MATH 571, Homework 7

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Solutions

Problem 1.

- (a) Pick a Δ -complex structure on the n-dimensional ball D^n . Note that this induces a Δ -complex structure on its boundary (n-1)-sphere S^{n-1} . Compute the simplicial relative homology $H_n(D^n, S^{n-1})$ by using the definition of relative homology.
- (b) Verify that your answer is correct by using the LES for a pair of spaces (on pages 115 and 117 of Hatcher).

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Proof.

(a) Let Δ^n be the *n*-simplex and note that simplicial complexes are also Δ -complexes. Also we have that $\Delta^n \cong D^n$ and $\partial \Delta^n \cong S^{n-1}$. Then we have that $C_n(\Delta^n, \partial \Delta^n) \cong C_n(D^n, S^{n-1})$. Now note that for m > n we have

$$C_m(\Delta^n, \partial \Delta^n) \cong C_m(\Delta^n)/C_{n+1}(\partial \Delta^n) \cong 0/0 \cong 0$$

since there are no higher dimensional simplicies than n in Δ^n . Also,

$$C_n(\Delta^n, \partial \Delta^n) \cong C_n(\Delta^n)/C_n(\partial \Delta^n) \cong \mathbb{Z}/0 \cong \mathbb{Z}.$$

Finally, for l < n,

$$C_l(\Delta^n, \partial \Delta^n) = \cong C_l(\Delta^n)/C_l(\partial \Delta^n) \cong \mathbb{Z}/\mathbb{Z} \cong 0$$

since Δ^n and $\partial \Delta^n$ contain all the same l < n dimensional simplicies. Then we have

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1}(\Delta^n, \partial \Delta^n) \xrightarrow{\partial_{n+1}} C_n(\Delta^n, \partial \Delta^n) \xrightarrow{\partial_n} C_{n-1}(\Delta^n, \partial \Delta^n) \xrightarrow{\partial_{n-1}} \cdots$$

With the above work, we have

$$\cdots \xrightarrow{\partial_{n+2}} 0 \xrightarrow{\partial_{n+1}} \mathbb{Z} \xrightarrow{\partial_n} 0 \xrightarrow{\partial_{n-1}} \cdots$$

Now

$$H_n(\Delta^n, \partial \Delta^n) \cong \operatorname{Ker}(\partial_n)/\operatorname{Im}(\partial_{n+1} \cong \mathbb{Z}/0 \cong \mathbb{Z}.$$

(b) Now we have the following LES

$$\cdots \to H_n(S^{n-1}) \to H_n(D^n) \xrightarrow{\alpha} H_n(D^n, S^{n-1}) \xrightarrow{\beta} H_{n-1}(S^{n-1}) \xrightarrow{\gamma} H_{n-1}(D^n) \to \cdots$$

Note that

$$H_n(S^{n-1}) \cong 0$$

$$H_n(D^n) \cong 0$$

$$H_{n-1}(S^{n-1}) \cong \mathbb{Z}$$

$$H_{n-1}(D^n) \cong 0.$$

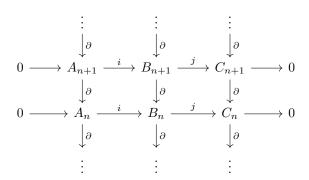
By exactness, we must have that β is an isomorphism and hence $H_n(D^n, S^{n-1}) \cong \mathbb{Z}$.

Problem 2. The proof of Theorem 2.16 in Hatcher contains 6 verification of inclusions. Prove them. For one of the last three such inclusions (ker $j_* \subseteq \operatorname{im} i_*$ or ker $\partial \subseteq \operatorname{im} j_*$ or ker $i_* \subseteq \operatorname{im} \partial$, your choice of which one), draw a diagram of the SES of chain complexes $0 \to A_{\bullet} \to B_{\bullet} \to C_{\bullet} \to 0$, and show where all the elements you consider live in this diagram.

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Proof. We have that $\operatorname{im} i_* \subset \ker j_*$ since ji = 0 means that $j_*i_* = 0$. Then $\operatorname{im} j_* \subset \ker \partial$ since $\partial j_* = 0$ by definition of ∂ . Specifically we have c = j(b) for some $b \in B_n$ and then we have $j(\partial b) = \partial j(b)$. Then we also have $\operatorname{im} \partial \subset \ker i_*$ since $i_*\partial = 0$.

 $\ker j_* \subset \operatorname{im} i_*$. A homology class in $\ker j_*$ is represented by a cycle $b \in B_n$ with j(b) a boundary. We can then write $j(b) = \partial c'$ for some $c' \in C_{n+1}$. By surjectivity of j, c' = j(b') for some $b' \in B_{n+1}$. This means that $j(b - \partial b') = j(b) - j(\partial b') = j(b) - \partial j(b') = 0$. This is because $\partial j(b') = \partial c' = j(b)$ and we defined j(b) as a boundary. It follows that $b - \partial b' = i(a)$ for some $a \in A_n$ and in fact this a is a cycle since $i(\partial(a) = \partial i(a) = \partial(b - \partial b') = \partial b = 0$ since $\partial^2 = 0$ and since b is a cycle. Then we have that $i_*[a] = [b - \partial b'] = [b]$, which shows that i_* maps onto $\ker j_*$. See the diagram below!



 $\ker \partial \subset \operatorname{im} j_*$. We let c be a homology class in $\ker \partial$, then we know that $a = \partial a'$ for $a' \in A_n$. Then b - i(a') is a cycle since $\partial (b - i(a')) = \partial b - \partial i(a') = \partial b - i(\partial a') = \partial b - i(a) = 0$ since by definition we have $\partial b = i(a)$. Also, j(b - i(a')) = j(b) - ji(a') = j(b) = c, hence j_* maps [b - i(a')] to [c].

 $\ker i_* \subset \operatorname{im} \partial$. Take a cycle $a \in A_{n-1}$ so that $i(a) = \partial b$ for some $b \in B_n$. Then we have that j(b) is a cycle since $\partial j(b) = j(\partial b) = ji(a) = 0$ again by definition. This means that $\partial [j(b)] = [a]$.

Problem 3. Exercise 16 on page 132 of Hatcher:

- (a) Show that $H_0(X, A) = 0$ iff A meets each path-component of X.
- (b) Show that $H_1(X, A) = 0$ iff $H_1(A) \to H_1(X)$ is surjective and each path-component of X contains at most one path-component of A.

Remark: For (b), Exercise 15 of Hatcher is useful. It says that if $A \to B \to C \to D \to E$ is an exact sequence, then C = 0 iff the map $A \to B$ is surjective and $D \to E$ is injective.

Proof.

(a) It's worth noting that we can reduce this problem by noting

$$H_n(X,A) \cong \bigoplus_{i \in I} H_n(X_i,A_i)$$

where each X_i is a path component of X with the index set I. Now this specifically means that we need to show

$$H_0(X_i, A_i) \cong 0$$

For the backward direction, suppose that A_i is empty, then we have

$$\cdots \to H_0(A_i) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X_i, A_i) \xrightarrow{\partial} 0$$

where $H_0(A_i) \cong 0$ and $H_0(X) \cong \mathbb{Z}$. Then exactness shows that $H_0(X_i, A_i) \cong \mathbb{Z}$. Hence we must have that A_i is nonempty.

For the forward direction, we suppose that $H_0(X_i, A_i) = 0$ for each i. Then we have

$$\cdots \to H_0(A_i) \xrightarrow{i_*} H_0(X_i) \xrightarrow{j_*} H_0(X,A) \xrightarrow{\partial} 0$$

where we see that we have the exact sequence

$$H_0(A_i) \xrightarrow{i_*} \mathbb{Z} \xrightarrow{j_*} 0.$$

But exactness here implies that i_* is surjective, and hence $H_0(A_i) \not\cong 0$ and hence A_i is nonempty.

(b) Suppose that $H_1(X,A) \cong 0$. We have

$$\rightarrow H_1(A) \xrightarrow{i_*} H_1(X) \xrightarrow{j_*} H_1(X,A) \xrightarrow{\partial} H_0(A) \xrightarrow{i_*} H_0(X) \rightarrow$$

is an exact sequence. But here, we must have that $H_1(A) \to H_1(X)$ is surjective and $H_0(A) \to H_0(X)$ is injective. We can decompose A into the path components A_i for $i \in I$ and do the same for X by decomposing X into X_j with $j \in J$. Then note that injectivity of $H_0(A) \to H_0(X)$ means that $\bigoplus_{i \in I} H_0(A_i) \to \bigoplus_{j \in J} H_0(X_j)$ is injective for each component and so there must only be at most one path component of A in each path component of X or this map fails to be injective.

For the other direction, we have that $H_1(A) \to H_1(X)$ is surjective. Then if each path component of X contains at most a single path component of A, then $H_0(A) \to H_0(X)$ is injective by the argument above. These two maps being surjective and injective, respectively, implies that $H_1(X,A) \cong 0$.

Problem 4. Choose any midterm (or old homework) problem. Clearly state the problem. Write a solution that is as clear as possible.

I will redo problem 3 from the midterm.

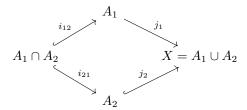
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Proof. Note that A_1 and A_2 are each a torus with a disk removed and so each is homotopy equivalent to a wedge sum of two circles. This means

$$\pi_1(A_1) \cong \langle a, b \rangle$$

 $\pi_1(A-2) \cong \langle c, d \rangle.$

We have that $A_1 \cap A_2$ is path connected and this gives the following diagram:



Specifically, we have that

$$\pi_1(A_1 \cup A_2) \cong \pi_1(X) \cong (\pi_1(A_1) * \pi_1(A_2))/N$$

where N is the normal subgroup generated by all elements of the form $i_{12}(w)i_{21}(w)^{-1}$ with $w \in \pi_1(A_1 \cap A_2)$. So we find by looking at $A_1 \cap A_2$ in the identification polygon for A_1 that $i_{12}(w) = aba^{-1}b^{-1}$ and for A_2 that $i_{21}(w)cdc^{-1}d^{-1}$. Hence we get

$$\pi_1(X) \cong \langle a, b, c, d \rangle / N$$

$$\cong \langle a, b, c, d | aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle.$$