MATH 271, Worksheet 2, *Solutions*
Ordinary Differential Equations

Problems 1-6 are related.

Problem 1. (Newton's law of cooling) Write down a differential equation that models the following scenario:

The temperature of a substance in an ambient environment changes over time proportionally to the difference of the substance from the ambient environment.

Let $T(t)$ be the temperature of the substance, T_a be the ambient temperature, and k be the constant of proportionality.

Solution 1. We have

$$T' = k(T_a - T).$$

How do we know we have this correct? If we assume $k > 0$ (which it is, in reality), then if $T > T_a$ we have $T_a - T < 0$ and so we will also have $T' < 0$. This makes sense as if we have a hotter object in a colder room, the object will cool down over time.

Problem 2. With the equation found above, find a general solution.

Solution 2. The equation

$$T' = k(T_a - T)$$

is separable. So we can find the general solution by

$$\begin{aligned}\frac{dT}{dt} &= k(T_a - T)\\ \frac{dT}{T_a - T} &= kdt.\end{aligned}$$

Now we can integrate

$$\begin{aligned}\int \frac{dT}{T_a - T} &= \int kdt\\ -\ln(T_a - T) &= kt + C\\ \ln(T_a - T) &= -kt - C.\end{aligned}$$

If we take the exponential of both sides, we can solve for T like so

$$\begin{aligned}T_a - T &= e^{-kt-C}\\ T &= T_a - e^{-kt-C}.\end{aligned}$$

Problem 3. With the parameter values $T_a = 100$, $k = 1$, and initial data $T(0) = 50$, find the particular solution.

Solution 3. Now, take our general solution with these values, and we have

$$T = 100 - e^{-t-C}.$$

Now we can solve for C by noting we have

$$50 = T(0) = 100 - e^{-0-C} = 100 - e^{-C}$$

which means that $e^{-C} = 50$ and hence, we have the particular solution

$$T(t) = 100 - 50e^{-t}.$$

Problem 4. Plot the particular solution to the problem and interpret what this means for the temperature over time.

Solution 4. Here is a plot of the solution.

Worksheet_2/desmos-graph(20).png

Now we can see the temperature of the object approaches the temperature of the ambient environment as time goes on. This is what we expect!

Problem 5. What happens instead if the initial temperature is equal to the ambient temperature? Does your solution reflect this? Does this make physical sense?

Solution 5. If the initial temperature was equal to the ambient temperature then we don't expect the object to change temperature over time. One would find that we have the equation

$$100 = T(0) = 100 - e^{-0-C} = 100 - e^{-C}$$

and so $e^{-C} = 0$ and thus

$$T(t) = 100,$$

which is what we expect.

Problem 6. Let $\delta = (T_a - T)$. Show that the equation you found in Problem 1 reduces to

$$\delta' = -k\delta.$$

What is this equation describing physically?

Solution 6. Well, note that $\delta' = (T_a - T)' = -T'$. Thus we can write

$$\delta' = -T' = -k(T_a - T) = -k\delta.$$

This equation is describing how the *difference* in temperature of the object versus the ambient environment will change over time. If you were to solve this, you would find that $\delta \to 0$ as $t \to \infty$.

Problems 7-8 are related.

Problem 7. Show that $x = c_1 \sin(t) + c_2 \cos(t)$ is a general solution to the equation

$$x'' + x = 0.$$

Solution 7. We plug in our guess for x into the left hand side of the equation and we'll check to see that we get zero. So,

$$\begin{aligned} x'' + x &= (c_1 \sin(t) + c_2 \cos(t))'' + (c_1 \sin(t) + c_2 \cos(t)) \\ &= -c_1 \sin(t) - c_2 \cos(t) + c_1 \sin(t) + c_2 \cos(t) \\ &= 0. \end{aligned}$$

Indeed, this x does solve our equation.

Problem 8. Find the particular solution if $x(0) = 1$ and $x'(0) = 2$. Plot your solution in the t, x-plane and in the x, x'-plane.

Solution 8. Now, we use our initial data with our general solution above

$$1 = x(0) = c_1 \sin(0) + c_2 \cos(0) = c_2$$

so $c_2 = 1$. Also, we have

$$0 = x'(0) = c_1 \cos(0) - \sin(0) = c_1$$

so $c_1 = 0$.

Problem 9. Consider the differential equation

$$x' = \frac{x^2 + tx + t^2}{tx}.$$

(a) Let $f(x, t) = x^2 + tx + t^2$. Show that $f(\lambda x, \lambda t) = f(x, t)$.

(b) Use the substitution $u = \frac{x}{t}$ in order to make the original equation separable.

(c) Find the general solution to this separable equation in terms of u and t. You may use Wolfram Alpha to compute the necessary integral.

(d) Find the solution to the original equation using the substitution $u = \frac{x}{t}$ and your solution from (c).

Solution 9.

(a) Take

$$\begin{aligned} f(\lambda x, \lambda t) &= \frac{(\lambda x)^2 + (\lambda t)(\lambda x) + (\lambda t)^2}{(\lambda t)(\lambda x)} \\ &= \frac{\lambda^2 x^2 + \lambda^2 tx + \lambda^2 t^2}{\lambda^2 tx} \\ &= \frac{x^2 + tx + t^2}{tx}. \end{aligned}$$

Indeed we have that $f(\lambda x, \lambda t) = f(x, t)$.

(b)