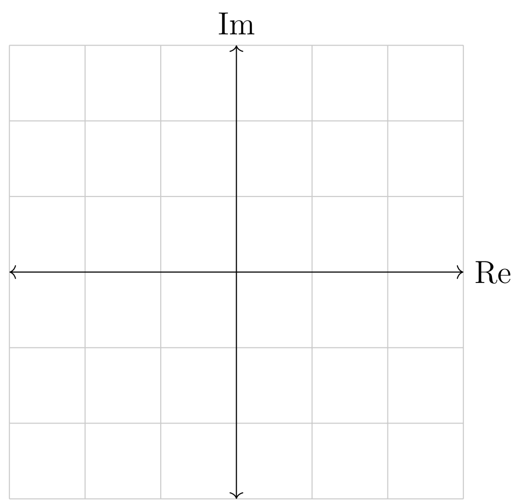
MATH 271, Worksheet 1
Complex Numbers

Problem 1. Add and multiply all possible pairs of the complex numbers

$$z_1 = 3 - 2i \qquad z_2 = -1 - i \qquad z_3 = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \qquad z_4 = -\pi + \pi i.$$

Problem 2. Plot and label the above points on the graph below. Pick two points and draw their sum geometrically.

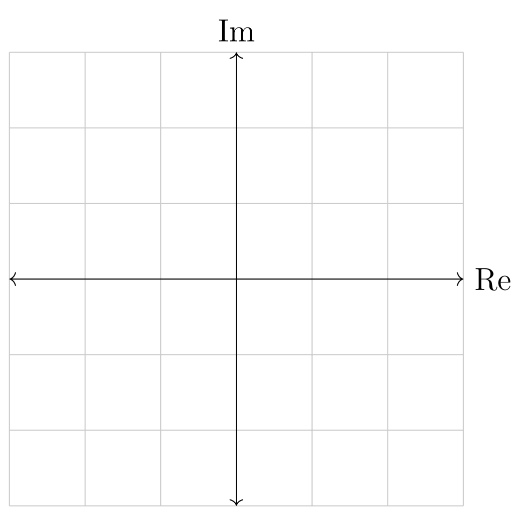


Problem 3. Convert all the above complex numbers in Cartesian form to polar form.

Problem 4. Multiply all possible pairs of complex numbers

$$w_1 = e^{i\pi} \qquad w_2 = -\sqrt{2}e^{i\frac{\pi}{4}} \qquad w_3 = 2e^{-i\frac{\pi}{3}} \qquad w_4 = 3e^{i\frac{\pi}{2}}.$$

Problem 5. Plot and label the above points in the graph below. Draw the product $w_1 w_2$ geometrically.



Problem 6. Show by a geometrical argument that $re^{i\theta} = re^{i(\theta+2n\pi)}$ for any integer value of n. Can you also show this by the typical conversion from polar to Cartesian?

Problem 7. Let $z = a+bi$. What is z^*? What is z in polar coordinates? How about z^*? Can you explain why zz^* will always be real using a geometrical (polar coordinate) argument?

Problem 8. Show that the functions $\sin(x)$ and $\cos(x)$ are periodic and determine the period. Is the function $\tan(x)$ periodic?

Problem 9. (Roots of unity) Given that $i^2 = -1$ we can factor equations in ways that are totally new to us. Solve the following. Geometrical reasoning may really help you here!

(a) Find all solutions to $z^2 = 1$.

(b) Find all solutions to $z^3 = 1$.

(c) Find all solutions to $z^4 = 1$.

(d) (Challenge) Find all solutions to $z^n = 1$.

Hint: each of the above has n solutions for z^n.

Problem 10. (Rotational symmetries) Consider a complex number $z = a+bi = re^{i\theta}$. What happens to this point as we repeatedly multiply by i? Pick a specific z (i.e., fix a and b or r and θ) and multiply by i until you see a pattern. What is happening? *Note: this is an example of a symmetry group or a discrete dynamical system!*

Problem 11. (Why i?) Consider the following (differential) equation

$$\frac{d^2}{dt^2}x(t) = -x(t).$$

Now, let $x(t) = e^{it}$. Show that the above expression is true.

Problem 12. (Characteristic polynomial) Consider the following (differential) equation

$$ax''(t) + bx'(t) + cx(t) = 0.$$

This equation can be converted to a quadratic equation

$$a\lambda^2 + b\lambda + c = 0.$$

What are the roots to this equation?

Problem 13. (Challenge) In fact, show that you get the above polynomial by taking the guess $f(t) = Ae^{\lambda t}$ where $A, \lambda \in \mathbb{R}$ are real constants.

Problem 14. (Bonus) The complex numbers can be thought of as a 2-dimensional number system. The **quaternions** are a 4-dimensional number system that are of the form

$$q = a + bi + cj + dk.$$

We define $i^2 = j^2 = k^2 = ijk = -1$. Pick another quaternion $\tilde{q} = \tilde{a} + \tilde{b}i + \tilde{c}j + \tilde{d}k$ and multiply each together. Is multiplication commutative? *Note: quaternions are extremely useful in 3-dimensional systems. They are used for describing computer graphics, robotics, relativity, spinning objects, or they also provide an algebraic way for formalizing the vector calculus in $\mathbb{R}^3$ which we will see in Math 272.*