σ, we have the gradient $\boldsymbol{\nabla}_\Sigma$ inherited from $\boldsymbol{\nabla}$ on Ω. In particular, we have the relationship

$$\boldsymbol{\nabla}_\Sigma \phi = \mathrm{P}_{I_\Sigma}(\boldsymbol{\nabla}\phi),$$

which is accessible with our knowledge of ϕ and Σ. The boundary current is then

$$\mathbf{J}|_\Sigma = \boldsymbol{\nabla}_\Sigma \phi + \Lambda(\phi).$$

Though we do not have access to u^ϕ directly, we do know that $\Delta u^\phi = \rho$ and as such we have the boundary four current by

$$J|_\Sigma = \Delta u^\phi|_\Sigma \gamma_0 + \mathbf{J}|_\Sigma$$

as well as the interior four current $J = \mathbf{J}$ since the interior is free of charges. Defining the the four vector potential as before, we arrive at the extra equation $\Delta \mathbf{A} = \mathbf{J}$ in Ω. Once again define the magnetic bivector field $b = \boldsymbol{\nabla} \wedge \mathbf{A}$ and we note that Ohm's law implies $\boldsymbol{\nabla} \cdot b = -\boldsymbol{\nabla} \wedge u^\phi$ in Ω and so the parabivector field $f = u^\phi + b$ is spatially monogenic since we also have $\boldsymbol{\nabla} \wedge b = 0$. This all holds assuming that we can solve the electromagnetic Neumann boundary value problem

$$\begin{cases} \Delta A = \mathbf{J} & \text{in } \Omega \\ A = A_\Sigma & \text{on } \Sigma \end{cases}$$

Show that we can determine the magnetic potential A_Σ on the boundary. This may also show that the two notions of the DN operator are equivalent. That'd be nice.

If we show there is always a unique monogenic conjugate b for any harmonic u then this must be what we are doing here. Is this gauranteed by the Cauchy integral?

Though briefly we mentioned Ω as a Riemannian manifold, we now take Ω to be a region in $\mathbb{R}^n$ for brevity. Using the DN operator, one can define a *Hilbert transform* by

$$T\phi = d\Lambda^{-1}\phi,$$

as in belishev_dirichlet_2008 [1]. It has yet to be shown that this definition coincides with the definition in brackx_hilbert_2008 [6], but there is reason to believe they are related. The classical Hilbert transform on $\mathbb{C}$ inputs a harmonic function and outputs another harmonic function v such that $u+iv$ is holomorphic. Essentially, this translates into finding a conjugate bivector field b to u^ϕ such that $u^\phi + b$ is monogenic. First, we require ϕ satisfies

This statement should come from the lagrangian perspective hopefully.

$$\left(\Lambda + (-1)^n d\Lambda^{-1} d\right)\phi = 0, \tag{31}$$

eq:conjuga

where d is the exterior derivative on forms. They show how to find the image of this, perhaps I can show what the kernel is. As shown earlier in Section subsec:diff_forms 2.3, d amounts to $\boldsymbol{\nabla}\wedge$ on the multivector field constituent of a form. When condition eq:conjugate_requirement 31 is met, there exists a *conjugate form* $\epsilon \in \Omega^{n-2}(M)$. As well, ϵ is also coclosed in that $\delta\epsilon = 0$. To retrieve the constituent $(n-2)$-vector E, we just note $\epsilon = E \cdot dX_k$. Given Hodge duality, we have a 2-form β such that $\star\beta = \epsilon$ and the corresponding bivector $b^\star = E$. Combining the fields u^ϕ and b into the parabivector $f = u^\phi + b \in \mathcal{G}_n^{0+2}(\Omega)$. We then note that f is monogenic if and only if

$$\boldsymbol{\nabla}\wedge u = -\boldsymbol{\nabla}\cdot b \qquad \text{and} \qquad \boldsymbol{\nabla}\wedge b = 0.$$

Lemma 5.1. *Given the fields u^ϕ and b as above, the corresponding parabivector field*

$$f = u^\phi + b$$

is monogenic.

Proof. Let $\star\beta^\psi = \epsilon$ as before and note that

$$du^\phi = \star d\epsilon = \star d \star \beta^\psi, \tag{32}$$

eq:conjuga

as shown in Theorem 5.1 in belishev_dirichlet_2008 [1]. The multivector equivalent of the right hand side of Equation eq:conjugate_belishev [?] yields

$$\begin{aligned}
(\boldsymbol{\nabla}\wedge b^\star)^\star &= [(\boldsymbol{\nabla}\cdot b^\dagger)I]^\star \\
&= [I^{-1}((\boldsymbol{\nabla}\cdot b^\dagger)I)]^\dagger \\
&= ((\boldsymbol{\nabla}\cdot b^\dagger)I)^\dagger I \\
&= \boldsymbol{\nabla}\cdot b^\dagger && \text{since } \dagger \text{ of a vector is trivial} \\
&= -\boldsymbol{\nabla}\cdot b. && \text{since } \dagger \text{ of a bivector is -1}
\end{aligned}$$

Perhaps I should just show this property in the differntial forms section. Thus, we have $\boldsymbol{\nabla}\wedge u + \boldsymbol{\nabla}\cdot b = 0$. Since ϵ is coclosed we have

$$\begin{aligned}
0 = \boldsymbol{\nabla}\cdot b^\star &= \boldsymbol{\nabla}\cdot (I^{-1}b)^\dagger \\
&= \boldsymbol{\nabla}\cdot (b^\dagger I) \\
&= (\boldsymbol{\nabla}\wedge b^\dagger)I \\
\implies \quad 0 &= \boldsymbol{\nabla}\wedge b.
\end{aligned}$$

Perhaps I should just show this property in the differntial forms section. Thus $\boldsymbol{\nabla} f = 0$ and F is monogenic. □

We have shown that conjugate forms give rise to monogenic fields. We now seek to determine for what boundary conditions ϕ we have at our disposal. Let $E^{\parallel} := \mathrm{P}_{I_\Sigma}(E)$, with I_Σ the boundary pseudoscalar satisfying $\nu I_\Sigma = I$. Hence by Equation eq:projection_rejection_v ?? we have $E^{\parallel} = \mathrm{R}_\nu(E)$ then in investigating the requirement from Equation eq:conjugate_requirement 31 we find the multivector equivalent

$$(\Lambda + (-1)^n(\boldsymbol{\nabla}\wedge)\Lambda^{-1}(\boldsymbol{\nabla}\wedge))\phi = E^{\perp} + (-1)^n T E^{\parallel}$$

so we arrive at the fact that we must have

$$E^{\perp} = (-1)^{n-1} T E^{\parallel}.$$

In other words,

$$T\,\mathrm{R}_\nu(E) = (-1)^{n-1}\,\mathrm{P}_\nu(E).$$

Thus, the Hilbert transform maps tangential components of $\boldsymbol{\nabla} u^{\phi} = E$ to nontangential boundary components on the boundary.

6 Conclusion

Write a conclusion.

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