Pendulum Project Report

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PHY324: Practical Physics II

December 2024

Abstract

This paper goes over a study which analyzes the physics governing a simple 2-dimensional pendulum. The authors construct a model to minimize external influence and asymmetry, record and track the motion of the pendulum, and analyze the influence of the length, mass, and angle of release of the pendulum on its period of oscillation and exponential decay. **TBD: Final data and verification**

1 Introduction

The pendulum can be used to understand many phenomena observed in classical mechanics, from harmonics and oscillations to chaos, center of mass or gravity, and superposition of forces. Thus, it is worth exploring to verify and strengthen our understanding of these phenomena. Additionally, it is a simple experiment to setup and acquire data for, so it makes for a good experiment for practicing experimental procedure and design.

If we take a video of the pendulum, and track the position of its center of mass using a tracking software. Then if we have the horizontal deviation from the position of rest, we can do two things: first, track the peaks (where the deviation reaches a local extrema) to get the period of oscillation, and second, fit it to the theoretical equation of decaying motion to get the constant of exponential decay.

Background Theory: Simple Pendulum

The time period, T, of an ideal undamped pendulum can be derived from its length, ℓ, and acceleration due to gravity, g, using a well known equation,

$$T = 2\pi\sqrt{\frac{\ell}{g}}. \tag{1}$$

For the construction using a string of length L and a weight with distance to its center of mass D, we can re-orient equation (1) to get a relation:

$$T \propto \sqrt{(L+D)} = \sqrt{\ell}. \tag{2}$$

We propose that an ideal pendulum's time period follows this, and should be independent of the mass and release angle.

No pendulum is ideal, so we expect some decay in the amplitude of the pendulum. Considering angles, we can obtain an exponentially decaying or damped equation for angle over time $\theta(t)$,

$$\theta(t) = \theta_0 \exp(-\frac{t}{\tau}) \cdot \cos(2\pi\frac{t}{T} + \phi_0), \tag{3}$$

where θ_0 is the release angle, t is time since release, T is the time period, ϕ_0 is the phase constant of release, and τ is the time constant of decay. The data we collect is the horizontal displacement from the mean position of the pendulum x. Using, trigonometry, substituting $\gamma = 1/\tau$, length ℓ, and equation (3), we can get a more suitable equation for analyzing data as:

$$x(t) = \ell \cdot \sin\left(\theta_0 \cdot e^{-\gamma t} \cdot \cos(2\pi\frac{t}{T} + \phi_0)\right). \tag{4}$$

Additionally, instead of fitting to the entire positional data, we simplify our job by choosing the extrema of the pendulum, which we know happens when $2\pi\frac{t}{T} + \phi_0 = 2\pi n$ $(\star)$ for some integer n, which is the number of oscillations t/T with $\phi_0 = 0$ (See Figure 10 for a visual explanation in the Appendix). Then for specific values t' which satisfy the above relation $(\star)$, we have:

$$x(t') = \ell \cdot \sin\left(\theta_0 \cdot e^{-\gamma t'}\right). \tag{5}$$

2 Methods

Experimental Setup

We used the following tools for measurements and building the setup: iGaging EZCal IP54 calipers, a thin plastic ruler, a spirit level, Stanley 3m/10' Tylon measuring tape, and an Acculab Sartorius Group VICON digital weighing scale. For recording the videos, we attached a phone to a metal stand. The videos were recorded on an Android 15 phone in 30 frames per second.

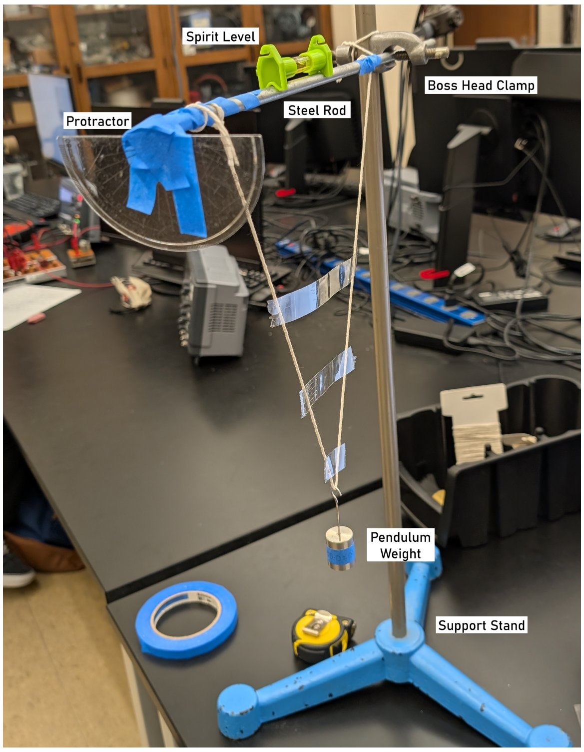

Figure 1: Experimental Setup for the pendulum model. We cut a hole in the tape to make sure the markings on the protractor were visible to measure the release angle.

Building the Pendulum

We used a three-footed metal support stand as a base for building the pendulum. Then, we used a boss head clamp to link another steel rod perpendicular to it at a reasonable height for video recording. We used a spirit level to level the rod and ensure that the pendulum oscillates as two-dimensionally as possible. This balanced the direction of gravity and the net force from the tension of the strings and also made the video analysis more accurate.

We took a long enough twine and tied it to two separate ends chosen randomly on the horizontal steel rod. We used tape to secure one end tightly. Then, we added a weight to create tension in the twine while holding the other end. According to the trial, we adjusted the second end to get the desired L value and secured the end with tape. This ensured that the ends did not slip and the length of the pendulum remained constant.

The goal of this two-end approach was to eliminate the asymmetry using tension from two strings. The tension in the string would self-adjust and move the weight of the mass to the middle. This prevented the pendulum from swinging in the plane of the strings as it would need to displace the strings or lift the weight up, which is highly unlikely in this system due to gravity and centrifugal forces. The system had a potential well in the linear plane of motion along the middle of the two strings ($\sim 2D$), which is what we wanted for the experiment.

Then we attached a protractor to the edge of the horizontal rod in a way that the center of measurement was collinear with the two pivots of the twine using tape and ensured the rest position of the pendulum had both strings in a plane with the 90-degree mark on the same (see Figure 1). This allowed us to measure the angle of release as per the trial being conducted.

We chose to add a few pieces of tape to the pendulum above the weight without stretching the strings. This ensured that the two pieces of twine did not twist. To complete the whole process, we measured the weight of each of the weights we used, then used a ruler to balance and calculate the vertical placement of the center of mass for them, and placed a layer of tape evenly around the weight at that height to make it easier for the tracking software while not affecting its center of mass. We hung the mass by the twine, started the video, and released the mass from the desired angle for each trial.

Data Collection

This paper aimed to see how the period and decay were affected by the construction of the pendulum. So, we varied the mass m, angle of release θ_0, and length of the pendulum $L + D$ (string L) individually while keeping the others fixed.

We used a harmonized value for D (See Discussion: Uncertainties). The exact weights (Table 1) and the trials for different L, m, θ_0 (Table 2) are available in the Appendix. We took roughly a minute of data for each trial to measure period and exponential decay. We tried to release the pendulum as closely to the center as possible to reduce any back-and-forth oscillations at the pivoting point where the weight hooks onto the twine.

We used the Physlets Tracker software (6.2.0) [1] for Windows 11 to analyze the motion of the pendulum. We recorded the videos and marked a portion of the weight from the first frame, which we adjusted to be the release frame. Then, we used the autotracker to track the mass and filled in the frames manually where the tracker was unsure. We used this data to get the period of oscillation and exponential decay. After the trials, we cut and measured the weight of twine and tape that swung with the pendulum.

3 Results

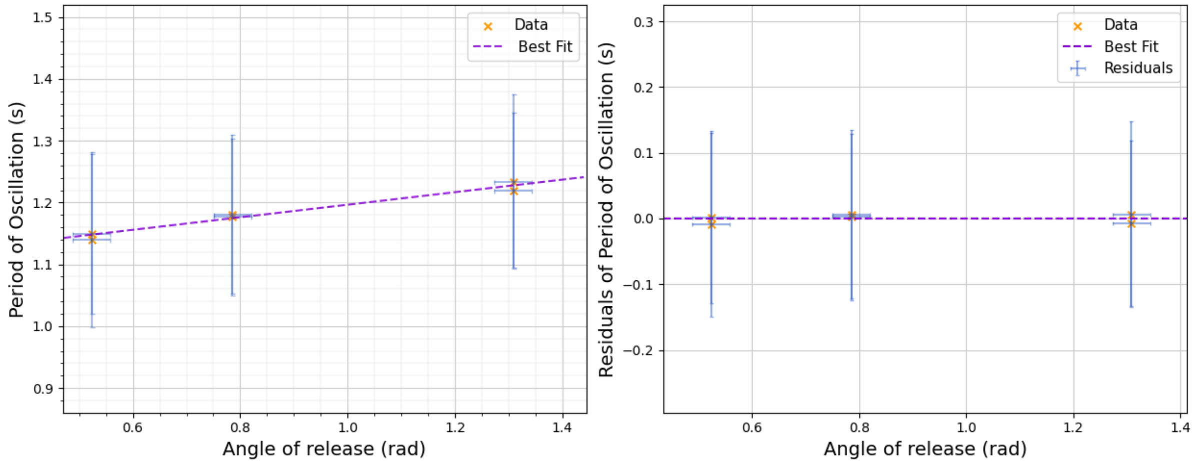

Figure 2: Plot of period of oscillation (T) vs angle of release (θ_0) for a pendulum of fixed mass $50.0g$ and length $32.5 \pm 0.4cm$, with residuals. Data (yellow) is presented with the linear best fit (purple). Line of best-fit has slope $0.1 \pm 0.2\ s/rad$ with offset $1.1 \pm 0.2(s)$. Plot has reduced $\chi^2 = 0.004$ with probability 1.00. Absolute angles were plotted for a general linear approximation. Residuals are unpatterned but small due to the low χ^2 value. Overall, the fit is decent and flat-slope indicates null-hypothesis.

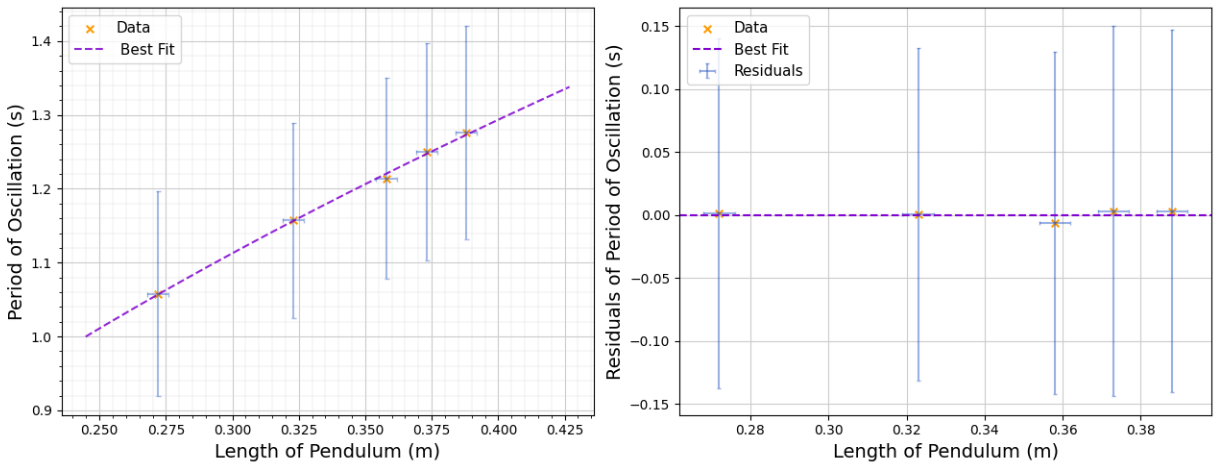

Figure 3: Plot of period of oscillation (T) vs length ($L + D$) for a pendulum of fixed mass $50.0g$ and release angle $45 \pm 2\ deg$, with residuals. Data (yellow) is presented with the square-root best fit (purple) following the prediction of equation (2) as $y = a\sqrt{x} + b$. Line of best-fit has parameters $a = 2.1 \pm 1.7\ s/\sqrt{m}$ with offset $b = -0.1 \pm 1.0(s)$. Plot has reduced $\chi^2 = 5 \times 10^{-5}$ with probability 1.00. Residuals are un-patterned and extremely small due to the low χ^2 value. Overall, the fit is good and validates our prediction for the relation.

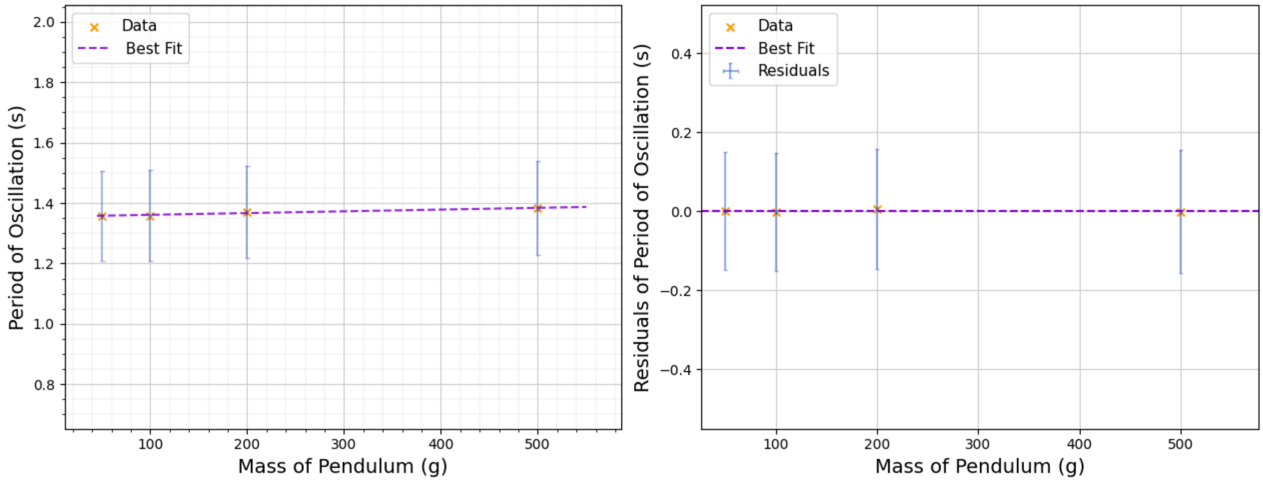

Figure 4: Plot of period of oscillation (T) vs mass (m) for a pendulum of fixed length 48.1 ± 0.4 cm and release angle 45 ± 2 deg, with residuals. Data (yellow) is presented with the linear best fit (purple). Line of best-fit has slope $(6 \pm 40) \times 10^{-5}$ s/g with offset $1.4 \pm 0.1(s)$. Plot has reduced $\chi^2 = 6 \times 10^{-5}$ with probability 1.00. Residuals are unpatterned and extremely small due to the low χ^2 value. Overall, the fit is good and very small slope is highly indicative of the null hypothesis.

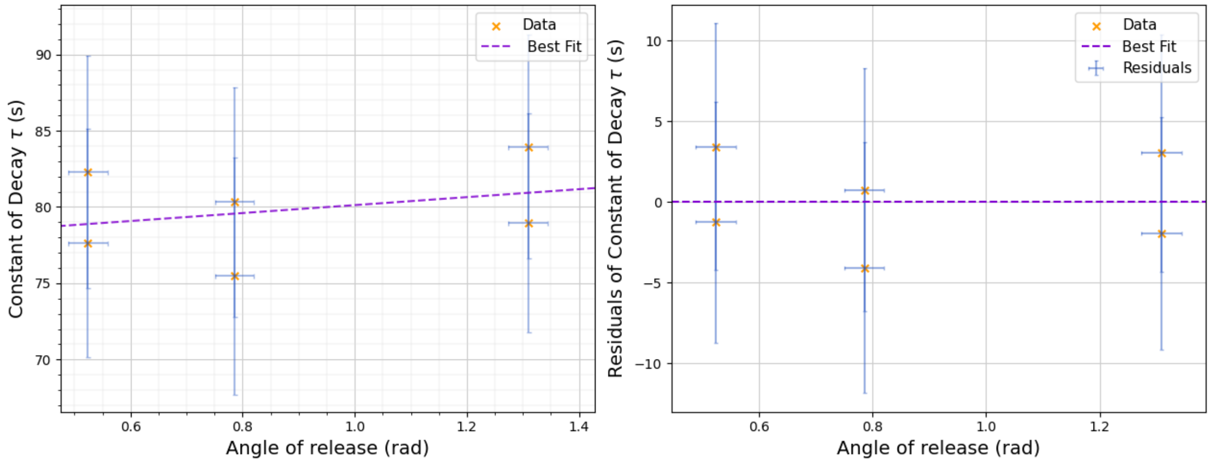

Figure 5: Plot of (τ) vs release angle for a pendulum fixed mass $50.0g$ and length $32.5 \pm 0.4cm$, with residuals. Data (yellow) is presented with the linear best fit (purple). Line of best-fit has slope 4.6 ± 9.2 s/rad with offset $77 \pm 9(s)$. Plot has reduced $\chi^2 = 0.20$ with probability 0.9. Absolute values of angles were plotted for a generalized approach. Residuals are unpatterned so the fit is good and the relation seems to be linear.

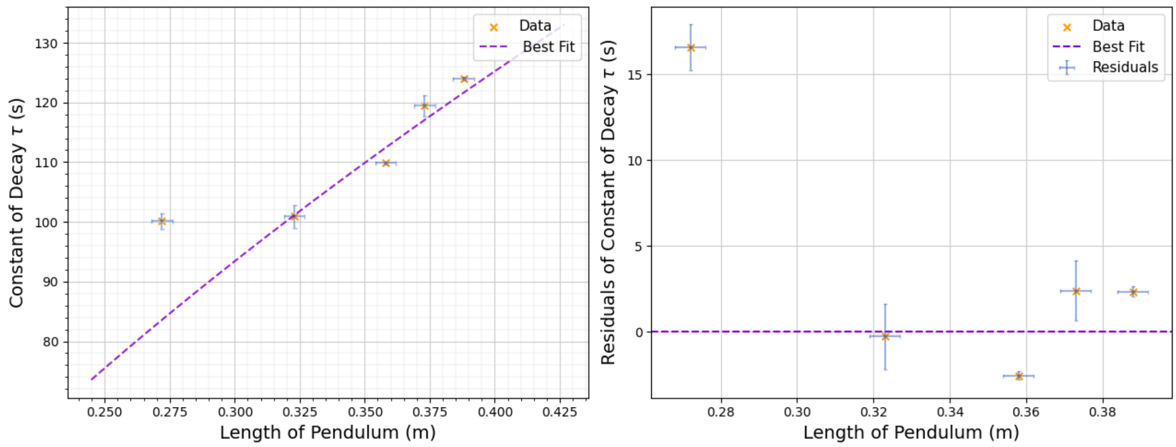

Figure 6: Plot of (τ) vs length for a pendulum fixed mass $50.0g$ and release angle 45 ± 2 deg, with residuals. Data (yellow) is presented with the square-root best fit (purple) following the prediction of equation (2) as $y = a\sqrt{x} + b$. Line of best-fit has parameters $a = 380 \pm 10$ $s/\sqrt{m}$ with offset $b = -112 \pm 6(s)$. Plot has reduced $\chi^2 = 0.86$ with probability 0.49. Residuals are un-patterned and except for the one outlier (video was not very long, hence inaccuracy in the fit due to high initial amplitude). Good chi-squared values and residuals indicate the fit is good and highlights a similar relation for the τ as period.

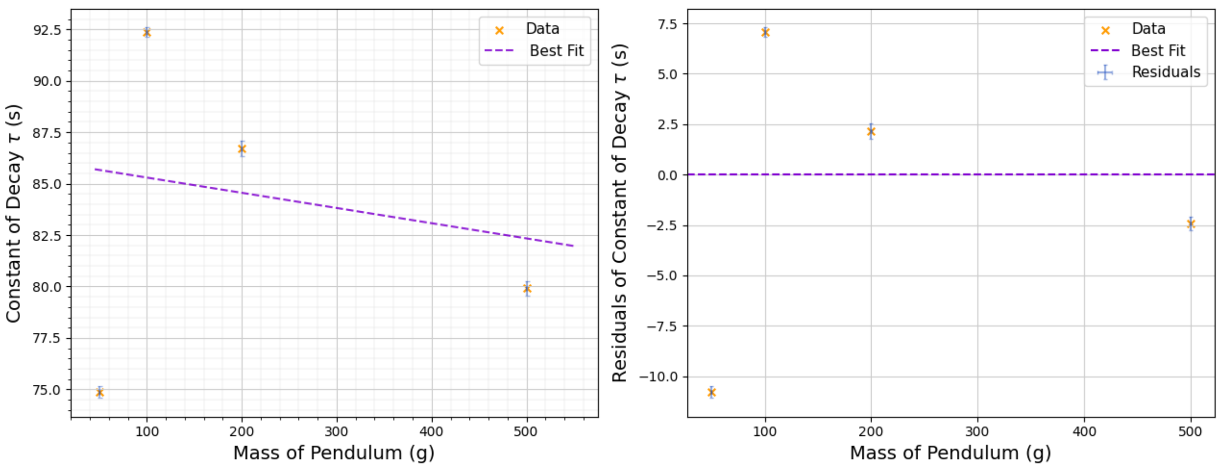

Figure 7: Plot of (τ) vs mass for a pendulum of fixed length 48.1 ± 0.4 cm and release angle 45 ± 2 deg, with residuals. Data (yellow) is presented with the linear best fit (purple). Line of best-fit has slope $(74 \pm 9) \times 10^{-4}$ s/g with offset $86 \pm 1(s)$. Plot has reduced $\chi^2 = 0.69$ with probability 0.56. Residuals are unpatterned. Despite good chi-sq. values and residuals, we don't have enough data points to classify the fit well, so for the time being, the fit is bad and gives us little information.

See Individual Data in Appendix for how each dataset looked. Figures 2 to 4 show the influence of the angle of release (θ_0), length ($L + D$) and mass m of the pendulum on the time period, while Figures 5 to 7 show it for the constant of decay. Time period was calculated as the average of the distance between consecutive peaks as seen in Figure 8, with uncertainty taken as standard error. The constant τ was calculated as the reciprocal of γ from the analysis method in Figure 8 using equation (5). Uncertainty was propagated from the standard error of γ, calculated using the covariance matrix using scipy curvefit, to get uncertainty for τ using the equation: $\frac{\Delta\tau}{\tau} = \frac{\Delta\gamma}{\gamma}$.

Uncertainties

The measurement uncertainties (See Table 2 in Appendix) we took for L, D, and the dimension of the weights come from the least count errors of the measuring device (± 0.1 cm for the measuring tape). We tried to eliminate parallax to the best of our capabilities.

The weights we used had a listing, but we re-measured them on a digital scale to detect any defects or change in weight from wear and tear from use. We used the scale's uncertainty from its user manual, which was calculated to be ± 0.1 g (See Table 1 in Appendix), as the uncertainty in m.

Table 1 (Appendix) shows that all weights had a very similar D_0 value. So, we took an overall uncertainty for simplicity: 50 ± 2 mm, which covered all values along their error margins. The weight of the twine and tape was measured to be around 1.0 ± 0.1 g. The lengths of pendulum (L) tested were between 22.5 cm and 32.5 cm. Using that, our calculations show the center of mass of the pendulum is shifted by at least 1 mm and at most 3 mm for the various weights. The offset thus is -2 ± 1 mm (minus sign as towards the pivots), and as a result, we set the final value of D, or the actual length of the pendulum, to be $= 48 \pm 3$ mm (adding them linearly).

We tried to minimize the error in the angle of release, θ_0, by ensuring the two pivots and the center of the protractor were collinear. From there, as the tweed had some thickness and to account for human error, we took the uncertainty to be twice the least count, or $\pm 2°$.

For the autotracker in the software, we used the junction of the metal of the weight and the tape attached to it for the best results. This was also roughly the center of mass of the pendulum. The experiment uses this choice as the autotracker uses the change in RGB values of the key region to track the object, meaning that the more variety and specificity is there in the key region, the more reliably the autotracker is able to find it in subsequent frames [2].

For position uncertainty, we took half the length of the largest side of the region of tape on the weight, as that is the most our measurement could be off by in each frame, this was the width of each weight as mentioned in Setup Details. For uncertainty in the measurement of time, we took half of the time interval between frames.

We plotted the obtained value from the tracking software (decay constant or time period) against the controlled variable (length, mass, release angle) to see the relationship of said value with the variable. Then, we used a curve fit to fit the best function (linear or otherwise by comparing visual cues with known distributions) and then used the errors from the best fit to get a root-mean-squared error on the final value from the covariance matrix.

Sources of Error

Two main things contribute to errors in our measurement: incorrect measurements (measured value doesn't match the physical phenomenon) and incorrect modeling (the measured value is incomplete for analysis).

For the first kind, there is probably some error in the measurement for the setup in the form of parallax for the length and angle of release. These are accounted for in the uncertainty.

There is also the video recording itself. It is impossible to have a perfect recording unless the sensor is as large as the setup. There is some angle and warping as a result of the lens. Then, there is the possibility of an error in the tracking software if any frames are bad or auto-tracked improperly.

For the second kind of error, there may be factors of the pendulum not considered in our model. One of these is the asymmetry of the pendulum. If the pendulum oscillates in the 2D plane xy, the mass may oscillate from the contact point with the twine string in the yz plane. This would take away energy from the system and give us an inaccurate/fluctuating measurement like a double pendulum.

The point of taking multiple measurements, and with positive and negative values for the same magnitudes, was to statistically minimize the influence of these variables using the law of large numbers .

4 Conclusion

Our experimental design seemed to work well and modeled the pendulum to a good extent. The asymmetry of the pendulum was low as the plots fitting to the extrema were mostly symmetrical and had constants that lied close to each other. The experiment was able to verify of the prediction of equation (4) of motion (Figure 8) and the direct proportionality of period T and length $\sqrt{(L+D)}$ as seen in Figure 3.

Our results gave us good reason to believe the null hypothesis for the mass and angle of release on their influence on period of oscillation for the pendulum as the slopes for their graphs were near zero (Figures 2 and 4).

Release angle seemed to have a similar effect on the decay constant, having a very flat slope, but it was not as concrete as for period, and could be a flat linear relation. Surprisingly, the length and the decay constant shared the same square-root proportionality as with period to a reliable extent. Our data for mass and τ seemed to be inconclusive. We suspect this might be possibly due to mixing data from two different video sources, sacrificing consistency of the setup for more data to use for statistical modeling. As mentioned before, there may be some warping of the videos which we couldn't account for properly. We were also limiter by the amount of weights we had for this experiment, having a variety of weights could help improve our data.

The relation between the decay constant and the length prompted us to consider the *Q-Factor* of the pendulum, which essentially tells us the time it takes for the pendulum to come to a stop. It is directly proportional to mass, angular frequency and frictional force at the pivot. While our mass data was inconclusive, the length data saw similar proportionality as of the angular frequency to the QF of a pendulum. It would also be an interesting idea to try and measure the frictional force at the pivot using scales to measure force needed to start the pendulum's oscillation and known constants for twine, metal and similar materials.

References

[1] Physlets Tracker v6.2.0 for Windows: *https://physlets.org/tracker/* - Accessed December 12, 2024

[2] Autotracker Tutorial: *https://www.youtube.com/watch?v=Dn0Zz7rtkZw* - Accessed December 12, 2024

[3] PHY324 Pendulum Project Instructions: *Instruction Manual* - Accessed December 12, 2024

Acknowledgment of Content Usage

The author of this paper came up with the setup with inputs from Tadhg Hearne and Larry Avramdis. Data was collected by the author for all the trials in the appendix. This data was shared only with Sen Yuan. The author would also like to credit Tosin Sanwoolu and Anushka Noble for sharing their data, which was not used in the measurements.

Acknowledgment of AI Use

The author used Grammarly for removing any spelling and grammatical errors in the content. The author of this paper did not use any AI tools for generating the content, data analysis, the experimental design, or any other aid other than the one listed above.

Appendix

Setup Details

All weights were shaped as near-cylindrical with a smaller cylindrical cavity on the underside for space for a hook.

Material	Weight (g)	Height (mm)	Outer Diameter (mm)	D_0 (mm)
Stainless Steel	50.0 ± 0.1	25 ± 1	20 ± 1	50 ± 1
Brass	100.0 ± 0.1	36 ± 1	22 ± 1	49 ± 1
Brass	200.2 ± 0.1	44 ± 1	29 ± 1	51 ± 1
Brass	500.9 ± 0.1	–	–	50 ± 1

Table 1: Weight, dimensions, and distance of center of mass from hook (D_0) for the weights used for the experiment. We did not have the data for height and diameter for the 500g weight, as that was only used by Tosin and Anushka

Different trial setups are presented in the table below. All trials were recorded on an android 15 device in 30 frames per second for roughly 2 minutes.

Trial	Length L (cm)	Weight m (g)	Release Angle θ_0 (deg)
1.	27.5 ± 0.1	50.0 ± 0.1	$+30 \pm 2$
2.	27.5 ± 0.1	50.0 ± 0.1	$+45 \pm 2$
3.	27.5 ± 0.1	50.0 ± 0.1	$+75 \pm 2$
4.	27.5 ± 0.1	50.0 ± 0.1	-30 ± 2
5.	27.5 ± 0.1	50.0 ± 0.1	-45 ± 2
6.	27.5 ± 0.1	50.0 ± 0.1	-75 ± 2
7.	43.3 ± 0.1 ($\star$)	50.0 ± 0.1	$+45 \pm 2$
8.	43.3 ± 0.1	100.0 ± 0.1	$+45 \pm 2$
9.	43.3 ± 0.1	200.2 ± 0.1	$+45 \pm 2$
10.	43.3 ± 0.1 ($\star$)	500.9 ± 0.1	$+45 \pm 2$
19.	22.5 ± 0.1	100.0 ± 0.1	$+45 \pm 2$
20.	27.5 ± 0.1	100.0 ± 0.1	$+45 \pm 2$
21.	31.0 ± 0.1 ($\star$)	100.0 ± 0.1	$+45 \pm 2$
21.	32.5 ± 0.1	100.0 ± 0.1	$+45 \pm 2$
21.	34.0 ± 0.1 ($\star$)	100.0 ± 0.1	$+45 \pm 2$

Table 2: Length of string, weight of mass, and release angle of pendulum for the different trials performed for the experiment. ($\star$) Indicates the dataset was from the ones provided by Tosin and Anushka.

Individual Data

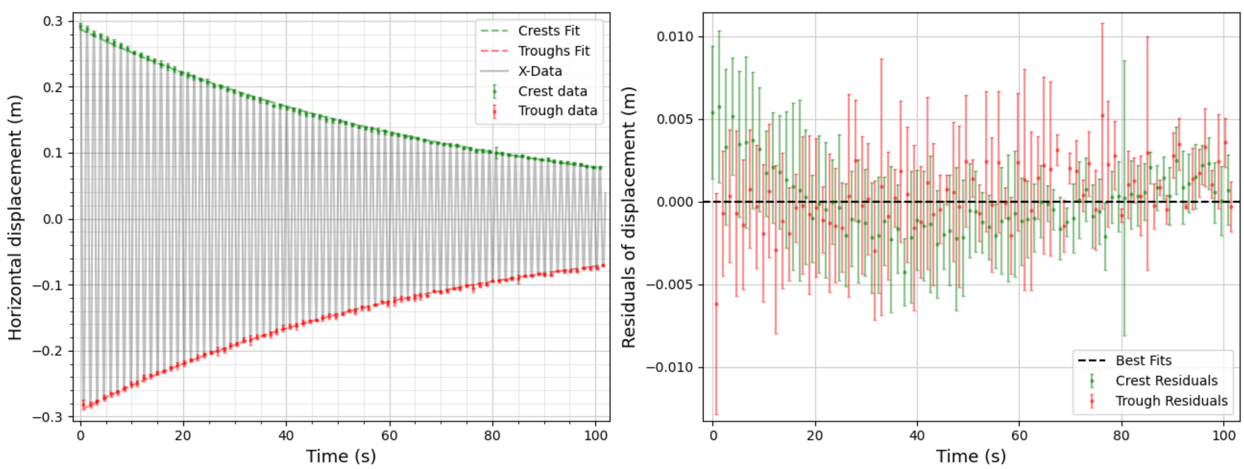

Figure 8: Plot of x-displacement vs time with residuals for a pendulum of $L = 34 \pm 1cm$, angle 45 ± 2 deg and mass $100.0 + 0.1g$. Data is an exponentially decaying gray line. Green error bars show location of crests, red error bars show location of troughs, and respective dotted lines show the best-fit with equation (5).

For Figure 8, Line of best-fit for crests approximates $\ell = 30 \pm 10cm$, $\theta_0 = 40 \pm 10\ deg$ and $\gamma = 1/\tau = 0.013 \pm 0.001(s^{-1})$. Line of best-fit for troughs also approximates (slightly different, but identical when rounded up) $\ell = 30 \pm 10cm$, $\theta_0 = 40 \pm 10\ deg$ and $\gamma = 1/\tau = 0.013 \pm 0.001(s^{-1})$. Both fits have reduced χ^2 value around 2×10^{-5} with probability 1.00, indicating an over-fit. The residuals are slightly patterned in the beginning but random afterwards. Overall, we claim that the fit is decent.

The overfitting indicates that our function estimates the motion of the pendulum well. The residuals are the way they are due to the decay not being perfectly exponential for larger angles, and then smoothing out for smaller angles (and consequentially x-displacement) as the small angle approximation becomes more and more accurate. Additionally, similar parameters for the crests and troughs indicates that the pendulum's construction and data collection both are quite symmetric.

Additional Figures

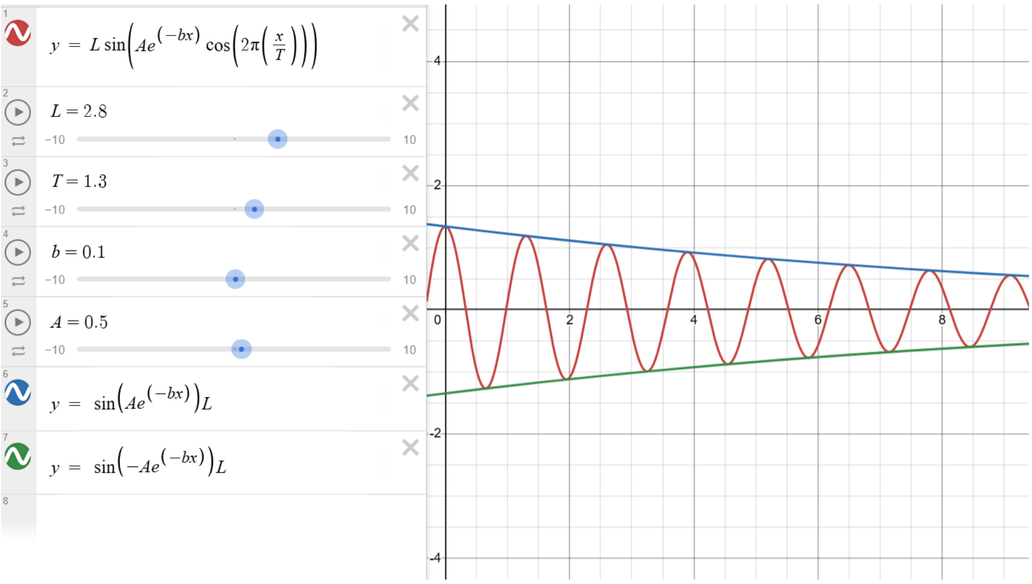

Figure 9: Proposed equation for exponential decay with a functions fitting to the extrema.

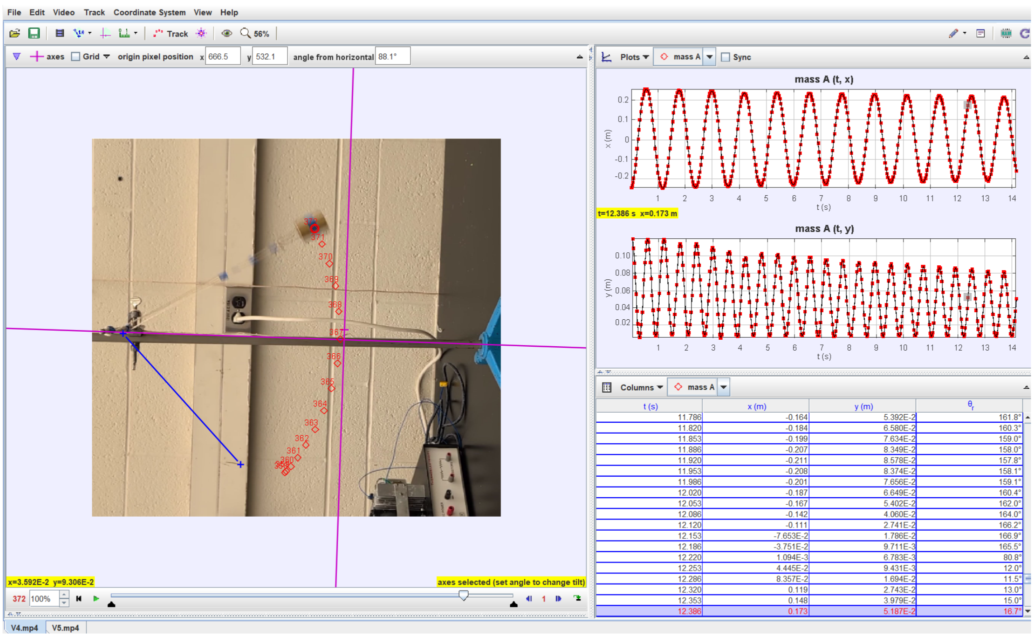

Figure 10: Setup of the physics tracker. Note that the axis were adjusted with the orientation of the video and to have the y axis pass directly through the pendulum pivots to make sure symmetry along x-axis is preserved.