Lecture 8: Convex Optimization

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Outline

- A convex optimization problem (or just a convex problem) is a problem consisting of minimizing a convex function over a convex set:

$$\begin{aligned} \min \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \in C \end{aligned} \tag{1}$$

  - C - convex set.
  - f - convex function over C.

- A functional form of a convex problem can be written as

$$\begin{aligned} \min \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & g_i(\mathbf{x}) \leq 0, \quad i = 1, 2, \cdots, m \\ & h_j(\mathbf{x}) = 0, \quad j = 1, 2, \cdots, p \end{aligned}$$

  $f, g_1, \cdots, g_m : \mathbb{R}^n \to \mathbb{R}$ are convex functions and $h_1, h_2, \cdots, h_p : \mathbb{R}^m \to \mathbb{R}$ are affine functions.

- Note that the functional form does fit into the general formulation (1)

“Convex Problems are Easy” - Local Minima are Global Minima

Theorem

Let $f : C \to \mathbb{R}$ be a convex function defined on the convex set $C \subseteq \mathbb{R}^n$. Let $\mathbf{x}^* \in C$ be a local minimum of f over C. Then $\mathbf{x}^*$ is a global minimum of f over C.

Theorem

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Proof.

- $\mathbf{x}^*$ is a local minimum of f over C $\Rightarrow$ $\exists r > 0$ such that $f(\mathbf{x}) \geq f(\mathbf{x}^*)$ for any $\mathbf{x} \in C \cap B[\mathbf{x}^*, r]$.
- Let $\mathbf{x}^* \neq \mathbf{y} \in C$. We will show that $f(\mathbf{y}) \geq f(\mathbf{x}^*)$.
- Let $\lambda \in (0, 1)$ be such that $\mathbf{x}^* + \lambda(\mathbf{y} - \mathbf{x}^*) \in B[\mathbf{x}^*, r]$.
- Since $\mathbf{x}^* + \lambda(\mathbf{y} - \mathbf{x}^*) \in B[\mathbf{x}^*, r]$, it follows that $f(\mathbf{x}^*) \leq f(\mathbf{x}^* + \lambda(\mathbf{y} - \mathbf{x}^*))$ and hence by Jensen's inequality: $f(\mathbf{x}^*) \leq f(\mathbf{x}^* + \lambda(\mathbf{y} - \mathbf{x}^*)) \leq (1 - \lambda)f(\mathbf{x}^*) + \lambda f(\mathbf{y})$.
- Thus, the desired inequality $f(\mathbf{x}^*) \leq f(\mathbf{y})$ follows.

$\square$

A small variation of the proof of the last theorem yields the following.

Theorem

Let $f : C \to \mathbb{R}$ be a strictly convex function defined on the convex set C. Let $\mathbf{x}^* \in C$ be a local minimum of f over C. Then $\mathbf{x}^*$ is a strict global minimum of f over C.

Another important and easily deduced property of convex problems is that the set of optimal solutions is also convex.

Theorem

Let $f : C \to \mathbb{R}$ be a convex function defined on the convex set $C \subseteq \mathbb{R}^n$. Then the set of optimal solutions of the problem

$$\min\{f(\mathbf{x}) : \mathbf{x} \in C\}$$

is convex. If, in addition, f is strictly convex over C, then there exists at most one optimal solution of the problem.

Proof. In class

Outline

- A Convex Problem:

$$\begin{aligned} \min \quad & -2x_1 + x_2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 \le 3 \end{aligned}$$

- A Nonconvex Problem:

$$\begin{aligned} \min \quad & x_1^2 - x_2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 = 3 \end{aligned}$$

$$
\text{(LP)}: \quad \begin{array}{ll} \min & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \le \mathbf{b} \\ & \mathbf{B}\mathbf{x} = \mathbf{g} \end{array}
$$

- A convex optimization problem (constraints and objective function are linear/affine and hence convex).
- It is also equivalent to a problem of maximizing a convex (linear) function subject to a convex constraints set. Hence, if the feasible set is compact and nonempty, then there exists at least one optimal solution which is an extreme point = basic feasible solution.
- A more general result drops the compactness assumption and is often called the fundamental theorem of linear programming.

- Convex quadratic problems are problems consisting of minimizing a convex quadratic function subject to affine constraints.
- The general form is

$$\begin{aligned} \min \quad & \mathbf{x}^\top \mathbf{Q}\mathbf{x} + 2\mathbf{b}^\top \mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{c} \end{aligned}$$

$\mathbf{Q} \in \mathbb{R}^{n\times n}$ is positive semidefinite, $\mathbf{b} \in \mathbb{R}^n, \mathbf{A} \in \mathbb{R}^{m\times n}, \mathbf{c} \in \mathbb{R}^m$.

Quadratically Constrained Quadratic Problems:

$$
\text{(QCQP)} \quad \begin{array}{ll} \min & \mathbf{x}^\top \mathbf{A}_0 \mathbf{x} + 2\mathbf{b}_0^\top \mathbf{x} + c_0 \\ \text{s.t.} & \mathbf{x}^\top \mathbf{A}_i \mathbf{x} + 2\mathbf{b}_i^\top \mathbf{x} + c_i \leq 0, \quad i = 1, 2, \cdots, m \\ & \mathbf{x}^\top \mathbf{A}_j \mathbf{x} + 2\mathbf{b}_j^\top \mathbf{x} + c_j = 0, \quad j = m+1, m+2, \cdots, m+p \end{array}
$$

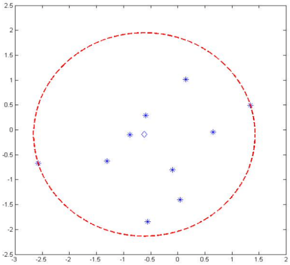
$\mathbf{A}_0, \cdots, \mathbf{A}_{m+p}$ - $n \times n$ symmetric, $\mathbf{b}_0, \cdots, \mathbf{b}_{m+p} \in \mathbb{R}^n$, $c_0, \cdots, c_{m+p} \in \mathbb{R}$.

- QCQPs are not necessarily convex problems.
- When there are no equality constrainers ($p = 0$) and all the matrices are positive semidefinite: $\mathbf{A}_i \succeq 0, i = 0, 1, \cdots, m$, the problem is convex, and is therefore called a convex QCQP.

Chebyshev Center Problem. Given m points $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_m$ in $\mathbb{R}^n$. The objective is to find the center of the minimum radius closed ball containing all the points.

- This ball is called the Chebyshev ball and the center is the Chebyshev center.
- In mathematical terms, the problem can be written as (r is radius and $\mathbf{x}$ is center):

$$\begin{array}{ll} \min\limits_{\mathbf{x},r} & r \\ \text{s.t.} & \mathbf{a}_i \in B[\mathbf{x}, r], \quad i = 1, 2, \cdots, n \end{array}$$



- or:

$$\begin{array}{ll} \min\limits_{\mathbf{x},r} & r \\ \text{s.t.} & \|\mathbf{x} - \mathbf{a}_i\| \le r, \quad i = 1, 2, \cdots, n \end{array}$$

- We are given n assets numbered as $1, 2, \cdots, n$. Let $Y_j\,(j = 1, 2, \cdots, n)$ be the random variable representing the return from asset j.
- Assume the expected returns are known:

$$\mu_j = E(Y_j),\ j = 1, 2, \cdots, n$$

and that the covariances of all the pairs of variables are also known:

$$\sigma_{i,j} = \mathrm{COV}(Y_i, Y_j),\ i, j = 1, 2, \cdots, n$$

- $x_j\ (j = 1, 2, \cdots, n)$ - the proportion of budget invested in asset j. The decision variables are constrained to satisfy $\mathbf{x} \in \Delta_n$.
- The overall return is the random variable:

$$R = \sum_{j=1}^{n} x_j Y_j.$$

whose expectation and variance are given by:

$$E(R) = \mu^\top \mathbf{x}, \quad V(R) = \mathbf{x}^\top \mathbf{C}\mathbf{x}$$

$\mu = (\mu_1, \mu_2, \cdots, \mu_n)^\top$ and $\mathbf{C}$ is the covariance matrix: $C_{i,j} = \sigma_{i,j}$

The Markowitz Model

- There are several formulations of the portfolio optimization problem, all referred to as "Markowitz model" after Harry Markowitz (1952).
- Minimizing the risk under a minimal return level:

$$\begin{aligned} \min \quad & \mathbf{x}^\top \mathbf{C} \mathbf{x} \\ \text{s.t.} \quad & \mu^\top \mathbf{x} \geq \alpha, \quad \mathbf{e}^\top \mathbf{x} = 1, \quad \mathbf{x} \geq 0 \end{aligned}$$

- Maximize the expected return subject to a bounded risk constraint:

$$\begin{aligned} \max \quad & \mu^\top \mathbf{x} \\ \text{s.t.} \quad & \mathbf{x}^\top \mathbf{C} \mathbf{x} \leq \beta, \quad \mathbf{e}^\top \mathbf{x} = 1, \quad \mathbf{x} \geq 0 \end{aligned}$$

- A penalty approach:

$$\begin{aligned} \min \quad & -\mu^\top \mathbf{x} + \gamma(\mathbf{x}^\top \mathbf{C} \mathbf{x}) \\ \text{s.t.} \quad & \mathbf{e}^\top \mathbf{x} = 1, \quad \mathbf{x} \geq 0 \end{aligned}$$

Definition

Given a nonempty closed convex set C, the orthogonal projection operator $P_C : \mathbb{R}^n \to C$ is defined by

$$P_C(\mathbf{x}) = \operatorname{argmin}\{\|\mathbf{y} - \mathbf{x}\|^2 : \mathbf{y} \in C\}$$

The first important result is that the orthogonal projection exists and is unique.

Theorem (The First Projection Theorem)

Let $C \subseteq \mathbb{R}^n$ be a nonempty closed and convex set. Then for any $\mathbf{x} \in \mathbb{R}^n$, the orthogonal projection $P_C(\mathbf{x})$ exists and is unique.

Proof. In class

Examples

- $C = \mathbb{R}^n_+$.

$$P_{\mathbb{R}^n_+}(\mathbf{x}) = [\mathbf{x}]_+$$

where $[\mathbf{v}]_+ = (\max\{v_1, 0\}, \max\{v_2, 0\}, \cdots, \max\{v_n, 0\})^\top$

- A box is a subset of $\mathbb{R}^n$ of the form

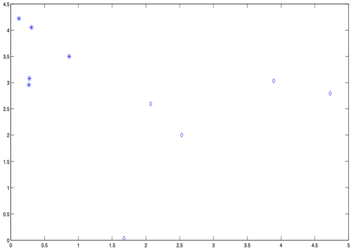
$$B = [l_1, u_1] \times [l_2, u_2] \times \cdots \times [l_n, u_n] = \{\mathbf{x} \in \mathbb{R}^n : l_i \leq x_i \leq u_i\}$$

where $l_i \leq u_i$ for all $i = 1, 2, \cdots, n$.

$$[P_B(\mathbf{x})]_i = \begin{cases} u_i & x_i \geq u_i \\ x_i & l_i < x_i < u_i \\ l_i & x_i \leq l_i \end{cases}$$

- $C = B[0, r]$

$$P_{B[0,r]} = \begin{cases} \mathbf{x} & \|\mathbf{x}\| \leq r \\ r\dfrac{\mathbf{x}}{\|\mathbf{x}\|} & \|\mathbf{x}\| > r \end{cases}$$



- Suppose that we are given two types of points in $\mathbb{R}^n$: type A and type B points.
- $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_m \in \mathbb{R}^n$ - type A.
- $\mathbf{x}_{m+1}, \mathbf{x}_{m+2}, \cdots, \mathbf{x}_{m+p} \in \mathbb{R}^n$ - type B.

The objective is to find a linear separator, which is a hyperplane of the form

$$H(\mathbf{w}, \beta) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{w}^\top \mathbf{x} + \beta = 0\}$$

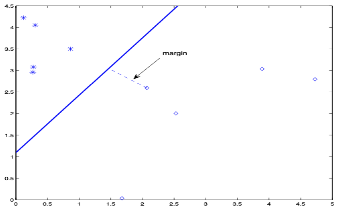
for which the type A and type B points are in its opposite sides:

$$\begin{aligned} \mathbf{w}^\top \mathbf{x}_i + \beta < 0, &\quad i = 1, 2, \cdots, m \\ \mathbf{w}^\top \mathbf{x}_i + \beta > 0, &\quad i = m+1, m+2, \cdots, m+p \end{aligned}$$

Underlying Assumption: the two sets of points are linearly separable, meaning that the set of inequalities has a solution.

Maximizing the Margin

The margin of the separator is the distance of the hyperplane to the closest point.



The separation problem will thus consist of finding the separator with the largest margin.

Lemma

Let $H(\mathbf{a}, b) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^\top \mathbf{x} = b\}$, where $\mathbf{0} \neq \mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Let $\mathbf{y} \in \mathbb{R}^n$. Then the distance between $\mathbf{y}$ and the set H is given by

$$d(\mathbf{y}, H(\mathbf{a}, b)) = \frac{|\mathbf{a}^\top \mathbf{y} - b|}{\|\mathbf{a}\|}$$

Proof. Later on.

$$\begin{aligned} \max \quad & \left\{ \min_{i=1,2,\cdots,m+p} \frac{|\mathbf{w}^\top \mathbf{x}_i + \beta|}{\|\mathbf{w}\|} \right\} \\ \text{s.t.} \quad & \mathbf{w}^\top \mathbf{x}_i + \beta < 0, \quad i = 1, 2, \cdots, m \\ & \mathbf{w}^\top \mathbf{x}_i + \beta > 0, \quad i = m+1, m+2, \cdots, m+p \end{aligned}$$

Nonconvex formulation $\Rightarrow$ difficult to handle.

- The problem has a degree of freedom in the sense that if $(\mathbf{w}, \beta)$ is an optimal solution, then so is any nonzero multiplier of it, that is, $(\alpha\mathbf{w}, \alpha\beta)$ for $\alpha \neq 0$. We can therefore decide that

$$\min_{i=1,2,\cdots,m+p} |\mathbf{w}^\top \mathbf{x}_i + \beta| = 1$$

- Thus, the problem can be written as

$$
\begin{aligned}
\max \quad & \frac{1}{\|\mathbf{w}\|} \\
\text{s.t.} \quad & \min_{i=1,2,\cdots,m+p} |\mathbf{w}^\top \mathbf{x}_i + \beta| = 1 \\
& \mathbf{w}^\top \mathbf{x}_i + \beta < 0, \quad i = 1, 2, \cdots, m \\
& \mathbf{w}^\top \mathbf{x}_i + \beta > 0, \quad i = m+1, m+2, \cdots, m+p
\end{aligned}
$$

- $$\begin{aligned} \min \quad & \frac{1}{2}\|\mathbf{w}\|^2 \\ \text{s.t.} \quad & \min_{i=1,2,\cdots,m+p} |\mathbf{w}^\top \mathbf{x}_i + \beta| = 1 \\ & \mathbf{w}^\top \mathbf{x}_i + \beta \le -1, \quad i = 1, 2, \cdots, m \\ & \mathbf{w}^\top \mathbf{x}_i + \beta \ge 1, \quad i = m+1, m+2, \cdots, m+p \end{aligned}$$

- The first constraint can be dropped (why?)

$$\begin{aligned} \min \quad & \frac{1}{2}\|\mathbf{w}\|^2 \\ \text{s.t.} \quad & \mathbf{w}^\top \mathbf{x}_i + \beta \le -1, \quad i = 1, 2, \cdots, m \\ & \mathbf{w}^\top \mathbf{x}_i + \beta \ge 1, \quad i = m+1, m+2, \cdots, m+p \end{aligned}$$

Convex Formulation.

$$\text{(TRS)}: \min\{\mathbf{x}^\top \mathbf{A}\mathbf{x} + 2\mathbf{b}^\top \mathbf{x} + c : \|\mathbf{x}\|^2 \leq 1\}$$

where $\mathbf{b} \in \mathbb{R}^n$, $c \in \mathbb{R}$ and $\mathbf{A}$ is an $n \times n$ symmetric matrix. In general, this is a nonconvex problem.

- By the spectral decomposition theorem, there exist an orthogonal matrix $\mathbf{U}$ and a diagonal matrix $\mathbf{D} = \mathrm{diag}(\mathrm{d_1}, \mathrm{d_2}, \cdots, \mathrm{d_n})$ such that $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^\top$, and hence (TRS) can be rewritten as

$$\min\{\mathbf{x}^\top \mathbf{U}\mathbf{D}\mathbf{U}^\top \mathbf{x} + 2\mathbf{b}^\top \mathbf{U}\mathbf{U}^\top \mathbf{x} + c : \|\mathbf{U}^\top \mathbf{x}\|^2 \leq 1\}$$

- Making the linear change of variables $\mathbf{y} = \mathbf{U}^\top \mathbf{x}$, the problem reduces to

$$\min\{\mathbf{y}^\top \mathbf{D}\mathbf{y} + 2\mathbf{b}^\top \mathbf{U}\mathbf{y} + c : \|\mathbf{y}\|^2 \leq 1\}$$

- Denoting $\mathbf{f} = \mathbf{U}^\top \mathbf{b}$, we obtain

$$\min \quad \sum_{i=1}^n d_i y_i^2 + 2\sum_{i=1}^n f_i y_i + c, \quad \text{s.t.} \quad \sum_{i=1}^n y_i^2 \leq 1. \tag{2}$$

Lemma

Let $\mathbf{y}^*$ *be an optimal solution of* (2). *Then* $f_i y_i^* \leq 0$ *for all* $i = 1, 2, \cdots, n$.

Hidden Convexity in Trust Region Subproblems

Lemma

Let $\mathbf{y}^*$ be an optimal solution of (2). Then $f_i y_i^* \le 0$ for all $i = 1, 2, \cdots, n$.

Proof.

- Denote the objective function of (2) by $g(\mathbf{y}) \equiv \sum_{i=1}^n d_i y_i^2 + 2\sum_{i=1}^n f_i y_i + c$.
- Let $i \in \{1, 2, \cdots, n\}$. Define $\tilde{\mathbf{y}}$ as

$$\tilde{y}_j = \begin{cases} y_j^* & j \neq i \\ -y_i^* & j = i \end{cases}$$

- $\tilde{\mathbf{y}}$ is feasible and $g(\mathbf{y}^*) \le g(\tilde{\mathbf{y}})$
- $\sum_{i=1}^n d_i (y_i^*)^2 + 2\sum_{i=1}^n f_i y_i^* + c \le \sum_{i=1}^n d_i (\tilde{y}_i)^2 + 2\sum_{i=1}^n f_i \tilde{y}_i + c$
- After cancelleation of terms, $2f_i y_i^* \le 2f_i(-y_i^*)$, implying $f_i y_i^* \le 0$.

$\square$

Back to the TRS problem –

- Make the change of variable $y_i = -\text{sgn}(f_i)\sqrt{z_i}$ ($z_i \geq 0$)
- Problem (2) becomes

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} d_i z_i - 2 \sum_{i=1}^{n} |f_i| \sqrt{z_i} + c \\ \text{s.t.} \quad & \sum_{i=1}^{n} z_i \leq 1 \\ & z_1, z_2, \cdots, z_n \geq 0 \end{aligned}$$

- Convex optimization problem