Tensor Structures on Manifolds

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Colin Roberts

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Abstract

Tensors are useful objects for understanding transformations of vector spaces, comparisons of vectors, and more. Smooth manifolds naturally provide a parameterized family of vector spaces which can be glued together using the underlying manifold structure. Specifically, the topology of the manifold induces a topology on the collection of vector spaces which allows us to create vector bundles over manifolds. We seek to combine the notion of tensors with vector bundles to build richer tools which we refer to as tensor fields. Formally, tensor fields can be realized as sections of tensor products of bundles or equivalently by tensor products of sections of bundles. Defining tensor fields is a fruitful endeavor as it proves useful in characterizing geometrical properties of a manifold. It also provides a framework for analysis on manifolds which allows for defining partial differential equations and dynamical systems on manifolds.

Keywords– manifold, smooth map, vector bundle, section, tensor field

1 Introduction

1.1 Motivation

Manifolds are spaces that locally behave like $\mathbb{R}^m$. In essence, they seek to generalize some of the properties of $\mathbb{R}^m$ but with nontrivial topology. This nontrivial topology is important in many applications. For example, consider the convective flow in the Earth's atmosphere or even a bead sliding on a circular loop. Both are examples of dynamical problems on manifolds.

The goal of differential geometry is to understand the local geometrical features of manifolds via calculus. One idea is to study real valued functions on manifolds in hopes of noticing differences in the geometric structure. An example of this would be a Morse function which can detect topological features. To investigate local geometry, we work with more general functions called tensors. Smooth manifolds admit some natural tensors which can give the manifold shape. For example, attaching a metric tensor allows one to distinguish the unit 2-sphere from the surface of a bumpy potato.

The metric tensor is a specific type of tensor field attached to a Riemannian manifold. Tensor fields are smoothly varying tensors defined on vector spaces attached at each point on a manifold. These vector spaces can be glued together in a structure called a vector bundle using the smooth structure on the underlying manifold. With the notion of a smooth map between manifolds, one can then define tensor fields as smooth sections of an appropriate tensor product of vector bundles.

1.2 Preliminaries

To create smooth tensor fields we must first build the notion of a smooth manifold. The first step is to define a *topological manifold*. These are topological spaces which are locally equivalent (*homeomorphic*) to $\mathbb{R}^m$.

Definition 1.1: Homeomorphism

Let $M \equiv (M, \mathcal{O}_M)$ and $N \equiv (N, \mathcal{O}_N)$ be two topological spaces. We say that M is ***homeomorphic*** to N if there exists a continuous bijection $f\colon M \to N$ with continuous inverse.

With this notion, we can define topological manifolds. The addition of smooth structure will follow.

Definition 1.2: Topological Manifolds

Let $M \equiv (M, \mathcal{O})$ be a (second countable) Hausdorff topological space. We say that M is an ***m-dimensional topological manifold*** if M is locally homeomorphic to $\mathbb{R}^m$. That is, each $U \in \mathcal{O}$ is homeomorphic to some open set $X \subset \mathbb{R}^m$ with the metric topology.

Remark 1.1: Other Notions of Manifolds

Often, manifolds come with other properties. However, we really don't need any others. The Hausdorff property isn't asking much and prevents some pathological spaces such as the line with two origins. The second countable property allows for integration.

This definition gives us the notion of a manifold that is familiar. However, we wish to add a *smooth structure* to the space so that we can develop a richer geometry. Roughly speaking, we must upgrade the notion of a homeomorphism to that of a *diffeomorphism*. This allows us to use the smoothness of $\mathbb{R}^m$ to do calculus on a manifold.

Definition 1.3: Smooth Structure

Let M be a an m-dimensional topological manifold. We call $M \equiv (M, \mathcal{O}, \mathcal{A})$ a ***smooth manifold*** if we can place a smooth structure on M (the addition of $\mathcal{A}$). The smooth structure is given by the following.

- ***Coordinate charts***. Given $U_\alpha \in \mathcal{O}$ we have a $(U_\alpha, \varphi_\alpha)$ with $\varphi \colon U \subset M \to X \subset \mathbb{R}^m$.
- Smooth transitions. That is, given overlapping charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) with $U_\alpha \cap U_\beta = U_{\alpha\beta} \neq \emptyset$ we have that the map

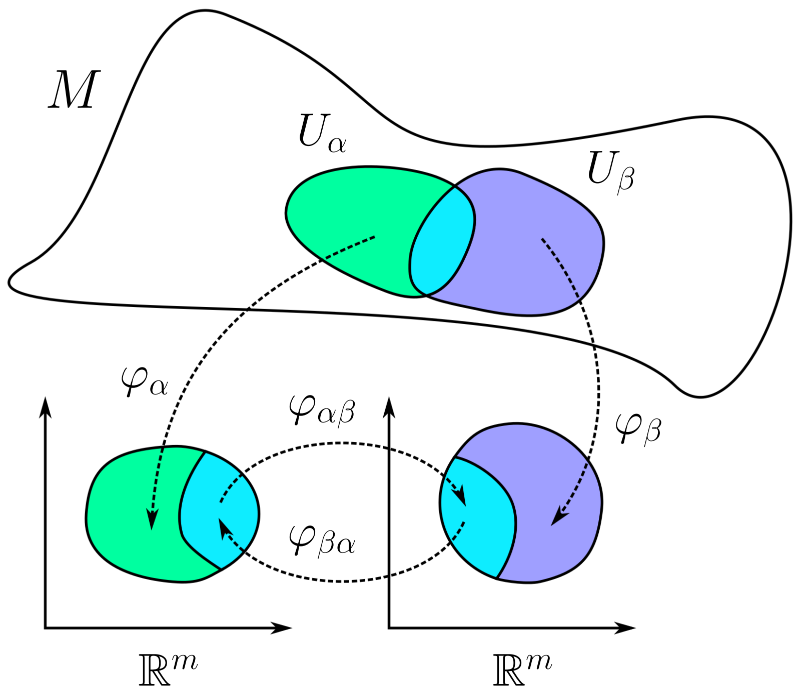
  $$\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} \colon \varphi(U_{\alpha\beta}) \subset \mathbb{R}^m \to \mathbb{R}^m$$

  is smooth as a map from $\mathbb{R}^m$ to $\mathbb{R}^m$.
- We collect all possible compatible charts into a collection called an ***atlas*** $\mathcal{A}$ which gives us the smooth structure.

The degree of smoothness depends on the differentiability class of the transitions. That is, they could be C^k for $k = 0, \ldots, \infty$, or analytic. Here we choose C^∞ and by manifold refer to C^∞-smooth manifolds for the remainder of this talk.

The provided definition of a smooth manifold may not seem intuitive at first glance. However, the naming scheme used is rather helpful as we can picture charts as pages of an atlas of Earth. The interpretation of the charts can be seen pictorially.

Figure 1.1: Coordinate Charts



A manifold M with overlapping charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) and transition functions $\varphi_{\alpha\beta}$ and $\varphi_{\beta\alpha}$.

Finally, we need to define a more general notion of smoothness if we wish to do calculus on manifolds. Since we understand calculus with functions from $\mathbb{R}^m$ to $\mathbb{R}^n$, we build from there.

Definition 1.4: Smooth Map

Let $f\colon M \to N$ be a map from an m-dimensional manifold M to an n-dimensional manifold N. Let (φ, U) and (ψ, V) be charts on M and N respectively with $f\colon U \to V$. We say that f is a ***smooth map*** if

$$\psi \circ f \circ \varphi^{-1}\colon \varphi(U) \subset \mathbb{R}^m \to \psi(V) \subset \mathbb{R}^n$$

is smooth.

2 Gluing Together Tensors

2.1 Vector Bundles

One way to add more structure to a manifold is to consider *bundles* and their respective *sections*. Vector bundles seek to glue together vector spaces over a manifold and sections of bundles play the role of what are referred to as fields or global functions. For example, one may wish to attach a vector field on a manifold to define a dynamical system. In this language, a vector field is a sections of the tangent bundle. *My apologies to those offended by the name "field" being used here!*

Definition 2.1: Vector Bundles

A (finite dimensional) ***vector bundle on M*** is a triple $E \equiv (E, M, \pi)$.

- E is the ***total space***.
- M is a smooth manifold called the ***base space***.
- π is the ***projection map*** $\pi \colon E \to M$.

The relevant vector space is fibered over E as $V = \pi^{-1}(x)$ for $x \in M$. Locally, we have that E looks like a product $U \times V$ with $U \in \mathcal{O}$.

Warning: this does not mean every vector bundle is a product $M \times V$. If it is, we call the bundle ***trivial***.

Note: When referring to vector bundles, I will restrict to the case where the fibers, V, are finite dimensional.

Remark 2.1: Bundles are Manifolds

Any vector bundle E is itself a smooth manifold. This fact allows us to define *smooth sections* by using the definition of a smooth map.

Every smooth manifold comes equipped with a natural bundle. This bundle is the *tangent bundle*. It is a collection of the *tangent spaces* that are attached at each point of a manifold.

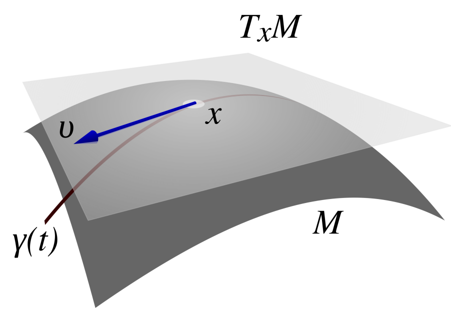
Definition 2.2: Tangent Space

Consider a curve $\boldsymbol{\gamma} \colon (-\epsilon, \epsilon) \to M$ with $\boldsymbol{\gamma}(0) = x$. Then we define the ***tangent space at x***, $T_x M$, to be the set of equivalence classes of tangent vectors $\dot{\boldsymbol{\gamma}}(0)$ to curves $\boldsymbol{\gamma}$.

The tangent space is made up of differential operators. The dimension of the tangent space describes the different directions one can differentiate in. If M is an m-dimensional manifold, then T_xM is an m-dimensional vector space for each $x \in M$.

It's worth seeing what this looks like pictorially. One can imagine a curve $\boldsymbol{\gamma}$ that passes through the point x which has a tangent vector $\dot{\boldsymbol{\gamma}}(0) = \mathbf{v}$. If we were to take every possible curve through x, then the set of velocity vectors at x defines the tangent space T_xM.

Figure 2.1: Tangent Space T_xM



A curve $\boldsymbol{\gamma}$, its tangent vector $\dot{\boldsymbol{\gamma}} = \mathbf{v}$, and the tangent space T_xM.

Then we wish to collect each of the tangent spaces along with the point at which they are attached. Keeping track of the position we glue these vector spaces is very necessary as we will want to refer to elements in a specific tangent space.

Definition 2.3: Tangent Bundle

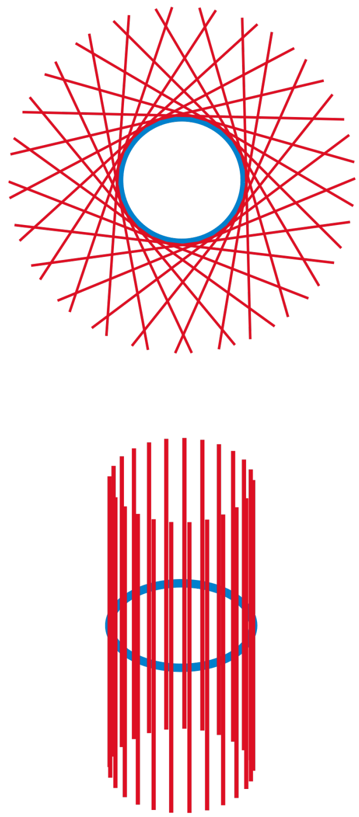
The ***tangent bundle***, TM, is the disjoint union of all the tangent spaces

$$TM := \bigsqcup_{x \in M} T_xM$$

along with a projection map π to the base space M. We define π so that for any $\mathbf{v} \in T_xM$ we have $\pi(\mathbf{v}) = x$.

Seeing an explicit example of a tangent bundle can help build intuition. However, the tangent bundle can only be visualized easily on 1-dimensional manifolds as $\dim(TM) := 2\dim(T_xM) = 2\dim(M)$. So even for 2-dimensional manifolds, $\dim(TM) = 4$.

Figure 2.2: Building the Tangent Bundle



(Top) Many different tangent spaces on the circle S^1. (Bottom) Reorienting the tangent spaces of S^1 to realize $TS^1 = S^1 \times \mathbb{R}$.

We would like to also define the dual bundle to the tangent bundle so that we can later build tensors. The idea used here will be helpful as well.

Definition 2.4: Cotangent Bundle

Let TM be the tangent bundle. Then we have that $T_xM = \pi^{-1}(x)$ which has has a dual vector space T_x^*M we call the ***cotangent space***. Then the ***cotangent bundle***, T^*M, is defined to be

$$T^*M \coloneqq \bigsqcup_{x \in M} T_x^*M$$

which sits over M with projection π.

2.2 Sections of Bundles

The point of defining sections is to properly generalize the notion of functions to manifolds. Specifically, sections are useful for defining vector (or tensor) valued functions on a manifold. The key issue is that, in general, manifolds may have not have global coordinates. Thus we must be careful to build global functions in the correct way.

Definition 2.5: Sections of Bundles

Let E be a vector bundle. Then a (smooth) ***section*** σ of E is a (smooth) map

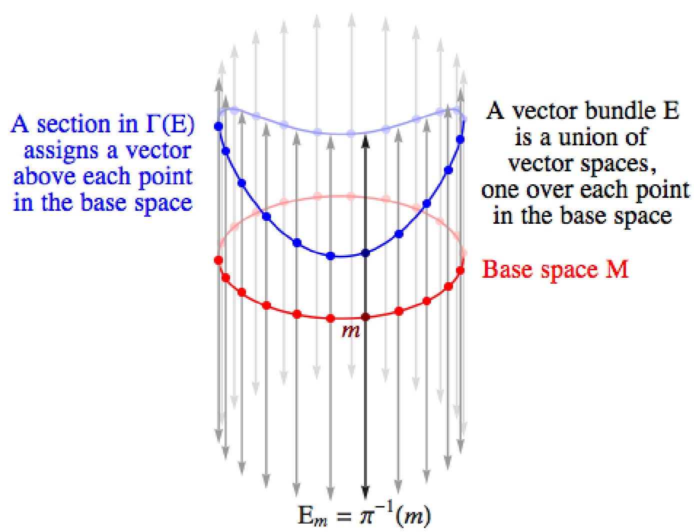
$$\sigma \colon M \to E$$

satisfying $\pi \circ \sigma = \mathrm{Id}_M$. We then define the space of smooth sections

$$\Gamma(E) := \{\sigma \mid \sigma \text{ is a smooth section}\}.$$

Visualizing a section of a bundle is not much more difficult. In fact, a section is just a nice generalization of functions we have all seen in multivariate calculus. Below is an example taking $M = S^1$ as our manifold.

Figure 2.3: Sections of the Tangent Bundle



Orienting the tangent spaces vertically, we can create the tangent bundle for S^1 as a product $TS^1 = S^1 \times \mathbb{R}$. A section of the tangent bundle is simply a smooth closed curve on TS^1 that moves around the circle component in a monotone way.

Remark 2.2: Vector Fields are Sections

The way we define a vector field on a manifold has essentially already been written down. A ***vector field*** is a section of the tangent bundle. We usually define the space of vector fields $\mathfrak{X}(M)$ by putting

$$\mathfrak{X}(M) := \Gamma(TM).$$

2.3 Tensor Product of Bundles

Geometers like to make measurements on spaces. In the real world, all measurements we make come in the form of a real number. Hence, this definition that follows should seem reasonably motivated. Another specific application of tensors on manifolds is to give a manifold shape. If we just care about differentiable manifolds, we are working in the realm of differential topology. In that respect, the unit sphere S^2 in $\mathbb{R}^3$ is no difference than the surface of a bumpy potato.

Definition 2.6: (p,q)-Tensor

A (p,q)***-tensor*** τ is a multilinear map

$$\tau \colon \underbrace{V^* \times \cdots \times V^*}_{p \text{ copies}} \times \underbrace{V \times \cdots \times V}_{q \text{ copies}} \rightarrowtail \mathbb{R}.$$

Then we collect all of the tensors on a vector space and place them in their own vector space.

Definition 2.7: Tensor Space

We define the vector space V_q^p of (p,q)-tensors on V by writing

$$\underbrace{V \otimes \cdots \otimes V}_{p \text{ copies}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{q \text{ copies}} := \left\{\tau \mid \tau \text{ is a } (p,q)\text{−tensor}\right\}.$$

The question is then how to define tensor products of vector bundles. Given a vector bundle E, we know that for each $x \in M$, $\pi^{-1}(x) = V$ is a vector space. This leads us to a construction along each fiber.

Definition 2.8: Tensor Product of Bundles

Let (E, M, π_E) and (F, M, π_F) be vector bundles over the same base manifold M. Then we have for $x \in M$, $\pi_E^{-1}(x) = V$ and $\pi_F^{-1}(x) = W$ are the vector space fibers over the point x in the respective bundles. The ***tensor product of E and F*** is defined by

$$E \otimes F := \bigsqcup_{x \in M} \pi_E^{-1}(x) \otimes \pi_F^{-1}(x).$$

3 Parameterized Tensors

With the framework built up, we can now realize tensors that are parameterized by a manifold. Then are a few salient facts that should be mentioned.

Definition 3.1: Tensor Field

A ***(p,q)-tensor field*** is a (smooth) section

$$\tau \colon M \to \underbrace{T^*M \otimes \cdots T^*M}_{p \text{ copies}} \otimes \underbrace{TM \otimes \cdots \otimes TM}_{q \text{ copies}}.$$

Equivalently, one can write

$$\tau \in \Gamma(T^*M \otimes \cdots T^*M \otimes TM \otimes \cdots \otimes TM).$$

In other words, at each point $x \in M$, $\tau(x)$ is a (p,q)-tensor.

Note: We could have tensor fields that are constructed over different vector bundles, but the natural bundles on M are TM and T^*M.

The act of taking sections of a bundle $\Gamma(\cdot)$ plays nicely with the tensor product of bundles. In fact, Γ can be thought of as a functor in an appropriate sense. This can be summed up in the following proposition and theorem.

Proposition 3.1: The Space of Sections is a Module

Let E be a vector bundle. Then $\Gamma(E)$ is a $C^\infty(M)$-module.

It's worth noting that $\Gamma(E)$ is infinite dimensional. However, we will find that we still retain reflexivity.

Since we can realize the space of sections $\Gamma(E)$ as a module, this leads us to believe we can alternatively define tensor fields via taking tensor products of the modules of sections themselves. This means of construction is potentially different.

Theorem 3.1: Tensor Products of Sections

Let E and F be vector bundles over M. Then we have

$$\Gamma(E) \otimes_{C^\infty(M)} \Gamma(F) = \Gamma(E \otimes F).$$

This shows that the two different constructions of a tensor field are equivalent and thus either are valid to work with. At times, each has an advantage. One last final result is in order.

Theorem 3.2: Modules of Sections are Reflexive

Let E be a vector bundle over M and $\Gamma(E)$ be sections of the bundle. Then

$$\Gamma(E)^* = \Gamma(E^*).$$

Hence, it follows that $\Gamma(E)$ is reflexive. That is,

$$\Gamma(E)^{**} = \Gamma(E^{**}) = \Gamma(E)$$

since E is finite dimensional.

With all of this constructed, let us consider an extremely important example of all of this.

Example 3.1: Riemannian Manifold

A ***Riemannian manifold*** $M \equiv (M, g)$ is a smooth manifold with a positive definite symmetric $(0,2)$-tensor field g. M has many nice properties.

- M has an inner product on each T_xM.
- TM and T^*M are naturally isomorphic via Riesz representation.
- M is a metric space.
- M has a canonical volume form.
- M has a natural connection on TM.
- M has a well defined Laplacian on k-forms.

Riemannian manifolds are extremely useful in generalizing calculus on $\mathbb{R}^m$ as they essentially mimic all of the structure in $\mathbb{R}^m$ aside from potentially having nontrivial topology or curvature. On top of that, it is not asking much to build a Riemannian manifold since every smooth manifold admits a Riemannian metric.