MATH 560, Homework 3

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Solutions

Problem 1. The discrete Fourier transform of a fector $f \in \mathscr{C}^n$ may be written

$$\hat{f}_j = \sum_{k=0}^{n-1} f_k \exp(-2\pi i jk/n), \quad j = 0, \dots, n-1 \tag{1}$$

while the inverse transform is given by

$$f_k = \frac{1}{n}\sum_{j=0}^{n-1} \hat{f}_j \exp(2\pi i jk/n), \quad k = 0, \dots, n-1 \tag{2}$$

Define the Fourier basis vector to be

$$v_j = (1, z^j, \dots, z^{(n-1)j})^T$$

The Fourier basis expansion can be written

$$\hat{f} = \sum_{k=0}^{n-1} f_k \bar{v}_k \tag{3}$$

and the inverse

$$f = \frac{1}{n}\sum_{j=0}^{n-1} \hat{f}_j v_j \tag{4}$$

where $z = \exp(2\pi i/n)$ and $\bar{v}_k$ is the complex conjugate of v.

Show that formulas (1) and (2) can be obtained from (3) and (4), respectively.

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Proof. For (1) to (3) we have

$$\begin{aligned} \hat{f}_j &= \sum_{k=0}^{n-1} f_k \exp(-2\pi i jk/n), \quad j = 0, \dots, n-1 \\ &= \sum_{k=0}^{n-1} f_k \bar{v}_k \quad j = 0, \dots, n-1 \\ \implies \hat{f} &= \sum_{k=0}^{n-1} f_k \bar{v}_k. \end{aligned}$$

For (2) to (4) we have

$$\begin{aligned} f_k &= \frac{1}{n}\sum_{k=0}^{n-1} \hat{f}_j \exp(2\pi i jk/n), \quad k = 0, \dots, n-1 \\ &= \frac{1}{n}\sum_{j=0}^{n-1} \hat{f}_j v_j \quad k = 0, \dots, n-1 \\ \implies f &= \frac{1}{n}\sum_{j=0}^{n-1} \hat{f}_j v_j. \end{aligned}$$

□

Problem 2. Compute the Discrete Fourier Transform of f where

(a) $f = v_3$ and $n = 8$.

(b) $f = (1, 2, -1, 4)$.

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Solution (Part (a)). Using $v_j^T \bar{v}_k = n\delta_{jk}$ we have that

$$\begin{aligned}\hat{f} &= \sum_{k=0}^{7} f_k \bar{v}_k \\ &= \sum_{k=0}^{7} v_3 \bar{v}_k \\ &= (0, 0, 1, 0, 0, 0, 0, 0)\end{aligned} \qquad \blacksquare$$

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Solution (Part (b)).

$$\begin{aligned}\hat{f} &= \sum_{k=0}^{3} f_k \bar{v}_k \\ &= \bar{v}_0 + 2\bar{v}_1 - \bar{v}_2 + 4\bar{v}_3\end{aligned} \qquad \blacksquare$$

Problem 3. (§2.2 Problem 2a.) Let β and γ be the standard ordered bases for $\mathbb{R}^n \to \mathbb{R}^m$, compute $[T]_\beta^\gamma$.

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Solution. $[T]_\beta^\gamma$ can be found by,

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \\ T_{31} & T_{32} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} T_{11}a_1 + T_{12}a_2 \\ T_{21}a_1 + T_{22}a_2 \\ T_{31}a_1 + T_{32}a_2 \end{bmatrix} = \begin{bmatrix} 2a_1 - a_2 \\ 3a_1 + 4a_2 \\ a_1 \end{bmatrix}$$

$$\implies \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{bmatrix} = [T]_\beta^\gamma$$

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Problem 4. (§2.2 Problem 4.) Define

$$T\colon M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})\text{by} \qquad T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a+b) + (2d)x + bx^2$$

Let

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \quad \text{and} \quad \gamma = \{1, x, x^2\}.$$

Compute $[T]_\beta^\gamma$.

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Solution.

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a+b \\ 2d \\ b \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 \end{bmatrix} = [T]_\beta^\gamma \qquad \blacksquare$$

Problem 5. (§2.2 (Problem 8.) Let V be n-dimensional vector space with an ordered basis β. Define $T\colon V \to \mathbb{F}^n$ by $T(x) = [x]_\beta$. Prove that T is linear.

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Proof. To show T is linear, consider $u, v \in V$ and $a \in \mathbb{F}$. Then

$$\begin{aligned} T(au+v) &= [au+v]_\beta \\ &= [au]_\beta + [v]_\beta \\ &= a[u]_\beta + [v]_\beta \\ &= aT(u) + T(v) \end{aligned}$$

□

Problem 6. (§2.2 Problem 15.) Let V and W be vector spaces, and let S be a subset of V. Define $S^0 = \{T \in \mathscr{L}(V,W) | T(x) = 0 \text{ for all } x \in S\}$. Prove the following statements.

(a) S^0 is a subspace of $\mathscr{L}(V,W)$.

(b) If S_1 and S_2 are subsets of V and $S_1 \subseteq S_2$, then $S_2^0 \subseteq S_1^0$.

(c) If V_1 and V_2 are subspaces of V, then $(V_1+V_2)^0 = V_1^0 \cap V_2^0$.

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Proof (Part (a)). To show S^0 is a subspace we need to show closure under addition and scalar multiplication as well as the existence of the zero vector.

Surely $T = 0$ is in S^0 as $0(x) = 0$ for any $x \in V$ so for any $y \in S$, $0(y) = 0$. Then let $T_1, T_2 \in S^0$ and $x \in S$, then $(T_1 + T_2)(x) = T_1(x) + T_2(x) = 0 + 0 = 0$. So S^0 is closed under addition. Finally, let $a \in \mathbb{F}$, $T \in S^0$, and $x \in S$, then $(aT)(x) = aT(x) = a0 = 0$. So S^0 is closed under scalar multiplication. □

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Proof (Part (b)). First, suppose that $S_2^0 \supset S_1^0$. Then we have that $\exists T \in S_2^0$ so that $T \notin S_1^0$. Thus $\exists x \in S_1$ so that $T(x) = 0$. But $S_1 \subseteq S_2$ which implies that $x \in S_2$ and we know that $T \in S_2^0$ so $T(x) = 0$ which is a contradiction. Thus we have that $S_2^0 \subseteq S_1^0$. □

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Proof (Part (c)). For $(V_1+V_2)^0 \subseteq V_1^0 \cap V_2^0$ we let $v \in (V_1+V_2)^0 = V_1^0 + V_2^0$. Suppose for a contradiction that $v \notin V_1^0 \cap V_2^0$ and thus v is an element in $V_1 \cup V_2$ so that $T(v) \neq 0$. But this contradicts $v \in (V_1+V_2)^0$ and thus $v \in V_1^0 \cap V_2^0$. For the other inclusion, let $v \in V_1^0 \cap V_2^0$. Thus we have $v \in V_1^0$ and $v \in V_2^0$. Thus suppose we have that $v \notin V_1^0 + V_2^0$. Then we can write $v = v_1 + v_2$ with $v_1 \in V_1$ and $v_2 \in V_2$ with $T(v_1+v_2) \neq 0$. But we have that $T(v) = T(v_1+v_2) = 0$ since $v \in V_1^0 \cap V_2^0$. Thus $V_1^0 \cap V_2^0 \subseteq (V_1+V_2)^0$. So both containments imply that $(V_1+V_2)^0 = V_1^0 \cap V_2^0$. □

Problem 7. (§2.5 Problem 2d.) For each of the following pairs of ordered bases β and β' for $\mathbb{R}^2$, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.

$$\beta = \{(-4,3),(2,-1)\} \text{ and } \beta' = \{(2,1),(-4,1)\}$$

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Proof.

$$\begin{aligned}
(-4,3) &= a(2,1) + b(-4,1) \\
(2,-1) &= c(2,1) + d(-4,1) \\
\implies a &= \frac{8}{6}, b = \frac{10}{6}, c = \frac{1}{3}, d = -23
\end{aligned}$$

So we have

$$Q = \begin{bmatrix} \frac{4}{3} & \frac{1}{3} \\ \frac{5}{3} & \frac{-2}{3} \end{bmatrix}$$

□

Problem 8. (§2.5 Problem 6a.) For each matrix A and ordered basis β, find $[L_A]_\beta$. Also, find an invertible matrix Q such that $[L_A]_\beta = Q^{-1}AQ$.

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \text{ and } \beta = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

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Proof.

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$
$$Q^{-1} = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}$$

So we have

$$\begin{aligned} [L_A]_\beta = Q^{-1}AQ &= \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 11 \\ -2 & 4 \end{bmatrix} \end{aligned}$$

□

Problem 9. (§2.5 Problem 10.) Prove that if A and B are similar $n \times n$ matrices, then $\operatorname{tr}(A) = \operatorname{tr}(B)$. *Hint:* Use Exercise of §2.3.

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Proof. We have from Exercise 13, $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ so now

$$\begin{aligned}\operatorname{tr}(B) &= \operatorname{tr}(Q^{-1}AQ)\\ &= \operatorname{tr}((Q^{-1}Q)A)\\ &= \operatorname{tr}(A)\end{aligned}$$

□