The Framework of Quantum Mechanics

A C^*-Algebraic Approach

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Algebras

Definition

An *algebra* $\mathcal{A}$ over a field $\mathbb{F}$ is a vector space $\mathcal{A}$ over $\mathbb{F}$ with an binary operation such that if $\alpha \in \mathbb{F}$ and $A, B \in \mathcal{A}$ then $\alpha(AB) = (\alpha A)B = A(\alpha A)$.

Example

Consider an n-dimensional vector space V over any $\mathbb{F}$. Then the elements in $\mathcal{L}(V)$ form an algebra.
Then $\mathbb{C}$ is a matrix algebra over $\mathbb{R}$, and the quaternions $\mathcal{Q}$ are a matrix algebra over $\mathbb{C}$.

Banach Algebras

Definition

A *Banach algebra* is an algebra $\mathcal{A}$ over $\mathbb{F}$ that has a norm $\|\cdot\|$ relative to which $\mathcal{A}$ is a Banach space and such that for all $A, B \in \mathcal{A}$ we have

$$\|AB\| \leq \|A\|\|B\|.$$

Example

If X is a compact Hausdorff space, then $\mathcal{A} = \mathcal{C}_0(X, \mathbb{C})$ form a Banach algebra if we define multiplication for $f, g \in \mathcal{A}$ by $(fg)(x) = f(x)g(x)$ for $x \in X$. This is a commutative unital algebra.

Banach Algebras

We can safely assume that the algebras we look at are unital by the following.

Proposition

If $\mathcal{A}$ is a Banach algebra without identity, then $\tilde{\mathcal{A}} = \mathcal{A} \times \mathbb{F}$ with operations

- $(A, \alpha) + (B + \beta) = (A + B, \alpha + \beta)$;
- $\beta(A, \alpha) = (\beta A, \beta\alpha)$;
- $(A, \alpha)(B, \beta) = (AB + \alpha B + \beta A, \alpha\beta)$
- $\|(A, \alpha)\| = \|A\| + |\alpha|$

give us that $\tilde{\mathcal{A}}$ is a Banach algebra with identity $(0, 1)$ and $A \mapsto (A, 0)$ is an isometric isomorphism (linear bijective isometry) of $\mathcal{A}$ into $\tilde{\mathcal{A}}$.

$*$-algebras

Definition

A $*$-algebra $\mathcal{A}$ is an algebra together with an *involution* that for $A, B \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, satisfies

- $A^{**} = A$,
- $(A + B)^* = A^* + B^*$ and $(AB)^* = B^* A^*$,
- $(\lambda A)^* = \overline{\lambda} A^*$.

C^*-Algebras

Definition

A C^*-algebra $\mathcal{A}$ is a Banach algebra over $\mathbb{C}$ with an involution $*$ so that $\mathcal{A}$ is also a $*$-algebra and $\forall a \in \mathcal{A}$

$$\|A^* A\| = \|A\| \; \|A^*\|.$$

The extra requirement is called the C^**-condition*, and shows $\|AA^*\| = \|A\|^2$.

Examples

Example

$\mathbb{C}$ itself is a C^*-algebra with $*$ being the complex conjugate.

Example

If H is a Hilbert space, $\mathcal{A} = \mathscr{B}(H)$ is a C^*-algebra where $*$ denotes the adjoint.

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$*$-Isomorphisms

Definition

If $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, then a bounded linear map $\pi\colon \mathcal{A} \to \mathcal{B}$ is a *$*$-homomorphism* if

- For $A, B \in \mathcal{A}$, we have $\pi(AB) = \pi(A)\pi(B)$.
- For $A \in \mathcal{A}$, we have $\pi(A^*) = \pi(A)^*$.

For C^*-algebras, any $*$-homomorphism is bounded with norm ≤ 1. We also have that injective $*$-homomorphisms are isometries. A bijective $*$-homomorphism is a *C^*-isomorphism* and we say $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic*.

States

Definition

Let $\mathcal{A}$ be a C^*-algebra. Then a *state* is a positive linear functional $S\colon \mathcal{A} \to \mathbb{R}$ with norm 1.

Specifically, we will care about the following.

Definition

Let $\mathcal{A}$ be a C^*-algebra of bounded operators on a corresponding Hilbert space H, then the linear functional $S_x\colon \mathcal{A} \to \mathbb{R}$ is given by

$$S_x(A) := \langle Ax, x\rangle$$

for $A \in A$. Note $S_x(1) = \|x\|^2$ thus S_x is a state if $\|x\| = 1$. Specifically S_x is a *vector state*.

GN Theorems

Theorem (GN Theorem for Commutative C^*-Algebras)

A commutative (unital) C^*-algebra $\mathcal{A}$ is isomorphic to the C^*-algebra of bounded continuous functions on a compact Hausdorff space X.

Theorem (GN Theorem for Non-Commutative C^*-Algebras)

An arbitrary C^*-Algebra $\mathcal{A}$ is isomorphic to a C^*-algebra of bounded operators on a Hilbert space H.

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Goal

Now, with some machinery defined (and more to come), we want to relate operator algebras in classical mechanics to those in quantum mechanics.

Goal: Provide a way to realize the axioms for quantum mechanics.

Classically, we think of *phase space* (think position and momentum as coordinates) of a system to be a compact (symplectic) manifold Γ. Then:

Definition (Classical Observables)

The *classical observables* are the continuous real-valued functions acting on the phase space. Namely, $\mathcal{C}^0(\Gamma, \mathbb{R})$. We will denote all classical observables on Γ as $\mathcal{O} = \mathcal{C}^0(\Gamma, \mathbb{R})$.

Why is it the case that Γ is compact and that the observables are continuous?

First Result

If we let $S = (p, q) \in \Gamma$, $A, B \in \mathcal{O}$, and $\lambda \in \mathbb{R}$, then we define

- $(A + B)(S) := A(S) + B(S)$,
- $(\lambda A)(S) := \lambda A(S)$,
- $(AB)(S) := A(S)B(S)$,
- $\|A\| := \sup\{|A(S)| \ : \ S \in \Gamma\}$,
- $(A^*)(S) = \overline{A(S)}$.

Notice that A is a real valued function, and thus $\overline{A(S)} = A(S)$. This leads us to believe that elements of $\mathcal{O}$ are self-adjoint.

First Result

What can be stated in the classical case as a theorem will eventually become an axiom for the quantum case. Namely:

Theorem (Properties of Classical Observables)

The set of observables $\mathcal{O}$ of a classical system are the self-adjoint elements of a separable commutative C^*-algebra $\mathcal{A}$.

Classical States

- It is a result of the Reisz-Markov-Kakutani Representation Theorem that we can write a classical state as a linear functional $S \colon \mathcal{A} \to \mathbb{C}$ by

$$S(A) = \int_\Gamma A d\mu_S$$

  where μ_S is a uniquely defined Borel probability measure.
- We can then think of $S(A)$ as the *expected value of the observable A with the particle in the state S*.

- We can then define variance from this by

$$\Delta_S(A)^2 := S[(A - S(A))^2].$$

- Yet we find that for classical states and observables that $\Delta_S(A) = 0$.
- This brings to light that the algebra of observables for quantum systems must not be commutative so that $\Delta_S(P)\Delta_S(Q) \geq \frac{\hbar}{2}$.
- Digression on uncertainty.

- From the earlier theorem for classical systems, we consider just removing commutivity.
- We will see now that this coupled Heisenberg's commutation relation

$$[Q, P] = \alpha\hbar\mathbf{1},$$

with $\alpha \in \mathbb{C}$ and $|\alpha| = 1$ forces the Heisenberg uncertainty principle

$$\Delta_S(P)\Delta_S(Q) \geq \frac{\hbar}{2},$$

to hold.

Heisenberg's Uncertainty

Non-Commuting Observables Imply Uncertainty Principle.

Let $A, B \in \mathcal{O}$ and fix a state S. We can assume $S(A) = S(B) = 0$ since we could take the observables $A - S(A)$ and $B - S(B)$. Then

$$\Delta_S(A)^2 \Delta_S(B)^2 = S(A^2)S(B^2).$$

Since $(A - i\lambda B)(A + i\lambda B) \geq 0$, $\forall \lambda \in \mathbb{R}$, positivity of S implies

$$S(A^2) + |\lambda|^2 S(B^2) + i\lambda S([A, B]) \geq 0,$$

where $[A, B] = AB - BA$.

Heisenberg's Uncertainty

Continued.

Now, define

$$M = \begin{bmatrix} S(A^2) & \frac{1}{2}S(i[A,B]) \\ \frac{1}{2}S(i[A,B]) & S(B^2) \end{bmatrix} \quad \text{and} \quad \vec{\alpha} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

and we see that the inequality

$$(A - i\lambda B)(A + i\lambda B) \geq 0$$

becomes

$$\vec{\alpha}^T M \vec{\alpha} \geq 0.$$

This means M is positive semi-definite.

Heisenberg's Uncertainty

Continued.

Hence

$$\det M = S(A^2)S(B^2) - \frac{1}{4}S(i[A,B])^2 \geq 0$$

and thus

$$\Delta_S(A)\Delta_S(B) \geq \frac{1}{2}|S([A,B])|.$$

We then find that if $[P,Q] = \alpha\hbar\mathbf{1}$ with $\alpha \in \mathbb{C}$ and $|\alpha| = 1$,

$$\Delta_S(P)\Delta_S(Q) \geq \frac{\hbar}{2}.$$

□

Conclusions

With what we've shown above, we can conclude the two major axioms for quantum mechanics.

Axiom (Quantum Observables)

The observables of a quantum system are the self adjoint elements of a separable Hilbert space.

Axiom (Quantum States)

The set of states $\mathcal{S}$ of a quantum system is the set of all positive linear functionals ψ on $\mathcal{A}$ such that $\psi(\mathbf{1}) = 1$. We think of the functional $\psi(A)$ as $\langle A\psi, \psi\rangle$.

Future

These results start to bleed into other specific areas of research surrounding quantum mechanics. Just to list a few,

- second quantization,
- (local) quantum field theory,
- deformation quantization,
- geometrical quantization.

Main Sources

If you're interested to read more, my main sources were

- *The C^*-Algebraic Formalism of Quantum Mechanics*, Jonathan Gleason,
- *An Introduction to the Mathematical Structure of Quantum Mechanics*, Franco Strocchi.