MATH 517, Homework 1

Colin Roberts

September 1, 2017

Solutions

Problem 1. Let E be a nonempty subset of an ordered set. Assume α is a lower bound for E and β is an upper bound. Show that $\alpha \leq \beta$.

:

Proof. Since α is a lower bound, necessarily $\alpha \leq x \ \forall x \in E$ and similarly $\beta \geq x \ \forall x \in E$ since E is nonempty and ordered. Thus we have

$$\begin{aligned} &\alpha \leq x \leq \beta \\ \implies &\alpha \leq \beta \end{aligned}$$

∎

Problem 2. If $x \in \mathbb{C}$, show that there exists $r \geq 0$ and $w \in \mathbb{C}$ with $\|w\| = 1$ so that $z = rw$. Are r and w uniquely determined by z.

:

Proof. Let $r = |z|$ and then we have that $w = \frac{z}{r} = \frac{z}{|z|}$ so that $|w| = 1$. Then we have $wr = \frac{z}{r}r = z$. But note that z does not uniquely define r and w since $|-z| = |z|$. $\square$

Problem 3. Let $x, y \in \mathbb{R}^n$. Show that

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

What does this mean geometrically as a statement about parallelograms?

:

Solution.

$$\begin{aligned} |x+y|^2 + |x-y|^2 &= (x+y)\cdot(x+y) + (x-y)\cdot(x-y) \\ &= x\cdot x + 2x\cdot y + y\cdot y + x\cdot x - 2x\cdot y + y\cdot y \\ &= 2x\cdot x + 2y\cdot y \\ &= 2|x|^2 + 2|y|^2 \end{aligned}$$

This statement relates the length of the diagonals of the parallelograms to the side lengths. ■

Problem 4. Let X be an infinite set with the trivial metric.

(a) Prove that d is a metric X.

(b) What are the open sets of X?

(c) What are the closed sets?

(d) What are the compact sets?

:

Proof (a). First note that $(p,q)=0$ if and only if $p=q$ for $p,q\in X$. With $p\neq q\in X$ we have $(p,q)=1>0$. Second, if $p=q$ then $(p,q)=0=(q,p)$. With $p\neq q$, $(p,q)=1$ and by the trivial metric $(q,p)=1$ since $p\neq q$. Finally let $p=q$ then for $p,q\in X$ we have $(p,q)=0\leq(p,r)+(r,q)\leq 2$ with $0\leq(p,r)+d(r,q)$ if $r=q=p$ else $(p,r)+(r,q)=2$ if $r\neq q=p$. If $p\neq q$ and $p,q,r\in X$ then $(p,q)=1\leq(p,r)+(r,q)\leq 2$ with $1=(p,r)+(r,q)$ if $r=q$ or $r=p$ and $(p,r)+(r,q)=2$ if $r\neq q$ and $r\neq p$. □

:

Solution (b).

The open sets of X are any subset. We can separate any singleton from the others with an open ball of radius $r<1$ due to the trivial metric and we can make any subset from a union of the singletons. ■

:

Solution (c). The closed sets are all sets by the properties of compliments of open sets. ■

:

Solution (d). The compact sets are any finite set. Since any finite set will have a finite open subcover and an infinite set won't if we choose the open sets that form an open cover to be singletons. ■

Problem 5. Consider $\mathbb{Q}$ as a metric space with $(x, y) = |x - y|$. Let

$$E = \{x \in \mathbb{Q} | 2 < x^2 < 3\}$$

Show that E is both open and closed in $\mathbb{Q}$.

:

Proof (E is open). Let $x \in E$ so that $2.5 \leq x^2 < 3$ and fix $r > 0$. Then let $\frac{\delta}{2} = x^2 + 3$ with $\delta \in \mathbb{Q}$ and $\delta < r$. Then $|x^2 - 3| = |3 - \frac{\delta}{2} - 3| = \frac{\delta}{2} < \frac{r}{2} < r$. Then note that $N_{3-\frac{\delta}{2}}(x) \subseteq E$ and is open in E. Again let $x \in E$ so that $2 < x^2 \leq 2.5 \in E$ and fix $r > 0$. Then let $\frac{\delta}{2} = x^2 + 2$ with $\delta \in \mathbb{Q}$ and $\delta < r$. Then $|x^2 - 2| = |2 - \frac{\delta}{2} - 2| = \frac{\delta}{2} < \frac{r}{2} < r$. Then note that $N_{2-\frac{\delta}{2}}(X) \subseteq E$ and is open in E. So every point in E has an open neighborhood about that point contained in E. □

:

Proof (E is closed). To show E is closed, suppose there exists a limit point x of E such that $x^2 < 2$ or $x^2 > 3$. Then $\forall r > 0$ we have that $N_r(x^2) \cap E \neq \emptyset$. Suppose $x^2 > 3$ then let $|x^2 - 3| = \delta$, then let $r < \frac{\delta}{2}$ and note that $N_r(x^2) \cap E = \emptyset$. Finally, suppose that $x^2 < 2$ then let $|x^2 - 2| = \delta$, then let $r < \frac{\delta}{2}$ and note that $N_r(x^2) \cap E = \emptyset$. □