MATH 272, Worksheet 5

Coordinate systems.

Problem 1. Consider a description of the plane $\mathbb{R}^2$ in both Cartesian coordinates (x, y) and polar coordinates (r, θ). Recall the coordinate transformations

$$x(r,\theta) = r\cos\theta$$
$$y(r,\theta) = r\sin\theta.$$

Let $f(x,y) = \frac{1}{\sqrt{1-x^2-y^2}}$ and let us attempt to integrate

$$\iint_{\Sigma} f(x,y)d\Sigma,$$

where Σ is the unit disk defined by $x^2 + y^2 \leq 1$.

(a) Convert $f(x, y)$ in Cartesian coordinates to a function $f(r, \theta)$ in polar coordinates.

(b) Note that we can convert the area form $d\Sigma = dxdy$ in Cartesian coordinates to an area form in polar coordinates. Think of the function

$$\vec{\text{Pol}}(r,\theta) = \begin{pmatrix} x(r,\theta) \\ y(r,\theta) \end{pmatrix},$$

as the function that coverts Cartesian coordinates into polar coordinates. Then,

$$[J]_{\vec{\text{Pol}}} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix},$$

is the Jacobian of this transformation. The magnitude of determinant of a matrix describes the stretching that the matrix does to the space at a point, and thus the magnitude of the determinant will be a function that depends on each point that describes the local stretching behavior.

Compute $[J]_{\vec{\text{Pol}}}$ and compute the determinant of this matrix as well. Simplify this expression as much as possible.

(c) Now, to find the area form in polar coordinates, we simply take

$$\left|\det\left([J]_{\vec{\text{Pol}}}\right)\right| drd\theta.$$

Confirm that you have the area form $rdrd\theta$.

(d) Draw a picture of a small segment (sides dr and $d\theta$) in the plane. Can you see why the area of this segment depends on the radius? Can you also see why it does <u>not</u> depend on the angle?

(e) We know we have this correct if

$$\int_0^{2\pi} \int_0^R r dr d\theta = \pi R^2,$$

since this is the area contained inside a circle of radius R. Show that the above integral is true.

(f) Now, set up the integral posed initially in polar coordinates and evaluate this integral using a change of variables (do not use WolframAlpha).

Problem 2. Repeat a very similar argument to show that the volume element $d\Omega$ in cylindrical coordinates is $\rho d\rho d\theta dz$. *Hint: The coordinate transformations for x and y are analogous. But, we have the addition of the z-coordinate, but this is identical to the Cartesian z-coordinate.*

Problem 3. Provide an explicit or implicit description (both if possible) of the following regions in $\mathbb{R}^3$ in Cartesian coordinates, cylindrical coordinates, and spherical coordinates.

(a) A solid box with side lengths a, b, and c.

(b) A solid cylinder of height h.

(c) A solid ball of radius R.

Problem 4. Likewise, provide an explicit or implicit description (both if possible) of the following surfaces in $\mathbb{R}^3$ in Cartesian coordinates, cylindrical coordinates, and spherical coordinates.

(a) A the surface of a box with side lengths a, b, and c.

(b) A the surface of a cylinder of height h including endcaps.

(c) A the surface of a ball of radius R (i.e., the sphere).

Problem 5. One can also take a look at curves in each coordinate system. Plot the following curves by hand. Play around with different choices of parameterizations. Hold certain variables constant. See how this affects your curve!

(a) $x(t) = t$, $y(t) = t$, $z(t) = t$.

- What happens if we hold $x(t) = C$ constant?
- What if we held both $x(t) = C_1$, and $y(t) = C_2$ constant?
- Choose some other functions (including constants) for $x(t)$, $y(t)$, and $z(t)$, and plot these as well.

(b) $\rho(t) = t$, $\theta(t) = t$, $z(t) = t$.

- What happens if we hold $\rho(t) = C$ constant?
- What if we held both $\rho(t) = C_1$, and $\theta(t) = C_2$ constant? Or held $\rho(t)$ and $z(t)$ constant?
- Choose some other functions (including constants) for $\rho(t)$, $\theta(t)$, and $z(t)$, and plot these as well.

(c) $r(t) = t$, $\theta(t) = t$, $\phi(t) = t$.

- What happens if we hold $r(t) = C$ constant?
- What if we held both $r(t) = C_1$, and $\theta(t) = C_2$ constant? Or held $r(t)$ and $\phi(t)$ constant?
- Choose some other functions (including constants) for $\rho(t)$, $\theta(t)$, and $z(t)$, and plot these as well.

(d) ** Find a parameterization of a straight line passing through a point (x_0, y_0, z_0) in cylindrical and spherical coordinates. *Hint: This isn't too hard in those coordinate systems if the curve passes through the origin. But, it can be a bit difficult otherwise!*

Problem 6. Integrate the following functions in their relevant coordinate systems over the given region Ω.

(a) $f(x, y, z) = xy + yz$ over Ω which is the solid unit cube.

(b) $f(\rho, \theta, z) = \frac{z}{\rho} \sin(\theta)$ over Ω which is the cylinder of radius 1 and of height 2. Align this cylinder so the z-axis runs through the core of the cylinder and so the height is split at the xy-plane.

(c) $f(r, \theta, \phi) = \frac{1}{r^2} \sin(\theta) \sin(\phi)$ over the unit ball centered at the origin.

Problem 7. In Cartesian coordinates, one can consider the curves given when two of the Cartesian coordinates are held constant. That is,

- Define $\vec{\gamma}_x$ by $x(t) = t$, $y(t) = y_0$, and $z(t) = z_0$;

- Define $\vec{\gamma}_y$ by $x(t) = x_0$, $y(t) = t$, and $z(t) = z_0$;
- Define $\vec{\gamma}_z$ by $x(t) = x_0$, $y(t) = y_0$, and $z(t) = t$.

If we consider the unit tangent vectors to these curves, we will recover our basis vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ at whichever point we wish. Let's see how.

(a) Compute the tangent vectors $\dot{\vec{\gamma}}_x$, $\dot{\vec{\gamma}}_y$, and $\dot{\vec{\gamma}}_z$. Normalize these vectors if need be.

(b) For example, does choosing values of t, y_0, and z_0 for $\dot{\vec{\gamma}}_x(t)$ change the tangent vector? That is, do the tangent vectors change based on the point at which they are based? Make sure to consider all the different tangent vectors!

(c) To see what these unit vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ look like, we can also compute, for example,

$$\hat{\boldsymbol{x}}(x,y,z) = \frac{\vec{\boldsymbol{\nabla}} x(x,y,z)}{\left|\vec{\boldsymbol{\nabla}} x(x,y,z)\right|}.$$

Note that $x(x,y,z) = x$. This is just saying the x-position only depends on the x-value of the point we are at.

(d) Show that these vectors are orthogonal at every point (x,y,z).

(e) Plot the vector fields $\hat{\boldsymbol{x}}(x,y,z)$, $\hat{\boldsymbol{y}}(x,y,z)$, and $\hat{\boldsymbol{z}}(x,y,z)$.

For the following two problems, you will want to plot these curves by converting back to Cartesian coordinates. One will also notice that there is nothing to surprising happening. For example, the unit tangent vector (field) $\hat{\boldsymbol{\theta}}$ will always point in the positive θ direction at any given point.

Problem 8. * In cylindrical coordinates, one can consider the curves given when two of the cylindrical coordinates are held constant. That is,

- Define $\vec{\gamma}_\rho$ by $\rho(t) = t$, $\theta(t) = \theta_0$, and $z(t) = z_0$;
- Define $\vec{\gamma}_\theta$ by $\rho(t) = \rho_0$, $\theta(t) = t$, and $z(t) = z_0$;
- Define $\vec{\gamma}_z$ by $\rho(t) = \rho_0$, $\theta(t) = \theta_0$, and $z(t) = t$.

If we consider the unit tangent vectors to these curves, we will recover a new set of basis vectors $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{z}}$. In fact, these basis vectors depend on the point at which they are based, and hence they are actually defining a field of vectors. One should emphasize this by putting $\hat{\boldsymbol{\rho}}(\rho,\theta,z)$, $\hat{\boldsymbol{\theta}}(\rho,\theta,z)$, and $\hat{\boldsymbol{z}}(\rho,\theta,z)$.

(a) Compute the tangent vectors $\dot{\vec{\gamma}}_\rho$, $\dot{\vec{\gamma}}_\theta$, and $\dot{\vec{\gamma}}_z$. Normalize these vectors if need be.

(b) For example, does choosing values of t, θ_0, and z_0 for $\dot{\vec{\gamma}}_\rho(t)$ change the tangent vector? That is, do the tangent vectors change based on the point at which they are based? Make sure to consider all the different tangent vectors!

(c) To find out what these vectors $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{z}}$ look like in terms of the unit vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$, we can also compute, for example,

$$\hat{\boldsymbol{\rho}}(x,y,z) = \frac{\vec{\boldsymbol{\nabla}}\rho(x,y,z)}{\left|\vec{\boldsymbol{\nabla}}\rho(x,y,z)\right|}.$$

Show that

$$\begin{aligned}
\hat{\boldsymbol{\rho}}(x,y,z) &= \frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{x}} + \frac{y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{y}}, \\
\hat{\boldsymbol{\theta}}(x,y,z) &= \frac{-y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{x}} + \frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{y}}, \\
\hat{\boldsymbol{z}}(x,y,z) &= \hat{\boldsymbol{z}}.
\end{aligned}$$

(d) Show that these vectors are orthogonal at every point (x,y,z).

(e) Plot the vector fields $\hat{\boldsymbol{\rho}}(x,y,z)$, $\hat{\boldsymbol{\theta}}(x,y,z)$, and $\hat{\boldsymbol{z}}(x,y,z)$.

Problem 9. * In spherical coordinates, one can consider the curves given when two of the spherical coordinates are held constant. That is,

- Define $\vec{\gamma}_r$ by $r(t) = t$, $\theta(t) = \theta_0$, and $\phi(t) = \phi_0$;
- Define $\vec{\gamma}_\theta$ by $r(t) = r_0$, $\theta(t) = t$, and $\phi(t) = \phi_0$;
- Define $\vec{\gamma}_\phi$ by $r(t) = r_0$, $\theta(t) = \theta_0$, and $\phi(t) = t$.

If we consider the unit tangent vectors to these curves, we will recover a new set of basis vectors $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$. In fact, these basis vectors depend on the point at which they are based, and hence they are actually defining a field of vectors. One should emphasize this by putting $\hat{\boldsymbol{r}}(r,\theta,\phi)$, $\hat{\boldsymbol{\theta}}(r,\theta,\phi)$, and $\hat{\boldsymbol{\phi}}(r,\theta,\phi)$.

(a) Compute the tangent vectors $\dot{\vec{\gamma}}_r$, $\dot{\vec{\gamma}}_\theta$, and $\dot{\vec{\gamma}}_\phi$. Normalize these vectors if need be.

(b) For example, does choosing values of t, θ_0, and ϕ_0 for $\dot{\vec{\gamma}}_r(t)$ change the tangent vector? That is, do the tangent vectors change based on the point at which they are based? Make sure to consider all the different tangent vectors!

(c) To find out what these vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ look like in terms of the unit vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$, we can also compute, for example,

$$\hat{\boldsymbol{r}}(x,y,z) = \frac{\vec{\boldsymbol{\nabla}} r(x,y,z)}{\left|\vec{\boldsymbol{\nabla}} r(x,y,z)\right|}.$$

Show that

$$\begin{aligned}
\hat{\boldsymbol{r}}(x,y,z) &= \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{x}} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{y}} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{z}}, \\
\hat{\boldsymbol{\theta}}(x,y,z) &= \frac{-y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{x}} + \frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{y}}, \\
\hat{\boldsymbol{\phi}}(x,y,z) &= \frac{xz}{\sqrt{x^2+y^2}(x^2+y^2+z^2)}\hat{\boldsymbol{x}} + \frac{yz}{\sqrt{x^2+y^2}(x^2+y^2+z^2)}\hat{\boldsymbol{y}} + \frac{-\sqrt{x^2+y^2}}{x^2+y^2+z^2}\hat{\boldsymbol{z}}
\end{aligned}$$

(d) Show that these vectors are orthogonal at every point (x,y,z).

(e) Plot the vector fields $\hat{\boldsymbol{r}}(x,y,z)$, $\hat{\boldsymbol{\theta}}(x,y,z)$, and $\hat{\boldsymbol{\phi}}(x,y,z)$.

The following four problems are used to determine how the gradient vector is found in various coordinate systems.

Problem 10. * Suppose that we have the vector $\vec{\boldsymbol{v}} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$ in the plane $\mathbb{R}^2$. If I move this vector infinitesimally in each component, then how much length is swept out? That is, what is the differential $d\vec{\boldsymbol{v}}$? We can compute this differential by taking,

$$d\vec{\boldsymbol{v}} = \frac{\partial \vec{\boldsymbol{v}}}{\partial x}dx + \frac{\partial \vec{\boldsymbol{v}}}{\partial y}dy.$$

Then we can refer to, for example, the value

$$h_x = \left|\frac{\partial \vec{\boldsymbol{v}}}{\partial x}\right|,$$

as the scale factor for x.

(a) Show that in Cartesian coordinates that

$$d\vec{\boldsymbol{v}} = h_x\hat{\boldsymbol{x}}dx + h_y\hat{\boldsymbol{y}}dy = \hat{\boldsymbol{x}}dx + \hat{\boldsymbol{y}}dy.$$

In other words, show the scale factors h_x and h_y are one.

(b) In general, the gradient $\vec{\boldsymbol{\nabla}}$ is computed by swapping dx for $\frac{\partial}{\partial x}$ and likewise dy for $\frac{\partial}{\partial y}$. We also invert the scale factors. Thus,

$$\vec{\boldsymbol{\nabla}} = \frac{1}{h_x}\hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \frac{1}{h_y}\hat{\boldsymbol{y}}\frac{\partial}{\partial y} = \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y}$$

Problem 11. * Likewise, suppose that we have the vector $\vec{\boldsymbol{v}} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$ in the plane $\mathbb{R}^2$. Now, use the polar coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$

If I move this vector infinitesimally in each component, then how much length is swept out? That is, what is the differential $d\vec{\boldsymbol{v}}$? We can compute this differential by taking,

$$d\vec{\boldsymbol{v}} = \frac{\partial \vec{\boldsymbol{v}}}{\partial \rho}d\rho + \frac{\partial \vec{\boldsymbol{v}}}{\partial \theta}d\theta.$$

Then we can refer to, for example, the value

$$h_\rho = \left|\frac{\partial \vec{\boldsymbol{v}}}{\partial \rho}\right|,$$

as the scale factor for ρ.

(a) Show that in polar coordinates that

$$h_r = 1 \qquad \text{and} \qquad h_\theta = r.$$

This means that

$$d\vec{\boldsymbol{v}} = \hat{\boldsymbol{\rho}}d\rho + r\hat{\boldsymbol{\theta}}d\theta.$$

(b) Argue why the gradient in polar coordinates is then

$$\vec{\boldsymbol{\nabla}} = \hat{\boldsymbol{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta}.$$

Problem 12. * Repeat the previous arguments for cylindrical coordinates.

The following two problems derive the divergence and Laplacian in cylindrical and spherical coordinates.

Problem 13. Consider a description of the plane $\mathbb{R}^2$ in both Cartesian coordinates (x, y) and polar coordinates (r, θ). Recall the coordinate transformations

$$\begin{aligned} x(r, \theta) &= r\cos\theta \\ y(r, \theta) &= r\sin\theta. \end{aligned}$$

Let $f(x, y) = \frac{1}{\sqrt{1-x^2-y^2}}$ and let us attempt to integrate

$$\iint_{\Sigma} f(x, y) d\Sigma,$$

where Σ is the unit disk defined by $x^2 + y^2 \leq 1$.

(a) Convert $f(x, y)$ in Cartesian coordinates to a function $f(r, \theta)$ in polar coordinates.

(b) Note that we can convert the area form $d\Sigma = dxdy$ in Cartesian coordinates to an area form in polar coordinates. Think of the function

$$\vec{\text{Pol}}(r, \theta) = \begin{pmatrix} x(r, \theta) \\ y(r, \theta) \end{pmatrix},$$

as the function that coverts Cartesian coordinates into polar coordinates. Then,

$$[J]_{\vec{\text{Pol}}} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix},$$

is the Jacobian of this transformation. The magnitude of determinant of a matrix describes the stretching that the matrix does to the space at a point, and thus the magnitude of the determinant will be a function that depends on each point that describes the local stretching behavior.

Compute $[J]_{\vec{\text{Pol}}}$ and compute the determinant of this matrix as well. Simplify this expression as much as possible.

(c) Now, to find the area form in polar coordinates, we simply take

$$\left|\det\left([J]_{\vec{\text{Pol}}}\right)\right| drd\theta.$$

Confirm that you have the area form $rdrd\theta$.

(d) Draw a picture of a small segment (sides dr and $d\theta$) in the plane. Can you see why the area of this segment depends on the radius? Can you also see why it does <u>not</u> depend on the angle?

(e) We know we have this correct if

$$\int_0^{2\pi} \int_0^R rdrd\theta = \pi R^2,$$

since this is the area contained inside a circle of radius R. Show that the above integral is true.

(f) Now, set up the integral posed initially in polar coordinates and evaluate this integral using a change of variables (do not use WolframAlpha).

Problem 14. Repeat a very similar argument to show that the volume element $d\Omega$ in cylindrical coordinates is $\rho d\rho d\theta dz$. *Hint: The coordinate transformations for x and y are analogous. But, we have the addition of the z-coordinate, but this is identical to the Cartesian z-coordinate.*

Problem 15. Provide an explicit or implicit description (both if possible) of the following regions in $\mathbb{R}^3$ in Cartesian coordinates, cylindrical coordinates, and spherical coordinates.

(a) A solid box with side lengths a, b, and c.

(b) A solid cylinder of height h.

(c) A solid ball of radius R.

Problem 16. Likewise, provide an explicit or implicit description (both if possible) of the following surfaces in $\mathbb{R}^3$ in Cartesian coordinates, cylindrical coordinates, and spherical coordinates.

(a) A the surface of a box with side lengths a, b, and c.

(b) A the surface of a cylinder of height h including endcaps.

(c) A the surface of a ball of radius R (i.e., the sphere).

Problem 17. One can also take a look at curves in each coordinate system. Plot the following curves by hand. Play around with different choices of parameterizations. Hold certain variables constant. See how this affects your curve!

(a) $x(t) = t$, $y(t) = t$, $z(t) = t$.

- What happens if we hold $x(t) = C$ constant?
- What if we held both $x(t) = C_1$, and $y(t) = C_2$ constant?
- Choose some other functions (including constants) for $x(t)$, $y(t)$, and $z(t)$, and plot these as well.

(b) $\rho(t) = t$, $\theta(t) = t$, $z(t) = t$.

- What happens if we hold $\rho(t) = C$ constant?
- What if we held both $\rho(t) = C_1$, and $\theta(t) = C_2$ constant? Or held $\rho(t)$ and $z(t)$ constant?
- Choose some other functions (including constants) for $\rho(t)$, $\theta(t)$, and $z(t)$, and plot these as well.

(c) $r(t) = t$, $\theta(t) = t$, $\phi(t) = t$.

- What happens if we hold $r(t) = C$ constant?
- What if we held both $r(t) = C_1$, and $\theta(t) = C_2$ constant? Or held $r(t)$ and $\phi(t)$ constant?
- Choose some other functions (including constants) for $\rho(t)$, $\theta(t)$, and $z(t)$, and plot these as well.

(d) ** Find a parameterization of a straight line passing through a point (x_0, y_0, z_0) in cylindrical and spherical coordinates. *Hint: This isn't too hard in those coordinate systems if the curve passes through the origin. But, it can be a bit difficult otherwise!*

Problem 18. Integrate the following functions in their relevant coordinate systems over the given region Ω.

(a) $f(x, y, z) = xy + yz$ over Ω which is the solid unit cube.

(b) $f(\rho, \theta, z) = \frac{z}{\rho}\sin(\theta)$ over Ω which is the cylinder of radius 1 and of height 2. Align this cylinder so the z-axis runs through the core of the cylinder and so the height is split at the xy-plane.

(c) $f(r, \theta, \phi) = \frac{1}{r^2}\sin(\theta)\sin(\phi)$ over the unit ball centered at the origin.

Problem 19. **** Picture the surface of the unit sphere. If we have a (tangent) vector field defined along the surface (i.e., solely vectors that are a combination of $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$), is it possible that this vector field is nonzero everywhere? This is known as the *Hairy Ball Theorem* since it is analogous to the idea of "combing a fuzzy tennis ball with no cowlicks." *Hint: You can certainly comb a hairy torus, though!*