MATH 517, Homework 12

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Solutions

Problem 1. Give an example of an equicontinuous sequence of functions that converges pointwise but not uniformly.

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Proof. Consider the sequence of functions $f_n = \frac{x}{n}$ defined on all of $\mathbb{R}$. Then note that this is an equicontinuous sequence of functions. To see this, fix $\epsilon > 0$ and let $0 < \delta < \epsilon$. Then we have for any n and $|x - y| < \delta$

$$\begin{aligned} |f_n(x) - f_n(y)| &= \left| \frac{x}{n} - \frac{y}{n} \right| \\ &= \left| \frac{x - y}{n} \right| \\ &< |x - y| < \epsilon. \end{aligned}$$

Thus we have that this sequence is in fact equicontinuous.

To see that f_n converges pointwise, fix x and $\epsilon > 0$ then let $N \in \mathbb{N}$ be such that $N \geq \frac{|x|}{\epsilon}$. Then for $n > N$ we have

$$\begin{aligned} |f_n(x) - 0| &= \left| \frac{x}{n} \right| \\ &< \epsilon. \end{aligned}$$

Now, for a contradiction, suppose that f_n does in fact converge uniformly. Then $\exists N \in \mathbb{N}$ such that for $n > N$ we have

$$|f_n(x) - 0| < 1,$$

for all $x \in \mathbb{R}$. However, $f_n(n) = 1$ and we have a contradiction. Thus, f_n does not converge uniformly. We then have that f_n is an equicontinuous sequence of functions converging pointwise but not uniformly. $\square$

Problem 2. (Rudin 7.15) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and we define $f_n(t) = f(nt)$ for each $n = 1, 2, \dots$. If $\{f_n\}$ is equicontinuous on $[0, 1]$, what can you conclude about f?

We claim that f is a constant function on $[0, \infty)$. :

Proof. Suppose that f is not a constant function. So we have that for some $x, y \in [0, \infty)$ that $f(x) \neq f(y)$. Let us then say that $|f(x) - f(y)| = \epsilon$. Now, notice that we have for sufficiently large n that $x/n, y/n \in [0, 1]$.

$$|f(x) - f(y)| = |f_n(x/n) - f_n(y/n)| = \epsilon.$$ □

So now notice that $|x/n - y/n| \to 0$ as $n \to \infty$, and this means that $\{f_n\}$ was not equicontinuous on $[0, 1]$. This contradiction shows that f is in fact constant on $[0, \infty)$.

Problem 3. (Rudin 7.16) Suppose $\{f_n\}$ is an equicontinuous sequence of functions on a compact set K. Show that $\{f_n\}$ converges pointwise on K if and only if it converges uniformly on K.

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Proof. For the forward direction, we have that $\{f_n\}$ converges to f pointwise. Fix $\epsilon > 0$ and fix a $\delta > 0$ so that we have $|f_n(p) - f_n(q)| < \epsilon/3$ for $|p - q| < \delta$, which we can do by equicontinuity. Note that by compactness of K, we have that for finitely many x_i for $i = 1, ..., M$ that $\cup_{i=1}^{M} N_\delta(x_i) \supseteq K$. Then by pointwise convergence, we can also find an $N \in \mathbb{N}$ sufficiently large so that for some $x_i \in K$ and $n, m > N$ we have $|f_n(x_i) - f_m(x_i)| < \epsilon/3$. Then for an arbitrary $x \in K$ we have that $x \in N_\delta(x_i)$ for some i. It then follows that

$$\begin{aligned}|f_n(x) - f_m(x)| &\leq |f_m(x) - f_m(x_i)| + |f_m(x_i) - f_n(x_i)| + |f_n(x_i) - f_n(x)| \\ &\leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.\end{aligned}$$

For the converse, this is trivial, as uniform continuity is strictly stronger than pointwise. $\square$