MATH 272, Homework 5

Due March 15th

Problem 1. Let $\vec{\boldsymbol{V}}$ be a vector field in the plane $\mathbb{R}^2$ defined by

$$\vec{\boldsymbol{V}}(x, y) = \begin{pmatrix} \frac{1}{2}x - y \\ x + \frac{1}{2}y \end{pmatrix},$$

and let $\vec{\boldsymbol{x}}(t) = \begin{pmatrix} e^{\frac{1}{2}t}(-c_1 \sin(t) + c_2 \cos(t)) \\ e^{\frac{1}{2}t}(c_1 \cos(t) + c_2 \sin(t)) \end{pmatrix}$ for $t \in [0, \pi]$ where c_1 and c_2 are yet undetermined constants.

(a) Show that a flow of $\vec{\boldsymbol{V}}$ yields a linear system of equations.

(b) Show that $\vec{\boldsymbol{x}}(t)$ is a flow of the vector field $\vec{\boldsymbol{V}}$.

(c) Let $\vec{\boldsymbol{x}}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Determine the particular solution to the initial value problem.

(d) [**MATLAB**] Plot the $\vec{\boldsymbol{V}}$ and your particular solution $\vec{\boldsymbol{x}}$ simultaneously by modifying

```
vector_field_2d.m
```

and

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curve.m
```

Then enter the following into the command window

```
vector_field_2d
```

followed by

```
curve
```

and finally to get the correct view enter

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view(0,90)
```

Choose good bounds for your plot so that the whole curve is visible.

Problem 2. Let us consider the discrete heat equation for n equally spaced particles on a line segment for which we have the following picture

Let $u_j(t) := u(x_j, t)$ denote the temperature of particle j at time t, let k_j be the thermal transport coefficient between particles j and $j+1$, and let $f_j(t) = f(x_j, t)$ be the thermal energy source on particle j.

(a) For the boundary particles x_1 and x_n, we have

$$\dot{u}_1 = -k_1 u_1 + k_1 u_2 + f_1 \qquad \text{and} \qquad \dot{u}_n = -k_n u_n + k_1 u_{n-1} + f_n,$$

which correspond to *Neumann type boundary conditions.* Explain each term in the above equations.

(b) If we attached x_1 to x_n with a material with a thermal transport coefficent of k_0 the above equations would need modification. Write these new equations. These are the *periodic boundary conditions.*

(c) Explain why periodic boundary conditions are the same as working with a circular domain.

(d) If we force u_1 and u_n to be constant, what will the equations for the boundary particles be? These would be the *Dirichlet type boundary conditions.*

(e) For the interior particles, we have the relationship

$$\dot{u}_j = -k_{j-1} u_j - k_j u_j + k_{j-1} u_{j-1} + k_j u_{j+1} + f_j \qquad \text{for } j = 2, \dots, n-1.$$

Explain what each term describes in the above equation.

(f) In the limit as $n \to \infty$, we then have that k is described as a function of position, x. The source free heat equation then reads

$$\frac{\partial}{\partial t} u(x,t) = \frac{\partial}{\partial x} \left(k(x) \frac{\partial}{\partial x} u(x,t) \right) + f(x,t).$$

Explain how this equation differs from the equation

$$\frac{\partial}{\partial t} u(x,t) = k(x) \frac{\partial^2}{\partial x^2} u(x,t) + f(x,t).$$

Problem 3. Consider the 1-dimensional homogeneous Laplace equation given by

$$\frac{\partial^2}{\partial x^2} u_E(x) = 0,$$

with the domain Ω as the unit interval on the x-axis. Take the Dirichlet boundary conditions $u_E(0) = T_0$ and $u_E(L) = T_L$. Think of these values as the ambient temperature at the endpoints of the rod. These temperatures are constant since the ambient environment is so large.

(a) Find the particular solution to this Laplace equation.

(b) Suppose that $v(x,t)$ is a solution to the 1-dimensional source free isotropic heat equation with zero Dirichlet boundary values. Show that

$$u(x,t) = v(x,t) + u_E(x),$$

is a solution to the 1-dimensional source free isotropic heat equation with Dirichlet boundary values $u(0,t) = T_0$ and $u(L,t) = T_L$.

(c) From Problem 1, we know that $\lim_{t\to\infty} v(x,t) = 0$. Hence, show that the long time limit of a solution to the source free heat equation yields a solution to the Laplace equation.

(d) Argue why the equilibrium temperature profile of a rod can be found without solving the heat equation.

Problem 4. Using intuition from the previous problem, explain how one could solve the heat equation with a nonzero source term that only depends on x. In other words, how could one try to solve

$$\left(-k\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\right) u(x,t) = f(x),$$

Problem 5. Consider the 2-dimensional source free isotropic heat equation given by

$$\left(-k\Delta + \frac{\partial}{\partial t}\right) u(x,y,t) = 0,$$

with the domain Ω as the unit square in the xy-plane. Take as well the Dirichlet boundary conditions $u(x,y,t) = 0$ for x and y on the boundary of Ω.

(a) Show that $u_{mn}(x,y,t) = \sin(m\pi x)\sin(n\pi y)e^{-k(n^2+m^2)\pi^2 t}$ is a solution to the PDE and Dirichlet boundary conditions for any non-negative integers m and n.

(b) Show that a linear combination of solutions u_{mn} and u_{pq} is also a solution.

(c) For $m = n = 1$ and $k = 1$, plot the solution for the values $t = 0$, $t = 0.01$, $t = 0.1$ and $t = 1$. Explain what is physically happening as time moves forward.

(d) Explain what varying the value for the conductivity k does to the solution. Feel free to use plots to support your hypothesis.

(e) Explain the mathematical reason why increasing m and n causes the solution to converge to zero more quickly.

(f) Explain the physical reason why increasing m and n causes the solution to converge to zero more quickly. Plots may help support your reasoning.