MATH 571, Homework 4

Colin Roberts

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Solutions

Problem 1. Do Exercise 12 on page 80 of Hatcher: "Let a and b be the generators of $\pi_1(S^1 \vee S^1)$ corresponding to the two S^1 summands. Draw a picture of the covering space of $S^1 \vee S^1$ corresponding to the normal subgroup generated by a^2, b^2, and $(ab)^4$, and prove that this covering space is indeed the correct one."

Remark: Let N be the normal subgroup in $\langle a, b\rangle$ generated by a^2, b^2, and $(ab)^4$; note that N is larger than the group $\langle a^2, b^2, (ab)^4\rangle$. To prove that your covering space $\tilde{X}$ is correct, you need to show that $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = N$. Showing $(\supseteq)$ doesn't require too much paper. To get $(\subseteq)$, it suffices to show that $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is generated by conjugates of a^2, b^2, and $(ab)^4$.

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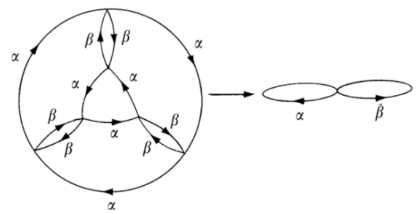
Proof. We'll draw the covering space below:

Now, to see that $H \coloneqq p_*(\pi_1(\tilde{X}, \tilde{x_0})) \supseteq N$ we note that N is the smallest normal subgroup generated by the elements a^2, b^2, and $(ab)^4$. Since $\tilde{X}$ is a normal covering space, H is normal and thus $H \supseteq N$. We note that $\tilde{X}$ is normal since any deck transformation is just a rotation of this very symmetric space.

To see that $H \subseteq N$, we note that $\pi_1(\tilde{X}, \tilde{x_0})$ is generated by 9 elements created by conjugating a^2, b^2, and $(ab)^4$ with each other. Since $H \cong \pi_1(\tilde{X}, \tilde{x_0})$, we have that H is generated by these 9 elements which must be elements of N since N is a normal subgroup generated by a^2, b^2, and $(ab)^4$ and N thus contains these 9 conjugates. So $H \subseteq N$ as well and so $H = N$. $\square$

Problem 2. Let $\tilde{X}$ be the 6-fold cover of $S^1 \vee S^1$ drawn below.



(a) Use Proposition 1.32 and Proposition 1.39 to deduce the size of the group $G(\tilde{X})$ of deck transformations. Use the symmetries of $\tilde{X}$ to identify the group $G(\tilde{X})$.

(b) Alter $\tilde{X}$ by reversing the direction of the three α arrows on the inner circle only. What is the size of the group $G(\tilde{X})$ of deck transformations? What is the group $G(\tilde{X})$?

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Proof. By proposition 1.32 we have that the index of $p_*(\pi_1(\tilde{X}, \tilde{x_0})$ in $\pi_1(X, x_0)$ is 6. Note that the only two groups of order 6 up to isomorphism are $D_3 \cong S_3$ and $\mathbb{Z}_6$. Let's take a look at our 6 fold cover:

We see that this has the presentation $H \cong \langle \alpha^3, \beta^2, (\alpha\beta)^2, \beta\alpha^3\beta, \alpha^2\beta^2\alpha, \alpha\beta^2\alpha^2, \alpha\beta^2\alpha^2, (\beta\alpha)^2 \rangle$. It's not hard to see that $N(H) \cong \pi_1(X, x_0)$ since we have $\alpha H \alpha^{-1} = H$ and $\beta H \beta^{-1}$ by searching a bit for elements of H that commute with α and β. We then have

$$\begin{aligned} G(\tilde{X}) \cong N(H)/H &\cong \langle \alpha, \beta \rangle / H \\ &\cong \langle \alpha, \beta \mid \alpha^3, \beta^2, \alpha\beta\alpha\beta \rangle \\ &\cong D_3 \cong S_3. \end{aligned}$$

Next, when we switch the arrows we get the following 6 fold cover:

We see that this has the presentation $H \cong \langle \alpha^3, \beta^2, \alpha\beta\alpha^{-1}\beta^{-1}, \beta\alpha^3\beta, alpha^2\beta^2\alpha, \alpha\beta^2\alpha^2, \beta\alpha\beta\alpha^2 \rangle$. Again, we find that $N(H) \cong \pi_1(X, x_0)$. With that, we get

$$\begin{aligned} G(\tilde{X}) \cong N(H)/H &\cong \langle \alpha, \beta \rangle / H \\ &\cong \alpha^3, \beta^2, \alpha\beta\alpha^{-1}\beta^{-1} \rangle \\ &\cong \mathbb{Z}_6. \end{aligned} \qquad \square$$