MATH 272, Worksheet 7

Vector fields, and associated calculus.

Problem 1. Plot the following vector fields. Then compute the associated Jacobian matrix as well as the curl and divergence.

(a) $\vec{\boldsymbol{U}} = \begin{pmatrix} xyz \\ xyz \\ xyz \end{pmatrix}$.

(b) $\vec{\boldsymbol{V}} = \begin{pmatrix} e^{x+y+z} \\ e^{x+y+z} \\ e^{x+y+z} \end{pmatrix}$.

(c) $\vec{\boldsymbol{W}} = \begin{pmatrix} x\sin(y) \\ x\cos(y) \\ xz \end{pmatrix}$.

(d) $\vec{\boldsymbol{F}} = \begin{pmatrix} 5+yz \\ -5-xz \\ xy \end{pmatrix}$.

Problem 2. Explain (geometrically) why the field $\vec{\boldsymbol{U}} = \begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix}$ has nonzero curl everywhere. Reason why the direction of the curl is solely along the z-axis. Do this <u>without</u> computing the curl.

Problem 3. Explain (geometrically) why the field $\vec{\boldsymbol{V}} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$ has nonzero divergence everywhere. Reason why the divergence is a scalar quantity as opposed to a vector quantity. Do this <u>without</u> computing the curl.

Problem 4. Compute the Laplacian of the following fields.

(a) $f(x,y) = (x+y)^2$

(b) $g(x,y) = (x+y)e^{x^2+y^2}$.

(c) $\vec{\boldsymbol{U}} = \begin{pmatrix} \cos(x)\cos(y)\cos(z) \\ \sin(x)\sin(y)\sin(z) \\ xyz \end{pmatrix}$.

(d) $\vec{\boldsymbol{V}} = \begin{pmatrix} \frac{y}{z} \\ \frac{x}{z} \\ \frac{x}{y} \end{pmatrix}$.

Problem 5. Suppose that $\vec{\boldsymbol{V}} = \vec{\boldsymbol{\nabla}}\phi$ for some scalar field ϕ. Explain why if $\vec{\boldsymbol{V}}$ is divergence free (i.e., if $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{V}} = 0$) that $\vec{\boldsymbol{\Delta}}\vec{\boldsymbol{V}} = \vec{\boldsymbol{0}}$.

Problem 6. One of Maxwell's equations states for a magnetic field $\vec{\boldsymbol{B}}$ that

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{B}} = 0,$$

is an identity. Does this mean that $\vec{\boldsymbol{\Delta}}\vec{\boldsymbol{B}} = 0$?

Problem 7. Compute the flux of the vector fields given in Problem 1 through the following surfaces.

(a) Σ_1 is the unit square in the xy-plane.

(b) Σ_2 is the unit square in the xz-plane.

(c) Σ_3 is the unit square in the yz-plane.

(d) Σ_4 is the surface of the unit cube.

Problem 8. Compute the line integral of the vector fields given in Problem 1 along the following curves.

(a) $\vec{\gamma}_1$ is the boundary unit square in the xy-plane.

(b) $\vec{\gamma}_2$ is the unit circle in the xy-plane.

(c) $\vec{\gamma}_3$ is the curve $\vec{\gamma}_3(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$ from time $t = 0$ to $t = 1$.

Problem 9. Compare the result for integrating $\vec{\boldsymbol{U}}$ from Problem 1 around $\vec{\gamma}_1$ from Problem 8 with computing the flux of the curl of $\vec{\boldsymbol{U}}$ through Σ_1 from Problem. That is, check to see

$$\int_{\vec{\gamma}_1} \vec{\boldsymbol{U}} \cdot d\vec{\gamma}_1 \stackrel{?}{=} \iint\limits_{\Sigma_1} \left(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{U}}\right) \cdot \hat{\boldsymbol{n}} d\Sigma_1.$$

This result is typically referred to as Stokes' theorem. It is another way of relating an integral in a region to an integral along its boundary.