Clifford Analysis and a Noncommutative Gelfand Representation

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Outline

- Introduce geometric algebra and calculus.
- Describe the toolbox in comparison to differential forms.
- Prove a multivector version of the Hodge-Morrey decomposition.
- Prove a noncommutative version of the Gelfand representation.

Section 1

Introduction

Subsection 1

Motivation

Electrical Impedance Tomography

Electrical Impedance Tomography (EIT) asks whether one can determine the conductivity of a medium based on measurements along the boundary.

Calderón problem

- Let M be a smooth, connected, oriented Riemannian manifold with boundary ∂M with metric g.
- Conductivity is represented by g.
- Forward problem: Let $\Delta u = 0$ in M and $u = \phi$ on ∂M.
- Inverse problem: Given the ***Dirichlet-to-Neumann map*** $\Lambda\phi = \frac{\partial u}{\partial \nu}$, can we recover (M, g)?

Subsection 2

Preliminaries

- ***Clifford algebra*** originated in 1878 with William Kingdon Clifford's work that extends Hermann Grassmann's ***exterior algebra***.
- ***Clifford analysis*** arrived in the 1980's due to Hestenes, Sobczyk, Sommen, Brackx, and Delenghe in order to enrich Éllie Cartan's ***differential forms***.
  - Atiyah-Singer Dirac operator and spin manifolds.

Clifford algebras

Let V be a vector space over a field $\mathbb{F}$ with quadratic form Q.

- Define the tensor algebra

$$\mathcal{T}(V) \coloneqq \bigoplus_{j=0}^{\infty} V^{\otimes j} = \mathbb{F} \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots.$$

- Form the ***Clifford algebra*** via a quotient

$$C\ell(V, Q) \coloneqq \mathcal{T}(V) / \langle \mathbf{v} \otimes \mathbf{v} - Q(V) \rangle.$$

Geometric and exterior algebras

Let V be a vector space over a field $\mathbb{F}$ with quadratic form Q.

- Given a (pseudo) inner product g, we set $Q(\cdot) = g(\cdot, \cdot)$ and define a ***geometric algebra***

$$\mathcal{G} \coloneqq C\ell(V, g).$$

- The ***exterior algebra*** is given by

$$\bigwedge(V) \coloneqq C\ell(V, 0).$$

Algebra structure

We define a multiplication in $\mathcal{G}(V)$ by noting how the product $\otimes$ acts in the quotient.

- Given $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{G}(V)$ we can take the product

$$\boldsymbol{u}\boldsymbol{v} = \underbrace{\boldsymbol{u} \cdot \boldsymbol{v}}_{\text{scalar}} + \underbrace{\boldsymbol{u} \wedge \boldsymbol{v}}_{\text{bivector}} .$$

  - The scalar part is symmetric: $\boldsymbol{u} \cdot \boldsymbol{v} = g(\boldsymbol{u}, \boldsymbol{v})$.
  - The bivector part is antisymmetric: $\boldsymbol{u} \wedge \boldsymbol{v} = -\boldsymbol{v} \wedge \boldsymbol{u}$.

Multivectors

- $\mathcal{G}$ is graded and of dimension 2^n.
  - There are $\binom{n}{r}$ elements of grade r called ***r-vectors***.
  - Those that are exterior products of r independent vectors are ***r-blades***.

    E.g., $\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_r$.
- The most general elements are ***multivectors*** and are given by

$$A = \sum_{r=0}^{n} \langle A \rangle_r,$$

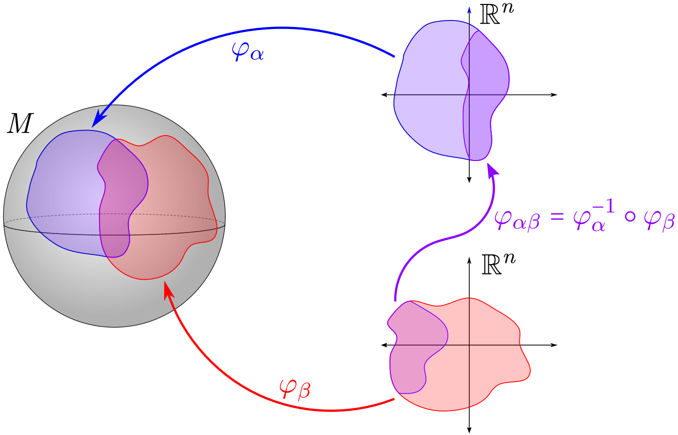
where $\langle A \rangle_r$ extracts the grade r part of A.

Examples

$\mathbb{C}$ and quaternions, even subalgebras

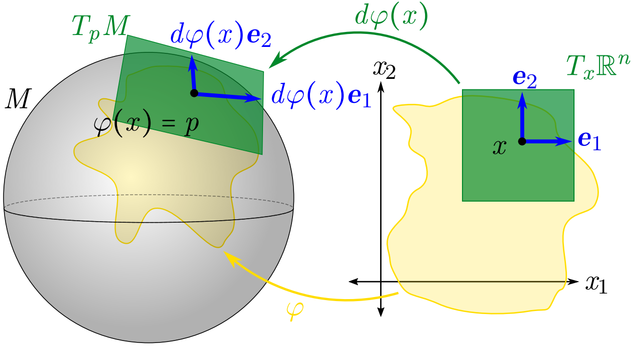
The playing field

We let M be a smooth, compact, connected, and oriented n-dimensional Riemannian manifold with metric g (unless otherwise stated).



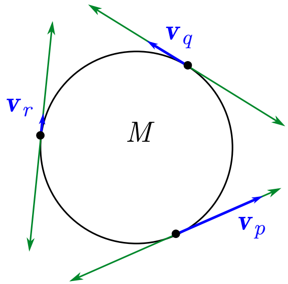
The playing field

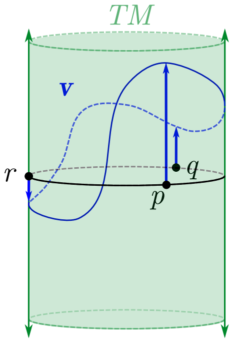
At each point on M, we have the tangent space T_pM.



The playing field

From M, we create the tangent bundle TM whose sections are vector fields.





Idea: On each tangent space, let us construct a manner in which to multiply vectors.

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Idea: Glue together geometric tangent spaces.

- Each $C\ell(T_pM, g_p)$ is a ***geometric tangent space*** which we glue together to form
$$C\ell(TM, g) \coloneqq \bigsqcup_{p \in M} C\ell(T_pM, g_p).$$
- The space of ***(smooth) multivector fields*** is
$$\mathcal{G}(M) \coloneqq \{C^\infty\text{-smooth sections of } C\ell(TM, g)\}.$$

Clifford Algebraic Structure

- How do we add and multiply vector fields.
- Extend this to products on multivectors

Subsection 3

Preliminaries