MATH 271, Worksheet 9

Linear independence, span, and bases. Matrix determinants and traces.

Problem 1. Consider the following three vectors $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}} \in \mathbb{R}^3$ given by

$$\vec{\boldsymbol{u}} = \hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}, \qquad \vec{\boldsymbol{v}} = 2\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + 2\hat{\boldsymbol{z}}, \qquad \vec{\boldsymbol{w}} = -2\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}.$$

(a) We can write a linear combination of these vectors by taking

$$\alpha\vec{\boldsymbol{u}} + \beta\vec{\boldsymbol{v}} + \gamma\vec{\boldsymbol{w}},$$

where $\alpha, \beta, \gamma \in \mathbb{R}$. Write this linear combination as a matrix times a vector.

(b) Are these vectors linearly independent?

(c) Does this list of vectors form a basis for $\mathbb{R}^3$? *Hint: use the above work. Can any vector in $\mathbb{R}^3$ be written as a linear combination of these vectors?*

Problem 2. Consider the following three vectors $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}, \vec{\boldsymbol{w}} \in \mathbb{R}^3$ given by

$$\vec{\boldsymbol{u}} = \hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}, \qquad \vec{\boldsymbol{v}} = \hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}, \qquad \vec{\boldsymbol{w}} = 2\hat{\boldsymbol{x}} + 2\hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}.$$

(a) We can write a linear combination of these vectors by taking

$$\alpha\vec{\boldsymbol{u}} + \beta\vec{\boldsymbol{v}} + \gamma\vec{\boldsymbol{w}},$$

where $\alpha, \beta, \gamma \in \mathbb{R}$. Write this linear combination as a matrix times a vector.

(b) Are these vectors linearly independent?

(c) Does this list of vectors form a basis for $\mathbb{R}^3$? *Hint: use the above work. Can any vector in $\mathbb{R}^3$ be written as a linear combination of these vectors?*

Problem 3. Compute the determinants of the matrices you found in Problems 1 and 2. Explain how this gives insight on your ability to find solutions to inhomogeneous and homogeneous equations with those matrices.

Problem 4. Suppose we have a matrix $[A]$ such that $[A]\vec{\boldsymbol{u}} = \lambda\vec{\boldsymbol{u}}$ for some constant λ. Suppose as well that $\vec{\boldsymbol{v}}$ satisfies the same equation in that $[A]\vec{\boldsymbol{v}} = \lambda\vec{\boldsymbol{v}}$. Finally, suppose there exists a vector $\vec{\boldsymbol{w}}$ that satisfies a similar equation $[A]\vec{\boldsymbol{w}} = \eta\vec{\boldsymbol{w}}$ but with $\eta \neq \lambda$.

(a) Show that any vector in the span of $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ also satisfies the same equation as $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$.

(b) Show that the span of $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{w}}$ does not solve either of the given equations.

Problem 5. Consider the matrix

$$[J] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which acts as a counter clockwise rotation by $\pi/2$ in the xy-plane.

(a) Show that $\det([J]) = 1$.

(b) Explain why $[J]$ does not distort areas using what you know about the determinant.

(c) Consider a new matrix $[J] - \lambda[I]$ where $[I]$ is the 2×2 identity matrix and λ is a scalar variable. Compute $\det([J] - \lambda[I])$. This is called the *characteristic polynomial.*

(d) Find the roots of the characteristic polynomial.

Problem 6. Consider the matrices

$$[A] = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}, \quad [B] = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}, \quad [C] = \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}.$$

(a) Compute the determinant of each matrix.

(b) For each matrix, draw the vectors $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ and draw the transformed vectors (the matrix applied to $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$). Explain how the matrices transform areas and relate this back to the determinant of the matrices. Do this in a different plane for each matrix to avoid making this look messy.

Problem 7. Consider the vectors in $\mathbb{R}^3$, $\vec{\boldsymbol{u}} = 3\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}} + 4\hat{\boldsymbol{z}}$ and $\vec{\boldsymbol{v}} = -\hat{\boldsymbol{y}} - 2\hat{\boldsymbol{z}}$. Show that $\operatorname{tr}(\vec{\boldsymbol{u}}\vec{\boldsymbol{v}}^\top) = \vec{\boldsymbol{u}}^\top\vec{\boldsymbol{v}}$.

Problem 8. Prove the previous problem for two arbitrary vectors in $\mathbb{R}^n$.

Problem 9. Is it true that $\operatorname{tr}([A]^\top) = \operatorname{tr}([A])$ for any matrix? Why or why not?

Problem 10. Consider the linear transformations on $\mathbb{R}^3$ to $\mathbb{R}^3$ given by

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Fact: These matrices are generators for the *group of rotations* $\mathrm{SO}(3)$ of $\mathbb{R}^3$.

(a) Let $\theta = \pi/2$. Show that $R_x(\pi/2)$ rotates a vector counter clockwise by $\pi/2$ radians around the x-axis.

(b) Show that the determinant of each of these matrices is 1 for any value of θ.

(c) Using properties of determinants, show that the determinant of a product of rotation matrices is also 1.

(d) Explain geometrically why a rotation matrix must have a determinant of 1.

(e) Show that $R_x(\theta)R_x(\theta)^\top = I$. This in fact true for any rotation matrix.

Problem 11. Consider the matrix

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

(a) Compute $\mathrm{tr}(M)$.

(b) Compute $M^{R_x} = R_x(\pi/2)MR_x(\pi/2)^\top$.

(c) What is the trace of M^{R_x}?

(d) Can you see why you have the answer in (c) from properties of the trace?