Determining the Torque Ratings for DC Motors

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1 Introduction

In this report we aim to derive the formula for determining the ratings of the motor needed to be installed on the AGV. Special focus is placed on the additional requirements imposed on the motors when AGV is making a turn, in comparison to when the AGV is moving in a straight line. In order to do this, we will need to derive the dynamic model of the AGV while it is taking a turn. A dynamic model is the relationship of the forces and torques on a mechanical system with its speeds, accelerations and its inertial parameters such as masses and inertias. A number of different methods are there to determine this dynamic model. We will use Kane's method for the job.

2 Turning Dynamics

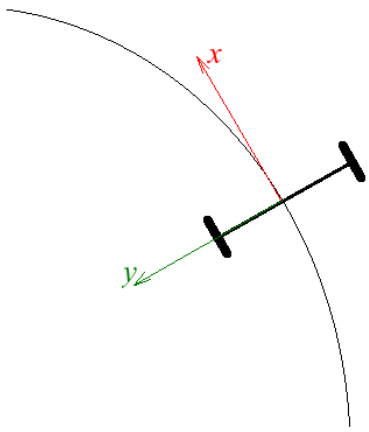

Figure 1: Taking a turn

Here is a list of symbols used in our derivation:

$\dot{x}$ Forward speed of the AGV

$\dot{\psi}$ Rotation speed of AGV

R Radius of the wheel

L Distance between the wheels

$\mathbf{MS}_b = \begin{bmatrix}\mathbf{MX}_b & \mathbf{MY}_b & \mathbf{MZ}_b\end{bmatrix}^T$ is the center of mass of the AGV body

$J_b = \begin{bmatrix}\mathbf{XX}_b & \mathbf{XY}_b & \mathbf{XZ}_b \\ \mathbf{XY}_b & \mathbf{YY}_b & \mathbf{YZ}_b \\ \mathbf{XZ}_b & \mathbf{YZ}_b & \mathbf{ZZ}_b\end{bmatrix}$ is the inertia matrix of the body

$J_w = \begin{bmatrix}\mathbf{XX}_w & 0 & 0 \\ 0 & \mathbf{YY}_w & 0 \\ 0 & 0 & \mathbf{ZZ}_w\end{bmatrix}$ is the inertia matrix of the wheel

$\bar{F} = \begin{bmatrix}F_x & F_y & F_z\end{bmatrix}^T$ is the tugging force (reaction) applied on the AGV

$E = \begin{bmatrix}E_x & E_y & E_z\end{bmatrix}^T$ are the coordinates of the point at which tugging force is being applied

τ_R, τ_L Torques being applied on right and left wheels

θ_R, θ_L Rotation of right and left wheels

2.1 Defining Generalized Velocities

It is easier to derive the dynamic model of the system in terms of the generalized velocities: $\{\dot{q}\} = \{\ \dot{x},\ \dot{\psi}\}$. These two velocities can take arbitrary values all of whom will be kinematically admissible. In other words, they represent our two degrees of freedom. We will now derive two dynamic equations in terms of these generalized velocities.

Since we are dealing with quasi-velocities, we will use Kane's method to derive the dynamic equations.

2.2 Introduction to Kane's formulation

The Kane's formulation is as follows:

$$\sum_k \left[m_k \bar{a}_{Gk} \cdot (\bar{v}_{Gk})_j + \left(\frac{d\bar{H}_{Gk}}{dt}\right) \cdot (\bar{\omega}_k)_j \right] = \sum_n \bar{F}_n \cdot (\bar{v}_n)_j + \sum_n \bar{M}_m \cdot (\bar{\omega}_m)_j \quad j = 1...K \tag{1}$$

where

j is the unique number identifying each generalized co-ordinate in the system

k is the unique number identifying each rigid body in the system

n is the unique number identifying each external force acting on the system

m is the unique number identifying each external torque acting on the system

m_k is the mass of the kth body

$\bar{a}_{Gk}$ is the acceleration of the center of mass of kth body

$\bar{v}_{Gk}$ is the velocity of the center of mass of the kth body

$\bar{H}_{Gk}$ is the angular momentum of body k about its center of mass

$\bar{\omega}_k$ is the angular velocity of the body k

F_n is the nth external force

M_m is the mth external moment

$\bar{v}_n$ is the velocity of the point at which external Force F_n is acting

$\bar{\omega}_m$ is the angular velocity of the body on which torque is acting relative to the actuator applying the torque

$()_j$ $= \frac{\partial()}{\partial \dot{q}_j}$ the partial derivative of the quantity in brackets () with respect to the generalized velocity $\dot{q}_j$

2.3 Kane's Left-Hand Side

The left hand side of the Kane's equation containes a sum whose range is equal to the number of bodies in the system. We have three bodies: Left-wheel (L), right wheel (R) and the body of robot (B). Each term in the sum consists of the acceleration ($\bar{a}_{Gk}$), velocity ($\bar{v}_{Gk}$), angular momentum ($\bar{H}_{Gk}$) of the center of mass and the body's angular velocity ($\bar{\omega}_k$). And then some partieal derivatives wrt to the generalized coordinates ($(\bar{\omega}_k)_j = \frac{\partial \bar{\omega}_k}{\partial \dot{q}_j}$ and $(\bar{v}_{Gk})_j = \frac{\partial \bar{v}_{Gk}}{\partial \dot{q}_j}$). We will have two equations corresponding to each generalized coordinate $\{\dot{q}_j\} = \{\dot{x}, \dot{\psi}\}$.

2.3.1 Left Wheel

This evaluation takes place in the $x_L y_L z_L$ frame fixed to the left wheel such that it is parallel to frame $x_0 y_0 z_0$ when $\theta_L = 0$. So $\bar{i}_0 = cos\theta_L \bar{i}_L + sin\theta_L \bar{k}_L$, $\bar{j}_0 = \bar{j}_L$ and $\bar{k}_0 = -sin\theta_L \bar{i}_L + cos\theta_L \bar{k}_L$. Angular velocity:

$$\begin{aligned}
\bar{\omega}_L &= \dot{\psi}\bar{k}_0 + \dot{\theta}_L \bar{j}_0 \\
&= \dot{\psi}\bar{k}_0 + \left(\frac{1}{R}\dot{x} - \frac{L}{2R}\dot{\psi}\right)\bar{j}_0 \\
&= -\dot{\psi} sin\theta_L \bar{i}_L + \left(\frac{1}{R}\dot{x} - \frac{L}{2R}\dot{\psi}\right)\bar{j}_L + \dot{\psi} cos\theta_L \bar{k}_L
\end{aligned} \tag{2}$$

The terms that follow are also similarly to be expressed in frame $x_L y_L z_L$ but that step is skipped for brevity. Velocity:

$$\begin{aligned}
\bar{v}_{GL} &= \bar{v}_0 + \bar{\omega}_0 \times \bar{r}_{L/O} \\
&= \dot{x}\bar{i}_0 + \dot{\psi}\bar{k}_0 \times \frac{L}{2}\bar{j}_0 \\
&= \left(\dot{x} - \frac{L}{2}\dot{\psi}\right)\bar{i}_0
\end{aligned} \tag{3}$$

Linear acceleration:

$$\begin{aligned}\bar{a}_{GL} &= \bar{a}_0 + \bar{\alpha}_0 \times \bar{r}_{L/O} + \bar{\omega}_0 \times (\omega_0 \times \bar{r}_{L/O}) \\ &= \ddot{x}\bar{i}_0 + \dot{x}\left(\dot{\psi}\bar{k}_0 \times \bar{i}_0\right) + \ddot{\psi}\bar{k}_0 \times \frac{L}{2}\bar{j}_0 + \dot{\psi}\bar{k}_0 \times \left(\dot{\psi}\bar{k}_0 \times \frac{L}{2}\bar{j}_0\right) \\ &= \left(\ddot{x} - \frac{L}{2}\ddot{\psi}\right)\bar{i}_0 + \left(\dot{x}\dot{\psi} - \frac{L}{2}\dot{\psi}^2\right)\bar{j}_0 \end{aligned} \tag{4}$$

Angular momentum and its derivative:

$$\begin{aligned}\bar{H}_{GL} &= I_w\bar{\omega}_L \\ \frac{d\bar{H}_{GL}}{dt} &= \frac{\partial\bar{H}_{GL}}{\partial t} + \bar{\omega}_L \times \bar{H}_{GL}\end{aligned} \tag{5}$$

where $I_w = \begin{bmatrix} \mathbf{ZZ}_w & 0 & 0 \\ 0 & \mathbf{YY}_w & 0 \\ 0 & 0 & \mathbf{ZZ}_w \end{bmatrix}$. Due to symmetry the off-diogonal terms in the inertia matrix vanish, and the inertia about x_L-axis and z_L-axis are both equal (signified by $\mathbf{ZZ}_w$).

2.3.2 Right Wheel

This evaluation takes place in the $x_Ry_Rz_R$ frame fixed to the right wheel such that it is parallel to frame $x_0y_0z_0$ when $\theta_R = 0$. So $\bar{i}_0 = cos\theta_R\bar{i}_R + sin\theta_R\bar{k}_R$, $\bar{j}_0 = \bar{j}_R$ and $\bar{k}_0 = -sin\theta_R\bar{i}_R + cos\theta_R\bar{k}_R$. Angular velocity:

$$\begin{aligned}\bar{\omega}_R &= \dot{\psi}\bar{k}_0 + \dot{\theta}_R\bar{j}_0 \\ &= \dot{\psi}\bar{k}_0 + \left(\frac{1}{R}\dot{x} + \frac{L}{2R}\dot{\psi}\right)\bar{j}_0 \\ &= -\dot{\psi}sin\theta_L\bar{i}_R + \left(\frac{1}{R}\dot{x} + \frac{L}{2R}\dot{\psi}\right)\bar{j}_R + \dot{\psi}cos\theta_L\bar{k}_R\end{aligned} \tag{6}$$

The terms that follow are also similarly to be expressed in frame $x_Ry_Rz_R$ but that step is skipped for brevity. Velocity:

$$\begin{aligned}\bar{v}_{GR} &= \bar{v}_0 + \bar{\omega}_0 \times \bar{r}_{R/O} \\ &= \dot{x}\bar{i}_0 + \dot{\psi}\bar{k}_0 \times \left(-\frac{L}{2}\bar{j}_0\right) \\ &= \left(\dot{x} + \frac{L}{2}\dot{\psi}\right)\bar{i}_0\end{aligned} \tag{7}$$

Linear acceleration:

$$\begin{aligned}\bar{a}_{GR} &= \bar{a}_0 + \bar{\alpha}_0 \times \bar{r}_{R/O} + \bar{\omega}_0 \times (\omega_0 \times \bar{r}_{R/O}) \\ &= \ddot{x}\bar{i}_0 + \dot{x}\left(\dot{\psi}\bar{k}_0 \times \bar{i}_0\right) + \ddot{\psi}\bar{k}_0 \times \left(-\frac{L}{2}\bar{j}_0\right) + \dot{\psi}\bar{k}_0 \times \left(\dot{\psi}\bar{k}_0 \times \left(-\frac{L}{2}\bar{j}_0\right)\right) \\ &= \left(\ddot{x} + \frac{L}{2}\ddot{\psi}\right)\bar{i}_0 + \left(\dot{x}\dot{\psi} + \frac{L}{2}\dot{\psi}^2\right)\bar{j}_0\end{aligned} \tag{8}$$

Angular momentum and its derivative:

$$\begin{aligned}\bar{H}_{GR} &= I_w \bar{\omega}_R \\ \frac{d\bar{H}_{GR}}{dt} &= \frac{\partial \bar{H}_{GR}}{\partial t} + \bar{\omega}_0 \times \bar{H}_{GR}\end{aligned} \tag{9}$$

2.3.3 Body

We will evaluate the quantities in frame $x_0 y_0 z_0$.
Angular velocity:

$$\bar{\omega}_B = \dot{\psi}\bar{k}_0 \tag{10}$$

Velocity:

$$\begin{aligned}\bar{v}_{GB} &= \bar{v}_0 + \bar{\omega}_B \times \bar{r}_{B/O} \\ &= \dot{x}\bar{i}_0 + \bar{\omega}_B \times \frac{1}{m_B}\mathbf{MS}_B\end{aligned} \tag{11}$$

isa Linear acceleration:

$$\bar{a}_{GB} = \bar{a}_0 + \bar{\alpha}_B \times \bar{r}_{B/O} + \bar{\omega}_B \times \left(\bar{\omega}_B \times \bar{r}_{B/O}\right) \tag{12}$$

where

$$\begin{aligned}\bar{a}_0 &= \frac{d\bar{v}_0}{dt} = \frac{d\left(\dot{x}\bar{i}_0\right)}{dt} = \ddot{x}\bar{i}_0 + \dot{x}\left(\dot{\psi}\bar{k}_0 \times \bar{i}_0\right) \\ \bar{\alpha}_B &= \frac{d\bar{\omega}_B}{dt} = \frac{d\left(\dot{\psi}\bar{k}_0\right)}{dt} = \ddot{\psi}\bar{k}_0\end{aligned} \tag{13}$$

Angular momentum and its derivative:

$$\begin{aligned}\bar{H}_{GB} &= I_B \bar{\omega}_B \\ \frac{d\bar{H}_{GB}}{dt} &= \frac{\partial \bar{H}_{GB}}{\partial t} + \bar{\omega}_B \times \bar{H}_{GB}\end{aligned} \tag{14}$$

2.3.4 Left-Hand Side Final Evaluation

The Kanes' formulation (Eq. 1) are in fact two equations each corresponding to a different generalized velocity $\dot{q}_j = \dot{x}, \dot{\psi}$. Each of these three equation is a summation of the given expression evaluated for each body $k = L, R, B$ i.e. Left-wheel, right-wheel and body. When the expressions that we evaluated in eqs. 2-14 are substituted in Kanes' equations (Eq. 1) and the results evaluted (using MATLAB code listed in the Appendix section A.1), we get the following expression:

$$\mathbf{A}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}}$$

where

$$\begin{aligned}\mathbf{A} &= \begin{bmatrix} m_B + 2m_w + \frac{2\mathbf{YY}_w}{R^2} & -\mathbf{MY}_b \\ -\mathbf{MY}_b & \frac{m_w L^2}{2} + \frac{\mathbf{YY}_w L^2}{2R^2} + 2\mathbf{ZZ}_w + \frac{\mathbf{MX}_b^2}{m_B} + \frac{\mathbf{MY}_b^2}{m_b} + \mathbf{ZZ}_b \end{bmatrix} \\ \mathbf{C} &= \begin{bmatrix} 0 & -\mathbf{MX}_b\dot{\psi} \\ \frac{1}{2}\mathbf{MX}_b\dot{\psi} & \frac{1}{2}\mathbf{MX}_b\dot{x} \end{bmatrix}\end{aligned} \tag{15}$$

2.4 Kane's Right Hand Side

The right hand side of the Kane's forumulation 1 is the sum of some dot product terms. Each term is either the dot product of:

- force applied on the system $\bar{F}_n$
- the linear velocity $\bar{v}_n$ of the point differentiated partially wrt the the unique gerneralized speed $\dot{q}_j$ corresponding to each equation i.e. $\frac{\partial \bar{v}_n}{\partial \dot{q}_j}$

or the dot product of:

- torque applied on the system $\bar{\tau}_n$
- the angular velocity $\bar{\omega}_n$ of the body differentiated partially wrt the the unique gerneralized speed $\dot{q}_j$ corresponding to each equation i.e. $\frac{\partial \bar{\omega}_n}{\partial \dot{q}_j}$

So, in order to analyse the right-hand side of the equation, we need to list down all the forces and torques applied to the system and the points at which they are being applied. They are as follows:

$\bar{\tau}_L, \bar{\tau}_R$ Torques applied by wheel motors (in the body) on the right wheel and left wheel at points R and L on the respective wheels

$\bar{F}$ The reaction force from the load being tugged by the AVG acting at point E

Let $\bar{F} = \begin{bmatrix} F_x & F_y & F_z \end{bmatrix}^T$ be the components of forces/torques defined in the frame $x_0y_0z_0$ and $\bar{r}_{E/O} = \begin{bmatrix} -E_x & 0 & E_z \end{bmatrix}^T$ are the coordinates of point E in the $x_0y_0z_0$ frame. Also $\bar{F}_g = \begin{bmatrix} 0 & 0 & -m_b g \end{bmatrix}^T$. Also $\bar{r}_{G/O}$ is to be expressed in the $x_0y_0z_0$ which will be $\bar{r}_{G/O} = \frac{1}{m_B} \begin{bmatrix} \mathbf{MX}_B \\ \mathbf{MY}_B \\ \mathbf{MZ}_B \end{bmatrix}$.

We now give closed form expressions of terms contributed on the RHS by these forces and torques:

1. Torque on the wheel at L will contribute the following terms

$$\tau_L \bar{j}_0 \cdot \frac{\partial \bar{\omega}_L}{\partial \dot{q}_j} \tag{16}$$

where $\bar{\omega}_L = \dot{\psi}\bar{k}_0 + \left(\frac{1}{R}\dot{x} - \frac{L}{2R}\dot{\psi}\right)\bar{j}_0$

2. Torque on the wheel at R will contribute the following terms

$$\tau_R \bar{j}_0 \cdot \frac{\partial \bar{\omega}_R}{\partial \dot{q}_j} \tag{17}$$

where $\bar{\omega}_R = \dot{\psi}\bar{k}_0 + \left(\frac{1}{R}\dot{x} + \frac{L}{2R}\dot{\psi}\right)\bar{j}_0$

3. The force $\bar{F}$ contributes the following term:

$$\bar{F} \cdot \frac{\partial \bar{v}_E}{\partial \dot{q}_j} \tag{18}$$

where

$$\begin{aligned} \bar{v}_E &= \bar{v}_0 + \bar{\omega}_B \times \bar{r}_{E/O} \\ &= \dot{x}\bar{i}_0 + \dot{\psi}\bar{k}_0 \times \left(-E_x\bar{i}_0 + E_z\bar{k}_0\right) \\ &= \dot{x}\bar{i}_0 - E_x\dot{\psi}\bar{j}_0 \end{aligned} \tag{19}$$

The overall contribution of the forces and torques on the right-hand side of the Kane's equations is:

$\dot{x}$: $\frac{1}{R}\left(\tau_R + \tau_L\right) + F_x$

$\dot{\psi}$: $\frac{L}{2R}\left(\tau_R - \tau_L\right) - E_x F_y$

2.5 Final Expression for the Turning Dynamics

$$\left(m_B + 2m_w + \frac{2\mathbf{YY}_w}{R^2}\right)\ddot{x} - \mathbf{MY}_b\ddot{\psi} - \mathbf{MX}_b\dot{\psi}^2 = \frac{1}{R}\left(\tau_R + \tau_L\right) + F_x \tag{20}$$

$$-\mathbf{MY}_b\ddot{x} + \left(\frac{m_w L^2}{2} + \frac{\mathbf{YY}_w L^2}{2R^2} + 2\mathbf{ZZ}_w + \frac{\mathbf{MX}_b^2}{m_B} + \frac{\mathbf{MY}_b^2}{m_b} + \mathbf{ZZ}_b\right)\ddot{\psi} + \mathbf{MX}_b\dot{x}\dot{\psi} = \frac{L}{2R}\left(\tau_R - \tau_L\right) - E_x F_y \tag{21}$$

3 Torque Requirements

The dynamic equation 20-21 can be used to find out expressions for τ_R and τ_L. We get:

$$\tau_R = \left(\mathbb{M}_1 - \frac{R}{L}\mathbf{MY}_b\right)\ddot{x} + \left(-\frac{R}{2}\mathbf{MY}_b + \mathbb{M}_2\right)\ddot{\psi} - \frac{R}{2}\mathbf{MX}_b\dot{\psi}^2 + \frac{R}{L}\mathbf{MX}_b\dot{x}\dot{\psi} - \frac{R}{2}F_x + \frac{R}{L}E_x F_y \tag{22}$$

$$\tau_L = \left(\mathbb{M}_1 + \frac{R}{L}\mathbf{MY}_b\right)\ddot{x} + \left(-\frac{R}{2}\mathbf{MY}_b - \mathbb{M}_2\right)\ddot{\psi} - \frac{R}{2}\mathbf{MX}_b\dot{\psi}^2 - \frac{R}{L}\mathbf{MX}_b\dot{x}\dot{\psi} - \frac{R}{2}F_x - \frac{R}{L}E_x F_y \tag{23}$$

where

$$\mathbb{M}_1 = \frac{R}{2}\left(m_B + 2m_w + \frac{2\mathbf{YY}_w}{R^2}\right) \tag{24}$$

$$\mathbb{M}_2 = \frac{R}{L}\left(\frac{m_w L^2}{2} + \frac{\mathbf{YY}_w L^2}{2R^2} + 2\mathbf{ZZ}_w + \frac{\mathbf{MX}_b^2}{m_B} + \frac{\mathbf{MY}_b^2}{m_b} + \mathbf{ZZ}_b\right) \tag{25}$$

If the mass distribution of the AGV body is symmetrical then $\mathbf{MX}_b = \mathbf{MY}_b = 0$ then:

$$\tau_R = \mathbb{M}_1\ddot{x} + \mathbb{M}_2\ddot{\psi} - \frac{R}{2}F_x + \frac{R}{L}E_x F_y \tag{26}$$

$$\tau_L = \mathbb{M}_1\ddot{x} - \mathbb{M}_2\ddot{\psi} - \frac{R}{2}F_x - \frac{R}{L}E_x F_y \tag{27}$$

If the system is moving on a straight path then there is no rotation of the body about its center $\dot{\psi} = \ddot{\psi} = 0$ and the y-component of the tug force is also zero meaning $F_y = 0$. The equations become:

$$\tau_R = \mathbb{M}_1 \ddot{x} - \frac{R}{2} F_x \tag{28}$$

$$\tau_L = \mathbb{M}_1 \ddot{x} - \frac{R}{2} F_x \tag{29}$$

In order to estimate toque limits, let's insert estimates of various parameters that appear in the above equations:

$$\begin{aligned}
R &= 20in = 10 \times 0.0254 = 0.254m \\
\mathbb{M}_1 &= \frac{R}{2} \times 300kg = 38.1kgm \\
\ddot{x}_{max} &= \frac{18m/min}{1s} = 0.3m/s^2 \\
F_x &= 1000kg \times 0.3m/s^2 = 300N \\
L &= 3ft = 0.9m \\
W &= 4ft = 1.2m \\
\mathbf{ZZ}_b &= \frac{1}{12} m_b \left(L^2 + W^2\right) = \frac{1}{12} \times 300 \left(1.2^2 + 0.9^2\right) = 56.25kgm^2 \\
\mathbb{M}_2 &= \frac{R}{L} \mathbf{ZZ}_b = 15.625kgm^2 \\
\ddot{\psi}_{max} &= 15rad/s^2 \\
\tau_{straight} &= 49.53Nm \\
P_{straight} &= \tau_{straight} \frac{\dot{x}_{max}}{R} = 58.5Watts \\
\tau_{turning} &= 234.37Nm \\
P_{turning} &= \tau_{turning} \frac{\dot{x}_{max}}{R} = 276.82Watts
\end{aligned}$$

4 Conclusion

Equations 26-27 represent dynamics while the robot is turning, while equations 28-29 represent dynamics while the robot is moving in a straight line. Upon comparison it is clear that there are two additional terms in the turning motion which will increase the torque requirements of the robot: $\frac{1}{2} E_x F_y$ and $\mathbb{M}_2 \ddot{\psi}$. The first of these two terms is simply the torque exerted by the load being tugged. To analyse the second term however, we will need to look at the definition of $\mathbb{M}_2$ in eq. 25. The most dominant term defining $\mathbb{M}_2$ (eq. 25) is $\mathbf{ZZ}_b$ i.e. the inertia of the AGV body about vertical (or z-) axis. This is because the rest of the terms are either zero ($\mathbf{MX}_b, \mathbf{MY}_b$) or small in comparison ($m_w, \mathbf{YY}_w$). This inertia terms ($\mathbf{ZZ}_b$) is multiplying with rotational accelration of the body about the vertical ($\ddot{\psi}$). Now, as the AGV encounters a turn in its path, this rotational acceleration jumps from zero to a positive value. And the second term comes into play. The control algorithm should try to minimize this acceleration in order to minimize this term.

A MATLAB code for Evaluation of Kane's Equaiton

A.1 Kane's LHS

This is the code listing for the evaluation of Kane's equation LHS (Eq. 1) for our robot:

```
%% Definitions:

% define frames:
% x0y0z0: Origin wheels mid, z0 always up, x0 always along heading
% x1y1z1: Origin on wheels mid, x1 along wheels connect (L->R), y1 along
% the base (at angle -q_imu from the x0 axis)

clear all
syms x psii dpsi dx ddpsi ddx L g mw mb real
syms XXw XYw XZw YYw YZw ZZw real
syms XXb XYb XZb YYb YZb ZZb real
syms tau_R tau_L R real
syms MXb MYb MZb real

syms t X(t) PSI(t) dX dPSI ddX ddPSI real
dX=diff(X,t);dPSI=diff(PSI,t);
ddX=diff(dX,t);ddPSI=diff(dPSI,t);
q = [x psii]'; dq = [dx dpsi]'; ddq = [ddx ddpsi]';

mydiff = @(H) formula(subs(diff(symfun(subs(H,...
    [x,psii,dx,dpsi,ddx,ddpsi],...
    [X, PSI,dX,dPSI,ddX,ddPSI]),t),t),...
    [X, PSI,dX,dPSI,ddX,ddPSI],...
    [x,psii,dx,dpsi,ddx,ddpsi]));

%% Left Wheel


thetaL = x/R - psii*L/(2*R);
dthetaL = dx/R - dpsi*L/(2*R);
iL = [1 0 0]'; jL = [0 1 0]'; kL = [0 0 1]';
i0 = [cos(thetaL) 0 sin(thetaL)]'; j0 = [0 1 0]'; k0 = [-sin(thetaL) 0 cos(thetaL)]';
w0 = dpsi*k0;
v0 = dx*i0;
alpha0 = ddpsi*k0;
a0 = ddx*i0 + dx*(cross(dpsi*k0,i0));
r0L = (L/2)*j0;
Iw=[ZZw 0 0;0 YYw 0;0 0 ZZw];
wL = w0 + dthetaL*j0;
vGL = v0 + cross(w0, r0L);
aGL = a0 + cross(alpha0, r0L) + cross(w0, cross(w0, r0L));
HGL = Iw*wL;
p = mydiff(HGL);
dHGL = p + cross(wL,HGL);

% %% Right Wheel


thetaR = x/R + psii*L/(2*R);
dthetaR = dx/R + dpsi*L/(2*R);
iR = [1 0 0]'; jR = [0 1 0]'; kR = [0 0 1]';
i0 = [cos(thetaR) 0 sin(thetaR)]'; j0 = [0 1 0]'; k0 = [-sin(thetaR) 0 cos(thetaR)]';
w0 = dpsi*k0;
v0 = dx*i0;
alpha0 = ddpsi*k0;
a0 = ddx*i0 + dx*(cross(dpsi*k0,i0));
r0R = -(L/2)*j0;
wR = w0 + dthetaR*j0;
vGR = v0 + cross(w0, r0R);
aGR = a0 + cross(alpha0, r0R) + cross(w0, cross(w0, r0R));
HGR = Iw*wR;
dHGR = mydiff(HGR)+cross(wR,HGR);

%% Body

i0 = [1 0 0]'; j0 = [0 1 0]'; k0 = [0 0 1]';
w0 = dpsi*k0;
v0 = dx*i0;
a0 = ddx*i0 + dx*(cross(dpsi*k0,i0));
IB=[XXb XYb XZb;XYb YYb YZb; XZb YZb ZZb];
wB = w0;
alphaB = ddpsi*k0;
r0B = [MXb MYb MZb]'/mb;
vGB = v0 + cross(wB,r0B);
aGB = a0 + cross(alphaB, r0B) + cross(wB, cross(wB, r0B));
HGB = IB*wB;
dHGB = mydiff(HGB)+cross(wB,HGB);

%% Kanes LHS

KL = sym(zeros(3,1)); KR = sym(zeros(3,1)); KB = sym(zeros(3,1));
for i=1:2
    KL(i)=mw*aGL'*diff(vGL,dq(i))+dHGL'*diff(wL,dq(i));
```

```
    KR(i)=mw*aGR'*diff(vGR,dq(i))+dHGR'*diff(wR,dq(i));
    KB(i)=mb*aGB'*diff(vGB,dq(i))+dHGB'*diff(wB,dq(i));
end
Kw = KL + KR;
K = Kw + KB;

%% Equations

AA = sym(zeros(2,2)); CC = sym(zeros(2,2));
for i=1:2
    for j=1:2
        AA(i,j)=getcoeff(K(i),ddq(j),1);
        % This divides the coefficient of (dqj)(dqk) equally in all colum
        % j and k
        CC(i,j)=getcoeff(K(i), dq(j),2)*dq(j);
        ccc = getcoeff(K(i),dq(j),1);
        CC(i,j) = CC(i,j)+ccc;
        for k=1:2
            CC(i,j) = CC(i,j) - 0.5*(getcoeff(ccc,dq(k),1))*dq(k);
        end
    end
end

AA=simplify(AA);
CC=simplify(CC);
```

A.2 Kane's RHS

This is the code listing for the evaluation of Kane's equation RHS (Eq. 1) for our robot:

```
syms mB g dx dpsi dphi MXb MYb MZb real
syms tau_l tau_r R L qimu real
syms Fx Fy Fz Ex Ey Ez real

dq = [dx dpsi]';
a = sym(zeros(2,1)); b = sym(zeros(2,1));
c = sym(zeros(2,1));


% Torques at point R

wR = dpsi*k0 + (dx/R + (L*dpsi)/(2*R))*j0;

Tau_r = [0 tau_r 0]';
for i=1:2
    a(i) = Tau_r'*diff(wR,dq(i));
end

% Torques at point L
wL = dpsi*k0 + (dx/R - (L*dpsi)/(2*R))*j0;
Tau_l = [0 tau_l 0]';

for i=1:2
    b(i) = Tau_l'*diff(wL,dq(i));
end

% Force F
F = [Fx Fy Fz]';
rOE = [-Ex 0 Ez];
vE = dx*i0 - Ex*dpsi*j0;

for i=1:2
    c(i) = F'*diff(vE,dq(i));
end
```