MATH 271, Worksheet 5
Sequences and Series

Problem 1. Write down the first few terms in the sequence for the following:

(a) $a_n = n$;

(b) $b_n = \frac{1}{n^2}$;

(c) $c_n = 2^{-n}$.

Problem 2. For the above sequences, state whether each converges or diverges. If they converge, state the limit.

Problem 3. Consider the recursive sequence

$$a_n = \frac{1}{2}a_{n-1} + 1$$

with $a_1 = 1$.

(a) Write the first few terms in the sequence.

(b) Can you write a_n as a function $f(n)$? If so, what is $f(n)$?

(c) Does this sequence converge or diverge? Can you show why with a limit $\lim_{n\to\infty} f(n)$?

(d) Can you show that this is a Cauchy sequence?

Problem 4. Consider the sequence

$$a_n = ar^n.$$

(a) If $|r| < 1$, show that this sequence $\{a_n\}_{n=0}^{\infty}$ converges to zero.

(b) Consider now the *geometric series*

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} ar^n.$$

Show that the N^{th} partial sum for this series satisfies

$$\sum_{n=0}^{N} ar^n = a\left(\frac{1-r^{N+1}}{1-r}\right).$$

(c) Does the geometric series converge for all r? For $|r| < 1$? When it converges, what does it converge to?

Problem 5. Often we wish to think about functions being represented by series. For example, we can consider the function

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

where $n!$ is read as "n-factorial" and

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1.$$

Then $1! = 1$ and we define $0! = 1$ as well.

(a) Consider $f(1)$. Use a tool like WolframAlpha to compute the series

$$f(1) = \sum_{n=0}^{\infty} \frac{1}{n!}$$

(b) For any value of x, this series converges. So this defines a function on all real numbers. In fact, the series converges even for complex numbers. Simplify the series into its real and imaginary parts. Note,

$$f(ix) = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}.$$

(c) We can take derivatives of the function $f(x)$ by differentiating the series *term by term.* That is,

$$\frac{d}{dx} f(x) = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{d}{dx} \left(\frac{x^n}{n!} \right).$$

(d) Show that $\frac{d}{dx} f(x) = f(x)$.

(e) What is your guess for what function $f(x)$ is?

Problem 6. Find the radius of convergence for the following power series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$;

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$;

(c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^n$;

(d) Repeat (c) but for $z \in \mathbb{C}$ instead of $x \in \mathbb{R}$.

(e) How does the convergence in (a) and (b) compare with the convergence of the typical p-series?