MATH 517, Homework 6

Colin Roberts

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Solutions

Problem 1. (Rudin 5.14) Let $f\colon (a,b)\to\mathbb{R}$ be differentiable.

(a) Show that f is convex if and only if f' is monotone increasing.

(b) If f'' exists on all of (a,b), show that f is convex if and only if $f''(x)\geq 0$ for all $x\in(a,b)$.

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Proof (a). For the forward direction we let $f\colon (a,b)\to\mathbb{R}$ be differentiable and convex. Then for $a<s<t<u<v<w<b$ we have

$$\frac{f(t)-f(s)}{t-s}\leq\frac{f(u)-f(t)}{u-t}\leq\frac{f(v)-f(u)}{v-u}\leq\frac{f(w)-f(v)}{w-v}$$

Then if we let $t\to s^+$ and $v\to w^-$ we have

$$\begin{aligned} f'(s)=\lim_{t\to s^+}\frac{f(t)-f(s)}{t-s}&\leq\lim_{v\to w^-}\frac{f(w)-f(v)}{w-v}\\ &\implies f'(s)\leq f'(w). \end{aligned}$$

Thus we have that f' is monotone increasing.
For the converse, suppose that f' is monotone increasing. Then for $x<y<z\in(a,b)$ we have $f'(x)\leq f'(y)\leq f'(z)$. By the mean value theorem we have $x<c<y$ and $y<d<z$ so that $f'(c)=\frac{f(y)-f(x)}{y-x}$ and $f'(d)=\frac{f(z)-f(y)}{z-y}$. By assumption, $f'(c)\leq f'(d)$ which means that

$$\frac{f(y)-f(x)}{y-x}\leq\frac{f(z)-f(y)}{z-y}.$$

Then necessarily $y=\lambda x+(1-\lambda)z$ for $\lambda\in(0,1)$. So we have

$$\begin{aligned} &\frac{f(\lambda x+(1-\lambda)z)-f(x)}{(\lambda x+(1-\lambda)z)-x}\leq\frac{f(z)-f(\lambda x+(1-\lambda)z)}{z-(\lambda x+(1-\lambda)z)}\\ \iff&\lambda(z-x)(f(\lambda x+(1-\lambda)z)-f(x))\leq(1-\lambda)(z-x)(f(z)-f(\lambda x+(1-\lambda)z))\\ \iff&f(\lambda x+(1-\lambda)z)\leq\lambda f(x)+(1-\lambda)f(z) \end{aligned}$$

Hence, f is convex. □

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Proof (b). For the forward direction, suppose that f is convex. Thus for $x<y\in(a,b)$ we have $f'(x)<f'(y)$. Then note that $y=x+h$ for $h>0$ and thus

$$\begin{aligned} &0\leq f'(y)-f'(x)\\ \implies&0\leq f'(x+h)-f'(x)\\ \implies&0\leq\frac{f'(x+h)-f'(x)}{h}\\ \implies&0\leq\lim_{h\to0}\frac{f'(x+h)-f'(x)}{h}=f''(x). \end{aligned}$$

So $f''(x)\geq0$ for any $x\in(a,b)$. We know the last implication is true since the limit must exist by the fact f'' exists for every $x\in\mathbb{R}$ and since the set $[0,\infty)$ is closed.
The converse is immediate by Theorem 5.11. □

Problem 2. Assume $f : \mathbb{R} - \mathbb{R}$ is continuous and satisfies $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

(a) Prove that if f is differentiable, then f' is constant.

(b) Prove that f is differentiable by showing $f(x) = cx$ for some $c \in \mathbb{R}$.

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Proof (a). Let $x, y \in \mathbb{R}$ be arbitrary disctinct elements and $y \neq 0$. Then,

$$\begin{aligned} f'(x+y) &= \lim_{h\to 0} \frac{f(x+y+h) - f(x+y)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) + f(y) - f(x) - f(y)}{h} = f'(x) \end{aligned}$$

Since $f'(x+y) = f'(x)$ and $x + y \neq x$ we have that f' must be constant. □

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Proof (b). First, let $f(1) = c$ and $q =\neq 0 \in \mathbb{Q}$. Then $q = \frac{m}{n}$ with $m, n \in \mathbb{Z}$. It follows that

$$f(q) = f\left(\frac{m}{n}\right) = f\left(\frac{1}{n} + \ldots + \frac{1}{n}\right) = f\left(\frac{1}{n}\right) + \ldots + f\left(\frac{1}{n}\right) = mf\left(\frac{1}{n}\right),$$

and it follows that if $q = 1$ then $q = \frac{n}{n}$ so

$$\begin{aligned} f(1) &= nf\left(\frac{1}{n}\right) \\ \implies \frac{1}{n}f(1) &= f\left(\frac{1}{n}\right). \end{aligned}$$

It follows that for any $q \in Q$ $f(q) = cq$. In other words, f is a linear function if the inputs are rational (including $q = 0$). So now let $\{x_i\}$ be a sequence of rationals converging to a real number x, then by continuity of f we have that $\lim_{i\to\infty} f(x_i)$ converges to $f(x)$. So we have

$$\begin{aligned} f(x) &= \lim_{i\to\infty} f(x_i) \\ &= \lim_{i\to\infty} cx_i \\ &= cx. \end{aligned}$$

So $f(x) = cx$, which is linear for all reals. Thus f is differentiable. □

Problem 3. Let $a, h \in \mathbb{R}$ with $h > 0$. Suppose f is twice differentiable on $[a-h, a+h]$ so that f'' is continuous at a.

(a) If $f'(a) = 0$ and $f''(a) < 0$, show that f has a strict local maximum at a; i.e., $f(x) < a$ for all x in a neighborhood of a. (*Hint:* Use Taylor's theorem)

(b) Is the assumption that f'' is continuous at a necessary? Justify your answer with a proof or counterexample.

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Proof (a). Since $f''(a) < 0$ we have that for some $\epsilon > 0$ that any $p \in (a-\epsilon, a+\epsilon)$ satisfies $f''(p) < 0$. Then let $\delta = \min(h, \epsilon)$ and let $x \neq a \in (a-\delta, a+\delta)$, then for y between a and x we have

$$\begin{aligned} f(x) &= P(x) + \frac{f''(y)}{2!}(x-a)^2 \\ &= (f(a) + f'(a)(x-a)) + \frac{f''(y)}{2}(x-a)^2. \end{aligned}$$

So then we have

$$f(x) - \frac{f''(y)}{2}(x-a)^2 = f(a) + f'(a)(x-a).$$

But we have that $\frac{f''(y)}{2}(x-a)^2 < 0$ and thus

$$f(x) < f(a).$$

Hence we have a strict local max at $f(a)$. □

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Proof (b). We can show that f'' need be continuous at $x = a$. We have that $f''(a) = \lim_{t \to a} \frac{f'(t) - f'(a)}{t-a} = \lim_{t \to a} \frac{f'(t)}{t-a} < 0$. Then for some $r > 0$ and for some $\delta > 0$ we have that $d(t, a) < \delta$ implies that $\frac{f'(t)}{t-a} < -r$. Then for a p such that $d(p, a) < \delta$ we want to show that for all $\epsilon > 0$ that

$$\begin{aligned} &|f''(p) - f''(a)| < \epsilon \\ \iff &\lim_{h \to 0} \left| \frac{f'(p+h) - f'(p) - f'(a+h)}{h} \right| < \epsilon, \end{aligned}$$

which would show that f'' is continuous at $x = a$. Note that $\lim_{h \to 0} f'(p+h) - f(a+h) = f'(p) - f'(a)$ which can be made as small as we would like by continuity of f' and choice of δ. Then we are left with showing $f'(p)$ can be as small as we would like, but $f'(a) = 0$ and f' continuous would allow us to do this. The last check would be that dividing by h would not destroy this result. I.e.,

$$\iff \left| \frac{f'(p)}{h} \right| < \epsilon. \qquad \square$$

I believe with a smart choice of δ we can show that this is indeed less than ϵ.