MATH 570, Homework 11

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Solutions

Problem 1.

(a) Let X and Y be topological spaces with $f\colon X \to Y$ continuous. Suppose $p_n \in X$ is a sequence of points converging to $p \in X$. Prove that $f(p_n)$ converges to $f(p)$.

(b) The following is related to pages 344–345 of our book. A sequence of abelian groups and group homomorphisms $\ldots \to G_{p+1} \xrightarrow{\alpha_{p+1}} G_p \xrightarrow{\alpha_p} G_{p-1} \to \ldots$ is *exact* if $\mathrm{im}(\alpha_{p+1}) = \ker(\alpha_p)$ for each p. A 5-term exact sequence of the form

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

is called a *short exact sequence*. Here the abelian groups "0" on either end are the trivial group. Prove in a short exact sequence that α is injective, that β is surjective, and that there is a group isomorphism $C \cong B/\alpha(A)$.

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Proof.

(a) Suppose $f(p_n)$ does not converge to $f(p)$. Then there exists an open neighborhood $N_f(f(p))$ such that $N_f(f(p)) \cap f(p_n) = \emptyset$. Note that $f^{-1}(N(f(p))) = N(p)$ is an open neighborhood containing p, and thus $N(p) \cap p_n \neq \emptyset$. This is a contradiction since we would necessarily have that $f(N(p)) \subseteq N(f(p))$ contains points of $f(p_n)$, yet we supposed the contrary. Thus, $f(p_n)$ converges to $f(p)$.

(b) Denote e_X as the identity element of which ever group X. Then, define $\varphi\colon 0 \to A$ and note that the exact sequence implies that $\ker(\alpha) = \mathrm{im}(\varphi) = e_A$. Then this means that e_A is the only element in A such that α maps to e_B. This implies injectivity of α. Hence α is injective. Similarly define $\psi\colon C \to 0$ and note that $\mathrm{im}(\beta) = \ker(\psi) = C$. Since $\mathrm{im}(\beta) = C$ we have that β is surjective. Note that the first isomorphism theorem implies that $C \cong B/\alpha(A)$ since $\alpha(A) = \ker(\beta)$.

Problem 2. In class on Monday 11/13 we will show that a continuous map $f\colon X \to Y$ produces a homomorphsim of singular homology groups $f_*\colon H_p(X) \to H_p(Y)$, and furthermore that this produces a p-dimensional singular homology functor $H_p\colon \mathrm{Top} \to \mathrm{Ab}$ from the category of topological spaces to the category of abelian groups (Proposition 13.2). If $A \subseteq X$ is a retract of X, then prove that one can have an injective group homomorphism $H_p(A) \to H_p(X)$ and a surjective group homomorphism $H_p(X) \to H_p(A)$.

Remark: This is essentially Corollary 13.4 in our book.

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Proof. Note that for a retract we have $r\colon X \to A$ is such that $r|_A = \mathrm{Id}_A$ and $\iota\colon A \to X$ is such that $r \circ \iota = \mathrm{Id}_A$. Then $r_*\colon H_p(X) \to H_p(A)$ and $\iota_*\colon H_p(A) \to H_p(X)$, and since H_p is a functor, we have that r_* and ι_* are group homomorphisms. Now fix $a_1, a_2 \in H_p(A)$ and consider

$$\begin{aligned} \iota_*(a_1) &= \iota_*(a_2) \\ \iota_*(a_1)\iota_*(a_2^{-1}) &= e_{H_p(X)} \\ \iff r_*(\iota_*(a_1 a_2^{-1})) &= r_*(e_{H_p(X)}) \\ a_1 a_2^{-1} &= e_{H_p(A)} \\ a_1 &= a_2. \end{aligned}$$

So ι_* is an injective group homomorphism. Now consider any $a \in H_p(A)$. Then

$$\begin{aligned} a &= r_* \circ \iota_*(a) \\ a &= r_*(\iota_*(a)), \end{aligned}$$

which shows that we have an element $x = \iota_*(a) \in H_p(X)$ so that $r_*(x) = a$. Thus r_* is a surjective group homomorphism. $\square$

Problem 3. Let $n \geq 0$ be an integer. The Brouwer fixed point theorem states that every continuous map $f\colon \overline{B^n} \to \overline{B^n}$ has a fixed point, i.e. a point $x \in \overline{B^n}$ with $f(x) = x$. Prove the Brouwer fixed point theorem, as follows.

(a) Suppose for a contradiction that a continuous map $f\colon \overline{B^n} \to \overline{B^n}$ has no fixed points. Use f to define a continuous retract $g\colon \overline{B^n} \to S^{n-1}$. If you like you can define this map precisely with English words and a picture (instead of a formula). You do not need to prove that your map g is continuous.

(b) Use the facts $H_{n-1}(\overline{B^n}) = 0$ and $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$ (which we'll prove later) to derive a contradiction.

Remark: See problems 13-7 and 8-6 in our book if you like. The proof outline in 8-6 (for $n = 2$ only) is slightly different; note that the book's map $\phi\colon \overline{B^2} \to S^1$ need not be a retract. You could also use this proof outline if you so choose.

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Proof. (a) Suppose for a contradiction that we have a continuous map $f\colon \overline{B^n} \to \overline{B^n}$ with no fixed points, i.e. each $f(x) = y \in \overline{B^n}$ with $y \neq x$. Specifically this means that we have for any $f(x)$, a unique line from $f(x)$ to a point $f(x') \in \partial\overline{B^n} = S^{n-1}$. Let $g\colon f(\overline{B^n}) \to S^{n-1}$ be the continuous function taking any point $f(x) \in B^n$ to $\partial\overline{B^n}$. This $g \circ f$ is then a retract from $\overline{B^n}$ to S^{n-1}.

(b) We have that $f_*\colon H_{n-1}(\overline{B^n}) \to H_{n-1}(S^{n-1})$ is surjective by Problem 2. However, there does not exist a surjective group homomorphism from 0 to $\mathbb{Z}$ which shows that f_* was not a retract. Thus, by this contradiction, we must have that f had at least one fixed point. $\square$