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A GENERALIZATION OF MOSES RANK TEST

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Moses (1963) proposed a rank test to test for the equality of scale parameters when location parameters are possibly different in the two-sample problem. In this paper, we consider a generalization of Moses test for the two-sample problem and investigate their asymptotic properties. It is found that the proposed tests are useful in the practical applications.

1. Introduction

For the testing hypothesis in the two-sample problem, rank tests are useful when the underlying distributions are unknown or the standard assumptions of, e.g., normality, exponentiality and so on are doubtful. The rank tests are quite efficient if we can select suitable scores, see Hodges & Lehmann (1956) and Chernoff & Savage (1958). When the sample sizes are small, the rank tests are easily applied and the percentage points of well known rank statistics are already tabulated. However, if the sample sizes are not small, it is very cumbersome to calculate the exact sample significances and, in some cases, it is troublesome to get the statistics themselves. For example, if we want to use the normal score test when the total number of observations is larger than 50, then we can not get the normal scores from Biometrika Table (1962). As the number of observations increases, the number of ties increases and it becomes more troublesome to get the statistic and its variance.

Further, usual rank tests require the equality of the scale parameters of the samples in the location problem and the equality of locations in the scale problem. Thus, it is desirable to construct tests which are easy to apply and are not inefficient even if there are unequal nuisance parameters.

In this paper, we propose a generalization of Moses (1963). We believe that the proposed methods satisfy the above properties. Note that our generalization can be applied to multi-sample problem, test for symmetry, analysis of variance problems and so on.

2. Proposed method

Let $X_1, \cdots, X_{km}(Y_1, \cdots, Y_{kn})$ be a random sample from a population with continuous distribution function $F(x)(G(x))$. Suppose that we want to test whether F and G are the same in some sense.

Let $h(x_1, \cdots, x_k)$ be a function such that it is not constant and symmetric in its arguments. Consider

$$V_i = h(X_{(i-1)k+1}, \cdots, X_{ik}) \qquad i=1, \cdots, m$$

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and

$$W_j=h(Y_{(j-1)k+1}, \cdots, Y_{jk}) \qquad j=1, \cdots, n .$$

Denote by $F^*(x)$ and $G^*(x)$ the distribution functions of V_i's and W_j's, respectively. We assume that the transformation function $h(x_1, \cdots, x_k)$ are chosen so that F^* and G^* are continuous and our null hypothesis is H_0: $F^*(x)=G^*(x)$. Put $N=m+n$ and $M=km+kn$. Let $a_N(1), \cdots, a_N(N)$ be scores constants such that they satisfy

$$\int_0^1 \{a_N(1+[uN])-L(u)\}^2 du \to 0 \quad \text{as } N\to\infty$$

for some non-constant square integrable function $L(u)$. Let R_i be the rank of V_i among $\{V_1, \cdots, V_m, W_1, \cdots, W_n\}$ and the statistic we want to consider is

$$S=a_N(R_1)+\cdots+a_N(R_m) .$$

If the model to be considered has nuisance parameters but allows adequate function h, then the distribution of S under the null hypothesis will be independent of $F(x)$ and $G(x)$ even if the hypothesis does not necessarily imply $F(x)=G(x)$. The cases are given in Sections 4 and 5.

3. Asymptotic properties of S and its efficiency

In this section, let us consider the asymptotic normality of S. Assume that $F^*(x)$ and $G^*(x)$ has density functions $f^*(x)$ and $g^*(x)$, respectively and that $g^*(x)=f^*(x;\theta)$ and $f^*(x)=f^*(x;0)$ for a one parameter family of density functions. Thus our null hypothesis is given by H_0: $\theta=0$. Furthermore, it is assumed that $f^*(x;\theta)$ has finite Fisher information

$$0<\int\left(\frac{\partial}{\partial\theta}\log f^*(x;\theta)\right)^2 f^*(x;\theta)\,dx\,\bigg|_{\theta=0}<\infty .$$

Now, let us consider a sequence of alternative hypotheses

$$H_M:\quad \theta=\theta_M \tag{3.1}$$

and suppose

$$M\theta_M^2 \to b^2 \qquad \text{as } M\to\infty$$

and

$$0<b^2<\infty .$$

Then, along with the well known method of Hajek & Sidak (1967, Ch. 6), we can show that

$$S-E_0(S)\sim N\left(m_s, \frac{mn}{N}\sigma^2(L)\right)$$

where the expectation E_0 is taken under H_0 and

$$m_s=\frac{mn}{N}\theta_M\int_0^1 L(u)L(u;f^*)\,du ,$$

and

$$\sigma^2(L)=\int_0^1 L^2(u)\,du-\left(\int_0^1 L(u)\,du\right)^2$$

$$L(u;f^*)=\frac{\partial}{\partial\theta}\log f^*(x;\theta)\Big|_{x=F^{*-1}(u),\,\theta=0}$$

As a competitor of S, let us consider usual linear rank statistics. Let Q_i be the rank of X_i among all the observations $\{X_1, \cdots, X_{km}, Y_1, \cdots, X_{kn}\}$. Let $\{b_M(1), \cdots, b_M(M)\}$ be a set of constants such that it satisfy

$$\int_0^1\{b_M(1+[uM]-K(u)\}^2du\to 0 \quad \text{as } M\to\infty$$

for some non-constant square integrable function $K(u)$ and consider usual rank statistic

$$T=b_M(Q_1)+\cdots+b_M(Q_{km}) .$$

We consider the case that $F(x)$ and $G(x)$ have density functions $f(x)$ and $g(x)$, respectively and that $g(x)=f(x;\theta)$ so that $f(x)=f(x;0)$ i.e., there are no nuisance parameters. Assume that

$$0<\int\left(\frac{\partial}{\partial\theta}\log f(x;\theta)\right)^2 f(x;\theta)\,dx\Big|_{\theta=0}<\infty$$

and consider the sequence of alternatives H_M given in (3.1). Then we have

$$T-E_0(T)\sim N\left(m_T,\ \frac{kmn}{N}\sigma^2(K)\right)$$

where

$$m_T=\frac{kmn}{N}\theta_M\int_0^1 K(u)L(u;f)\,du ,$$

$$\sigma^2(K)=\int_0^1 K^2(u)\,du-\left(\int_0^1 K(u)\,du\right)^2$$

and

$$L(u;f)=\frac{\partial}{\partial\theta}\log f(x;\theta)\Big|_{x=F^{-1}(u),\,\theta=0} .$$

From the above results, the Pitman efficiency of the test based on S with respect to that based on T is given by

$$e(S,T)=\frac{\sigma^2(K)}{k\sigma^2(L)}\left(\int_0^1 L(u)L(u,f^*)du\Big/\int_0^1 K(u)L(u;f)\,du\right)^2 . \tag{3.2}$$

In the rest of this paper we investigate the performance of the test based on S.

4. Location problem

4.1 Common scale parameters case

Consider the case that $f(x;\theta)=f(x-\theta)$. Then a reasonable function $h(x_1, \cdots, x_k)$ will be

$$h(x_1,\cdots,x_k)=\sum_{i=1}^k x_i . \tag{4.1}$$

Denote by $f_k(x)$ the k-times convolution of $f(x)$. Then we have

$$f^*(x;\theta)=f_k(x-k\theta) .$$

Put

$$\psi(u;f)=-f'(F^{-1}(u))/f(F^{-1}(u)) .$$

Then we have

$$e(S,T)=\frac{k\sigma^2(K)}{\sigma^2(L)}\left(\int_0^1 L(u)\psi(u,f_k)\,du\Big/\int_0^1 K(u)\psi(u;f)\,du\right)^2 .$$

It is easily shown that when $L(u)=K(u)$ and the population distribution is normal, $e(S, T)=1$. Thus, if we use the same scores, S is asymptotically equally efficient with T for the normal distribution.

If both S and T are the Wilcoxon statistic i.e., $L(u)=K(u)=u$, then

$$e(S,T)=K\left(\int_{-\infty}^{\infty} f_k^2(x)\,dx\Big/\int_{-\infty}^{\infty} f^2(x)\,dx\right)^2 . \tag{4.2}$$

If we can get $f_k(x)$, (4.2) can be computed. For example, when the population distribution is Double-exponential i.e.,

$f(x)=\frac{1}{2}\exp(-|x|)$, we have

$$f_k(x)=\frac{1}{\Gamma^2(k)2^k}\sum_{j=0}^{k-1}\begin{bmatrix}k-1\\ j\end{bmatrix}\Gamma(k+j)2^{-j}|x|^{k-1-j}\exp(-|x|) .$$

In this case, we have $e(S, T)=.781$ for $k=2$, $e(S, T)=.727$ for $k=3$ and the efficiency decreases as k increases.

The function

$$h(x,\cdots,x_k)=\text{med}\,x_i \tag{4.3}$$

will be also reasonable for $k>2$. When $L(u)=K(u)=u$ and $k=3$,

$$e(S,T)=432\left(\int_{-\infty}^{\infty} F^2(x)(1-F(x))^2 f^2(x)dx\Big/\int_{-\infty}^{\infty} f^2(x)\,dx\right)^2 . \tag{4.4}$$

In the Double-exponential case, (4.4) is .9075 and is larger than .727. Thus, for the distributions with heavy tails, (4.3) will be more efficient than (4.1).

4.2 Different scale parameters case

Consider the model

$$g(x)=f((x-\theta)/\sigma) , \tag{4.5}$$

and the null hypothesis: $\theta=0$ then (4.1) and (4.3) are inadequate. In this case,

$$h(x_1,\cdots,x_k)=\bar{x}/S_x \tag{4.6}$$

or

$$h(x_1,\cdots,x_k)=\text{med}\,x_i/(x_{(k-j+1)}-x_{(j)}) \qquad \text{for some } j<\left[\frac{k}{2}\right]+1 \tag{4.7}$$

will be reasonable. Here, $\bar{x}=\Sigma x_i/k$, $S_x^2=\Sigma(x_k-\bar{x})^2/(k-1)$ and med x_i is the median of $x_{(1)}<\cdots<x_{(k)}$ which are the ordered $x_1,\cdots,x_k$. In these cases, it is very difficult to compute the Pitman efficiency.

Now, let us consider the performance of our rank tests based on (4.6) or (4.7). The case $m=n=10$, $k=3$ is considered and we choose Normal distribution, Logistic distribution and Double-exponential distribution as the population distributions. As rank statistics, the normal score statistic and the Wilcoxon statistics are investigated. As a parametric competitor, we consider the Welch test. Fixing the model (4.5) with $\sigma=1/2$ we performed computer simulations for various θ. The critical points of the normal score statistic are taken from Klotz (1964) and that of the Wilcoxon statistic are taken from Hollander & Wolfe (1973) and Yanagawa (1981). The critical points of the Welch test can be approximated by t-distribution with adequate degrees of freedom.

The computations are repeated 1,000 times for selected values of θ and the alternative considered is: $\theta>0$. Some of the results are given in Table 1. The entries of the table are the number of times that the null hypothesis is rejected. Since the distributions of rank statistics are discrete, we used the randomization method in order to get the exact significance level of the normal score test and the Wilcoxon test. In Table 1, Wilc. 1 and NS. 1 refer to the Wilcoxon test and the normal score test, respectively for the function (4.6) and Wilc. 2 and NS. 2 correspond to (4.7). The notations N, L and DE means Normal, Logistic and Double-exponential, respectively. The case $\theta=0$ corresponds to the null hypothesis. From Table 1, we can conclude that the t-approximation of the critical points of the Welch test is not accurate and hence the Welch test is inadequate. It is also found that the function (4.6) gives nice powers.

We performed further simulations by using other series of random numbers in order to check upon the reliability of our simulation and we got similar results.

Table 1. 5% powers of Moses type tests.

		$\theta=0$		$\theta=.1$	.2	.3	$\theta=.2$	.4	.5
	N	L	DE	N	L	DE	N	L	DE
Welch	89	90	87	112	130	143	155	165	186
Wilc. 1	53	58	57	91	110	148	153	187	292
Wilc. 2	54	43	42	79	81	112	111	140	227
NS. 1	58	56	56	85	106	146	149	178	297
NS. 2	56	40	40	76	85	111	111	145	235
	$\theta=.3$	.6	.6	$\theta=.4$	.8	.8	$\theta=.5$	1	1
	N	L	DE	N	L	DE	N	L	DE
Welch	196	199	245	237	241	324	287	301	407
Wilc. 1	234	288	468	349	427	625	465	549	741
Wilc. 2	176	221	364	249	310	527	330	413	659
NS. 1	227	287	476	333	423	625	448	551	754
NS. 2	170	234	395	254	334	548	336	451	685

5. Scale problem

Let us consider the case

$$f(x;\theta)=\exp(-\theta)f(\exp(-\theta)x)\,, \qquad -\infty<x<\infty\,.$$

Here, for simplicity, we assume that the samples have the same location parameters, although the functions given below do not require the common location parameters. In this case, the function

$$h(x_1, \cdots, x_k)=\sum_{i=1}^{k}(x_i-\bar{x})^2\ , \qquad \bar{x}=\sum_{i=1}^{k}x_i/k \tag{5.1}$$

given by Moses (1963) will be reasonable. Put $f_{1k}(x)$ the density function given by $f(x)=f(x;0)$. Then we have

$$f_{1k}(x;\theta)=\exp(-2\theta)f_{1k}(\exp(-2\theta)x))\ ,$$

$$L(u;f_{1k})=-2-2xf_{1k}(x)/f_{1k}(x)\Big|_{x=F_{1k}^{-1}(u)}\ ,$$

and

$$L(u;f)=-1-F^{-1}(u)f'(F^{-1}(u))/f(F^{-1}(u))\ .$$

Here, $F_{1k}(x)$ is the distribution function of $f_{1k}(x)$. The efficiency (3.2) can be calculated as (4.2).

If $f(x)$ is the density function of $N(0, 1)$, then (5.1) induces chi-squared distribution χ^2_{k-1} with degrees of freedom $k-1$. Hence we have

$$L(u;f_{1k})=F_{1(k)}^{-1}(u)-k+1\ .$$

Here, $F_{1(k)}$ is the distribution function of χ^2_{k-1}. Consider the case $L(u)=(\Phi^{-1}(u))^2$ and $K(u)=L(u;f_{1k})$. These give the optimal scores for S and T. Then

$$e(S,T)=\frac{k-1}{k}\ .$$

Thus, when k, m and n are sufficiently large, the information loss due to the use of (5.1) is not serious. Further, the distribution of S is not affected from the discrepancy between the locations parameters, but that of T is not the case. Moses (1963) proposed to adopt $L(u)=K(u)=u$. Shorack (1969) investigated the Moses test and the estimator based on the test, and he called the procedures useful inefficient method. It can be said that Moses method is not inefficient if we use optimal score generating function.

The functions

$$h(x_1, \cdots, x_k)=\max x_i-\min x_i$$

and

$$h(x_1 \cdots, x_k)=x_{(k+1-j)}-x_{(j)} \qquad \text{for some } j<\left[\frac{k}{2}\right]+1$$

will be also reasonable. These functions give location-free procedures.

6. Scale slippage problem

Consider the model

$$\begin{aligned} f(x;\theta)&=\exp(-\theta)f(\exp(-\theta)x) && \text{if } x>0 \\ &=0 && \text{otherwise}\ . \end{aligned}$$

Then the function

$$h(x_1, \cdots, x_k) = \sum_{i=1}^{k} \log x_i \tag{6.1}$$

converts the scale-slippage problem to the location problem. Thus, we can use the results in Section 4. Hence the efficiency $e(S, T)$ can be obtained from (4.2) with replacing $f_k(x)$ with $f_{X(k)}(x)$ the density function given by the k-times convolution of the density function $\exp(x) f(\exp(x))$, $-\infty < x < \infty$. Therefore, if we use $L(u) = K(u)$ and the population distributions are Log-normal, then it always holds that

$$e(S, T) = 1 .$$

If the population distribution is exponential, then the function

$$h(x_1, \cdots, x_k) = \max x_i \tag{6.2}$$

will give powerful test. Shanubhogue & Gore (1985) used (6.2) and constructed U-statistics.

7. Remarks

In the previous sections, k is fixed and we applied the asymptotic results with $m, n \to \infty$ and computed the Pitman efficiency. For the practical data, it is desired that m and n are so large that the asymptotic normality holds but small to make possible to compute the exact sample significance. For example, when we want to use the probability tables in Hollander & Wolfe (1973) or Yanagawa (1981), m and n are smaller than 10 but should be as large as possible. If one of m and n are small then it is possible to compute the exact sample significance. However, in this case, the asymptotic results can not be applied.

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