Math 272:

Higher Dimensional ODEs:

We have studied equations of the form:

$$\dot{x}(t) = \underline{f(x,t)}.$$

The right hand side comes from a model. It tells us how x changes over time.

which we referred to as a first-order ODE. In this case, x represents a value we measure (e.g., position, velocity, quantity, concentration, etc.), and the independent variable t is typically thought of as time (though we could have, say, quantity depend on position).

Now, we could consider how many quantities change over time. Truth is, we saw this when we looked at reacting chemical species. In this case, we would then allow our dependent variable x to be a vector $\vec{x}$. Indeed, we must then have that our right hand side follows suit so that we get

$$\dot{\vec{x}} = \vec{f}(\vec{x},t).$$

This equation essentially says that our vector $\vec{x}$ follows the vector field $\vec{f}$ that depends on both the current time and position. For example, $\vec{x}$ could be the position of a particle and $\vec{f}$ could be the wind vector at position $\vec{x}$ and time t.

Ex: A particle in constant wind

Let $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$ describe the position of a particle at time t, let $\vec{f}(\vec{x}, t) = \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$ be the wind at position $\vec{x}$ and time t (in this case, it is constant so we see no apparent dependence on $\vec{x}$ or t). Then let

$\vec{x}(0) = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ be the initial position of the particle. We have that the particle evolves over time by:

$$\dot{\vec{x}} = \vec{f}.$$

Specifically, $\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$ which gives us three ODEs:

$$\dot{x}(t) = V_x$$
$$\dot{y}(t) = V_y$$
$$\dot{z}(t) = V_z.$$

The solutions are independent of one another and we can simply integrate to find:

$$x(t) = V_x t + x_0$$
$$y(t) = V_y t + y_0$$
$$z(t) = V_z t + z_0$$

and thus,

$$\vec{x}(t) = t \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} + \vec{x}(0).$$

Exercise: Make sure you can find the three solutions yourself via integration. Also, plot $\vec{f}$ and plot your solution $\vec{x}$.

It's possible that we may want to look at higher order systems such as:

$$\ddot{\vec{x}} = \vec{f}(\vec{x}, \dot{\vec{x}}, t),$$

but, alas, we have the following (VERY IMPORTANT) theorem.

Theorem: All higher order ODEs are equivalent to ~some~ first order system $\dot{\vec{x}} = \vec{f}(\vec{x}, t)$.

The system may look very different! But you can always unpack it and find what you originally wanted!

Aside: This could lead you to things like: Phase space, Hamiltonian systems, Lagrangian systems, and numerical methods!

The moral of the story? It suffices to just study equations in the form:

$$\dot{\vec{x}} = \vec{f}$$

as far as ODEs go. However, there is one last point I'd like to make. Let's look at it through example.

Ex: A High Dimensional Coupled. Linear ODE

Let, $\vec{u}(t)$ be the temperature of n different particles at time t, i.e.,

$$\vec{u} = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{n-1}(t) \\ u_n(t) \end{pmatrix}, \quad \text{w/ } u_i \text{ the temp of particle } i.$$

Then we can let $\vec{f}$ describe their interaction by

$$\vec{f}(\vec{u},t) = \begin{pmatrix} K_n u_n - 2K_1 u_1 + K_2 u_2 \\ K_1 u_1 - 2K_2 u_2 + K_3 u_3 \\ \vdots \\ K_{n-2} u_{n-2} - 2K_{n-1} u_{n-1} + K_n u_n \\ K_{n-1} u_{n-1} - 2K_n u_n + K_1 u_1 \end{pmatrix}$$

This equation $\dot{\vec{u}} = \vec{f}(\vec{u},t)$ can be written as:

$$\dot{\vec{u}} = [A]\vec{u}, \quad \text{where } [A] = \begin{pmatrix} -2K_1 & K_2 & 0 & 0 & \cdots & 0 \\ K_1 & -2K_2 & K_3 & 0 & \cdots & 0 \\ 0 & K_2 & -2K_3 & K_4 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ K_1 & 0 & 0 & \cdots & K_{n-1} & -2K_n \end{pmatrix}$$

This equation is <u>linear</u> since $\vec{f}$ can be represented by a matrix. The most important thing to note is that this equation is <u>coupled</u> since:

$$\dot{u}_j = K_{j-1} u_{j-1} - 2K_j u_j + K_{j+1} u_{j+1}$$

The j^{th} particle depends on the $j-1^{st}$ and $j+1^{st}$ particle!

→ Coupled here just tells us that some u_j depends on the other values of different particles u_q.

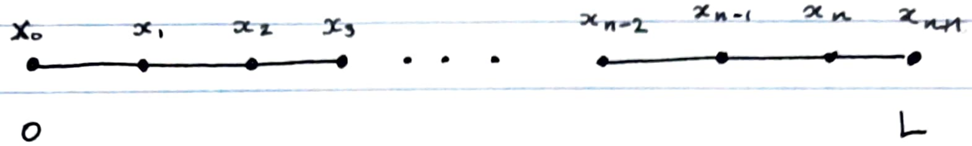
In particular, each u_j depends on its neighbor particles u_{j-1} and u_{j+1} suggesting that proximity allows for some type of interaction.

<u>The Derivation of the 1-Dimensional Heat Eqn. on a Ring.</u>

The previous work has us considering an interacting system of particles u_j. Let us consider two questions:

1. Where did this equation come from?
2. What if, instead of a discrete list of particles, we looked at the interaction and evolution of a continuous material?

Consider a circular rod of length L broken into equal segments



(Let $x_0 = x_{n+1}$)

Newton's law of cooling says that the temp. of x_j, given by the value u_j, satisfies the ODE:

$$\begin{aligned} \dot{u}_j &= -(K_j u_j - K_{j-1} u_{j-1}) - (K_j u_j - K_{j+1} u_{j+1}) \\ &= K_{j-1} u_{j-1} - 2K_j u_j + K_{j+1} u_{j+1} \end{aligned}$$

where K_j is the conductivity of the rod at position x_j.

Thus, the system follows,

$$\dot{\vec{u}} = [A]\,\vec{u}, \quad \text{with } [A] \text{ defined previously.}$$

This system merely approximates a continuous material. If we take the limit as $n \to \infty$, we then get that

$$\vec{u}(t) \mapsto u(x,t).$$

That is, we can replace a vector with n distinct components to a function that has a value at EACH position x (as opposed to just having values at discrete x_j).

Exercise: Think hard about that substitution. Make sure that it seems reasonable.

Now our task is to determine what happens to [A] as we let $n \to \infty$.

To simplify this process, let us take all $K_j = K_0$ so that the conductivity of the rod is constant. It turns out that the distance between particles affects conductivity and so we must have

$$K = \frac{K_0}{(\delta x)^2},$$

where $\delta x = x_j - x_{j-1}$, is the distance between particles. This means that,

$$\dot{u}_j = \frac{u_{j-1} - 2u_j + u_{j+1}}{(\delta x)^2}.$$

Thus, as $n \to \infty$, $\delta x \to 0$ and this tells us

$$\dot{u}_j \approx K_0 \partial^2/\partial x^2 \, u(x_j, t).$$

Moreover, we get the PDE (Partial Differential Equation):

$$\frac{\partial}{\partial t} u(x,t) = K_0 \frac{\partial^2}{\partial x^2} u(x,t).$$

We often rewrite this equation as:

$$\boxed{\left(-K_0\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\right)u(x,t) = 0}$$

This is known as the 1-Dimensional isotropic source-free linear heat equation.

- 1-Dimensional : There is only 1 spatial variable (x).
- Isotropic : Conductivity (K_0) is constant in the material.
- Source-free : The right hand side is 0 meaning that no heat is generated IN the material.
- Linear : The operator $\left(-K_0\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\right)$ is linear.

This work has answered both questions 1 & 2.

Exercise: Using the definition of the derivative as a limit, show

$$\frac{d^2f}{dx^2} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - 2f(x) + f(x-\delta x)}{(\delta x)^2}$$

in order to show the limit we took earlier is correct.

Solution:

First, note that we have the equation

$$\left(-K_0\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\right)u(x,t) = 0$$

as well as $u(0,t) = u(L,t)$ since we forced $x_0 = x_{n+1}$ in our original derivation.

Exercise: Argue why $u(0,t) = u(L,t)$ follows from $x_0 = x_{n+1}$. We also refer to the condition that $u(0,t) = u(L,t)$ as <u>periodic boundary conditions</u>.

Also, our rod must start with some initial temperature profile and so we must specify that $u(x,0) = f(0)$. This is our <u>initial condition</u> (which is just like the initial condition for an ODE.)

<u>Ex</u>: Let $f(x) = \sin\left(\frac{\pi x}{L}\right)$ and $K_0 = 1$. Then the function $u(x,t) = \sin\left(\frac{\pi x}{L}\right) e^{-\pi^2 t}$ solves the heat equation with periodic boundary conditions and the given initial conditions.

- I.C. : $u(x,0) = \sin\left(\frac{\pi x}{L}\right) = f(x)$ ✓
- B.C. : $u(0,t) = 0 = u(L,t)$ ✓
- PDE :

$$\left(-\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\right) u(x,t) = +\pi^2 \sin\left(\frac{\pi x}{L}\right) e^{-\pi^2 t} + (-\pi^2) \sin\left(\frac{\pi x}{L}\right) e^{-\pi^2 t}$$
$$= 0 \quad ✓.$$

Exercise: Plot $u(x,t)$ for $x \in [0,L]$ and $t \in [0,\infty)$.

One should interpret the result above. Essentially, we are forcing the rod to always be at zero temperature when $x=0$ and $x=L$ (which are identified). All the heat in the rod dissipates as it leaves through this region, so we see the rod cool to 0 as $t \to \infty$.

Questions :

1. Are there other boundary conditions that we may want?

   A: Yes.

2. How do we generalize to higher dimensions?

   A: Next up.

3. What if K_0 isn't constant? In other words, what if our material has an anisotropic conductivity?

   A : Next up.

4. How do we find solutions?

   A: Separation of variables.

5. Can we do this for any (physically reasonable) initial conditions? How?

   A: Yes. Fourier series.