MATH 517, Homework 9

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Solutions

Problem 1. (Rudin 9.5) Prove that every $A \in L(\mathbb{R}^n, \mathbb{R})$ corresponds to a unique $\vec{y} \in \mathbb{R}^n$ so that $A\vec{x} = \vec{x} \cdot \vec{y}$. Also prove that $\|A\| = |\vec{y}|$.

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Proof. Choose the standard orthonormal euclidean basis, and we have a matrix representation of A given by

$$A_\beta = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_2 \end{bmatrix}.$$

Again (assuming $\vec{x}$ was given in the standard basis) $\vec{x} = \vec{x}_\beta = \begin{bmatrix} x_1, & x_2, & \dots, & x_n \end{bmatrix}$ when written in the standard basis. We have then that $A_\beta \vec{y}_\beta = A_1 x_1 + \cdots + A_n x_n$. So then let $\vec{y} = \begin{bmatrix} A_1, & A_2, & \dots, & A_n \end{bmatrix}$ and we have that $A_\beta \vec{x} = \vec{x} \cdot \vec{y}$.

To see $\vec{y}$ is unique, consider another vector $\vec{z}$ such that

$$\begin{aligned} \vec{y} \cdot \vec{x} &= \vec{z} \cdot \vec{x} \\ \iff A_1 x_1 + \cdots + A_n x_n &= z_1 x_1 + \cdots + z_n x_n \\ \iff A_i &= z_i \quad \forall i. \end{aligned}$$

So $\vec{y} = \vec{z}$ and we have that $\vec{y}$ is unique.

For the second part of this proof we have that $\|A\| = \sup_{|\vec{x}|=1} |Ax|$. So

$$\sup_{|\vec{x}|=1} |Ax| = \sup_{|\vec{x}|=1} |\vec{x} \cdot \vec{y}|.$$

Note that $|\vec{x} \cdot \vec{y}| \leq |x||y| = |y|$, which implies that $\sup_{|\vec{x}|=1} |\vec{x} \cdot \vec{y}| = |\vec{y}|$. $\square$

Problem 2. (Rudin 9.6) Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) := \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Prove that $(D_1 f)(x,y)$ and $(D_2 f)(x,y)$ exist at every point of $\mathbb{R}^2$, but that f is not even continuous at $(0,0)$.

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Proof. First, we show that $(D_1 f)(x,y)$ exists at every point of $\mathbb{R}^2$. So consider for $(x,y) \neq (0,0)$

$$\begin{aligned}
(D_1 f)(x,y) &= \lim_{t\to 0} \frac{f(x+t,y) - f(x,y)}{t} \\
&= \lim_{t\to 0} \frac{\frac{(x+t)y}{(x+t)^2+y^2} - \frac{xy}{x^2+y^2}}{t} \\
&= \lim_{t\to 0} \frac{1}{t}\left(\frac{xy+ty}{x^2+2tx+y^2} - \frac{xy}{x^2+y^2}\right) \\
&= \lim_{t\to 0} \frac{1}{t}\left(\frac{(x^2+y^2)((xy+ty) - xy(x^2+2tx+y^2)}{(x^2+y^2)(x^2+2tx+y^2)}\right) \\
&= \lim_{t\to 0} \frac{1}{t}\left(\frac{x^3y+tx^2y+xy^3+ty^3-x^3y-2tx^2y-xy^3}{x^4+2tx^3+2x^2y^2+2txy^2+y^4}\right) \\
&= \lim_{t\to 0} \frac{1}{t}\left(\frac{tx^2y+ty^3-2tx^2y}{x^4+2tx^3+2x^2y^2+2txy^2+y^4}\right) \\
&= \lim_{t\to 0} \frac{x^2y+y^3-2x^2y}{x^4+2tx^3+2x^2y^2+2txy^2+y^4} \\
&= \frac{x^2y+y^3-2x^2y}{x^4+2x^2y^2+y^4}.
\end{aligned}$$

Now consider $(D_1 f)(0,0)$, which we find by

$$\begin{aligned}
(D_1 f)(0,0) &= \lim_{t\to 0} \frac{f(t,0)}{t} \\
&= \lim_{t\to 0} \frac{1}{t}\left(\frac{0}{t^2}\right) \\
&= 0.
\end{aligned}$$

The argument for $(D_2 f)(x,y)$ is exactly analogous. Just swap x for y in the above proof. Doing this shows that both $(D_1 f)(x,y)$ and $(D_2 f)(x,y)$ exist at each point of $\mathbb{R}^2$.

To see that f is not continuous at $(0,0)$ we do the following: For a contradiction, assume f is continuous at $(0,0)$ and then $\epsilon = \frac{1}{4}$. Continuity implies that $\exists\delta > 0$ such that $d_{\mathbb{R}^2}((x,y),(0,0)) < \delta$ means that $d_{\mathbb{R}}(f(x,y),0) < \epsilon = \frac{1}{4}$. It follows that

$$\begin{aligned}
\left|\frac{xy}{x^2+y^2}\right| &= \left|\frac{x^2}{2x^2}\right| && \text{letting } x = y\text{, but forcing } d_{\mathbb{R}^2}((x,x),(0,0)) < \delta \\
&= \frac{1}{2} > \frac{1}{4} = \epsilon.
\end{aligned}$$

Hence we have a contradiction, and f is not continuous at $(0,0)$. $\square$

Problem 3. **(Rudin 9.8)** Suppose $E \subseteq \mathbb{R}^n$ is open and that $f\colon E \to \mathbb{R}$ is differentiable on E. Prove that if f has a local maximum at $\vec{x} \in E$, then $f'(\vec{x}) = 0$ (Remember that 0 here really means the constant linear map that sends everything to 0).

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Proof. Since E is open we have that for some $\delta_{i_1} > 0$ we have $N(\vec{x}, \delta_{i_1}) \in E$. Next, define for each $i = 1, \dots, n$, the functions $g_i\colon (-\delta_{i_1}, \delta_{i_1}) \to \mathbb{R}$ by $g_i(h) = f(\vec{x} + he_i)$ where e_i denotes the ith standard orthonormal basis vector. Note that $g_i(0) = f(\vec{x})$ and thus for each g_i we have $g_i(0) \geq g_i(h)$ for any $h \neq 0 \in (-\delta_{i_1}, \delta_{i_1})$. Since f is differentiable, each partial derivative exists, and we have $\frac{\partial f}{\partial x_i} = \lim_{h\to 0} \frac{g_i(h) - g_i(0)}{h} = g_i'(0)$. By the way g_i is defined, $g_i(0)$ is a local maximum, and by Theorem 5.8 we have that $g_i'(0) = 0$. Of course, this implies that $0 = \frac{\partial f}{\partial x_i}(\vec{x})$ since $g_i'(0) = \frac{\partial f}{\partial x_i}(\vec{x})$. It follows that this is true for each $i = 1, \dots, n$ and thus each partial derivative is identically 0 at $\vec{x}$ and this implies that

$$f'(\vec{x}) = \nabla f(\vec{x}) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) = (0, \dots, 0) = 0$$

□

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The proof is done above, but I was curious to see if what I have below does work. If you don't want to read it, I won't be offended! Otherwise it's just checking my intuition.

(Following from the sentence before I mentioned Theorem 5.8...) Differentiability of f implies f is also continuous and it follows that each g_i is as well. So, for some $h_i \in (-\delta_{i_1}, 0)$ and $h_i' \in (0, \delta_{i_1})$ we have that $g_i(h_i) = g_i(h_i')$. Consider then a sequence $\{\delta_{i_j}\}_{j\in\mathbb{N}}$ that converges monotonically to 0 and has δ_{i_1} defined as above. Certainly for each δ_{i_j} we have an $h_{i_j} \in (-\delta_{i_j}, 0)$ and $h_{i_j}' \in (0, \delta_{i_j})$ satisfying $g_i(h_{i_j}) = g_i(h_{i_j})$. By the mean value theorem applied to h_{i_j} and h_{i_j}', we have that for each j there exists a point c_j such that $g_i'(c_j) = 0$. Note that $(-\delta_{i_j}, \delta_{i_j})$ converges to $\{0\}$ which implies that the sequence $\{c_j\}_{j\in\mathbb{N}}$ converges to $\{0\}$ as well. Hence, $g_i'(0) = \frac{\partial f}{\partial x_i}(\vec{x}) = 0$. This is true for each i as well and since each partial derivative is 0 at $\vec{x}$, we have that $f'(\vec{x}) = \nabla f(\vec{x}) = 0$.

Problem 4. (Rudin 9.13) Suppose $f\colon \mathbb{R} \to \mathbb{R}^3$ is differentiable, and that $|f(t)| = 1$ for every $t \in \mathbb{R}$. Explain why $f'(t)$ can be interpreted as an element of $\mathbb{R}^3$ for each t, and prove that $f'(t) \cdot f(t) = 0$ for all t. Interpret this result geometrically.

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Proof. First, $f'(t)$ is defined to be the unique linear map that satisfies

$$\lim_{h \to 0} \frac{|f(t+h) - f(t) - f'(t)h|}{|h|} = 0.$$

The numerator is vector substraction of elements in $\mathbb{R}^3$ and $h \in \mathbb{R}$, which means that $f'(t) \in \mathbb{R}^3$.

Now, given $|f(t)| = 1$ we have that $f(t) \cdot f(t) = 1$ as well. Then, taking the derivative of both sides,

$$\begin{aligned}
&D(f(t) \cdot f(t)) = 0 \\
\iff\ & D\left(\sum_{i=1}^{3} f_i(t) f_i(t)\right) = 0 \\
\iff\ & \sum_{i=1}^{3} \frac{d}{dt}(f_i(t) f_i(t)) = 0 \\
\iff\ & \sum_{i=1}^{3} (f_i'(t) f_i(t) + f_i(t) f_i'(t)) = 0 \\
\iff\ & 2\sum_{i=1}^{3} f_i'(t) f_i(t) = 0 \\
\iff\ & 2f'(t) \cdot f(t) = 0 \\
\iff\ & f'(t) \cdot f(t) = 0.
\end{aligned}$$

Geometrically we are looking at a function that is a curve that lies on the sphere for every $t \in \mathbb{R}$. When we look at the derivative of f, $f'(t)$, we are looking at the tangent vector to the curve. The tangent plane to the sphere, in which $f'(t)$ lives, is perpendicular to $f(t)$ for every t. In fact, from what I know, $f(t)$ defines the tangent plane in that every vector in the tangent plane is orthogonal to $f(t)$. $\square$