

MATH 271, HOMEWORK 6
DUE OCTOBER 19TH

Problem 1. Consider the differential equation

$$f'(x) = \frac{1}{\sqrt{1-x^2}} f(x).$$

- (a) Write down the 2nd order Taylor approximation to $\frac{1}{\sqrt{1-x^2}}$ centered at zero.
- (b) Using this second order approximation, find the general solution to the differential equation using separation.
- (c) The solution you find using the approximation doesn't have an issue at $x = 1$, but I claim the original equation does. What is wrong at $x = 1$? Our approximation is then only reasonable in the window $[0, 1)$ (and really isn't that accurate near 1 either).

Problem 2. Consider the differential equation

$$f'(x) = xf(x)$$

with initial condition $f(0) = 1$.

- (a) Find the particular solution to this separable differential equation.
- (b) What is the Taylor series centered at zero for this solution?
- (c) Now, assume that the solution $f(x)$ can be written as a power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Determine all of the coefficients a_n which will give us the power series representation for $f(x)$. *Hint: use your solution from (a) to help you.*

Problem 3. Consider the differential equation

$$(x-1)f'(x) + f(x) = 0$$

with initial condition $f(0) = 1$.

- (a) Find the solution to this equation using separation.
- (b) Find the Taylor series centered at zero for your solution in (a).
- (c) Again, suppose that the solution can be written as a power series and determine all the coefficients a_n so that we find the power series representation for $f(x)$. *Hint: use your solution from (a) to help you.*

Problem 4. We derived two linearly independent (even and odd) solutions to *Legendre's equation*

$$(1 - x^2)f''(x) - 2xf'(x) + \alpha(\alpha + 1)f(x) = 0$$

which were

$$f(x) = \sum_{n=0}^{\infty} a_{2n}x^{2n} \quad \text{and} \quad f(x) = \sum_{n=0}^{\infty} a_{2n+1}x^{2n+1}.$$

- (a) Look up where this equation shows up in quantum mechanics and write it down.
- (b) If we add initial conditions then we get a finite polynomial for each choice of $\alpha = 0, 1, 2, 3, \dots$. Using this, the first four polynomials are

$$\begin{aligned} f_0(x) &= 1 & f_1(x) &= x \\ f_2(x) &= 1 - 3x^2 & f_3(x) &= x - \frac{5x^3}{3}. \end{aligned}$$

Show that these above polynomials are *orthogonal* by showing

$$\int_{-1}^1 f_i(x)f_j(x)dx = 0$$

when $i \neq j$.