MATH 272, Worksheet 8

Potentials and surfaces.

Problem 1. Note that the identity $\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\nabla}} f) = \vec{\mathbf{0}}$ always holds for any smooth scalar field f.

- Pick a few functions $f(x,y)$ of your own and plot the graphs $z = f(x,y)$ and plot the vector field $\vec{\boldsymbol{\nabla}} f$ as well. Can you reason why the identity must be true from these plots?
- If you plot the vector field $\vec{\boldsymbol{V}} = \begin{pmatrix} 0 \\ x \end{pmatrix}$ (which has nonzero curl), could this have come from the gradient of some function? What would the surface have to look like in order to have this as a gradient? Could it even be a valid function/surface?

Problem 2. Decide whether the following fields have potentials. If so, determine what they are. Plot the fields as well.

(a) $\vec{\boldsymbol{U}}(x,y,z) = \begin{pmatrix} 2x+2y+2z \\ 2x+2y+2z \\ 2x+2y+2z \end{pmatrix}$.

(b) $\vec{\boldsymbol{V}}(x,y,z) = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}$.

(c) $\vec{\boldsymbol{W}}(x,y,z) = \begin{pmatrix} e^y \\ e^x \\ \sin(x)\sin(y) \end{pmatrix}$.

Problem 3. For the fields in Problem 2, take the curves $\vec{\boldsymbol{\gamma}}_1(t) = \begin{pmatrix} t \\ t \\ t \end{pmatrix}$ and $\vec{\boldsymbol{\gamma}}_2(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$ from time $t=0$ to time $t=1$ and integrate

$$\int_{\vec{\boldsymbol{\gamma}}_i} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{\gamma}}_i.$$

For which fields should this integral <u>not</u> depend on the choice of curve? Recall we refer to these fields whose that are independent of the choice of path from point a to point b as conservative.

Problem 4. *** Given a conservative vector field $\vec{\boldsymbol{V}}$ and a curve $\vec{\boldsymbol{\gamma}}\colon [a,b] \to \mathbb{R}^3$, we know that

$$\int_{\vec{\boldsymbol{\gamma}}} \vec{\boldsymbol{V}} \cdot d\vec{\boldsymbol{\gamma}},$$

only depends on the start and end points of the curve $\vec{\boldsymbol{\gamma}}$. That is, if we fix $\vec{\gamma}(a)$ and $\vec{\gamma}(b)$, the path between those two points does not change the integral.

If $\vec{\boldsymbol{V}}$ is conservative, then $\vec{\boldsymbol{V}} = \vec{\boldsymbol{\nabla}} f$ for some scalar field f. This yields the identity,

$$\int_{\vec{\gamma}} \left(\vec{\boldsymbol{\nabla}} f\right) \cdot d\vec{\gamma} = f(\vec{\gamma}(b)) - f(\vec{\gamma}(a)). \tag{1}$$

This is, once again, some type of generalization of the Fundamental Theorem of Calculus (FTC) via the very general Stokes' theorem.

(a) Show that the above identity in Equation (1) is nothing but FTC. *Hint: take your (1-dimensional) curve* $\vec{\gamma}(x) = x$ *so that* $\vec{\gamma}(a) = a$ *and* $\vec{\gamma}(b) = b$. *Finally, note* $\vec{\boldsymbol{\nabla}} = \frac{d}{dx}$.

(b) Consider now a different curve $\vec{\boldsymbol{\eta}}\colon [\tilde{a}, \tilde{b}] \to \mathbb{R}$. So long as $\vec{\boldsymbol{\eta}}(\tilde{a}) = a$ and $\vec{\boldsymbol{\eta}}(\tilde{b}) = b$, our identity states that the integral should output the same value. Realize this as the u-substitution (or change of variables) that you learned in Calc. 1.

(c) Now, in 3-dimensions, we can discretize any curve to n small movements in the x-, y-, or z-direction, and in each direction FTC will hold. Thus, summing up the n integrals (one from each movement) will cancel off many contributions and leave you only with the beginning and end point of the whole curve as a contribution. Taking the limit that these movements are dx, dy, and dz, one can see that a choice of path for a smooth curve will not matter. Draw a picture of this argument and clarify the approach to a proof.

Problem 5. For the following scalar fields f, decide whether the level set is a surface or not. If the level set is a surface, what portions (if any) can be described as a graph $z = g(x, y)$? What about as graphs $y = h(x, z)$ or $x = p(y, z)$? Plot after your analysis to see if your predictions are correct.

(a) $f(x, y, z) = xyz$ with $C = 1$.

(b) $f(x, y, z) = xyz$ with $C = 0$.

(c) $f(x, y, z) = z(x - y) + z(e^x - e^y)$ with $C = 1$.

(d) $f(x, y, z) = z + \cos(xy)$ with $C = 0$.

(e) $f(x, y, z) = z \sin(x) \sin(y) \sin(z)$ with $C = 1$.

Problem 6. For the sets above that do indeed describe surfaces, find the surface normal $\hat{\boldsymbol{n}}$ and the area form $d\Sigma$.

Problem 7. Consider the surface of the unit cube in $\mathbb{R}^3$ which we'll call Σ. Does the normal $\hat{\boldsymbol{n}}$ and the area form $d\Sigma$ change in smooth way as we move along the cube? What is the (right handed) area form on each portion of the cube? Is there a well defined $\hat{\boldsymbol{n}}$ along edges or corners? Why does this not matter when we consider the total flux of some vector field $\vec{\boldsymbol{V}}$,

$$\iint_{\Sigma} \vec{\boldsymbol{V}} \cdot \hat{\boldsymbol{n}} d\Sigma?$$

Problem 8. Consider the surface defined by the graph of $z = \sqrt{1-x^2-y^2}$ for $x^2+y^2 \leq 1$. Become familiar with this example!

(a) Plot this surface.

(b) Plot the image of following curves on the surface Σ.

- $\vec{\gamma}_1(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}$.
- $\vec{\gamma}_2(t) = \begin{pmatrix} 0 \\ t \end{pmatrix}$.
- $\vec{\gamma}_3(t) = \begin{pmatrix} \frac{1}{2}\cos(t) \\ \frac{1}{2}\sin(t) \end{pmatrix}$.

Which (if any) correspond to a line of longitude or a line of latitude?

(c) Find an equation for the tangent plane at the point $(0,0,1)$.

(d) Note that the point $(0.1, 0.1, 1)$ is on this tangent plane. How close is this point to corresponding point on the sphere? That is, how close is $(0.1, 0.1, 1)$ to the point $(0.1, 0.1, \sqrt{1-(0.1)^2-(0.1)^2})$? Is the tangent plane a reasonable approximation? What if instead we take $(0.01, 0.01, 1)$ instead?

(e) Find the surface normal $\hat{\boldsymbol{n}}$.

(f) What is the area form $d\Sigma$?

(g) Set up an integral that computes the surface area of Σ.

(h) Set up an integral that computes the total flux of the vector field $\vec{\boldsymbol{V}}(x,y,z) = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}$ through the surface.

Problem 9. Consider the surface defined by the graph of $z = \sin(x)\sin(y)$ for $0 \leq x \leq \pi$ and $0 \leq y \leq \pi$.

(a) Consider as well the curve $\vec{\gamma}(t) = \begin{pmatrix} t \\ t \end{pmatrix}$. What is the area under the curve if we take its image on the surface?

(b) Find the area form $d\Sigma$.

(c) If we have as well a scalar function $f(x,y,z) = x^2+y^2+z^2$, compute the integral

$$\iint_{\Sigma} f d\Sigma.$$

Problem 10. Our surfaces have been frozen in time. However, essentially every physical phenomenon evolves over time. There are a few ways surfaces arise when time is involved. Let us consider two examples.

(a) Consider the two variable scalar field $T(x,t) = \sin(x)e^{-t}$ with $0 \leq x \leq L$ and $t \geq 0$.

- Show that this function satisfies $\left(-\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\right) T(x,t) = 0$. This is known as the 1-dimensional heat equation. Here $T(x,t)$ models the temperature of point x at time t on a rod of length L.
- Plot the graph $z = T(x,t)$ for $0 \leq x \leq L$ and $t \geq 0$. What can we say about the temperature of the rod as $t \to \infty$?

(b) Consider the three variable scalar field $u(x,y,t) = \sin(mx)\sin(ny)\sin(t)$ with $0 \leq x \leq \pi$, $0 \leq y \leq \pi$, $t \geq 0$, and m and n are positive integers.

- Show that this function satisfies $\left(-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial t^2}\right) u(x,y,t) = 0$. This is known as the 2-dimensional wave equation. Here, $u(x,y,t)$ models the height of a membrane at the point (x,y) and time t.
- Plot the graph of the surface $u(x,y,t_0)$ for various values of t_0, m and n. Or, visit `https://www.geogebra.org/3d/y55rd83m` to have full freedom with this surface (and watch it move over time).