Commutative Algebras of Multivectors from the Scalar Dirichlet-to-Neumann Operator

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Abstract

TODO

1 Outline

1. Show that we can determine the metric on M from axial monogenic paravectors at least in the case for $\mathbb{R}^3$ or something.
2. Recover a space of monogenic paravectors from Λ.
3. Show that this space of monogenic paravectors contains the axial subalgebras.
4. Use this fact to recover g from Λ.

2 Introduction

mention that we will only work with positive definite g and mention what holds in that case so I don't have to again. Refer to the boundary ∂M as Σ the whole time as well.

Key citations for geometric algebra/calculus [9, 11] and for the Hilbert transform [6, 1].

In 1980, Alberto Calderón proposed an inverse problem in his paper *On an inverse boundary value problem* [7] where he asks if one can determine the conductivity matrix of a medium from Cauchy data supplied on the boundary. In practice, one determines the Cauchy data from measurements of voltage and current which often leads to this being referred to as the Electrical Impedance Tomography (EIT) problem. In this landscape, this mapping is known as the voltage-to-current map. Very much related is the Calderón problem for Riemannian manifolds which is introduced in [10] and [12]. Rather than determining the conductivity matrix from the voltage-to-current map, one attempts to reconstruct the Riemannian metric from the Dirichlet-to-Neumann (DN) map. These probably have to be smooth to some extent... The DN map takes any given Dirichlet boundary values and outputs the corresponding Neumann data in order to generate the Cauchy data. In dimension $n = 2$, the EIT and Calderón problem are not equivalent but in dimensions $n \neq 2$ the problems are equivalent via a change of variables seen in [17].

One approach to reconstructing the Riemannian metric in dimension $n = 2$ appears in [2], where Belishev uses the Boundary–Control (BC) method to determine the metric up to conformal class. The BC method takes an algebraic approach to determining the conformal class of the metric by first constructing the algebra of holomorphic functions on M from the DN map and realizing M as the topologized spectrum of this algebra this is probably not exactly right. In dimension $n = 2$, the Laplace-Beltrami operator is conformally invariant, and this result cannot be improved. Belishev and Vakulenko attempts to move towards generalizing this approach to dimension $n = 3$ in [3, 4] by replacing the complex structure with a quaternionic structure but this has not lead to a complete solution.

To extend the BC method to higher dimensions, one essentially needs to generalize two important pieces. First, from the DN map, recover the algebra of holomorphic functions on M. This algebra provides means for constructing the complex structure on the manifold. The complex structure on a surface, one can then determine the Hodge star operator. Then, using theory from Gelfand for commutative Banach algebras, M is shown to be homeomorphic to the topologized spectrum of the algebra of holomorphic functions on M with the complex structure determining the metric g. In this paper, we provide a means for generalizing the first component of the BC method to manifolds of arbitrary dimension as the following theorem. give more background on the BC method

Theorem 2.1. *Let Λ be the scalar Dirichlet-to-Neumann map on some unknown smooth, connected, and oriented Riemannian manifold (M,g), then Λ determines a unique Banach $*$-algebra of even monogenic Clifford fields on M.*

Given an inner product, the complex structure is naturally isomorphic to the even sub-Clifford algebra of scalars and bivectors in dimension $n=2$ with the inner product acting as the quadratic form. Fortunately, this can be generalized for arbitrary dimension. The notion of holomorphicity is then replaced by monogenicity as the Wirtinger derivative $\frac{\partial}{\partial \bar{z}}$ is exchanged by the more general Dirac operator D. The DN map allows one to recover an algebra of conjugate scalars and bivectors whose sum are monogenic functions. From this set, we generate a Banach $*-$algebra of even monogenic Clifford fields on M.

3 Clifford algebras

Let (V,Q) be an n-dimensional vector space V over some field K with quadratic form Q. Then, we can construct the tensor algebra as the space

$$\mathcal{T}(V) \coloneqq \bigoplus_{j=0}^{\infty} V^{\otimes j} = K \bigoplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots .$$

From the tensor algebra $\mathcal{T}(V)$, we can quotient by the ideal generated by $v \otimes v - Q(v)$ to define *Clifford algebra* $C\ell(V,Q)$. That is,

$$C\ell(V,Q) = \mathcal{T}(V) \ / \ \langle v \otimes v - Q(v) \rangle.$$

Let $e_1, \dots, e_n$ be an arbitrary basis for V, then we can consider the tensor product of basis elements $e_i \otimes e_j$. The tensor product in $\mathcal{T}(V)$ induces a product in the quotient $C\ell(V,Q)$ which we refer to as the *Clifford multiplication*. We write this product as concatenation $e_i e_j$ and define the multiplication by

$$e_i e_j = \begin{cases} Q(e_i) & \text{if } i=j, \\ e_i \wedge e_j & \text{if } i \neq j, \end{cases}$$

where $\wedge$ is the exterior product satisfying $v \wedge w = -w \wedge v$ for all $v,w \in V$. As a consequence, the exterior algebra $\bigwedge(V)$ can be realized as a subalgebra of any Clifford algebra over V or as a Clifford algebra with a trivial quadratic form $Q=0$.

Note that $C\ell(V,Q)$ is a $\mathbb{Z}$-graded algebra with elements of grade-0 up to elements of grade-n. We refer to grade-0 elements as scalars, grade-1 elements as vectors, grade-2 elements as *bivectors*, grade-k elements as *k-vectors*, and grade-n elements as *pseudoscalars*. For each grade, there is a basis of $\binom{n}{k}$ simple k-vectors. For example, if $\dim(V)=4$, then there are $\binom{4}{3}=4$ 3-vectors that form a basis. In particular, one may choose the following list

$$e_1 \wedge e_2 \wedge e_3, \quad e_1 \wedge e_2 \wedge e_4, \quad e_1 \wedge e_3 \wedge e_4, \quad e_2 \wedge e_3 \wedge e_4.$$

In general, an element $A \in C\ell(V,Q)$ is written as a linear combination of basis elements of all possible grades and we refer to M as a *multivector*. To extract the grade-k components of A, we use the notation

$$\langle A \rangle_k$$

to denote the grade-k components of the multivector M. For example, we could let $A \in C\ell(\mathbb{R}^4, \|\cdot\|)$ be given by

$$A = 1 + 2e_1 + e_3 + 3e_1e_3e_4$$

and we have

$$\langle A \rangle_0 = 1, \quad \langle A \rangle_1 = 2e_1 + e_3, \quad \langle A \rangle_2 = 0, \quad \langle A \rangle_3 = 3e_1e_3e_4, \quad \langle A \rangle_4 = 0.$$

Thus, a general multivector A can be given by

$$A = \sum_{k=0}^{n} \langle A \rangle_k.$$

If A contains only grade-k components, then we say that A is *homogeneous*. For example, we can think of vectors as homogeneous grade-1 multivectors.

The Clifford multiplication of vectors can be extended to multiplication of vectors with homogeneous grade-k multivectors. In particular, given a vector $v \in C\ell(V,Q)$ and a homogeneous grade-k multivector $A_k \in C\ell(V,Q)$, we have

$$aA_k = \langle aA_k \rangle_{k-1} + \langle aA_k \rangle_{k+1}, \tag{1}$$

eq:vector_r

which decomposes the multiplication into a grade lowering *interior product* and a grade raising *exterior product*. This allows us to extend the Clifford multiplication further. Given a homogeneous grade-s multivector B_s, we have

$$A_kB_s = \langle A_rB_s \rangle_{|r-s|} + \langle A_rB_s \rangle_{|r-s|+2} + \cdots + \langle A_rB_s \rangle_{r+s}. \tag{2}$$

eq:general_

Some specific graded elements of the product are worth noting here, Add in $\times$ for bivectors as the part that comes out with the same grade. This is special.

$$A_k \cdot B_s := \langle A_kB_s \rangle_{|k-s|} \tag{3}$$

$$A_k \wedge B_s := \langle A_kB_s \rangle_{k+s} \tag{4}$$

$$A_k \rfloor B_s := \langle A_kB_s \rangle_{s-k} \tag{5}$$

eq:left_co

$$A_k \lfloor B_s := \langle A_kB_s \rangle_{k-s}. \tag{6}$$

eq:right_c

This rule for multiplication then allows for the multiplication of two general multivectors in $C\ell(V,Q)$. Then we also have the identities

$$A_r \cdot B_s = A_r \rfloor B_s \qquad \text{if } k \leq s \tag{7}$$

eq:left_co

$$A_r \cdot B_s = A_r \lfloor B_s \qquad \text{if } k \geq s. \tag{8}$$

eq:right_c

For homogeneous k-vectors A_k and B_k, the products above simplify to

$$A_k \lfloor B_k = A_k \rfloor B_k = A_k \cdot B_s. \tag{9}$$

dot_equival

Finally, for a vector α we have

$$\alpha A_k = \alpha \rfloor A_k + \alpha \wedge A_k, \tag{10}$$

so the $\cdot$ and $\lfloor$ notation coincide for left multiplication by vectors. Proofs and more details for these identities can be found in chisolm_geometric_2012 [8]. The key reasoning behind the extra multiplication symbols $\rfloor$ and $\lfloor$ is to avoid needing to pay special attention to the specific grade of each multivector in a geometric product. The product $\cdot$ on A_k and B_s depends on k and s and as such given by either $\rfloor$ or $\lfloor$ but one must know k and s in order to define this product exactly.

Talk about when we will switch notations. For vectors, for example, it's worth switching it to conform to our typical thoughts of inner products and what not.

If a homogeneous grade-k multivector A_k is an exterior product of k vectors so $A_k = v_1 \wedge \cdots \wedge v_k$, we say that A_k is a *k-blade*. For example, the basis vectors given in 4_dim_basis ?? are all blades. We refer to an $(n-1)$-blade as a *pseudovector* and one can note that every $(n-1)$-vector is a pseudovector.. In other literature, some will refer to a k-blade as *simple* or *decomposable*. If we are given two k-blades $A_k = \alpha_1 \wedge \cdots \wedge \alpha_k$ and $B_k = \beta_1 \wedge \cdots \wedge \beta_k$ we have

$$A_k \cdot B_k = \det(\alpha_i \cdot \beta_j)_{i,j=1}^k, \tag{11}$$

eq:dot_pro

which is equivalent to $A_k \rfloor B_k$ and $A_k \lfloor B_k$ through dot_equivalent_contraction 9.

fix this and replace the later version.

$C\ell(V,Q)$ is naturally a $\mathbb{Z}/2\mathbb{Z}$-graded algebra since we can partition the grades of elements into either even or odd. We say that A is an *even* (resp. *odd*) *multivector* if A is a sum of only even (resp. odd) grade elements. Taking note of the multiplication defined in eq:general_clif 2, one can see that the multiplication of a an even (resp. odd) multivector with another even (resp. odd) multivector results in an even (resp. odd) multivector. Thus, the even and odd multivectors form closed subalgebras of $C\ell(V,Q)$.

Example 3.1. Let $V = \mathbb{R}^2$ and let the quadratic form Q be given by the Euclidean norm $Q(\cdot) = \|\cdot\|$. Let e_1 and e_2 be the standard unit vectors and note that we have 1 as the basis scalar, and e_1e_2 as the basis pseudoscalar. Thus, a general multivectors m and r can be written as

$$m = m_0 + m_1e_1 + m_2e_2 + m_{12}e_1e_2, \qquad r = r_0 + r_1e_1 + r_2e_2 + r_{12}e_1e_2.$$

We can then multiply mr and find

$$\langle mr \rangle_0 = m_0r_0 + m_1r_1 + m_2r_2 - m_{12}r_{12},$$

$$\langle mr \rangle_1 = (m_0r_1 + m_1r_0 - m_2r_{12} + m_{12}r_2)e_1 + (m_0r_2 + m_2r_0 + m_1r_{12} - m_{12}r_1)e_2,$$

and

$$\langle mr \rangle_2 = (m_1r_2 - m_2r_1)e_1e_2.$$

Most notably, we see that $(e_1e_2)^2 = -1$ and this allows us to consider a multivector

$$z = x + ye_1e_2$$

as a representation of the complex number $\zeta = x + iy$. Thus, the even subalgebra of this Clifford algebra is indeed isomorphic to the complex numbers $\mathbb{C}$. Show that there is a nice way to represent complex numbers sitting inside of every clifford algebra dim¿2

uaternions

Example 3.2. Let $V = \mathbb{R}^3$ and $Q(\cdot) = \|\cdot\|$. Then, let

$$b_1 = e_2e_3, \quad b_2 = e_3e_1, \quad b_3 = e_1e_2,$$

and note that we can write a even multivector as

$$q = q_0 + q_1b_1 + q_2b_2 + q_3b_3.$$

Note as well that

$$b_1^2 = b_2^2 = b_3^2 = -1,$$

and

$$b_1b_2b_3 = +1.$$

In this case, this even subalgebra is extremely close to being a copy of the quaternion algebra. Indeed, one can arrive at a representation of the quaternions by taking

$$\boldsymbol{i} \leftrightarrow B_1, \quad \boldsymbol{j} \leftrightarrow -B_2, \quad \boldsymbol{k} \leftrightarrow B_3,$$

and noting that we then have $ijk = -1$ as well as $i^2 = j^2 = k^2 = 1$. A more rigorous explanation is again provided in doran_geometric_2003 [9].

In the case where V has a (pseudo) inner $(\cdot,\cdot)$, we can define the quadratic form Q for a clifford algebra by $Q(v) = (v,v)$. In this case, we refer to the Clifford algebra $C\ell(V,Q)$ as a *geometric algebra* and we generally put $\mathcal{G}(V)$ and assume the inner product will be given alongside. For example, when $V = \mathbb{R}^n$ and we define Q using the Euclidean inner product, we have $C\ell(V,Q) = \mathcal{G}(\mathbb{R}^n)$ and moreover we put $\mathcal{G}(R^n) = \mathcal{G}_n$. For more information on the topic, the text doran_geometric_2003 [9] gives an extremely thorough treatment of geometric algebra as well as a wide range of applications to physics problems.

3.1 Duality and pseudoscalars

eudoscalars

In a geometric algebra, one has access to an inner product and hence there is a natural isomorphism between V and V^*. Namely, given an arbitrary basis e_i for V there exists the basis f_i for V^* such that $f_i(e_j) = \delta_{ij}$. There is then a unique map $\sharp\colon V^* \to V$ with $f \mapsto f^\sharp$ such that

$$f_i^\sharp \cdot e_j = \delta_{ij}.$$

For sake of ease, we put $e^i := f_i^\sharp$. In terms of a generic basis for V, if the coefficients for the inner product is given by $g_{ij} = e_i \cdot e_j$, we can put $e^i = g^{ij}e_j$ where g^{ij} is the coefficients schindler_geometric_2020

to matrix inverse of g_{ij}. One can see this definition taken in [14]. Similarly, there is the isomorphism $\flat \colon V \to V^*$ given by $e \mapsto e^\flat$ satisfying

$$e_i^\flat(e_j) = \delta_{ij}.$$

Thus, there is no need to distinguish between the vector space V and its dual V^* as it suffices to consider V itself with reciprocal basis elements e^i.

Based on the inner product, a volume element can be defined by $\mu = e_1 \wedge e_2 \wedge \cdots \wedge e_n = \sqrt{|g|} I$ where $\sqrt{|g|}$ is the square root of the determinant of the matrix g_{ij} and I us the unit pseudoscalar. Note that $I = \frac{1}{\sqrt{|g|}} e_1 \wedge e_2 \wedge \cdots e_n$. We can define μ^{-1} such that $\mu^{-1}\mu = 1 = \mu\mu^{-1}$ and analogously I^{-1}. One can equivalently put $e^j = (-1)^{j-1} e_1 \wedge e_2 \wedge \cdots \wedge \check{e_j} \wedge \cdots \wedge e_n \mu^{-1}$ and note that this gives $\mu^{-1} = e^n \wedge \cdots \wedge e^1$.

maybe mention the reverse here.

Conveniently, the unit pseudoscalar satisfies the relation

$$IA_k = (-1)^{k(n-1)} A_k I.$$

Thus, I commutes with the even subalgebra, and anticommutes with the odd subalgebra. Moreso, the pseudoscalar allows one to exchange the interior and exterior products as

$$(A_k \wedge B_s) I = A_k \cdot (B_s I) \tag{12}$$

eq:wedge_t

for homogeneous k and s-vectors A_k and B_s. The above holds true if we replace I with I^{-1} when working in spaces where g is positive definite due to the fact that I^{-1} differs only by a sign. If $B_s = C_{n-s} I$ then,

$$(A_k \cdot B_s) I^{-1} = A_k \cdot (C_{n-s} I) = (A_k \wedge C_{n-s}) I = (A_k \wedge (B_s I)) I,$$

and in particular

$$(A_k \cdot B_s) I^{-1} = A_k \wedge (B_s I). \tag{13}$$

eq:dot_to_

This shows the duality between the inner and exterior products.

define reverse operator

Example 3.3. Consider $\mathcal{G}_3$ with the standard orthonormal vector basis $e_1, \dots, e_n$. Then, we can define the *cross product* of two vectors u and v by

$$u \times v = (u \wedge v) I^{-1}.$$

The special fact of $\mathcal{G}_3$ is that vectors and bivectors are Hodge dual to one another. That is to say, bivectors are pseudovectors when the underlying vector space is dimension 3. One can also note that the vector $w = u \times v$ is sometimes refered to as axial and in other cases the pseudovector $u \wedge v$ is referred to as axial.

3.1.1 Projection onto subspaces

There is a direct relationship between unit k-blades and k-dimensional subspaces. Indeed, each unit k-blade B_k corresponds to a k-dimensional subspace. That is, each point in $Gr(k, n)$ corresponds to a unit k-blade. Since blades represent subspaces, they also give us a compact way of projecting vectors into subspaces. In particular, given a vector a, the projection onto the subspace spanned by the k-blade B_k is given by

$$(a \cdot B_k) B_k^{-1}.$$

4 Multivector fields

We want to generalize the setting of geometric algebra to include a smooth structure. One can take the work above for $\mathcal{G}_n$ and consider a C^∞-module structure as opposed to the $\mathbb{R}$-algebra structure in the proceeding section. For brevity, we utilize the same notation $\mathcal{G}_n$ for the C^∞-module and $\mathbb{R}$-algebra as the structure will be clear from context. This smooth setting simply makes the coefficients of the global basis blades given by C^∞ functions as opposed to $\mathbb{R}$ scalars. In this case, we refer to a generic element in the C^∞-module $\mathcal{G}_n$ as a *multivector field*.

4.1 Directional derivative and gradient

remove hodge star from here and put it in the integration section. Get rid of stuff with exterior derivative and codifferential

Note that $\mathbb{R}^n$ has global coordinates and thus we can choose a global vector basis $e_1, \ldots, e_n$ and generate $\mathcal{G}_n$ from this basis. Multiplication of fields is computed pointwise. The directional derivative ∇_ω is defined in the usual sense, and we can develop the gradient as $\boldsymbol{\nabla} = \sum_i e^i \nabla_{e_i}$. We will adopt the Einstein summation convention when needed (e.g., the repeated indices in $\boldsymbol{\nabla} = e^i \nabla_{e_i}$ indicates summation over i). Note then that $\omega \cdot \boldsymbol{\nabla} = \nabla_\omega$ defines the directional derivative via the gradient.

This structure defined above is typically referred to as *geometric calculus*. The setting for geometric calculus extends the setting of differential forms and reduces some of the complexity with tensor computations. The gradient operator acts on a homogeneous k-vector A_k by

$$\boldsymbol{\nabla} A_k = \langle \boldsymbol{\nabla} A_k \rangle_{k-1} + \langle \boldsymbol{\nabla} A_k \rangle_{k+1} \coloneqq \boldsymbol{\nabla} \cdot A_k + \boldsymbol{\nabla} \wedge A_k.$$

Thus, the gradient splits into two operators $\boldsymbol{\nabla}\cdot$ and $\boldsymbol{\nabla}\wedge$. Here, $\boldsymbol{\nabla}\wedge$ can be identified with the exterior derivative d and $\boldsymbol{\nabla}\cdot$ can be identified with the codifferential δ on differential forms (see schindler_geometric_2020 [14]). This of course means the standard properties that apply to d and δ apply to $\boldsymbol{\nabla}\wedge$ and $\boldsymbol{\nabla}\cdot$. Namely, we have

$$(\boldsymbol{\nabla}\wedge)^2 = 0 \qquad (\boldsymbol{\nabla}\cdot)^2 = 0 \tag{14}$$ eq:differer

and likewise $\delta = (-1)^{n(k-1)+1} \star \boldsymbol{\nabla} \wedge \star$ and thus

$$\boldsymbol{\nabla}\cdot = (-1)^{\cdot}{..} \star \boldsymbol{\nabla} \wedge \star \tag{15}$$

when acting on a homogeneous k-vector. Indeed, this property follows from wedge_to_dot ?? and one can realize the $\pm$ term as arising from the commutivity properties of the unit pseudoscalar. Since eq:differential_properties 14 holds, the gradient operator gives rise to the grade preserving Laplace-Beltrami or Hodge-Laplacian operator

$$\Delta = \boldsymbol{\nabla}\boldsymbol{\nabla} = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \wedge + \boldsymbol{\nabla} \wedge \boldsymbol{\nabla} \cdot,$$

which is manifestly coordinate invariant by definition. It also motivates the use of the physicist notation $\boldsymbol{\nabla}^2 = \Delta$.

4.2 Differential forms

Introduce the directed measure and hodge duality. Then prove greens formula. Give an example of the 3-ball in spherical coordinates and surface integrals and stuff

Naturally, we would also like to be able to integrate multivectors. In order to do so, we appeal to the language of differential forms and build a path from multivectors to forms. Given the coordinate system x^i on $\mathbb{R}^n$, we form the basis of tangent vector fields $\partial_i = \frac{\partial}{\partial x^i}$ with the reciprocal 1-forms dx^i which are gradients of the coordinate functions. Thinking of 1-forms as linear functions on tangent vectors, we have $dx^i \partial_j = \delta^i_j$. The benefit of this definition is that the 1-forms dx^i carry a natural measure and we can form product measures via the exterior product. For example, we have $d\Sigma = e_i \wedge e_j dx^i dx^j$. Then,we have $(e^j \wedge e^i) \cdot d\Sigma = dx^i dx^j - dx^j dx^i$ which retains the antisymmetry of the differential forms.

Is dX_k really just a k-density? Good answers on stack exchange

In an n-dimensional space with a position dependent inner product g, we have the n-dimensional volume measure $d\Omega = \sqrt{|g|}dx^1 \dots dx^n$. If we then define $dX_n = e^n \wedge \cdots \wedge e^1 dx^1 \dots dx^n$ we then find that $d\Omega = I^\dagger \cdot dX_n$ as

$$I^\dagger \cdot dX_n = \sqrt{|g|}(e_n \wedge \cdots \wedge e_1) \cdot (e^n \wedge \cdots \wedge e^1) dx^1 \cdots dx^n.$$

Similarly, for $k < n$, we can define the k-dimensional volume measure as

$$dX_k = \frac{1}{k!}(e^{i_k} \wedge \cdots \wedge e^{i_1}) dx^{i_1} \cdots dx^{i_k}.$$

We can now write a k-form α_k as $\alpha_k = A_k \cdot dX_k$. In this sense, a differential form is made up of two essential components namely the multivector field and the k-dimensional volume measure. This decomposition is important when the underlying space has interesting topological or geometrical features. In $\mathbb{R}^n$, this distinction is less important.

For example, if we wish to write a 2-form α_2 we take $dX_2 = \frac{1}{2!} e^j \wedge e^i dx^i dx^j$ and $A_2 = a_{ij} e_i \wedge e_j$ to yield

$$\alpha_2 = A_2 \cdot dX_2 = \frac{a_{ij}}{2!}(e_i \wedge e_j) \cdot (e^j \wedge e^i) dx^i dx^j = \frac{a_{ij}}{2!}(dx^i dx^j - dx^j dx^i)$$

Thus, we arrive at an isomorphism between k-forms and k-vectors as a contraction with the k-dimensional volume measure dX_k since

$$\alpha_k = A_k \cdot dX_k.$$

Hence, we can see now how a differential form simply appends the measure attached to the underlying space. We can also see how this generalizes the musical isomorphisms $\sharp$ and $\flat$ by taking a vector field a and noting

$$a \cdot dX_1 = a^i e_i \cdot e^j dx^j = a^i dx^i, \tag{16}$$ eq:line_el

corresponds to the usual $\flat$ map on vector fields.

The exterior algebra of differential forms comes with an addition $+$ and exterior multiplication $\wedge$. We note that the sum of two k-forms α_k and β_k that $\alpha_k + \beta_k$ is also a k-form which we can see by letting $\alpha_k = A_k \cdot dX_k$ and $\beta_k = B_k \cdot dX_k$ and putting

$$\alpha_k + \beta_k = (A_k \cdot dX_k) + (B_k \cdot dX_k) = (A_k + B_k) \cdot dX_k,$$

due to the linearity of $\cdot$. If instead had an s form β_s then we have the exterior product

$$\alpha_k \wedge \beta_s = (A_k \wedge B_k) \cdot dX_{k+s},$$

where $dX_{k+s} = 0$ if $k + s > n$.

With differential forms one also has the exterior derivative d, the Hodge star $\star$, and the codifferential δ. Given we can write a differential k-form as $\alpha_k = A_k \wedge dX_k$, we wish to define d, $\star$, δ by their actions on the k-vector A_k. In particular, we have

$$d\alpha_k = (\boldsymbol{\nabla} \wedge A_k) \cdot dX_{k+1},$$

which realizes the exterior derivative as the grade raising component of the gradient $\boldsymbol{\nabla}$. Of course, for scalars, this returns the gradient as desired. The Hodge star inputs a k-form and outputs a $(n-k)$-form and we define $\star$ so that for two k-forms α_k and β_k we have $\alpha_k \wedge \star\beta_k = (A_k \cdot B_k^\dagger)d\Omega$. This is since

$$A_k \cdot B_k^\dagger = \langle A_K, B_K \rangle \sqrt{|g|},$$

where $\langle A_K, B_K \rangle$ is the typical inner product on k-vectors extended through to exterior algebra. Thus, a coordinate expression for $\star$ acting on multivectors is given by $B_k^\star = (I^{-1}B_k)^\dagger$ so that $\star\beta = (I^{-1}B_k)^\dagger \cdot dX_{n-k}$. Indeed, we have

$$\begin{aligned}
\alpha_k \wedge \star\beta_k &= (A_k^\dagger \wedge B_k^\star) \cdot dX_n \\
&= (A_k \wedge (I^{-1}B_k)^\dagger) \cdot dX_n \\
&= (A_k \wedge (B_k^\dagger I)) \cdot dX_n \\
&= (A_k \cdot B_k^\dagger) I^{-1} \cdot dX_n \\
&= A_k \cdot B_k^\dagger d\Omega,
\end{aligned}$$

since $I^{-1} = I^\dagger$ in spaces with g positive definite.

Cite Hestenes.

Then, in the typical fashion we define the codifferential $\delta = (-1)^{n(k-1)+1} \star d \star$ when acting on k-forms. Then,

$$\begin{aligned}
\delta\alpha_k &= (-1)^{n(k-1)+1} \star d \star \alpha_k \\
&= (-1)^{n(k-1)+1} \star d[(I^{-1}A_k)^\dagger \cdot dX_{n-k}] \\
&= (-1)^{n(k-1)+1} \star [\boldsymbol{\nabla} \wedge (A_k^\dagger I)] \cdot dX_{n-k+1} \\
&= (-1)^{n(k-1)+1} \star [(\boldsymbol{\nabla} \cdot A_k^\dagger) I^{-1}] \cdot dX_{n-k+1} \\
&= (-1)^{n(k-1)+1} \star [
\end{aligned}$$

introduce the $\Omega^k(M)$ notation here at some point. Show the Green's and Stokes' theorems in terms of multivectors.

4.3 Riemannian manifolds

4.3.1 Geometric algebra structure

Here we really just need to show a Riemannian manifold is locally a vector manifold then show integration on M is well defined. Then show the general greens and stokes' theorems.

Given the natural invariance of the differential operator $\boldsymbol{\nabla}$, extending geometric calculus to non-Euclidean spaces follows readily. For two other introductions to the topic, see schindler_geometric_2020 [14] which we follow more closely and `https://math.uchicago.edu/~amathew/dirac.pdf` which is more general. In order to build a geometric algebra structure parameterized by smooth manifolds, we need a smoothly varying inner product defined on the tangent bundle As such, we will be working with Riemannian manifolds.

Let (M, g) be an n-dimensional smooth, compact, and oriented Riemannian manifold with boundary ∂M. Then at each point $p \in M$, we have the tangent space T_pM and the metric g_p which can be combined to yield a geometric algebra structure. We can glue together the geometric algebras $\mathcal{G}(T_pM, g_p)$ the bundle

$$\mathcal{G}(M,g) \coloneqq \dot{\bigcup}_{p\in M} \mathcal{G}(T_pM, g_p),$$

which we refer to as the *geometric algebra bundle*. We can then define the space of C^∞-smooth sections of the geometric algebra bundle, $\mathcal{G}(M)$, and refer to its elements as the *multivector fields* on M. The space of multivector fields then forms a both a $\mathbb{Z}$- and $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{R}$-algebra with an inherited geometric multiplication from the pointwise product. The $\mathbb{Z}$-grading allows us to define the space of homogeneous k-vector fields which we denote by $\mathcal{G}^k(M)$. The $\mathbb{Z}/2\mathbb{Z}$-grading allows us to define the space of even (resp. odd) multivector fields denoted by $\mathcal{G}^{[0]}(M)$ (resp. $\mathcal{G}^{[1]}(M)$).

Given the metric g, we have an isomorphism between T^*M and TM which extends to an isomorphism between the multivector fields and the corresponding differential forms $\mathcal{G}^*(M)$. Indeed, this is entirely encapsulated by the musical isomorphisms $\sharp$ and $\flat$ seen in Subsection subsection:duality_and_pseudoscalars 3.1 when a choice of local coordinates is made. As before, it suffices to work with $\mathcal{G}(M)$ as one always has access to reciprocal elements.

4.3.2 Differential operators

Locally, all n-dimensional manifolds take coordinates in $\mathbb{R}^n$ by $\phi\colon \mathcal{O} \subset M \to \mathcal{U} \subset \mathbb{R}^n$ with open sets $\mathcal{O}$ and $\mathcal{U}$. Then a point $p \in M$ corresponds to $\phi(p) = (x^1(p), \dots, x^n(p)) = x \in \mathbb{R}^n$. We put $e_i = \frac{\partial}{\partial x^i}$ as a local vector field basis. We put $e_i(p)$ to denote a tangent vector T_pM in local coordinates. We also have the dual 1-forms $dx^1, \dots, dx^n$ which satisfy $dx^i(e_j) = \delta^i_j$. We have shown that $dx^i = e_i \cdot e^j dx^j$ and hence we identify e^i as the multivector equivalent to dx^i as $e^i \cdot e_j = \delta^i_j$ as desired. This produces the line element seen in eq:line_element 16.

One then extends the directional derivative ∇_ω in $\mathbb{R}^n$ to the covariant derivative ∇_ω that acts on multivector fields on M. We use the same notation, but the context will make the distinction clear. This is done in usual manner; start with the unique Levi-Civita connection on M and form the coordinate independent covariant derivative on M. In schindler_geometric_2020 [14] one will find the construction of ∇_ω and a list of properties. Following that, we have the gradient ∇ given in local coordinates by $e^i \boldsymbol{\nabla}_{e_i}$ which decomposes into the $\boldsymbol{\nabla}\wedge$ and $\boldsymbol{\nabla}\cdot$. Similarly, we define the Hodge-Laplacian $\Delta = \boldsymbol{\nabla}^2$. Using the terminology from differential forms, we have the following definition.

Definition 4.1. Let $\alpha \in \mathcal{G}^k(M)$, $\beta \in \mathcal{G}^{k+1}(M)$, and $\gamma \in \mathcal{G}^{k-1}(M)$. Then

- α is *closed* if $\boldsymbol{\nabla} \wedge \alpha = 0$.
- α is *exact* if $\alpha = \boldsymbol{\nabla} \wedge \gamma$ for some γ.
- α is *coclosed* if $\boldsymbol{\nabla} \cdot \alpha = 0$.
- α is *coexact* if $\alpha = \boldsymbol{\nabla} \cdot \beta$ for some β.
- α is *harmonic* if $\Delta\alpha = 0$.

Geometric calculus includes another definition for multivectors that is a big motivation for those who study Clifford analysis.

Definition 4.2. Let $f \in \mathcal{G}(M)$. Then we say that f is *monogenic* if $f \in \ker(\boldsymbol{\nabla})$.

Monogenic fields are of utmost importance as they have many beautiful properties. For example, in regions of Euclidean spaces, a monogenic field f can be completely determined by its Dirichlet boundary values. This is the exact analog of the Cauchy integral formula in complex analysis. More can be seen include some sources.

go through this stuff and clean it up using the pseudoscalar definitions and what not above. It will be much easier. Add stokes theorem in with directed measures. These will work for coordinate patches and I'll have to show it converges using a partition of unity.

Fix this remaining stuff. Intuition from this part can be spread through the multivector parts of this paper.

The metric also induces the Riemannian volume form $\mu \in \mathcal{G}^n(M)$ on M which is of top degree and hence a pseudoscalar as well as a inner product $\langle \cdot, \cdot \rangle_{L_2(\Sigma)}$ on each fiber of $\mathcal{G}^k(M)$. Let ω and η be homogeneous Clifford fields, then the *Hodge star* operator is then defined to be the unique operator $\star\colon \mathcal{G}^k(M) \to \mathcal{G}^{n-k}(M)$ satisfying

$$\omega_p \wedge \star\eta_p = \langle \omega_p, \eta_p \rangle_{L_2(\Sigma)} \mu_p,$$

at any point $p \in M$ and for any $\omega, \eta \in \mathcal{G}^k(M)$. Note that this extends to all of M as

$$\langle\!\langle \omega, \eta \rangle\!\rangle = \int_M \omega \wedge \star\eta.$$

For example, let M be a submanifold of $\mathbb{R}^3$ with the Euclidean inner product. Thus, μ is the standard volume form inherited from $\mathbb{R}^3$. Then, M has a global orthonormal coordinates x^1, x^2, and x^3 which induce the orthonormal set of basis 1-forms dx^1, dx^2, and dx^3. One should think of 1-forms and vectors on M as being equivalent. Of course, this is made rigorous by the isomorphism given by the Riesz representation theorem with the given inner product. maybe this is worth showing. With these coordinates, the Hodge star on 1-forms is given explicitly by

$$\star dx^1 = dx^2 \wedge dx^3, \quad \star dx^2 = dx^3 \wedge dx^1, \quad \star dx^3 = dx^1 \wedge dx^2.$$

We can see that for 1-forms on M, the Hodge star outputs a 2-form that represents an oriented plane element that is perpendicular to the original 1-form. This plane element is also scaled by the magnitude equal to that of the original 1-form due to the linearity in the original definition. Working this way allows us to recover the same notion of an inner product of vectors in $\mathbb{R}^3$ but with 1-forms instead. Indeed, if we have the 1-forms

$$\alpha = \alpha_1 dx^1 + \alpha_2 dx^2 + \alpha_3 dx^3, \qquad \beta = \beta_1 dx^1 + \beta_2 dx^2 + \beta_3 dx^3,$$

then

$$\alpha \wedge \star\beta = (\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3) dx^1 \wedge dx^2 \wedge dx^3.$$

Note that the coefficient on $dx^1 \wedge dx^2 \wedge dx^3$ is exactly the inner product if we utilized the $\sharp$ map on the 1-forms and applied the Euclidean inner product. That is, $\alpha^\sharp \cdot \beta^\sharp$.

Since M is a manifold with boundary ∂M, we have the map $\iota \colon \partial M \hookrightarrow M$ as the inclusion of the boundary into M. This map induces the pullback $\iota^* \colon T^*M \to T^*\partial M$ on forms. Of particular interest will be pulling back 0- and $(n-1)$-forms to the boundary. For 0-forms u, $\iota^*(u)$ is simply the boundary values for a smooth function. In the case for a 1-form α on M, we have that $\star\alpha$ is a $(n-1)$-form everywhere perpendicular to α. Thus, the pullback $\iota^*(\star\alpha)$ is an $(n-1)$-form on the $(n-1)$-dimensional boundary manifold. Indeed, since $\star\alpha$ is perpendicular to α, $\iota^*(\star\alpha)$ represents the normal component of the 1-form α at the boundary ∂M. Letting $\mu_{\partial M}$ be the boundary volume form, the total flux of the field α through ∂M is given by

$$\int_\Sigma \iota^*(\star\alpha) \tag{17}$$ eq:flux

This equation is not right and I should write it in terms of the multivectors first. This should be easy: Just project the multivector field into the subspace $T_\xi \partial M$. It would be

$$(A_k(p) \cdot I_\partial(p)) I_\partial^{-1}(p)$$

Define the outward normal vector first. Define a pullback for fields? Does the above not just do that? One could prove this maybe pulling back tensors then taking the necessary quotient. This is where clifford algebras may help.

Let I_Σ be oriented pseudoscalar on Σ then we can define the unit normal vector ν by requiring $\nu = (-1)^{n-1} I_\Sigma I^{-1}$. Then the *tangential component* of A_k on ∂M by $A_k^\parallel = (A_k \rfloor I_\partial) \rfloor I_\partial^{-1}$ and the *normal component* $A_k^\perp = (A_k \rfloor \nu) \rfloor \nu$. These are exactly **t** and **n** from schwarz_hodge_1995 [15]. For a vector field a we arrive at

$$(a \rfloor \nu) d\Sigma = ((aI) \rfloor I_\Sigma) I_\Sigma^{-1} \cdot dX_{n-1}.$$

Taking $\alpha = a \cdot dX_1$ to be the 1-form corresponding to a vector field a, if we then want to integrate to find the total flux of a through Σ, we have

$$\int_\Sigma a \cdot \nu d\Sigma = \int_\Sigma (aI) \cdot I_\Sigma d\Sigma = \int_\Sigma \iota^*(\star\alpha).$$

Intuitively, aI takes the orthogonal complement of a and we then project this onto I_Σ. Note both I and I_Σ is are unit pseudoscalars, and thus $(aI) \cdot I_\Sigma$ measures the strength of $a^\perp$ and the measure $d\Sigma$ takes into account the local geometry of Σ.

use projection from Chisolm eqn 94).

We could also define ν such that $\nu \wedge I_\Sigma = I$ then $\nu \cdot (I_\Sigma I^{-1}) = 1$. We want whatever makes the above integral work though. In some sense, the direction of μ doesn't matter we just want it to behave nicely with the other pseudoscalars. I think the $(-1)^{n-1}$ just considers if we take $I_\Sigma \wedge \nu = I$.

5 Gelfand representation

5.1 Axial monogenic fields

Rephrase in terms of hilbert transform? Copy stuff from other paper here.

For this section, let $n = \dim(M) = 3$. Supposing that ϕ satisfies eq:conjugate_requirement 21 (I dropped this requirement for now) one can generate paravectors $f = u + b$ and define the space of *monogenic paravectors*

$$\mathcal{M}(M) = \{f \mid \boldsymbol{\nabla} f = 0\}$$

The original requirement that $\Delta u^\phi = 0$ is obtained since f is monogenic. We can then generate an algebra from this set by

$$\mathcal{A}(M) = \{fg \mid f, g \in \mathcal{M}(M)\},$$

but, as mentioned in belishev_algebras_2019 [5], this algebra generated by these monogenic fields in $\mathcal{M}(M)$ produce fields that are not monogenic. Indeed, this is a well known fact in Clifford analysis mentioned in schepper_introductory_nodate [13]. Fundamentally, however, this fact that the product of monogenics is no longer monogenics makes the direct approach in belishev_calderon_2003 [2] intractable. This issue comes down to the lack of commutivity of paravectors in dimensions higher than 2. However, for certain so-called axial fields, commutivity is regained. In fact, the construction of these fields was done in belishev_algebra [3] in order to create a closed commutative algebra of monogenic fields. These axial fields will relate directly to complex holomorphic functions.

In belishev_algebras_2017, belishev_algebras_2019 [3, 5], the definition of axial is defined for quaternion fields and the properties are discussed. It is evident from the Example ex:quaternions 3.2 that quaternion fields are analogous to paravector fields via the given identification. This identification is key in connecting the relevant algebras to the DN map. So we proceed by following the definitions in place.

Definition 5.1. Let $F = U + B$ be a paravector and let ω be a unit vector. We then say that F is ω*-axial* if $\nabla_\omega F = 0$.

Make sure I define the covariant derivative and stuff

From the grade preserving nature of ∇, we see that the requirement $\nabla_\omega f = 0$ reduces to a grade-wise requirement

$$\nabla_\omega U = 0 \qquad \text{and} \qquad \nabla_\omega B = 0.$$

Thus, we can write $B = \beta\omega I = \beta B$ for a smooth scalar field β satisfying $\nabla_\omega \beta = 0$. So long as ω-axial monogenics are closed under multiplication, we can recover a sub-algebra of holomorphic functions inside of the larger algebra $\mathcal{M}(M)$ generated by monogenic paravectors. If we take two ω-axial monogenic fields $f = u_f + \beta_f B$ and $g = u_g + \beta_g B$, then we have

$$fg = u_f u_g - \beta_f \beta_g + B(u_f b_g + u_g b_f). \tag{18}$$

eq:axial_mu

Namely, this follows from the fact that

$$B^2 = (\omega I)^2 = -1.$$

This fact is essential. In essence, we now have a direct representation of a holomorphic function if we let $i = B$. One should then realize that an ω-axial monogenic f is built by translating a holomorphic function along the direction defined by ω since f has no dependence on this direction. Moreover, it is clear that B is a 2-blade. Note that for some unit vectors r and p, we have $\omega = r \times p$. Thus, $B = (r \times p)I^{-1}$. Indeed, this fits with the interpretation above in that B is acting as a pseudoscalar in some manner. To say this fully, B is the pseudoscalar for the plane spanned by r and p. Another way of rephrasing f being ω-axial is then to say that f is constant on all translations of the rp-plane. In this case, f depends solely on two variables and is exactly a holomorphic function. This is simply dual to the notion of being constant along straight lines in a 3-dimensional space. One can think of ω as a member of the Grassmanian $Gr(1,3)$ whereas its dual $B = \omega I$ lies in $Gr(2,3)$ which is isomorphic. Indeed, I gives a natural isomorphism between $Gr(1,3)$ and $Gr(2,3)$.

If f is an ω-axial monogenic, then we can recall the Cauchy-Riemann equations yield

$$\boldsymbol{\nabla} u = (\omega \wedge \boldsymbol{\nabla}\beta)I \qquad \text{and} \qquad -B \wedge \boldsymbol{\nabla}\beta B = 0. \tag{19}$$

eq:axial_ca

On this plane given by the blade B, we want to realize B acting as i for a holomorphic function. In particular, this means we need the Dirac operator to respect multiplication by constant paravectors (which is analogous to scaling complex functions by a complex number). If one has an ω-axial monogenic f, we wish that for a constant paravector $k = k_1 + k_2 B$ that $\boldsymbol{\nabla}(kf) = 0$ as well. $\boldsymbol{\nabla}$ is clearly $\mathbb{R}$-linear, so it sufficies to show the following.

_monogenic **Lemma 5.1.** *Let $f = u + \beta B$ be an ω-axial monogenic paravector, then Bf is ω-axial and monogenic.*

Proof.

I can use equations 82 from Chisolm to avoid the use of the cross product

It is clear that Bf is ω-axial due to the grade preserving linearity of the covariant derivative.

To see that Bf is monogenic, we take $Bf = Bu - \beta$. Then,

$$\boldsymbol{\nabla}(Bf) = \boldsymbol{\nabla}(Bu) - \boldsymbol{\nabla}\beta,$$

where we have the graded components

$$\langle \boldsymbol{\nabla}(Bf) \rangle_1 = (\boldsymbol{\nabla} \cdot Bu) - \boldsymbol{\nabla}\beta$$
$$\langle \boldsymbol{\nabla}(Bf) \rangle_3 = (\boldsymbol{\nabla} \wedge Bu).$$

Note that

$$\boldsymbol{\nabla}\cdot(Bu) = -\omega\times(\boldsymbol{\nabla}\wedge u) = -\omega\times(\omega\times\boldsymbol{\nabla}\beta) = -\omega(\nabla_\omega\beta)+\boldsymbol{\nabla}\beta = \boldsymbol{\nabla}\beta$$

by eq:axial_cauchy_riemann 19 and thus $\langle\boldsymbol{\nabla}(Bf)\rangle_1 = 0$.

For the grade-3 component,

$$\boldsymbol{\nabla}\wedge(Bu) = \omega\cdot(\boldsymbol{\nabla}\wedge B)II^{-1}u = I^{-1}\nabla_\omega u = 0$$

since u is ω-axial. Thus we have $\boldsymbol{\nabla}(Bf)=0$ is monogenic. □

The point here is that we have now effectively found functions that can be scaled by $\alpha+\beta B$ and remain monogenic. This is the constant multiple rule for the Wirtinger derivative for complex functions. Generically, if I take some multivector A times a monogenic field f, Af need not be monogenic.

Proposition 5.1. *Let f and g be monogenic and ω-axial. Then $fg=gf$, fg is ω-axial, and fg is monogenic.*

Proof.

Clean this up with better notation

- First, it is clear that $fg=gf$ by Equation eq:axial_multiplication 18.
- The product fg is ω-axial simply by the product rule of the multivector covariant derivative. That is,
$$\nabla_\omega(fg) = (\nabla_\omega f)g + f(\nabla_\omega g) = 0.$$
- To see that the product is monogenic, we have
$$\boldsymbol{\nabla}(fg) = \boldsymbol{\nabla}(u_f u_g - b_f b_g + B(u_f b_g + u_g b_f)).$$
  Then the grade-1 components are
$$\langle\boldsymbol{\nabla}(fg)\rangle_1 = \boldsymbol{\nabla}\wedge(u_f u_g - b_f b_g) + \boldsymbol{\nabla}\cdot B(u_f b_g + u_g b_f),$$
  and the grade-3 components are
$$\langle\boldsymbol{\nabla}(fg)\rangle_3 = \boldsymbol{\nabla}\wedge B(u_f b_g + u_g b_f).$$
  For the grade-1 components, we have
$$\begin{aligned}\boldsymbol{\nabla}(u_f u_g - b_f b_g) &= (\boldsymbol{\nabla}u_f)u_g + u_f(\boldsymbol{\nabla}u_g) - (\boldsymbol{\nabla}b_f)b_g - b_f(\boldsymbol{\nabla}b_g)\\ \boldsymbol{\nabla}\cdot I\omega(u_f b_g + u_g b_f) &= (\boldsymbol{\nabla}\cdot I\omega u_f)b_g + u_f(\boldsymbol{\nabla}\cdot Bb_g) + b_f(\boldsymbol{\nabla}\cdot Bu_g) + (\boldsymbol{\nabla}\cdot Bb_f)u_g,\end{aligned}$$
  and since f and g are both monogenic we have
$$\langle\boldsymbol{\nabla}(fg)\rangle_1 = (\boldsymbol{\nabla}\cdot Bu_f - \boldsymbol{\nabla}b_f)b_g + (\boldsymbol{\nabla}\cdot B)u_g - \boldsymbol{\nabla}b_g)b_f.$$

Then, note that
$$\langle \nabla B f \rangle_1 = \nabla \cdot B u_f - \nabla b_f = 0$$
by Lemma lem:mult_by_i_monogenic 5.1 and likewise for $\langle \nabla B g \rangle_1$. Thus,
$$\langle \nabla (fg) \rangle_1 = 0.$$

Likewise, for the grade-3 component of the gradient
$$\langle \nabla (fg) \rangle_3 = I^{-1} \nabla_\omega (u_f b_g + u_g b_f) = 0,$$
by the product rule for the covariant derivative and the fact that f and g are ω-axial.

□

Add in power series stuff here. We can write $f = u + ib$ as a power series of $x + yB$?

As we move through the different axial vectors, it's as if we're doing some tomography on 2d slices of the domain.

Now describe how to do the rest of the algebra stuff here.

Theorem 5.1. *(2D Gelfand) For any $\mu \in \mathcal{M}$ there is a point $z^\mu \in D$ such that $\mu = \delta_{z^\mu}$. The map*
$$\gamma \colon \mathcal{M} \to D, \quad \mu \mapsto z^\mu$$
is a homemorphism so that $\mathcal{M} \cong D$. The Gelfand transform
$$\Gamma \colon \mathcal{A}(D) \to C^{\mathbb{C}}(\mathcal{M}), \quad (\Gamma f)(\mu) = \mu(f), \quad \mu \in \mathcal{M}$$
is an isometric isomorphism onto its image, so that $\mathcal{A}(D) \cong \Gamma(\mathcal{A}(D))$.

In local coordinates the following definition works...

Definition 5.2. Let B be a unit 2-blade then we say that a paravector $f = u + \beta B$ is B-planar if $f((x \cdot B)B^{-1}) = f(x)$ for all x.

Theorem 5.2. *In $\mathbb{R}^3$, if $\omega I = B$, then B-planar is equivalent to ω-axial.*

Proof.

finish

□

Definition 5.3. The we denote by the space $\mathcal{P}$ to be the constant paravectors and by $\mathcal{P}(M)$ to be the paravector fields on M.

The space $\mathcal{P}$ can act on $\mathcal{P}(M)$ to produce a left module structure. In particular, we let $p = p_0 + p_2 \in \mathcal{P}$ act on a paravector field $f = u + b$ by
$$p \curvearrowright f = p_0 f + p_2 u + p_2 \cdot b + p_2 \times b.$$
This is distinct from the geometric multiplication as we would also produce the grade-4 element $p_2 \wedge b$ when the dimension is greater than three. In particular,
$$\langle p \curvearrowright f \rangle_0 = p_0 f + p_2 \cdot b \qquad \text{and} \qquad \langle p \curvearrowright f \rangle_2 = p_2 u + p_2 \times b.$$

Proposition 5.2. *The space $\mathcal{M}(M)$ is a left $\mathcal{P}$-module. I should probably distinguish between the constants and the fields in the notation for $\mathcal{G}_n$.*

Proof. I'm not sure this is really necessary to prove. □

Proposition 5.3. *An ω-axial field f is harmonic if and only if*

$$\dots$$

This was shown earlier?

Lemma 5.2. *(Density) (Monogenics are in the span of axial monogenics) We have that*

$$\overline{\operatorname{Span}\{\mathcal{A}_\omega(M) \mid \omega \in \operatorname{Gr}(2,n)\}} = \mathcal{M}(M).$$

(Lemma 1 from B.S.)

Theorem 5.3. *(Stone-Weierstrass for Quaternions)*

Definition 5.4. Define the *$\mathcal{P}$-dual* $\mathcal{M}^\times(M)$ as

$$\mathcal{M}^\times(M) \coloneqq \{l \in \mathcal{L}(\mathcal{M}(M);\mathcal{P}) \mid l(pf) = pl(f),\ \forall p \in \mathcal{M}(M),\ p \in \mathcal{P}\}$$

and the *(multiplicative) $\mathcal{P}$-functionals* by

$$\mathfrak{M}^{\mathcal{P}} \coloneqq \{\mu \in \mathcal{M}^\times(M) \mid \mu(fg) = \mu(f)\mu(g),\ \forall f,g \in \mathcal{A}_\omega(M),\ \omega \in \operatorname{Gr}(2,n,\})$$

$\mathcal{M}^\times(M)$ are the paravector valued functionals (they are linear over paravectors) and in particular we have the functionals $\mathfrak{M}^{\mathcal{P}}$ that are multiplicative on ω-axial monogenics..

Theorem 5.4. *(Main result) For any $\mu \in \mathfrak{M}^{\mathcal{P}}$, there is a point $x^\mu \in M$ such that $\nu = \delta_x^{\mathcal{P}}$. The map*

$$\gamma \colon \mathfrak{M}^{\mathcal{P}} \to M, \quad \mu \mapsto x^\mu$$

is a homeomorphism, so that $\mathfrak{M}^{\mathcal{P}} \cong M$. The Gelfand transform

$$\Gamma \colon \mathcal{M}(M) \to C(\mathfrak{M}^{\mathcal{P}};\mathcal{P}), \quad (\Gamma f)(\mu) \coloneqq \mu(f), \quad \mu \in \mathfrak{M}^{\mathcal{P}},$$

is an isometry onto its image, so that $\mathcal{M}(M) \cong \Gamma(\mathcal{M}(M))$.

6 Dirichlet to Neumann Operator and Hilbert Transform

Let $u^\phi \in \Omega^0(M)$ be a smooth 0-form (scalar function) that is a solution to the following Dirichlet boundary value problem

$$\begin{cases} \Delta u^\phi = 0 & \text{in } M \\ \iota^*(u) = \phi. \end{cases}, \tag{20}$$

eq:dirichl

where Δ refers to the Laplace-Beltrami operator on differential forms. For the Calderón problem, the manifold M and metric g are unknown and one seeks to determine as much as possible about (M, g) from measurements on the boundary. Due to the relationship between the EIT and Calderón problem, we use the notation ϕ for the Dirichlet boundary values since ϕ should be thought of as the prescribed voltage along the boundary.

Rewrite this equation in terms of multivectors. Write down the general version for k-vector fields and specialize to the scalar version later after showing the B.S. lemmas hold.

For any given solution to the boundary value problem, there is the corresponding Neumann data $E = \iota^*(\star du)$. As with ϕ, the notation E is used as the Neumann data measured in the EIT problem corresponds to the electric field flux at the boundary. One attains the current J by multiplying with E by the boundary conductivity matrix. The set of both boundary conditions (ϕ, E) is the *Cauchy data* and the *Dirichlet-to-Neumann (DN) map* Λ is defined such that $\Lambda\phi = E$ and in particular this yields $\iota^*(\star du^\phi) = E$. Note that this map Λ is often referred to as the *scalar* DN map as $\Lambda\colon \Omega^0(\partial M) \to \Omega^{n-1}(\partial M)$ inputs a scalar Dirichlet condition. An extension of the DN map to forms can be found in [1, 16] belishev_dirichlet_20. The Calderón problem for Riemannian manifolds is then to recover the pair (M, g) up to isometry from complete knowledge of the DN map Λ.

Denote by $\mathcal{H}(M) = \{u \in \Omega^0(M) \mid du = 0\}$ the space of harmonic 0-forms on M. From the DN map, one can define the *Hilbert transform* $T\colon \iota^*\mathcal{H}(M) \to \iota^*\mathcal{H}(M)$. This function acts on traces of harmonic forms by

$$T\phi = d\Lambda^{-1}\phi,$$

and is defined in [1] belishev_dirichlet_2008. The authors show benefit to defining the Hilbert transform as it provides the ability to generate so called conjugate forms. When the condition

$$\left(\Lambda + (-1)^n d\Lambda^{-1} d\right)\phi = 0, \tag{21}$$

eq:conjuga

is met, then there exists a *conjugate form* $\epsilon^\psi \in \Omega^{n-2}(M)$ with boundary trace $\psi = \iota^*\epsilon$ satisfying $Td\phi = d\psi$. As well, ϵ is also coclosed in that $\delta\epsilon = 0$.

Now, there exists a 2-form b^ψ such that $\star b^\psi = \epsilon$. Using the isomorphism between forms and multivectors, we can let U be the scalar field corresponding to u^ϕ and we can let B be the bivector field corresponding to b^ψ. We can add these to yield the paravector $F = U + B \in \mathcal{G}(M)$. Recall that a multivector field is monogenic if $\boldsymbol{\nabla} F = 0$. Applying this to the paravector F yields the equations

$$\boldsymbol{\nabla} \wedge U = -\boldsymbol{\nabla} \cdot B \qquad \text{and} \qquad \boldsymbol{\nabla} \wedge B = 0.$$

The conjugacy relation $du^\phi = \star d\epsilon^\psi$ is equivalent to having the multivector F be monogenic.

Lemma 6.1. *Given the forms u^ϕ and b^ψ conjugate as above, the corresponding paravector field*

$$F = U + B$$

is monogenic.

Proof. Let $\star b^\psi = \epsilon$ and note that

$$du = \star d\epsilon = \star d \star b^\psi.$$

Now, writing the multivector equivalent of the right hand side yields

$$\begin{aligned}
(\boldsymbol{\nabla} \wedge B^\star)^\star &= [(\boldsymbol{\nabla} \cdot B^\dagger)I]^\star \\
&= [I^{-1}((\boldsymbol{\nabla} \cdot B^\dagger)I)]^\dagger \\
&= ((\boldsymbol{\nabla} \cdot B^\dagger)I)^\dagger I \\
&= I^\dagger(\boldsymbol{\nabla} \cdot B^\dagger)^\dagger I \\
&= \boldsymbol{\nabla} \cdot B^\dagger && \text{since } \dagger \text{ of a vector is trivial} \\
&= -\boldsymbol{\nabla} \cdot B. && \text{since } \dagger \text{ of a bivector is -1}
\end{aligned}$$

Thus, we have $\boldsymbol{\nabla} \wedge U + \boldsymbol{\nabla} \cdot B = 0$. Since ϵ is coclosed we have

$$\begin{aligned}
0 = \boldsymbol{\nabla} \cdot B^\star &= \boldsymbol{\nabla} \cdot (I^{-1}B)^\dagger \\
&= \boldsymbol{\nabla} \cdot (B^\dagger I) \\
&= (\boldsymbol{\nabla} \wedge B^\dagger)I \\
&= \boldsymbol{\nabla} \wedge B.
\end{aligned}$$

Thus $\boldsymbol{\nabla} F = 0$ and F is monogenic. □

Add about the 2D problem and generating algebras?

7 Other

7.1 Cauchy and Poisson integrals

In regions of $\mathbb{R}^n$, one can define a Cauchy integral operator and Hilbert transform for multivector fields. The details of these integral operators are laid out in brackx_hilbert_2008 [6]. Note that the authors there take the opposite signature to $\mathcal{G}_n$ and define the gradient operator as $\underline{\partial} = e_j \nabla_{e_j}$. Thus, we have $\boldsymbol{\nabla} = g^{ij}\underline{\partial}$. Nonetheless, the fundamental solution to $\boldsymbol{\nabla}$ is a vector field given by

$$E(x) = \frac{1}{a_m}\frac{x}{|x|^m},$$

for $x \in \mathbb{R}^n$. This is clear to see if we take e_i to be a (local) orthonormal basis

$$\begin{aligned}
\boldsymbol{\nabla} \wedge E &= \frac{1}{a_m}\left(\frac{1}{|x|^n}\nabla_{e_i}x^j + x^j\nabla_{e_i}\frac{1}{|x|^n}\right)e^i \wedge e_j \\
&= \left(\frac{1}{|x|^n}\delta_i^j - \frac{3x^ix^j}{|x|^{n+2}}\right)e^i \wedge e_j \\
&= -\frac{3x^ix^j}{|x|^{n+2}}e^i \wedge e_j && \text{since } e^i \wedge e_i = 0 \\
&= 0 && \text{since } e^j \wedge e_i = -e^i \wedge e_j \text{ for an orthonormal basis.}
\end{aligned}$$

(see `https://math.stackexchange.com/questions/811248/wedge-product-between-nonorthogonal-`
This is also clear since E is a radial field and thus has no curl. Then, let B_ϵ be the n-ball of radius ϵ centered at the origin and we have

$$\begin{aligned}\int_{B_\epsilon} \boldsymbol{\nabla} \cdot E d\Omega &= \int_{S_\epsilon} E \cdot \nu d\Sigma \\ &= \frac{1}{a_n} \int_{S_\epsilon} \frac{x \cdot \frac{x}{|x|}}{|x|^n} d\Sigma \\ &= \frac{1}{a_n} \int_{S_\epsilon} \frac{1}{\epsilon^{n-1}} d\Sigma \\ &= \frac{1}{a_n} \int_{S_\epsilon} \frac{1}{\epsilon^{n-1}} \epsilon^{n-1} d\phi_1 d\phi_2 \cdots d\phi_{n-1} \\ &= 1.\end{aligned}$$

Let $\partial M = \Sigma$ and define now the $\mathcal{G}_n$ valued inner product on multivector fields $f, g \in L_2(\Sigma)$

$$\langle f, g \rangle_{L_2(\Sigma)} = \int_{\partial M} f(\zeta) g(\zeta) d\Sigma(\zeta).$$

We can then define the *Cauchy kernel* for $x \in M$ and $\zeta \in \partial M$ using the fundamental solution E as

$$C(\zeta, x) = -\frac{1}{a_n} \nu(\zeta) E(x - \zeta)$$

where $\nu(\zeta)$ is the outward normal vector to the hypersurface $\Sigma = \partial M$. Note the inclusion of the minus sign is due to the signature of the inner product g. The Cauchy integral for $\phi \in L_2(\partial M)$ is then

$$\mathcal{C}[\phi](x) = \langle C(\zeta, x), \phi(\zeta) \rangle_{L_2(\Sigma)} = \frac{1}{a_n} \int_\Sigma \frac{\zeta - x}{|x - \zeta|^n} \nu(\zeta) \phi(\zeta) d\Sigma(\zeta).$$

The most important properties of the Cauchy integral is that $\mathcal{C}[\phi]$ is monogenic in M and for a scalar function ϕ, $\mathcal{C}[\phi]$ is a paravector. Specifically,

$$\begin{aligned}\mathcal{C}[\phi](x) &= \frac{1}{a_n} \int_\Sigma \frac{\zeta - x}{|x - \zeta|^n} \nu(\zeta) \phi(\zeta) d\Sigma(\zeta) \\ &= \frac{1}{a_n} \left(\int_\Sigma \phi(\zeta) \frac{\zeta - x}{|x - \zeta|^n} \cdot \nu(\zeta) d\Sigma(\zeta) + \int_\Sigma \phi(\zeta) \frac{\zeta - x}{|x - \zeta|^n} \wedge \nu(\zeta) d\Sigma(\zeta) \right)\end{aligned}$$

Similarly, for the n-ball of radius r, $B_r \subset \mathbb{R}^n$, we have the *Poisson kernel*

$$P(\zeta, x) = \frac{1}{a_n} \frac{r^2 - |x|^2}{r|x - \zeta|^n}.$$

Notably, we have the Poisson integral

$$\mathcal{P}[\phi](x) = \langle P(\zeta, x), \phi(\zeta) \rangle_{L_2(\Sigma)} = \frac{1}{a_n} \int_\Sigma \phi(\zeta) \frac{r^2 - |x|^2}{r|x - \zeta|^n},$$

which is harmonic on B_r and extends continuously onto Σ. Briefly letting $g_{ij} = \delta_{ij}$, if we then consider the Cauchy integral over $\Sigma = \partial B_r = S_r$ then it is apparent that the Poisson integral deviates from the scalar part of the Cauchy integral as

$$\langle \mathcal{C}[\phi](x) \rangle_0 = \frac{1}{a_n} \int_\Sigma \phi(\zeta) \frac{r^2 - x \cdot \xi}{r|x-\xi|^n} d\Sigma(\zeta).$$

Sadly, this means that we do not have the boundary behavior of the Cauchy integral that we desire. Namely, the $\iota^* \langle \mathcal{C}[\phi](x) \rangle_0 \neq \phi$ in general. It is also worth noting that it is an open problem to determine a general form for the Poisson integral for other domains in $\mathbb{R}^n$. However, since $\mathcal{C}[\phi](x)$ is monogenic, we have that the components are harmonic.

- Prove that the Hilbert transforms are equivalent on traces of harmonic functions. Specifically, $Td\phi = d\psi$.
- Discuss hardy spaces as closure of $\mathcal{M}(M)$.
- $H^2 = 1$ on $L_2(\partial M)$ which should show we satisfy the theorem below.

Let $M \subset \mathbb{R}^3$ be a

7.2 Generating axial monogenics

The following questions remain for a domain in $\mathbb{R}^3$.

Question 7.1. For what boundary values $\varphi \in C_\infty(\Sigma)$ can we generate axial monogenics?

Question 7.2. Do these boundary values exhaust the whole axial algebra $\mathcal{A}_\omega(M)$?

Fix an axis ω which defines the blade $B = \omega I$ and thus defines the B-plane in $\mathbb{R}^3$. Then, let $f = u + \beta B$ be an ω-axial monogenic. We can then determine the boundary values for f on Σ by orthogonal projection onto the B-plane. That is, we care only about the components of f perpendicular to the axis ω and hence we take for $\zeta \in \Sigma$

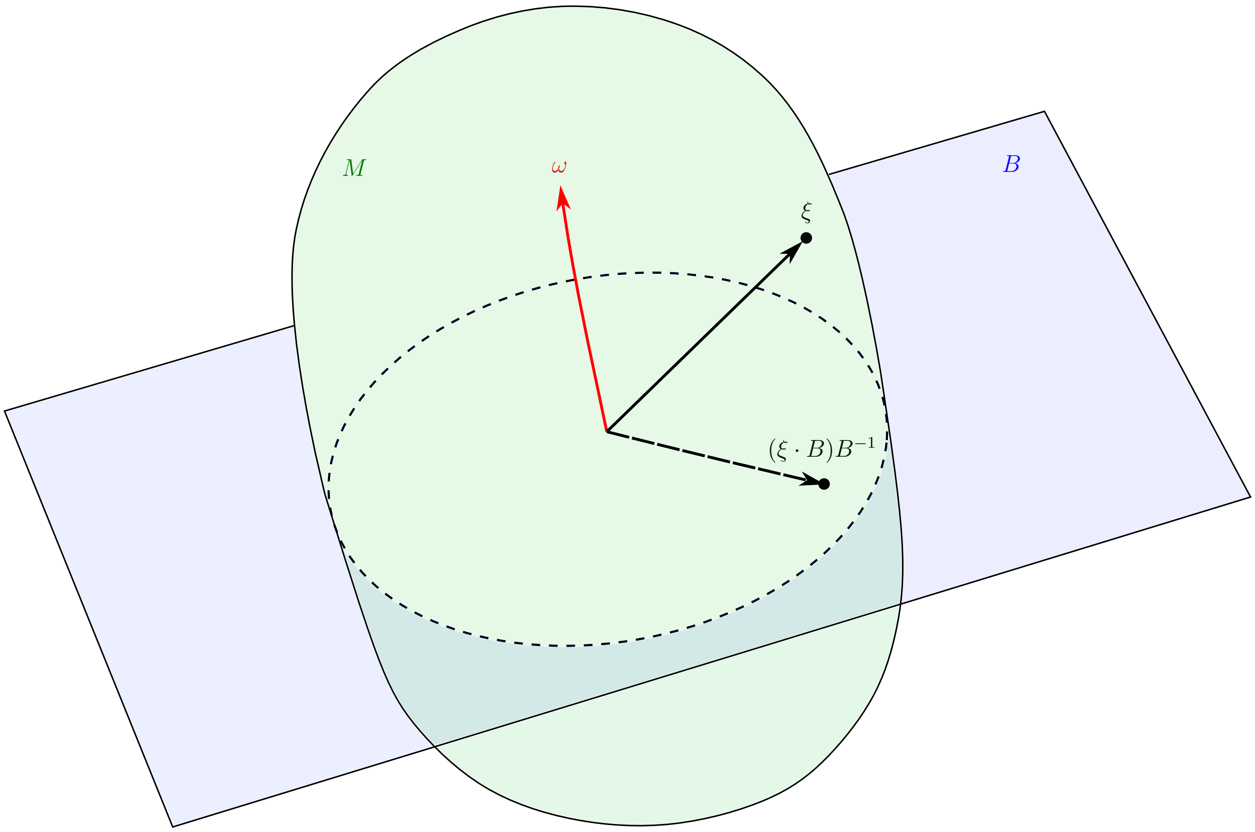
$$\zeta^\perp = \omega\omega \wedge \zeta = (x \cdot B)B^{-1}.$$

showing the relationship between projection onto a plane and being orthogonal to an axis in $\mathbb{R}^3$. Specifically, this means that the relationship $f(x) = f(x + t\omega)$ can be written as

$$f(x) = f((x \cdot B)B^{-1}),$$

in that we only care about the portion of x along the plane given by B. Thus, for $\xi \in \Sigma$ we have

$$f(\xi) = f((\xi \cdot B)B^{-1}).$$



So boundary values of axial monogenics are axial and...?.

Example 7.1. Consider the 3-dimensional example with $M = B_3$ and $\Sigma = S^2$. Let e_1, e_2, e_3 be a global orthonormal basis and let $g_{ij} = \delta_{ij}$. Then let $B = e_1 \wedge e_2$. Then the paravector field $f(x^1, x^2, x^3) = x^1 + x^2 B$ is e_3-axial. Clearly we can see that $f(x^1, x^2, x^3 + t) = f(x^1, x^2, x^3)$ for any t. f is also monogenic as one can show

$$\boldsymbol{\nabla} f = e_1 + (e_2 \wedge e_3)I = e_1 - e_1 = 0.$$

Indeed, this f is none other than the complex function $f(z) = z$ with B taking the role of the imaginary unit i.

Let $x = x^1 e_1 + x^2 e_2 + x^3 e_3$. Then,

$$B(x \cdot B) = (e_1 e_2)(x^1 e_2 - x^2 e_1) = x^1 e_1 + x^2 e_2.$$

Thus, for $\xi \in S^2$, we have $f(\xi) = \xi^1 + \xi^2 B$.

If we consider now every ω-axial monogenic can be written as a power series, if we can construct z we should be done...?

It is clear that we can define a monogenic field $f = u + b$ via the Cauchy integral, but we then require $\nabla_\omega f = 0$. Let $f = \mathcal{C}[\varphi](x)$, then we must have

$$\nabla_\omega \langle \mathcal{C}[\varphi](x) \rangle_0 = 0 \qquad \text{and} \qquad \nabla_\omega \langle \mathcal{C}[\varphi](x) \rangle_2 = 0.$$

The first condition yields

$$0 = \int_\Sigma \frac{(\nu(\zeta) \cdot x)(\omega \cdot x)}{|x - \zeta|^2} \phi(\zeta) d\Sigma(\zeta).$$

Theorem 7.1. *For any $\omega \in Gr(1,3)$ we have that $\mathcal{A}_\omega(M) \subset \mathcal{M}(M)$.*

Proof. This seems to be saying that we need boundary values in some hardy space or something. They defined this conjugacy thing as G. Fix a unit vector ω. We want to show that for any $f = u + b \in \mathcal{A}_\omega(M)$ that $\iota^* u = \phi$ satisfies eq:conjugacy_requirement ??. That is,

$$G\phi = (\Lambda - d\Lambda^{-1}d)\phi = 0.$$

Note that ϕ is the trace of a harmonic function, so this operator is well defined. Note that the equation

$$\Lambda\psi = d\phi$$

has a solution ☐

8 Radon transform and integral geometry

I feel like there is some way to go from projection onto subspaces as a map to grassmannians and reconstructing the manifold. It's like a morse function type of thing. Radon transforms also come to mind.

9 Relation to the BC Method

Describe how this process can lead to the BC method in dimension $n = 2$

10 Conclusion

A Appendix

Put axial condition for cauchy integral and some other quick proofs in here.

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