MATH 570, Homework 3

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Solutions

Problem 1. Let X be a topological space and let $A \subseteq X$ be a subset.

- The *interior of* A *in* X, denoted by $\operatorname{Int} A$, is the largest open set in X contained inside of A:

$$\operatorname{Int} A = \cup\{U \subseteq X \mid U \subseteq A \text{ and } U \text{ is open in } X\}.$$

- The *boundary of* A *in* X is $\partial A = \overline{A} \cap \overline{X \setminus A}$ (this is equivalent to the slightly different definition on page 24 of our book).

1. Prove that a point $x \in X$ is in ∂A if and only if every neighborhood of x contains both a point of A and a point of $X \setminus A$.
2. Prove that A is open in X if and only if A contains none of its boundary points, i.e. $A \cap \partial A = \emptyset$.

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Proof (Part (1)). For the forward direction, let $x \in \partial A$. Thus we have $x \in \bar{A} \cap X \bar{\setminus} A$. Which means $x \in \bar{A}$ and $x \in X \bar{\setminus} A$. By (4) on our last homework we have that all neighborhoods of x, N_x, necessarily satisfy $N_x \cap A \neq \emptyset$ and $N_x \cap X \setminus A \neq \emptyset$.

For the reverse direction, if for all neighborhoods of a point x, N_x, we have $N_x \cap A \neq \emptyset$ and $N_x \cap X \setminus A \neq \emptyset$, then we have $x \in \bar{A}$ and $x \in X \bar{\setminus} A$. Thus $x \in \bar{A} \cap X \bar{\setminus} A$. □

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Proof (Part (2)). For the forward direction, let $A \subseteq X$ be open. Suppose that $x \in \partial A$ and $x \in A$. But for all neighborhoods of the point x, N_x, $N_x \cap X \setminus A \neq \emptyset$ since $x \in \partial A$. Thus we contradict A being open.

For the reverse direction, suppose $A \cap \partial A = \emptyset$. Let $x \in A$ and consider an arbitrary neighborhood of x, N_x. If $N_x \subseteq A$ we have that A is open and we are done. Otherwise if $N_x \cap X \setminus A \neq \emptyset$ then also $N_x \cap A \neq \emptyset$ since we said $x \in A$. But this implies that $x \in \partial A$. Thus $N_x \subseteq A$ and we have that A is open. □

Problem 2. Let X be a topological space and let $S \subseteq X$. Prove that the subspace topology on S is indeed a topology (i.e. satisfies the definition on page 20 of our book).

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Proof. For the first requirement we have that S is an element since $S \cap X = S$. Also we have $\emptyset = S \cap \emptyset$ so the subset contains S and $\emptyset$.

For the second requirement, let $U_1, \ldots, U_n \subseteq S$, then each $U_i = S \cap V_i$ for $V_i \in X$. Then $U_1 \cap U_2 \cap \ldots \cap U_n = (S \cap V_1) \cap (S \cap V_2) \cap \ldots \cap (S \cap V_n) = S \cap (\cap_{i=1}^{n} V_i)$. Since $\cap_{i=1}^{n} V_i \subseteq X$ is open in X then we have that $S \cap (\cap_{i=1}^{n} V_i)$ is open in S.

Finally let $\{U_\alpha\}_{\alpha \in A}$ be an arbitrary collection of open sets of S. Again we have that $U_\alpha = S \cap V_\alpha$. Then, $\cup_{\alpha \in A} U_\alpha = \cup_{\alpha \in A} (S \cap V_\alpha) = S \cap (\cup_{\alpha \in A} V_\alpha)$. And we have that $\cup_{\alpha \in A} V_\alpha$ is open in X, so $S \cap (\cup_{\alpha \in A} V_\alpha)$ is open in S. $\square$

Problem 3. Read and understand Proposition 2.44 and its proof in our book, which describes when a collection $\mathscr{B}$ of subsets of a set X is a valid basis for some topology on X. Now, suppose that $X_1, \ldots, X_n$ are topological spaces. Use Proposition 2.44 to prove that the basis $\mathscr{B} = \{U_1 \times \ldots \times U_n \mid U_i$ is open in X_i for all $i\}$ is indeed a valied basis for some topology on $X_1 \times \ldots \times X_n$ (called the *product topology*).

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Proof. For the first requirement let $\mathscr{B} = \{U_1 \times ... \times U_n | U_i$ is open in $X_i \forall i\}$. Let $\mathscr{B}_i = \{U_i \subseteq X_i | U_i$ is open in $X_i\}$, then we have that $X_i = \cup U_i \in \mathscr{B}_i U_i$ and thus we have $\cup U_1 \in \mathscr{B}_1 U_1 \times ... \cup_{U_n \in \mathscr{B}_n} U_n = X_1 \times ... \times X_n$.

Let $A = A_1 \times A_2 \times ... \times A_n \in \mathscr{B}$ and $B = B_1 \times B_2 \times ... \times B_n \in \mathscr{B}$ then $A \cap B = (A_1 \cap B_1) \times ... \times (A_n \cap B_n)$. Then we have that there exists $C_i \subseteq A_i \cap B_i$ for $i = 1, ..., n$ since we contain every open set for each X_i in $\mathscr{B}$. Thus we have $C \subseteq A \cap B$ given by $C = C_1 \times ... \times C_n$. $\square$

Problem 4. The above images are of a 2D surface which is a 2-holed torus. On the left the holes appear to be linked, but on the right they do not. However, there is a way to bend and stretch this shape in $\mathbb{R}^3$ to get from the surface on the left to the one on the right (imagine the shape is made of a very flexible rubber or play-doh which you are allowed to bend or stretch but not tear). Draw a deformation (a sequence of pictures) showing how to do this.

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Proof.