Questions:

1. Are there other boundary conditions that we may want?

   A: Yes.

2. How do we generalize to higher dimensions?

   A: Next up.

3. What if K_0 isn't constant? In other words, what if our material has an ~~anisotropic~~ nonuniform conductivity?

   A: Next up.

4. How do we find solutions?

   A: Separation of variables.

5. Can we do this for any (physically reasonable) initial conditions? How?

   A: Yes. Fourier series.

More General Heat Equation

The equation we arrived at last time was for a very special case. Specifically, it was for heat flowing on a ring with a single heat sink and no internal heat sources. Moreover, we had constant conductivity.

In general, our spatial variables will be in 3-dimensions, so our temperature profile is given by $u(x, y, z, t) = u(\vec{x}, t)$.

When $K(x,y,z)$ depends on what position we look at, we can't treat the material as having constant conductivity. Ultimately, we arrive at

$$-\vec{\nabla}\cdot\left(K(\vec{x})\vec{\nabla}u(\vec{x},t)\right)+\frac{\partial}{\partial t}u(\vec{x},t)=f(\vec{x},t)$$

which is the most general form of the heat equation we will look at. Note that $\vec{\nabla}\cdot$, and $\vec{\nabla}$ only act as derivatives with respect to the spatial variables. Here as well we can think of $f(\vec{x},t)$ as a spatio-temporally varying heat source term. It could be that heat is supplied to our substance in a way that varies over both space and time.

Separation of Variables:

The technique we will cover for solving PDEs is known as separation of variables. Fundamentally, in 1-dimension, we will take the ansatz that our solution function u can be broken up as

$$u(x,t)=X(x)T(t).$$

This approach works very well for certain equations and domains. When using this ansatz, we will be able to separate our PDE into multiple ODEs. Hence the name. Explaining the method is not as easy as seeing it done, so we will work through an example to see the approach.

Ex: Consider the following problem.

$$\begin{cases} \text{Region}: & x \in \Omega = [0,1] \\ \text{PDE}: & \left(-K\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\right) u(x,t) = 0, \quad K>0 \\ \text{B.C}: & u(0,t) = u(1,t) = 0 \quad (\text{Dirichlet}) \\ \text{I.C}: & u(x,0) = \sin(\pi x). \end{cases}$$

Take the ansatz $u(x,t) = X(x)T(t)$. Then we can plug this into the PDE.

$$0 = \left(-K\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\right) X(x)T(t) = -KX''T + XT'$$

$$\Rightarrow \quad 0 = -KX''T + XT'$$

$$\Rightarrow \quad K\,X''/X = T'/T$$

Now, we have a function of x on the left and a function of t on the right. Thus, it must be that each side is equal to some (unknown) constant λ. Thus,

$$\Rightarrow K\,X''/X = \lambda \quad \text{and} \quad T'/T = \lambda.$$

We refer to this λ as the separation constant.

Both of these equations are ODEs we know how to solve. Let's solve the equation for T first.

$$T'/T = \lambda \quad \Rightarrow \quad T' = \lambda T$$

which has solutions $T(t) = Ce^{\lambda t}$. Note that unless $C = 0$, this function is always nonzero.

Next, we have $\kappa\, X''/X = \lambda \Rightarrow \kappa X'' - \lambda X = 0$ which is the ~~lin~~ second order linear ODE with constant coefficients we spent time analyzing. This has the solution

$$X(x) = Ae^{\sqrt{\frac{\lambda}{\kappa}}\,x} + Be^{-\sqrt{\frac{\lambda}{\kappa}}\,x}.$$

Thus, we have found that

$$\rightarrow \quad u(x,t) = e^{\lambda t}\left(Ae^{\sqrt{\frac{\lambda}{\kappa}}\,x} + Be^{-\sqrt{\frac{\lambda}{\kappa}}\,x}\right).$$

Next, we can apply the Dirichlet boundary conditions and determine more for our solution. Specifically,

$$0 = u(0,t) = e^{\lambda t}(A+B) \quad \Rightarrow \quad B = -A$$

Then

$$0 = u(1,t) = e^{\lambda t}\left(Ae^{\sqrt{\frac{\lambda}{\kappa}}} - Ae^{-\sqrt{\frac{\lambda}{\kappa}}}\right)$$

Now, ~~th~~ there are nontrivial solutions to this equation if $\lambda \leq 0$. If $\lambda > 0$, then A must be 0 and we fail to match our initial condition!

If $\lambda = 0$, we also fail to match our initial condition, hence $\lambda < 0$, and so we can rewrite our solution as,

$$X(x) = A\sin\left(\sqrt{\tfrac{\lambda}{k}}\,x\right) + B\cos\left(\sqrt{\tfrac{\lambda}{k}}\,x\right)$$

Applying boundry conditions again yields,

$$0 = u(0,t) = e^{\lambda t}\left(B\cos(0)\right) \Rightarrow B = 0$$

and

$$0 = u(1,t) = e^{\lambda t}\left(A\sin\left(\sqrt{\tfrac{\lambda}{k}}\right)\right) \Rightarrow \sqrt{\tfrac{\lambda}{k}} = n\pi,$$ for

any positive integer n. Solving for λ yields

$$\lambda = -k n^2 \pi^2 \quad (\text{since } \lambda < 0 \text{ and } k > 0)$$

and thus

$$u(x,t) = A e^{-kn^2\pi^2} \sin(n\pi x)$$

is a solution to the PDE and B.C.'s for any positive integer n (and indeed a sum of these solutions is also a solution.)

To match our initial condition, we need

$$u(x,0) = Ae^{0}\sin(n\pi x) = \sin(\pi x)$$

$$\Rightarrow n = 1. \quad \text{and} \quad A = 1.$$

Finally, it must be that our particular solution is

$$\boxed{u(x,t) = e^{-K\pi^2 t}\sin(\pi x)}$$

Poisson's (Laplace's) Equation

As time goes to infinity in a solution to the heat equation, we are met with a steady state solution. Take our previous example, we had

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} e^{-K\pi^2 t}\sin(\pi x) = 0 = u_E(x).$$

We can think of this limit $\lim_{t\to\infty} u(x,t) = u_E(x)$ as the <u>equilibrium solution</u>. This function $u_E(x)$ describes the temperature profile of the rod after we allow heat flow for a very long time.

Let us disect the heat equation a bit. Say we have a solution $v(x,t)$ and an equilibrium solution $u_E(x)$, then notice if we take

$$\left(-K\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\right)\left(v(x,t) + u_E(x)\right) = 0$$

then this means that

$$-K \frac{\partial^2}{\partial x^2} u_E(x) = 0,$$

which is the 1-dimensional Laplace equation. One should also be careful here as we have not included boundary or initial conditions, which do affect what the solutions $V(x,t)$ and $u_E(x)$ will look like. The salient fact here is that we can break up our solutions and force part of this ~~the~~ solution to solve an equation independent of time.

If we took this with our most general form,

$$\left(-\vec{\nabla}\cdot(K\vec{\nabla}) + \frac{\partial}{\partial t}\right) u(\vec{x},t) = f(\vec{x},t),$$

we can hope to break apart ~~the~~ $u(\vec{x},t)$ into two pieces again. Indeed, if we let $f(\vec{x})$ only depend on space, then substitute that on the right hand side, we can find a solution

$$u(\vec{x},t) = V(\vec{x},t) + u_E(\vec{x})$$

where

$$-\vec{\nabla}\cdot(K(\vec{x})\vec{\nabla}u_E(\vec{x})) = f(\vec{x}).$$

Now, if our equation comes with boundary conditions, we have that $u_E(\vec{x})$ satisfies those boundary conditions (if they are not time dependent) and $v(\vec{x},t)$ satisfies constant 0 ~~[illegible]~~ boundary conditions. We will talk about boundary conditions in the future.

Ex: Take the following problem with $\Omega = [0,1]$.

$$\begin{cases} \text{PDE:} & \left(-K \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\right) u(x,t) = 0 \\ \text{I.C.:} & u(x,0) = \cos(2\pi x) + 1 \\ \text{B.C:} & \frac{\partial u(0,t)}{\partial x} = 0 = \frac{\partial}{\partial x} u(1,t) \qquad \text{(Neumann.)} \end{cases}$$

We will suppose that we have

$$u(x,t) = v(x,t) + u_E(x)$$

where $v(x,t)$ satisfies the PDE and $v(0,t) = 0 = v(1,t)$ and ~~match the initial conditions~~. Then we let

$$-K \frac{\partial^2}{\partial x^2} u_E(x) = 0$$

where $u_E(x)$ satisfies the original given boundary conditions.

Note that $v(x,t) =$ ~~$v(x,t) = (\cos(2\pi x) - t)e$~~ satisfies the PDE, and not the zero boundary conditions, ~~and the [illegible]~~ then we can take

$$u_E(x) = Ax + B$$

as a general solution. Applying the Neumann boundary conditions, we get

$$u_E(x) = B$$

Then we know that we have

$$u(x,t) = e^{\lambda t}\left(A\sin\left(\sqrt{\tfrac{\lambda}{k}}\,x\right) + B\sin\left(\sqrt{\tfrac{\lambda}{k}}\,x\right)\right).$$

Applying the Neumann boundary conditions, we get that

$$u_n(x,t) = e^{-kn^2\pi^2 t}\left(a_n \cos(n\pi x)\right)$$

is a solution for each integer n. Hence, our solution can be written as:

$$u(x,t) = \sum_{n=0}^{\infty} a_n e^{-kn^2\pi^2 t} \cos(n\pi x).$$

Then we note $u(x,0) = \cos(2\pi x) + 1$ which means that $a_0 = 1$, $a_2 = 1$, and all other $a_j = 0$. Thus

$$\boxed{u(x,t) = e^{-k4\pi^2 t}\cos(2\pi x) + 1}$$

is our solution. Indeed, $v(x,t) = e^{-k\pi^2 t}\cos(2\pi x)$ satisfies 0 Neumann boundary conditions and $u_E(x) = 1$ satisfies our Neumann boundary conditions and

$$-k\frac{\partial^2}{\partial x^2}u_E(x) = 0.$$

<u>Exercise:</u> Plot and interpret this solution $u(x,t)$. How does it differ from our previous solution that had different boundary conditions?

Boundary Conditions:

Boundary conditions play a massive role in the structure of our solution to a PDE. The previous two examples both use the same underlying PDE, but the change in b.c. gives the solutions different characteristics. In those examples we had

1. (Dirichlet) $u(0,t) = 0, \quad u(1,t) = 0.$
2. (Neumann) $\frac{\partial}{\partial x} u(0,t) = 0, \quad$ and $\quad \frac{\partial}{\partial x} u(1,t) = 0.$

In 1, we specified a specific value the solution had to take on the boundary of the interval. In terms of the heat equation, this is like forcing a specific temperature at the ends of the rod. In particular we held the rod to 0 degrees.

In 2, we specified the value that the solution's derivative had to take on the boundary. This physically meant that we specified the heat flux at the boundary. In particular we forced this heat flux to be zero. Can you see why we ended up with the equilibrium solution that we got for both of these cases?

In general, we are given a region Ω in space with a boundary surface Σ. We then state that our solution $u(\vec{x},t)$ must satisfy the PDE inside of Ω and that $u(\vec{x},t)$ must satisfy conditions on the boundary surface Σ. These types of conditions come in two types. ~~&~~ In actuality, there are three types, but we will not study Robin conditions.

1. Dirichlet B.C.

- This type of B.C. forces our solution $u(\vec{x},t)$ to match a specific function on Σ. Specifically, on Σ, $u(\vec{x},t) = f(\vec{x},t)$.

- This is like forcing a specific temp on the boundary surface at each point.

- for our earlier example we forced ~~either~~ $u(x,t) = 0$ when $x=0$ or $x=1$ which are the boundary points for $\Omega = [0,1]$.

2. Neumann B.C.

- This type of B.C. forces our solution's $u(\vec{x},t)$ ~~data~~ flux through the boundary surface. Specifically, on Σ we force

$$\frac{\partial u}{\partial \hat{n}} := (\vec{\nabla} u) \cdot \hat{n} = g(\vec{x},t),$$

where $\hat{n}$ is the surface normal.

- This is like forcing heat flux (or flow) through the boundary.

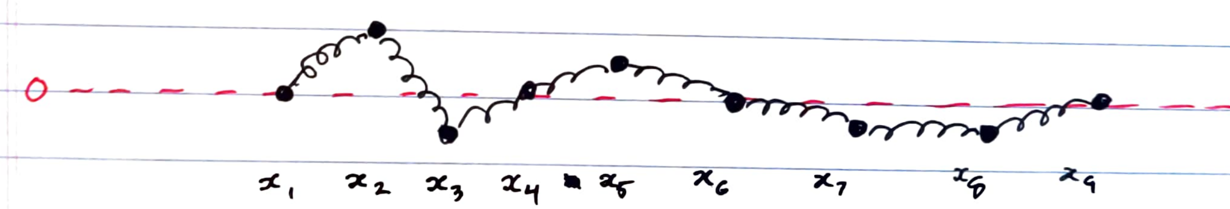
- In our previous example, we had

$$\frac{\partial}{\partial x} u(0,t) = 0 = \frac{\partial}{\partial x} u(1,t)$$

which ~~are~~ is the flux through the boundary points $x=0$, $x=1$. This specifically meant that we allowed <u>no</u> heat flow through the boundary of the rod.

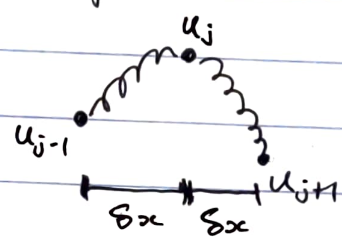
<u>Wave Equation:</u>

Suppose we have a system of n particles connected with springs. For simplicity, we will assume that each particle is of the same mass, and each spring is the same stiffness.



In the above picture, we let the horizontal position be given by x, and we choose each particle equally spaced. Then, ~~we wish for~~ we set the height of particle j to be u_j.

The force on the j^{th} particle will then tell us how that particle accelerates. Again, we will want to take the limit as $n \to \infty$ so that we end up with a continuous elastic material. In this case, the particles are very close in x position, which allows us to say the following.



δx is very small.

$\Rightarrow$ distance between particle x_{j-1} and x_j is approximately $|u_j - u_{j-1}|$.

Thus the force on particle x_j is

$$
\begin{aligned}
F_j &= -K(u_j - u_{j-1}) - K(u_j - u_{j+1}) \\
&= \cancel{m} K(u_{j-1} - 2u_j + u_{j+1})
\end{aligned}
$$

Thus, by Newton's 2^{nd} law states

$$m\ddot{u}_j = F_j = \cancel{m} K(u_{j-1} - 2u_j + u_{j+1})$$

$$\Rightarrow \ddot{u}_j = \frac{K}{m}(u_{j-1} - 2u_j + u_{j+1})$$

Assuming the total length is L, and we have n masses, then $L = n(\delta x)$. The total mass $M = nm$ and the total spring constant is $\bar{K} = K/n$. Thus we arrive at

$$\ddot{u}_j = \frac{KL^2}{M}\left(\frac{u_{j-1} - 2u_j - u_{j+1}}{(\delta x)^2}\right).$$

Taking $n \to \infty$ yields,

$$\frac{\partial^2}{\partial t^2} u(x,t) = \frac{KL^2}{M} \frac{\partial^2}{\partial x^2} u(x,t).$$

This is known as the <u>1-dimensional</u> <u>uniform</u> <u>linear</u> <u>wave equation</u>. Often, we write this as

$$\left(-\frac{\partial^2}{\partial x^2} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u(x,t) = 0$$

~~d'Alembertis~~ wave operator.
d'Alembert's (or d'Alembertian)

In higher dimensions, we have

$$\left(-\Delta + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u(\vec{x},t) = 0,$$

where Δ is the Laplace operator acting only on the spatial variables.

One could make this equation more complex by making the material non-uniform (i.e., different elastic behavior at different points) or add a forcing term to the right hand side.

<u>Ex</u>: ... in typed notes and HW.

Ex: 1-D wave equation w/ Mixed B.C.

Take the 1-dimensional wave equation

$$\left(-\frac{\partial^2}{\partial x^2} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u(x,t) = 0$$

on $\Omega = [0,1]$, with B.C. $u(0,t) = 0$ and $\frac{\partial}{\partial x} u(1,t) = 0$.
Note that we have both Dirichlet and Neumann conditions. We often refer to this as half open half closed conditions. Performing separation of variables, one can find the general solutions

$$u_n(x,t) = \left(A\sin\left(\tfrac{2n+1}{2}\pi c t\right) + B\cos\left(\tfrac{2n+1}{2}\pi c t\right)\right)\sin\left(\tfrac{2n+1}{2}\pi x\right)$$

for every integer n.

Now, if we were given initial conditions

$$u(x,0) = f(x)$$

and

$$\frac{\partial}{\partial t} u(x,0) = g(x),$$

we could find a particular solutions. Note that we do need two initial conditions here. One for the initial position of the elastic rod and the other for the initial velocity. This solution is what you would find for an elastic rod attached in place at one end and free to move on the other. This is also much like what a sound wave in a pipe organ is like.

d'Alembert's Idea:

If you have ever played with a slinky, you ~~ha~~ may have tried holding both ends and then quickly moved ~~to~~ one end up and down while holding the other steady. If you do this (possibly with a friend), you will see that a wave travels through the slinky and returns back while upside down. This is often called a traveling wave. There is also the ability to compress the slinky and see a compression wave travel through as well. Traffic and sound are other examples of compression waves.

If we take our 1-dimensional wave equation,

$$\left(-\frac{\partial^2}{\partial x^2} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u(x,t) = 0$$

then we can define new variables,

$$\xi = x - ct \quad \text{and} \quad \eta = x + ct.$$

Then the differential of each is

$$d\xi = dx - c\,dt \quad \text{and} \quad d\eta = dx + c\,dt.$$

From this, it follows that

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t} \quad \text{and} \quad \frac{\partial}{\partial \eta} = \frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}.$$

Then one can note that

$$\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} = -\frac{\partial^2}{\partial x^2} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

is our original wave operator! Thus, the wave equation is analogous to solving

$$\frac{\partial^2}{\partial \xi \partial \eta} u(\xi, \eta) = 0$$

which has the general solution

$$u(\xi, \eta) = F(\xi) + G(\eta).$$

{ Exercise: Show that the above solution is really a solution

In our original variables, this gives us that

$$u(x,t) = F(x-ct) + G(x+ct)$$

which shows that our solution $u(x,t)$ is broken into two components: $F(x-ct)$ represents a wave traveling to the right and $G(x-ct)$ represents a wave traveling to the left.

Ex: Wave solution on infinite elastic material

Let $\Omega = \mathbb{R}$ and consider

$$\left(-\frac{\partial^2}{\partial x^2} + \frac{1}{c^2}\frac{\partial^2}{\partial x^2}\right) u(x,t) = 0.$$

If we are given I.C.

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial}{\partial t} u(x,0) = g(x)$$

then d'Alembert's formula states that our solution is

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

Note that we do not have any boundaries on Ω so there are no boundary conditions.

For the sake of simplicity, suppose

$$f(x) = e^{-x^2} \quad \text{and} \quad g(x) = 0.$$

Then we have

$$u(x,t) = \frac{1}{2}\left(e^{-(x-ct)^2} + e^{-(x+ct)^2}\right)$$

is our particular solution.

Exercise: Plot the solution to the wave equation given above.

Domains Without Boundary:

In the last example we took our domain to be $\Omega = \mathbb{R}$. In that case, Ω has no boundaries! Thus, we cannot always expect a PDE to come with boundary conditions.

When we are considering problems defined over all space $\mathbb{R}^3$, this is fairly typical. One restriction we may impose is that our solution decays (quickly enough) to zero as we go to infinity.

Another option is a domain that is ring shaped or a domain that is the surface of a sphere. Here, we have a form of restrictions that look much like boundary conditions. For example, if we are looking to solve the heat equation on a ring (as we began with), if the ring has circumference L, then we require that

$$u(0,t) = u(L,t)$$

and possibly

$$\frac{\partial}{\partial x} u(0,t) = \frac{\partial}{\partial x} u(L,t).$$

We refer to these as periodic boundary conditions.
But, you should think of these as forcing a continuous function on all of the domain.