MATH 272, Homework 3

Due February 17th

Problem 1. Show that for any smooth (more than twice differentiable) fields $f(x,y,z)$ and $\vec{\boldsymbol{V}}(x,y,z)$ that

(a) $\vec{\boldsymbol{\nabla}} \times \left(\vec{\boldsymbol{\nabla}} f\right) = \vec{\mathbf{0}}$;

(b) $\vec{\boldsymbol{\nabla}} \cdot \left(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{V}}\right) = 0$.

Problem 2. Let

$$\vec{\boldsymbol{U}}(x,y,z) = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \qquad \text{and} \qquad \vec{\boldsymbol{V}}(x,y,z) = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix},$$

be vector fields.

(a) Explain why there exists no potential function $\phi(x,y,z)$ for the vector field $\vec{\boldsymbol{U}}$.

(b) Explain why there does exist a potential function $\phi(x,y,z)$ for the field $\vec{\boldsymbol{V}}$.

(c) Compute the potential function for $\vec{\boldsymbol{V}}$.

Problem 3. Consider the two dimensional scalar field $T(x,y) = x + y$ that describes the temperature on the square plate Ω given by the set $0 \leq x, y \leq 1$. Compare the two answers you get!

(a) Compute the integral

$$\int_{\Omega} T d\Omega.$$

(b) Let $\vec{\boldsymbol{\gamma}}$ be the curve that traverses the boundary of the square plate in the counterclockwise direction. Compute

$$\int_{\vec{\boldsymbol{\gamma}}} T d\vec{\boldsymbol{\gamma}}.$$

Problem 4. Let $f(x,y,z) = 2xy + e^{xz} + \sin(y)$ be a scalar field. Integrate f over the triangular prism Ω defined by taking the half triangle of the unit square in the xy-plane satisfying $x \leq y$ and with height 4 above the xy-plane.

Problem 5. Consider $f(x,y) = 3x^4 + x^3 - 18x^2y^2 - 3xy^2 + 3y^4$.

(a) Show $\Delta f = 0$.

(b) Find the surface normal to the graph of f.

Problem 6. Parameterize the following either implicitly or explicitly. In Cartesian coordinates, find the parameterization of the normal vector as well.

(a) The plane perpendicular to the vector $\vec{\boldsymbol{v}} = \hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}$ passing through the point $(1, 1, 1)$.

(b) The upper half of the unit disk in the xy-plane.

(c) The surface of the unit sphere in $\mathbb{R}^3$.

Problem 7. Consider the following vector field

$$\vec{\boldsymbol{E}}(x, y, z) = \begin{pmatrix} \frac{x}{(x^2+y^2+z^2)^{3/2}} \\ \frac{y}{(x^2+y^2+z^2)^{3/2}} \\ \frac{z}{(x^2+y^2+z^2)^{3/2}} \end{pmatrix},$$

which models the electric field of a proton (in units of of charge $q = 1$) placed at the origin.

(a) Show that $\vec{\boldsymbol{E}}(x, y, z) = -\vec{\boldsymbol{\nabla}}\phi(x, y, z)$ where $\phi(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$. We refer to $\phi(x, y, z)$ as the electrostatic potential (or voltage).

(b) Let Ω be a box with side lengths two centered at the origin. Compute the total flux of $\vec{\boldsymbol{E}}$ through the surface of the box Σ. That is,

$$\int_\Sigma \vec{\boldsymbol{E}} \cdot \hat{\boldsymbol{n}} d\Sigma.$$

(c) Using the provided argument, one can compute

$$\int_\Omega \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}} d\Omega.$$

- Compute $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}}$ and note that this is zero everywhere except at $(x, y, z) = (0, 0, 0)$.
- Note that the two integrals in this problem are equal. This is known as the *divergence theorem* and it is a special case of a more general theorem called *Stokes' theorem* which generalizes the fundamental theorem of calculus. Is it true that $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}} = 0$ everywhere?

(d) Does the total flux depend on the size or shape of the box?