MATH 519, Exam 1

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Solutions

Problem 1. Use the Cauchy Integral Formula to evaluate $\int_C \frac{\cos(z)}{z} dz$ where C is the unit circle.
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Proof. We have

$$\begin{aligned}
\int_C \frac{\cos(z)}{z} dz &= \int_C \frac{\exp(iz) - \exp(-iz)}{2z} dz \\
&= \frac{1}{2}\left(\int_C \frac{\exp(iz)}{z-0} dz + \int_C \frac{\exp(-iz)}{z-0} dz\right) \\
&= \frac{1}{2}\left(2\pi i \exp(i \cdot 0) + 2\pi i \exp(-i \cdot 0)\right) && \text{by Cauchy's Integral Formula} \\
&= 2\pi i.
\end{aligned}$$

□

Problem 2. Let $f(z) = \frac{1}{p(z)}$, where $p(z)$ is some degree k polynomial. What is the maximum number of different values for the integral of $f(z)$ around various closed, simple, positively-oriented contours C that do not pass through any of the roots of $p(z)$? (<u>NOTE:</u> If you can handle the combinatorics and write down the explicit answer, do so. Otherwise, describe how you might go about counting all the possible values.)
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Proof. Letting $\binom{n}{m} = \frac{n!}{m!(n-m)!}$, we have that there are $\binom{k}{0} = 1$ ways for a simple closed curve to inclose zero roots of $p(z)$, $\binom{k}{1}$ ways for a simple closed curve to inclose a single root of $p(z)$, and in general we have $\binom{k}{n}$ ways for a simple closed curve to inclose $n \leq k$ roots of $p(z)$. Since we only allow for positively-oriented curves, we then have that the total number of values for a contour integral around our simple closed positively-oriented curve is given by

$$\sum_{n=0}^{k} \binom{k}{n}.$$

□

Problem 3.

(a) Evaluate $\int_C \frac{\cos(z)}{z^5 - 1} dz$ where C is the circle $|z - 2i| = \frac{1}{2}$.

(b) Evaluate $\int_C \frac{e^z}{z^2 + 4} dz$ where C is the circle $|z - 2i| = \frac{1}{2}$.

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Proof. (a) Note that $\dfrac{\cos(z)}{z^5-1}$ is holomorphic on C and within the interior of C. Thus

$$\int_C \frac{\cos(z)}{z^5-1}dz = 0.$$

(b) We have that $z^2+4 = (z-2i)(z+2i)$. Then note that we have

$$\int_C \frac{e^z}{z+2i}\cdot\frac{1}{z-2i}dz$$

allows for the use of Cauchy's integral formula. Namely, $\dfrac{e^z}{z+2i}$ is holomorphic on C and in the interior of C as well, meaning that if we let $f(z) = \dfrac{e^z}{z+2i}$ then we have

$$\begin{aligned}\int_C \frac{e^z}{z^2+4}dz &= \int_C \frac{f(z)}{z-2i}dz \\ &= 2\pi i f(2i) \\ &= 2\pi i\frac{e^{2i}}{4i} \\ &= \frac{\pi e^{2i}}{2}.\end{aligned}$$

□

Problem 4. Suppose $f\colon \mathbb{C}\to\mathbb{C}$ is entire and $|f(z)| < e^{-|z|}$ for all $z\in\mathbb{C}$. What can you say about the image of $\mathbb{C}$ under $f(z)$?

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Proof. The image of f is a singleton since f must be constant. We have this by Liouville's theorem since f is entire and f is bounded. We're given f is entire, and to see that f is bounded, note that $\sup_{z\in\mathbb{C}} e^{-|z|} = 1$ and hence $|f(z)| < 1$ which shows f is bounded. □

Problem 5. Each of the following functions has an isolated singularity at $z = 0$. Determine which type of isolated singularity each on is AND

- if it is removable, define $f(0)$ so that $f(z)$ is analytic;
- if it is a pole, find the residue of $f(z)$ at $z = 0$; and
- if it is essential, decide which value (if any) is neglected from the range in a (any) neighborhood of $z = 0$.

(a) $\frac{\cos(z)}{z}$

(b) $\frac{\cos(z)-1}{z}$

(c) $e^{\frac{1}{z}} - 5$

(d) $\frac{z^2+1}{z(z-1)}$.

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Proof.

(a) At $z = 0$ residue should be 1. We show this by computing a_{-1} of the laurent expansion where

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-c)^{n+1}}dz.$$

Here γ is a closed path around the point we are doing the expansion on, c. In our case, we will choose γ to be the unit circle with the typical orientation. Then we have

$$\begin{aligned} a_{-1} &= \frac{1}{2\pi i}\int_\gamma \frac{\cos(z)/z}{z^{-1+1}}dz \\ &= \frac{1}{2\pi i}\int_\gamma \frac{\cos(z)}{z} \\ &= 1. \end{aligned}$$

This result is from Problem 1.

(b) At $z = 0$ residue should be 0. We compute by using γ as the unit circle with typical orientation again, and we find

$$\begin{aligned} a_{-1} &= \frac{1}{2\pi i}\int_\gamma \frac{\cos(z)}{z} - \frac{1}{z}dz \\ &= \frac{1}{2\pi i}\int_\gamma \frac{\cos(z)}{z}dz - \frac{1}{2\pi i}\int_\gamma \frac{1}{z}dz \\ &= 1 - 1. \end{aligned}$$

Note that the $\cos(z)/z$ integral is Problem 1 again and that the $1/z$ integral is easily seen by Cauchy's integral formula.

(c) This has an essential singularity at $z = 0$ and the neglected value is $z = -5$.

(d) At $z = 0$ residue should be -1. Here we choose γ to be the circle $|z| = \frac{1}{2}$ with the typical orientation so that we avoid the singularity at $z = 1$. Then letting $f(z) = \frac{z^2+1}{z-1}$ we have

$$\begin{aligned} a_{-1} &= \int_\gamma \frac{z^2+1}{z(z-1)}dz \\ &= \int_\gamma f(z)\cdot\frac{1}{z}dz \\ &= f(0) = -1. \end{aligned}$$

□