MATH 271, Worksheet 10

Inverse and similar matrices. Eigenvalue problem and diagonalization.
Hermitian matrices.

Problem 1. Consider the two matrices

$$[A] = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix} \qquad \text{and} \qquad [B] = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

(a) Argue why the matrix $[A]$ cannot be invertible. *Hint: you can use ideas from Problem 9 to show this.*

(b) Compute the inverse matrix $[B]^{-1}$ for $[B]$.

(c) Solve the system of equations $[B]\vec{\boldsymbol{x}} = \vec{\boldsymbol{y}}$ for the following vectors.

i. $\vec{\boldsymbol{y}} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

ii. $\vec{\boldsymbol{y}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

iii. $\vec{\boldsymbol{y}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Problem 2. Consider the matrices

$$[A] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad [B] = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

(a) Show that $[A]$ and $[B]$ are both invertible.

(b) Find $[A]^{-1}$ and $[B]^{-1}$.

(c) Show that $([A][B])^{-1} = [B]^{-1}[A]^{-1}$.

Problem 3. Simplify the following expressions.

(a) $([A][B])^{-1}[A][B]$.

(b) $[A]^2[B]^3[A]([A][B])^{-1}$.

(c) $([A][B][C]^{-1})^{-1}[A][B][C]^{-1}$.

Problem 4. Show that for any invertible matrix $[A]$ that $\det([A]^{-1}) = \frac{1}{\det([A])}$.

Problem 5. Let $[B]$ be similar to $[A]$ by the relationship $[B] = [P]^{-1}[A][P]$.

(a) Given that $[P]$ is invertible, show that $[P]$ transforms the standard basis $\hat{\boldsymbol{x}}_1, \hat{\boldsymbol{x}}_2, \dots, \hat{\boldsymbol{x}}_n$ into a new basis given by the columns of $[P]$.

(b) Show that $[P]^{-1}$ transforms the basis given by the columns of $[P]$ into the standard basis.

(c) Explain why $[B]$ performs the same transformation as $[A]$ but just on a different basis (e.g., different choices of coordinates).

Problem 6. Let $[B]$ be similar to $[A]$ by the relationship $[B] = [P]^{-1}[A][P]$.

(a) Show that the trace is invariant under similarity. That is, show $\mathrm{tr}([A]) = \mathrm{tr}([B])$.

(b) Show that the determinant is invariant under similarity. *Hint: you will need to use the result from Problem 4.*

(c) Show that $[A]$ and $[B]$ have the same eigenvalues. It may help to think that if we have $\vec{\boldsymbol{v}}$ as an eigenvector for $[A]$, then what is the corresponding eigenvector for $[B]$?

Problem 7. Compute the eigenvalues and eigenvectors for the following matrices.

(a) $[A] = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$.

(b) $[B] = \begin{pmatrix} 1 & 3 \\ 5 & 1 \end{pmatrix}$.

(c) $[C] = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

Problem 8. Diagonalize the above matrices (if possible).

Problem 9. Argue why the eigenvectors corresponding to a zero eigenvalue are elements of the nullspace.

Problem 10. Show that there must be at least one zero eigenvalue if the determinant of a matrix is zero. Explain what this means geometrically and relate it beck to the geometric interpretation of the determinant.

Problem 11. Given the matrix

$$[A] = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

(a) Using the definition of the adjoint and hermitian (self-adjoint), show that $[A]$ is not hermitian.

(b) Show that there exists only one eigenvector for $[A]$ (e.g., one linearly independent vector in $\mathrm{Null}([A] - \lambda[I])$.

(c) Show that there exists two linearly independent vectors in $\mathrm{Null}(([A] - \lambda[I])^2)$.