MATH 271, Worksheet 4, *Solutions*

Sequences and Series

Problem 1. Write down the first few terms in the sequence for the following:

(a) $a_n = n$;

(b) $b_n = \frac{1}{n^2}$;

(c) $c_n = 2^{-n}$.

Solution 1.

(a) We have
$$\{a_n\}_{n=1}^{\infty} = 1,\ 2,\ 3,\ 4,\ 5, \ldots.$$

(b) We have
$$\{b_n\}_{n=1}^{\infty} = 1,\ \frac{1}{4},\ ,\ \frac{1}{9},\ \frac{1}{16},\ \frac{1}{25}, \ldots.$$

(c) We have
$$\{c_n\}_{n=1}^{\infty} = 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \frac{1}{32}, \ldots.$$

Problem 2. For the above sequences, state whether each converges or diverges. If they converge, state the limit.

Solution 2.

(a) The sequence $\{a_n\}_{n=1}^{\infty}$ diverges as $a_n \to \infty$. In other words, the a_n continue to grow without bound.

(b) The sequence $\{b_n\}_{n=1}^{\infty}$ converges to zero. For any value $\epsilon > 0$, I can find an $N \in \mathbb{N}$ so that the term $0 < b_N < \epsilon$.

(c) Similarly, the sequence $\{c_n\}_{n=1}^{\infty}$ converges to zero.

Problem 3. Consider the recursive sequence
$$a_n = \frac{1}{2}a_{n-1} + 1$$
with $a_1 = 1$.

(a) Write the first few terms in the sequence.

(b) Can you write a_n as a function $f(n)$? If so, what is $f(n)$?

(c) Does this sequence converge or diverge? Can you show why with a limit $\lim_{n\to\infty} f(n)$?

(d) Can you show that this is a Cauchy sequence?

Solution 3.

(a) Given that $a_1 = 1$, we can get

$$\begin{aligned} a_2 &= \frac{1}{2}a_1 + 1 = \frac{3}{2} \\ a_3 &= \frac{1}{2}a_2 + 1 = \frac{7}{4} \\ a_4 &= \frac{1}{2}a_3 + 1 = \frac{15}{8} \\ a_5 &= \frac{1}{2}a_4 + 1 = \frac{31}{16}. \end{aligned}$$

(b) Yes, we can. Notice that we have

$$a_n = \frac{2^n - 1}{2^{n-1}} = f(n).$$

(c) Yes, this sequence converges to 2 since

$$\lim_{x\to\infty} f(x) = 2.$$

(d) Yes, so we need to show that for any $\epsilon > 0$ that we have for some $N \in \mathbb{N}$ that for $K \geq N$

$$|a_K - a_{K+1}| < \epsilon.$$

So we have

$$a_K = \frac{2^K - 1}{2^{K-1}} \qquad \text{and} \qquad a_{K+1} = \frac{2^{K+1} - 1}{2^K}.$$

Then

$$\begin{aligned} \left| \frac{2^K - 1}{2^{K-1}} - \frac{2^{K+1} - 1}{2^K} \right| &= \left| \frac{2^{K+1} - 2}{2^K} - \frac{2^{K+1} - 1}{2^K} \right| \\ &= \left| \frac{-1}{2^K} \right| \\ &= \frac{1}{2^K}. \end{aligned}$$

So now we wish to have $\frac{1}{2^K} < \epsilon$ and if we choose $K > \log_2\left(\frac{1}{\epsilon}\right)$, then we have shown this sequence is Cauchy.

Remark 1. Showing that a sequence is Cauchy is a bit of a stretch for us. Just know what it means when we say a sequence is Cauchy!

Problem 4. Consider the sequence

$$a_n = ar^n.$$

(a) If $|r| < 1$, show that this sequence $\{a_n\}_{n=0}^{\infty}$ converges to zero.

(b) Consider now the *geometric series*

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} ar^n.$$

Show that the N^{th} partial sum for this series satisfies

$$\sum_{n=0}^{N} ar^n = a\left(\frac{1-r^{N+1}}{1-r}\right).$$

(c) Does the geometric series converge for all r? For $|r| < 1$? When it converges, what does it converge to?

Solution 4.

(a) Let $|r| < 1$, then we consider the sequence

$$\{a_n\}_{n=0}^{\infty} = \{ar^n\}_{n=0}^{\infty} = a,\ ar,\ ar^2,\ ar^3, \dots.$$

Now, since $|r| < 1$, we have that $|r|^{n+1} < |r|^n$. Since this continues indefinitely, we have that $\lim_{n\to\infty} |r|^n = 0$. If $r < 0$, then the sequence just alternates in sign, but still converges to zero. Also, multiplication by a constant a does not affect convergence as well. That is, $\lim_{n\to\infty} a|r|^n = a \cdot \lim_{n\to\infty} |r|^n$.

(b) This is a bit tough, but I work through this in the notes. Consider

$$A_N = \sum_{n=0}^{N} ar^n = a + ar + ar^2 + \cdots ar^N = a(1 + r + r^2 + \cdots + r^N).$$

Then, if we subtract rA_N to both sides, we get

$$\begin{aligned}
A_N - rA_N &= a(1 + r + r^2 + \cdots + r^N)ra(1 + r + r^2 + \cdots + r^N) \\
(1-r)A_N &= a(1 + r + r^2 + \cdots + r^N) - a(r + r^2 + r^3 + \cdots + r^{N+1}) \\
(1-r)A_N &= a(1 - r^{N+1}) \\
A_N &= \frac{1-r^{N+1}}{1-r}.
\end{aligned}$$

(c) No, the sequence does not converge for all r as if we have $r = 1$ the series becomes

$$\sum_{n=0}^{\infty} a = a\sum_{n=0}^{\infty} 1,$$

which diverges unless $a = 0$. Similarly, if $r > 1$, then $|r|^{n+1} > |r|^n$ and the series will diverge as well which we can see by taking the limit of the sequence of partial sums. For $|r| < 1$, the limit of sequence of partial sums does converge, and specifically it will converge to

$$A_N \to \frac{a}{1-r}.$$

Problem 5. Often we wish to think about functions being represented by series. For example, we can consider the function

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

where $n!$ is read as "n-factorial" and

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1.$$

Then $1! = 1$ and we define $0! = 1$ as well.

(a) Consider $f(1)$. Use a tool like WolframAlpha to compute the series

$$f(1) = \sum_{n=0}^{\infty} \frac{1}{n!}$$

(b) For any value of x, this series converges. So this defines a function on all real numbers. In fact, the series converges even for complex numbers. Simplify the series into its real and imaginary parts. Note,

$$f(ix) = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!}.$$

(c) We can take derivatives of the function $f(x)$ by differentiating the series *term by term*. That is,

$$\frac{d}{dx} f(x) = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{d}{dx} \left(\frac{x^n}{n!} \right).$$

(d) Show that $\frac{d}{dx} f(x) = f(x)$.

(e) What is your guess for what function $f(x)$ is?

Solution 5.

(a) Using WolframAlpha one can write

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Sum[1/(n!),{n,0,infty}]
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which will output

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

(b) Note that $f(x) = e^x$ (although I don't tell you that here). Anyways, we take

$$\sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \cdots$$
$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right).$$

Thus we have

$$\mathrm{Re}(f(ix)) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \text{and} \qquad \mathrm{Im}(f(ix)) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Notice that these are the series for $\cos(x)$ and $\sin(x)$ respectively.

(c) If we take

$$\frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)$$
$$= 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

(d) Since $\frac{d}{dx} f(x) = f(x)$, we know that $f(x) = Ce^x$ for some constant C. In fact, it is true that $f(x) = e^x$.