MATH 271, Worksheet 3
Second Order Equations and Boundary Value Problems

Problem 1. Consider the following differential equation

$$x'' + x = 0.$$

(a) Find the general solution to this equation.

(b) Does the solution grow or decay over time?

(c) What is $\lim_{t\to\infty} x(t)$?

Problem 2. Next, consider a related equation

$$x'' + x = t.$$

that has an additional linear external force.

(a) What is the solution to the homogenous equation?

(b) Find the particular integral with the given forcing term.

(c) What is the specific solution to this equation?

(d) Does the solution grow or decay over time?

(e) What is $\lim_{t\to\infty} x(t)$?

Problem 3. Consider now the equation

$$x'' + x = F(t)$$

where the external force is $F(t) = \cos(t)$.

(a) Find the particular integral with the given forcing term.

(b) What is the specific solution to this equation?

(c) What is $\lim_{t\to\infty}$? What does this mean about the growth or decay of the solution over time?

Problem 4. Write down a second order linear differential equation that oscillates and also decays over time.

Problem 5. Consider the boundary value problem

$$x'' = g$$

with boundary values $x(0) = 0$, $x\left(-\frac{2}{g}\right) = 0$ and $g = -9.8[m/s^2]$. We can think of this as solving the *inverse problem* of one that we have seen in a homework. Specifically, think of this as knowing where a ball is launched and knowing where it lands and trying to find the speed it must have been thrown at. Another interpretation is the shape of a rod bending due to gravity. We call this *Poisson's equation.*

(a) Find the general solution. If you already know it from the homework, just write it down.

(b) Use the boundary values above to find the particular solution.

(c) Is the solution unique?

Problem 6. Consider the *time independent Schödinger equation* for a *free particle* constrained inside of a 1-dimensional box of length L. That is, we have the equation

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x)$$

on the unit interval $[0, L]$.

(a) Find the general solution to this equation with no constraint.

(b) Given the constraint, we have the boundary values $\psi(0) = \psi(L) = 0$. What are the general solutions given this constraint?

(c) Show that the sum of two solution $\psi_1(x)$ and $\psi_2(x)$ is also a solution. When we have a particle whose state (or *wavefunction*) ψ is a sum of general solutions, we say that ψ is in a *superposition state.*

(d) The wavefunction is not really a physically meaningful quantity. However, if we consider a region $[a, b]$ in the box $[0, L]$ the quantity

$$P([a, b]) = \int_a^b |\psi(x)|^2 dx$$

is meaningful. This expression tells us the *probability* that a particle will be observed in the region $[a, b]$. Take your general solutions you found in (b) (with the constraint) and solve for the constants that give you

$$\int_0^L |\psi(x)|^2 dx = 1.$$

We call this *normalization* and we must do so for each state so that we can interpret the integral $P([a, b])$ as a probability.