MATH 272, Exam 1

Oral Examination Problems

Due one hour before your exam time slot.

Instructions You are allowed a textbook, homework, notes, worksheets, material on our Canvas page. You can use online tools such as Desmos and Wolfram Alpha to check your work, but you will need to explain how you arrived at your answers. You can work with other students and this is, in fact, encouraged! However, I will not be giving out direct help for these problems but can answer questions about previous problems and notes, for example. Ambiguous or illegible answers will not be counted as correct. Scan your solutions and submit them as a pdf on Canvas under Oral Exam 1.

Note, there are four total problems.

Problem 1. Let $h(x, y)$ be the height of a sheet above the xy-plane at the point (x, y). Then, the level sets of h are exactly the lines you find in a topographic map. Below is a map of White River National Forest, CO including Snowmass Peak.

(a) Determine the level set (possibly multiple curves) for $h(x, y) = 12,000$.

(b) In the coordinates of this map, the town of Crystal and the famous Crystal Mill is located roughly at $x = 18$, $y = 25.5$. Locate this point and describe the local geometry (e.g., where the steep and shallow gradients are, which way the river should flow).

(c) Sheep Mountain is located at roughly $x = 16.5$ and $y = 26$. What is the max elevation of Sheep Mountain?

(d) You want to hike to Snowmass Peak (roughly $(21.5, 32)$) because you are very brave. Assuming you are dropped off at Lead King Basin (roughly $(20, 28)$). Assuming you can hike over any terrain (including water if need be), can you find a path that follows the steepest gradient from Lead King Basin to Snowmass Peak?

(e) Locate the flattest area that you can.

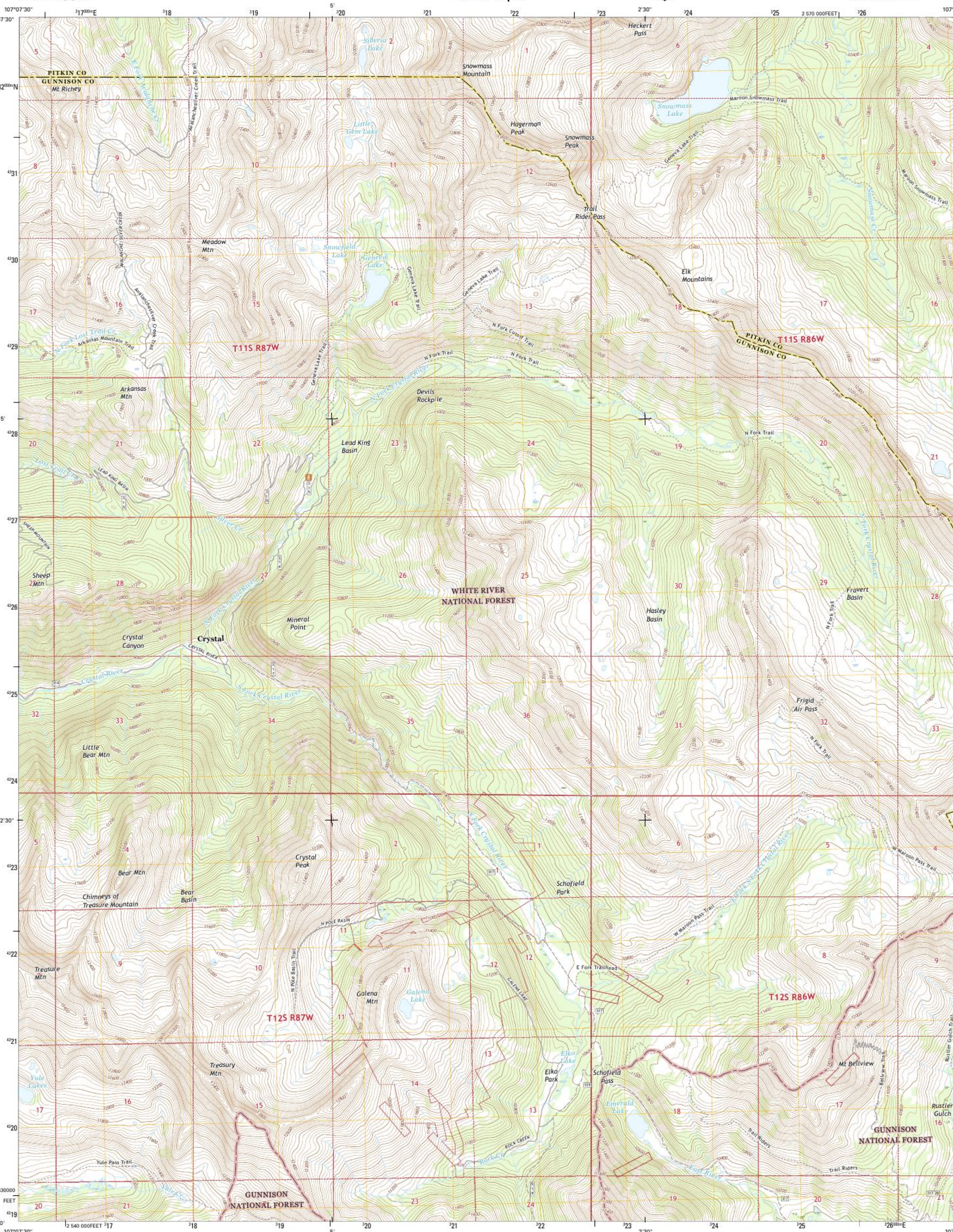
(f) Plot gradient vectors at each point on the grid.

(g) Follow Schofield Pass starting from roughly $(22.5, 19)$. How much elevation gain and loss do you experience over this curve? Can you think of how we could represent this via integration? (*Hint: you may be able to use both h and $\vec{\nabla} h$ to compute this in different ways.*)

U.S. DEPARTMENT OF THE INTERIOR
U.S. GEOLOGICAL SURVEY

The National Map
US Topo

SNOWMASS MOUNTAIN QUADRANGLE
COLORADO
7.5-MINUTE SERIES



Produced by the United States Geological Survey
North American Datum of 1983 (NAD83)
World Geodetic System of 1984 (WGS84). Projection and
1 000-meter grid: Universal Transverse Mercator, Zone 13S
10 000-foot ticks: Colorado Coordinate System of 1983 (central zone)

This map is not a legal document. Boundaries may be generalized for this map scale. Private lands within government reservations may not be shown. Obtain permission before entering private lands.

Imagery....................NAIP, September 2013
Roads....................U.S. Census Bureau, 2015 - 2016
Roads within US Forest Service Lands..........FSTopo Data with limited Forest Service updates, 2012 - 2016
Names....................GNIS, 2016
Hydrography..........National Hydrography Dataset, 2013
Contours..........National Elevation Dataset, 2004
Boundaries..........Multiple sources; see metadata file 1972 - 2016
Public Land Survey System..........BLM, 2011
Wetlands..........FWS National Wetlands Inventory 1977 - 2014

UTM GRID AND 2016 MAGNETIC NORTH DECLINATION AT CENTER OF SHEET

U.S. National Grid
100,000-m Square ID
CD
Grid Zone Designation
13S

SCALE 1:24 000

CONTOUR INTERVAL 40 FEET
NORTH AMERICAN VERTICAL DATUM OF 1988

This map was produced to conform with the National Geospatial Program US Topo Product Standard, 2011.
A metadata file associated with this product is draft version 0.6.19

QUADRANGLE LOCATION

1 Redstone
2 Capitol Peak
3 Highland Peak
4 Marble
5 Maroon Bells
6 Marcellina Mountain
7 Oh-be-joyful
8 Gothic

ADJOINING QUADRANGLES

ROAD CLASSIFICATION

Expressway
Secondary Hwy
Ramp
Local Connector
Local Road
4WD
Interstate Route
US Route
State Route
FS Primary Route
FS Passenger Route
FS High Clearance Route

Check with local Forest Service unit for current travel conditions and restrictions.

SNOWMASS MOUNTAIN, CO
2016

Problem 2. A *plasma* is a highly ionized gas. In fact, plasma is the most abundant form of matter in the universe. Plasmas can be treated as charged fluids whose motion is coupled to its own electromagnetic fields.

Because of this, in confined plasmas (e.g., Tokamak & Stellerator) you may see that the fluid velocity vector field $\vec{\boldsymbol{v}}$ and the fluid vorticity $\vec{\boldsymbol{\omega}}$ are parallel. In other words, $\vec{\boldsymbol{v}} \times \vec{\boldsymbol{\omega}} = 0$.

As a note for plasmas, the vorticity is coupled to the magnetic field and the velocity is coupled to the electric field.

(a) Suppose that $\vec{\boldsymbol{v}}$ is the fluid velocity. We say that $\vec{\boldsymbol{v}}$ is a *Beltrami field* if

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}} = \lambda \vec{\boldsymbol{v}}$$

for some scalar (eigenvalue) λ. Suppose further that $\vec{\boldsymbol{v}}$ is divergence free, i.e., $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}} = 0$. Show that

$$\vec{\boldsymbol{\Delta}} \vec{\boldsymbol{v}} = -\lambda^2 \vec{\boldsymbol{v}}$$

where $\vec{\boldsymbol{\Delta}}$ is the vector Laplacian.

(b) Fluid vorticity is the curl of the fluid velocity, i.e., $\vec{\boldsymbol{\omega}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{v}}$. Show that if $\vec{\boldsymbol{v}}$ is a Beltrami field, then the fluid vorticity and fluid velocity are parallel.

(c) For a specific example, let

$$\vec{\boldsymbol{v}}(x, y, z) = \begin{pmatrix} \sin(z) + \cos(y) \\ \sin(x) + \cos(z) \\ \sin(y) + \cos(x) \end{pmatrix}.$$

plot this vector field over a suitable range of input values.

(d) Show that $\vec{\boldsymbol{v}}$ is a Beltrami field. What is the eigenvalue?

(e) Show that $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{v}} = 0$ and find $\vec{\boldsymbol{\Delta}} \vec{\boldsymbol{v}}$ without computing any derivatives.

Problem 3. Consider a curve

$$\vec{\gamma}(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \\ \cos(2t) \end{pmatrix} \qquad t \in [0, 2\pi].$$

Imagine this curve is a wire that we dip into a soapy solution. What will the shape of the bubble that forms on this wire look like? Let's work through it.

(a) Consider the function $f(x, y) = x^2 - y^2$. Plot the graph of this function and the curve $\vec{\gamma}$ simultaneously.

(b) Show that for points (x, y) on the unit circle in the xy-plane that the graph of $f(x, y)$ matches the points along $\vec{\gamma}$. *Hints:*
- *You should be able to see this in your plot from (a). This isn't proof, but it can help you visualize the problem.*
- *Can you parameterize the unit circle as a curve* $\vec{\boldsymbol{\eta}}(t)$ *and consider* $f(\vec{\boldsymbol{\eta}}(t))$*?*

(c) Show that $f(x, y)$ is harmonic. That is, show $\Delta f = 0$. This shows the bubble is a *minimal surface.*

(d) Set up the integral

$$\int_\Sigma d\Sigma,$$

which computes the area of the soap film. Here, Σ is the graph of the function $f(x, y)$ with the domain given by the unit disk $x^2 + y^2 \leq 1$.

(e) (BONUS) Convert the integral in the previous part to cylindrical coordinates and find a numerical value for the integral. *You will not be able to get WolframAlpha to compute this integral in cartesian coordinates – this is another reason why using coordinate systems better suited to your problem is important!*

Problem 4. Let us explore our other coordinate systems.

(a) Define a function in cylindrical coordinates that is constant on the surface of an infinitely tall cylinder of radius 1 but is not constant in all of space.

(b) Set up the bounds of an integral in spherical coordinates that integrates over an eighth of a thick spherical shell with inner radius 5 and outer radius 25. Take the eighth that lies in the octant where x and y are both positive.

(c) Convert the following function

$$f(x, y, z) = \frac{x}{y(x^2 + y^2 + z^2)}$$

into spherical coordinates.