MATH 271, Exam 3

Oral Examination Problems

Due one hour before your exam time slot.

Instructions You are allowed a textbook, homework, notes, worksheets, material on our Canvas page. You can use online tools such as Desmos and Wolfram Alpha to check your work, but you will need to explain how you arrived at your answers. You can work with other students and this is, in fact, encouraged! However, I will not be giving out direct help for these problems but can answer questions about previous problems and notes, for example. Ambiguous or illegible answers will not be counted as correct. Scan your solutions and submit them as a pdf on Canvas under Oral Exam 3.

Note, there are four total problems.

Problem 1. Consider the following vectors in $\mathbb{R}^2$:

$$\vec{\boldsymbol{u}} = \hat{\boldsymbol{x}} - 3\hat{\boldsymbol{y}} \quad \vec{\boldsymbol{v}} = -2\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}} \quad \vec{\boldsymbol{w}} = -\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}.$$

(a) Draw all vectors $\vec{\boldsymbol{u}}$, $\vec{\boldsymbol{v}}$, and $\vec{\boldsymbol{w}}$ in the plane. Draw $\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}$ in the plane as well.
(b) Are any of these vectors orthogonal? Explain.
(c) Explain why $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ form a basis for $\mathbb{R}^2$.
(d) Given the vector $\vec{\boldsymbol{y}} = 13\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}$, write $\vec{\boldsymbol{y}}$ as a linear combination of $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$.

Problem 2. Consider the vectors $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \in \mathbb{R}^3$ given by

$$\vec{\boldsymbol{u}} = -\hat{\boldsymbol{x}} - 2\hat{\boldsymbol{y}} \qquad \text{and} \qquad \vec{\boldsymbol{v}} = 3\hat{\boldsymbol{z}}.$$

(a) Compute $\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}$.

(b) Given any $\vec{\boldsymbol{w}} \in \mathbb{R}^3$ with $\vec{\boldsymbol{w}} = w_1\hat{\boldsymbol{x}} + w_2\hat{\boldsymbol{y}} + w_3\hat{\boldsymbol{z}}$ we can create the matrix

$$[\vec{\boldsymbol{w}}] = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}.$$

Show that

$$[\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}] = [\vec{\boldsymbol{u}}][\vec{\boldsymbol{v}}] - [\vec{\boldsymbol{v}}][\vec{\boldsymbol{u}}].$$

Problem 3. Consider the matrices

$$[A] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad [B] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [C] = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

(a) Explain what the transformation $[A]$ does to the basis vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$.
(b) Find $[A]^{-1}$.
(c) Find the volume of the parallapiped generated by the columns of $[C]$.
(d) Find $\mathrm{Null}([B])$.

Problem 4. Consider the same matrix

$$[A] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(a) Is $[A]$ hermitian? Explain.

(b) Show that $\hat{\boldsymbol{x}}$ is an eigenvector with eigenvalue $\lambda_1 = 1$.

(c) Compute $\det([A])$ and $\mathrm{tr}([A])$ and using these quantities plus your knowledge from (b), show that the other two eigenvalues are $\lambda_2 = 1$ and $\lambda_3 = -1$. (DO NOT USE THE CHARACTERISTIC POLYNOMIAL!)

(d) Find the remaining eigenvectors for $[A]$ and diagonalize $[A]$.