MATH 570, Homework 8

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Solutions

Problem 1. The two parts of this problem are unrelated.

(a) Prove that the circle $S^1 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\}$ is not a retract of the closed disk $\overline{B^2} = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$.

(b) Suppose $f\colon S^1 \to S^1$ is a map which is not homotopic to the identity map on S^1. Prove that $f(x) = -x$ for some point $x \in S^1$.

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Proof (a). Suppose that S^1 is a retract of $\overline{B^2}$. Then $\iota_{S^1}\colon S^1 \to \overline{B^2}$ is the inclusion map which induces an injection on the fundamental groups given by $(\iota_{S^1})_*\colon \pi_1(S^1) \hookrightarrow \pi_1(\overline{B^2})$. But note that $\pi_1(S^1) = \mathbb{Z}$ and $\pi_1(\overline{B^2})$ is the trivial group. This is a contradiction since there is no injection from $\mathbb{Z}$ to the trivial group. $\square$

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Proof (b). Suppose that f is not homotopic to Id. Then, for a contradiction, suppose that $f(x) \neq -x$ for any point x. From the last homework, we know that if for any x, $f(x) \neq -x = -\mathrm{Id}(x)$, then $f(x) \simeq \mathrm{Id}$. This contradicts our supposition that $f(x) \neq -x$ at some point since otherwise f would be homotopic to Id. Thus $f(x) = -x$ for some point. $\square$

Problem 2. Let S^1 be the unit circle and let $C = S^1 \times [-1, 1]$ be a cylinder. Prove that $S^1 \simeq C$.

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Proof. Consider the following maps $f \colon S^1 \times [-1, 1] \to S^1 \times \{0\}$ and and $g \colon S^1 \times \{0\} \hookrightarrow S^1 \times [-1, 1]$ with $f(\theta, x) \mapsto (\theta, 0)$ and $g(\theta, 0) \mapsto (\theta, 0)$. Note that $S^1 \times \{0\} \cong S^1$ and we have $f \circ g = \mathrm{Id}_{S^1}$. Then consider $H \colon I \times [-1, 1] \to [-1, 1]$ defined by

$$H(t, x) = (1 - t)x.$$

So we have H continuous and $H(0, x) = Id_{[-1,1]}(x)$ and $H(1, x) = 0$. Then $\mathrm{Id}_{S^1} \times H$ is continuous and provides a homotopy equivalence between $g \circ f$ and $Id_{S^1 \times [-1,1]}$. So we have $g \circ f \simeq Id_{S^1 \times [-1,1]}$. Hence, $C \simeq S^1$. $\square$

Problem 3. A topological space X is *contractible* if $\mathrm{Id}_X : X \to X$ is homotopic to a constant map.

(a) Prove that X is contractible if and only if X is homotopy equivalent to a one-point space.

(b) Let X and Y be topological spaces. Prove that if either X or Y is contractible, then every continuous map from X to Y is homotopic to a constant map.

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Proof (a). For the forward direction, suppose that X is contractible. Thus Id_X is homotopic via $H(t,x)$ to a constant map. Then let $f\colon X \to \{p\}$ be defined by $f(x) = p$ and let $g\colon \{p\} \to X$ be defined by $g(p) = q$ for some specific $q \in X$. and note that $f \circ g = \mathrm{Id}_{\{p\}}$. Consider then $g \circ f$ and note $g \circ f \simeq \mathrm{Id}_X$ by the homotopy H given by the fact X is contractible. Thus $X \simeq \{p\}$, with $\{p\}$ a one point space.

For the converse, suppose that $X \simeq \{p\}$ with $\{p\}$ a one point space. Then there exists $f\colon X \to \{p\}$ and $g\colon \{p\} \to X$ with $f \circ g \simeq \mathrm{Id}_{\{p\}}$ and $g \circ f \simeq \mathrm{Id}_X$. Note that $g \circ f(x) = q$ for all $x \in X$ and some $q \in X$. This then implies that X is contractible since $g \circ f$ is a constant map that is homotopic to the identity map on X, Id_X. $\square$

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Proof (b). Without loss of generality, let Y be contractible. Thus Id_Y is homotopic to a constant map C. Let $f\colon X \to Y$ be a continuous map. Then notice that $f = \mathrm{Id}_Y \circ f \simeq C \circ f = C$. Thus we have f is homotopic to a constant map. $\square$

Problem 4. The fundamental theorem of algebra states that every non-constant single-variable polynomial with complex coefficients has at least one complex root. Give a proof of the fundamental theorem of algebra *using facts related to the fundamental group of the circle* (there are many other different proofs).

Let $f(x) = x^n + c_1 x^{n-1} + \cdots + c_{n-1}x + c_n$ be a polynomial with $n > 0$ and each $c_i \in \mathbb{C}$. We want to show that there are n complex numbers, including multiplicities, x_i such that $f(x_i) = 0$.

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Proof. We may assume without loss of generality that $p(z)$ is monic. So let

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + z^n.$$

Supposing $p(z)$ has no roots in $\mathbb{C}$, we will show p is constant. First, consider for a fixed $r \in \mathbb{C}$ the loop

$$f_r(s) = \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}$$

Indeed, by assumption the denominators are never zero, so this function is continuous for all $s \in [0,1]$. Further, each value $f_r(s)$ is on the unit circle. Finally, $f_r(0) = (p(r)/p(r))/|p(r)/p(r)| = 1$, and $f_r(1)$ yields the same value, so this is a closed path based at 1.

We note this function is continuous in both s and r (indeed, they are simply rational functions defined for all s, r), so that $f_r(s)$ is a homotopy of loops as r varies. If $r = 0$, then the function is constant for all s, and so for any fixed r, the loop $f_r(s)$ is homotopic to the constant loop.

Now fix a value of r which is larger than both $|a_0| + \cdots + |a_{n-1}|$ and 1. For $|z| = r$, we have

$$|z^n| = r \cdot r^{n-1} > (|a_0| + \cdots + |a_{n-1}|)|z^{n-1}|$$

And hence $|z^n| > |a_0 + a_1 z + \cdots + a_{n-1}z^{n-1}|$. It follows that the polynomial $p_t(z) = z^n + t(a_{n-1}z^{n-1} + \cdots + a_0)$ has no roots when both $|z| = r$ and $0 \le t \le 1$. Fixing this r, and replacing p with $p_t(z)$ in the formula for $f_r(s)$, we have a homotopy from $f_r(s)$ (when $t = 1$, nothing is changed) to the loop which winds around the unit circle n times, where n is the degree of the polynomial. Indeed, plug in $t = 0$ to get $f_r(s) = (r^n e^{2\pi i n s}/r^n)/|r^n e^{2\pi i n s}/r^n|$, which is the loop $\omega_n(s) = e^{2\pi i n s}$.

In other words, we have shown that the homotopy classes of f_r and ω_n are equal, but f_r is homotopic to the constant map. Translating this into fundamental groups, as $\pi_1(S^1, 1) = \mathbb{Z}$, we note that $[\omega_n] = [f_r] = 0$, but if $\omega_n = 0$ then it must be the case that $n = 0$, as $\mathbb{Z}$ is the free group generated by ω_1. Hence, the degree of p to begin with must have been 0, and so p must be constant. $\square$