MATH 546, Homework 4

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Solutions

Problem 1 (The space $H^{1/2}$, take 1: Via the Fourier transform). If one thinks of the space H^k as the space of functions that have k square integrable (weak) derivatives, then $H^{1/2}$ would be the space of functions that have half a derivative. This is hard to understand in terms of what such a half derivative should actually be, but we can define it as follows: Recall that the Fourier transformation satisfies the property

$$\begin{aligned}
\mathcal{F}\left[\frac{d^j}{dx^j}f(x)\right](k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{d^j}{dx^j}f(x)\right]e^{ikx}\,\mathrm{d}x \\
&= (-1)^j\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\left[\frac{d^j}{dx^j}e^{ikx}\right]\,\mathrm{d}x \\
&= (-1)^j\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)(ik)^j e^{ikx}\,\mathrm{d}x \\
&= (-ik)^j\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{ikx}\,\mathrm{d}x \\
&= (-ik)^j\mathcal{F}[f](k).
\end{aligned}$$

The key step here was simply the integration by parts from the first to the second line.

Because the Fourier transform is invertible, we also have

$$\frac{d^j}{dx^j}f = \mathcal{F}^{-1}\left(\mathcal{F}\left[\frac{d^j}{dx^j}f\right]\right) = \mathcal{F}^{-1}\left((-ik)^j\mathcal{F}[f]\right).$$

This formula is useful because we can now talk about what it means to take a half-derivative: we just choose $j = 1/2$ in this last formula – we can then compute $\left(\frac{d}{dx}\right)^{1/2} f$ by just doing the forward and inverse Fourier transform. Of course, this can also be done with any other j, whether it is an integer or not, and whether it is positive or not.

So let's come back to the original question: Is the step function

$$h(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases}$$

defined on the interval $\Omega = (-1, 1)$ in the space $H^{1/2}(\Omega)$? As usual, we will say that a function u is in $H^s(0, 1)$ if it is in L^2 and its sth derivative is square integrable.

To answer this question, you will need to figure out whether $\left(\frac{d}{dx}\right)^{1/2} h$ is square integrable. For this, you have to (i) find the Fourier transform of h (which here is really just a Fourier series, because your domain is finite), and (ii) use the Plancherel identity that says that the L^2 norm of a function and the L^2 norm of its (inverse) Fourier transform are equal (potentially up to a constant, depending on how exactly one defines the Fourier transform). The latter is useful because you won't have to do the awkward inverse Fourier transform.

While you're there, answer the following question:

- If your answer is that $h \in H^{1/2}$, then is h also in the spaces $H^{1/2+\varepsilon}$ for any $\varepsilon > 0$?
- If your answer is that $h \notin H^{1/2}$, then is h at least in the spaces $H^{1/2-\varepsilon}$ for any $\varepsilon > 0$?

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Solution. Note that we can take the Fourier representation of $h(x)$ by

$$h(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin((2n+1)\pi x).$$

Then, if we take

$$\mathcal{F}(h)(k) = \frac{2}{\pi(2k+1)}.$$

So then we have that

$$\mathcal{F}\left(\frac{d^{1/2}}{dx^{1/2}} h\right)(k) = (-ik)^{1/2} \frac{2}{\pi(2k+1)} \quad \text{for } k = 0, 1, \ldots.$$

Then using Plancherel (that $\mathcal{F}$ is unitary), we check that $\mathcal{F}\left(\frac{d^{1/2}}{dx^{1/2}} h\right)(k)$ is square integrable by

$$\int_{-\infty}^{\infty} \mathcal{F}\left(\frac{d^{1/2}}{dx^{1/2}} h\right)(k) dk = \sum_{k=0}^{\infty} \left| (-ik)^{1/2} \frac{2}{\pi(2k+1)} \right|^2 = \sum_{k=0}^{\infty} \frac{4k}{\pi^2(2k+1)^2} = \infty.$$

However, for $j = 1/2 - \epsilon$ we have

$$\sum_{k=0}^{\infty} \frac{4k^{-2\epsilon}}{\pi^2(2k+1)^2} < \infty$$

□

by a comparison test to a convergent p-series. The idea here is that the terms we sum over decay like $\frac{1}{j^{\epsilon}}$ and so the series converges.

Problem 2 (The space $H^{1/2}$, take 2: Slobodeckij's definition). Let's take a different definition of H^s where s may not be an integer: We say that a function $u \in L^2$ is in the space H^s if its H^s-norm is finite, where this norm is defined as follows:

$$\|u\|_{H^s(\Omega)} = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{2s+d}} \mathrm{d}y \, \mathrm{d}x.$$

Here, d is the dimension of the domain Ω. Again, this definition (originally given by Slobodeckij) is valid for arbitrary $0 < s < 1$. (If you wanted to define the space $H^{3/2}$ in this way, you'd check that $u \in H^1$ and that $\nabla u \in H^{1/2}$.)

Answer the same questions as for Problem 1:

- Is $h \in H^{1/2}$ using this definition of the space?
- If your answer is that $h \in H^{1/2}$, then is h also in the spaces $H^{1/2+\varepsilon}$ for any $\varepsilon > 0$?
- If your answer is that $h \notin H^{1/2}$, then is h at least in the spaces $H^{1/2-\varepsilon}$ for any $\varepsilon > 0$?

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Solution. We take

$$\begin{aligned}\|h\|_{H^{1/2}} &= \int_{-1}^{1} \int_{-1}^{1} \frac{|h(x) - h(y)|^2}{|x-y|^2} dy dx \\ &= \int_{-1}^{1} \left[\int_{-1}^{0} \frac{|h(x)|^2}{|x-y|^2} dy + \int_{0}^{1} \frac{|h(x) - 1|^2}{|x-y|^2} dy \right] dx.\end{aligned}$$

Note that we have $|x-y|^2 = x^2 - 2xy + y^2$ and thus we have

$$\begin{aligned}\|h\|_{H^{1/2}} &= \int_{-1}^{1} \frac{|h(x)|^2}{x^2 + x} dx + \int_{-1}^{1} \frac{|h(x) - 1|^2}{x^2 - x} dx \\ &= \int_{0}^{1} \frac{1}{x^2 + x} dx + \int_{-1}^{0} \frac{1}{x^2 - x} dx.\end{aligned}$$

Note that both of these integrals diverge (with positive value), thus we have that $h \notin H^{1/2}$ by this definition.

Now, if we consider $H^{1/2-\epsilon}$, then we check

$$\begin{aligned}\|h\|_{H^{1/2-\epsilon}} &= \int_{-1}^{1} \int_{-1}^{1} \frac{|h(x) - h(y)|^2}{|x-y|^{2-2\epsilon}} dy dx \\ &= \int_{-1}^{1} \left(\int_{-1}^{0} \frac{|h(x)|^2}{|x-y|^{2-2\epsilon}} dy \int_{0}^{1} \frac{|h(x) - 1|^2}{|x-y|^{2-2\epsilon}} dy \right) dx \\ &= \int_{0}^{1} \int_{-1}^{0} \frac{1}{|x-y|^{2-2\epsilon}} dy dx + \int_{-1}^{0} \int_{0}^{1} \frac{1}{|x-y|^{2-2\epsilon}} dy dx \\ &= \int_{0}^{1} \int_{-1}^{0} \frac{1}{(x-y)^{2-2\epsilon}} dy dx + \int_{-1}^{0} \int_{0}^{1} \frac{1}{(x-y)^{2-2\epsilon}} dy dx \qquad \text{since } x - y \text{ will be positive,} \\ &= \frac{2 - 4^\epsilon}{\epsilon - 2\epsilon^2}.\end{aligned}$$

Then note that we have

$$\frac{2 - 4^\epsilon}{\epsilon - 2\epsilon^2} < \infty$$

for any $\epsilon > 0$. $\square$

Problem 3 (The space $H^{1/2}$, take 3). The last way we'll consider here in which one could define the space $H^{1/2}(\Omega)$ on a one-dimensional domain Ω is a bit backward because it doesn't quite give a statement one can easily check. It assumes that the one-dimensional domain Ω is (a subset of) the boundary of a two-dimensional domain $\Sigma \subset \mathbb{R}^2$, i.e., $\Omega \subset \partial\Sigma$. Since we're considering $\Omega = (-1, 1)$, we can for example choose $\Sigma = (-1, 1)^2$.

Then take a close look at the following statement: We define $H^{1/2}(\Omega)$ as

$$H^{1/2}(\Omega) = \left\{\varphi \in L^2(\Omega) : \text{there exists } u \in H^1(\Sigma) \text{ so that } T_\Omega u = \varphi\right\}.$$

Here, T_Ω is the trace operator that takes the boundary values of u on the part of the boundary of Σ that is Ω. In other words, $H^{1/2}(\Omega)$ is the set of all possible boundary values that functions $u \in H^1(\Sigma)$ can have. You'll note that unlike the two other definitions, this one really is specific to an index of $1/2$ and can't easily be generalized to other fractional values.

As before, check whether $h \in H^{1/2}(\Omega)$ using this definition.

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Solution. Consider $\Sigma = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1,\ y \geq 0\} \setminus \{(0, 0)\}$ and note that $\Omega \subset \partial\Sigma$. Then we can view this with polar coordinates and consider the function

$$u(r, \phi) = \cos\left(\frac{\phi}{2}\right).$$

We then have that

$$Tu|_{(-1,0)} = \lim_{\phi \to \pi} u(r, \phi) = 0$$

and

$$Tu|_{(0,1)} = \lim_{\phi \to 0} u(r, \phi) = 1.$$

So we have

$$Tu = h \text{ almost everywhere.}$$

So it follows that $h \in H^{1/2}$.

However, this relies on the fact that the chosen $u \in H^1(\Sigma)$ which is not clear. Take

$$\nabla u = \begin{bmatrix} 0 \\ \frac{1}{2r}\sin\left(\frac{\phi}{2}\right) \end{bmatrix}.$$

Then

$$\begin{aligned} \int_\Sigma \|\nabla u\|^2 r dr d\phi &= \int_\Sigma \frac{1}{4r^2}\sin^2\left(\frac{\phi}{2}\right) r dr d\phi \\ &\propto \int_\Sigma \frac{1}{r} dr, \end{aligned}$$

which diverges. $\square$

Problem 4 (The spaces H^k and their Fourier basis). We briefly talked about this in class: Just like $\mathbb{R}^n$, one can give the spaces H^k a *basis.* There are of course many bases one could choose, but the simplest one (and in many situations the most *convenient* one) is the Fourier basis.

For this purpose, let's stay in 1d and for convenience choose $\Omega = (0, 2\pi)$. Then every function in $H_0^k(\Omega)$ can be written as

$$u(x) = \sum_{j=1}^{\infty} a_j \sin(jx) = a_1 \sin(x) + a_2 \sin(2x) + \dots$$

Prove the following statements:

1. If the coefficients a_j of a function $u(x)$ satisfy the condition $|a_j| = o\left(\frac{1}{\sqrt{j}}\right)$ – that is, if $\lim_{j\to\infty} \frac{|a_j|}{1/\sqrt{j}} = 0$ – then $u \in L^2(\Omega)$.
2. If the coefficients a_j of a function $u(x)$ satisfy the condition $|a_j| = o\left(\frac{1}{j^{3/2}}\right)$ then $u \in H^1(\Omega)$.
3. If the coefficients a_j of a function $u(x)$ satisfy the condition $|a_j| = o\left(\frac{1}{j^{k+1/2}}\right)$ for some $k \geq 0$, then $u \in H^k(\Omega)$.

This last condition also allows us to define membership in spaces H^k where k is not an integer. This is of course what we did in Problem 1.

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Solution.

1. Take

$$\begin{aligned}
\|u\|_{L^2}^2 = \int_{\Omega} |u(x)|^2 dx &= \int_0^{2\pi} \left| \sum_{j=1}^{\infty} a_j \sin(jx) \right|^2 dx \\
&\leq \int_0^{2\pi} \left| \sum_{j=1}^{\infty} a_j \right|^2 dx && \text{since } -1 \leq \sin(jx) \leq 1 \ \forall x, \\
&\leq \int_0^{2\pi} \sum_{j=1}^{\infty} |a_j|^2 dx \\
&= \sum_{j=1}^{\infty} |a_j|^2 \int_0^{2\pi} dx \\
&= 2\pi \sum_{j=1}^{\infty} |a_j|^2.
\end{aligned}$$

Then, since $|a_j| = o\left(\frac{1}{j^{1/2+\epsilon}}\right)$ we have that $|a_j|^2 = o\left(\frac{1}{j^{1+\epsilon}}\right)$. Thus we have a convergent p-series

$$\|u\|_{L^2}^2 \leq 2\pi \sum_{j=1}^{\infty} |a_j|^2 < \infty.$$

2. Take

$$\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2.$$

Note that $|a_j| = o\left(\frac{1}{j^{3/2+\epsilon}}\right)$ which by the previous work shows that

$$\|u\|_{L^2}^2 < \infty.$$

Hence, it suffices to show that

$$\|\nabla u\|_{L^2}^2 < \infty.$$

Now, since $u(x)$ is periodic and continuous since $u \in H_0^k(\Omega)$ we can compute

$$\nabla u(x) = \sum_{j=1}^{\infty} j a_j \cos(jx).$$

So we have

$$\begin{aligned}
\|\nabla u\|_{L^2}^2 &= \int_0^{2\pi} \left| \sum_{j=1}^{\infty} j a_j \cos(jx) \right|^2 dx \\
&\leq \int_0^{2\pi} \left| \sum_{j=1}^{\infty} j a_j \right|^2 dx \\
&\leq 2\pi \sum_{j=1}^{\infty} j^2 |a_j|^2.
\end{aligned}$$

Then, since $|a_j| = o\left(\frac{1}{j^{3/2+\epsilon}}\right)$ we have that

$$j^2 |a_j|^2 = o\left(\frac{j^2}{j^{3+\epsilon}}\right) = o\left(\frac{1}{j^{1+\epsilon}}\right)$$

and hence our series above is a convergent p-series. Thus it follows that $u \in H^1(\Omega)$.

3. By an inductive argument, one can see that the highest order derivative requires the most restriction on the decay rate of the coefficients. Hence, if we wish to show $u \in H^k(\Omega)$ it suffices to show

$$\|\nabla^k u\|_{L^2}^2 < \infty.$$

Assuming we can do term by term derivatives, we note that we have

$$\nabla^k u(x) \leq \sum_{j=1}^{\infty} j^k |a_j|$$

as there is either a $\sin(jx)$ or $\cos(jx)$ term but each is bounded by 1 in absolute value. So we have

$$\|\nabla^k u\|_{L^2}^2 \leq 2\pi \sum_{j=1}^{\infty} j^{2k} |a_j|^2,$$

and since we have $|a_j| = o\left(\frac{1}{j^{k+1/2+\epsilon}}\right)$ we have

$$j^{2k} |a_j|^2 = o\left(\frac{j^{2k}}{j^{2k+1+\epsilon}}\right) = o\left(\frac{1}{j^{1+\epsilon}}\right).$$

Again, we have a convergent p-series so $u \in H^k(\Omega)$. $\square$

Bonus problem (The space H^{-1} and its Fourier basis). The space of functions $H^{-1}(\Omega)$ is the set of all functions $u(x)$ so that

$$\int_\Omega u(x)v(x)\,\mathrm{d}x$$

is finite for all possible $v \in H^1$. Take again $\Omega = (0, 2\pi)$ and show the following generalization of the results of the previous problem to the case $k = -1$. In other words, show the following statement:

1. If the coefficients a_j of a function $u(x)$ satisfy the condition $|a_j| = o\left(\frac{1}{j^{-1+1/2}}\right)$, then $u \in H^{-1}(\Omega)$.

To get an understanding of how such functions look like, generate sequences of coefficients a_j (for example, chosen randomly in some way) that satisfy the conditions for H^1, L^2, H^{-1} and plot these functions. Do they look qualitatively different?

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Solution. If we take functions of the form

$$u(x) = \sum_{j=1}^{\infty} a_j \sin(jx)$$

and

$$v(x) = \sum_{k=1}^{\infty} b_k \sin(kx),$$

then we can consider

$$\int_{-1}^{1} u(x)v(x)dx.$$

We then have

$$\begin{aligned}
\int_{-1}^{1} uvdx &= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\int_{-1}^{1} a_j b_k \sin(jx)\sin(kx)dx \\
&= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\int_{-1}^{1} a_j b_k \delta_j k dx \\
&= \sum_{k=1}^{\infty}\int_{-1}^{1} a_k b_k dx \\
&= 2\sum_{k=1}^{\infty} a_k b_k < \infty
\end{aligned}$$

since we have that $|a_j| = o\left(\frac{1}{j^{-1+1/2}}\right)$ and $|b_j| = o\left(\frac{1}{j^{3/2+\epsilon}}\right)$ so $|a_j b_j| = o\left(\frac{1}{j^{1+\epsilon}}\right)$.

Let

$$f(x) = \sum_{j=1}^{\infty} j^{-7/4}\sin(jx)$$

so that $f \in H^1(\Omega)$. The plot is:

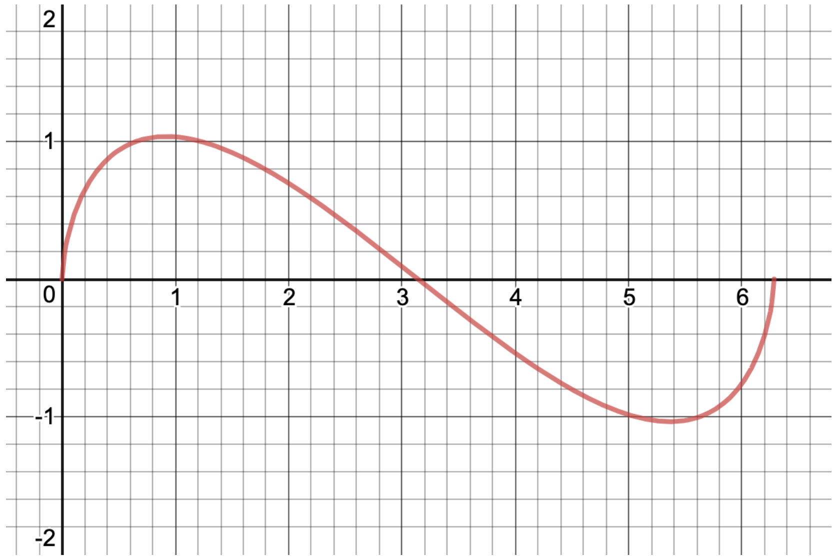

Figure 1: A “random" $H_0^1(\Omega)$ function.

Then let

$$g(x) = \sum_{j=1}^{\infty} j^{-3/4} \sin(jx)$$

so that $g \in L^2(\Omega)$. The plot is

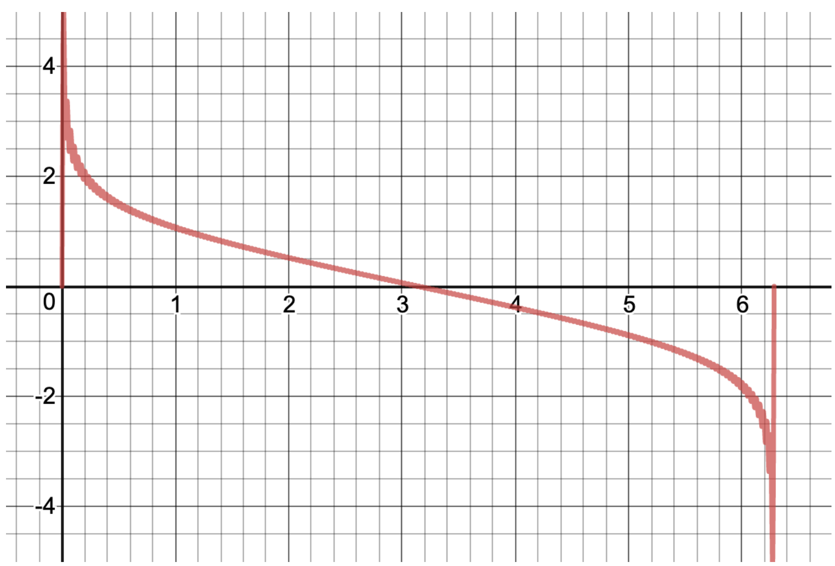

Figure 2: A “random" $L_0^2(\Omega)$ function.

Finally, let

$$h(x) = \sum_{j=1}^{\infty} j^{1/4} \sin(jx)$$

$\square$

so that $h \in H^1(\Omega)$. The plot is

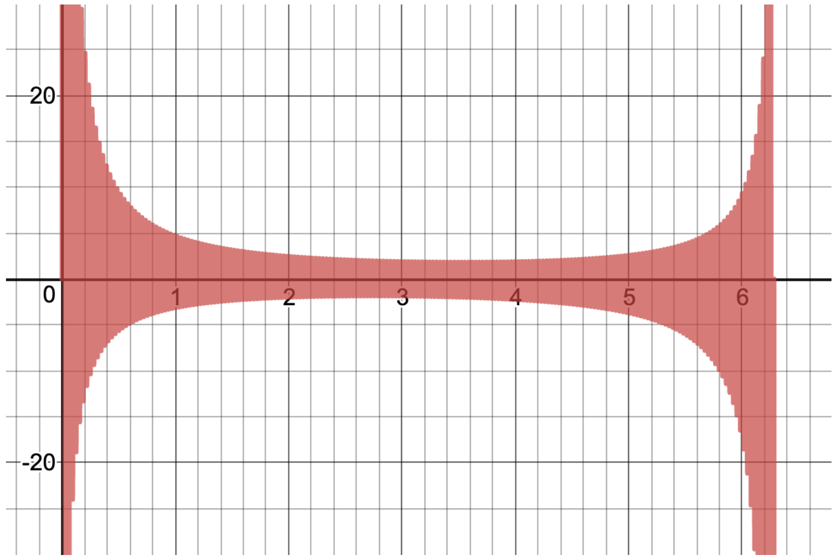

Figure 3: A “random" $H_0^{-1}(\Omega)$ function.