

MATH 271, WORKSHEET 4  
SEQUENCES AND SERIES

**Problem 1.** Write down the first few terms in the sequence for the following:

- (a)  $a_n = n$ ;
- (b)  $b_n = \frac{1}{n^2}$ ;
- (c)  $c_n = 2^{-n}$ .

**Problem 2.** For the above sequences, state whether each converges or diverges. If they converge, state the limit.

**Problem 3.** Consider the recursive sequence

$$a_n = \frac{1}{2}a_{n-1} + 1$$

with  $a_1 = 1$ .

- (a) Write the first few terms in the sequence.
- (b) Can you write  $a_n$  as a function  $f(n)$ ? If so, what is  $f(n)$ ?
- (c) Does this sequence converge or diverge? Can you show why with a limit  $\lim_{n \rightarrow \infty} f(n)$ ?
- (d) Can you show that this is a Cauchy sequence?

**Problem 4.** Consider the sequence

$$a_n = ar^n.$$

- (a) If  $|r| < 1$ , show that this sequence  $\{a_n\}_{n=0}^{\infty}$  converges to zero.
- (b) Consider now the *geometric series*

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} ar^n.$$

Show that the  $N^{\text{th}}$  partial sum for this series satisfies

$$\sum_{n=0}^N ar^n = a \left( \frac{1 - r^{N+1}}{1 - r} \right).$$

- (c) Does the geometric series converge for all  $r$ ? For  $|r| < 1$ ? When it converges, what does it converge to?

**Problem 5.** Often we wish to think about functions being represented by series. For example, we can consider the function

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

where  $n!$  is read as “ $n$ -factorial” and

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1.$$

Then  $1! = 1$  and we define  $0! = 1$  as well.

(a) Consider  $f(1)$ . Use a tool like WolframAlpha to compute the series

$$f(1) = \sum_{n=0}^{\infty} \frac{1}{n!}$$

(b) For any value of  $x$ , this series converges. So this defines a function on all real numbers. In fact, the series converges even for complex numbers. Simplify the series into its real and imaginary parts. Note,

$$f(ix) = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}.$$

(c) We can take derivatives of the function  $f(x)$  by differentiating the series *term by term*. That is,

$$\frac{d}{dx} f(x) = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{d}{dx} \left( \frac{x^n}{n!} \right).$$

(d) Show that  $\frac{d}{dx} f(x) = f(x)$ .

(e) What is your guess for what function  $f(x)$  is?