MATH 681, HW-2

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Problem 1

To prove $Y_t = B_t^2 - t$ is a martingale, i.e., $E[Y_t|\mathcal{F}_s] = Y_s$, it's equivalent to prove $E[Y_t - Y_s|\mathcal{F}_s] = 0$. Since

$$\begin{aligned}
E[Y_t|\mathcal{F}_s] = Y_s &\Leftrightarrow E[B_t^2 - t|\mathcal{F}_s] = B_s^2 - s \\
&\Leftrightarrow E[B_t^2 - t|\mathcal{F}_s] - B_s^2 - s = 0 \\
&\Leftrightarrow E[B_t^2 - t|\mathcal{F}_s] - E[B_s^2 - s|\mathcal{F}_s] = 0 \\
&\Leftrightarrow E[(B_t^2 - t) - (B_s^2 - s)|\mathcal{F}_s] = 0 \\
&\Leftrightarrow E[Y_t - Y_s|\mathcal{F}_s] = 0
\end{aligned}$$

Next, we derive $E[Y_t - Y_s|\mathcal{F}_s]$.

$$\begin{aligned}
E[Y_t - Y_s|\mathcal{F}_s] &= E[(B_t^2 - t) - (B_s^2 - s)|\mathcal{F}_s] \\
&= E[B_t^2 - B_s^2|\mathcal{F}_s] + (s - t) \\
&= E[B_t^2|\mathcal{F}_s] - E[B_s^2|\mathcal{F}_s] + (s - t) \\
&= E[(B_t - B_s + B_s)^2|\mathcal{F}_s] - E[B_s^2|\mathcal{F}_s] + (s - t) \\
&= E[(B_t - B_s)^2|\mathcal{F}_s] + 2B_sE[B_t - B_s|\mathcal{F}_s] + E[B_s^2|\mathcal{F}_s] - E[B_s^2|\mathcal{F}_s] + (s - t) \\
&= E[(B_t - B_s)^2] + 2B_sE[B_t - B_s] + (s - t) \\
&= t - s + s - t \\
&= 0
\end{aligned}$$

And $E[|Y_t|] = E[|B_t^2 - t|] < \infty$. Therefore, $(Y_t)_{t\geqslant 0}$ is a martingale with respect to $(\mathcal{F}_s)_{s\geqslant 0}$.

Problem 2

$$Q_\Pi = \sum_{i=1}^{n}(W_{t_i} - W_{t_{i-1}})^2$$

$$
\begin{aligned}
E[Q_\Pi] &= \sum_{i=1}^{n} E[(W_{t_i} - W_{t_{i-1}})^2] \\
&= \sum_{i=1}^{n} E[(B_{t_i} + \mu t_i - B_{t_{i-1}} - \mu t_{i-1})^2] \\
&= \sum_{i=1}^{n} E[(B_{t_i} - B_{t_{i-1}})^2] + \sum_{i=1}^{n} E[(B_{t_i} - B_{t_{i-1}})(\mu t_i - \mu t_{i-1})] + \sum_{i=1}^{n} E[(\mu t_i - \mu t_{i-1})^2] \\
&= \sum_{i=1}^{n} Var[(B_{t_i} - B_{t_{i-1}})] + \mu^2 \sum_{i=1}^{n} E[(t_i - t_{i-1})^2] \\
&= \sum_{i=1}^{n} (t_i - t_{i-1}) + \mu^2 \sum_{i=1}^{n} (t_i - t_{i-1})^2 \\
&= T + \mu^2 \sum_{i=1}^{n} (t_i - t_{i-1})^2 \\
&\leqslant T + \mu^2 \sum_{i=1}^{n} \sup(t_i - t_{i-1})(t_i - t_{i-1}) \\
&= T + \mu^2 \sum_{i=1}^{n} \|\Pi\| \, (t_i - t_{i-1}) \\
&\to T \qquad \text{as } \|\Pi\| \to \infty
\end{aligned}
$$

$$
\begin{aligned}
Var[Q_\Pi] &= \sum_{i=1}^{n} Var[(W_{t_i} - W_{t_{i-1}})^2] \\
&= \sum_{i=1}^{n} \left\{ E[(W_{t_i} - W_{t_{i-1}})^4] - \left[E[(W_{t_i} - W_{t_{i-1}})^2] \right]^2 \right\} \\
&= \sum_{i=1}^{n} \left\{ 3(t_i - t_{i-1})^2 - \left[(t_i - t_{i-1}) + \mu^2 (t_i - t_{i-1})^2 \right]^2 \right\} \\
&= \sum_{i=1}^{n} \left\{ 3(t_i - t_{i-1})^2 - (t_i - t_{i-1})^2 - 2\mu^2 (t_i - t_{i-1})^3 - \mu^4 (t_i - t_{i-1})^4 \right\} \\
&= \sum_{i=1}^{n} \left\{ 2(t_i - t_{i-1})^2 - 2\mu^2 (t_i - t_{i-1})^3 - \mu^4 (t_i - t_{i-1})^4 \right\} \\
&\leqslant \sum_{i=1}^{n} \sup(t_i - t_{i-1}) \left\{ 2(t_i - t_{i-1}) - 2\mu^2 (t_i - t_{i-1})^2 - \mu^4 (t_i - t_{i-1})^3 \right\} \\
&= \sum_{i=1}^{n} \|\Pi\| \left\{ 2(t_i - t_{i-1}) - 2\mu^2 (t_i - t_{i-1})^2 - \mu^4 (t_i - t_{i-1})^3 \right\} \\
&\to 0 \qquad \text{as } \|\Pi\| \to \infty
\end{aligned}
$$

We know the fact that $< W, W >_T$ is a r.v. such that $E[(< W, W >_T - Q_\Pi)^2] \to 0$ as $\|\Pi\| \to \infty$.

Here, we have

$$
\begin{aligned}
E[(T - Q_\Pi)^2] &= Var[Q_\Pi] + [E[T - Q_\Pi]]^2 \\
&= Var[Q_\Pi] + 0 \\
&\to 0 \qquad \text{as } \|\Pi\| \to \infty
\end{aligned}
$$

Thus, $< W, W >_T = T$.

Problem 3

(a) Let $B_s = X \sim N(0,s), B_t = Y \sim N(0,t)$, and $B_t - B_s = Y - X = Z \sim N(0, t-s)$. Since B_s and B_t are independent, then we will have

$$\begin{aligned}
P[B_s \leqslant x, B_t \leqslant y] &= P[X \leqslant x, Y \leqslant y] \\
&= P[X \leqslant x, X + Z \leqslant y] \\
&= \iint_{x+z \leqslant y} f_{X,Z}(x,z) \mathrm{d}z \mathrm{d}x \\
&= \iint_{x+z \leqslant y} f_X(x) f_Z(z) \mathrm{d}z \mathrm{d}x \\
&= \int_{-\infty}^{y-x} \int_{-\infty}^{x} f_X(x) \mathrm{d}x f_Z(z) \mathrm{d}z \\
&= \int_{-\infty}^{y-x} F_X(x) f_Z(z) \mathrm{d}z
\end{aligned}$$

Here, we have $f_X(x) = \frac{1}{\sqrt{2\pi s}} \exp\left[-\frac{1}{2}(\frac{x^2}{s})\right]$, $f_Z(z) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left[-\frac{1}{2}(\frac{z^2}{t-s})\right]$. Then,

$$P[B_s \leqslant x, B_t \leqslant y] = \int_{-\infty}^{y-x} \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi s}} \exp\left[-\frac{1}{2}(\frac{x^2}{s})\right] \frac{1}{\sqrt{2\pi(t-s)}} \exp\left[-\frac{1}{2}(\frac{z^2}{t-s})\right] \mathrm{d}x \mathrm{d}z$$

(b)

$$\begin{aligned}
f_{B_s,B_t}(x,y) &= \frac{\partial^2 (P[B_s \leqslant x, B_t \leqslant y])}{\partial y \partial x} \\
&= \frac{\partial (\int_{-\infty}^{y-x} f_X(x) f_Z(z) \mathrm{d}z)}{\partial y} \\
&= f_X(x) f_Z(y-x) \\
&= \frac{1}{\sqrt{2\pi s}} \exp\left[-\frac{1}{2}(\frac{x^2}{s})\right] \frac{1}{\sqrt{2\pi(t-s)}} \exp\left[-\frac{1}{2}(\frac{(y-x)^2}{t-s})\right] \\
&= \frac{1}{2\pi\sqrt{s(t-s)}} \exp\left[-\frac{1}{2}(\frac{x^2}{s} + \frac{(y-x)^2}{t-s})\right]
\end{aligned}$$

Problem 4

$$\begin{aligned}
E[B_s | B_t] &= \int_{-\infty}^{\infty} x \frac{f_X(x) f_Z(y-x)}{f_Y(y)} \mathrm{d}x \\
&= \int_{-\infty}^{\infty} x \frac{\frac{1}{\sqrt{2\pi s}} \exp\left[-\frac{1}{2}(\frac{x^2}{s})\right] \frac{1}{\sqrt{2\pi(t-s)}} \exp\left[-\frac{1}{2}(\frac{(y-x)^2}{t-s})\right]}{\frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{1}{2}(\frac{y^2}{t})\right]} \mathrm{d}x \\
&= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi(s - \frac{s^2}{t})}} \exp\left\{-\frac{1}{2}\left[\frac{(x - \frac{sy}{t})^2}{s - \frac{s^2}{t}}\right]\right\} \mathrm{d}x \\
&= \frac{sy}{t}
\end{aligned}$$

Problem 5

(a) We use *Python* to simulate a path of a standard Brownian motion and its relevant geometric Brownian motion. The code is in Appendix.

The graphs of the path versus time are as below.

(b) We use the result from Section 3.4.3 in the textbook.

$$\frac{1}{T_2 - T_1} \sum_{i=1}^{n} \left(\log \left(\frac{S_{t_i}}{S_{t_{i-1}}} \right) \right) \approx \sigma^2$$

The result computed from our simulation is 4.937322215118538, and it is quite close to the real $\sigma = 5$.

Problem 6

(a) Sine τ_B is the first hitting time of S to a level B, i.e., $\tau_B = \min\{t \geqslant 0; S_t = B\}$. So when $\tau_B > T$, we will have $S_T < B$.

$$\begin{aligned} S_T &= S_0 \exp(\sigma B_T) < B \\ \Leftrightarrow B_T &< \frac{1}{\sigma} \ln\left(\frac{B}{S_0}\right) \end{aligned}$$

Here, $M_T = \max_{s \leqslant T} B_s$, then $B_T < \frac{1}{\sigma} \ln\left(\frac{B}{S_0}\right) \Leftrightarrow M_T < \frac{1}{\sigma} \ln\left(\frac{B}{S_0}\right)$. Therefore,

$$E[(S_T - K)_+ \mathbb{1}_{\{\tau_B > T\}}] = E[(S_0 \exp(\sigma B_T) - K)_+ \mathbb{1}_{\{M_T < \frac{1}{\sigma} \ln\left(\frac{B}{S_0}\right)\}}]$$

(b) Since $M_t = \max_{s \leqslant t} B_t$, it's obviously $B_t \leqslant M_t$. Assume $B_t = w, M_t = m$, then $w \leqslant m$. From theorem 3.3.7, we have

$$f_{M(t),B(t)}(m, w) = \frac{2(2m - w)}{t\sqrt{2\pi t}} \exp\left(-\frac{(2m - w)^2}{2t}\right), \qquad w \leqslant m, m > 0$$

And

$$\mathbb{1}_{\{M_T < \frac{1}{\sigma} \ln\left(\frac{B}{S_0}\right)\}} = \mathbb{1}_{\{m < \frac{1}{\sigma} \ln\left(\frac{B}{S_0}\right)\}} = \begin{cases} 1 & \text{if } m < \frac{1}{\sigma} \ln\left(\frac{B}{S_0}\right) \\ 0 & \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned} E[(S_T - K)_+ \mathbb{1}_{\{\tau_B > T\}}] &= E[(S_0 \exp(\sigma B_T) - K)_+ \mathbb{1}_{\{M_T < \frac{1}{\sigma} \ln\left(\frac{B}{S_0}\right)\}}] \\ &= \int_0^{\frac{1}{\sigma} \ln\left(\frac{B}{S_0}\right)} \int_{-\infty}^{m} [S_0 \exp(\sigma w) - K]_+ \frac{2(2m - w)}{t\sqrt{2\pi t}} \exp\left(-\frac{(2m - w)^2}{2t}\right) dw dm \end{aligned}$$

(c)

$$\begin{aligned} &(S_T - K)_+ > 0 \\ \Leftrightarrow\ & S_T > K \\ \Leftrightarrow\ & S_0 \exp(\sigma B_T) > K \\ \Leftrightarrow\ & B_T > \frac{1}{\sigma} \ln\left(\frac{K}{S_0}\right) \\ \Leftrightarrow\ & w > \frac{1}{\sigma} \ln\left(\frac{K}{S_0}\right) \end{aligned}$$

Then we can simplify the integral as below:

$$E[(S_T-K)_+\mathbb{1}_{\{\tau_B>T\}}]=\int_{\frac{1}{\sigma}\ln(\frac{K}{S_0})}^{\frac{1}{\sigma}\ln(\frac{B}{S_0})}\int_{\frac{1}{\sigma}\ln(\frac{K}{S_0})}^{m}[S_0\exp(\sigma w)-K]\frac{2(2m-w)}{t\sqrt{2\pi t}}\exp\left(-\frac{(2m-w)^2}{2t}\right)\mathrm{d}w\mathrm{d}m$$

(d) Change the order of integration, we will have

$$E[(S_T-K)_+\mathbb{1}_{\{\tau_B>T\}}]=\int_{\frac{1}{\sigma}\ln(\frac{K}{S_0})}^{\frac{1}{\sigma}\ln(\frac{B}{S_0})}[S_0\exp(\sigma w)-K]\int_{w}^{\frac{1}{\sigma}\ln(\frac{B}{S_0})}\frac{2(2m-w)}{t\sqrt{2\pi t}}\exp\left(-\frac{(2m-w)^2}{2t}\right)\mathrm{d}m\mathrm{d}w$$

By the help of *Mathematica*, we can get the integration as follow.

$$\frac{1}{2}\left\{K\left[-2\Phi\left(\frac{\ln(\frac{B}{S_0})}{\sqrt{2t}\sigma}\right)+\Phi\left(\frac{\ln(\frac{K}{S_0})}{\sqrt{2t}\sigma}\right)-\Phi\left(\frac{-2\ln(\frac{B}{S_0})+\ln(\frac{K}{S_0})}{\sqrt{2t}\sigma}\right)\right]\right.$$
$$+e^{\sigma^2t/2}S_0\left[-\Phi\left(\frac{\sigma t-\frac{\ln(B/S_0)}{c}}{\sqrt{2t}}\right)+e^{2\ln(B/S_0)}\left(\Phi\left(\frac{\sigma t+\frac{\ln(B/S_0)}{c}}{\sqrt{2t}}\right)-\Phi\left(\frac{\sigma^2t+2\ln(\frac{B}{S_0})-\ln(\frac{K}{S_0})}{\sqrt{2t}\sigma}\right)\right)\right]$$
$$\left.+\Phi\left(\frac{\sigma t-\frac{\ln(B/S_0)}{c}}{\sqrt{2t}}\right)\right\}$$

Appendix

In [1]:
```
import numpy as np
import math
import random
from bokeh.plotting import figure
from bokeh.io import show, output_notebook
output_notebook(hide_banner=True)
```

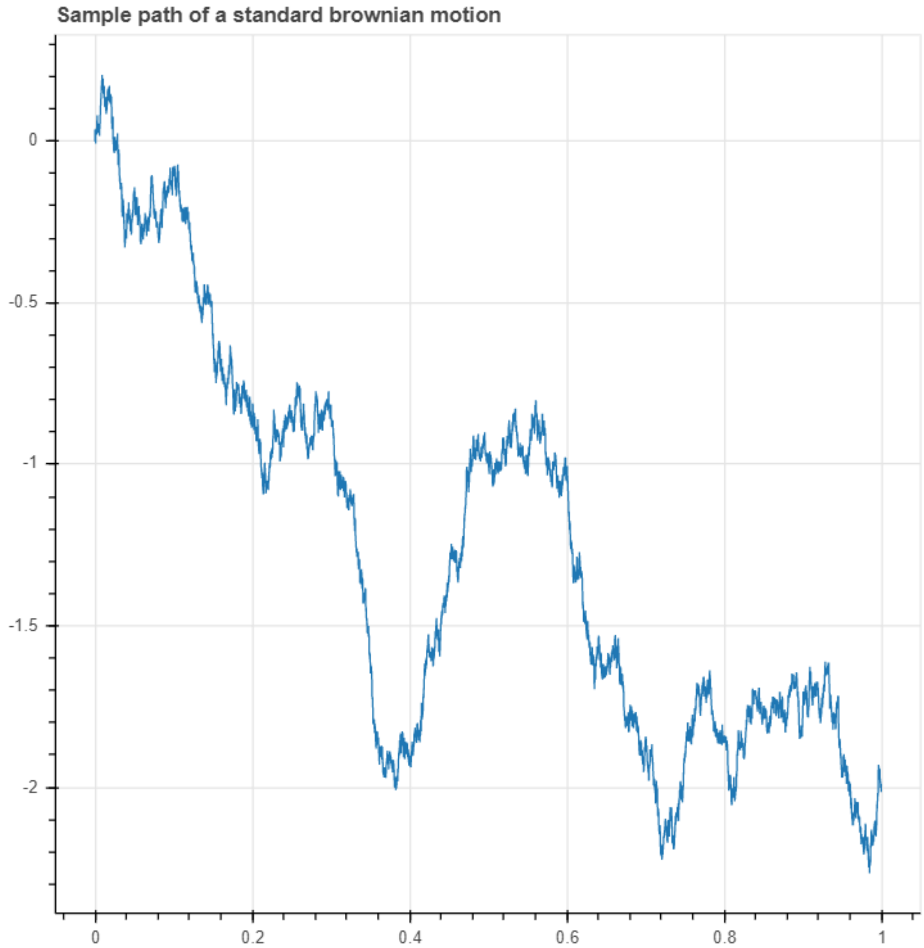
In [2]:
```
def std_brownian(n,T):
    partition=float(T)/n
    B=np.zeros(n + 1, dtype=np.float64)
    Z=np.random.randn(n + 1)

    B[0]=0
    for i in range(1,n + 1):
        B[i]=B[i-1]+math.sqrt(partition)*Z[i]

    return B
```

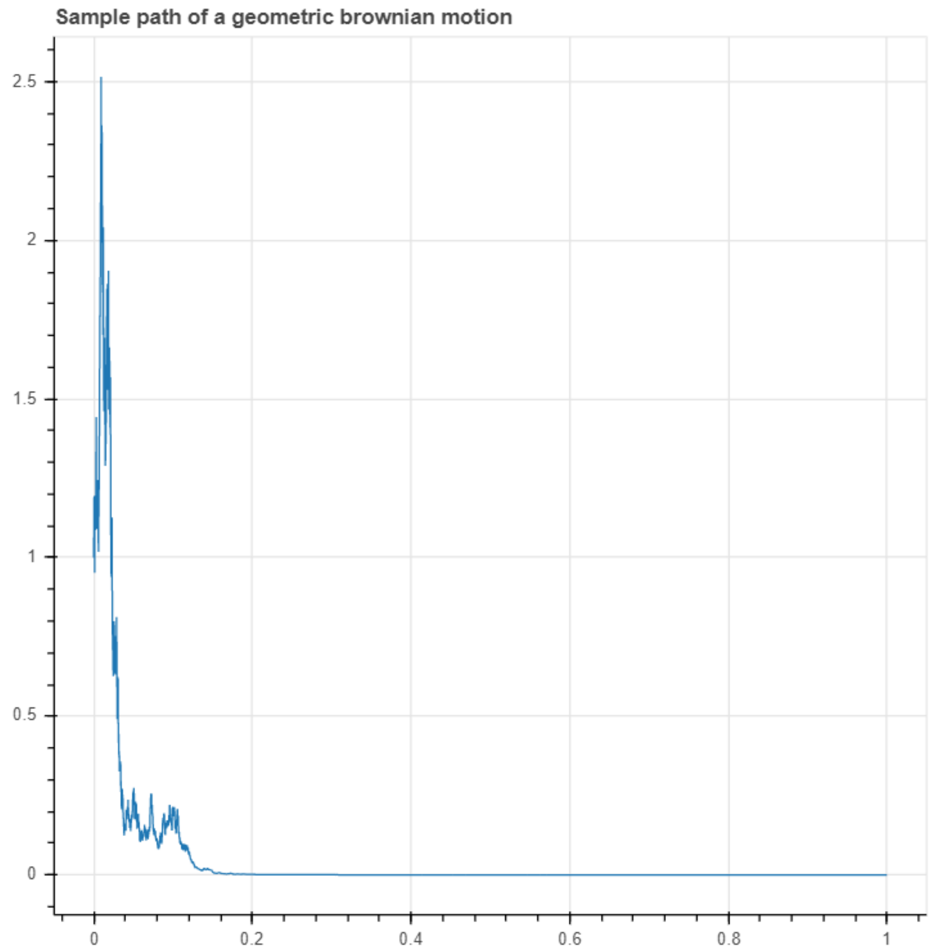
In [3]:
```
random.seed(20191021)
n = 10000
T = 1
t = [x / n for x in range(0, n + 1)]
B = std_brownian(n,T)
p = figure(title='Sample path of a standard brownian motion')
r = p.line(t, B)
result = show(p)
```

In [4]:
```
S = np.zeros(n + 1, dtype='float')
S[0] = 1

sigma = 5
alpha = 1

for i in range(n + 1):
    S[i] = S[0] * math.exp(sigma * B[i] +
    (alpha - (1 / 2) * (sigma ** 2)) * t[i])

G_p = figure(title='Sample path of a geometric brownian motion')
G_r = G_p.line(t, S)
G_result = show(G_p)
```

In [5]:
```
sigma_square = 0
for i in range(1, n + 1):
    sigma_square += (math.log(S[i] / S[i-1])) ** 2
sigma_simulation = math.sqrt(sigma_square)
print(sigma_simulation)
```

```
4.937322215118538
```


Sample path of a standard brownian motion
0
-0.5
-1
-1.5
-2
0
0.2
0.4
0.6
0.8
1


Sample path of a geometric brownian motion
2.5
2
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1

In[5]:=

```
Integrate[(S * E^(c * w) - K) * (2 * (2 * m - w) / (t * Sqrt[2 * Pi * t])
  ) * E^(-1 * (2 * m - w)^2 / (2 * t)), {m, w, (1 / c) * log[10, B / S]}]
```

Out[5]=

$$-\frac{\left(e^{-\frac{w^2}{2\,t}}-e^{-\frac{\left(w-\frac{2\,\log\left[10,\,\frac{B}{S}\right]}{c}\right)^2}{2\,t}}\right)\left(K-e^{c\,w}\,S\right)}{\sqrt{2\,\pi}\,\sqrt{t}}$$

In[9]:=

$$\text{Integrate}\left[-\frac{\left(e^{-\frac{w^2}{2\,t}}-e^{-\frac{\left(w-\frac{2\,\log\left[10,\,\frac{B}{S}\right]}{c}\right)^2}{2\,t}}\right)\left(K-e^{c\,w}\,S\right)}{\sqrt{2\,\pi}\,\sqrt{t}},\ \{w,\ (1/c)*\log[10,\ K/S],\ (1/c)*\log[10,\ B/S]\}\right]$$

Out[9]=

$$\frac{1}{2}\left(K\left(-2\,\text{Erf}\left[\frac{\log\left[10,\,\frac{B}{S}\right]}{\sqrt{2}\,c\,\sqrt{t}}\right]+\text{Erf}\left[\frac{\log\left[10,\,\frac{K}{S}\right]}{\sqrt{2}\,c\,\sqrt{t}}\right]-\text{Erf}\left[\frac{-2\,\log\left[10,\,\frac{B}{S}\right]+\log\left[10,\,\frac{K}{S}\right]}{\sqrt{2}\,c\,\sqrt{t}}\right]\right)+e^{\frac{c^2\,t}{2}}\,S\left(-\text{Erf}\left[\frac{c\,t-\frac{\log\left[10,\,\frac{B}{S}\right]}{c}}{\sqrt{2}\,\sqrt{t}}\right]+e^{2\,\log\left[10,\,\frac{B}{S}\right]}\left(\text{Erf}\left[\frac{c\,t+\frac{\log\left[10,\,\frac{B}{S}\right]}{c}}{\sqrt{2}\,\sqrt{t}}\right]-\text{Erf}\left[\frac{c^2\,t+2\,\log\left[10,\,\frac{B}{S}\right]-\log\left[10,\,\frac{K}{S}\right]}{\sqrt{2}\,c\,\sqrt{t}}\right]\right)+\text{Erf}\left[\frac{c\,t-\frac{\log\left[10,\,\frac{K}{S}\right]}{c}}{\sqrt{2}\,\sqrt{t}}\right]\right)\right)$$

The integration result