

MATH 271, WORKSHEET 7
VECTORS, VECTOR SPACES, AND LINEAR TRANSFORMATIONS.

Problem 1. Compare and contrast the structure of the complex numbers \mathbb{C} with the vector space \mathbb{R}^2 . Note any differences and similarities. Can you multiply vectors in \mathbb{R}^2 ?

Problem 2. Let $\vec{u}, \vec{v} \in \mathbb{R}^2$ be given by

$$\vec{u} = 2\hat{x} + 3\hat{y} \quad \text{and} \quad \vec{v} = -\hat{x} + \hat{y}.$$

- (a) Draw \vec{u} , \vec{v} , and $\vec{u} + \vec{v}$ in the plane.
- (b) Compute $\|\vec{u}\|$ and $\|\vec{v}\|$.
- (c) Compute $\vec{u} \cdot \vec{v}$.
- (d) Find a vector orthogonal to \vec{u} .

Problem 3. Let $\vec{u}, \vec{v} \in \mathbb{R}^3$ be given by

$$\vec{u} = \hat{x} - \hat{y} + \hat{z} \quad \text{and} \quad \vec{v} = -\hat{x} + \hat{y} - \hat{z}.$$

- (a) Are \vec{u} and \vec{v} orthogonal?
- (b) Normalize \vec{u} and \vec{v} to get \hat{u} and \hat{v} .
- (c) Compute the projection of \vec{v} onto the direction defined by \vec{u} .

Problem 4. Let $\vec{u}, \vec{v} \in \mathbb{R}^3$ be given by

$$\vec{u} = -3\hat{x} - 2\hat{y} + \hat{z} \quad \text{and} \quad \vec{v} = \hat{x} - 2\hat{y} + \hat{z}.$$

- (a) Compute the angle between \vec{u} and \vec{v} .
- (b) Without computing the cross product, compute the area of the parallelogram generated by \vec{u} and \vec{v} . *Hint: you know the angle between the vectors, use this fact.*
- (c) Without computing the cross product, what component of the product $\vec{u} \times \hat{x}$ must be zero?
- (d) Compute $\vec{u} \times \vec{v}$.
- (e) Give a geometrical interpretation of the cross product $\vec{u} \times \vec{v}$. Explain why $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$.

Problem 5. Recall that the states found in the solution to the free particle in a 1-dimensional box of length L were $\psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$. Let S denote the set of all solutions to the free particle in a 1-dimensional box boundary value problem. Show that a superposition of states (with coefficients in \mathbb{C}) is **NOT** a solution. That is, if we let $\Psi(x) = \alpha_j(x)\psi_j + \alpha_k\psi_k(x)$, then $\Psi(x)$ is **NOT** solution to the boundary value problem

$$-\frac{\hbar^2}{2m} \frac{d^2\Psi}{dx^2} = E\Psi$$

with boundary values $\Psi(0) = 0$ and $\Psi(L) = 0$.

Problem 6. Consider the transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ given by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ x + y \end{pmatrix}.$$

- (a) Show that this transformation is linear.
- (b) Write down a matrix for this linear transformation.
- (c) Can you draw a picture of the output of this transformation? What kind of object is it?

Problem 7. Consider the system of linear equations:

$$\begin{aligned} 3x + 2y + 0z &= 5 \\ 1x + 1y + 1z &= 3 \\ 0x + 2y + 2z &= 4. \end{aligned}$$

- (a) Write the augmented matrix M for this system of equations.
- (b) Use row reduction to get the augmented matrix in row-echelon form.
- (c) Determine the solution to the system of equations.

Problem 8. Consider the equation

$$\begin{pmatrix} 1 & 3 & 4 \\ 2 & 9 & 9 \\ 1 & 5 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 20 \\ 11 \end{pmatrix}.$$

Does this equation have a solution or not? If so, determine the solution.