MATH 571, Homework 6

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Solutions

Problem 1. Hatcher Exercise 17 on page 80: Given a group G and a normal subgroup N, show that there exists a normal covering space $\tilde{X} \to X$ with $\pi_1(X) \cong G$, $\pi_1(\tilde{X}) \cong N$, and deck transformation group $G(\tilde{X}) \cong G/N$.

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Proof. Note that we can construct a space X as a CW-complex so that $\pi_1(X) \cong G$. We construct X with a single 0-cell, a 1-cell for each generator with endpoints identified with the 0-cell, and finally by gluing in a 2-cell along the proper 1-cells for each relation in the group presentation. Then the classification theorem states that there exists a covering space $\tilde{X}$ for each subgroup of $\pi_1(X)$, and hence we specifically have an $\tilde{X}$ so that $\pi_1(\tilde{X}) \cong N \subseteq G$. We also know that $\tilde{X}$ is a normal covering space since $\pi_1(\tilde{X})$ is a normal subgroup of $\pi_1(X)$. This then yields the desired deck transformation group $G(\tilde{X}) \cong G/N \cong \pi_1(X)/\pi_1(\tilde{X})$ since $\pi_1(\tilde{X})$ is normal. $\square$

Problem 2. Hatcher Exercise 4 on page 131: Compute the 0-, 1-, 2-, and 3-dimensional simplicial homology groups of the "triangular parachute" obtained from Δ^2 by identifying its three vertices to a single point.

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Proof. Denote this triangular parachute space as X and label our single vertex as v, the three edges as a, b and c, and the single face as f. See the drawing below.

Now we have that $\Delta_0(X) \cong \mathbb{Z}$, $\Delta_1(X) \cong \mathbb{Z}^2$, $\Delta_2(X) \cong \mathbb{Z}$, $\Delta_3(X) \cong 0$, and $\Delta_4(X) \cong 0$. We then have

$$\partial_0(\Delta_0(X)) \cong 0 \implies \ker \partial_0 \cong \mathbb{Z} \quad \mathrm{im}\partial_0 \cong 0.$$

Now we also get

$$\partial_1(\Delta_1(X)) \cong 0,$$

since each edge is mapped to the identity since, for example, $a \mapsto v - v = 0$. This then implies

$$\ker \partial_1 \cong \mathbb{Z}^3 \quad \mathrm{im}\partial_1 \cong 0.$$

Next, we have

$$\partial_2(\Delta_2(X)) = a - b + c,$$

which tells us specifically that

$$\ker \partial_2 \cong 0 \quad \mathrm{im}\partial_2 \cong \mathbb{Z}.$$

More easily, we finally see that

$$\begin{aligned} \ker \partial_3 \cong 0 \quad \mathrm{im}\partial_3 \cong 0 \;\; \text{and} \\ \ker \partial_4 \cong 0 \quad \mathrm{im}\partial_4 \cong 0. \end{aligned}$$

We then compute the homology groups per usual. So we have

$$\begin{aligned} H_0^\Delta(X) &\cong Z \\ H_1^\Delta(X) &\cong \mathbb{Z}^2 \\ H_2^\Delta(X) &\cong 0 \\ H_3^\Delta(X) &\cong 0. \end{aligned}$$

This shows we have a single connected components, two 1-dimensional holes, and no 2- (or higher) dimensional holes. □

Problem 3. Hatcher Exercise 5 on page 131: Compute the 0-, 1-, and 2-dimensional simplicial homology groups of the Klein bottle using the Δ-complex structure on page 102.

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Proof. We first write down the chain groups:

$$\begin{aligned} \Delta_0(X) &\cong \mathbb{Z} \\ \Delta_1(X) &\cong \mathbb{Z}^3 \\ \Delta_2(X) &\cong \mathbb{Z}^2 \\ \Delta_3(X) &\cong 0. \end{aligned}$$

Investigating each boundary map, we find for ∂_0 that

$$v \mapsto 0.$$

For ∂_1,

$$\begin{aligned} a &\mapsto v - v = 0 \\ b &\mapsto v - v = 0 \\ c &\mapsto v - v = 0. \end{aligned}$$

For ∂_2,

$$\begin{aligned} U &\mapsto b - c + a \\ L &\mapsto a - b + c. \end{aligned}$$

Finally, for ∂_3, there are no 3-simplicies so $\text{im}\partial_3 \cong \ker \partial_3 \cong 0$. We then compute homology,

$$H_0^{\Delta}(X) \cong \mathbb{Z},$$
$$H_1^{\Delta}(X) \cong Z \oplus \mathbb{Z}/2\mathbb{Z},$$

since for $\text{im}\partial_1$ we have a basis $\{a, b, c\}$ and for $\ker \partial_2$ we have a basis $\{b - c + a, a - b + c\}$ which can be rewritten as $\{b - c + a, 2c\}$. When we take $H_1^{\Delta}(X) \cong \text{im}\partial_1 / \ker \partial_2$ we mod out a copy of $\mathbb{Z}$ for the element $b - c + a$ and we get a single $\mathbb{Z}/2\mathbb{Z}$ term since $2c$ is an identity in $H_1^{\Delta}(X)$. The other copy of $\mathbb{Z}$ left was from $\ker \partial_1$. Finally

$$H_2^{\Delta}(X) \cong 0. \qquad \square$$