3.4 :

- Calcul de $\int \frac{\operatorname{ch} x}{\operatorname{sh} x + \operatorname{ch} x} \cdot dx$.

Il est parfois bon de revenir aux exponentielles :

$$\int \frac{\operatorname{ch} x}{\operatorname{sh} x+\operatorname{ch} x} dx = \int \frac{e^{x}+e^{-x}}{2} \cdot \frac{1}{e^{x}} dx$$

$$= \int \frac{1}{2}(1+e^{-2x})\, dx$$

$$= \frac{1}{2}x - \frac{1}{4}e^{-2x}$$

- Calcul de $\int \frac{dx}{\operatorname{sh}^2 x+2}$

  . Si on avait à calculer

$$\int \frac{dx}{\sin^2 x+2}.$$

en remarquant que si $F(x) = \frac{1}{\sin^2 x+2}$, alors $F(x+\pi) = F(x)$, on effectuerait le changement de variable $t = \tan x$.
On effectue donc le changement de variable $t = \operatorname{th} x$ sur notre calcul d'origine :

$$\int \frac{dx}{\operatorname{sh}^2 x+2} = \dots$$

$t = \operatorname{th} x$
(Gros film intello de Georges Lucas avant Star Wars).

$dt = (1-\operatorname{th}^2 x)dx$
$dt = (1-t^2)dx$

$\operatorname{ch}^2 x - \operatorname{sh}^2 x = 1$
donc $\operatorname{ch}^2 x = 1+\operatorname{sh}^2 x$
donc $\operatorname{th}^2 x = \frac{\operatorname{sh}^2 x}{\operatorname{ch}^2 x} = \frac{\operatorname{sh}^2 x}{1+\operatorname{sh}^2 x}$
donc $\operatorname{th}^2 x(1+\operatorname{sh}^2 x) = \operatorname{sh}^2 x$
donc $\operatorname{sh}^2 x = \frac{\operatorname{th}^2 x}{1-\operatorname{th}^2 x}$.

Donc $\displaystyle\int \frac{dx}{sh^2x+2} = \int \frac{dt}{1-t^2} \cdot \frac{1}{\frac{t^2}{1-t^2}+2}$

$$= \int \frac{dt}{t^2+2(1-t^2)} = \int \frac{dt}{2-t^2}.$$

$$= -\int \frac{dt}{t^2-2}$$

Or $\displaystyle\frac{1}{X^2-2} = \frac{1}{(X-\sqrt{2})(X+\sqrt{2})}$

Il existe donc $\alpha, \beta \in \mathbb{R}$ tels que

$$\frac{1}{(X-\sqrt{2})(X+\sqrt{2})} = \frac{\alpha}{X-\sqrt{2}} + \frac{\beta}{X+\sqrt{2}}$$

On multiplie par $X-\sqrt{2}$ et on évalue en $\sqrt{2}$: $\alpha = 1/(2\sqrt{2})$
$X+\sqrt{2}$ ... $-\sqrt{2}$: $\beta = -1/2\sqrt{2}$.

Donc $\displaystyle\frac{1}{X^2-2} = \frac{1}{2\sqrt{2}} \cdot \left(\frac{1}{X-\sqrt{2}} - \frac{1}{X+\sqrt{2}}\right)$

Donc $\displaystyle\int \frac{dx}{sh^2x+2} = \frac{1}{2\sqrt{2}} \int \left(\frac{1}{t+\sqrt{2}} - \frac{1}{t-\sqrt{2}}\right) dt$

$$= \frac{1}{2\sqrt{2}} \cdot \left(\ln|t+\sqrt{2}| - \ln|t-\sqrt{2}|\right)$$

$$= \frac{1}{2\sqrt{2}} \cdot \ln\left|\frac{t+\sqrt{2}}{t-\sqrt{2}}\right|$$

$$= \frac{1}{2\sqrt{2}} \ln\left|\frac{th\,x+\sqrt{2}}{th\,x-\sqrt{2}}\right|$$

$\displaystyle\int \frac{dx}{sh^2x+2} = \frac{1}{2\sqrt{2}} \ln\left(\frac{\sqrt{2}+th\,x}{\sqrt{2}-th\,x}\right)$ car $th\,t \in]-1,1[$

3.5)

$$\int \frac{dx}{x\sqrt{x+1}}$$

$\forall x \in \mathbb{R} \quad x+1>0 \iff x>-1$

On se place donc sur un intervalle I de $]-1,0[\cup]0,+\infty[$.
On a :

$$\int \frac{dx}{x\sqrt{x+1}} = \int \frac{2\cancel{u}\,du}{(u^2-1).\cancel{u}} \qquad \begin{aligned} u &= \sqrt{x+1} \\ u^2 &= x+1 \\ 2u\,du &= dx. \end{aligned}$$

$$= 2\int \frac{du}{u^2-1}$$

On pose $F = \frac{1}{X^2-1} = \frac{1}{(X-1)(X+1)}$. Il existe $\alpha, \beta \in \mathbb{R}$ tels que :

$$\frac{1}{(X-1)(X+1)} = \frac{\alpha}{X-1} + \frac{\beta}{X+1}$$

On multiplie par $X-1$ et on évalue en 1 : $\alpha = 1/2$
$X+1$ ... -1 : $\alpha = -1/2$.

Donc

$$\frac{1}{X^2-1} = \frac{1}{2}\cdot\left(\frac{1}{X-1} - \frac{1}{X+1}\right)$$

Donc $\displaystyle\int \frac{dx}{x\sqrt{x+1}} = \int \left(\frac{1}{u-1} - \frac{1}{u+1}\right) du$

$$= \ln|u-1| - \ln|u+1|$$

$$= \ln\left|\frac{u-1}{u+1}\right| = \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right|$$

Donc $\displaystyle\int \frac{dx}{x\sqrt{x+1}} = \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right|$

- $\displaystyle\int \arctan\sqrt{1+x^2}\,dx = \int \underbrace{\arctan\sqrt{1+x^2}}_{\downarrow} \cdot \overset{\uparrow}{1}\, .dx.$

$$= x\arctan\sqrt{1+x^2} - \int x \cdot 2x \cdot \frac{1}{2}(1+x^2)^{-1/2} \frac{dx}{1+(\sqrt{1+x^2})^2}$$

$$= x\arctan\sqrt{1+x^2} - \int \frac{x^2\,dx}{\sqrt{1+x^2}\,(2+x^2)}$$

On pose $x = \operatorname{sh} t$. Alors $dx = \operatorname{ch} t . dt$. Donc.

$$\int \frac{x^2\,dx}{\sqrt{1+x^2}\,(2+x^2)} = \int \frac{\operatorname{sh}^2 t . \operatorname{ch} t\, dt}{\sqrt{1+\operatorname{sh}^2 t}\,(2+\operatorname{sh}^2 t)}.$$

$$= \int \frac{\operatorname{sh}^2 t . \operatorname{ch} t}{\operatorname{ch} t . (2+\operatorname{sh}^2 t)}\, dt$$

$$= \int \frac{\operatorname{sh}^2 t}{2+\operatorname{sh}^2 t} . dt$$

On effectue le changement de variable $u = \operatorname{th} t$. Alors $du = (1-u^2)dt$ donc.

$$\int \frac{x^2\,dx}{\sqrt{1+x^2}\,(2+x^2)} = \int \frac{\frac{u^2}{1-u^2}}{2+\frac{u^2}{1-u^2}} . \frac{du}{(1-u^2)}$$

$$= \int \frac{u^2}{(1-u^2)(2(1-u^2)+u^2)}\, du$$

$$= \int \frac{u^2 . du}{(1-u^2)(2-u^2)} = \int \frac{u^2\,du}{(u^2-1)(u^2-2)}.$$

On pose $F = \frac{X^2}{(X^2-1)(X^2-2)} = \frac{X^2}{(X-1)(X+1)(X-\sqrt{2})(X+\sqrt{2})}$

Il existe donc $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ tels que :

$$F = \frac{X^2}{(X^2-1)(X^2-2)} = \frac{\alpha}{X-1} + \frac{\beta}{X+1} + \frac{\gamma}{X-\sqrt{2}} + \frac{\delta}{X+\sqrt{2}}.$$

Or $F(-X) = F(X)$ donc.

$$F(X) = F(-X) = \frac{-\beta}{X-1} + \frac{-\alpha}{X+1} + \frac{-\delta}{X-\sqrt{2}} + \frac{-\gamma}{X+\sqrt{2}}.$$

Par unicité de la décomposition en éléments simples, $\alpha = -\beta$ et $\gamma = -\delta$.
On pose $P = (X^2-1)(X^2-2) = X^4 - 3X^2 + 2$. Donc
$P' = 4X^3 - 6X = X(4X^2-6)$.

On multiplie par $X-1$ et on évalue en 1 : $\alpha = \frac{1^2}{P'(1)} = -\frac{1}{2}$

———————— $X-\sqrt{2}$ ———————— $\sqrt{2}$: $\gamma = \frac{(\sqrt{2})^2}{P'(\sqrt{2})} = \frac{\cancel{2}}{\sqrt{2}\,(\cancel{2})} = \frac{1}{\sqrt{2}}$

Donc

$$F(X) = \frac{1}{2}\left(\frac{1}{X+1} - \frac{1}{X-1}\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{X-\sqrt{2}} - \frac{1}{X+\sqrt{2}}\right)$$

Donc

$$\int \frac{x^2\,dx}{\sqrt{1+x^2}\,(2+x^2)} = \frac{1}{2}\ln\left|\frac{u+1}{u-1}\right| + \frac{1}{\sqrt{2}}\ln\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right|$$

$$= \frac{1}{2}\ln\left|\frac{\operatorname{th} t+1}{\operatorname{th} t-1}\right| + \frac{1}{\sqrt{2}}\ln\left|\frac{\operatorname{th} t-\sqrt{2}}{\operatorname{th} t+\sqrt{2}}\right|$$

$$= \frac{1}{2}\ln\left(\frac{1+\operatorname{th} t}{1-\operatorname{th} t}\right) + \frac{1}{\sqrt{2}}\cdot\ln\left(\frac{\sqrt{2}-\operatorname{th} t}{\sqrt{2}+\operatorname{th} t}\right)$$

car $-1 < \operatorname{th} t < 1$

Or, $x = \operatorname{sh} t$ donc $\operatorname{th} t = \frac{\operatorname{sh} t}{\operatorname{ch} t} = \frac{\operatorname{sh} t}{\sqrt{1+\operatorname{sh}^2 t}} = \frac{x}{\sqrt{1+x^2}}$

Donc

$$\int \arctan\sqrt{1+x^2}\,dx = x\arctan\sqrt{1+x^2} + \frac{1}{2}\ln\left(\frac{\sqrt{1+x^2}-x}{\sqrt{1+x^2}+x}\right) + \frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{2(1+x^2)}+x}{\sqrt{2(1+x^2)}-x}\right)$$