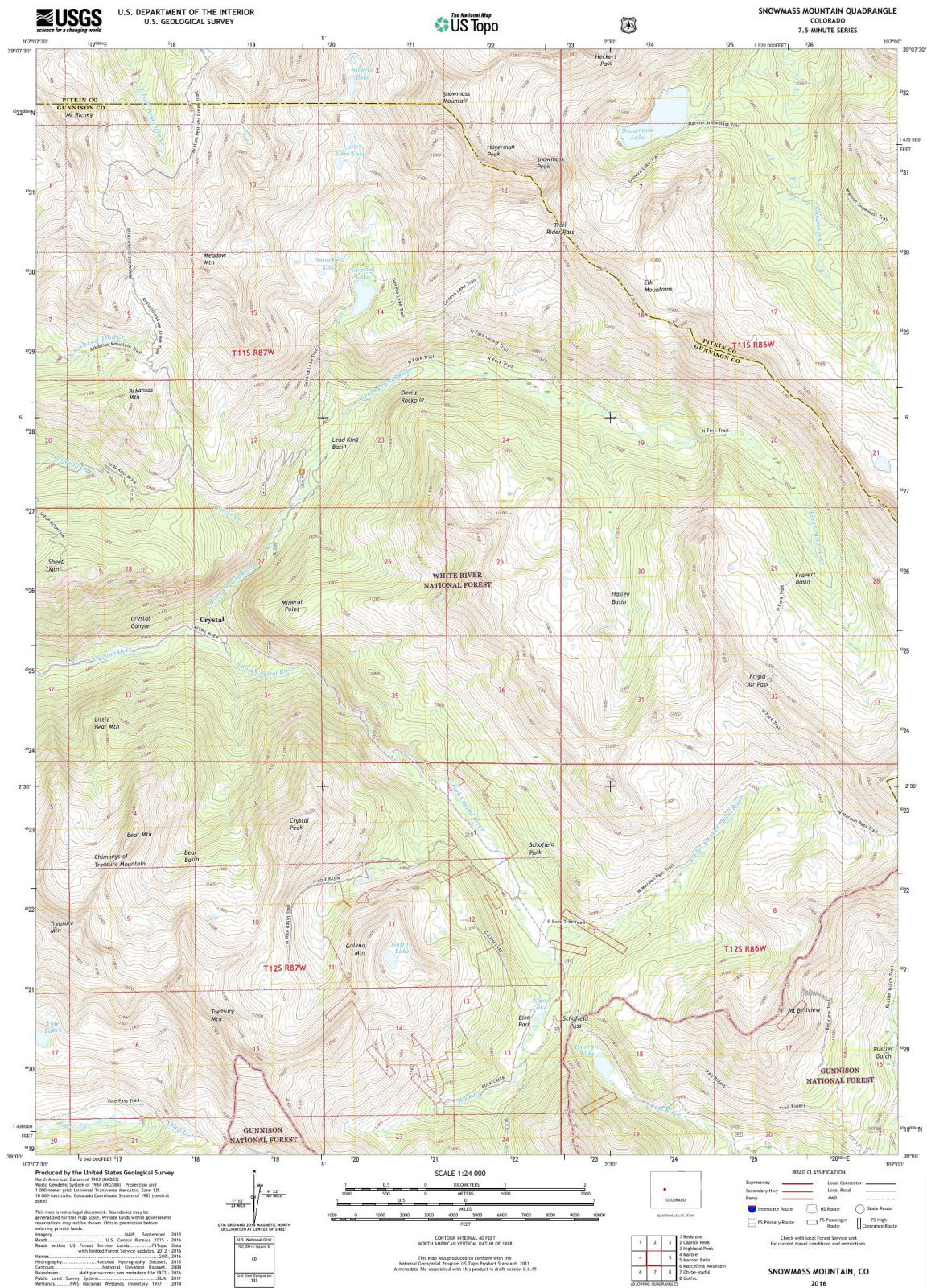# MATH 272, WORKSHEET 2
VECTOR FIELDS AND DIFFERENTIAL CALCULUS OF FIELDS.

Problem 1. These same functions appeared on Worksheet 1 for which you plotted the level surfaces. Now, to analyze these functions further, compute and plot the gradient vector fields. Compare your understanding of the gradient field with the level surfaces; what is their relationship?

(a) $E(x, y, z) = \frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9}$.

(b) $f(x, y, z) = xyz$.

(c) $g(x, y, z) = e^x - y^2 - z^2$.

(d) $h(x, y, z) = \sin(x) + \cos(y) - \tanh(z)$.

(e) $p(x, y, z) = \sin^2(x) + \sin^2(y) - \frac{1}{2}\sin(z)$.

(f) $q(x, y, z) = x^2 + xy + y^2 + \sin(yz)$.

(g) One of your own choosing.

Problem 2. Let $h(x, y)$ be the height of a sheet above the xy-plane at the point (x, y). Then, the level sets of h are exactly the lines you find in a topographic map. Below is a map of White River National Forest, CO including Snowmass Peak.

(a) Determine the level set (possibly multiple curves) for $h(x, y) = 12,000$.

(b) In the coordinates of this map, the town of Crystal and the famous Crystal Mill is located roughly at $x = 18$, $y = 25.5$. Locate this point and describe the local geometry (e.g., where the steep and shallow gradients are, which way the river should flow).

(c) Sheep Mountain is located at roughly $x = 16.5$ and $y = 26$. What is the max elevation of Sheep Mountain?

(d) You want to hike to Snowmass Peak (roughly $(21.5, 32)$) because you are very brave. Assuming you are dropped off at Lead King Basin (roughly $(20, 28)$). Assuming you can hike over any terrain (including water if need be), can you find a path that follows the steepest gradient from Lead King Basin to Snowmass Peak?

(e) Locate the flattest area that you can.

(f) Plot gradient vectors at each point on the grid.

(g) Follow Schofield Pass starting from roughly $(22.5, 19)$. How much elevation gain and loss do you experience over this curve? Can you think of how we could represent this via integration? (*Hint: you may be able to use both h and $\vec{\nabla}h$ to compute this in different ways; we'll return to this idea in the future.*)

U.S. DEPARTMENT OF THE INTERIOR
U.S. GEOLOGICAL SURVEY

The National Map
US Topo

SNOWMASS MOUNTAIN QUADRANGLE
COLORADO
7.5-MINUTE SERIES

PITKIN CO
GUNNISON CO

Mt Richey

Snowmass
Mountain

Heckert
Pass

Hagerman
Peak

Snowmass
Peak

Trail
Rider Pass

Meadow
Mtn

Elk
Mountains

PITKIN CO
GUNNISON CO

T11S R86W

T11S R87W

Arkansas
Mtn

Devils
Rockpile

Lead King
Basin

Sheep
Mtn

WHITE RIVER
NATIONAL FOREST

Hasley
Basin

Fravert
Basin

Mineral
Point

Crystal
Canyon

Crystal

Frigid
Air Pass

Little
Bear Mtn

Crystal
Peak

Schofield
Park

Bear Mtn

Chimneys of
Treasure Mountain

Bear
Basin

Mt Bellview

T12S R86W

Treasure
Mtn

Galena
Mtn

Galena
Lake

Treasury
Mtn

Elko
Park

Schofield
Pass

Rustler
Gulch

T12S R87W

GUNNISON
NATIONAL FOREST

Yule Lake

GUNNISON
NATIONAL FOREST

SCALE 1:24 000

UTM GRID AND 2015 MAGNETIC NORTH
DECLINATION AT CENTER OF SHEET

U.S. National Grid

100,000-m Square ID

CD

Grid Zone Designation
13S

CONTOUR INTERVAL 40 FEET
NORTH AMERICAN VERTICAL DATUM OF 1988

This map was produced to conform with the
National Geospatial Program US Topo Product Standard, 2011.
A metadata file associated with this product is draft version 0.6.19

COLORADO

QUADRANGLE LOCATION

1	2	3
4		5
6	7	8

ADJOINING QUADRANGLES

1 Redstone
2 Capitol Peak
3 Highland Peak
4 Marble
5 Maroon Bells
6 Marcelline Mountain
7 Oh-be-joyful
8 Gothic

SNOWMASS MOUNTAIN, CO

2016

2

Problem 3. Plot the following vector fields.

(a) $\vec{U}(x, y, z) = \begin{pmatrix} xyz \\ xyz \\ xyz \end{pmatrix}$.

(c) $\vec{W}(x, y, z) = \begin{pmatrix} x\sin(y) \\ x\cos(y) \\ xz \end{pmatrix}$.

(b) $\vec{V}(x, y, z) = \begin{pmatrix} e^{x+y+z} \\ e^{x+y+z} \\ e^{x+y+z} \end{pmatrix}$.

(d) $\vec{F}(x, y, z) = \begin{pmatrix} 5 + yz \\ -5 - xz \\ xy \end{pmatrix}$.

Problem 4. Compute the divergence, curl, and Jacobian matrix of the vector fields in Problem 3.

Problem 5. Given a function, a vector, and a point, compute the directional derivatives at that point.

(a) $f(x, y) = x^2/y$, $\vec{u} = 3\hat{x} - \hat{y}$, and $(x, y) = (2, 2)$.

(b) $g(x, y, z) = x\cos(yz^2)$, $\vec{v} = \frac{1}{2}\hat{x} + \frac{1}{2}\hat{y} + \hat{z}$, and $(x, y, z) = (0, -1, 2)$.

Problem 6. Give a physical argument for why the field $\vec{U} = \begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix}$ has nonzero curl everywhere. Reason why the direction of the curl is solely along the z-axis. Do this <u>without</u> computing the curl.

Problem 7. Give a physical argument why the field $\vec{V} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$ has nonzero divergence everywhere. Reason why the divergence is a scalar quantity as opposed to a vector quantity. Do this <u>without</u> computing the curl.

Problem 8. Compute the Laplacian of the following fields.

(a) $f(x, y) = (x + y)^2$

(b) $g(x, y) = (x + y)e^{x^2+y^2}$.

(c) $\vec{U} = \begin{pmatrix} \cos(x)\cos(y)\cos(z) \\ \sin(x)\sin(y)\sin(z) \\ xyz \end{pmatrix}$.

(d) $\vec{V} = \begin{pmatrix} \frac{y}{z} \\ \frac{x}{z} \\ \frac{x}{y} \end{pmatrix}$.

Problem 9. Suppose that $\vec{V} = \vec{\nabla}\phi$ for some scalar field ϕ. Explain why if $\vec{V}$ is divergence free (i.e., if $\vec{\nabla} \cdot \vec{V} = 0$) that $\vec{\Delta}\vec{V} = \vec{0}$.

Problem 10. One of Maxwell's equations states for a magnetic field $\vec{B}$ that

$$\vec{\nabla} \cdot \vec{B} = 0,$$

is an identity. Does this mean that $\vec{\Delta}\vec{B} = 0$?