MATH 571, Homework 8

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March 30, 2018

Solutions

Problem 1. Pick a Δ-complex structure on the pair of spaces $(S^1 \times S^1, S^1 \times \{1\})$ – probably the first Δ-complex structure you think of on $S^1 \times S^1$ will work. Compute the simplicial relative homology $H_n(S^1 \times S^1, S^1 \times \{1\})$ for all n.

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Proof. Let me draw the Δ-complex for $S^1 \times S^1$ and for $S^1 \times \{1\}$ below.

Now we look at the chain complex

$$\cdots \to \Delta_3(X)/\Delta_3(A) \xrightarrow{\partial_3} \Delta_2(X)/\Delta_2(A) \xrightarrow{\partial_2} \Delta_1(X)/\Delta_1(A) \xrightarrow{\partial_1} \Delta_0(X)/\Delta_0(A) \xrightarrow{\partial_0} 0.$$

Note that for $i \geq 3$, $\Delta_i(X)/\Delta_i(A) \cong 0$ since there are no simplicies of dimension 3 or higher. We have that $\Delta_2(X)/\Delta_2(A) \cong \mathbb{Z}^2$ is generated by T, U and that under ∂_2 we have

$$T \mapsto b - c + a = b - c$$
$$U \mapsto b - c + a = b - c.$$

Then $\Delta_1(X)/\Delta_1(A) \cong \mathbb{Z}^2$ is generated by b and c and under ∂_1 we have

$$b \mapsto 0$$
$$c \mapsto 0.$$

Then since $\Delta_0(X) \cong \mathbb{Z}$ and $\Delta_0(A) \cong Z$ we have $\Delta_0(X)/\Delta_0(A) \cong 0$. We then compute homology to find that

$$H_0(X, A) \cong 0$$
$$H_1(X, A) \cong \mathbb{Z}$$

since we have that $\ker \partial_1$ is generated by $\{b, b - c\}$ with a change of basis and $\mathrm{im}\partial_1$ is generated by $\{b - c\}$. Then

$$H_2(X, A) \cong \mathbb{Z}.$$

Finally for $i \geq 3$ we have $H_i(X, A) \cong 0$. $\square$

Problem 2. Hatcher exercise 9(a) on page 155: Compute the homology groups of the quotient of S^2 obtained by identifying the north and south poles to a point.

Remark: I recommend using the long exact sequence for the singular homology of a pair (S^2, S^0).
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Proof. First note that we have S^2/S^0 as our desired space since S^0 as a subspace of S^2 can be taken to be the north and south poles. Then also we have that S^2 and S^0 form a good pair since S^0 is a closed subspace that is also a deformation retract of open neighborhoods about the north and south pole of S^2. Now, this means $\tilde{H}_i(S^2/S^0) \cong H_i(S^2, S^0)$ so we can use the long exact sequence in the above remark. Namely, we have

$$\begin{aligned} \cdots \to H_3(S^0) \cong 0 \to H_3(S^2) \cong 0 \to H_3(S^2, S^0) \\ \to H_2(S^0) \cong 0 \to H_2(S^2) \cong \mathbb{Z} \to H_2(S^2, S^0) \\ \to H_1(S^0) \cong 0 \to H_1(S^2) \cong 0 \to H_1(S^2, S^0) \\ \to H_0(S^0) \cong \mathbb{Z}^2 \to H_0(S^2) \cong \mathbb{Z} \to H_0(S^2, S^0) \cong 0. \end{aligned}$$ □

Note that we have $H_0(S^2, S^0)$ by our previous homework problem since S^0 meets the connected component of S^2. Now, this exact sequence gives us that for $i \geq 3$, $H_i(S^2, S^0) \cong 0$. We also have $H_2(S^2, S^0) \cong \mathbb{Z}$ by exactness above. Exactness again implies that $H_1(S^2, S^0) \cong H_0(S^0)/H_0(S^2)$ and hence $H_1(S^2, S^0) \cong \mathbb{Z}$. □