Geometric Algebra and Calculus on Riemannian Manifolds

Colin Roberts

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1 Eigenfields of Laplace-Beltrami Operator

Consider the clifford-laplace-beltrami operator $\boldsymbol{\Delta}_g$ and the eigen equation

$$\boldsymbol{\Delta}_g F = \lambda F.$$

1.1 Spinors on 2-dimensional Manifold

If we let $F = f_0 + f_2$ be an even clifford field on a 2-dimensional manifold, we are interested in the spectral equation

$$\boldsymbol{\Delta}_g F = \lambda F$$

which is then

$$\boldsymbol{\Delta}_g f_0 + \boldsymbol{\Delta}_g f_2 = \lambda f_0 + \lambda f_2,$$

hence we are simultaneously diagonalizing the laplace operator. For functions f_0, we have

$$\delta d f_0 = \lambda f_0$$

and for bivectors we simultaneously have

$$(d\delta + \delta d) f_2 = \lambda f_2.$$

Working in coordinates, we locally have $F(x, y)$ (use the definition of laplace beltrami in coordinates?)

2 Clifford Bundles

What are the eigenspinors of laplace beltrami on a manifold? Can one show that with some Q the Clifford sections on M form a Banach algebra?

`https://en.wikipedia.org/wiki/Clifford_algebra` Look at "basis and dimension" to show that we will work with orthonormal frames. Let (M, Q) be an m-dimensional smooth manifold with a quadratic form Q on the tangent bundle. By polarisation, this induces

$$g(u, v) = \frac{1}{2}(Q(u + v) - Q(u) - Q(v)).$$

Thus any quadratic manifold is a Riemannian manifold (M, g). Then we can consider an induced Clifford bundle $C\ell(TM, g)$ on M with the quadratic form $Q(v) = g(v, v)$. Other sources make the choice of $Q(v) = -g(v, v)$ in order to mimic the complex and quaternionic fields.

Specifically, we define $C\ell(T_pM, g_p)$ to be the Clifford algebra on the tangent space T_pM and let

$$C\ell(TM, g) \coloneqq \dot{\bigcup}_{p\in M} C\ell(T_pM, g_p)$$

be the Clifford bundle.

We can then define the space of Clifford sections by noting we have a natural projection

$$\pi \colon C\ell(TM, g) \to M$$

that maps a Clifford element to the point at which it is based and putting

$$\Gamma C\ell(TM, g) \coloneqq \{\sigma \colon M \to C\ell(TM, g) \mid \pi \circ \sigma = \mathrm{Id}_M\}.$$

How do we define smoothness here? Maybe just consider a section of the tensor algebra. Then we can show the smoothness and algebra properties from a quotient map from there.

Definition 2.1. Define a *k-blade field* as a smooth section of

$$\bigodot^k(TM)$$

Proposition 2.1. *The space $\Gamma C\ell(TM, g)$ forms an algebra.*

Proposition 2.2. *There exists a norm on $\Gamma C\ell(TM, g)$ via involution (multiplication by the pseudoscalar).*

Proposition 2.3. *With this norm, $\Gamma C\ell(TM, g)$ is a Banach algebra.*

Proposition 2.4. *With (blah) we have that $\Gamma C\ell(TM, g)$ is a Banach *-algebra.*

We need that the algebra of clifford sections is a Banach *-algebra. `https://en.wikipedia.org/wiki/Gelfand%E2%80%93Naimark_theorem`

`https://en.wikipedia.org/wiki/Banach_algebra#Spectral_theory` This ties up the knot for the relationship of different notions of spectra.

Can we define an inner product on the clifford algebra in a natural way? `https://math.stackexchange.com/questions/2606319/is-the-natural-norm-on-the-exterior-algebra-submult` That is, multiply the components of rank n in a meaningful way like in the link.

Relationship to frame bundles and flags?

Two versions of a Clifford norm and mention of $C\ell(p, q, r)$ `https://math.stackexchange.com/questions/1128844/about-the-definition-of-norm-in-clifford-algebra`

Also this `https://mathoverflow.net/questions/176140/norms-on-clifford-algebra-c-norm`

I feel like a good Ph.D. project would be to fucking make a single source that has information on clifford algebras on manifolds with all the relations to function algebras and such.

This is cool `https://www.jstor.org/stable/pdf/1970397.pdf`

Clifford homology/cohomology/elliptic complexes with dirac operator (derivative)

How do we define this dirac operator in a coordinate free way? Will it be able to give us Im $\subseteq$ Ker? Maybe there is some way to say "yes, up to non-harmonic functions" as D^2 is a k-blade laplacian.

This definition is essentially defined throughout [1]

Definition 2.2. Define

$$D_L = \sum_{i=1}^{n} e_i \frac{\partial}{\partial x^i}$$

and

$$D_R = \sum_{i=1}^{n} \frac{\partial}{\partial x^i} e_i$$

Definition 2.3. Define the *exterior derivative* $D\wedge$ by

$$\frac{1}{2}\left(D_L - D_R\right)$$

and the *interior derivative* $D\cdot$ by

$$\frac{1}{2}\left(D_L + D_R\right).$$

And again, in [1], it's shown that

$$(D\cdot)^2 = 0 = (D\wedge)^2.$$

$$\begin{tikzcd}
\cdots \arrow[r, bend right] & \odot^{k-2}V \arrow[l, bend right] \arrow[r, bend right] & \odot^{k-1}V \arrow[l, bend right] \arrow[r, bend right] & \odot^{k}V \arrow[l, bend right, "D\cdot"'] \arrow[ll, bend right=50, "(D\cdot)^2"'] \arrow[r, bend right, "D\wedge"'] \arrow[rr, bend right=50, "(D\wedge)^2"'] & \odot^{k+1}V \arrow[l, bend right] \arrow[r, bend right] & \odot^{k+2}V \arrow[l, bend right] \arrow[r, bend right] & \cdots \arrow[l, bend right]
\end{tikzcd}$$

probabilistic (norm one) clifford sections?

left/right dirac operators

in lectures on clifford algebras - clifford analysis lecture, there is a generalization of cauchy integral formula

3 C^*-Algebras

References

[1] Chris Doran. *Geometric Algebra for Physicists.* Cambridge University Press, 2003.