MATH 271, Worksheet 8, *Solutions.*

Linear transformations, matrices, and linear systems.

Problem 1. Note that any linear transformation $T\colon \mathbb{R}^m \to \mathbb{R}^n$ is fully understood by its action on the vectors

$$\hat{\boldsymbol{x}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \hat{\boldsymbol{x}}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots \quad , \hat{\boldsymbol{x}}_m = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

and note that all these vectors $\hat{\boldsymbol{x}}_j \in \mathbb{R}^m$. In particular, we have

$$\begin{aligned} T(\hat{\boldsymbol{x}}_1) &= \vec{\boldsymbol{v}}_1 \\ T(\hat{\boldsymbol{x}}_2) &= \vec{\boldsymbol{v}}_2 \\ &\vdots \\ T(\hat{\boldsymbol{x}}_m) &= \vec{\boldsymbol{v}}_m, \end{aligned}$$

where the vectors $\vec{\boldsymbol{v}}_j \in \mathbb{R}^n$ and as such can be written as column vectors with n entries.

(a) As per usual, let $\hat{\boldsymbol{x}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and let $\hat{\boldsymbol{y}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Let $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$A(\hat{\boldsymbol{x}}) = 5\hat{\boldsymbol{x}} + 6\hat{\boldsymbol{y}} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

and

$$A(\hat{\boldsymbol{x}}) = 2\hat{\boldsymbol{x}} - 3\hat{\boldsymbol{y}} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

If I wanted to transform an arbitrary vector $\vec{\boldsymbol{u}} = u_1\hat{\boldsymbol{x}} + u_2\hat{\boldsymbol{y}} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, how can I use the definition of A acting on unit vectors?

(b) Determine a matrix of numbers $[A] = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ that captures this linear transformation through matrix-vector multiplication.

(c) How do the columns of $[A]$ relate to $A(\hat{\boldsymbol{x}})$ and $A(\hat{\boldsymbol{y}})$?

(d) Now, how can I think of $[A]\vec{\boldsymbol{u}}$ as describing a linear combination of the columns of $[A]$?

Solution 1.

(a) We apply A to $\vec{\boldsymbol{u}}$ and use the properties of linearity. Specifically,

$$\begin{aligned}
A(\vec{\boldsymbol{u}}) &= A(u_1\hat{\boldsymbol{x}} + u_2\hat{\boldsymbol{y}}) \\
&= A(u_1\hat{\boldsymbol{x}}) + A(u_2\hat{\boldsymbol{y}}) && \text{by property (i) of linearity} \\
&= u_1A(\hat{\boldsymbol{x}}) + u_2A(\hat{\boldsymbol{y}}) && \text{by property (ii) of linearity} \\
&= u_1(5\hat{\boldsymbol{x}} + 6\hat{\boldsymbol{y}}) + u_2(2\hat{\boldsymbol{x}} - 3\hat{\boldsymbol{y}}) && \text{by definition of } A \\
&= (5u_1 + 2u_2)\hat{\boldsymbol{x}} + (6u_1 - 3u_2)\hat{\boldsymbol{y}}.
\end{aligned}$$

(b) To determine a matrix $[A]$ that represents the transformation A, we can take

$$[A] = \begin{pmatrix} | & | \\ A(\hat{\boldsymbol{x}}) & A(\hat{\boldsymbol{y}}) \\ | & | \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 6 & -3 \end{pmatrix}.$$

(c) The columns of $[A]$ are exactly $A(\hat{\boldsymbol{x}})$ and $A(\hat{\boldsymbol{y}})$.

(d) If we perform $[A]\vec{\boldsymbol{u}}$ we can see this. In particular

$$[A]\vec{\boldsymbol{u}} = \begin{pmatrix} | & | \\ A(\hat{\boldsymbol{x}}) & A(\hat{\boldsymbol{y}}) \\ | & | \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u_1A(\hat{\boldsymbol{x}}) + u_2A(\hat{\boldsymbol{y}}),$$

which is exactly what we got in (a).

Problem 2. Repeat the steps in Problem 1 but with the transformation $B\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$B(\hat{\boldsymbol{x}}) = \hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}} \qquad \text{and} \qquad B(\hat{\boldsymbol{y}}) = -\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} - \hat{\boldsymbol{z}}.$$

Solution 2. The steps for this problem are analogous to the previous. There is no need to worry about the fact that the dimensions change with B. Just follow the same recipe!

(a) We apply B to $\vec{\boldsymbol{u}}$,

$$\begin{aligned}
B(\vec{\boldsymbol{u}}) &= B(u_1\hat{\boldsymbol{x}} + u_2\hat{\boldsymbol{y}}) \\
&= B(u_1\hat{\boldsymbol{x}}) + B(u_2\hat{\boldsymbol{y}}) && \text{by property (i) of linearity} \\
&= u_1B(\hat{\boldsymbol{x}}) + u_2B(\hat{\boldsymbol{y}}) && \text{by property (ii) of linearity} \\
&= u_1(\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}) + u_2(-\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} - \hat{\boldsymbol{z}}) && \text{by definition of } A \\
&= (u_1 - u_2)\hat{\boldsymbol{x}} + (u_1 + u_2)\hat{\boldsymbol{y}} + (u_1 - u_2)\hat{\boldsymbol{z}}.
\end{aligned}$$

(b) To determine a matrix $[B]$ that represents the transformation B, we can take

$$[B] = \begin{pmatrix} | & | \\ B(\hat{\boldsymbol{x}}) & B(\hat{\boldsymbol{y}}) \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

(c) The columns of $[B]$ are exactly $B(\hat{\boldsymbol{x}})$ and $B(\hat{\boldsymbol{y}})$.

(d) If we perform $[B]\vec{\boldsymbol{u}}$ we can see this. In particular

$$[B]\vec{\boldsymbol{u}} = \begin{pmatrix} | & | \\ B(\hat{\boldsymbol{x}}) & B(\hat{\boldsymbol{y}}) \\ | & | \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u_1B(\hat{\boldsymbol{x}}) + u_2B(\hat{\boldsymbol{y}}),$$

which is exactly what we got in (a).

Problem 3. Repeat the steps in Problem 1 but with the transformation $C\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$C(\hat{\boldsymbol{x}}) = \hat{\boldsymbol{y}}, \qquad C(\hat{\boldsymbol{y}}) = \hat{\boldsymbol{x}}, \qquad C(\hat{\boldsymbol{z}}) = \hat{\boldsymbol{x}}.$$

Solution 3. The steps for this problem are analogous to the previous. Just note that we should take $\vec{\boldsymbol{u}} \in \mathbb{R}^3$ as an input and we let $\vec{\boldsymbol{u}} = u_1\hat{\boldsymbol{x}} + u_2\hat{\boldsymbol{y}} + u_3\hat{\boldsymbol{z}}$.

(a) We apply C to $\vec{\boldsymbol{u}}$,

$$\begin{aligned}
C(\vec{\boldsymbol{u}}) &= C(u_1\hat{\boldsymbol{x}} + u_2\hat{\boldsymbol{y}} + u_3\hat{\boldsymbol{z}}) \\
&= C(u_1\hat{\boldsymbol{x}}) + C(u_2\hat{\boldsymbol{y}}) + C(u_3\hat{\boldsymbol{z}}) && \text{by property (i) of linearity} \\
&= u_1C(\hat{\boldsymbol{x}}) + u_2C(\hat{\boldsymbol{y}}) + u_3C(\hat{\boldsymbol{z}}) && \text{by property (ii) of linearity} \\
&= u_1(\hat{\boldsymbol{y}}) + u_2(\hat{\boldsymbol{x}}) + u_3(\hat{\boldsymbol{x}}) && \text{by definition of } A \\
&= (u_2 + u_3)\hat{\boldsymbol{x}} + (u_1)\hat{\boldsymbol{y}}.
\end{aligned}$$

(b) To determine a matrix $[C]$ that represents the transformation C, we can take

$$[C] = \begin{pmatrix} | & | & | \\ C(\hat{\boldsymbol{x}}) & C(\hat{\boldsymbol{y}}) & C(\hat{\boldsymbol{z}}) \\ | & | & | \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

(c) The columns of $[C]$ are exactly $C(\hat{\boldsymbol{x}})$, $C(\hat{\boldsymbol{y}})$, and $C(\hat{\boldsymbol{z}})$.

(d) If we perform $[C]\vec{\boldsymbol{u}}$ we can see this. In particular

$$[B]\vec{\boldsymbol{u}} = \begin{pmatrix} | & | & | \\ C(\hat{\boldsymbol{x}}) & C(\hat{\boldsymbol{y}}) & C(\hat{\boldsymbol{z}}) \\ | & | & | \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = u_1B(\hat{\boldsymbol{x}}) + u_2B(\hat{\boldsymbol{y}}) + u_3B(\hat{\boldsymbol{z}}),$$

which is exactly what we got in (a).

Problem 4. Let

$$[M] = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad [P] = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad [Q] = \begin{pmatrix} 2 & 1 \end{pmatrix} \quad [R] = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad [S] = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}$$

(a) Compute the following matrix products (when possible) and state which multiplications are not possible.

$$[M][M], \qquad [P][P], \qquad [Q][P], \qquad [M][S], \qquad [S][M].$$

(b) Compute the following:

i. $[A] = [P][Q]$;

ii. $[B] = [Q]^T[P]^T$. Is this equal to $([P][Q])^T$?

iii. $[C] = [M][R] - [R][M]$. Do these matrices commute?

Solution 4.

(a) We will go through these and compute the multiplications as necessary.

- $[M][M]$ is possible since $[M]$ is square and we are multiplying it times itself. We get
$$[M][M] = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}.$$
- $[P][P]$ is not possible since $[P]$ is a 2×1 matrix and we cannot multiply a 2×1 by a 2×1.
- $[Q][P]$ is possible since it is a 1×2 matrix multiplied times a 2×1 matrix. We get
$$[Q][P] = \begin{pmatrix} 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \end{pmatrix}.$$
Since $[Q][P]$ is a 1×1 matrix, we usually drop the parentheses and just write $[Q][P] = 3$.
- $[M][S]$ is possible since we are multiplying a 2×2 with a 2×3. We get
$$[M][S] = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 9 & 9 & 9 \\ 3 & 3 & 3 \end{pmatrix}.$$
- $[S][M]$ is not possible since it is a 2×3 times a 2×2.

(b) We will compute each of the next products taking into account that for some matrix $[T]$ with elements t_{ij} we have the transpose $[T]^T$ with components t_{ji}. That is, we swap rows for columns when we transpose a matrix.

i. We have

$$[A] = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}.$$

ii. First note

$$[Q]^T = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \text{and} \qquad [P]^T = \begin{pmatrix} 1 & 1 \end{pmatrix}.$$

Thus,

$$[Q]^T[P]^T = 3.$$

Yes, this is equal to $([P][Q])^T$, and this fact is true in general.

iii. We take

$$[M][R] = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$[R][M] = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}.$$

Then

$$[M][R] - [R][M] = \begin{pmatrix} 2 & -2 \\ 0 & -2 \end{pmatrix}.$$

Problem 5. The linear transformation $H\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$H(\hat{\boldsymbol{x}}) = \hat{\boldsymbol{y}} \qquad \text{and} \qquad H(\hat{\boldsymbol{y}}) = \hat{\boldsymbol{x}},$$

has some nice properties.

(a) In some sense, H is the square root of 1 in that $H^2 = H \circ H = 1$. Show that this is true.

(b) Write down a matrix representation for H and denote it by $[H]$.

(c) Consider a linear combination of matrices

$$[\eta] = x[I] + y[H],$$

where $[I]$ is the 2×2 identity matrix. Compute $[\eta]^2$.

Solution 5.

(a) Let $\vec{\boldsymbol{u}} = u_1\hat{\boldsymbol{x}} + u_2\hat{\boldsymbol{y}}$ be an arbitrary vector in $\mathbb{R}^2$. Then

$$H^2(\vec{\boldsymbol{u}}) = H(H(\vec{\boldsymbol{u}})) = H(H(u_1\hat{\boldsymbol{x}} + u_2\hat{\boldsymbol{y}})) = H(u_2\hat{\boldsymbol{x}} + u_1\hat{\boldsymbol{y}}) = u_1\hat{\boldsymbol{x}} + u_2\hat{\boldsymbol{y}},$$

so indeed H^2 acts like multiplication by 1.

(b) We can construct a matrix representation as we did in earlier problems. Namely,

$$[H] = \begin{pmatrix} | & | \\ H(\hat{\boldsymbol{x}}) & H(\hat{\boldsymbol{y}}) \\ | & | \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(c) We have

$$[\eta]^2 = (x[I] + y[H])^2 = x^2[I]^2 + xy[I][H] + xy[H][I] + y^2[H]^2.$$

Now, using what we know about $[I]$ in that $[I][H] = [H][I] = [H]$, $[I]^2 = [I]$, and that $[H]^2 = [I]$ by (a), we have

$$[\eta]^2 = (x^2 + y^2)[I] + 2xy[H].$$

Note that one can explicitly show $[H]^2 = [I]$ using the matrix representation, but the work from (a) suffices since the matrix behaves analogously to the original linear transformation (which is the point!).

Problem 6. Consider the system of linear equations:

$$\begin{aligned} x + 2y &= 3 \\ x + y &= 3 \end{aligned}$$

(a) Write this system in the form:

$$[A]\vec{\boldsymbol{x}} = \vec{\boldsymbol{y}}$$

(b) Row reduce to find a the solution $\vec{\boldsymbol{x}}$.

Solution 6.

(a) Let $\vec{\boldsymbol{x}} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\vec{\boldsymbol{y}} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$, and note that this yields $[A] = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$. One can explicitly determine $[A]$ by simply computing the elements a_{ij} by taking

$$[A]\vec{\boldsymbol{x}} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ x + y \end{pmatrix}.$$

(b) Now, we create the augmented matrix

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 1 & 1 & 3 \end{array} \right).$$

Subtract R1 from R2 to get

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -1 & 0 \end{array} \right).$$

Add 2R2 to R1 to get

$$\left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & -1 & 0 \end{array} \right).$$

Finally, we can multiply R2 by -1 (which is not totally necessary) and get

$$\left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 0 \end{array} \right).$$

The solution can now be read off from the last column to yield $x = 3$ and $y = 0$.

Problem 7. Consider the system of linear equations:

$$\begin{aligned} 3x + 2y + 0z &= 5 \\ 1x + 1y + 1z &= 3 \\ 0x + 2y + 2z &= 4. \end{aligned}$$

(a) Write the augmented matrix M for this system of equations.

(b) Use row reduction to get the augmented matrix in row-echelon form.

(c) Determine the solution to the system of equations.

Solution 7.

(a) The augmented matrix for this system of equations is

$$[M] = \left(\begin{array}{ccc|c} 3 & 2 & 0 & 5 \\ 1 & 1 & 1 & 3 \\ 0 & 2 & 2 & 4 \end{array}\right).$$

(b) To row reduce, we start by dividing row 3 (R3) by 2 to get

$$\left(\begin{array}{ccc|c} 3 & 2 & 0 & 5 \\ 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \end{array}\right).$$

Then, we can subtract R1 from R2 to get

$$\left(\begin{array}{ccc|c} 3 & 2 & 0 & 5 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right).$$

Next, we can subtract 3R2 from R1 to get

$$\left(\begin{array}{ccc|c} 0 & 2 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right).$$

Now, divide R1 by 2,

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right).$$

Subtract R1 from R3

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

This step is a bit unecessary, but we can swap R1 and R2

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

(c) Thus, from the row-echelon form, we arrive at the solution $x = 1$, $y = 1$, and $z = 1$.

Problem 8. Consider the equation

$$\begin{pmatrix} 1 & 3 & 4 \\ 2 & 9 & 9 \\ 1 & 5 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 20 \\ 11 \end{pmatrix}.$$

Does this equation have a solution or not? If so, determine the solution.

Solution 8. We can see whether this equation has a solution by row reducing. Later on, we will be able to use the *determinant* to determine this more quickly. At any rate, we have the augmented matrix

$$[M] = \left(\begin{array}{ccc|c} 1 & 3 & 4 & 8 \\ 2 & 9 & 9 & 20 \\ 1 & 5 & 5 & 5 \end{array} \right).$$

I skip the steps here, but one can row reduce to

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \text{-54} \\ 0 & 1 & 0 & \text{-10} \\ 0 & 0 & 1 & 23 \end{array} \right).$$

Since we achieved our goal of having the right half of the reduced matrix having 1's along the diagonal (it is an identity matrix), we do indeed have a solution. In particular, the solution is $x = -54$, $y = -10$, and $z = 23$. One can check this by performing the matrix multiplication posed in the statement of this problem.

Problem 9. Consider the matrix

$$[A] = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}.$$

Determine the nullspace of $[A]$.

Solution 9. The nullspace of $[A]$, $\mathrm{Null}([A])$ consists of all the vectors $\vec{\boldsymbol{x}}$ such that

$$[A]\vec{\boldsymbol{x}} = \vec{\mathbf{0}}.$$

Thus, we are seeking to solve the homogeneous equation, or, in other words, the system of equations

$$\begin{aligned} 3x + y &= 0 \\ 6x + 2y &= 0. \end{aligned}$$

So, we create the augmented matrix

$$\left(\begin{array}{cc|c} 3 & 1 & 0 \\ 6 & 2 & 0 \end{array}\right).$$

Notice that R2 is equal to 2R1 and so we can subtract 2R1 from R2 to get

$$\left(\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Indeed, we cannot further reduce this matrix. However, we can see that the new system of equations is

$$\begin{aligned} 3x + y &= 0 \\ 0x + 0y &= 0. \end{aligned}$$

The first equation says that $y = -3x$ while the second says x and y are free to be anything. So, we are free to choose x and y but we are subject to the constraint from the first equation. In particular, we can take $x = 1$ to yield $y = -3$, and so one solution in particular is

$$\vec{\boldsymbol{x}} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Notice that if I take a constant α, then $\alpha\vec{\boldsymbol{x}}$ is also a solution. To see this, we take

$$[A](\alpha\vec{\boldsymbol{x}}) = \alpha[A]\vec{\boldsymbol{x}} = 0,$$

due to the linearity of $[A]$. So, the nullspace is the set of vectors $\begin{pmatrix} \alpha \\ -3\alpha \end{pmatrix}$ for any constant $\alpha \in \mathbb{R}$.