MATH 272, Homework 2

Due February 8th

Problem 1. Consider the function

$$f(x,y) = \sin\left(\frac{2\pi x}{5}\right)\sin\left(\frac{2\pi y}{5}\right).$$

comes up when you want to find out how a square shaped drum head will vibrate when hit.

(a) Plot this function on the region Ω given by $0 \leq x \leq 5$ and $0 \leq y \leq 5$.

(b) What is the value the function $f(x,y)$ on the boundary of the given region Ω (i.e, when $x = 0$, $x = 5$, $y = 0$, and $y = 5$)?

(c) Show that $f(x,y)$ is an eigenfunction of the Laplacian Δ. That is, $\Delta f = \lambda f$ for some eigenvalue λ. What is the eigenvalue?

Problem 2. Consider the following vector field

$$\vec{\boldsymbol{B}} = -\frac{y}{2}\hat{\boldsymbol{x}} + \frac{x}{2}\hat{\boldsymbol{y}}.$$

Here, $\vec{\boldsymbol{B}}$ denotes the magnetic field. It may be helpful to plot the fields in this problem.

(a) Show that $\vec{\boldsymbol{B}}$ has no divergence. (This is one of Gauss's laws.)

(b) Show that $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}} = \vec{\boldsymbol{J}}$ (Ampére's law) where

$$\vec{\boldsymbol{J}} = \hat{\boldsymbol{z}}.$$

This vector field $\vec{\boldsymbol{J}}$ represents the electric current (moving charges) in space. One could argue that the current creates the magnetic field via Ampére's law.

(c) Magnetic fields induce a force $\vec{\boldsymbol{F}}$ on charged particles by the Lorentz force

$$\vec{\boldsymbol{F}} = \dot{\vec{\gamma}} \times \vec{\boldsymbol{B}} = \ddot{\vec{\gamma}}$$

Where $\dot{\vec{\gamma}}$ is the velocity of the particle (where we have chosen a mass $m = 1$ and charge $q = 1$). Let us do the following.

- Assume that $\dot{\vec{\gamma}} = \hat{\boldsymbol{x}}$, what is the force on the particle?
- Repeat the previous step for $\dot{\vec{\gamma}} = \hat{\boldsymbol{y}}$ and $\dot{\vec{\gamma}} = \hat{\boldsymbol{z}}$.
- Compare and contrast the forces you found.

(d) Can you argue why applying a magnetic field to a molecule may cause it to heat up? Can you compare this idea with your home microwave?

Problem 3. Set up but do not compute the integrals of the given fields over the given curves.

(a) $f(x,y) = xe^{x+y} + \cos(xy)$, $\vec{\gamma}(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$, $t_0 = 0$, $t_1 = 2\pi$.

(b) $g(x,y,z) = \frac{\ln(z^2)}{e^{xy}}$, $\vec{\gamma}(t) = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$, $t_0 = 1$, $t_1 = -1$.

(c) $\vec{\boldsymbol{U}}(x,y) = \begin{pmatrix} -y \\ x \end{pmatrix}$, $\vec{\gamma}(t) = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$, $t_0 = 5$, $t_1 = 10$.

(d) $\vec{\boldsymbol{V}}(x,y,z) = \begin{pmatrix} 2x \\ y \\ x \end{pmatrix}$, $\vec{\gamma}(t) = \begin{pmatrix} \cos(t) \\ t \\ \sqrt{t} \end{pmatrix}$, $t_0 = 0$, $t_1 = 1$.

Problem 4. Let $f(x,y,z) = x\cos(y) + yz$ be a scalar field and let $\vec{\gamma} = \begin{pmatrix} 1 \\ t \\ \sin(t) \end{pmatrix}$ be a curve from time $t_0 = 0$ to $t_1 = 2\pi$. Compute

$$\int_{\vec{\gamma}} f(\vec{\gamma}) d\vec{\gamma}.$$

Problem 5. Consider the following vector field

$$\vec{\boldsymbol{E}} = \frac{x}{(x^2+y^2+z^2)^{3/2}}\hat{\boldsymbol{x}} + \frac{y}{(x^2+y^2+z^2)^{3/2}}\hat{\boldsymbol{y}} + \frac{z}{(x^2+y^2+z^2)^{3/2}}\hat{\boldsymbol{z}},$$

which you can think of as the electric field of a positive point charge. We argued that this field $\vec{\boldsymbol{E}}$ is conservative in a previous homework problem. Specifically, $\vec{\boldsymbol{E}} = \vec{\boldsymbol{\nabla}}\phi$, for the scalar field

$$\phi(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

This follows from Faraday's law for static charges.

(a) Compute the integral

$$T = \int_{\vec{\gamma}} \vec{\boldsymbol{E}} \cdot d\vec{\gamma} \qquad \text{where} \qquad \vec{\gamma}(t) = \begin{pmatrix} t \\ t \\ t \end{pmatrix},$$

and $t \in [t_0, t_1]$ with t_0 and t_1 both greater than 0. Note that this integral T describes the gain in kinetic energy of a charged particle that moved along the path $\vec{\gamma}$.

(b) Equivalently, since $\vec{\boldsymbol{E}}$ is conservative, we have

$$T = \int_{\vec{\gamma}} \vec{\boldsymbol{E}} \cdot d\vec{\gamma} = \phi(\vec{\gamma}(t_1)) - \phi(\vec{\gamma}(t_0)).$$

Show that this is true for the given vector field and potential. This shows that the choice of path does not matter; only the endpoints $\vec{\gamma}(t_0)$ and $\vec{\gamma}(t_1)$ matter.

(c) Argue why the integral around any closed curve must be zero.

(d) Argue why it must be that $\vec{\boldsymbol{E}}$ has no curl.