MATH 272, Homework 6

Due April 29th

Problem 1. (11 pts) Are you sure you understand what constitutes a vector space? What about an inner product? Let's see a few examples. Please work through each part of the question.

Given an a vector space V with an inner product $\langle -, - \rangle$, we can always define a *norm* (or *energy*) by taking $v \in V$ and putting

$$\|v\|^2 = \langle v, v \rangle.$$

P.S. both l's seen here are short for Henri Lebesgue.

(a) **(3 pts)** (Finite dimensional inner product space) Consider the vector space $\mathbb{C}^3$ with the Hermitian inner product $\langle -, - \rangle$.

- Compute the norm of

$$\vec{\boldsymbol{u}} = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}.$$

- Compute the norm of

$$\vec{\boldsymbol{v}} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}.$$

- Compute the inner product $\langle \vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \rangle$.
- Provide an example of a basis for $\mathbb{C}^3$.

(b) **(4 pts)** (Countably infinite dimensional inner product space) Consider the space of *square summable* (*finite energy*) sequences $\ell^2(\mathbb{C})$. That is, an element of the $\ell^2(\mathbb{C})$ is a sequence of complex numbers $\{a_n\}_{n=0}^{\infty}$ with an inner product defined by

$$\langle \{a_n\}, \{b_n\} \rangle = \sum_{n=0}^{\infty} a_n^* b_n.$$

and we require the vectors to have finite energy which gives us the definition:

$$\ell^2(\mathbb{C}) := \{\{a_n\} \mid a_n \in \mathbb{C} \text{ such that } \|\{a_n\}\| < \infty\}.$$

- Show that the sequence $a_n = \frac{1}{n+1}$ is in $\ell^2(\mathbb{C})$. What is its norm? *Please use WolframAlpha!*
- Show that the sequence $b_n = \frac{1}{2^n}$ is in $\ell^2(\mathbb{C})$. What is its norm?
- Compute the inner product $\langle \{a_n\}, \{b_n\} \rangle$. *Please use WolframAlpha!*
- Provide an example of a basis for $\ell^2(\mathbb{C})$.

(c) **(4 pts)** (Functional inner product space) Consider the space of *square integrable* (*finite energy*) functions $L^2(\Omega)$ on the region $\Omega = [0,1]$. That is, an element of the $L^2(\Omega)$ is a complex-valued function $f\colon [0,1] \to \mathbb{C}$ of complex numbers with an inner product defined by

$$\langle f, g\rangle = \int_\Omega f^* g d\Omega = \int_0^1 f^*(x) g(x) dx.$$

and we require the vectors to have finite energy which gives us the definition:

$$L^2(\Omega) \coloneqq \{f\colon \Omega \to \mathbb{C} \mid \text{such that } \|f\| < \infty\}.$$

- Show that the function $f(x) = e^{i2\pi x}$ is in $L^2(\Omega)$. What is its norm?
- Show that the sequence $g(x) = \sin(2\pi x)$ is in $L^2(\Omega)$. What is its norm?
- Compute the inner product $\langle \{a_n\}, \{b_n\}\rangle$. *Please use WolframAlpha!*

Problem 2. (4 pts) Consider the real function $f(x) = 1 \in L^2(\Omega)$ on the domain $\Omega = [0, L]$.

(a) **(1 pts)** What is the norm of f, $\|f\|$?

(b) **(1 pts)** Normalize $f(x)$.

(c) **(2 pts)** Find a nonzero normalized polynomial of degree ≤ 1 that is orthogonal to $f(x)$.

Problem 3. (6 pts) A wavefunction Ψ for a particle in the 1-dimensional box $\Omega = [0,1]$ is a member of the space of finite energy functions $L^2(\Omega)$. Recall that Ψ could be written as a superposition of normalized states

$$\psi_n(x) = \sqrt{2}\sin(n\pi x).$$

That is,

$$\Psi(x) = \sum_{n=1}^{\infty} a_n \psi_n(x),$$

for some choice of the coefficients a_n.

(a) **(3 pts)** Let $a_n = \frac{\sqrt{6}}{n\pi}$. Show that $\Psi(x)$ is normalized. *Hint: first, use orthogonality of the states $\psi_n(x)$ to your advantage. Then you will need to know what an infinite series evaluates to. Use a tool like WolframAlpha to evaluate this series.*

(b) **(2 pts)** Note that we can approximate $\Psi(x)$ by taking a finite sum approximation up to some chosen N by

$$\Psi(x) \approx \sum_{n=1}^{N} a_n \psi_n(x).$$

Plot the approximation of $\Psi(x)$ for $N = 1, 5, 25, 50, 100$.

(c) **(1 pts)** Describe the wave function Ψ.

Problem 4. (9 pts) When making a measurement of the position of the particle, we will use the *position operator* x. This is the same as the variable x in the original problem statement, but it is also an operator! Similarly, we could measure the momentum of a particle using the *momentum operator* p. The potential $V(x)$ is a function of the position operator and it, itself, is an operator. Lastly, I should mention the *Hamiltonian operator* $H = \frac{p^2}{2M} + V$.

What I mean here by operator is that the operators defined above are linear transformations $\mathcal{L}\colon L^2(\Omega) \to L^2(\Omega)$. *Actually, I am lying to you. It is true that* $x\colon L^2(\Omega) \to L^2(\Omega)$ *but you have to be careful which spaces you are talking about when it comes to the momentum operator* p. *Do not worry, this understanding that the underlying space is* $L^2(\Omega)$ *is good enough!*

(a) **(1 pts)** True or false. A self-adjoint operator has a real-valued spectrum.

(b) **(1 pts)** Show that the position operator x is self-adjoint.

(c) **(2 pts)** We can compute the expected position of a particle with wavefunction $\Psi(x)$ by computing

$$\mathbb{E}[x] = \langle \Psi, x\Psi \rangle.$$

Let $\Psi(x) = \psi_1(x)$, compute $\mathbb{E}[x]$. This value $\mathbb{E}[x]$ tells you where we expect to find the particle on average.

(d) **(1 pts)** In fact, any real valued function $V(x)$ of the position operator x is also self-adjoint. Make a quick argument on why this must be true.

(e) **(2 pts)** We define the *momentum operator* $p = -i\hbar\frac{d}{dx}$. Using integration by parts, show that this operator is self-adjoint.

(f) **(2 pts)** Argue that the Hamiltonian operator is self-adjoint. *Hint: look at how H is defined. Don't show more work than you need to. This part should be short and sweet.*

Remark 1. The fact that all measurements in quantum mechanics are self-adjoint operators motivates the Dirac bra-ket notation which looks like this:

$$\mathbb{E}[x] = \langle \Psi |\, x \,| \Psi \rangle.$$

This is because x can act on either side!

Problem 5. (6 pts) Let's explore two subspaces of $L^2(\Omega)$ for $\Omega = [0, 1]$. One built by exponentials and one built by sines and cosines.

(a) **(3 pts)** Show that the set of functions $\{1, \sqrt{2}\cos(2n\pi x), \sqrt{2}\sin(2n\pi x)\}$ for integers $n \geq 1$ are orthonormal in $L^2(\Omega)$ with $\Omega = [0, 1]$.

(b) **(3 pts)** Recall Euler's formula: $e^{ix} = \cos(x) + i\sin(x)$. Argue that

$$\mathrm{Span}\{e^{i2n\pi x} \mid n \in \mathbb{Z}\} \qquad \text{and} \qquad \mathrm{Span}\{1, \sqrt{2}\cos(2n\pi x), \sqrt{2}\sin(2n\pi x) \mid m, n \in \mathbb{Z},\ n \geq 1\}$$

are the same subspace. *Hint: can you just show that a given $e^{i2n\pi x}$ corresponds to a pair $\cos(2n\pi x)$ and $\sin(2n\pi x)$ using Euler's formula? Also, you should use the fact that sine is an odd function and cosine is an even function.*

Problem 6. (10 pts) It turns out that the set of complex exponentials $\{e^{i2n\pi x} \mid n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\Omega)$ for $\Omega = [0, 1]$ (and thus by the previous problem, so are the sines and cosines). This is the foundational insight of the Fourier transform/series. Let us denote by $\phi_n(x) = e^{i2n\pi x}$.

Let $f \in L^2(\Omega)$, then the *Fourier transform* of f is the function $\hat{f}\colon \mathbb{Z} \to \mathbb{C}$ defined by

$$\hat{f}(n) = \langle f, \phi_n \rangle.$$

Then the *Fourier series of f* is the series

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)\phi_n(x)$$

(a) **(2 pts)** Let V be an n-dimensional vector space. Recall that you can write a vector $v \in V$ in terms of an orthonormal basis $v_1, \ldots, v_n$ for V by

$$v = \sum_{j=1}^{n} \langle v, v_j \rangle v_j.$$

Explain why the Fourier series is the same concept just for an infinite dimensional vector space.

(b) **(2 pts)** Compute the Fourier transform of the function

$$f(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{4} \\ 1, & \frac{1}{4} \leq x \leq \frac{3}{4} \\ 0, & \frac{3}{4} \leq x \leq 1 \end{cases}$$

Please note that you will want to consider the term $\hat{f}(0)$ separately from the others.

(c) **(1 pts)** Write out the Fourier series of f.

(d) **(2 pts)** Convert the Fourier series of f into a series of sine and cosine functions using Euler's formula. *Hint: you can also just use this from the beginning and put*

$$\hat{f}(n) = \langle f, \phi_n \rangle = \int_0^1 f(x)\cos(2n\pi x)dx + i\int_0^1 f(x)\sin(2n\pi x)dx.$$

(e) **(2 pts)** Plot an approximation of the Fourier series of sine and cosine functions up to $N = 1, 3, 5, 10, 50, 100$ only on the domain Ω. Please graph your approximations to the original f. Describe what is happening with your approximations.

(f) **(1 pts)** What happens if you plot your Fourier series over all of $\mathbb{R}$?

Problem 7. (10 pts) One advantage of the Fourier transform is that we can use it to solve differential equations in a new way that also allows us to consider far more general forcing terms. The basic idea is that the Fourier transform converts a differential equation into an algebraic equation. Let us see how this works.

First, let me say that we will be working over $L^2(\Omega)$ with $\Omega = [0, 1]$. Recall the Dirac delta $\delta(x)$ which satisfies the properties

$$\int_0^1 \delta(x - x_0)dx = 1 \qquad \text{and} \qquad \int_0^1 \delta(x - x_0)f(x)dx = f(x_0)$$

whenever $x_0 \in \Omega$. You can imagine the Dirac delta $\delta(x - x_0)$ as a probability distribution where all the mass is located at a single point x_0.

(a) **(2 pts)** Show that for any f satisfying the Dirichlet boundary conditions $f(0) = 0$ and $f(1) = 0$ that for $n \neq 0$

$$\left\langle \frac{d}{dx} f, \phi_n \right\rangle = i2\pi n \hat{f}(n).$$

Hint: use integration by parts.

(b) **(2 pts)** Consider the Poisson (elastic deformation) problem

$$\begin{cases} \frac{d^2}{dx^2} f(x) = \delta(x - x_0) \\ f(0) = 0 = f(1) & \text{as boundary conditions.} \end{cases}$$

Apply the Fourier transform to both sides to show that we have for $n \neq 0$

$$\hat{f}(n) = -\frac{e^{i2n\pi x_0}}{4\pi^2 n^2}.$$

(c) **(2 pts)** Hence, the solution to the problem is just the Fourier series

$$f(x) = c_0 + c_1 x + \sum_{n=-\infty}^{-1} -\frac{e^{in\pi x_0}}{4\pi^2 n^2}\phi_n + \sum_{n=1}^{\infty} -\frac{e^{in\pi x_0}}{4\pi^2 n^2}\phi_n.$$

Let $x_0 = 1/2$ and convert this Fourier series to a real valued Fourier series in terms of sines and cosines.

(d) **(1 pts)** Determine the constants c_0 and c_1 using the boundary conditions on f.

(e) **(2 pts)** Plot your approximation to the Fourier series for $N = 1, 3, 5, 10, 50, 100$.

(f) **(1 pts)** Explain your result given the following interpretation: You can imagine that $\delta(x - 1/2)$ is a point mass of mass 1 placed exactly at $x = 1/2$ on your elastic rod Ω. *Hint: does this look what what you'd imagine the deformation to look like?*

Problem 8. (Bonus 10 pts.) Show that the Fourier transform is an isomorphism between $\ell^2(\mathbb{C})$ and $L^2(\Omega)$ where $\Omega = [0, 1]$. For an extra 5 points, argue that this is true if Ω is any closed interval.