# MATH 271, Worksheet 10, *Solutions.*

## Inverse and similar matrices. Eigenvalue problem and diagonalization. Hermitian matrices.

**Problem 1.** Consider the two matrices

$$[A] = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix} \qquad \text{and} \qquad [B] = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

(a) Argue why the matrix $[A]$ cannot be invertible.

(b) Compute the inverse matrix $[B]^{-1}$ for $[B]$.

(c) Solve the system of equations $[B]\vec{\boldsymbol{x}} = \vec{\boldsymbol{y}}$ for the following vectors.

i. $\vec{\boldsymbol{y}} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

ii. $\vec{\boldsymbol{y}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

iii. $\vec{\boldsymbol{y}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

**Solution 1.**

(a) To see $[A]$ is not invertible we note that $\det([A]) = 0$. The columns of $[A]$ are linearly dependent, so it cannot be inverted.

(b) To find $[B]^{-1}$ we start with the augmented matrix

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right).$$

We can subtract 2R1 from R2 to get

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -3 & -2 & 1 \end{array}\right),$$

and then we can multiply R3 by -1/3 to get

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 2/3 & -1/3 \end{array}\right).$$

We can now subtract 2R2 from R1 to get

$$\left(\begin{array}{cc|cc} 1 & 0 & -1/3 & 2/3 \\ 0 & 1 & 2/3 & -1/3 \end{array}\right)$$

which tells us

$$[B]^{-1} = \begin{pmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix}.$$

(c) We can note that if $[B]\vec{\boldsymbol{x}} = \vec{\boldsymbol{y}}$ then we also have $\vec{\boldsymbol{x}} = [B]^{-1}\vec{\boldsymbol{y}}$. Thus, we find the following.

i. With this $\vec{\boldsymbol{y}}$ we get

$$\vec{\boldsymbol{x}} = \begin{pmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix}.$$

ii. Next, we have

$$\vec{\boldsymbol{x}} = \begin{pmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix}.$$

iii. Finally,

$$\vec{\boldsymbol{x}} = \begin{pmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

**Problem 2.** Consider the matrices

$$[A] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad [B] = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

(a) Show that $[A]$ and $[B]$ are both invertible.

(b) Find $[A]^{-1}$ and $[B]^{-1}$.

(c) Show that $([A][B])^{-1} = [B]^{-1}[A]^{-1}$.

**Solution 2.**

(a) To see both matrices are invertible we note that $\det([A]) = 1$ and $\det([B]) = 2$ are both nonzero.

(b) To find the inverses, we can row reduce. The work for $[A]$ is quick. We take the augmented matrix

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right).$$

We subtract R2 from R1 to get

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & \text{-1} \\ 0 & 1 & 0 & 1 \end{array}\right),$$

showing that

$$[A]^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Similarly, one can show

$$[B]^{-1} = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix},$$

and the work is very similar to Problem 1.

(c) We have

$$[A][B] = \begin{pmatrix} 3 & 3 \\ 1 & 2 \end{pmatrix}.$$

Then we also have

$$[B]^{-1}[A]^{-1} = \begin{pmatrix} 2/3 & -1 \\ 1/3 & 1 \end{pmatrix}.$$

If we multiply those two matrices together, we should get the identity $[I]$. Indeed, we have

$$([A][B])([B]^{-1}[A]^{-1}) = \begin{pmatrix} 3 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2/3 & -1 \\ 1/3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

**Problem 3.** Simplify the following expressions.

(a) $([A][B])^{-1}[A][B]$.

(b) $[A]^2[B]^3[A]([A][B])^{-1}$.

(c) $([A][B][C]^{-1})^{-1}[A][B][C]^{-1}$.

**Solution 3.**

(a) We have

$$\begin{aligned}([A][B])^{-1}[A][B] &= [B]^{-1}[A]^{-1}[A][B] \\ &= [B]^{-1}[I][B] \\ &= [B]^{-1}[B] \\ &= [I].\end{aligned}$$

(b) We have

$$[A]^2[B]^3[A]([A][B])^{-1} = [A]^2[B]^3[A][B]^{-1}[A],$$

which cannot be further simplified.

(c) We have

$$\begin{aligned}([A][B][C]^{-1})^{-1}[A][B][C]^{-1} &= [C][B]^{-1}[A]^{-1}[A][B][C]^{-1} \\ &= [I].\end{aligned}$$

**Problem 4.** Show that for any invertible matrix $[A]$ that $\det([A]^{-1}) = \frac{1}{\det([A])}$.

**Solution 4.**

(a) We have

$$1 = \det([I]) = \det([A][A]^{-1}) = \det([A])\det([A]^{-1}).$$

Thus, it must be that

$$\det([A]^{-1}) = \frac{1}{\det([A])}.$$

**Problem 5.** Let $[B]$ be similar to $[A]$ by the relationship $[B] = [P]^{-1}[A][P]$.

(a) Given that $[P]$ is invertible, show that $[P]$ transforms the standard basis $\hat{\boldsymbol{x}}_1, \hat{\boldsymbol{x}}_2, \ldots, \hat{\boldsymbol{x}}_n$ into a new basis given by the columns of $[P]$.

(b) Show that $[P]^{-1}$ transforms the basis given by the columns of $[P]$ into the standard basis.

(c) Explain why $[B]$ performs the same transformation as $[A]$ but just on a different basis (e.g., different choices of coordinates).

**Solution 5.**

(a) We know that $[P]$ is square, so let's say $[P]$ is $n \times n$. Let the $i$th column of $[P]$ be given by the vector $\vec{\boldsymbol{p}}_i$ so that

$$[P] = \begin{pmatrix} | & | & & | \\ \vec{\boldsymbol{p}}_1 & \vec{\boldsymbol{p}}_2 & \cdots & \vec{\boldsymbol{p}}_n \\ | & | & & | \end{pmatrix}.$$

Now, by construction, we have

$$[P]\hat{\boldsymbol{x}}_i = \vec{\boldsymbol{p}}_i.$$

(b) Since $[P]$ is invertible, we left multiply the above equation by $[P]^{-1}$ and we get

$$\hat{\boldsymbol{x}}_i = [P]^{-1}\vec{\boldsymbol{p}}_i.$$

(c) Since $[B] = [P]^{-1}[A][P]$ we have

$$[P][B] = [A][P].$$

Thus, for the standard basis vector $\hat{\boldsymbol{x}}_i$ we get

$$[P][B]\hat{\boldsymbol{x}}_i = [A]\vec{\boldsymbol{p}}_i,$$

from the above equation. $[B]\hat{\boldsymbol{x}}_i$ gives us a linear combination in the standard basis and we then multiply by $[P]$ to transform the standard basis to the basis given by the vectors $\vec{\boldsymbol{p}}_i$. So we see that $[B]$ transforms $\hat{\boldsymbol{x}}_i$ vectors in the same way that $[A]$ transforms the $\vec{\boldsymbol{p}}_i$ vectors.

**Problem 6.** Let $[B]$ be similar to $[A]$ by the relationship $[B] = [P]^{-1}[A][P]$.

(a) Show that the trace is invariant under similarity. That is, show $\mathrm{tr}([A]) = \mathrm{tr}([B])$.

(b) Show that the determinant is invariant under similarity. *Hint: you will need to use the result from Problem 4.*

(c) Show that $[A]$ and $[B]$ have the same eigenvalues. It may help to think that if we have $\vec{\boldsymbol{v}}$ as an eigenvector for $[A]$, then what is the corresponding eigenvector for $[B]$?

**Solution 6.**

(a) We have

$$\mathrm{tr}([B]) = \mathrm{tr}([P]^{-1}[A][P]) = \mathrm{tr}([A][P][P]^{-1}) = \mathrm{tr}([A]).$$

(b) Similarly,

$$\det([B]) = \det([P]^{-1}[A][P]) = \det([P]^{-1})\det([A])\det([P]) = \det([A]),$$

once we note $\det([P]^{-1}) = \frac{1}{\det([P])}$.

(c) Let $\vec{\boldsymbol{v}}$ be an eigenvector of $[B]$ with eigenvalue $\lambda$ which means

$$[B]\vec{\boldsymbol{v}} = \lambda\vec{\boldsymbol{v}}.$$

Thus, we must have

$$[P]^{-1}[A][P]\vec{\boldsymbol{v}} = \lambda\vec{\boldsymbol{v}}.$$

In particular,

$$[A][P]\vec{\boldsymbol{v}} = \lambda[P]\vec{\boldsymbol{v}},$$

and we realize $[P]\vec{\boldsymbol{v}}$ is the eigenvector of $[A]$ with eigenvalue $\lambda$.

**Problem 7.** Compute the eigenvalues and eigenvectors for the following matrices.

(a) $[A] = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$.

(b) $[B] = \begin{pmatrix} 1 & 3 \\ 5 & 1 \end{pmatrix}$.

(c) $[C] = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

**Solution 7.**

(a) To find the eigenvalues of $[A]$ we first find the characteristic polynomial $\det([A] - \lambda[I])$. We have

$$[A] - \lambda[I] = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{pmatrix}.$$

Thus, the characteristic polynomial is $\det([A] - \lambda[I]) = (2-\lambda)(3-\lambda)$. The roots to the characteristic polynomial are the eigenvalues so we have

$$\lambda_1 = 2 \qquad \text{and} \qquad \lambda_2 = 3.$$

We can now find the corresponding eigenvectors.

**For $\lambda_1 = 2$:** We note the eigenvectors are elements of $\mathrm{Null}([A] - \lambda_1[I])$. We have

$$[A] - \lambda_1[I] = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

We get the augmented matrix

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right),$$

and we can take subtract R1 from R2 to get

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

This corresponds to two equations

$$\begin{aligned} 0x + y &= 0 \\ 0x + 0y &= 0. \end{aligned}$$

Thus, we must have $y = 0$, but $x$ is free to be anything. Note that $\vec{\mathbf{0}}$ is never an eigenvector (we don't allow it) but we can simply choose $x = 1$ and get the eigenvector

$$\vec{\boldsymbol{e}}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

**For $\lambda_2 = 3$:** We note the eigenvectors are elements of Null($[A] - \lambda_2[I]$). We have

$$[A] - \lambda_2[I] = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}.$$

We get the augmented matrix

$$\left(\begin{array}{cc|c} \text{-1} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

This corresponds to two equations

$$\begin{aligned} -x + y &= 0 \\ 0x + 0y &= 0. \end{aligned}$$

So, $x$ and $y$ are free to be anything so long as $x = y$. Choose $x = 1$ so that $y = 1$ as well and the eigenvector is

$$\vec{\boldsymbol{e}}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(b) The process is the same. We have

$$\det([B] - \lambda[I]) = (1 - \lambda)(1 - \lambda) - 15.$$

The roots to this polynomial are then

$$\lambda_1 = 1 - \sqrt{15} \qquad \text{and} \qquad \lambda_2 = 1 + \sqrt{15}.$$

**For $\lambda_1 = 1 - \sqrt{15}$:** We note the eigenvectors are elements of Null($[B] - \lambda_1[I]$). We have

$$[B] - \lambda_1[I] = \begin{pmatrix} \sqrt{15} & 3 \\ 5 & \sqrt{15} \end{pmatrix}.$$

We get the augmented matrix

$$\left(\begin{array}{cc|c} \sqrt{15} & 3 & 0 \\ 5 & \sqrt{15} & 0 \end{array}\right),$$

which can be reduced to

$$\left(\begin{array}{cc|c} 1 & \sqrt{\frac{3}{5}} & 0 \\ 0 & 0 & 0 \end{array}\right),$$

and so we see that $x = -\sqrt{\frac{3}{5}}y$. If we choose $y = 1$ then $x = -\sqrt{\frac{3}{5}}$ giving us the eigenvector

$$\vec{\boldsymbol{e}}_1 = \begin{pmatrix} -\sqrt{\frac{3}{5}} \\ 1 \end{pmatrix}.$$

**For $\lambda_2 = 1 + \sqrt{15}$:** We note the eigenvectors are elements of Null($[B] - \lambda_1[I]$). We have

$$[B] - \lambda_2[I] = \begin{pmatrix} -\sqrt{15} & 3 \\ 5 & -\sqrt{15} \end{pmatrix}.$$

We get the augmented matrix

$$\left(\begin{array}{cc|c} -\sqrt{15} & 3 & 0 \\ 5 & -\sqrt{15} & 0 \end{array}\right),$$

which can be reduced to

$$\left(\begin{array}{cc|c} 1 & -\sqrt{\frac{3}{5}} & 0 \\ 0 & 0 & 0 \end{array}\right),$$

and so we see that $x = \sqrt{\frac{3}{5}}y$. If we choose $y = 1$ then $x = \sqrt{\frac{3}{5}}$ giving us the eigenvector

$$\vec{\boldsymbol{e}}_2 = \begin{pmatrix} \sqrt{\frac{3}{5}} \\ 1 \end{pmatrix}.$$

(c) Note that $[C]$ is real and symmetric, so we expect the eigenvalues to be real and the eigenvectors to be orthogonal. Indeed, we have the characteristic polynomial

$$\det([C] - \lambda[I]) = -\lambda^3 + 2\lambda^2 + \lambda - 2,$$

which as roots

$$\lambda_1 = -1, \quad \lambda_2 = 1, \quad \lambda_3 = 2.$$

These roots are the eigenvalues and each is real.

**For $\lambda_1 = -1$:** We note the eigenvectors are elements of $\mathrm{Null}([B] - \lambda_1[I])$. We have

$$[C] - \lambda_1[I] = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

We get the augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right),$$

which can be reduced to

$$\left(\begin{array}{ccc|c} 1 & 0 & \text{-1} & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

and we note $x = z$ and $y = -2z$. If we then choose $z = 1$ then $x = 1$ and $y = 2$ giving us the eigenvector

$$\vec{\boldsymbol{e}}_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

**For $\lambda_2 = 1$:** We note the eigenvectors are elements of $\text{Null}([B] - \lambda_2[I])$. We have

$$[C] - \lambda_2[I] = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We get the augmented matrix

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right),$$

which can be reduced to

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

and we note $x = -z$ and $y = 0$. If we then choose $x = 1$ then $z = -1$ giving us the eigenvector

$$\vec{\boldsymbol{e}}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

**For $\lambda_3 = 2$:** We note the eigenvectors are elements of $\text{Null}([B] - \lambda_3[I])$. We have

$$[C] - \lambda_3[I] = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

We get the augmented matrix

$$\left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right),$$

which can be reduced to

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

and we note $x = z$ and $y = -z$. If we then choose $x = 1$ then $z = 1$ and $y = -1$ giving us the eigenvector

$$\vec{\boldsymbol{e}}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

One can then show that $\vec{\boldsymbol{e}}_1$, $\vec{\boldsymbol{e}}_2$, and $\vec{\boldsymbol{e}}_3$ are orthogonal!

**Problem 8.** Diagonalize the above matrices (if possible).

**Solution 8.**

(a) Our eigenvectors for $[A]$ are $\vec{\boldsymbol{e}}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{\boldsymbol{e}}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus we can take our $[P]$ matrix as

$$[P] = \begin{pmatrix} | & | \\ \vec{\boldsymbol{e}}_1 & \vec{\boldsymbol{e}}_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then we have

$$[P]^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Thus we can find a matrix $[D]$ similar to $[A]$ that is diagonal by

$$[D] = [P]^{-1}[A][P] = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Note that the eigenvalues are in order along the diagonal based on how we ordered the eigenvectors in $[P]$.

(b) Repeating now for $[B]$ we take

$$[P] = \begin{pmatrix} | & | \\ \vec{\boldsymbol{e}}_1 & \vec{\boldsymbol{e}}_2 \\ | & | \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{3}{5}} & 1 \\ \sqrt{\frac{3}{5}} & 1 \end{pmatrix},$$

and

$$[P]^{-1} = \begin{pmatrix} -\frac{\sqrt{15}}{6} & \frac{1}{2} \\ \frac{\sqrt{15}}{6} & \frac{1}{2} \end{pmatrix}$$

Then

$$[D] = [P]^{-1}[B][P] = \begin{pmatrix} 1 - \sqrt{15} & 0 \\ 0 & 1 + \sqrt{15} \end{pmatrix}.$$

(c) Finally for $[C]$ we have

$$[P] = \begin{pmatrix} | & | & | \\ \vec{\boldsymbol{e}}_1 & \vec{\boldsymbol{e}}_2 & \vec{\boldsymbol{e}}_2 \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix},$$

and

$$[P]^{-1} = \begin{pmatrix} -\frac{1}{2} & -1 & -\frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 1 & 1 & 1 \end{pmatrix}.$$

Then

$$[D] = [P]^{-1}[B][P] = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

**Problem 9.** Argue why the eigenvectors corresponding to a zero eigenvalue are elements of the nullspace.

**Solution 9.** This is tautologically true. Let $A$ be some linear transformation and let $\vec{\boldsymbol{e}}$ be an eigenvector with eigenvalue $\lambda = 0$. Then

$$A\vec{\boldsymbol{e}} = \lambda\vec{\boldsymbol{e}} = 0\vec{\boldsymbol{e}} = \vec{\mathbf{0}}.$$

So we also have $\vec{\boldsymbol{e}} \in \mathrm{Null}(A)$.

**Problem 10.** Show that there must be at least one zero eigenvalue if the determinant of a matrix is zero. Explain what this means geometrically and relate it beck to the geometric interpretation of the determinant.

**Solution 10.** Consider an $n \times n$ matrix $[A]$ then we have $n$ (possibly repeated and complex) eigenvalues $\lambda_1, \dots, \lambda_n$. We note that $\det([A])$ is the product of the eigenvalues so

$$\det([A]) = \lambda_1 \lambda_2 \cdots \lambda_n.$$

Thus, the only possible way the determinant can be zero is if at least one $\lambda_i = 0$.

To see this geometrically, we note that for $\lambda_i = 0$ we have (at least one) corresponding eigenvector $\vec{\boldsymbol{e}}_i$ satisfying $[A]\vec{\boldsymbol{e}}_i = \lambda_i \vec{\boldsymbol{e}} = \vec{\mathbf{0}}$. We see now that the linear transformation "squishes" down $\operatorname{Span}\{\vec{\boldsymbol{e}}\}$ to $\vec{\mathbf{0}}$. For example, in 3-dimensions we have the transformation we've described before given by

$$[A] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for which $\det([A]) = 0$ and $\hat{\boldsymbol{z}}$ is an eigenvector with eigenvalue zero. We note that for any vector $\vec{\boldsymbol{v}} \in \operatorname{Span}\{\hat{\boldsymbol{z}}\}$ we can put $\vec{\boldsymbol{v}} = \alpha \hat{\boldsymbol{z}}$ and find

$$[A]\vec{\boldsymbol{v}} = [A]\alpha\hat{\boldsymbol{z}} = \alpha[A]\hat{\boldsymbol{z}} = 0.$$

The whole $z$-axis is squished down to zero, and all vectors in $\mathbb{R}^3$ are sent to their "shadow" in the $xy$-plane. Any matrix that has a zero eigenvalue will have a direction squished like this!

**Problem 11.** Given the matrix

$$[A] = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

(a) Using the definition of the adjoint and hermitian (self-adjoint), show that $[A]$ is not hermitian.

(b) Show that there exists only one eigenvector for $[A]$ (e.g., one linearly independent vector in $\mathrm{Null}([A] - \lambda[I])$.

(c) Show that there exists two linearly independent vectors in $\mathrm{Null}(([A] - \lambda[I])^2)$.

**Solution 11.**

(a) Let $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \in \mathbb{C}^2$ be complex vectors given by $\vec{\boldsymbol{u}} = u_1\hat{\boldsymbol{x}} + u_2\hat{\boldsymbol{y}}$ and $\vec{\boldsymbol{v}} = v_1\hat{\boldsymbol{x}} + v_2\hat{\boldsymbol{y}}$ with $u_1, u_2, v_1, v_2 \in \mathbb{C}$. We first note

$$\begin{aligned} \langle [A]\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \rangle &= \langle (2u_1 + u_2)\hat{\boldsymbol{x}} + 2u_2\hat{\boldsymbol{y}}, v_1\hat{\boldsymbol{x}} + v_2\hat{\boldsymbol{y}} \rangle \\ &= (2u_1 + u_2)v_1^* + 2u_2v_2^*. \end{aligned}$$

Now, if $[A]$ was hermitian we'd have $\langle [A]\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \rangle = \langle \vec{\boldsymbol{u}}, [A]\vec{\boldsymbol{v}} \rangle$. Instead,

$$\begin{aligned} \langle \vec{\boldsymbol{u}}, [A]\vec{\boldsymbol{v}} \rangle &= \langle u_1\hat{\boldsymbol{x}} + u_2\hat{\boldsymbol{y}}, (2v_1 + v_2)\hat{\boldsymbol{x}} + 2v_2\hat{\boldsymbol{y}} \rangle \\ &= u_1(2v_1^* + v_2^*) + 2u_2v_2^*, \end{aligned}$$

which is not equal to $\langle [A]\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \rangle$ in general.

(b) We first find the eigenvalue by

$$\det([A] - \lambda[I]) = (2 - \lambda)^2.$$

So the eigenvalue is $\lambda = 2$. Now we find the eigenvectors which are in $\mathrm{Null}([A] - \lambda[I])$ by taking the augmented matrix for $[A] - \lambda[I]$

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

We note that $y = 0$ and we can choose $x = 1$ so the only eigenvector is

$$\vec{\boldsymbol{e}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(c) We have

$$([A] - \lambda[I])^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We can note that $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ are both in $\mathrm{Null}(([A] - \lambda[I])^2)$ and they are linearly independent.