MATH 272, Exam 3

Oral Examination Problems

Due one hour before your exam time slot.

Instructions You are allowed a textbook, homework, notes, worksheets, material on our Canvas page. You can use online tools such as Desmos and Wolfram Alpha to check your work, but you will need to explain how you arrived at your answers. You can work with other students and this is, in fact, encouraged! However, I will not be giving out direct help for these problems but can answer questions about previous problems and notes, for example. Ambiguous or illegible answers will not be counted as correct. Scan your solutions and submit them as a pdf on Canvas under Oral Exam 3.

Note, there are three total problems and a bonus.

Problem 1. Given a self-adjoint operator (or *observable*) $\mathcal{L}$, we can consider the *expected value of the observable* $\mathcal{L}$ for a given wavefunction Ψ by computing

$$\mathbb{E}[\mathcal{L}] = \langle \Psi, \mathcal{L}\Psi \rangle.$$

For this problem, let $\Omega = [0, 1]$ and recall the orthonormal stationary states are

$$\psi_n(x) = \sqrt{2}\sin(n\pi x)$$

and the orthonormal time-dependent states are

$$\psi_n(x,t) = \sqrt{2}e^{-i\frac{E_n}{\hbar}t}\sin(n\pi x)$$

where the energies are

$$E_n = \hbar\omega_n = \frac{n^2\pi^2\hbar^2}{2m}.$$

(a) Consider first a stationary superposition wavefunction $\Psi(x) = \frac{1}{\sqrt{2}}\psi_1(x) + \frac{1}{\sqrt{2}}\psi_2(x)$. Now, compute the expected value of position by computing

$$\mathbb{E}[x] = \langle \Psi, x\Psi \rangle.$$

This value $\mathbb{E}[x]$ tells you where we expect to find the particle on average.

(b) Consider the time-dependent superposition wavefunction $\Psi(x,t) = \frac{1}{\sqrt{2}}\psi_1(x,t) + \frac{1}{\sqrt{2}}\psi_2(x,t)$. Compute $\mathbb{E}[x]$ again and note that $\mathbb{E}[x]$ tells you where we expect to find the particle on average, but it will now depend on time. Can you corroborate this with a graph? *Hint: perhaps a graph you have already made before?*

(c) Define the energy operator $T = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}$. Show that this operator is self-adjoint.

(d) Again, letting $\Psi(x,t) = \frac{1}{\sqrt{2}}\psi_1(x,t) + \frac{1}{\sqrt{2}}\psi_2(x,t)$, compute $\mathbb{E}[T]$. The expected value $\mathbb{E}[T]$ tells us what the observed energy will be on average as a function of time.

Problem 2. Consider the source-free heat equation

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}\right) u(x,t) = 0$$

on the unit interval $\Omega = [0,1]$.

(a) Using separation of variables, show that you get

$$u(x,t) = e^{-\lambda t}(ae^{i\sqrt{\lambda}x} + be^{-i\sqrt{\lambda}x})$$

is a solution for any λ and where a and b are just some constants.

(b) Show that if you take periodic boundary conditions $u(0,t) = u(1,t)$ then you get $\sqrt{\lambda} = 2n\pi$ for any integer n. (Note that this set of boundary conditions is equivalent to just looking at heat flow on a circle.)

(c) Hence, a general solution to the source heat equation on the circle is given by a Fourier series

$$u(x,t) = \sum_{n=-\infty}^{\infty} c_n e^{-4n^2\pi^2 t} e^{i2n\pi x}.$$

Suppose that you choose initial conditions whereby you attach one hot half-circle to another cooler half-circle, i.e.,

$$u(x,0) = \begin{cases} 1, & x \in \left[0, \frac{1}{2}\right] \\ 0, & x \in \left(\frac{1}{2}, 1\right] \end{cases}.$$

Determine the c_n for this initial condition. *Hint: this really is just the Fourier transform* $\hat{u}(n,0) = c_n$.

(d) Approximate your solution by summing up to $N = 100$. Plot your solution as an animation over time t using Desmos. Attach solutions for various values of t that show the time evolution of your $u(x,t)$ or just be ready to share your screen with the animation when we meet.

Problem 3. Suppose that we have a time-dependent particle in a box $\Omega = [0, 1]$ with initial condition

$$\Psi(x, 0) = \begin{cases} 0, & x \in \left[0, \frac{1}{4}\right) \cup \left(\frac{3}{4}, 1\right] \\ 1, & x \in \left[\frac{1}{4}, \frac{3}{4}\right]. \end{cases}$$

Recall that this must solve the time-dependent Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - i\hbar\frac{\partial}{\partial t}\right)\Psi(x, t) = 0.$$

(a) Is $\Psi(x, 0)$ normalized? If not, then normalize it.

(b) Using the fact that the $\psi_n(x)$ are an orthonormal basis for the solution space to this problem, write your normalized $\Psi(x, 0)$ as a series

$$\Psi(x, 0) = \sum_{n=1}^{\infty} a_n \psi_n(x).$$

(c) To get the time-dependent solution to the Schrödinger equation, we can just take your coefficents a_n you found from before and put

$$\Psi(x, t) = \sum_{n=1}^{\infty} a_n \psi_n(x, t).$$

Plot an approximation (say $N = 100$) of the real and imaginary parts simultaneously of your time-dependent wavefunction as an animation. Attach solutions for various values of t or just be ready to share your screen with the animation when we meet.

(d) Show that your time-dependent series solution is normalized for all values of t.

(e) (BONUS) We can Fourier transform with the t variable by

$$\hat{\Psi}(x, \omega) = \int_{\infty}^{\infty} \Psi(x, t) e^{i2\pi\omega t} dt.$$

Show that by applying the Fourier transform to the time-dependent Schrödinger equation you get the time-independent equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x, \omega) = E\Psi(x, \omega)$$

for a fixed value of ω (you can assume that $\Psi(x, t)$ vanishes when $t = -\infty$ and $t = +\infty$). Also, what is E in terms of ω and $\hbar$?

Problem 4. (BONUS) You and a friend wish to determine who has thicker or thinner hair than the other. Devise an experiment that involves the Fourier transform that would allow you to find out who has the thicker strand of hair.