# MATH 571, Homework 7

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## Solutions

**Problem 1.**

(a) Pick a $\Delta$-complex structure on the $n$-dimensional ball $D^n$. Note that this induces a $\Delta$-complex structure on its boundary $(n-1)$-sphere $S^{n-1}$. Compute the simplicial relative homology $H_n(D^n, S^{n-1})$ by using the definition of relative homology.

(b) Verify that your answer is correct by using the LES for a pair of spaces (on pages 115 and 117 of Hatcher).

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*Proof.*

(a) Let $\Delta^n$ be the $n$-simplex and note that simplicial complexes are also $\Delta$-complexes. Also we have that $\Delta^n \cong D^n$ and $\partial\Delta^n \cong S^{n-1}$. Then we have that $C_n(\Delta^n, \partial\Delta^n) \cong C_n(D^n, S^{n-1})$. Now note that for $m > n$ we have

$$C_m(\Delta^n, \partial\Delta^n) \cong C_m(\Delta^n)/C_{n+1}(\partial\Delta^n) \cong 0/0 \cong 0$$

since there are no higher dimensional simplicies than $n$ in $\Delta^n$. Also,

$$C_n(\Delta^n, \partial\Delta^n) \cong C_n(\Delta^n)/C_n(\partial\Delta^n) \cong \mathbb{Z}/0 \cong \mathbb{Z}.$$

Finally, for $l < n$,

$$C_l(\Delta^n, \partial\Delta^n) =\cong C_l(\Delta^n)/C_l(\partial\Delta^n) \cong \mathbb{Z}/\mathbb{Z} \cong 0$$

since $\Delta^n$ and $\partial\Delta^n$ contain all the same $l < n$ dimensional simplicies. Then we have

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1}(\Delta^n, \partial\Delta^n) \xrightarrow{\partial_{n+1}} C_n(\Delta^n, \partial\Delta^n) \xrightarrow{\partial_n} C_{n-1}(\Delta^n, \partial\Delta^n) \xrightarrow{\partial_{n-1}} \cdots.$$

With the above work, we have

$$\cdots \xrightarrow{\partial_{n+2}} 0 \xrightarrow{\partial_{n+1}} \mathbb{Z} \xrightarrow{\partial_n} 0 \xrightarrow{\partial_{n-1}} \cdots.$$

Now

$$H_n(\Delta^n, \partial\Delta^n) \cong \mathrm{Ker}(\partial_n)/\mathrm{Im}(\partial_{n+1} \cong \mathbb{Z}/0 \cong \mathbb{Z}.$$

(b) Now we have the following LES

$$\cdots \to H_n(S^{n-1}) \to H_n(D^n) \xrightarrow{\alpha} H_n(D^n, S^{n-1}) \xrightarrow{\beta} H_{n-1}(S^{n-1}) \xrightarrow{\gamma} H_{n-1}(D^n) \to \cdots.$$

Note that

$$\begin{aligned} H_n(S^{n-1}) &\cong 0 \\ H_n(D^n) &\cong 0 \\ H_{n-1}(S^{n-1}) &\cong \mathbb{Z} \\ H_{n-1}(D^n) &\cong 0. \end{aligned}$$

By exactness, we must have that $\beta$ is an isomorphism and hence $H_n(D^n, S^{n-1}) \cong \mathbb{Z}$. $\square$

**Problem 2.** The proof of Theorem 2.16 in Hatcher contains 6 verification of inclusions. Prove them. For one of the last three such inclusions ($\ker j_* \subseteq \text{im} i_*$ or $\ker \partial \subseteq \text{im} j_*$ or $\ker i_* \subseteq \text{im}\partial$, your choice of which one), draw a diagram of the SES of chain complexes $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$, and show where all the elements you consider live in this diagram.

$\vdots$

*Proof.* We have that $\text{im} i_* \subset \ker j_*$ since $ji = 0$ means that $j_* i_* = 0$. Then $\text{im} j_* \subset \ker \partial$ since $\partial j_* = 0$ by definition of $\partial$. Specifically we have $c = j(b)$ for some $b \in B_n$ and then we have $j(\partial b) = \partial j(b)$. Then we also have $\text{im}\partial \subset \ker i_*$ since $i_* \partial = 0$.

$\ker j_* \subset \text{im} i_*$. A homology class in $\ker j_*$ is represented by a cycle $b \in B_n$ with $j(b)$ a boundary. We can then write $j(b) = \partial c'$ for some $c' \in C_{n+1}$. By surjectivity of $j$, $c' = j(b')$ for some $b' \in B_{n+1}$. This means that $j(b - \partial b') = j(b) - j(\partial b') = j(b) - \partial j(b') = 0$. This is because $\partial j(b') = \partial c' = j(b)$ and we defined $j(b)$ as a boundary. It follows that $b - \partial b' = i(a)$ for some $a \in A_n$ and in fact this $a$ is a cycle since $i(\partial(a) = \partial i(a) = \partial(b - \partial b') = \partial b = 0$ since $\partial^2 = 0$ and since $b$ is a cycle. Then we have that $i_*[a] = [b - \partial b'] = [b]$, which shows that $i_*$ maps onto $\ker j_*$. See the diagram below!

$$\begin{tikzcd}
 & \vdots \arrow[d, "\partial"] & \vdots \arrow[d, "\partial"] & \vdots \arrow[d, "\partial"] & \\
0 \arrow[r] & A_{n+1} \arrow[r, "i"] \arrow[d, "\partial"] & B_{n+1} \arrow[r, "j"] \arrow[d, "\partial"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "i"] \arrow[d, "\partial"] & B_n \arrow[r, "j"] \arrow[d, "\partial"] & C_n \arrow[r] \arrow[d, "\partial"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

$\ker \partial \subset \text{im} j_*$. We let $c$ be a homology class in $\ker \partial$, then we know that $a = \partial a'$ for $a' \in A_n$. Then $b - i(a')$ is a cycle since $\partial(b - i(a')) = \partial b - \partial i(a') = \partial b - i(\partial a') = \partial b - i(a) = 0$ since by definition we have $\partial b = i(a)$. Also, $j(b - i(a')) = j(b) - ji(a') = j(b) = c$, hence $j_*$ maps $[b - i(a')]$ to $[c]$.

$\ker i_* \subset \text{im}\partial$. Take a cycle $a \in A_{n-1}$ so that $i(a) = \partial b$ for some $b \in B_n$. Then we have that $j(b)$ is a cycle since $\partial j(b) = j(\partial b) = ji(a) = 0$ again by definition. This means that $\partial[j(b)] = [a]$.

**Problem 3.** Exercise 16 on page 132 of Hatcher:

(a) Show that $H_0(X, A) = 0$ iff $A$ meets each path-component of $X$.

(b) Show that $H_1(X, A) = 0$ iff $H_1(A) \to H_1(X)$ is surjective and each path-component of $X$ contains at most one path-component of $A$.

*Remark: For (b), Exercise 15 of Hatcher is useful. It says that if $A \to B \to C \to D \to E$ is an exact sequence, then $C = 0$ iff the map $A \to B$ is surjective and $D \to E$ is injective.*

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*Proof.*

(a) It's worth noting that we can reduce this problem by noting

$$H_n(X, A) \cong \bigoplus_{i \in I} H_n(X_i, A_i)$$

where each $X_i$ is a path component of $X$ with the index set $I$. Now this specifically means that we need to show

$$H_0(X_i, A_i) \cong 0$$

For the backward direction, suppose that $A_i$ is empty, then we have

$$\cdots \to H_0(A_i) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X_i, A_i) \xrightarrow{\partial} 0$$

where $H_0(A_i) \cong 0$ and $H_0(X) \cong \mathbb{Z}$. Then exactness shows that $H_0(X_i, A_i) \cong \mathbb{Z}$. Hence we must have that $A_i$ is nonempty.

For the forward direction, we suppose that $H_0(X_i, A_i) = 0$ for each $i$. Then we have

$$\cdots \to H_0(A_i) \xrightarrow{i_*} H_0(X_i) \xrightarrow{j_*} H_0(X, A) \xrightarrow{\partial} 0$$

where we see that we have the exact sequence

$$H_0(A_i) \xrightarrow{i_*} \mathbb{Z} \xrightarrow{j_*} 0.$$

But exactness here implies that $i_*$ is surjective, and hence $H_0(A_i) \not\cong 0$ and hence $A_i$ is nonempty.

(b) Suppose that $H_1(X, A) \cong 0$. We have

$$\to H_1(A) \xrightarrow{i_*} H_1(X) \xrightarrow{j_*} H_1(X, A) \xrightarrow{\partial} H_0(A) \xrightarrow{i_*} H_0(X) \to$$

is an exact sequence. But here, we must have that $H_1(A) \to H_1(X)$ is surjective and $H_0(A) \to H_0(X)$ is injective. We can decompose $A$ into the path components $A_i$ for $i \in I$ and do the same for $X$ by decomposing $X$ into $X_j$ with $j \in J$. Then note that injectivity of $H_0(A) \to H_0(X)$ means that $\bigoplus_{i \in I} H_0(A_i) \to \bigoplus_{j \in J} H_0(X_j)$ is injective for each component and so there must only be at most one path component of $A$ in each path component of $X$ or this map fails to be injective.

For the other direction, we have that $H_1(A) \to H_1(X)$ is surjective. Then if each path component of $X$ contains at most a single path component of $A$, then $H_0(A) \to H_0(X)$ is injective by the argument above. These two maps being surjective and injective, respectively, implies that $H_1(X, A) \cong 0$. □

**Problem 4.** Choose any midterm (or old homework) problem. Clearly state the problem. Write a solution that is as clear as possible.

I will redo problem 3 from the midterm.

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*Proof.* Note that $A_1$ and $A_2$ are each a torus with a disk removed and so each is homotopy equivalent to a wedge sum of two circles. This means

$$\begin{aligned} \pi_1(A_1) &\cong \langle a, b \rangle \\ \pi_1(A-2) &\cong \langle c, d \rangle. \end{aligned}$$

We have that $A_1 \cap A_2$ is path connected and this gives the following diagram:

$$\begin{tikzcd} & A_1 \arrow[dr, "j_1"] & \\ A_1 \cap A_2 \arrow[ur, hook, "i_{12}"] \arrow[dr, hook, "i_{21}"] & & X = A_1 \cup A_2 \\ & A_2 \arrow[ur, hook, "j_2"] & \end{tikzcd}$$

Specifically, we have that

$$\pi_1(A_1 \cup A_2) \cong \pi_1(X) \cong (\pi_1(A_1) * \pi_1(A_2))/N$$

where $N$ is the normal subgroup generated by all elements of the form $i_{12}(w)i_{21}(w)^{-1}$ with $w \in \pi_1(A_1 \cap A_2)$. So we find by looking at $A_1 \cap A_2$ in the identification polygon for $A_1$ that $i_{12}(w) = aba^{-1}b^{-1}$ and for $A_2$ that $i_{21}(w)cdc^{-1}d^{-1}$. Hence we get

$$\begin{aligned} \pi_1(X) &\cong \langle a, b, c, d \rangle / N \\ &\cong \langle a, b, c, d | aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle. \end{aligned}$$ □