MATH 271, Worksheet 4
Second Order Linear Equations and Boundary Value Problems

Problem 1. Write down the characteristic polynomial for the following equations. Then, find the roots to the characteristic polynomial and write down the general solution.

(a) $x'' + x' + x = 0$.

(b) $x'' - x' - x = 0$.

(c) $x'' - x' + x = 0$.

(d) $x'' + x' - x = 0$.

Problem 2. For the above solutions, analyze their behavior qualitatively. That is, do the solutions oscillate, grow, decay, or some combination of these, or something else entirely?

Problem 3. Consider the equation

$$x'' + bx' + cx = 0.$$

The roots to the characteristic polynomial are then

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

(a) Explain why if $c > 0$ and $b = 0$ the solution $x(t)$ will be purely oscillatory.

(b) Explain why if $b > 0$ and $b^2 < 4c$, the solution will oscillate and decay.

(c) Explain why if $b < 0$ and $b^2 < 4c$, the solution will oscillate and grow.

Problem 4. Write down a second order linear differential equation that oscillates and also decays over time.

Problem 5. Consider the following differential equation

$$x'' + x = 0.$$

(a) Find the general solution to this equation.

(b) Given the initial conditions $x(0) = 1$ and $x'(0) = 1$, find the particular solution.

(c) Plot your particular solution.

(d) Does the solution grow or decay over time?

(e) What is $\lim_{t\to\infty} x(t)$?

Problem 6. Next, consider a related equation

$$x'' + x = t.$$

that has an additional linear external force.

(a) What is the solution to the homogenous equation?

(b) Find the particular integral with the given forcing term.

(c) What is the specific solution to this equation?

(d) Does the solution grow or decay over time?

(e) What is $\lim_{t\to\infty} x(t)$?

Problem 7. Consider now the equation

$$x'' + x = F(t)$$

where the external force is $F(t) = \cos(t)$.

(a) Find the particular integral with the given forcing term.

(b) What is the specific solution to this equation?

(c) What is $\lim_{t\to\infty}$? What does this mean about the growth or decay of the solution over time?

Problem 8. Consider the boundary value problem

$$x'' = g$$

with boundary values $x(0) = 0$, $x\left(-\frac{2}{g}\right) = 0$ and $g = -9.8[m/s^2]$. We can think of this as solving the *inverse problem* of one that we have seen in a homework. Specifically, think of this as knowing where a ball is launched and knowing where it lands and trying to find the speed it must have been thrown at.

Another interpretation is the shape of a rod bending due to gravity. x'' would measure the curvature of this rod, and this equation would say that the rod under the force of gravity would have a constant curvature. In this case, the dependent variable t should be thought of as spatial rather than temporal.

Finally, this equation above is referred to as *Poisson's equation*.

(a) Find the general solution. If you already know it from the homework, just write it down.

(b) Use the boundary values above to find the particular solution.

(c) Is the solution unique?

Problem 9. Consider the *time independent Schödinger equation* for a *free particle* constrained inside of a 1-dimensional box of length L. That is, we have the equation

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x)$$

on the unit interval $[0, L]$.

(a) Find the general solution to this equation with no constraint.

(b) Given the constraint, we have the boundary values $\psi(0) = \psi(L) = 0$. What are the general solutions given this constraint?

(c) Show that the sum of two solution $\psi_1(x)$ and $\psi_2(x)$ is also a solution. When we have a particle whose state (or *wavefunction*) ψ is a sum of general solutions, we say that ψ is in a *superposition state.*

(d) The wavefunction is not really a physically meaningful quantity. However, if we consider a region $[a, b]$ in the box $[0, L]$ the quantity

$$P([a, b]) = \int_a^b |\psi(x)|^2 dx$$

is meaningful. This expression tells us the *probability* that a particle will be observed in the region $[a, b]$. Take your general solutions you found in (b) (with the constraint) and solve for the constants that give you

$$\int_0^L |\psi(x)|^2 dx = 1.$$

We call this *normalization* and we must do so for each state so that we can interpret the integral $P([a, b])$ as a probability.