MATH 271, Worksheet 6
Vectors and Vector Spaces

Problem 1. Compare and contrast the structure of the complex numbers $\mathbb{C}$ with the vector space $\mathbb{R}^2$. Note any differences and similarities.

Problem 2. Let $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \in \mathbb{R}^2$ be given by

$$\vec{\boldsymbol{u}} = 2\hat{\boldsymbol{x}} + 3\hat{\boldsymbol{y}} \qquad \text{and} \qquad \vec{\boldsymbol{v}} = -\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}.$$

(a) Draw $\vec{\boldsymbol{u}}$, $\vec{\boldsymbol{v}}$, and $\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}$ in the plane.

(b) Compute $\|\vec{\boldsymbol{u}}\|$ and $\|\vec{\boldsymbol{v}}\|$.

(c) Compute $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}$.

(d) Find a vector orthogonal to $\vec{\boldsymbol{u}}$.

Problem 3. Let $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \in \mathbb{R}^3$ be given by

$$\vec{\boldsymbol{u}} = \hat{\boldsymbol{x}} - \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}} \qquad \text{and} \qquad \vec{\boldsymbol{v}} = -\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} - \hat{\boldsymbol{z}}.$$

(a) Are $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ orthogonal?

(b) Normalize $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ to get $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{v}}$.

(c) Compute the projection of $\vec{\boldsymbol{v}}$ onto the direction defined by $\vec{\boldsymbol{u}}$.

Problem 4. Let $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \in \mathbb{R}^3$ be given by

$$\vec{\boldsymbol{u}} = -3\hat{\boldsymbol{x}} - 2\hat{\boldsymbol{y}} + \hat{\boldsymbol{z}} \qquad \text{and} \qquad \vec{\boldsymbol{v}} = \hat{\boldsymbol{x}} - 2\hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}.$$

(a) Compute the angle between $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$.

(b) Without computing the cross product, compute the area of the parallelogram generated by $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$. *Hint: you know the angle between the vectors, use this fact.*

(c) Without computing the cross product, what component of the product $\vec{\boldsymbol{u}} \times \hat{\boldsymbol{x}}$ must be zero?

(d) Compute $\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}$.

(e) Give a geometrical interpretation of the cross product $\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}$. Explain why $\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}} = -\vec{\boldsymbol{v}} \times \vec{\boldsymbol{u}}$.

Problem 5. Recall that the states found in the solution to the free particle in a 1-dimensional box of length L were $\psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$. Let S denote the set of all solutions to the free particle in a 1-dimensional box boundary value problem. Show that a superposition of states (with coefficients in $\mathbb{C}$) is also a solution. That is, if we let $\Psi(x) = \alpha_j(x)\psi_j + \alpha_k\psi_k(x)$, then $\Psi(x)$ is also a solution to the boundary value problem

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} = E\Psi$$

with boundary values $\Psi(0) = 0$ and $\Psi(L) = 0$.

Note that this shows that the set S is a vector space over the complex numbers $\mathbb{C}$.