MATH 272, Worksheet 6

Curves, scalar fields, and calculus.

Problem 1. Consider the curve

$$\vec{\gamma}(t) = \begin{pmatrix} (5+3\cos(8t))\cos(t) \\ (5+3\cos(8t))\sin(t) \\ 3\sin(8t) \end{pmatrix}.$$

(a) Plot this curve from $t=0$ to $t=2\pi$.

(b) Compute the tangent (velocity) vector $\dot{\vec{\gamma}}(t)$.

(c) Compute the normal (acceleration) vector $\ddot{\vec{\gamma}}(t)$.

(d) Compute the following

$$\int_{\vec{\gamma}} \left| \ddot{\vec{\gamma}}(t) \right| dt,$$

which is closely related to the total curvature of the curve. Indeed, this would be the total force applied to an object of mass $m=1$.

Problem 2.

(a) Write an equation for a curve

$$\vec{\gamma} \colon \mathbb{R} \to \mathbb{R}^3,$$

satisfying:

- Starts with $\vec{\gamma}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.
- Ignoring the z-component, makes a spiral emanating from the origin.
- Moves upward at a constant rate in the z-direction.

Plot this curve that you made to verify that it is correct.

(b) Find the tangent vector $\dot{\vec{\gamma}}(t)$ to this curve.

(c) Find the normal vector $\ddot{\vec{\gamma}}(t)$ to this curve.

Problem 3. Consider the two dimensional scalar function

$$f(x,y) = e^{\frac{xy}{x^2+y^2-1}}.$$

(a) Plot the graph of this function $(x,y,f(x,y))$ for $x^2+y^2<1$.

(b) Plot the graph for $x^2+y^2>1$.

(c) Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

(d) Compute the gradient vector field $\vec{\boldsymbol{\nabla}} f$.

(e) Plot the vector field $\vec{\boldsymbol{\nabla}} f$.

Problem 4.

(a) Write the equation for a scalar function

$$f\colon \mathbb{R}^2 \to \mathbb{R},$$

satisfying,

- Has positive $\frac{\partial f}{\partial x}$ everywhere.
- Has negative $\frac{\partial f}{\partial y}$ everywhere.

(*Hint: it may help to try adding single variable functions together. That is, let* $f(x,y) = u(x) + v(y)$.)

(b) Find the gradient of the function you chose.

Problem 5. Using your curve $\vec{\gamma}$ from Problem 2 with time starting at $t_0 = 0$ and $t_1 = 2\pi$ and your scalar field f from Problem 4, compute the following.

$$\int_{\vec{\gamma}} f(\gamma) d\vec{\gamma} = \int_{t_0}^{t_1} f(\gamma(t)) \left|\dot{\vec{\gamma}}(t)\right| dt.$$

Problem 6. We have briefly discussed the idea of *work* (change in energy) before and wrote

$$W = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{r}},$$

where $\vec{\boldsymbol{F}}$ was a constant force and $\vec{\boldsymbol{r}}$ was a straight line displacement.

Now, we can write the real version of this. The work done on a particle moving along a curve $\vec{\gamma}(t)$ that starts at time t_0 and ends at time t_1 experiencing a (spatially dependent) force field $\vec{\boldsymbol{F}}(x,y,z)$ is

$$W = \int_{\vec{\gamma}} \vec{\boldsymbol{F}}(\vec{\gamma}) \cdot d\vec{\gamma} = \int_{t_0}^{t_1} \vec{\boldsymbol{F}}(\vec{\gamma}(t)) \cdot \dot{\vec{\gamma}}(t) dt.$$

Compute the work given the following

$$\vec{\boldsymbol{F}}(x,y,z) = \begin{pmatrix} x^2 \\ y \\ \sqrt{z} \end{pmatrix} \qquad \text{and} \qquad \vec{\gamma}(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}.$$

Problem 7. For the following functions, plot the level sets for $c = -1$, $c = 0$, and $c = 1$.

(a) For just $c = 1$, plot the level set for $E(x,y,z) = \frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9}$.

(b) $f(x,y,z) = xyz$.

(c) $g(x,y,z) = e^x - y^2 - z^2$.

(d) $h(x,y,z) = \sin(x) + \cos(y) - \tanh(z)$.

(e) $p(x,y,z) = \sin^2(x) + \sin^2(y) - \frac{1}{2}\sin(z)$.

(f) $q(x,y,z) = x^2 + xy + y^2 + sin(yz)$.

(g) One of your own choosing.

Problem 8. Compute the integral of the scalar fields given in Problem 7 over the following regions.

(a) Ω_1 is the unit cube.

(b) Ω_2 is the unit cube along with the rectangular prism given by $2 < x < 3$, $2 < y < 4$, and $-1 < z < 1$.

(c) Ω_3 is the portion of the unit cube left over after slicing diagonally by the plane given by the equation $x + y + z = \frac{1}{2}$. Specifically, take the region left <u>under</u> the plane and in the unit cube.