MATH 272, Homework 4

Due February 26^{th}

Problem 1. (7 pts.) Let $\vec{\boldsymbol{V}}$ be a vector field in the plane $\mathbb{R}^2$ defined by

$$\vec{\boldsymbol{V}}(x,y) = \begin{pmatrix} \frac{1}{2}x - y \\ x + \frac{1}{2}y \end{pmatrix},$$

and let $\vec{\boldsymbol{x}}(t) = \begin{pmatrix} e^{\frac{1}{2}t}(-c_1\sin(t) + c_2\cos(t)) \\ e^{\frac{1}{2}t}(c_1\cos(t) + c_2\sin(t)) \end{pmatrix}$ for $t \in [0,\pi]$ where c_1 and c_2 are yet undetermined constants.

(a) **(2 pts.)** Show that a flow of $\vec{\boldsymbol{V}}$ yields a linear system of equations.

(b) **(2 pts.)** Show that $\vec{\boldsymbol{x}}(t)$ is a flow of the vector field $\vec{\boldsymbol{V}}$.

(c) **(1 pts.)** Let $\vec{\boldsymbol{x}}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Determine the particular solution to the initial value problem.

(d) **(2 pts.)** Plot the $\vec{\boldsymbol{V}}$ and your particular solution $\vec{\boldsymbol{x}}$ simultaneously. Choose good bounds for your plot so that the whole curve is visible.

Problem 2. (7 pts.) Consider our model for a molecular crystal potential for which we took the scalar field

$$u(x,y) = \cos^2(x) + \cos^2(y).$$

(a) **(1 pts.)** Plot the graph of of $u(x,y)$ and the level curves. Feel free to use your old work.

(b) **(1 pts.)** Write down the differential equation for a curve given by gradient descent of the system. That is, the negative of the gradient flow.

(c) **(2 pts.)** Without solving the problem, where would particles end up if they follow gradient descent? What if particles start on a peak?

(d) **(3 pts.)** Recall the matrix $[J]$ given by

$$[J] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The *Hamiltonian flow* is given by

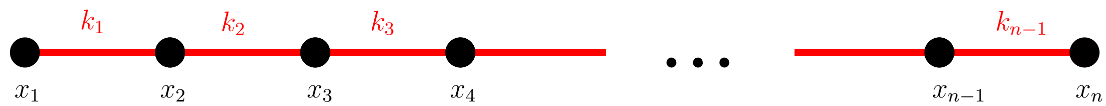
$$\dot{\vec{\boldsymbol{x}}}(t) = [J]\vec{\boldsymbol{\nabla}}u(\vec{\boldsymbol{x}}(t)).$$

The *Hamiltonian vector field* is the right hand side,

$$[J]\vec{\boldsymbol{\nabla}}u(x,y).$$

Plot the Hamiltonian vector field and explain why the level curves to $u(x,y)$ correspond to the Hamiltonian flows.

Problem 3. (10 pts.) Let us consider the discrete heat equation for n equally spaced particles on a line segment for which we have the following picture



Let $u_j(t) \coloneqq u(x_j, t)$ denote the temperature of particle j at time t, let k_j be the thermal transport coefficient between particles j and $j+1$, and let $f_j(t) = f(x_j, t)$ be the thermal energy source on particle j.

(a) **(2 pts.)** For the boundary particles x_1 and x_n, we have

$$\dot{u}_1 = -k_1 u_1 + k_1 u_2 + f_1 \qquad \text{and} \qquad \dot{u}_n = -k_n u_n + k_{n-1} u_{n-1} + f_n,$$

which correspond to *Neumann type boundary conditions*. Explain each term in the above equations.

(b) **(2 pts.)** If we attached x_1 to x_n with a material with a thermal transport coefficent of k_0 the above equations would need modification. Write these new equations. These are the *periodic boundary conditions.*

(c) **(1 pts.)** Explain why periodic boundary conditions are the same as working with a circular domain.

(d) **(1 pts.)** If we force u_1 and u_n to be constant, what will the equations for the boundary particles be? These would be the *Dirichlet type boundary conditions.*

(e) **(2 pts.)** For the interior particles, we have the relationship

$$\dot{u}_j = -k_{j-1} u_j - k_j u_j + k_{j-1} u_{j-1} + k_j u_{j+1} + f_j \qquad \text{for } j = 2, \ldots, n-1.$$

Explain what each term describes in the above equation.

(f) **(2 pts.)** In the limit as $n \to \infty$, we then have that k is described as a function of position, x. The source free heat equation then reads

$$\frac{\partial}{\partial t} u(x,t) = \frac{\partial}{\partial x}\left(k(x) \frac{\partial}{\partial x} u(x,t) \right) + f(x,t).$$

Explain how this equation differs from the equation

$$\frac{\partial}{\partial t} u(x,t) = k(x) \frac{\partial^2}{\partial x^2} u(x,t) + f(x,t).$$

Problem 4. (8 pts.) Consider the 1-dimensional homogeneous Laplace equation given by

$$\frac{\partial^2}{\partial x^2} u_E(x) = 0,$$

with the domain Ω as the unit interval on the x-axis. Take the Dirichlet boundary conditions $u_E(0) = T_0$ and $u_E(L) = T_L$. Think of these values as the ambient temperature at the endpoints of the rod. These temperatures are constant since the ambient environment is so large.

(a) **(2 pts.)** Find the particular solution to this Laplace equation.

(b) **(2 pts.)** Suppose that $v(x,t)$ is a solution to the 1-dimensional source free isotropic heat equation with zero Dirichlet boundary values. Show that

$$u(x,t) = v(x,t) + u_E(x),$$

is a solution to the 1-dimensional source free isotropic heat equation with Dirichlet boundary values $u(0,t) = T_0$ and $u(L,t) = T_L$.

(c) **(2 pts.)** From Problem 1, we know that $\lim_{t\to\infty} v(x,t) = 0$. Hence, show that the long time limit of a solution to the source free heat equation yields a solution to the Laplace equation.

(d) **(2 pts.)** Argue why the equilibrium temperature profile of a rod can be found without solving the heat equation.

Problem 5. (3 pts.) Using intuition from the previous problem, explain how one could solve the heat equation with a nonzero source term that only depends on x. In other words, how could one try to solve

$$\left(-k\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\right) u(x,t) = f(x),$$

Problem 6. (13 pts.) Consider the 2-dimensional source free isotropic heat equation given by

$$\left(-k\Delta + \frac{\partial}{\partial t}\right) u(x,y,t) = 0,$$

with the domain Ω as the unit square in the xy-plane. Take as well the Dirichlet boundary conditions $u(x,y,t) = 0$ for x and y on the boundary of Ω.

(a) **(2 pts.)** Show that $u_{mn}(x,y,t) = \sin(m\pi x)\sin(n\pi y)e^{-k(n^2+m^2)\pi^2 t}$ is a solution to the PDE and Dirichlet boundary conditions for any non-negative integers m and n.

(b) **(2 pts.)** Show that a linear combination of solutions u_{mn} and u_{pq} is also a solution.

(c) **(3 pts.)** For $m = n = 1$ and $k = 1$, plot the solution for the values $t = 0$, $t = 0.01$, $t = 0.1$ and $t = 1$. Explain what is physically happening as time moves forward.

(d) **(2 pts.)** Explain what varying the value for the conductivity k does to the solution. Feel free to use plots to support your hypothesis.

(e) **(2 pts.)** Explain the mathematical reason why increasing m and n causes the solution to converge to zero more quickly.

(f) **(2 pts.)** Explain the physical reason why increasing m and n causes the solution to converge to zero more quickly. Plots may help support your reasoning.

Problem 7. (11 pts.) Consider the 1-dimensional wave equation given by

$$\left(-\frac{\partial^2}{\partial x^2} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u(x,t) = 0.$$

We'll consider two distinct scenarios. First, we'll take an infinitely long elastic rod and second we'll take a rod of finite length with Dirichlet boundary conditions.

(a) **(2 pts.)** For a rod of infinite length, consider the initial conditions

$$u(x,0) = \begin{cases} x+1 & -1 \leq x \leq 0 \\ 1-x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \frac{\partial}{\partial t} u(x,0) = 0.$$

Find and plot the portion of the wave that moves to the right with $c = 1$.

(b) **(2 pts.)** Let $u_R(x,t)$ be your solution from (a), show that this satisfies the right-moving wave equation

$$\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right) u_R(x,t) = 0.$$

(c) **(1 pts.)** Why is it that we can ignore the points where your function $u_R(x,t)$ is not differentiable even though we are considering this as a solution to a PDE?

(d) **(2 pts.)** For an elastic rod Ω of finite length, $\Omega = [0,1]$, assume that we take the Dirichlet conditions $u(0,t) = 0 = u(1,t)$. With the initial conditions

$$u(x,0) = \sin(\pi x) \qquad \text{and} \qquad \frac{\partial}{\partial t} u(x,0) = 0,$$

find the solution using d'Alembert's formula.

(e) **(2 pts.)** Let $w(x,t)$ be your solution for (d), show that it is indeed equal to

$$w(x,t) = \sin(\pi x)\cos(\pi c t).$$

(f) **(2 pts.)** With your result from (e), explain how we can decompose a standing wave into a linear combination of two waves; one moving towards the left and one moving towards the right and both reflecting off the boundary.

Problem 8. (8 pts.) Consider the wave problem on the region $\Omega = [0,1]$ given by

$$\begin{cases} \left(-\frac{\partial^2}{\partial x^2} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u(x,t) = 0, & \text{in } (0,1), \\ u(0,t) = 0 \text{ and } u(1,t) = 0, & \text{as boundary conditions,} \\ u(x,0) = \sin(\pi x), & \text{as the initial condition.} \end{cases}$$

This problem corresponds to taking a plucked elastic string fixed at the endpoints.

(a) **(1 pts.)** Use the separation of variables ansatz $u(x,t) = X(x)T(t)$ to get a new separation constant. This will give two ODEs: one will be in terms of $X(x)$ and the other will be in terms of $T(t)$.

(b) **(2 pts.)** Use the boundary conditions and solve the ODE that is in terms of $X(x)$ which will simultaneously find the allowed values for the separation constant.

(c) **(2 pts.)** Using these allowed values for the separation constant, find the solution for $T(t)$.

(d) **(1 pts.)** Find the particular solution for $u(x,t)$ by matching the initial condition.

(e) **(2 pts.)** Plot your solution for $x \in [0,1]$ and $t \in [0,\infty)$ (i.e., just plot up to a large value of t). In this case, compare your plots for $c = 1/2$ and $c = 1$.

Problem 9. (9 pts.) Consider the heat flow problem on the region $\Omega = [0,1]$ given by

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) - 1, & \text{in } (0,1), \\ u(0,t) = 0 \text{ and } u(1,t) = 1, & \text{as boundary conditions,} \\ u(x,0) = \sin(\pi x) + \frac{1}{2}(x^2 + x), & \text{as the initial condition.} \end{cases}$$

This corresponds to a rod kept at fixed temperatures at the endpoints that starts with a warm center initially.

(a) **(3 pts.)** As with the previous problem, take an ansatz

$$u(x,t) = v(x,t) + u_E(x)$$

where $v(x,t)$ solves the following problem

$$\begin{cases} \frac{\partial}{\partial t}v(x,t) = \frac{\partial^2}{\partial x^2}v(x,t), & \text{in } (0,1), \\ v(0,t) = 0 \text{ and } v(1,t) = 0, & \text{as boundary conditions.} \end{cases}$$

Find the general solution $v(x,t)$ using separation of variables. *Hint: feel free to use the work in the notes (Example "Solving the Heat Equation" and Example "Particular Solution to the 1D Heat Equation").*

(b) **(2 pts.)** Show that for $u(x,t)$ to be a solution that

$$\frac{\partial^2}{\partial x^2}u_E(x) = 1.$$

(c) **(2 pts.)** Find the solution $u_E(x)$ to the following problem

$$\begin{cases} \frac{\partial^2}{\partial x^2}u_E(x) = 1, & \text{in } (0,1), \\ u_E(0) = 0 \text{ and } u_E(1) = 1, & \text{as boundary conditions.} \end{cases}$$

(d) **(2 pts.)** All is left in determining the function $u(x,t)$ is to determine the particular solution that satisfies the initial condition. Using our ansatz $u(x,t) = v(x,t) + u_E(x)$, determine the particular solution.