Lecture 7: Convex Functions

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Outline

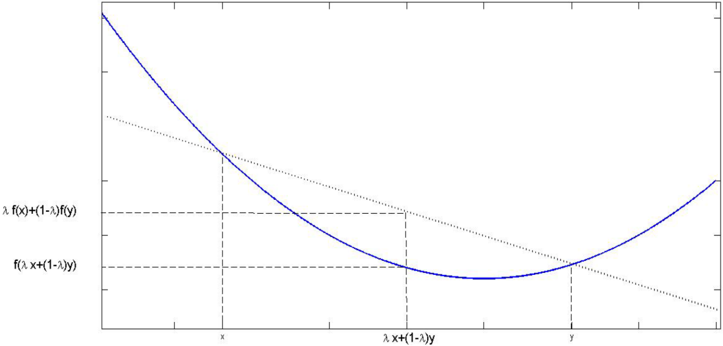
Outline

Convex Functions

Definition

A function $f : C \to \mathbb{R}$ defined on a convex set $C \subseteq \mathbb{R}^n$ is called convex (or convex over C) if

$$f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \le \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}) \text{ for any } \mathbf{x}, \mathbf{y} \in C, \lambda \in [0,1].$$



- In case where no domain is specified, naturally assume f is defined over the entire space $\mathbb{R}^n$.
- A function $f : C \to \mathbb{R}$ defined on a convex set $C \subseteq \mathbb{R}^n$ is called strictly convex if

$$f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) < \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}) \text{ for any } \mathbf{x} \neq \mathbf{y} \in C, \lambda \in (0,1).$$

- A function f is called concave if $-f$ is convex. Similarly, f is called strictly concave if $-f$ is strictly convex.
- We can also define concavity directly: a function f is concave if and only if for any $\mathbf{x}, \mathbf{y} \in C$ and $\lambda \in [0,1]$,

$$f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \geq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}).$$

- Affine Functions. $f(\mathbf{x}) = \mathbf{a}^\top \mathbf{x} + b$, where $\mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$.
- Norms. $g(\mathbf{x}) = \|\mathbf{x}\|$.
- Convexity of f: Take $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\lambda \in [0, 1]$. Then

$$
\begin{aligned}
f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) &= \mathbf{a}^\top(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) + b \\
&= \lambda(\mathbf{a}^\top \mathbf{x}) + (1-\lambda)(\mathbf{a}^\top \mathbf{y}) + \lambda b + (1-\lambda) b \\
&= \lambda(\mathbf{a}^\top \mathbf{x} + b) + (1-\lambda)(\mathbf{a}^\top \mathbf{y} + b) \\
&= \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}).
\end{aligned}
$$

- Convexity of g : Take $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\lambda \in [0, 1]$. Then

$$
\begin{aligned}
g(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) &= \|\lambda \mathbf{x} + (1-\lambda)\mathbf{y}\| \\
&\leq \|\lambda \mathbf{x}\| + \|(1-\lambda)\mathbf{y}\| \\
&= \lambda\|\mathbf{x}\| + (1-\lambda)\|\mathbf{y}\| \\
&= \lambda g(\mathbf{x}) + (1-\lambda) g(\mathbf{y}).
\end{aligned}
$$

Theorem

Let $f : C \to \mathbb{R}$ be a convex function where $C \subseteq \mathbb{R}^n$ is a convex set. Then for any $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k \in C$ and $\boldsymbol{\lambda} \in \Delta_k$, the following inequality holds:

$$f\left(\sum_{i=1}^{k} \lambda_i \mathbf{x}_i\right) \leq \sum_{i=1}^{k} \lambda_i f(\mathbf{x}_i).$$

Proof is very similar to the proof that any convex combination of points in a convex set is in the set – see the proof of Theorem 7.5 on pages 118, 119 of the textbook.

The Gradient Inequality

Theorem

Let $f : C \to \mathbb{R}$ be a continuously differentiable function defined on a convex set $C \subseteq \mathbb{R}^n$. Then f is convex over C if and only if

$$f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}) \leq f(\mathbf{y}) \text{ for any } \mathbf{x}, \mathbf{y} \in C. \tag{1}$$

Theorem

Let $f : C \to \mathbb{R}$ be a continuously differentiable function defined on a convex set $C \subseteq \mathbb{R}^n$. Then f is convex over C if and only if

$$f(\mathbf{x}) + \nabla f(\mathbf{x})^\top(\mathbf{y} - \mathbf{x}) \leq f(\mathbf{y}) \text{ for any } \mathbf{x}, \mathbf{y} \in C. \quad (1)$$

Proof.

- Suppose first that f is convex. Let $\mathbf{x}, \mathbf{y} \in C$ and $\lambda \in (0, 1]$. If $\mathbf{x} = \mathbf{y}$, then (1) trivially holds. We will therefore assume that $\mathbf{x} \neq \mathbf{y}$.
- $\frac{f(\mathbf{x}+\lambda(\mathbf{y}-\mathbf{x}))-f(\mathbf{x})}{\lambda} \leq f(\mathbf{y}) - f(\mathbf{x})$.
- Taking $\lambda \to 0^+$, we obtain $f'(\mathbf{x}; \mathbf{y} - \mathbf{x}) \leq f(\mathbf{y}) - f(\mathbf{x})$.
- Since f is continuously differentiable, $f'(\mathbf{x}; \mathbf{y} - \mathbf{x}) = \nabla f(\mathbf{x})^\top(\mathbf{y} - \mathbf{x})$, and (1) follows.

Proof Contd.

- To prove the reverse direction, assume the gradient inequality holds.
- Let $\mathbf{z}, \mathbf{w} \in C$ and $\lambda \in (0, 1)$. We will show that $f(\lambda \mathbf{z} + (1-\lambda)\mathbf{w}) \leq \lambda f(\mathbf{z}) + (1-\lambda) f(\mathbf{w})$.
- Let $\mathbf{u} = \lambda \mathbf{z} + (1-\lambda)\mathbf{w} \in C$. Then

$$\mathbf{z} - \mathbf{u} = \frac{\mathbf{u} - (1-\lambda)\mathbf{w}}{\lambda} - \mathbf{u} = -\frac{1-\lambda}{\lambda}(\mathbf{w} - \mathbf{u}).$$

- We have

$$f(\mathbf{u}) + \nabla f(\mathbf{u})^\top (\mathbf{z} - \mathbf{u}) \leq f(\mathbf{z}),$$

$$f(\mathbf{u}) - \frac{\lambda}{1-\lambda} \nabla f(\mathbf{u})^\top (\mathbf{z} - \mathbf{u}) \leq f(\mathbf{w}).$$

- Thus,

$$f(\mathbf{u}) \leq \lambda f(\mathbf{z}) + (1-\lambda) f(\mathbf{w}).$$

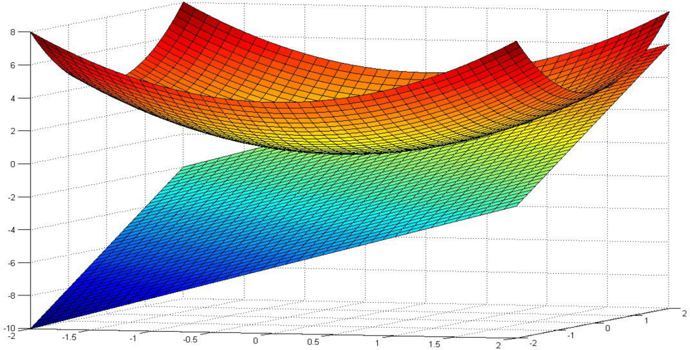
□

The Gradient Inequality for Strictly Convex Functions

Proposition

Let $f : C \to \mathbb{R}$ be a continuously differentiable function defined on a convex set $C \subseteq \mathbb{R}^n$. Then f is strictly convex over C if and only if

$$f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}) < f(\mathbf{y}) \text{ for any } \mathbf{x}, \mathbf{y} \in C \text{ satisfying } \mathbf{x} \neq \mathbf{y}.$$



A direct result of the gradient inequality is that the first order optimality condition $\nabla f(\mathbf{x}^*) = 0$ is sufficient for global optimality.

Proposition

Let f be a continuously differentiable function which is convex over a convex set $C \subseteq \mathbb{R}^n$. Suppose that $\nabla f(\mathbf{x}^*) = 0$ for some $\mathbf{x}^* \in C$. Then $\mathbf{x}^*$ is the global minimizer of f over C.

Proof. In class

Convexity of Quadratic Functions with Positive Semidefinite Matrices

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be the quadratic function given by $f(\mathbf{x}) = \mathbf{x}^\top \mathbf{A}\mathbf{x} + 2\mathbf{b}^\top \mathbf{x} + c$ where $\mathbf{A} \in \mathbb{R}^{n\times n}$ is symmetric, $\mathbf{b} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then f is (strictly) convex if and only if $\mathbf{A} \succeq \mathbf{0}$ ($\mathbf{A} \succ \mathbf{0}$).

Proof.

- The convexity of f is equivalent to

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}), \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

- Same as

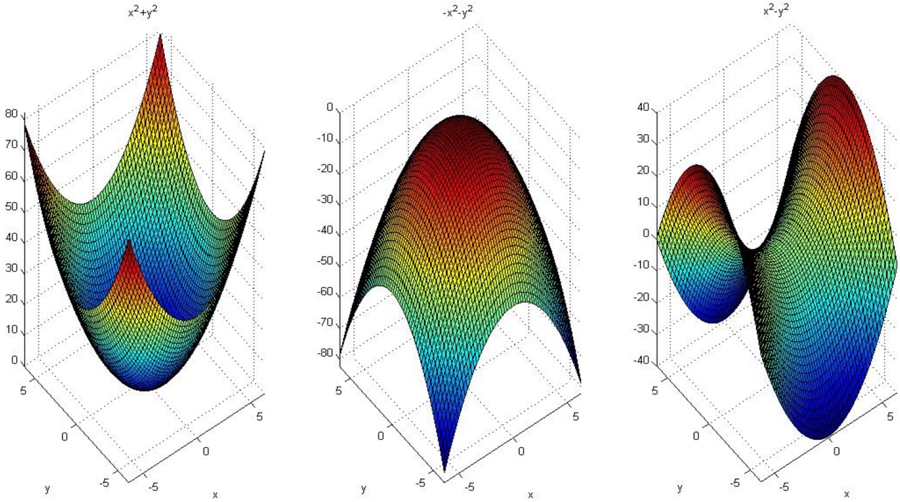
$$\mathbf{y}^\top \mathbf{A}\mathbf{y} + 2\mathbf{b}^\top \mathbf{y} + c \geq \mathbf{x}^\top \mathbf{A}\mathbf{x} + 2\mathbf{b}^\top \mathbf{x} + c + 2(\mathbf{A}\mathbf{x} + \mathbf{b})^\top (\mathbf{y} - \mathbf{x}), \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

- Or $(\mathbf{y} - \mathbf{x})^\top \mathbf{A}(\mathbf{y} - \mathbf{x}) \geq 0$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
- Equivalent to the inequality $\mathbf{d}^\top \mathbf{A}\mathbf{d} \geq 0$ for any $\mathbf{d} \in \mathbb{R}^n$.
- Same as $\mathbf{A} \succeq \mathbf{0}$.
- Similar arguments show that strict convexity is equivalent to

$$\mathbf{d}^\top \mathbf{A}\mathbf{d} > 0 \text{ for any } \mathbf{0} \neq \mathbf{d} \in \mathbb{R}^n,$$

namely to $\mathbf{A} \succ \mathbf{0}$.

□


x^2+y^2
-x^2-y^2
x^2-y^2
80
70
60
50
40
30
20
10
0
-10
-20
-30
-40
-50
-60
-70
-80
5
0
-5
y
x

Theorem

Suppose that f is a continuously differentiable function over a convex set $C \subseteq \mathbb{R}^n$. Then f is convex over C if and only if

$$(\nabla f(\mathbf{x}) - \nabla f(\mathbf{y}))^\top (\mathbf{x} - \mathbf{y}) \geq 0 \text{ for any } \mathbf{x}, \mathbf{y} \in C.$$

See the proof of Theorem 8.11 on pages 122, 123 of the textbook.

Second-Order Characterization of Convexity

Theorem

Suppose that f is a twice continuously differentiable function over an open convex set $C \subseteq \mathbb{R}^n$. Then f is convex over C if and only if $\nabla^2 f(\mathbf{x}) \succeq \mathbf{0}$ for any $\mathbf{x} \in C$.

Theorem

Suppose that f is a twice continuously differentiable function over an open convex set $C \subseteq \mathbb{R}^n$. Then f is convex over C if and only if $\nabla^2 f(\mathbf{x}) \succeq \mathbf{0}$ for any $\mathbf{x} \in C$.

Proof.

- Suppose that $\nabla^2 f(\mathbf{x}) \succeq \mathbf{0}, \forall \mathbf{x} \in C$. Let $\mathbf{x}, \mathbf{y} \in C$, then $\exists \mathbf{z} \in [\mathbf{x}, \mathbf{y}] \subseteq C$:

$$f(\mathbf{y}) = f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}) + \frac{1}{2}(\mathbf{y} - \mathbf{x})^\top \nabla^2 f(\mathbf{z})(\mathbf{y} - \mathbf{x}).$$

- $(\mathbf{y} - \mathbf{x})^\top \nabla^2 f(\mathbf{z})(\mathbf{y} - \mathbf{x}) \geq 0 \Rightarrow f(\mathbf{y}) \geq f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}) \Rightarrow f$ convex.

Proof Contd.

- Suppose that f is convex over C. Let $\mathbf{x} \in C$ and let $\mathbf{y} \in \mathbb{R}^n$.
- C is open $\Rightarrow \exists \epsilon > 0$ such that $\mathbf{x} + \lambda \mathbf{y} \in C, \forall \lambda \in (0, \epsilon)$.

$$f(\mathbf{x} + \lambda \mathbf{y}) \geq f(\mathbf{x}) + \lambda \nabla f(\mathbf{x})^\top \mathbf{y}.$$

- $f(\mathbf{x} + \lambda \mathbf{y}) = f(\mathbf{x}) + \lambda \nabla f(\mathbf{x})^\top \mathbf{y} + \frac{\lambda^2}{2} \mathbf{y}^\top \nabla^2 f(\mathbf{x}) \mathbf{y} + o(\lambda^2 \|\mathbf{y}\|^2)$.
- Thus, $\frac{\lambda^2}{2} \mathbf{y}^\top \nabla^2 f(\mathbf{x}) \mathbf{y} + o(\lambda^2 \|\mathbf{y}\|^2) \geq 0$ for any $\lambda \in (0, \epsilon)$.
- Dividing by λ^2, $\frac{1}{2} \mathbf{y}^\top \nabla^2 f(\mathbf{x}) \mathbf{y} + \frac{o(\lambda^2 \|\mathbf{y}\|^2)}{\lambda^2} \geq 0$.
- Taking $\lambda \to 0^+$, we have $\mathbf{y}^\top \nabla^2 f(\mathbf{x}) \mathbf{y} \geq 0, \forall \mathbf{y} \in \mathbb{R}^n$.
- Hence $\nabla^2 f(\mathbf{x}) \succeq \mathbf{0}$ for any $\mathbf{x} \in C$.

$\square$

$$f(\mathbf{x}) = \log(e^{x_1} + e^{x_2} + ... + e^{x_n}), \mathbf{x} \in \mathbb{R}^n.$$

- $\frac{\partial f}{\partial x_i}(\mathbf{x}) = \frac{e^{x_i}}{\sum_{j=1}^n e^{x_j}}, \quad i = 1, 2, ..., n,$
- $\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}) = \begin{cases} -\frac{e^{x_i} e^{x_j}}{(\sum_{j=1}^n e^{x_j})^2}, i \neq j, \\ -\frac{e^{x_i} e^{x_j}}{(\sum_{j=1}^n e^{x_j})^2} + \frac{e^{x_i}}{\sum_{j=1}^n e^{x_j}}, i = j \end{cases}$

- We can thus write the Hessian matrix as

$$\nabla^2 f(\mathbf{x}) = \mathrm{diag}(\mathbf{w}) - \mathbf{w}\mathbf{w}^\top, \quad \mathbf{w} = \left(\frac{e^{x_i}}{\sum_{j=1}^n e^{x_j}}\right)_{i=1}^n \in \Delta_n.$$

- For any $\mathbf{v} \in \mathbb{R}^n : \mathbf{v}^\top \nabla^2 f(\mathbf{x}) \mathbf{v} = \sum_{i=1}^n w_i {v_i}^2 - (\mathbf{v}^\top \mathbf{w})^2 \geq 0$ since defining $s_i = \sqrt{w_i} v_i$, $t_i = \sqrt{w_i}$, we have

$$(\mathbf{v}^\top \mathbf{w})^2 = (\mathbf{s}^\top \mathbf{t})^2 \leq \|\mathbf{s}\|^2 \|\mathbf{t}\|^2 = \left(\sum_{i=1}^n w_i {v_i}^2\right)\left(\sum_{i=1}^n w_i\right) = \sum_{i=1}^n w_i {v_i}^2.$$

- Thus, $\nabla^2 f(\mathbf{x}) \succeq \mathbf{0}$ and hence f is convex over $\mathbb{R}^n$.

Convexity of Quad-over-lin

$$f(x_1, x_2) = \frac{{x_1}^2}{x_2}$$

defined over $\mathbb{R} \times \mathbb{R}_{++} = \{(x_1, x_2) : x_2 > 0\}$.
In class

Outline

Convexity is preserved under several operations such as summation, multiplication by positive scalars and affine change of variables.

Theorem

- *Let f be a convex function defined over a convex set $C \subseteq \mathbb{R}^n$ and let $\alpha \geq 0$. Then αf is a convex function over C.*
- *Let $f_1, f_2, \ldots, f_p$ be convex functions over a convex set $C \subseteq \mathbb{R}^n$. Then the sum function $f_1 + f_2 + \ldots + f_p$ is convex over C.*
- *Let f be a convex function defined over a convex set $C \subseteq \mathbb{R}^n$. Let $\mathbf{A} \in \mathbb{R}^{n \times m}$ and $\mathbf{b} \in \mathbb{R}^n$. Then the function g defined by*

$$g(\mathbf{y}) = f(\mathbf{A}\mathbf{y} + \mathbf{b})$$

is convex over the convex set $D = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{A}\mathbf{y} + \mathbf{b} \in C\}$.

See the proofs of Theorems 7.16 and 7.17 of the textbook.

Example: Generalized Quadratic-over-linear

The generalized quad-over-lin function

$$g(\mathbf{x}) = \frac{\|\mathbf{A}\mathbf{x} + \mathbf{b}\|^2}{\mathbf{c}^\top \mathbf{x} + d} \quad (\mathbf{A} \in \mathbb{R}^{m\times n}, \mathbf{b} \in \mathbb{R}^m, \mathbf{c} \in \mathbb{R}^n, d \in \mathbb{R})$$

is convex over $D = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{c}^\top \mathbf{x} + d > 0\}$.
In class

Examples of Convex Functions

- $$f(x_1, x_2) = {x_1}^2 + 2x_1x_2 + 3{x_2}^2 + 2x_1 - 3x_2 + e^{x_1}$$
- $$f(x_1, x_2, x_3) = e^{x_1 - x_2 + x_3} + e^{2x_2} + x_1$$
- $$f(x_1, x_2) = -\log(x_1x_2)$$

  over $\mathbb{R}^2_{++}$

In class

Preservation of Convexity under Composition

- In general, convexity is not preserved under composition of convex functions.
- Example:

$$g(t) = t^2, \quad h(t) = t^2 - 4, \quad s(t) = g(h(t)).$$

In class

Preservation of Convexity under Composition

Theorem

Let $f : C \to \mathbb{R}$ be a convex function defined over the convex set $C \subseteq \mathbb{R}^n$. Let $g : I \to \mathbb{R}$ be a one-dimensional nondecreasing convex function over the interval $I \subseteq \mathbb{R}$. Assume that the image of C under f is contained in $I : f(C) \subseteq I$. Then the composition of g with f defined by $h(\mathbf{x}) \equiv g(f(\mathbf{x}))$ is convex over C.

Proof.

Let $\mathbf{x}, \mathbf{y} \in C$ and let $\lambda \in [0, 1]$. Then

$$\begin{aligned}
h(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) &= g(f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y})) \\
&\leq g(\lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y})) \\
&\leq \lambda g(f(\mathbf{x})) + (1-\lambda) g(f(\mathbf{y})) \\
&= \lambda h(\mathbf{x}) + (1-\lambda) h(\mathbf{y}),
\end{aligned}$$

thus establishing the convexity of h. □

Examples

- $h(\mathbf{x}) = e^{\|\mathbf{x}\|^2}$
- $h(\mathbf{x}) = (\|\mathbf{x}\|^2 + 1)^2$

In class

Point-Wise Maximum of Convex Functions

Theorem

Let $f_1, f_2, ..., f_p : C \to \mathbb{R}$ be p convex functions over the convex set $C \subseteq \mathbb{R}^n$. Then the maximum function

$$f(\mathbf{x}) \equiv \max_{i=1,2,...,p}\{f_i(\mathbf{x})\}$$

is convex over C.

Proof.

Let $\mathbf{x}, \mathbf{y} \in C$ and let $\lambda \in [0, 1]$. Then

$$\begin{aligned} f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) &= \max_{i=1,2,...,p} f_i(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \\ &\leq \max_{i=1,2,...,p}\{\lambda f_i(\mathbf{x}) + (1-\lambda) f_i(\mathbf{y}))\} \\ &\leq \lambda \max_{i=1,2,...,p} f_i(\mathbf{x}) + (1-\lambda)\max_{i=1,2,...,p} f_i(\mathbf{y}) \\ &= \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}). \end{aligned}$$

□

Point-Wise Maximum of Convex Functions

Examples.

- $f(\mathbf{x}) = \max\{x_1, x_2, ..., x_n\}$ is convex.
- For a given vector $\mathbf{x} = (x_1, x_2, ..., x_n)^\top \in \mathbb{R}^n$, let $x_{[i]}$ denote the i-th largest value in $\mathbf{x}$. For any $k \in \{1, 2, ..., n\}$ the function

$$h_k(\mathbf{x}) = x_{[1]} + x_{[2]} + ... + x_{[k]}$$

is convex. why?

Preservation of Convexity Under Partial Minimization

Theorem

Let $f : C \times D \to \mathbb{R}$ be a convex function defined over the set $C \times D$ where $C \subseteq \mathbb{R}^m$ and $D \subseteq \mathbb{R}^n$ are convex sets. Let

$$g(\mathbf{x}) = \min_{\mathbf{y} \in D} f(\mathbf{x}, \mathbf{y}), \quad \mathbf{x} \in C$$

where we assume that the minimum is finite. Then g is convex over C.

Proof.

Let $\mathbf{x}_1, \mathbf{x}_2 \in C$ and $\lambda \in [0, 1]$. Take $\epsilon > 0$. Then $\exists \mathbf{y}_1, \mathbf{y}_2 \in D$:

$$f(\mathbf{x}_1, \mathbf{y}_1) \le g(\mathbf{x}_1) + \epsilon,\ f(\mathbf{x}_2, \mathbf{y}_2) \le g(\mathbf{x}_2) + \epsilon.$$

By the convexity of f, we have

$$\begin{aligned} f(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2, \lambda\mathbf{y}_1 + (1-\lambda)\mathbf{y}_2) &\le \lambda f(\mathbf{x}_1, \mathbf{y}_1) + (1-\lambda) f(\mathbf{x}_2, \mathbf{y}_2) \\ &\le \lambda(g(\mathbf{x}_1) + \epsilon) + (1-\lambda)(g(\mathbf{x}_2) + \epsilon) \\ &= \lambda g(\mathbf{x}_1) + (1-\lambda) g(\mathbf{x}_2) + \epsilon. \end{aligned}$$

Hence $g(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \le \lambda g(\mathbf{x}_1) + (1-\lambda) g(\mathbf{x}_2) + \epsilon$ for any $\epsilon > 0$. It follows that
$g(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \le \lambda g(\mathbf{x}_1) + (1-\lambda) g(\mathbf{x}_2)$. □

Example: The distance function from a convex set $d_C(\mathbf{x}) \equiv \inf_{\mathbf{y} \in C} \|\mathbf{x} - \mathbf{y}\|$ is convex.

Definition

Let $f : S \to \mathbb{R}$ be a function defined over a set $S \subseteq \mathbb{R}^n$. Then the level set of f with level α is given by

$$\text{Lev}(f, \alpha) = \{\mathbf{x} \in S : f(\mathbf{x}) \leq \alpha\}.$$

Theorem

Let $f : C \to \mathbb{R}$ be a convex function over the convex set $C \subseteq \mathbb{R}^n$. Then for any $\alpha \in \mathbb{R}$ the level set $\mathrm{Lev}(f, \alpha)$ *is convex.*

Proof.

- Let $\mathbf{x}, \mathbf{y} \in \mathrm{Lev}(f, \alpha)$ and let $\lambda \in [0, 1]$.
- Then $f(\mathbf{x}), f(\mathbf{y}) \leq \alpha$. Hence,

$$f(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1 - \lambda)f(\mathbf{y}) \leq \lambda\alpha + (1 - \lambda)\alpha = \alpha,$$

- $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in \mathrm{Lev}(f, \alpha)$, and we have established the convexity of $\mathrm{Lev}(f, \alpha)$.

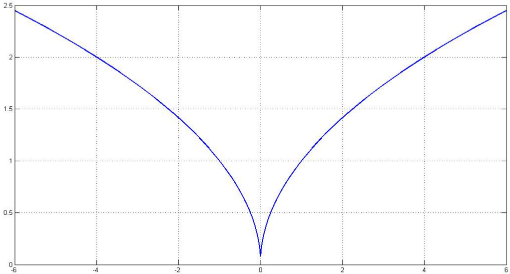
$\square$

Definition

A function $f : C \to \mathbb{R}$ defined over the convex set $C \subseteq \mathbb{R}^n$ is called quasi-convex if for any $\alpha \in \mathbb{R}$ the set $\mathrm{Lev}(f, \alpha)$ is convex.

Examples.

- $f(x) = \sqrt{|x|}$.
- $f(\mathbf{x}) = \frac{\mathbf{a}^\top \mathbf{x} + b}{\mathbf{c}^\top \mathbf{x} + d}$ over $C = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{c}^\top \mathbf{x} + d > 0\}$ where $\mathbf{a}, \mathbf{c} \in \mathbb{R}^n$ and $b, d \in \mathbb{R}$.



Outline

Convex functions are not necessarily continuous when defined on nonopen sets, but always local Lipschitz continuous at interior points of their domain.

Theorem

Let $f : C \to \mathbb{R}$ be a convex function defined over a convex set $C \subseteq \mathbb{R}^n$. Let $\mathbf{x}_0 \in \mathrm{int}(C)$. Then there exist $\epsilon > 0$ and $L > 0$ such that $B[\mathbf{x}_0, \epsilon] \subseteq C$ and

$$|f(\mathbf{x}) - f(\mathbf{x}_0)| \leq L\|\mathbf{x} - \mathbf{x}_0\| \text{ for any } \mathbf{x} \in B[\mathbf{x}_0, \epsilon].$$

Proof.

- Take $\epsilon > 0$ such that $B_\infty[\mathbf{x}_0, \epsilon] \equiv \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{x}_0\|_\infty \leq \epsilon\} \subseteq C$.
- Let $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ be the 2^n extreme points of $B_\infty[\mathbf{x}_0, \epsilon]$.
- For any $\mathbf{x} \in B_\infty[\mathbf{x}_0, \epsilon]$ there exists $\boldsymbol{\lambda} \in \Delta_{2^n}$ such that $\mathbf{x} = \sum_{i=1}^{2^n} \lambda_i \mathbf{v}_i$ (Krein-Milman). By Jensen's inequality,

$$f(\mathbf{x}) = f(\sum_{i=1}^{2^n} \lambda_i \mathbf{v}_i) \leq \sum_{i=1}^{2^n} \lambda_i f(\mathbf{v}_i) \leq M,$$

  where $M = \max_{i=1,2,...,2^n} f(\mathbf{v}_i)$.
- $B_2[\mathbf{x}_0, \epsilon] = B[\mathbf{x}_0, \epsilon] = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{x}_0\|_2 \leq \epsilon\} \subseteq B_\infty[\mathbf{x}_0, \epsilon]$.
- We therefore conclude that $f(\mathbf{x}) \leq M$ for any $\mathbf{x} \in B[\mathbf{x}_0, \epsilon]$.
- Let $\mathbf{x} \in B[\mathbf{x}_0, \epsilon]$ be such that $\mathbf{x} \neq \mathbf{x}_0$. Define

$$\mathbf{z} = \mathbf{x}_0 + \frac{1}{\alpha}(\mathbf{x} - \mathbf{x}_0), \quad \alpha = \frac{1}{\epsilon}\|\mathbf{x} - \mathbf{x}_0\|.$$

Proof Contd.

- Then obviously $\alpha \le 1$ and $\mathbf{z} \in B[\mathbf{x}_0, \epsilon]$, and in particular $f(\mathbf{z}) \le M$.
- $\mathbf{x} = \alpha \mathbf{z} + (1-\alpha)\mathbf{x}_0$.
- Consequently,
  $f(\mathbf{x}) \le \alpha f(\mathbf{z}) + (1-\alpha) f(\mathbf{x}_0) \le f(\mathbf{x}_0) + \alpha(M - f(\mathbf{x}_0)) =$
  $f(\mathbf{x}_0) + \frac{M - f(\mathbf{x}_0)}{\epsilon}\|\mathbf{x} - \mathbf{x}_0\|$.
- Thus, $f(\mathbf{x}) - f(\mathbf{x}_0) \le L\|\mathbf{x} - \mathbf{x}_0\|$ where $L = \frac{M - f(\mathbf{x}_0)}{\epsilon}$.
- We need to show that $f(\mathbf{x}) - f(\mathbf{x}_0) \ge -L\|\mathbf{x} - \mathbf{x}_0\|$.
- Define $\mathbf{u} = \mathbf{x}_0 + \frac{1}{\alpha}(\mathbf{x}_0 - \mathbf{x})$. Since $\mathbf{u} \in B[\mathbf{x}_0, \epsilon]$, then $f(\mathbf{u}) \le M$.
- $\mathbf{x} = \mathbf{x}_0 + \alpha(\mathbf{x}_0 - \mathbf{u})$. Therefore,

$$
\begin{aligned}
f(\mathbf{x}) = &\quad f(\mathbf{x}_0 + \alpha(\mathbf{x}_0 - \mathbf{u})) \ge f(\mathbf{x}_0) + \alpha(f(\mathbf{x}_0) - f(\mathbf{u})) \\
\ge &\quad f(\mathbf{x}_0) - \alpha(M - f(\mathbf{x}_0)) = f(\mathbf{x}_0) - \frac{M - f(\mathbf{x}_0)}{\epsilon}\|\mathbf{x} - \mathbf{x}_0\| \\
= &\quad f(\mathbf{x}_0) - L\|\mathbf{x} - \mathbf{x}_0\|.
\end{aligned}
$$

□

Convex functions are not necessarily differentiable, but all the directional derivatives at interior points exist.

Theorem

Let $f : C \to \mathbb{R}$ be a convex function over the convex set $C \subseteq \mathbb{R}^n$. Let $\mathbf{x} \in \mathrm{int}(C)$. Then for any $\mathbf{d} \neq \mathbf{0}$, the directional derivative $f'(\mathbf{x}; \mathbf{d})$ exists.

Proof.

- Let $\mathbf{x} \in \mathrm{int}(C)$ and let $\mathbf{d} \neq \mathbf{0}$. Then the directional derivative (if exists) is the limit
$$\lim_{t \to 0^+} \frac{f(\mathbf{x} + t\mathbf{d}) - f(\mathbf{x})}{t} \tag{2}$$
- Defining $h(t) = \frac{f(\mathbf{x}+t\mathbf{d})-f(\mathbf{x})}{t}$, (2) is the same as $\lim_{t\to 0^+} h(t)$.
- We will take an $\epsilon > 0$ for which $\mathbf{x} + t\mathbf{d}, \mathbf{x} - t\mathbf{d} \in C$ for all $t \in [0, \epsilon]$.
- Let $0 < t_1 < t_2 \leq \epsilon$. Then $f(\mathbf{x} + t_1\mathbf{d}) \leq (1 - \frac{t_1}{t_2})f(\mathbf{x}) + \frac{t_1}{t_2} f(\mathbf{x} + t_2\mathbf{d})$.
- Consequently, $\frac{f(\mathbf{x}+t_1\mathbf{d})-f(\mathbf{x})}{t_1} \leq \frac{f(\mathbf{x}+t_2\mathbf{d})-f(\mathbf{x})}{t_2}$.
- Thus, $h(t_1) \leq h(t_2) \Rightarrow h$ is monotone nondecreasing over $\mathbb{R}_{++}$. All that is left is to show that it is bounded below over $(0, \epsilon]$.

Proof Contd.

- Take $0 < t \leq \epsilon$. Note that

$$\mathbf{x} = \frac{\epsilon}{\epsilon + t}(\mathbf{x} + t\mathbf{d}) + \frac{t}{\epsilon + t}(\mathbf{x} - \epsilon\mathbf{d}).$$

- Hence,

$$f(\mathbf{x}) \leq \frac{\epsilon}{\epsilon + t} f(\mathbf{x} + t\mathbf{d}) + \frac{t}{\epsilon + t} f(\mathbf{x} - \epsilon\mathbf{d}).$$

- After some rearrangement of terms,

$$h(t) = \frac{f(\mathbf{x} + t\mathbf{d}) - f(\mathbf{x})}{t} \geq \frac{f(\mathbf{x}) - f(\mathbf{x} - \epsilon\mathbf{d})}{\epsilon}.$$

- h is bounded below over $(0, \epsilon]$.
- Since h is nondecreasing and bounded below over $(0, \epsilon]$, the limit $\lim_{t \to 0^+} h(t)$ exists $\Rightarrow$ the directional derivative $f'(\mathbf{x}; \mathbf{d})$ exists.

$\square$

Outline

- Until now we have discussed functions that are real-valued, meaning that they take values in $\mathbb{R} = (-\infty, \infty)$.
- We will now consider functions taking values in $\mathbb{R} \cup \{\infty\} = (-\infty, \infty]$. Such functions are called extended real-valued functions.
- Example: the indicator function: given a set $S \subseteq \mathbb{R}^n$, the indicator function $\delta_S : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is given by

$$\delta_S(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in S, \\ \infty & \text{if } \mathbf{x} \notin S. \end{cases}$$

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- The effective domain of an extended real-valued function is the set of vectors for which the function takes a real value:

$$\mathrm{dom}(f) = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) < \infty\}.$$

- An extended real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is called proper if is not always equal to infinity, meaning that there exists $\mathbf{x}_0 \in \mathbb{R}^n$ such that $f(\mathbf{x}_0) < \infty$.

- An extended real-valued function is convex if for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\lambda \in [0,1]$ the following inequality holds:

$$f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}),$$

where we use the usual arithmetic rules with ∞ such as

$$a + \infty = \infty \text{ for any } a \in \mathbb{R},$$

$$a \cdot \infty = \infty \text{ for any } a \in \mathbb{R}_{++}.$$

In addition, we have the much less obvious rule that $0 \cdot \infty = 0$.

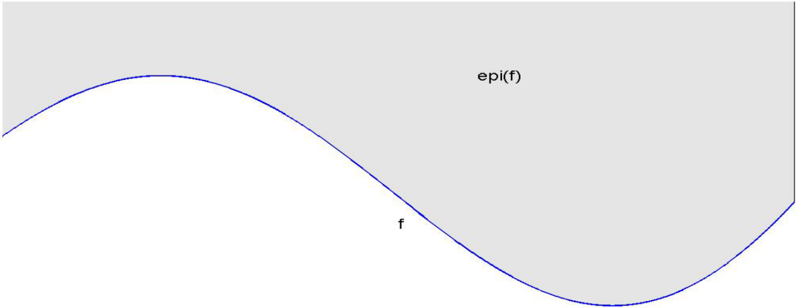
- It is easy to show that an extended real-valued function is convex iff $\mathrm{dom}(f)$ is a convex set and the restriction of f to its effective domain is a convex real-valued function over $\mathrm{dom}(f)$.
- As an example, the indicator function $\delta_C(\cdot)$ of a set $C \subseteq \mathbb{R}^n$ is convex if and only if C is a convex set.

Definition

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$. Then its epigraph $\text{epi}(f) \in \mathbb{R}^{n+1}$ is defined to be the set

$$\text{epi}(f) = \{(\mathbf{x}; t) : f(\mathbf{x}) \leq t\}.$$

It is not difficult to show that an extended real-valued function f is convex if and only if its epigraph set $\text{epi}(f)$ is convex.



Theorem

Let $f_i : \mathbb{R}^n \to \mathbb{R}$ be an extended real-valued convex function for any $i \in I$ (I being an arbitrary index set). Then the function $f(\mathbf{x}) = \sup_{i\in I} f_i(\mathbf{x})$ is an extended real-valued convex function.

Proof.

f_i convex for all $i \Rightarrow \text{epi}(f_i)$ convex $\Rightarrow \text{epi}(f) = \cap_{i\in I}\text{epi}(f_i)$ convex $\Rightarrow f(\mathbf{x}) = \sup_{i\in I} f_i(\mathbf{x})$ is convex. □

- Support Functions. Let $S \subseteq \mathbb{R}^n$. The support function of S is the function

$$\sigma_S(\mathbf{x}) = \sup_{\mathbf{y}\in S} \mathbf{x}^\top \mathbf{y}.$$

The support function is a convex function (regardless of whether S is convex or not).

Maximum of a Convex Function over a Compact Convex Set

Theorem

Let $f : C \to \mathbb{R}$ be convex and continuous over the nonempty convex and compact set $C \subseteq \mathbb{R}^n$. Then there exists at least one maximizer of f over C that is an extreme point of C.

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Let $f : C \to \mathbb{R}$ be convex and continuous over the nonempty convex and compact set $C \subseteq \mathbb{R}^n$. Then there exists at least one maximizer of f over C that is an extreme point of C.

Proof.

- Let $\mathbf{x}^*$ be a maximizer of f over C. If $\mathbf{x}^*$ is an extreme point of C, then the result is established. Otherwise,
- By Krein-Milman, $C = \mathrm{conv}(\mathrm{ext}(C)) \Rightarrow \exists \mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k \in \mathrm{ext}(C)$ and $\boldsymbol{\lambda} \in \Delta_k$ s.t. $\mathbf{x}^* = \sum_{i=1}^k \lambda_i \mathbf{x}_i$.
- By convexity of f, $f(\mathbf{x}^*) \leq \sum_{i=1}^k \lambda_i f(\mathbf{x}_i)$
- $\sum_{i=1}^k \lambda_i (f(\mathbf{x}_i) - f(\mathbf{x}^*)) \geq 0 \Rightarrow f(\mathbf{x}_i) = f(\mathbf{x}^*)$ (why?)

$\square$