MATH 517, Homework 5

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Solutions

Problem 1. (Rudin 4.6) Let $E \subseteq \mathbb{R}$ be compact and suppose that $f: E \to \mathbb{R}$. Define the *graph* of f by

$$G_f \coloneqq \{(x, f(x)) | x \in E\} \subseteq \mathbb{R}^2$$

Prove that f is continuous if and only if G_f is compact.

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Proof. We can say that $G_f = (E, f(E))$. For the forward direction, note that E and $f(E)$ are compact since we are assuming E is compact and f continuous. Since E is compact and $\mathbb{R}$ is a metric space, E is sequentially compact so that for any infinite sequence $\{x_i\} \in E$ we have that $\{x_{i_j}\} \to (x, f(x))$ is a convergent subsequence. Then note since f is continuous, $\{f(x_{i_j})\}$ is also convergent. If we consider $\{(x_{i_j}, f(x_{i_j}))\}$ we have that this cartesian product of sequences converges in $(E, f(E))$ and so $(E, f(E))$ is sequentially compact and therefore compact since it is a subset of $\mathbb{R}$.

For the reverse direction, suppose that G_f is compact. Thus G_f is also sequentially compact. Let $\{(x_i, f(x_i))\}$ be a sequence in G_f and note that we have a convergent subseqence $\{(x_{i_j}, f(x_{i_j}))\} \to (x, f(x))$. Since $\{(x_i, f(x_i))\}$ was an arbitrary sequence, $\{(x_{i_j}, f(x_{i_j}))\}$ was an arbitrary sequence converging to $(x, f(x))$. Thus since we also have $\lim_{j\to\infty} x_{i_j} = x$ and $\lim_{j\to\infty} f(x_{i_j}) = f(x)$, we know that f is continuous. $\square$

Problem 2. Let $E \subseteq X$, where X is a metric space, let Y be a complete metric space, and let $f : E \to Y$ be uniformly continuous. Prove that f has a continuous extension to $\bar{E}$.
(In particular, when E is dense in X, this gives a continuous extension to all of X, which is a generalization of Rudin Problem 4.13.)

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Proof. Let $\{x_i\}$ be an arbitrary sequence in E converging to $x \in \bar{E}$. Then we have that $f(x_i)$ defined for all x_i since f is uniformly continuous on E. By uniform continuity of f, if we fix $\epsilon > 0$ there is a corresponding $\delta > 0$ so that for $p, q \in E$ and $d_x(p,q) < \delta$ we have $d_y(f(p), f(q)) < \epsilon$. Since $\{x_i\} \to x$, we know that $\{x_i\}$ is Cauchy, and for $m, n > N \in \mathbb{N}$ we have that $d_x(x_m, x_n) < \delta$ and thus $d_y(f(x_m), f(x_n)) < \epsilon$. This implies that $\{f(x_i)\}$ is also a Cauchy sequence, and thus by completeness of Y we have that $\{f(x_i)\}$ converges to $y \in Y$. Now, define $g(x) = f(x)$ for $x \in E$ and $g(x) = y$ for $x \in \bar{E} \setminus E$ (i.e., each $x \in \bar{E} \setminus E$ corresponds to a $y \in Y$ where $g(x) = y$). Obviously if $E = \bar{E}$ then $g(x)$ was a continuous extension since f itself was continuous. Otherwise, let $\{x_i\} \to x \in \bar{E} \setminus E$ be an arbitrary sequence and each $x_i \in E$. Then we have $\lim_{i\to\infty} |g(x_i) - y| = 0$ and thus $g(x)$ is continuous on $\bar{E}$. □

Problem 3. (Rudin 4.18) Every rational x can be written uniquely in the form $x = m/n$ where m and n are relatively prime, $n > 0$, and we choose $n = 1$ when $x = 0$. Define *Thomae's function*

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{n} & \text{if } x \text{ is rational} \end{cases}$$

Show that f is continuous at every irrational number, and that f has a simple discontinuity at every rational number.

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Proof. To show that f is continuous at each irrational number x', we note that $x' = P + x$ for $P \in \mathbb{Z}$ and $x \in (0,1)$ irrational. Hence it suffices to show that f is continuous at each irrational $x \in (0,1)$. Consider let $\delta_N > 0$ so that $(x - \delta_N, x + \delta_N) \subseteq (0,1)$. Suppose, for a contradiction, that there does not exist an $N \in \mathbb{N}$ so that every rational q can be written as $q = \frac{Q}{M}$ for a natural number $M > N$. Since there are infinitely many rationals in $(x - \delta_N, x + \delta_N)$, there are infinitely many rationals written as $q' = \frac{Q'}{M'} \in (x - \delta_N, x + \delta_N)$ for $M' \leq N$. But if this is the case, then some rational $q' > 1$ since there are only finitely many natural numbers $1, \ldots M'$ for the denominator and thus we have $q' \notin (x - \delta_N, x + \delta_N)$, which contradicts $q' \in (x - \delta_N, x + \delta_N)$. Thus every rational in $(x - \delta_N, x + \delta_N)$ will have a denominator greater than N. Then fix $\epsilon > 0$ and let $N > \frac{1}{\epsilon}$. Then, for $y \in (x - \delta_N, x + \delta_N)$ we have $|y - x| < \delta_N$ that $|f(y) - f(x)| = |f(y)| \leq |\frac{1}{N}| < \epsilon$.

To show that f has a simple discontinuity at every rational, it suffices to consider a sequence of irrational numbers $\{x_i\} \to q \in \mathbb{Q}$. Then note that $f(q) = \frac{1}{N}$ for some N yet $f(x_i) = 0$ for each i. Thus $\{f(x_i)\}$ does not converge to $\frac{1}{N}$. Hence, since q was an arbitrary rational, we have that f has a simple discontinuity at every rational. $\square$

Problem 4. (Rudin 4.23) $f\colon (a,b) \to \mathbb{R}$ is *convex* if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$$

whenever $a < x,\ y < b$ and $0 < \lambda < 1$.

(a) Prove that every convex function is continuous.

(b) Prove that every increasing convex function of a convex function is convex (e.g., e^f is convex f is.)

(c) If f is convex and $a < s < t < u < b$, show that

$$\frac{f(t)-f(s)}{t-s} \le \frac{f(u)-f(s)}{u-s} \le \frac{f(u)-f(t)}{u-t}$$

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Proof (Part (a)). Consider an arbitrary sequence $\{\lambda_i\} \to 1$ for $\lambda_i \in (0,1)$. Then we have that $x_i = \lambda_i x + (1-\lambda_i)y$ for $x, y \in (a,b)$. Note that $x_i \in (a,b)$ for each i and that $\{x_i\} \to x$ is an arbitrary convergent sequence. Also, we know that $|x-y| = \pm(x-y)$ for either $+$ or $-$.

$$\begin{aligned} 0 \le \lim_{i\to\infty} |f(x_i) - f(x)| &= \pm(f(x_i) - f(x)) \\ &\le \pm(\lim_{i\to\infty} f(\lambda_i x + (1-\lambda_i)y) - f(x)) \\ &\le \pm(\lim_{i\to\infty} \lambda_i f(x) + (1-\lambda_i) f(y) - f(x)) \\ &= \pm(f(x) - f(x)) \\ &= 0. \end{aligned}$$

Hence, f is continuous. □

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Proof (Part (b)). Let g be continuous and defined on the range of f which we let be convex. Then for $x, y \in (a,b)$ and $\lambda \in (0,1)$

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &\le \lambda f(x) + (1-\lambda) f(y) \\ \implies g(f(\lambda x + (1-\lambda)y) &\le g(\lambda f(x)) + g((1-\lambda) f(y)) && \text{since } g \text{ increasing} \\ \implies &\le \lambda g(f(x)) + (1-\lambda) g(f(y)) && \text{since } g \text{ is convex} \end{aligned}$$

Thus $g \circ f$ is convex. □

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Proof (Part (c)). First let $\lambda = \frac{t-s}{u-s}$ and then $1-\lambda = \frac{u-t}{u-s}$. Then we also know that $\lambda u + (1-\lambda)s = \frac{u(t-s)+s(u-t)}{u-s} = t$. Then

$$\begin{aligned} f(\lambda u + (1-\lambda)s) &= f(t) \\ \implies f(t) &\le \lambda f(u) + (1-\lambda) f(s) \\ \implies f(t) - f(s) &\le \lambda(f(u) - f(s)) \\ \frac{f(t)-f(s)}{t-s} &\le \frac{f(u)-f(s)}{u-s}. \end{aligned}$$

Finally we get

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\begin{aligned}
0 &\leq \lambda f(u) + (1-\lambda) f(s) - f(t) \\
-(1-\lambda) f(s) &\leq \lambda f(u) - f(t) \\
(1-\lambda) f(u) - (1-\lambda) f(s) &\leq f(u) - f(t) \\
\implies \frac{f(u)-f(s)}{u-s} &\leq \frac{f(u)-f(t)}{u-t}.
\end{aligned}
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Thus

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\frac{f(t)-f(s)}{t-s} \leq \frac{f(u)-f(s)}{u-s} \leq \frac{f(u)-f(t)}{u-t} \qquad \square
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