MATH 271, Worksheet 8

Linear transformations, matrices, and linear systems.

Problem 1. Note that any linear transformation $T\colon \mathbb{R}^m \to \mathbb{R}^n$ is fully understood by its action on the vectors

$$\hat{\boldsymbol{x}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \hat{\boldsymbol{x}}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots \quad , \hat{\boldsymbol{x}}_m = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

and note that all these vectors $\hat{\boldsymbol{x}}_j \in \mathbb{R}^m$. In particular, we have

$$\begin{aligned} T(\hat{\boldsymbol{x}}_1) &= \vec{\boldsymbol{v}}_1 \\ T(\hat{\boldsymbol{x}}_2) &= \vec{\boldsymbol{v}}_2 \\ &\vdots \\ T(\hat{\boldsymbol{x}}_m) &= \vec{\boldsymbol{v}}_m, \end{aligned}$$

where the vectors $\vec{\boldsymbol{v}}_j \in \mathbb{R}^n$ and as such can be written as column vectors with n entries.

(a) As per usual, let $\hat{\boldsymbol{x}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and let $\hat{\boldsymbol{y}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Let $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$A(\hat{\boldsymbol{x}}) = 5\hat{\boldsymbol{x}} + 6\hat{\boldsymbol{y}} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

and

$$A(\hat{\boldsymbol{x}}) = 2\hat{\boldsymbol{x}} - 3\hat{\boldsymbol{y}} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

If I wanted to transform an arbitrary vector $\vec{\boldsymbol{u}} = u_1\hat{\boldsymbol{x}} + u_2\hat{\boldsymbol{y}} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, how can I use the definition of A acting on unit vectors?

(b) Determine a matrix of numbers $[A] = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ that captures this linear transformation through matrix-vector multiplication.

(c) How do the columns of $[A]$ relate to $A(\hat{\boldsymbol{x}})$ and $A(\hat{\boldsymbol{y}})$?

(d) Now, how can I think of $[A]\vec{\boldsymbol{u}}$ as describing a linear combination of the columns of $[A]$?

Problem 2. Repeat the steps in Problem 1 but with the transformation $B\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$B(\hat{\boldsymbol{x}}) = \hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}} \qquad \text{and} \qquad B(\hat{\boldsymbol{y}}) = -\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} - \hat{\boldsymbol{z}}.$$

Problem 3. Repeat the steps in Problem 1 but with the transformation $C\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$C(\hat{\boldsymbol{x}}) = \hat{\boldsymbol{y}}, \qquad C(\hat{\boldsymbol{y}}) = \hat{\boldsymbol{x}}, \qquad C(\hat{\boldsymbol{z}}) = \hat{\boldsymbol{x}}.$$

Problem 4. Let

$$[M] = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad [P] = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad [Q] = \begin{pmatrix} 2 & 1 \end{pmatrix} \quad [R] = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad [S] = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}$$

(a) Compute the following matrix products (when possible) and state which multiplications are not possible.

$$[M][M], \qquad [P][P], \qquad [Q][P], \qquad [M][S], \qquad [S][M].$$

(b) Compute the following:

i. $[A] = [P][Q]$;

ii. $[B] = [Q]^T[P]^T$. Is this equal to $([P][Q])^T$?

iii. $[C] = [M][R] - [R][M]$. Do these matrices commute?

Problem 5. The linear transformation $H\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$H(\hat{\boldsymbol{x}}) = \hat{\boldsymbol{y}} \qquad \text{and} \qquad H(\hat{\boldsymbol{y}}) = \hat{\boldsymbol{x}},$$

has some nice properties.

(a) In some sense, H is the square root of 1 in that $H^2 = H \circ H = 1$. Show that this is true.

(b) Write down a matrix representation for H and denote it by $[H]$.

(c) Consider a linear combination of matrices

$$[\eta] = x[I] + y[H],$$

where $[I]$ is the 2×2 identity matrix. Compute $[\eta]^2$.

Problem 6. Consider the system of linear equations:

$$\begin{aligned} x + 2y &= 3 \\ x + y &= 3 \end{aligned}$$

(a) Write this system in the form:

$$[A]\vec{\boldsymbol{x}} = \vec{\boldsymbol{y}}$$

(b) Row reduce to find a the solution $\vec{\boldsymbol{x}}$.

Problem 7. Consider the system of linear equations:

$$\begin{aligned} 3x + 2y + 0z &= 5 \\ 1x + 1y + 1z &= 3 \\ 0x + 2y + 2z &= 4. \end{aligned}$$

(a) Write the augmented matrix M for this system of equations.

(b) Use row reduction to get the augmented matrix in row-echelon form.

(c) Determine the solution to the system of equations.

Problem 8. Consider the equation

$$\begin{pmatrix} 1 & 3 & 4 \\ 2 & 9 & 9 \\ 1 & 5 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 20 \\ 11 \end{pmatrix}.$$

Does this equation have a solution or not? If so, determine the solution.

Problem 9. Consider the matrix

$$[A] = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}.$$

Determine the nullspace of $[A]$.