MATH 272, Worksheet 3

Integration over curves and potential functions. *Solutions.*

Problem 1. Draw a picture explaining what an integral of a scalar field over a curve is computing. That is, explain the reasoning behind the definition

$$\int_{\vec{\gamma}} f(\gamma) d\vec{\gamma} = \int_{t_0}^{t_1} f(\gamma(t)) \left| \dot{\vec{\gamma}}(t) \right| dt,$$

where f is a scalar field and $\vec{\gamma}$ is some curve starting at time t_0 and ending at time t_1.

Solution 1.

Problem 2. Compute the integrals of the scalar field $f(x, y, z) = 2 - x + y - z$ over the following curves.

(a) $\vec{\gamma}_1$ is the boundary of the unit square in the xy-plane.

(b) $\vec{\gamma}_2$ is the unit circle in the xy-plane.

(c) $\vec{\gamma}_3$ is the curve $\vec{\gamma}_3(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$ from time $t = 0$ to $t = 1$.

Solution 2.

Problem 3. Draw a picture explaining what an integral of a vector field over a curve is computing. That is, explain the reasoning behind the definition

$$\int_{\vec{\gamma}} \vec{\boldsymbol{V}}(\vec{\gamma}) \cdot d\vec{\gamma} = \int_{t_0}^{t_1} \vec{\boldsymbol{V}}(\vec{\gamma}(t)) \cdot \dot{\vec{\gamma}}(t) dt,$$

where $\vec{\boldsymbol{V}}$ is a vector field and $\vec{\gamma}$ is some curve starting at time t_0 and ending at time t_1.

Solution 3.

Problem 4. We have briefly discussed the idea of *work* (change in energy) before and wrote

$$W = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{r}},$$

where $\vec{\boldsymbol{F}}$ was a constant force and $\vec{\boldsymbol{r}}$ was a straight line displacement.

Now, we can write the real version of this. The work done on a particle moving along a curve $\vec{\gamma}(t)$ that is experiencing a spatially dependent force field $\vec{\boldsymbol{F}}(x, y, z)$ is

$$W = \int_{\vec{\gamma}} \vec{\boldsymbol{F}}(\vec{\gamma}) \cdot d\vec{\gamma}.$$

Compute the work given the following

$$\vec{\boldsymbol{F}}(x, y, z) = \begin{pmatrix} x^2 \\ y \\ \sqrt{z} \end{pmatrix} \qquad \text{and} \qquad \vec{\gamma}(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}.$$

Solution 4.

Problem 5. Note that the identity $\vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\nabla}} f) = \vec{\mathbf{0}}$ always holds for any smooth scalar field f.

- Pick a few functions $f(x, y)$ of your own and plot the graphs $z = f(x, y)$ and plot the vector field $\vec{\boldsymbol{\nabla}} f$ as well. Can you reason why the identity must be true from these plots?
- If you plot the vector field $\vec{\boldsymbol{V}} = \begin{pmatrix} 0 \\ x \end{pmatrix}$ (which has nonzero curl), could this have come from the gradient of some function? What would the surface have to look like in order to have this as a gradient? Could it even be a valid function/surface?

Solution 5.

Problem 6. Decide whether the following fields have potentials. Explain your reasoning. If they do have a potential, determine what it is. Plot the vector fields as well.

(a) $\vec{\boldsymbol{U}}(x,y,z) = \begin{pmatrix} 2x+2y+2z \\ 2x+2y+2z \\ 2x+2y+2z \end{pmatrix}.$

(b) $\vec{\boldsymbol{V}}(x,y,z) = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}.$

(c) $\vec{\boldsymbol{W}}(x,y,z) = \begin{pmatrix} e^y \\ e^x \\ \sin(x)\sin(y) \end{pmatrix}.$

Solution 6.

Problem 7. Consider the vector fields in Problem 6.

(a) Take the curves $\vec{\gamma}_1(t) = \begin{pmatrix} t \\ t \\ t \end{pmatrix}$ and $\vec{\gamma}_2(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$ from time $t = 0$ to time $t = 1$ and integrate

$$\int_{\vec{\gamma}_i} \vec{\boldsymbol{F}} \cdot d\vec{\gamma}_i.$$

for the given vector fields.

(b) For which fields should this integral <u>not</u> depend on the choice of curve? In other words, which of the vector fields are conservative?

(c) Compute

$$\int_{\vec{\gamma}_2} (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{W}}) \cdot d\vec{\gamma}_2.$$

(d) Compute

$$\int_{\vec{\gamma}_1} (\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{U}})(\vec{\gamma}) d\vec{\gamma}_1.$$

Solution 7.

Problem 8. Consider the vector field $\vec{\boldsymbol{V}}(x,y,z) = \begin{pmatrix} yz\cos(xyz) \\ xz\cos(xyz) \\ xy\cos(xyz) \end{pmatrix}$.

(a) Show that $\vec{\boldsymbol{V}}$ is conservative.

(b) Find the potential function f for $\vec{\boldsymbol{V}}$.

(c) Let $\gamma(t) = \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}$ running from $t_0 = 0$ to $t_1 = 1$. Show that

$$\int_{\vec{\gamma}} \vec{\boldsymbol{V}} \cdot d\vec{\gamma} = f(\vec{\gamma}(t_1)) - f(\vec{\gamma}(t_0)).$$

(d) If you knew (c) was true, does this prove that $\vec{\boldsymbol{V}}$ is conservative? Why or why not?

Solution 8.

Problem 9. *** Given a conservative vector field $\vec{\boldsymbol{V}}$ and a curve $\vec{\boldsymbol{\gamma}}\colon [t_0, t_1] \to \mathbb{R}^3$, we know that

$$\int_{\vec{\gamma}} \vec{\boldsymbol{V}} \cdot d\vec{\gamma},$$

only depends on the start and end points of the curve $\vec{\boldsymbol{\gamma}}$. That is, if we fix $\vec{\boldsymbol{\gamma}}(a)$ and $\vec{\boldsymbol{\gamma}}(b)$, the path between those two points does <u>not</u> change the integral.

If $\vec{\boldsymbol{V}}$ is conservative, then $\vec{\boldsymbol{V}} = \vec{\boldsymbol{\nabla}} f$ for some scalar field f. This yields the identity,

$$\int_{\vec{\gamma}} \left(\vec{\boldsymbol{\nabla}} f\right) \cdot d\vec{\gamma} = f(\vec{\gamma}(t_1)) - f(\vec{\gamma}(t_0)). \tag{1}$$

This is, once again, some type of generalization of the Fundamental Theorem of Calculus (FTC) via the very general Stokes' theorem.

(a) Show that the above identity in Equation (1) is nothing but FTC. *Hint: take your (1-dimensional) curve $\vec{\boldsymbol{\gamma}}(x) = x$ so that $\vec{\boldsymbol{\gamma}}(t_0) = t_0$ and $\vec{\boldsymbol{\gamma}}(t_1) = t_1$. Finally, note $\vec{\boldsymbol{\nabla}} = \frac{d}{dx}$.*

(b) Consider now a different curve $\vec{\boldsymbol{\eta}}\colon [\tilde{t_0}, \tilde{t_1}] \to \mathbb{R}$. So long as $\vec{\boldsymbol{\eta}}(\tilde{t_0}) = t_0$ and $\vec{\boldsymbol{\eta}}(\tilde{t_1}) = t_1$, our identity states that the integral should output the same value. Realize this as the u-substitution (or change of variables) that you learned in Calc. 1.

(c) Now, in 3-dimensions, we can discretize any curve to n small movements in the x-, y-, or z-direction, and in each direction FTC will hold. Thus, summing up the n integrals (one from each movement) will cancel off many contributions and leave you only with the beginning and end point of the whole curve as a contribution. Taking the limit that these movements are dx, dy, and dz, one can see that a choice of path for a smooth curve will not matter. Draw a picture of this argument and clarify the approach to a proof.

Solution 9.