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# An Ising spin state explanation for financial asset allocation

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## HIGHLIGHTS

- Continuous spin Ising model.
- Identification of fractional asset holding as spin degree of freedom.
- Partial moments produce continuously or discretely varying spins.
- Partial moments determined by Hamiltonian.

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## ABSTRACT

We build on the developments in the application of statistical mechanics, notably the identity of the spin degree of freedom in the Ising model, to explain asset price dynamics in financial markets with a representative agent. Specifically, we consider the value of an individual spin to represent the proportional holdings in various assets. We use partial moment arguments to identify asymmetric reactions to information and develop an extension of a plunging and dumping model. This unique identification of the spin is a relaxation of the conventional discrete state limitation on an Ising spin to accommodate a new archetype in Ising model-finance applications wherein spin states may take on continuous values, and may evolve in time continuously, or discretely, depending on the values of the partial moments.



## 1. Introduction: Ising spin and parallels to financial decision

The Ising model [1,2] is a theoretical construction centered around a fundamental, yet simple and generally applicable, mode of interaction between two particles. It was first constructed in an attempt to mimic the temperature-dependent ferromagnetic phase transition, but has found application in a wide variety of fields wherein the interaction between fundamental entities can be realistically modeled by the simple Ising construct. In Ising's original work, the energy $E$ of interaction between two magnetic particles, $i$ and $j$, is given by

$$E = -Js_i s_j, \tag{1}$$

where $s$ represents a particle's quantum mechanical intrinsic spin, and the proportionality constant $J$ is an element- and position-dependent "exchange energy". Due to the quantized nature of intrinsic spin, $s = \{-1, +1\}$, where $-1$ corresponds to a "spin down" state, and $+1$ a "spin up" state. Thus, the energy of interaction is minimized when the spins of two

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interacting particles are parallel; that is, when they both have the same value of $s$. Furthermore, magnetic particles can interact with an external magnetic field $H$ with energy

$$E = -Hs_i, \tag{2}$$

where $s_i$ is the spin of the $i$th particle. This energy is minimized if the particle's spin lies in the same direction as the applied field. If we consider the total energy of interaction between all the particles in a system, and include an external field, we have

$$E_{tot} = -J\sum_{i,j} s_i s_j - H\sum_i s_i, \tag{3}$$

where the double sum over $i$ and $j$ can, in one extreme, include only nearest neighbors (for short-range interactions); or, in the other extreme, can include interactions between every pair of spins in the system.

Obviously, a necessary step in applying an Ising-type model to solve a problem in a non-physics field is the identification of some property analogous to the intrinsic spin. The Ising model has in the last few decades been applied to explain certain aspects of financial markets [3–14], and the most commonly used analog to spin is the market sentiment of a rational agent. The quantization of spin/sentiment states is maintained by letting bullish sentiment be equivalent to a state $s = +1$, and bearish sentiment equivalent to $s = -1$. A three-state system, including $s = 0$ for neutral sentiment, has also been studied [15].

The identification of parallels between physical systems with a large number of degrees of freedom and financial markets, such as the quantization of market sentiment, has paved the way for the collective wisdom of years of application of the Ising model in physics to be suitable for describing market dynamics. For example, some of the most successful computational tools, that have been borrowed from physics and have enabled the extraction of useful results from Ising-type models applied to finance, are Monte Carlo algorithms that efficiently navigate through the complicated phase space of a complex system, and produce a Boltzmann distribution in equilibrium. The most popular of these types of Monte Carlo simulations is the Metropolis [16] algorithm. In the Logit model of decision theory, the probability of choosing decision $x$ over all alternatives has the same form as the Boltzmann distribution [3] with energy replaced by the negative of the decision maker's normative utility function, thus providing an important justification for the use of the Metropolis algorithm in the modeling of financial markets.

Spin model applications in finance have produced insightful results, but to date their buyer/seller characterizations allow for only a limited, discrete number of states. In this work, in breaking with Ising-finance tradition, we forward the prospect of extending the applicability of Ising-type models in finance by letting the value of an individual Ising spin represent the optimized fraction of holdings in a simple two-asset portfolio. This identification of the spin differs from previous Ising-finance models in two fundamental ways: (i) the optimized asset holdings fraction, which we denote as $R^*$, is not a direct measure of agent sentiment, but does maintain some indirect, utility-based measure of investment sentiment; and (ii) the value of the agent's spin, i.e. $R^*$, is allowed to take on continuous, as well as discrete, values. The preferential differences between two agents who have the same utility function would cause them to distribute their investment capital differently amongst two assets. This difference in holdings is likened to the Ising spin, while the differences in accessibility and processing ability of information may be likened to the Hamiltonian that governs the spin dynamics. Our approach is built on the development of preferences of direction of Scott and Horvath [17], the plunging and dumping model of Horvath and Scott [18], and the asymmetry of preferences of Horvath and Sinha [19]. This spin identification, $s = R^*$ allows for continuous changes in spin, while also allowing for discrete "jumps" which may ultimately explain rapid changes in market direction, reflected in observed bubbles and crashes. In this paper, we limit the description of this new type of spin degree of freedom to its general dependence on an agent's utility.

## 2. Cubic utility with single investor/particle

Similar to Horvath and Scott [18], consider an investor facing a decision regarding the fraction of holding between two risky assets, $X$ and $Y$. The investor chooses the fraction $R^*$, from the possible proportions $R$ $(R : [0, 1])$, of his investable funds to hold in asset $X$ and $(1 - R^*)$ in asset $Y$. The $R^*$ is selected by the investor to maximize his expected utility of investment $E(U)$, where $E(U) = f(R : Q)$, and $Q$ is a vector of risk factors proxied by statistical moments, partial or total, but beyond the control of the investor. $R$ is determined by the investor; $R$ determined by the investor as his holding in the risky assets, represents the spin state in the Ising model, such that:

$$R^* \equiv \max_R E(U) = f(R : Q). \tag{4}$$

Horvath and Scott [18] show that there is a possibility of discontinuous behavior when there is more than one solution to (4). When there are two moments in a quadratic utility function they show that discontinuities cannot happen. Quadratic utility is given by

$$U = a_0 + a_1 Z + a_2 Z^2, \tag{5}$$

where $\alpha_1 > 0$ for positive marginal utility, $\alpha_2 < 0$ for diminishing marginal utility, and $Z = RX + (1-R)Y$, with $X$ and $Y$ independent. They further demonstrate that

$$U = a_0 + a_1\mu_Y + a_2\mu_Y + a_2\sigma_Y + [a_1(\mu_X - \mu_Y) + 2a_2\sigma_Y(\mu_X - \mu_Y) + 2a_2(\sigma_X^2 - \sigma_Y^2)]R + [a_2(\mu_X - \mu_Y)^2 + a_2(\sigma_X^2 - \sigma_Y^2)]R^2, \tag{6}$$

which is quadratic in $R$, and has only one maximum with no possibility of a discontinuous change in $R$.

Horvath and Scott [18] then show that for a cubic utility function

$$U = a_0 + a_1 Z + a_2 Z^2 + a_3 Z^3, \tag{7}$$

with coefficients $\alpha_1 > 0, \alpha_2 < 0$, and $\alpha_3 > 0$, per Scott and Horvath [17]. With the introduction of $\lambda_Z = R^3\lambda_X + (1-R)^3\lambda_Y$ one obtains the cubic expected utility

$$E(U) = b_0 + b_1 R + b_2 R^2 + b_3 R^3, \tag{8}$$

where

$$\begin{aligned} b_0 &= a_0 + a_1\mu_Y + a_2(\sigma_Y^2 + \mu_Y^2) + a_3(\lambda_Y - 3\mu_Y\sigma_Y^2 - \mu_Y^3), \\ b_1 &= a_1(\mu_X - \mu_Y) + 2a_2(\mu_X\mu_Y - \sigma_Y^2 - \mu_Y^2) + 3a_3[-\lambda_Y - \sigma_Y^2(\mu_X + \mu_Y) - \mu_Y^2(\mu_X - \mu_Y)], \\ b_2 &= a_2(\sigma_X^2 + \sigma_Y^2 + (\mu_X - \mu_Y)^2) + 3a_3[\lambda_Y - \mu_Y(\sigma_X^2 - \sigma_Y^2) + (\mu_X - \mu_Y)\sigma_Y^2 + \mu_Y(\mu_X - \mu_Y)^2], \quad \text{and} \\ b_3 &= a_3[(\lambda_X - \lambda_Y) - 3(\mu_X - \mu_Y)(\sigma_X^2 + \sigma_Y^2) + (\mu_X - \mu_Y)^3]. \end{aligned} \tag{9}$$

Expected utility in (8) is cubic in $R$, and reflects the potential for discontinuous changes in $R$ even with continuous changes in the parameters.

## 3. Utility and partial moments

The two asset approach is intuitive and continued here. If one asset, say $X$, is sold the amount received does not just disappear. It will be placed in another asset such as cash, an asset with different parameters, or consumed, say asset $Y$. If asset $X$ is purchased, the purchase is funded from cash or the divestment of asset $Y$. It is this dynamic that we maintain is the basis for the continual or discrete jumps in $R$, the analog of the spin in the Ising model. For the Ising framework depicted in (3), we consider and extend the moment preference order of Scott and Horvath [17] by incorporating the upper and lower partial moments of Bawa [20], Fishburn [21], and Horvath and Sinha [19]. The expected utility is parameterized with $\omega$, the sum of the constant initial wealth and a random variable $Z$, as defined by Horvath and Sinha [19], up to and including the third moment: $\mu_{\omega X,l}$, $\mu_{\omega X,u}$, the first lower and upper partial moments of $\omega X$ respectively; and $\mu_{\omega Y,l}$, $\mu_{\omega Y,u}$, the first lower and upper partial moments of $\omega Y$. $\sigma^2_{\omega X,l}$ and $\sigma^2_{\omega X,u}$ are the second lower and upper partial moments of $\omega X$, and $\sigma^2_{\omega Y,l}$ and, $\sigma^2_{\omega Y,u}$ are the second lower and upper partial moments of $\sigma Y$. The notations of $\omega X$ and $\omega Y$ are interpreted as $\omega$ for $X$ and an $\omega$ for $Y$, with $\lambda_{\omega X,l}$ and $\lambda_{\omega X,u}$ providing the lower and upper third partial moments for $X$ and, finally, $\lambda_{\omega Y,l}$ and $\lambda_{\omega Y,u}$ representing the lower and upper third partial moments of asset $\omega Y$. We consider utility to be

$$E[U_\omega] = U[E(\omega_l)] + U[E(\omega_u)] + \frac{U^2_{\omega_l}}{2}m_l^2 + \frac{U^2_{\omega_u}}{2}m_u^2 + \frac{U^3_{\omega_l}}{6}m_l^3 + \frac{U^3_{\omega_u}}{6}m_u^3, \tag{10}$$

where $U^i = \partial^i U(W)/\partial W^i$ is the $i$th partial derivative of $U$, and $m^i = E[W - E(W)]^i$ is the $i$th central moment. The subscripts $l$ and $u$ denote lower and upper partial moments, respectively. Following Scott and Horvath [17] and Horvath and Sinha [19]

$$\begin{aligned} &\left|U^2_{\omega_l}\right| > \left|U^2_{\omega_u}\right|, \\ &U^2_{\omega_l} < 0, \\ &U^2_{\omega_u} < 0, \quad \text{and} \\ &U^3_{\omega_l} > U^3_{\omega_u}. \end{aligned} \tag{11}$$

We extend (4) to take into account partial moments,

$$R^* \equiv \max_R E(U) = f(R : Q_l, Q_u). \tag{12}$$

For $X_l$, $Y_l$, $X_u$, and $Y_u$ all independent, we then have the following utility

$$\begin{aligned} E[U_\omega] = {} & U(\mu_{\omega X,l}R + \mu_{\omega Y,l}(1-R)) + U(\mu_{\omega X,u}R + \mu_{\omega Y,u}(1-R)) \\ & + \frac{1}{2}[U^2_{\omega X_l}\sigma^2_{\omega X,l}R^2 + U^2_{\omega Y_l}\sigma^2_{\omega Y,l}(1-R)^2] + \frac{1}{2}[U^2_{\omega X_u}\sigma^2_{\omega X,u}R^2 + U^2_{\omega Y_u}\sigma^2_{\omega Y,u}(1-R)^2] \\ & + \frac{1}{6}[U^3_{\omega X_l}\lambda_{\omega X,l}R^3 + U^3_{\omega Y_l}\lambda_{\omega Y,l}(1-R)^3] + \frac{1}{6}[U^3_{\omega X_u}\lambda_{\omega X,u}R^3 + U^3_{\omega Y_u}\lambda_{\omega Y,u}(1-R)^3]. \end{aligned} \tag{13}$$

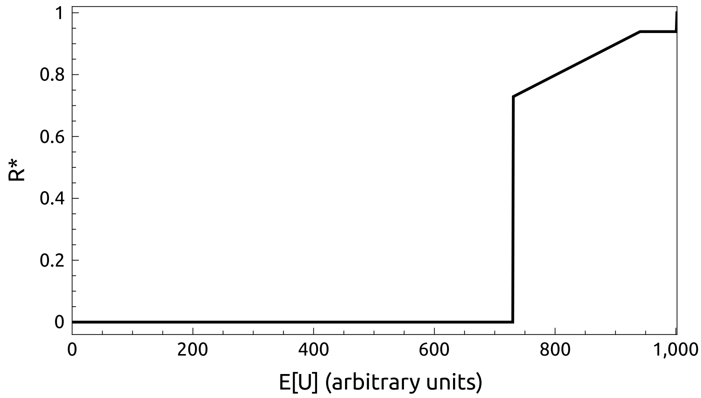


**Fig. 1.** $R^*$ as a function of $E(U)$ generated from (17). $Y$ is a certain cash payment (=\$2.00), and $X$ is an uncertain cash payment, with $X_u = \$35.00$ and $X_l = \$0.10$. $Pr(X_u) = 0.1$ and $Pr(X_l) = 0.9$, using $W^a$ as a von Neumann–Morgenstern [22] Utility function.

Eq. (13) can be cast into a different form,

$$E[U] = b_0 + b_1 R + b_2 R^2 + b_3 R^3, \tag{14}$$

where

$$\begin{aligned} b_0 &= U^3_{\omega Yl}\lambda_{\omega Y,l} - U^2_{\omega Yl}\sigma^2_{\omega Y,l} - U^2_{\omega Yu}\sigma^2_{\omega Y,u} + U(\mu_{\omega X,l}R + \mu_{\omega Y,l}(1-R)) + U(\mu_{\omega X,u}R + \mu_{\omega Y,u}(1-R)), \\ b_1 &= U^2_{\omega Yl}\sigma^2_{\omega Y,l} + U^2_{\omega Yu}\sigma^2_{\omega Y,u} - U^3_{\omega Yl}\lambda_{\omega Y,l} - U^3_{\omega Yu}\lambda_{\omega Y,u}, \\ b_2 &= U^3_{\omega Yl}\lambda_{\omega Y,l} - U^2_{\omega Xl}\sigma^2_{\omega X,l} - U^2_{\omega Xu}\sigma^2_{\omega X,u} + U^3_{\omega Yu}\lambda_{\omega Y,u} - U^2_{\omega Yl}\sigma^2_{\omega Y,l} - U^2_{\omega Yu}\sigma^2_{\omega Y,u}, \quad \text{and} \\ b_3 &= U^3_{\omega Xu}\lambda_{\omega X,u} + U^3_{\omega Xl}\lambda_{\omega X,l} - U^3_{\omega Yu}\lambda_{\omega Y,u} - U^3_{\omega Yl}\lambda_{\omega Y,l}. \end{aligned} \tag{15}$$

Eqs. (14) and (15) are cubic in the decision variable $R$ and, therefore, have the potential for discrete "jumps" in allocation or spin. Let $P_X$ and $P_Y$ denote the prices of assets $\omega X$ and $\omega Y$, respectively, and let

$$\begin{aligned} &\frac{\partial P_X}{\partial R} > 0, \quad \text{and} \\ &\frac{\partial P_Y}{\partial (1-R)} > 0. \end{aligned} \tag{16}$$

Following (16), prices of the alternative assets follow the von Neumann–Morgenstern utility [22] maximizing selection of $R$ and thus $(1-R)$.

## 4. Example

The generalized spin model may explain why markets move continuously or in discrete jumps depending on the relative values of the parameters of the model in terms of upper partial moments as may be seen in this example. Consider a simple power utility function and two assets, $X$ and $Y$. Let $Y$ be a cash asset and $X$ be an asset with uncertain outcomes (similar to Ref. [18]). Eliminating the variance and skew, and other extraneous terms for $Y$ from (14), and developing (15) in terms of $R$, we have

$$E(U) = U(Y)(1-R) + U(\mu_{\omega X,u})R + U(\mu_{\omega X,l})R + \frac{1}{2}(U^2_{\omega X_l}\sigma^2_{\omega X,l} + U^2_{\omega X_u}\sigma^2_{\omega X,u})R^2 + \frac{1}{6}(U^3_{\omega X_l}\lambda_{\omega X,l} + U^3_{\omega X_u}\lambda_{\omega X,u})R^3. \tag{17}$$

Utility-maximizing $R^*$ is plotted as a function of expected utility in Fig. 1. For these parameter values, the utility-maximizing $R$ starts at zero and remains so until a discrete jump to approximately 72% of investment in asset $X$ occurs. During this initial $R^* = 0$ the utility is such that all of the investment is in asset $Y$ with $R = 0$ and $(1-R) = 1$. Eventually, the higher variance and skew increases of the utility contribution of asset $X$ overwhelms the utility provided by $Y$, at which time the investor discretely revises the mix of assets $X$ and $Y$ from $R = 0$ in $X$ to approximately $R = 0.72$ in $X$.

In Fig. 2 the combination of moments in (17) produces a linear $R^*$ behavior without discontinuous jumps, reflecting normal, continuous changes in utility-maximizing allocation to $X$.

The differences in the behavior of $R^*$, as observed in Figs. 1 and 2, i.e. discontinuously vs. continuously increasing $R^*$, are due to the differences in the partial moments. In the framework of the Ising model, in particular the total energy expression of (3), the values of these moments would be directly determined by either, or both, of the interaction terms, the agent–agent exchange $J$, and by any influences $H$ on the market *ab extra*.

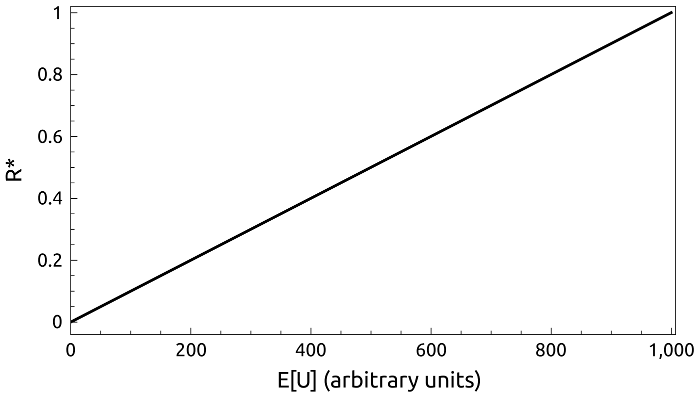


**Fig. 2.** $R^*$ as a function of $E(U)$ generated from (17). $Y$ is a certain cash payment (=\$2.00), and $X$ is an uncertain cash payment, with $X_u = \$30.00$ and $X_l = \$0.10$. $Pr(X_u) \approx 1$ and $Pr(X_l) \approx 0 (=1 \times 10^{-8})$, using $W^a$ as a von Neumann–Morgenstern [22] Utility function.

## 5. Conclusion

We have identified a candidate, $s = R^*$, where $R^*$ is the optimized fraction of agent holdings in a two-asset portfolio, for the spin degree of freedom in a continuous Ising-finance model, and have demonstrated the dependence of the spin on the characteristic moments on which the agent utility depends. We have allowed for asymmetric spin reactions, by including partial upper and lower moments, to the plunging and dumping model set forth by Horvath and Scott [18]. We have applied the Ising spin framework to the asset allocation of a representative agent, and demonstrated the possibility for continuous spin states within the extended plunging and dumping model [18]. Our example shows that, depending on the value of the moments, spin states may also change continuously, or discretely. It now remains to develop a model of the connection between the interaction terms of the spin system energy and the partial moments of agents' utility function. This work can also be further extended by considering the dynamical behavior of a large number of market participants instead of a representative agent, and by including a large number of financial assets instead of just two, as we have done in this paper.

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