MATH 272, Homework 8

Due April 6^{th}

Problem 1. Plot each of the following vector fields.

(a) $\hat{\boldsymbol{r}} = \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{x}} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{y}} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{z}}$.

(b) $\hat{\boldsymbol{\theta}} = \frac{-y}{\sqrt{x^2+y^2}}\hat{\boldsymbol{x}} + \frac{x}{\sqrt{x^2+y^2}}\hat{\boldsymbol{y}}$.

(c) $\hat{\boldsymbol{\phi}} = \frac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{x}} + \frac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{y}} + \frac{-\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{z}}$.

Problem 2. Consider the following vector field

$$\vec{\boldsymbol{E}} = \frac{x}{(x^2+y^2+z^2)^{3/2}}\hat{\boldsymbol{x}} + \frac{y}{(x^2+y^2+z^2)^{3/2}}\hat{\boldsymbol{y}} + \frac{z}{(x^2+y^2+z^2)^{3/2}}\hat{\boldsymbol{z}},$$

which you can think of as the electric field of a positive point charge. We argued that this field $\vec{\boldsymbol{E}}$ is conservative in a previous homework problem. Specifically, $\vec{\boldsymbol{E}} = \vec{\boldsymbol{\nabla}}\phi$, for some scalar field ϕ. This follows from Faraday's law for static charges.

(a) Compute the integral

$$T = \int_{\vec{\gamma}} \vec{\boldsymbol{E}} \cdot d\vec{\gamma} \qquad \text{where} \qquad \vec{\gamma}(t) = \begin{pmatrix} t \\ t \\ t \end{pmatrix},$$

and $a \leq t \leq b$ with a and b both greater than 0. Note that this integral T describes the gain in kinetic energy of a charged particle that moved along the path $\vec{\gamma}$.

(b) Equivalently, since $\vec{\boldsymbol{E}}$ is conservative, we have

$$T = \int_{\vec{\gamma}} \vec{\boldsymbol{E}} \cdot d\vec{\gamma} = \phi(\vec{\gamma}(b)) - \phi(\vec{\gamma}(a)).$$

Show that this is true for the given vector field and potential. This shows that the choice of path does not matter; only the endpoints $\vec{\gamma}(a)$ and $\vec{\gamma}(b)$ matter.

(c) Argue why the integral around any closed curve must be zero.

Problem 3. Let us see some of the benefit of using spherical coordinates.

(a) Using the fact that

$$\hat{\boldsymbol{r}} = \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{x}} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{y}} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{\boldsymbol{z}},$$

convert the vector field $\vec{\boldsymbol{E}}$ into spherical coordinates (i.e., only a function of r, θ, ϕ, and $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$).

(b) Parameterize the surface of a sphere of radius R (which we'll call Σ) as well as the outward normal vector $\hat{\boldsymbol{n}}$ and in spherical coordinates.

(c) Compute the following integral using spherical coordinates that we have found:

$$\iint_{\Sigma} \vec{\boldsymbol{E}} \cdot \hat{\boldsymbol{n}} d\Sigma,$$

where $d\Sigma$ will be the area form in spherical coordinates.

Problem 4. Note that the Laplacian Δ in cylindrical coordinates is given by

$$\Delta f(\rho, \theta, z) = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}.$$

Compute the Laplacian of

$$f(\rho, \theta, z) = \sqrt{\rho^2 + z^2} z \cos(\theta).$$

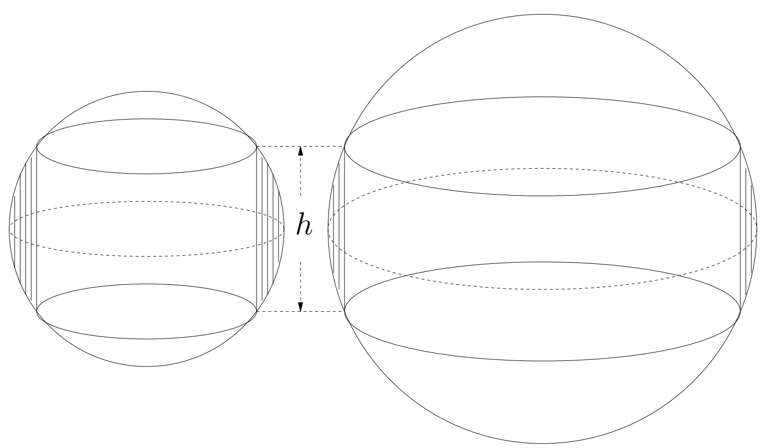
Problem 5. Note that the Laplacian Δ in spherical coordinates is given by

$$\Delta f(r, \theta, \phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin^2 \phi}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2 \sin \phi}\frac{\partial}{\partial \phi}\left(\sin \phi \frac{\partial f}{\partial \phi}\right).$$

Compute the Laplacian of

$$f(r, \theta, \phi) = r^2 \cos(\theta) \cos(\phi).$$

Problem 6. (BONUS) The following problem is a somewhat pop-culture math paradox known as the *napkin ring problem* (see Vsauce for more). Consider the following problem. We want to compute the volume inside a ball of radius R after drilling out an inscribed cylinder of height h. See the following picture.



The question is, does the left over volume (of the napkin ring) depend on the radius R of the sphere. You have your choice of working in spherical or cylindrical coordinates. Use whichever helps you most.