# MATH 272, Worksheet 2

## Vector fields and differential calculus of fields.

**Problem 1.** Plot the following vector fields.

(a) $\vec{\boldsymbol{U}}(x, y, z) = \begin{pmatrix} xyz \\ xyz \\ xyz \end{pmatrix}$.

(b) $\vec{\boldsymbol{V}}(x, y, z) = \begin{pmatrix} e^{x+y+z} \\ e^{x+y+z} \\ e^{x+y+z} \end{pmatrix}$.

(c) $\vec{\boldsymbol{W}}(x, y, z) = \begin{pmatrix} x\sin(y) \\ x\cos(y) \\ xz \end{pmatrix}$.

(d) $\vec{\boldsymbol{F}}(x, y, z) = \begin{pmatrix} 5 + yz \\ -5 - xz \\ xy \end{pmatrix}$.

**Problem 2.** Compute the divergence, curl, and Jacobian matrix of the vector fields in Problem 1.

**Problem 3.** For the following functions, compute and plot the gradient vector fields.

(a) For just $c = 1$, plot the level set for $E(x, y, z) = \frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9}$.

(b) $f(x, y, z) = xyz$.

(c) $g(x, y, z) = e^x - y^2 - z^2$.

(d) $h(x, y, z) = \sin(x) + \cos(y) - \tanh(z)$.

(e) $p(x, y, z) = \sin^2(x) + \sin^2(y) - \frac{1}{2}\sin(z)$.

(f) $q(x, y, z) = x^2 + xy + y^2 + sin(yz)$.

(g) One of your own choosing.

**Problem 4.** Given a function, a vector, and a point, compute the directional derivatives at that point.

(a) $f(x, y) = x^2/y$, $\vec{\boldsymbol{u}} = 3\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}$, and $(x, y) = (2, 2)$.

(b) $g(x, y, z) = x\cos(yz^2)$, $\vec{\boldsymbol{v}} = \frac{1}{2}\hat{\boldsymbol{x}} + \frac{1}{2}\hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}$, and $(x, y, z) = (0, -1, 2)$.

**Problem 5.** Give a physical argument for why the field $\vec{\boldsymbol{U}} = \begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix}$ has nonzero curl everywhere. Reason why the direction of the curl is solely along the $z$-axis. Do this <u>without</u> computing the curl.

**Problem 6.** Give a physical argument why the field $\vec{\boldsymbol{V}} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$ has nonzero divergence everywhere. Reason why the divergence is a scalar quantity as opposed to a vector quantity. Do this <u>without</u> computing the curl.

**Problem 7.** Compute the Laplacian of the following fields.

(a) $f(x,y) = (x+y)^2$

(b) $g(x,y) = (x+y)e^{x^2+y^2}$.

(c) $\vec{\boldsymbol{U}} = \begin{pmatrix} \cos(x)\cos(y)\cos(z) \\ \sin(x)\sin(y)\sin(z) \\ xyz \end{pmatrix}$.

(d) $\vec{\boldsymbol{V}} = \begin{pmatrix} \frac{y}{z} \\ \frac{x}{z} \\ \frac{x}{y} \end{pmatrix}$.

**Problem 8.** Suppose that $\vec{\boldsymbol{V}} = \vec{\boldsymbol{\nabla}}\phi$ for some scalar field $\phi$. Explain why if $\vec{\boldsymbol{V}}$ is divergence free (i.e., if $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{V}} = 0$) that $\vec{\boldsymbol{\Delta}}\vec{\boldsymbol{V}} = \vec{\boldsymbol{0}}$.

**Problem 9.** One of Maxwell's equations states for a magnetic field $\vec{\boldsymbol{B}}$ that

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{B}} = 0,$$

is an identity. Does this mean that $\vec{\boldsymbol{\Delta}}\vec{\boldsymbol{B}} = 0$?