MATH 271, Worksheet 7
Linear Transformations and Matrices

Problem 1. Consider the transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ x+y \end{pmatrix}.$$

(a) Show that this transformation is linear.

(b) Write down a matrix for this linear transformation.

(c) Can you draw a picture of the output of this transformation? What kind of object is it?

Problem 2. Consider the system of linear equations:

$$\begin{aligned} 3x + 2y + 0z &= 5 \\ 1x + 1y + 1z &= 3 \\ 0x + 2y + 2z &= 4. \end{aligned}$$

(a) Write the augmented matrix M for this system of equations.

(b) Use row reduction to get the augmented matrix in row-echelon form.

(c) Determine the solution to the system of equations.

Problem 3. Consider the equation

$$\begin{pmatrix} 1 & 3 & 4 \\ 2 & 9 & 9 \\ 1 & 5 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 20 \\ 11 \end{pmatrix}.$$

Does this equation have a solution or not? If so, determine the solution.

Problem 4. Consider the linear transformations on $\mathbb{R}^3$ to $\mathbb{R}^3$ given by

$$\begin{aligned} R_x(\theta) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \\ R_y(\theta) &= \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \\ R_z(\theta) &= \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Fact: These matrices are generators for the *group of rotations* $\mathrm{SO}(3)$ of $\mathbb{R}^3$.

(a) Let $\theta = \pi/2$. Show that $R_x(\pi/2)$ rotates a vector by $\pi/2$ radians in the xy-plane.

(b) Show that the determinant of each of these matrices is 1 for any value of θ.

(c) Using properties of determinants, show that the determinant of a product of rotation matrices is also 1.

(d) Explain geometrically why a rotation matrix must have a determinant of 1.

(e) Show that $R_x(\theta)R_x(\theta)^\dagger = I$. This in fact true for any rotation matrix.

Problem 5. Consider the matrix

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

(a) Compute $\mathrm{tr}(M)$.

(b) Compute $M^{R_x} = R_x(\pi/2) M R_x(\pi/2)^\dagger$.

(c) What is the trace of M^{R_x}?

(d) Can you see why you have the answer in (c) from properties of the trace?

Problem 6. Consider the following three vectors $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}} \in \mathbb{R}^3$ given by

$$\vec{\boldsymbol{u}} = \hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}, \qquad \vec{\boldsymbol{v}} = 2\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + 2\hat{\boldsymbol{z}}, \qquad \vec{\boldsymbol{w}} = -2\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}.$$

(a) We can write a linear combination of these vectors by taking

$$\alpha\vec{\boldsymbol{u}} + \beta\vec{\boldsymbol{v}} + \gamma\vec{\boldsymbol{w}},$$

where $\alpha, \beta, \gamma \in \mathbb{R}$. Write this linear combination as a matrix times a vector.

(b) Does this list of vectors form a basis for $\mathbb{R}^3$? *Hint: use the above work. Can any vector in* $\mathbb{R}^3$ *be written as a linear combination of these vectors?*