MATH 571, Homework 2

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Solutions

Problem 1. Give an example of a connected space X, a subspace $A \subseteq X$, and a map $r\colon X \to A$ that is a retract but which does not come from a deformation retraction. Prove your answer is correct.

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Proof. Let $X = S^1$ and let $A = \{p\} \in S^1$ be a single point. Then let $r\colon X \to A$ be a retraction. Note this map is continuous since $r^{-1}(p) = S^1$ and is a retraction since $r|_A = \mathrm{Id}_A$ and $r(X) = A$. Yet, this map is not a deformation retraction since $S^1 \not\simeq \{p\}$. □

Problem 2. The *Euler characteristic* of a finite CW complex X is $\chi(X) = \sum_i (-1)^i c_i$, where c_i is the number of i-cells of X. Alternatively see also the definition on page 6 of our book.

(a) If X is S^n with a CW structure of a single 0-cell e^0 and a single n-cell e^n, then what is $\chi(X)$?

(b) If X is S^n with a CW structure of two 0-cells, two 1-cells, ..., two n-cells, then what is $\chi(X)$?

(c) Explain why Theorem 2.44 on page 146 of our book, which we haven't covered yet, says that you should have expected to get the same answer in (a) and (b).

(d) Any simplicial complex has a CW complex structure with one i-cell for each i-simplex. Find the Euler characteristic of the n-simplex Δ^n by counting c_i for each i, and then computing the alternating sum $\chi(\Delta^n) = \sum_i (-1)^i c_i$. How in the world do you simplify that long alternating sum?

(e) Instead, now find the Euler characteristic of the n-simplex by remarking that Δ^n is homotopy equivalent to a simpler space (no proof needed), and then computing the Euler characteristic of that simpler space.

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Proof.

(a) $\chi(X) = \sum_i (-1)^i c_i = (-1)^0 + (-1)^n = 1 + (-1)^n$. So $\chi(X) = 0$ if n is even and $\chi(X) = 2$ if n is odd.

(b) $\chi(X) = \sum_i (-1)^i c_i = 2(-1)^0 + 2(-1)^1 + \cdots + 2(-1)^{n-1} + 2(-1)^n$. Then if n is even we can pair off these quantities by $2(-1)^0 + 2(-1)^1 = 0$, $2(-1)^2 + 2(-1)^3 = 0$, up to $2(-1)^{n-1} + 2(-1)^n = 0$ and hence $\chi(X) = 0$ if n is even. If n is odd, then we can pair off and cancel out terms up to the nth term, which will give us $\chi(X) = 2(-1)^n = 2$.

(c) Since S^n has a given homology, we don't expect the way we construct S^n to change the rank of any homology group of S^n. Otherwise, we would not have a consistent homology theory.

(d) Notice that we have $c_0 = n+1$ for Δ^n. Also, we have that $c_i = \binom{n+1}{i+1}$ for Δ^n as well. This gives that $\chi(X) = n+1+\sum_{i=1}^{n}(-1)^i\binom{n+1}{i+1}$. I think this will have the sum collpase into giving us $\chi(X) = 1$.

(e) Since each Δ^n is contractible, we have that $\Delta^n \simeq \Delta^0$. Then the long alternating sum just becomes $c_0 = 1$. □

Problem 3. Show that S^∞ is contractible. I encourage you to consult Example 1B.3 on page 88 in our book!

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Proof. First consider the homotopy $f_t \colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ given by $f_t(x_1, x_2, \dots) = (1-t)(x_1, x_2, \dots) + t(0, x_1, x_2, \dots)$ and note that for all $t \in [0,1]$, f_t takes nonzero vectors to nonzero vectors. This means that we can let $\frac{f_t}{|f_t|} \colon S^\infty \to S^\infty$ be a homotopy. We can then define $g_t \colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ by $g_t(x_1, x_2, \dots) = (1-t)(0, x_1, x_2, \dots) + (1, 0, 0, \dots)$, which is again nonzero for all $t \in [0,1]$. Hence, we can define a homotopy $\frac{g_t}{|g_t|} \colon S^\infty \to S^\infty$ where g_1 a constant map. Now, if we consider $h_t = \frac{g_{2t-1}}{|g_{2t-1}|} \circ \frac{f_{2t}}{|f_{2t}|}$ we have a homotopy from S^∞ to a point, showing that S^∞ is contractible. □