ColoState Spring 2018 MATH 617 Assignment 1

Due Wed. 01/31/2018

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(15 points) *Problem 1.* Let X, Y be fixed nonempty sets, f be a mapping from X to Y, $A, A_n (n \in \mathbb{N})$ subsets of X, and $B, B_n (n \in \mathbb{N})$ subsets of Y.

(i) Prove that $f(\bigcap_{n\in\mathbb{N}} A_n) \subseteq \bigcap_{n\in\mathbb{N}} f(A_n)$.

(ii) Give an example for (i) such that it is indeed a proper subset.

(iii) Prove that $f^{-1}(\bigcap_{n\in\mathbb{N}} B_n) = \bigcap_{n\in\mathbb{N}} f^{-1}(B_n)$.

(iv) Give an example such that $A \subsetneq f^{-1}(f(A))$.

(15 points) *Problem 2.* Study set operations

(i) Let $A_n = \left[-1+\frac{1}{n}, 1-\frac{1}{n}\right]$ for $n \in \mathbb{N}$. Find $\bigcup_{n\in\mathbb{N}} A_n$ and $\bigcap_{n\in\mathbb{N}} A_n$.

(ii) Let $f(x)$ be a real-valued function defined on a subset E of $\mathbb{R}$. Prove that

$$\{x \in E : f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \left\{x \in E : f(x) \geq \alpha + \frac{1}{n}\right\}$$

holds for any $\alpha \in \mathbb{R}$.

(20 points) *Problem 3.* Give an example of an uncountable null set.

(15 points) *Problem 4.* Prove that a countable union of null sets is still a null set.

(15 points) *Problem 5.* Let f be a nonnegative continuous function defined on $[a, b]$. Prove that if the Riemann integral $\int_a^b f(x)dx = 0$, then $f(x) \equiv 0$.

(20 points) *Problem 6.* Prove that any bounded monotone function on a closed finite interval is Riemann integrable.

Problem 1. Let X, Y be fixed nonempty sets, f be a mapping from X to Y, $A, A_n (n \in \mathbb{N})$ subsets of X, and $B, B_n (n \in \mathbb{N})$ subsets of Y.

(i) Prove that $f(\bigcap_{n\in\mathbb{N}} A_n) \subseteq \bigcap_{n\in\mathbb{N}} f(A_n)$.

(ii) Give an example for (i) such that it is indeed a proper subset.

(iii) Prove that $f^{-1}(\bigcap_{n\in\mathbb{N}} B_n) = \bigcap_{n\in\mathbb{N}} f^{-1}(B_n)$.

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Proof.

(i) Let $y \in f\left(\bigcap_{n\in\mathbb{N}} A_n\right)$. Thus there exists (possibly many) $x \in \bigcap_{n\in\mathbb{N}} A_n$ so that $f(x) = y$. By definition of the intersection, this implies that $x \in A_n$ for every $n \in \mathbb{N}$ and hence $y \in f(A_n)$ for every n. Thus $y \in \bigcap_{n\in\mathbb{N}} f(A_n)$ and we have $f\left(\bigcap_{n\in\mathbb{N}} A_n\right) \subseteq \bigcap_{n\in\mathbb{N}} f(A_n)$.

(ii) For all $n \in \mathbb{N}$ let $A_n = \left(\frac{-1}{n}, \frac{1}{n}\right)$ so that $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$. Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = 1$$

Then we have that $f\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \emptyset$ and $\bigcap_{n\in\mathbb{N}} f(A_n) = \{1\}$ and hence $f\left(\bigcap_{n\in\mathbb{N}} A_n\right) \not\supseteq \bigcap_{n\in\mathbb{N}} f(A_n)$.

(iii) Let $x \in f^{-1}\left(\bigcap_{n\in\mathbb{N}} B_n\right)$. Then there exists $y \in B_n$ for all $n \in \mathbb{N}$ so that $f^{-1}(y) = x$ and hence $x \in f^{-1}(B_n)$ for every n. Thus we have $x \in \bigcap_{n\in\mathbb{N}} f^{-1}(B_n)$ showing that $f^{-1}\left(\bigcap_{n\in\mathbb{N}} B_n\right) \subseteq \bigcap_{n\in\mathbb{N}} f^{-1}(B_n)$. For the other containment we let $x \in \bigcap_{n\in\mathbb{N}} f^{-1}(B_n)$. This means that we have $y \in B_n$ satisfying $f^{-1}(y) = x$ for every $n \in \mathbb{N}$. Hence, $y \in \bigcap_{n\in\mathbb{N}} B_n$ and we have that $x \in f^{-1}\left(\bigcap_{n\in\mathbb{N}} B_n\right)$. Thus $f^{-1}\left(\bigcap_{n\in\mathbb{N}} B_n\right) \subseteq \bigcap_{n\in\mathbb{N}} f^{-1}(B_n)$. Both containments then show $f^{-1}(\bigcap_{n\in\mathbb{N}} B_n) = \bigcap_{n\in\mathbb{N}} f^{-1}(B_n)$.

(iv) Let $f\colon \mathbb{R} \to \mathbb{R}$ and let $A = [0, 1]$. Note that $f(A) = [0, 1]$ and $f^{-1}(f(A)) = f^{-1}([0, 1]) = [-1, 1]$. Thus $A \subsetneq f^{-1}(f(A))$. $\square$

Problem 2. Study set operations

(i) Let $A_n = \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right]$ for $n \in \mathbb{N}$. Find $\bigcup\limits_{n\in\mathbb{N}} A_n$ and $\bigcap\limits_{n\in\mathbb{N}} A_n$.

(ii) Let $f(x)$ be a real-valued function defined on a subset E of $\mathbb{R}$. Prove that

$$\{x \in E : f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \left\{x \in E : f(x) \geq \alpha + \frac{1}{n}\right\}$$

holds for any $\alpha \in \mathbb{R}$.

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Proof.

(i) First we consider $\bigcup_{n\in\mathbb{N}} A_n$. Let $x \in (-1, 1)$. Then let $\delta = \min(|x-1|, |x+1|)$ and we have that there $\exists N \in \mathbb{N}$ so that $1/N < \delta$ and hence $x \in \bigcup_{n=1}^{N} A_n = \left[-1 + \frac{1}{N}, 1 - \frac{1}{N}\right]$. Next, suppose x with $|x| \geq 1$ is in $\bigcup_{n\in\mathbb{N}} A_n$. Then $\exists N \in \mathbb{N}$ so that $x \in \left[-1 + \frac{1}{N}, 1 - \frac{1}{N}\right]$. But, for any N, we have that $\left|-1 + \frac{1}{N}\right| \leq 1$ and $\left|1 - \frac{1}{N}\right| \leq 1$ which means that $x \notin \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right]$ for any n and hence $\bigcup_{n\in\mathbb{N}} A_n = (-1, 1)$.

Next, note that $0 \in A_1$ and since $A_n \supset A_{n+1}$ for every $n \in \mathbb{N}$, we have that $0 \in \bigcap_{n\in\mathbb{N}} A_n$. Suppose some $x \neq 0$ is in $\bigcap_{n\in\mathbb{N}} A_n$, then $x \notin A_1$ since $A_1 = \{0\}$ and hence $\bigcap_{n\in\mathbb{N}} A_n = \{0\}$.

(ii) If we let $p \in \{x \in E\colon f(x) > \alpha\}$ then we have that $f(p) - \alpha > 0$. Hence by the archimedean property, $\exists N \in \mathbb{N}$ so that $f(p) - \alpha > \frac{1}{N}$. Thus we have that

$$\{x \in E\colon f(x) > \alpha\} \subseteq \bigcup_{n=1}^{\infty} \left\{x \in E\colon f(x) \geq \alpha + \frac{1}{n}\right\}.$$

Next, let $p \in \bigcup_{n=1}^{\infty} \left\{x \in E\colon f(x) \geq \alpha + \frac{1}{n}\right\}$. Then we have that $f(p) - \alpha \geq \frac{1}{N}$ for $N \in \mathbb{N}$. This means that $f(p) > \alpha$ and hence

$$\{x \in E\colon f(x) > \alpha\} \supseteq \bigcup_{n=1}^{\infty} \left\{x \in E\colon f(x) \geq \alpha + \frac{1}{n}\right\}.$$

Finally, both containments show equality between sets. $\square$

Problem 3. Give an example of an uncountable null set.

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Proof. Consider the Cantor set K taken by removing the middle $1/3$ open interval from $[0, 1]$ as the first step, and repeating this process for the remaining closed intervals for each step. We then let the number of steps done, N, go to infinity to get the Cantor set K. We will show two major qualities: the Cantor set is uncountable and this set is also a null set.

To see that the Cantor set K is uncountable, note that we represent a point in the Cantor set by using a ternary representation. Put K_n as the Cantor set at the nth step. Then, let $x \in [0, 1]$, then the ternary representation is given by the sequence $\{a_n\}$ with each $a_n \in \{0, 1, 2\}$. The a_n are chosen by labeling the intervals at the nth step with the numbers 0 or 2 if $x \in K_n$ or 1 if $x \notin K_n$. If $x \notin K$, then at some finite $N \in \mathbb{N}$, a_N for the ternary representation is 1 and all subsequent a_n are 1 as well. However, if $x \in K$, then $x \in K_n$ for all $n \in N$, and hence the ternary representation of x is an infinite sequence of 0 and 2. It turns out that the set of every possible infinite sequence consisting of only 0 and 2 is exactly the power set of a countable set. Hence K is uncountable since there are as many members in x as there are the power set of some countable set, and the power set of a countable set is, by definition, uncountable.

Now, to see that K is a null set. Let $I_i^{(n)}$ be the ith remaining interval at step n. Note that there will be 2^n intervals $I_i^{(n)}$ at step n. Then $K = \cap_{n=1}^{\infty} \cup_{i=1}^{2^n} I_i^{(n)}$. Then, the length of all of these intervals at the Nth step is given by

$$\sum_{n=1}^{2^N} \left(\frac{1}{3}\right)^N = \left(\frac{2}{3}\right)^N.$$

Finally, for any $\epsilon > 0$ there exists an $N \in \mathbb{N}$ sufficiently large so that we have $(2/3)^N < \epsilon$. Hence, K is a null set since it is covered by open intervals with arbitrarily small length. $\square$

Problem 4. Prove that a countable union of null sets is still a null set.

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Proof. Let $\{A_n\}_{n\in\mathbb{N}}$ be a countable collection of null sets and fix $\epsilon > 0$. Then for each A_n we have a countable union of intervals $\{I_{n_i}\}_{i\in\mathbb{N}}$ such that $A_n \subseteq \bigcup_{i\in\mathbb{N}} I_{n_i}$ and that $\sum_{i=1}^{\infty} \lambda(I_{n_i}) < \frac{\epsilon}{2^i}$ by choosing I_{n_i} sufficiently small for each i. Then we have that

$$\sum_{n=1}^{\infty}\sum_{i=1}^{\infty} \lambda(I_{n_i}) = \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} \leq \epsilon,$$

which shows that $\{A_n\}_{n\in\mathbb{N}}$ is a null set. $\square$

Problem 5. Let f be a nonnegative continuous function defined on $[a, b]$. Prove that if the Riemann integral $\int_a^b f(x)dx = 0$, then $f(x) \equiv 0$.

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Proof. We will show the contrapositive. Suppose that $f(x) \neq 0$ for some x_0. Then since f is nonnegative, we have that $f(x_0) > 0$ and since f is continuous, there exists a $\delta > 0$ so that for $x \in (x_0 - \delta, x_0 + \delta) \subset [a, b]$ we have $f(x) > 0$. Then note that continuity of f implies that f is Riemann integrable and we have that $\int_{x_0-\delta}^{x_0+\delta} f(x)dx > 0$. It then follows that

$$0 < \int_{x_0-\delta}^{x_0+\delta} f(x)dx \leq \int_a^b f(x)dx.$$

Thus if $\int_a^b f(x)dx = 0$ we must have that $f(x) \equiv 0$. $\square$

Problem 6. Prove that any bounded monotone function on a closed finite interval is Riemann integrable.

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Proof. Let f be a bounded monotone function on the closed finite interval $[a, b]$. Without loss of generality, assume that f is monotone increasing so that $f(x) \geq f(y)$ when $x > y$. We will see where this extra assumption is used and explain why it's safe to do this. Let P_n be a regular partition of $[a, b]$ so that $x_i - x_{i-1} = (b-a)/n$. Then

$$U(P_n, f) = \sum_{k=1}^{n} f\left(a + \frac{k(b-a)}{n}\right)\left(\frac{b-a}{n}\right)$$
$$L(P_n, f) = \sum_{k=1}^{n} f\left(a + \frac{(k-1)(b-a)}{n}\right)\left(\frac{b-a}{n}\right).$$

Note, if f was monotone decreasing, we just switch the k for the $k-1$ in the two sums previously (this was the assumption made without losing generality). Now, we have that

$$\begin{aligned}\lim_{n\to\infty} U(P_n, f) - L(P_n, f) &= \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{\infty} f\left(a + \frac{k(b-a)}{n}\right) - \left(a + \frac{(k-1)(b-a)}{n}\right) \\ &= \lim_{n\to\infty} \frac{b-a}{n} f(b) - f(a) = 0.\end{aligned}$$

Thus f is Riemann integrable. $\square$