Geometric Calculus and a Noncommutative Gelfand Representation

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Outline

- Introduce geometric algebra and calculus.
- Describe the toolbox in comparison to differential forms.
- Prove a multivector version of the Hodge-Morrey decomposition.
- Prove a noncommutative version of the Gelfand representation.

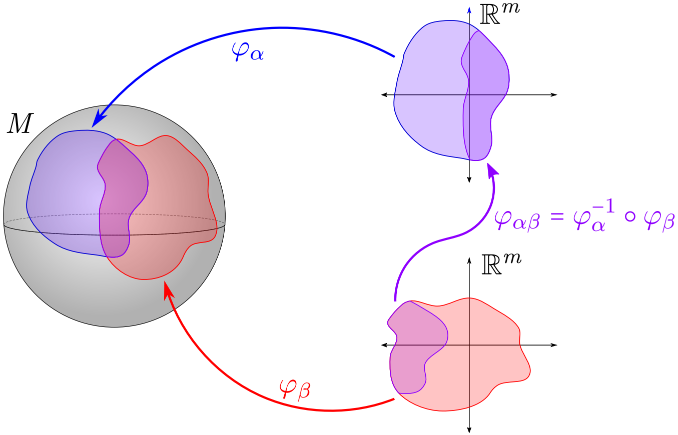
Section 1

Introduction

- ***Geometric algebra*** originated in 1878 with William Kingdon Clifford's work that extends Hermann Grassmann's ***exterior algebra***.
- ***Geometric calculus*** arrived in 1984 due to David Hestenes and Garrett Sobczyk in order to enrich Éllie Cartan's ***differential forms***.

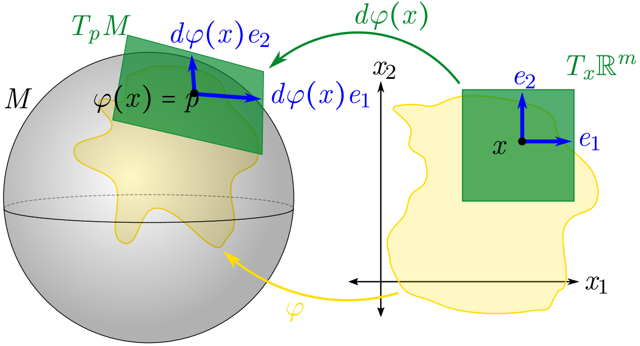
The playing field

We let M be a smooth, compact, connected, and oriented n-dimensional Riemannian manifold with metric g (unless otherwise stated).



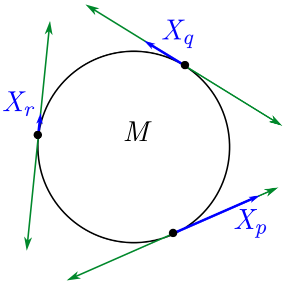
The playing field

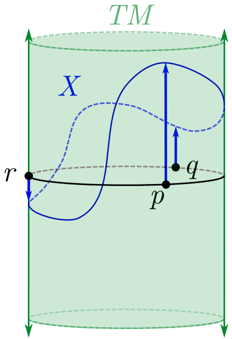
At each point on M, we have the tangent space T_pM.



The playing field

From M, we create the tangent bundle TM whose sections are vector fields.





<u>Idea:</u> On each tangent space, let us construct a manner in which to multiply vectors.

- Take the tensor algebra

$$\mathcal{T}(T_pM) \coloneqq \bigoplus_{j=0}^{\infty} T_pM^{\otimes j} = \mathbb{R} \oplus (T_pM \otimes T_pM) \oplus (T_pM \otimes T_pM \otimes T_pM) \oplus \cdots.$$

- Form the quotient algebra

$$\mathcal{G} = \mathcal{T}(T_pM)/\langle \boldsymbol{v}$$

Section 2

The Calderón Problem on Riemannian Manifolds

Subsection 1

Preliminaries

Geometry

- ***Smooth n-dimensional manifold***: A space that locally looks like (is C^∞ diffeomorphic to) an open subset of $\mathbb{R}^n$.
- ***Riemannian metric***: A smoothly varying inner product defined on Ω. In coordinates, g takes the form of a symmetric and positive definite matrix with entries g_{jk} with inverse g^{jk}.
- ***Exterior algebra***: Differential forms with the wedge product $\wedge$.
- ***Hodge Star***: Attached to the exterior algebra when we also have a Riemannian metric. Gives an isomorphism between k and $n-k$-forms.