MATH 271, Worksheet 7, *Solutions*

Vectors, vector spaces, and linear transformations.

Problem 1. Compare and contrast the structure of the complex numbers $\mathbb{C}$ with the vector space $\mathbb{R}^2$. Note any differences and similarities. Can you multiply vectors in $\mathbb{R}^2$?

Solution 1. The algebraic structure of $\mathbb{C}$ and the vector space $\mathbb{R}^2$ are quite similar. In particular, it was convenient to think of real axis as the x-axis and the imaginary axis as the y-axis in the plane. A complex number $z = x + iy$ can then be drawn as an arrow based at 0 with a tip at the point (x, y) in the plane. This is analogous to the point $\vec{\boldsymbol{u}} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$ in $\mathbb{R}^2$.

Adding two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ yields

$$x_1 + x_2 + i(y_1 + y_2),$$

which is a componentwise addition. This addition is the same as taking $\vec{\boldsymbol{u}}_1 = x_1\hat{\boldsymbol{x}} + y_1\hat{\boldsymbol{y}}$ and $\vec{\boldsymbol{u}}_2 = x_2\hat{\boldsymbol{x}} + y_2\hat{\boldsymbol{y}}$ and computing

$$\vec{\boldsymbol{u}}_1 + \vec{\boldsymbol{u}}_2 = (x_1 + x_2)\hat{\boldsymbol{x}} + (y_1 + y_2)\hat{\boldsymbol{y}}.$$

Thus, we can also think of addition in both $\mathbb{C}$ and $\mathbb{R}^2$ as attaching the tail z_1 to the tip of z_2 much like we can do the same with adding $\vec{\boldsymbol{u}}_2$ to $\vec{\boldsymbol{u}}_2$ in a geometrical sense.

Sadly, there is no (nontrivial to explain) notion of multiplication of vectors in **ANY** vector space. The closest we achieve is in $\mathbb{R}^3$ with a cross product (though you should not think of this as vector multiplication). But, we can multiply $z_1z_2 = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$. It turns out that we can represent this multiplication (somehow) in $\mathbb{R}^2$, but it requires us to use matrices or something called *geometric algebra*. I'm a huge proponent of geometric algebra, but it is not a common method of instruction (yet!).

Problem 2. Let $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \in \mathbb{R}^2$ be given by

$$\vec{\boldsymbol{u}} = 2\hat{\boldsymbol{x}} + 3\hat{\boldsymbol{y}} \qquad \text{and} \qquad \vec{\boldsymbol{v}} = -\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}.$$

(a) Draw $\vec{\boldsymbol{u}}$, $\vec{\boldsymbol{v}}$, and $\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}$ in the plane.

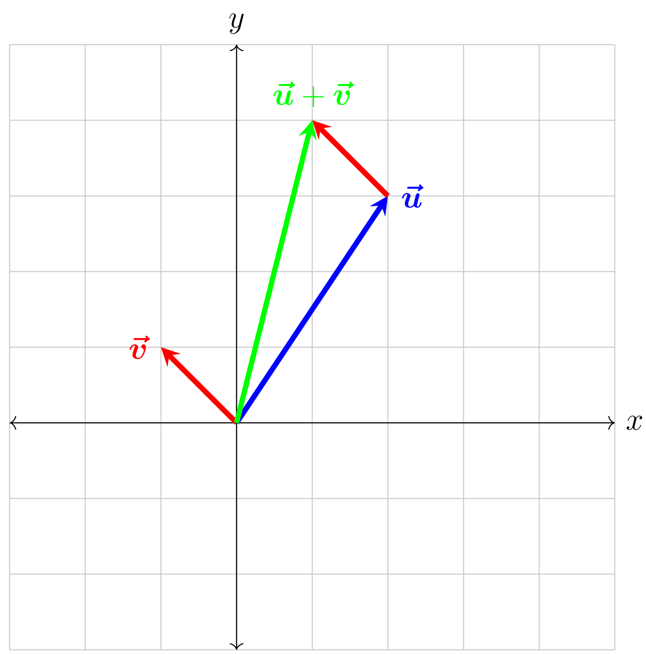
(b) Compute $\|\vec{\boldsymbol{u}}\|$ and $\|\vec{\boldsymbol{v}}\|$.

(c) Compute $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}}$.

(d) Find a vector orthogonal to $\vec{\boldsymbol{u}}$.

Solution 2.

(a) We have the following:



(b) To compute the lengths we can use the dot product (which amounts to using the pythagorean theorem). We have

$$\|\vec{\boldsymbol{u}}\| = \sqrt{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}}} = \sqrt{2^2 + 3^2} = \sqrt{13}$$

and

$$\|\vec{\boldsymbol{v}}\| = \sqrt{\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}}} = \sqrt{(-1)^2 + 1^2} = \sqrt{2}.$$

(c) We have

$$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = 2(-1) + 3(1) = 1.$$

(d) Consider some vector $\vec{\boldsymbol{w}} = w_1\hat{\boldsymbol{x}} + w_2\hat{\boldsymbol{y}}$. Then, we want $\vec{\boldsymbol{w}}$ to bo orthogonal to $\vec{\boldsymbol{u}}$ which means

$$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{w}} = 0.$$

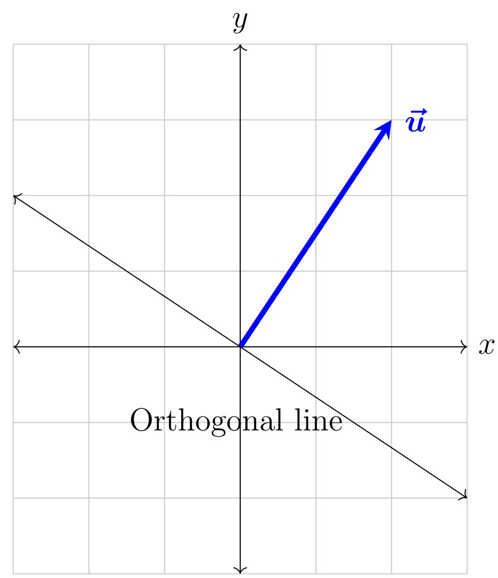
Hence, we arrive at an (underdetermined) equation

$$0 = 2w_1 + 3w_2.$$

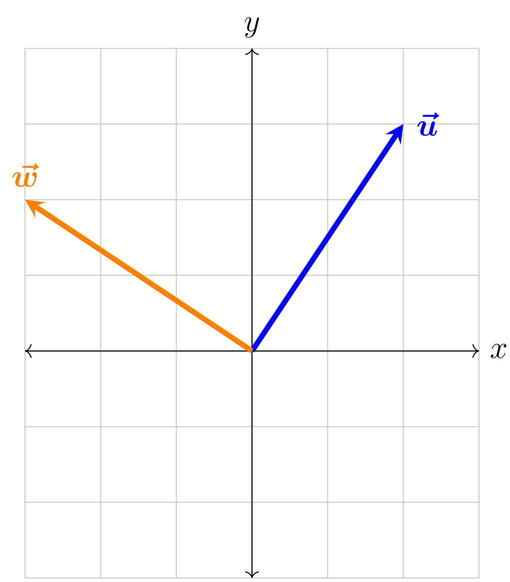
Clearly, we can take $w_1 = 0$ and $w_2 = 0$ to get a trivial answer. One should note that this yields the $\vec{\mathbf{0}}$ and $\vec{\mathbf{0}}$ is always orthogonal to every vector! To get a nontrivial answer, we note

$$2w_1 = -3w_2 \implies \frac{w_1}{w_2} = \frac{-3}{2},$$

so finding an orthogonal vector $\vec{\boldsymbol{w}}$ to $\vec{\boldsymbol{u}}$ in the plane amounts to just determining the angle that the vector $\vec{\boldsymbol{w}}$ must have. We can plot a line in the plane with slope $\frac{-3}{2}$ and see that any vector that lies on this line is orthogonal to $\vec{\boldsymbol{u}}$.



So it us up to us to simply choose some vector on this line. Perhaps the most obvious choice from looking at our equation above yeilds $w_1 = -3$ and $w_2 = 2$ which amounts to the orange vector $\vec{\boldsymbol{w}}$ shown below.



Problem 3. Let $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \in \mathbb{R}^3$ be given by

$$\vec{\boldsymbol{u}} = \hat{\boldsymbol{x}} - \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}} \qquad \text{and} \qquad \vec{\boldsymbol{v}} = -\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} - \hat{\boldsymbol{z}}.$$

(a) Are $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ orthogonal?

(b) Normalize $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ to get $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{v}}$.

(c) Compute the projection of $\vec{\boldsymbol{v}}$ onto the direction defined by $\vec{\boldsymbol{u}}$.

Solution 3.

(a) We can check this by taking the dot product

$$\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = -1 - 1 - 1 = -3,$$

so, no, the vectors are not orthogonal.

(b) To normalize $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ we first compute $\|\vec{\boldsymbol{u}}\|$ and $\|\vec{\boldsymbol{v}}\|$. We have

$$\|\vec{\boldsymbol{u}}\| = \sqrt{\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}}} = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$$

and

$$\|\vec{\boldsymbol{v}}\| = \sqrt{\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{v}}} = \sqrt{(-1)^2 + 1^2 + (-1)^2} = \sqrt{3}.$$

Then

$$\hat{\boldsymbol{u}} = \frac{1}{\|\vec{\boldsymbol{u}}\|}\vec{\boldsymbol{u}} = \frac{1}{\sqrt{3}}\hat{\boldsymbol{x}} - \frac{1}{\sqrt{3}}\hat{\boldsymbol{y}} + \frac{1}{\sqrt{3}}\hat{\boldsymbol{z}}$$

and

$$\hat{\boldsymbol{v}} = \frac{1}{\|\vec{\boldsymbol{v}}\|}\vec{\boldsymbol{v}} = -\frac{1}{\sqrt{3}}\hat{\boldsymbol{x}} + \frac{1}{\sqrt{3}}\hat{\boldsymbol{y}} - \frac{1}{\sqrt{3}}\hat{\boldsymbol{z}}.$$

(c) We can compute the projection of $\vec{\boldsymbol{v}}$ in the direction of $\vec{\boldsymbol{u}}$ by computing

$$\vec{\boldsymbol{v}} \cdot \hat{\boldsymbol{u}} = \frac{1}{\|\vec{\boldsymbol{u}}\|}\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{u}} = \frac{-3}{\sqrt{3}}.$$

Problem 4. Let $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \in \mathbb{R}^3$ be given by

$$\vec{\boldsymbol{u}} = -3\hat{\boldsymbol{x}} - 2\hat{\boldsymbol{y}} + \hat{\boldsymbol{z}} \qquad \text{and} \qquad \vec{\boldsymbol{v}} = \hat{\boldsymbol{x}} - 2\hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}.$$

(a) Compute the angle between $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$.

(b) Without computing the cross product, compute the area of the parallelogram generated by $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$. *Hint: you know the angle between the vectors, use this fact.*

(c) Without computing the cross product, what component of the product $\vec{\boldsymbol{u}} \times \hat{\boldsymbol{x}}$ must be zero?

(d) Compute $\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}$.

(e) Give a geometrical interpretation of the cross product $\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}$. Explain why $\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}} = -\vec{\boldsymbol{v}} \times \vec{\boldsymbol{u}}$.

Solution 4.

(a) To compute the angle θ between $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ we need to compute their lengths as well as the dot product between them since $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = \|\vec{\boldsymbol{u}}\|\|\vec{\boldsymbol{v}}\| \cos\theta$. So, we get

$$\|\vec{\boldsymbol{u}}\| = \sqrt{14}, \quad \|\vec{\boldsymbol{v}}\| = \sqrt{6}, \quad \vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}} = 2.$$

Hence,

$$2 = \sqrt{14}\sqrt{6}\cos\theta$$

and thus

$$\theta = \arccos\left(\frac{2}{\sqrt{14}\sqrt{6}}\right) \approx 1.351 \text{ radians}.$$

Problem 5. Recall that the states found in the solution to the free particle in a 1-dimensional box of length L were $\psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$. Let S denote the set of all solutions to the free particle in a 1-dimensional box boundary value problem. Show that a superposition of states (with coefficients in $\mathbb{C}$) is **NOT** a solution. That is, if we let $\Psi(x) = \alpha_j(x)\psi_j + \alpha_k\psi_k(x)$, then $\Psi(x)$ is **NOT** solution to the boundary value problem

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} = E\Psi$$

with boundary values $\Psi(0) = 0$ and $\Psi(L) = 0$.

Solution 5. To see that a superposition of states $\Psi = \alpha_j\psi_j + \alpha_k\psi_k$ is not a solution we must see that Ψ must not satisfy the ODE since Ψ does satisfy the boundary conditions. In particular,

$$\begin{aligned}-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} &= -\frac{\hbar^2}{2m}\left(\alpha_j\frac{d^2\psi_j}{dx^2} + \alpha_k\frac{d^2\psi_k}{dx^2}\right)\\ &= \alpha_j E_j\psi_j + \alpha_k E_k\psi_k.\end{aligned}$$

So, Ψ does not solve the (time independent) Schödinger equation! Only the states do! That is why we refer to these as *stationary states*. One must add time dependence in order for a superposition to be a solution!

Problem 6. Consider the transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ x+y \end{pmatrix}.$$

(a) Show that this transformation is linear.

(b) Write down a matrix for this linear transformation.

(c) Can you draw a picture of the output of this transformation? What kind of object is it?

Solution 6.

(a) To see that this transformation is linear, we consider two vectors $\vec{\boldsymbol{u}} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\vec{\boldsymbol{v}} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and a scalar $\alpha \in \mathbb{R}$. There are two key parts to showing the transformation is linear.

- We show $T(\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}) = T(\vec{\boldsymbol{u}}) + T(\vec{\boldsymbol{v}})$.

$$\begin{aligned}
T(\vec{\boldsymbol{u}} + \vec{\boldsymbol{v}}) &= T\begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix} \\
&= \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_1 + v_1 + u_2 + v_2 \end{pmatrix} \\
&= \begin{pmatrix} u_1 \\ u_2 \\ u_1 + u_3 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_1 + v_2 \end{pmatrix} \\
&= T(\vec{\boldsymbol{u}}) + T(\vec{\boldsymbol{v}}).
\end{aligned}$$

  So T satisfies the first requirement.

- Next we show $T(\alpha\vec{\boldsymbol{u}}) = \alpha T(\vec{\boldsymbol{u}})$.

$$\begin{aligned}
T(\alpha\vec{\boldsymbol{u}}) &= T\begin{pmatrix} \alpha u_1 \\ \alpha u_2 \end{pmatrix} \\
&= \begin{pmatrix} \alpha u_1 \\ \alpha u_2 \\ \alpha u_1 + \alpha u_2 \end{pmatrix} \\
&= \begin{pmatrix} \alpha u_1 \\ \alpha u_2 \\ \alpha(u_1 + u_2) \end{pmatrix} \\
&= \alpha T(\vec{\boldsymbol{u}}).
\end{aligned}$$

  So T satisfies the second requirement and thus T is linear.

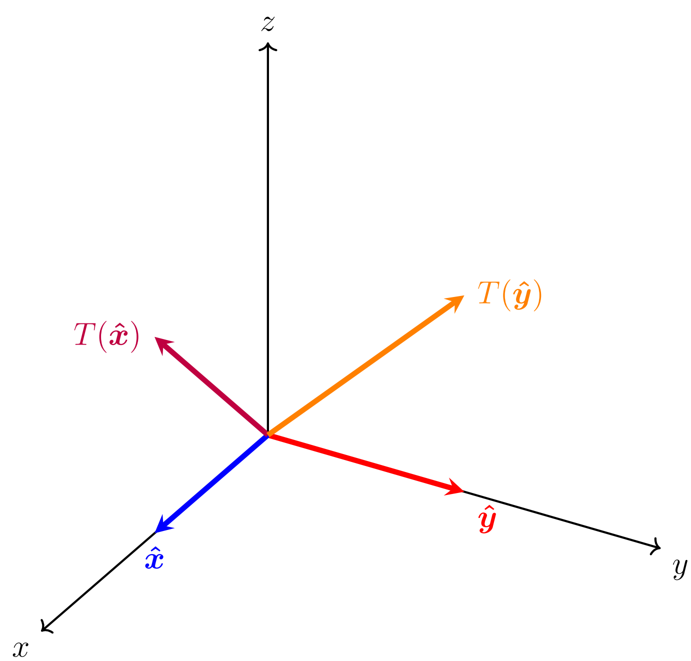
(b) To find the matrix for T, we see how T acts on the vectors $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$. We have

$$T(\hat{\boldsymbol{x}}) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad T(\hat{\boldsymbol{y}}) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Thus, the columns for the matrix $[T]$ are given by $T(\hat{\boldsymbol{x}})$ and $T(\hat{\boldsymbol{y}})$. Specifically,

$$[T] = \begin{pmatrix} | & | \\ T(\hat{\boldsymbol{x}}) & T(\hat{\boldsymbol{y}}) \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

(c) The picture could look something like this.



The output of this transformation is a vector in $\mathbb{R}^3$. So we are taking planar vectors and converting them to new vectors that live in space.