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Importance of positive feedbacks and overconfidence in a self-fulfilling Ising model of financial markets

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Abstract

Following a long tradition of physicists who have noticed that the Ising model provides a general background to build realistic models of social interactions, we study a model of financial price dynamics resulting from the collective aggregate decisions of agents. This model incorporates imitation, the impact of external news and private information. It has the structure of a dynamical Ising model in which agents have two opinions (buy or sell) with coupling coefficients, which evolve in time with a memory of how past news have explained realized market returns. We study two versions of the model, which differ on how the agents interpret the predictive power of news. We show that the stylized facts of financial markets are reproduced only when agents are overconfident and mis-attribute the success of news to predict return to herding effects, thereby providing positive feedbacks leading to the model functioning close to the critical point. Our model exhibits a rich multifractal structure characterized by a continuous spectrum of exponents of the power law relaxation of endogenous bursts of volatility, in good agreement with previous analytical predictions obtained with the multifractal random walk model and with empirical facts.

Keywords: Ising model; Overconfidence; Imitation and herding; Econophysics

1. Introduction

Many works borrow concepts from the theory of the Ising models and of phase transitions to model social interactions and organization (see, e.g., Refs. [1,2]). In particular, Orléan [3–8] has captured the paradox of combining rational and imitative behavior under the name "mimetic rationality," by developing models of

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mimetic contagion of investors in the stock markets which are based on irreversible processes of opinion forming. As recalled in the Appendix, the dynamical updating rules of the Ising model are obtained in a natural way to describe the formation of decisions of boundedly rational agents [9]. The Ising model is one of the simplest models describing the competition between the ordering force of imitation or contagion and the disordering impact of private information or idiosyncratic noise [10]. In the same class of minimal models of emergent social behaviors is the model of herding based on percolation clusters proposed by Cont and Bouchaud [11].

Starting with a framework suggested by Blume [12,13], Brock [14], Durlauf [15–20], and Phan et al. [21] summarize the formalism starting with different implementation of the agents' decision processes whose aggregation is inspired from statistical mechanics to account for social influence in individual decisions. Lux and Marchesi [22,23], Brock and Hommes [24], and Kirman and Teyssiere [25] have also developed related models in which agents' successful forecasts reinforce the forecasts. Such models have been found to generate swings in opinions, regime changes and long memory. An essential feature of these models is that agents are wrong for some of the time, but, whenever they are in the majority they are essentially right. Thus, they are not systematically irrational [26].

Here, we study a model of interacting agents buying and selling a single financial asset based on an extension of the Ising model. The agents make their decision based upon the combination of three different information channels: mutual influences or imitation, external news and idiosyncratic judgements. Agents update their willingness to extract information from the other agents' behavior based on their assessment of how past news have explained market returns. Agents update their propensity to herding according to what degree the news have been successful in predicting returns. We distinguish between two possible updating rules: rational and irrational. In the rational version, agents decrease their propensity to imitate if news have been good predictors of returns in the recent past. In the irrational version, agents mis-attribute the recent predictive power of news to their collective action, leading to positive self-reinforcement of imitation. We show that the model can reproduce the major empirical stylized facts of financial stock markets only when the updating of the strength of imitation is irrational, providing a direct test and the evidence for the importance of misjudgement of agents biased toward herding.

Section 2 specifies the model and compares it with previous related versions. Section 3 presents the results of exhaustive searches in the space of the major parameters of the two versions of the model. We describe in turn the distributions of returns at multiple time scales, the autocorrelations of the returns and of the volatility (absolute value of the returns) at different time scales, the multifractal properties of the structure functions of the absolute values of returns and their consequences in the characteristic relaxation of the volatility after bursts of endogenous versus exogenous origins. Section 4 concludes.

2. Model of imitation versus news impact

2.1. Definition of the model

We study the following model of N agents interacting within a network $\mathcal{N}$ (taken here for simplicity as the set of nodes linked by nearest-neighbor bonds on the square lattice; this implies that an agent sitting at a node interacts directly only with her four neighbors). At each time step t, each agent i places a buy ($s_i(t) = +1$) or sell ($s_i(t) = -1$) order. Her decision $s_i(t)$ is determined by the following process:

$$s_i(t) = \mathrm{sign}\left[\sum_{j\in\mathcal{N}} K_{ij}(t)E[s_j](t) + \sigma_i G(t) + \varepsilon_i(t)\right], \tag{1}$$

where $E[s_j](t)$ is the expectation formed by agent i on what will be the decision of agent j at the same time t. The left- and right-hand sides are, in principle, evaluated simultaneously at the same time to capture the anticipation by a given agent i of the actions of the other agents, which are going to determine the change of the market price from $t-1$ to t. Indeed, we assume that the decisions $s_i(t)$ are formed slightly before t, in the period from $t-1$ to t, when the news $G(t)$ has become available in the interval from $t-1$ to t, and are then converted into price at t by the market clearing process. In principle, the best strategy for agent i is indeed to base her action for the next investment period on her best guess of the *present* action of all other agents for the next investment period (see below).

Expression (1) embodies three contributions to the decision making process of agent i.

- *Imitation through the term* $\sum_{j\in\mathcal{N}} K_{ij}(t)E[s_j](t)$. The kernel K_{ij} is the relative propensity of the trader i to be contaminated by the sentiment of her friend j (coefficient of influence of j on i). In other words, $K_{ij}(t)$ quantifies the strength of the influence of agent j's expected decision on the decision of agent i, which evolves with time as we soon specify. Due to the heterogeneity of the traders, $K_{ij} \neq K_{ji}$, generally. The sum $\sum_{j\in\mathcal{N}}$ is carried over all agents j who are in direct contact with agent i.
- *Impact of external news through the term* $\sigma_i G(t)$. Here $G(t)$ quantifies the impact of the external news $I(t)$ on the decision of agent i. We follow the specification of the artificial stock market model formulated by Gonç alves [27], and assume that $I(t)$ follows a standard normal distribution and

$$G(t) = \begin{cases} 1 & \text{if } I(t) > 0, \\ -1 & \text{if} I(t) \leqslant 0. \end{cases} \tag{2}$$

  The agent-dependent parameter σ_i, which is uniformly distributed in $[0, \sigma_{\max}]$ and quenched at the beginning of the simulation, quantifies the relative impact at time t of the news' positive or negative outlook on the decision process of agent i. In other words, σ_i is the relative sensitivity of agent's sentiment to the news.
- *Idiosyncratic judgement associated with private information*. $\varepsilon_i(t)$ embodies the idiosyncratic content of the decision of agent i accounting for the interpretation of her own private information. The idiosyncratic judgement $\varepsilon_i(t)$ is time-dependent and assumed to be normally distributed around zero with an agent-dependent standard deviation equal to the sum of a common constant CV and of a uniform random variable in the interval $[0, 0.1]$ again to capture the heterogeneity of agents.

The market price is updated according to

$$p(t) = p(t-1)\exp[r(t)], \tag{3}$$

so that $p(t)$ and $r(t)$ are known at the end of the interval from $t-1$ to t. The return $r(t)$ is determined according to

$$r(t) = \frac{\sum_{i\in\mathcal{N}} s_i(t)}{\lambda N}, \tag{4}$$

where N is the number of traders in $\mathcal{N}$ and λ measures the market depth or liquidity and is taken constant. In expression (4), the decisions $s_i(t)$ are formed slightly before t, in the period from $t-1$ to t and are then converted into price at t by the market clearing process.

We account for the adaptive nature of agents and their learning abilities by updating the coefficient of influence of agent j on agent i according to the following rule:

$$K_{ij}(t) = b_{ij} + \alpha_i K_{ij}(t-1) + \beta r(t-1)G(t-1). \tag{5}$$

In this, we follow the large literature on the rationality of imitation and of adaptive behavior when lacking sufficient information or when this information seems unreliable (see, e.g., Refs. [14,22–24,28–41]). The coefficients b_{ij} quantify the intrinsic imitation influence of agent j on agent i in the absence of other effects. For $\alpha_i = \beta = 0$, we recover a constant coefficient of influence, which derives from a simple argument of bounded rationality recalled in the Appendix with (23). The coefficient $\alpha_i > 0$ (possibly different from one agent to another) quantifies the progressive loss of memory of past influences on the present. The last term with $\beta \neq 0$ quantifies how agent i updates her propensity for imitation based on the role of the exogenous news $G(t)$ in determining the sign and amplitude of the observed return in the preceding time period. This update depends upon whether the news $G(t-1)$ known between times $t-2$ and $t-1$ has the same sign as the price variations from $t-2$ to $t-1$, i.e., the same sign as the return $r(t-1)$ defined by (3). In addition to the sign, the amplitude of the return is also taken into account in the updating rule of the coefficient $K_{ij}(t)$ quantifying the propensity to imitate: indeed, a small amplitude of the return has low psychological as well as financial consequences and should not count as much as a large amplitude of the return. The simplest specification is to take into account the impact of the amplitude of the return linearly in its size, hence the form $\beta r(t)G(t-1)$ in (5). Stronger nonlinear dependence is probably more relevant [42–44] but is not considered further here to keep the discussion simple. Krawiecki et al. [45] have considered a simpler version in which each $K_{ij}(t)$ is purely random and is constructed as the sum of two random noises, one which is common to all coupling coefficients and one which is specific to it. They are able to reproduce volatility clustering and a power law distribution of returns at a single fixed time scale.

Gonçlaves [27] considered this model (5) with $\alpha_i = 0$, i.e., with no memory of past influence on present influence, which leads the time series of $K_{ij}(t)$ looking like a white-noise process. Our addition of the memory effect modifies this white-noise structure into a Ornstein-Uhlenbeck-type noise, tending to a random-walk-like process for $\alpha_i = 1$. With the new term $\alpha_i \neq 0$, $K_{ij}(t)$ keeps a memory of past successes that the news had on predicting the stock market moves over approximately $1/(1-\alpha_i)$ time steps. α_i thus characterizes the strength of the persistence of the links of agent i with other agents.

The sign of the coefficient β is crucial.

(1) For $\beta < 0$, agent i is less and less influenced by other agents, the better has been the success of the news in determining the direction and amplitude of the market return. This process is self-reinforcing since, as K_{ij} decreases, the dominant term becomes $\sigma_i(t)G(t)$, which further ensures that the news correctly predict the decision of agents, and therefore the direction of the market move, thus decreasing further the coefficient of influence K_{ij}. Reciprocally, agents tend to be more influenced by others when the news seems to incorrectly predict the direction of the market. The news being not reliable, the agents turn to other agents, believing that others may have useful information (see below and the Appendix for an elaboration of this argument).

Taking a negative β corresponds to agents behaving according to standard rational expectations with respect to the flow of external news. If the stock market is in agreement with the news most of the time, a rational agent would conclude that the impact of imitation, of herding, of trend following and of other endogenously generated positive feedbacks, is minor and the news are the dominating factor. Indeed, standard economics views the stock market as a machine transforming news into prices and the market is efficient when all news have been correctly incorporated and are continuously incorporated into the market prices. In our framework, this situation arises when the strategy $s_i(t) = \mathrm{sign}[G(t)]$ consisting in following the news is found at least as good as or better than (1). The agent would in this case decrease its propensity K to imitate, and continue to do so as long as the news are most of the time in agreement with the stock market moves. We can thus summarize this case $\beta < 0$ as describing ''boundedly rational'' agents.

(2) For $\beta > 0$, the more the news predict the direction of the market, the more the agents imitate other agents. In other words, there is a reinforcement of the influence between agents when the news and the stock market return match at the previous period. This is the ''irrational'' case where agents either mis-attribute the origin of the market moves to herding rather than to the impact of news, or misinterpret the exogenous character of news in terms of endogenous herding or infer that other agents will be following more eagerly as a group the direction given by the news.

The regime $\beta > 0$ may result from several mechanism.

- *Mutually reinforcing optimism.* When the market is rising ($r(t-1) > 0$) and the news are good ($G(t-1) = 1$), the agents may seek each other in order to determine if the rise can be sustained: if the agent's neighbors are all bearish, then the agent interprets this as a sign that the rise cannot be sustained due to a lack of majority support; if the friends are bullish, the agent is encouraged to feel optimistic, in the sense that more good news may be on the way. It is thus not so much the price rise or fall that agents try to predict and reflect but rather the continuation of good news (or bad news) in the future and whether they expect or not the good news to continue. Since a rise in the market is the result of more people feeling optimistic that more good news are on the way, then the optimism is spreading like an epidemic and the market rises, reinforcing the influence coefficient K_{ij} (called propensity to be influenced by the felling of others by Gonçalves [27]).

- *Overconfidence*. Another mechanism is the tendency for humans to exhibit overconfidence in their abilities, either individually or as a group. If they see that the stock market has moved in the same direction as the news indicated, they may conclude that the information provided by the other agents has been valuable, since the stock market is supposed to follow the rule of the majority. Agents may thus be tempted to increase their imitation behavior as long as news and stock market continue to match. Or said differently, they attribute a larger value in the prediction of the news which they interpret as being influenced by the majority opinion. Indeed, experiments committed by Darke and Freedman [46] show that a lucky event can lead to overconfidence. Heath and Gonzalez [47] have also compared decisions made alone to decisions made following interactions with others, and shown that, while interaction did not increase decision accuracy or meta-knowledge, subjects frequently showed stable or increasing confidence when they interacted with others, even with those who disagreed with them (see also Refs. [48–51]). A possible interpretation is that the interaction serves the role of rationalizing the subjects' decisions rather than collecting valuable information. In the same spirit, Sieck and Yates [52] have shown that exposition to others of the rationale behind decisions increase markedly subjects' confidence that their choices were appropriate.

In summary, the model contains the following general ingredients: (i) the agents make decisions based on a combination of three ingredients: imitation, news and private information; they are boundedly rational; (ii) traders are heterogeneous (K_{ij} and σ_i); (iii) The propensity to imitate and herd is evolving adaptively as an interpretation that the agents make of past successes of the news to predict the direction of the market.

2.2. *Specification of the updates of expectations of other agents' decisions*

The model is completely specified once the algorithm, used to construct how an agent estimates her expectation $E[s_j](t)$ of other agents' decisions in expression (1), is given. Three possibilities can be considered.

- $E[s_j](t) = s_j(t-1)$: agent i expects a persistence of the other agents' decisions, similar to a martingale condition; in the absence of any information other than the past actions, the next predicted action is the last one. This leads to transform (1) into the prescription
$$s_i(t) = \mathrm{sign}\left[\sum_{j\in\mathcal{N}} K_{ij}(t)s_j(t-1) + \sigma_i G(t) + \varepsilon_i(t)\right]. \tag{6}$$
Without the term $\sigma_i G(t)$, this was the model adopted by Johansen et al. [53] to develop a theory of herding to explain financial bubbles as regimes of strong imitation between agents. Kaizokji et al. [54] have also studied the dynamics of a stock market with heterogeneous agents in the framework of a recently proposed spin model for the emergence of bubbles and crashes. The Appendix recalls the derivation by Roehner and Sornette [9] showing how this specification is a natural consequence of bounded rationality of agents.
- $E[s_j](t) = s_j(t)$: each agent i has access to the information of the action of other traders instantaneously and she cannot do better than use it. Computationally, the solution $E[s_j](t) = s_j(t)$ is self-consistent since $s_i(t)$ depends on $s_j(t)$ which itself depends on the former. Such self-consistent formulation can be solved by using an iterative algorithm as follows:
$$E[s_j](t) = \lim_{k\to\infty} s_j^{[k]}(t), \tag{7}$$
where
$$s_i^{[k+1]}(t) = \mathrm{sign}\left[\sum_{j\in\mathcal{N}} K_{ij}(t)s_i^{[k]}(t) + \sigma_i G(t) + \varepsilon_i(t)\right] \tag{8}$$
with, for instance, the starting condition $s_i^{[0]}(t) = s_i(t-1)$.
- Information cascade along a given path $i = 1, 2, \ldots, N$ where the ordered sequence of indices i encodes the curvilinear abscissa along a path linking all agents, in the spirit of the information cascades [55,56]. In such

a scheme, $E[s_j](t) = s_j(t-1)$ if $j > i$ and $E[s_j](t) = s_j(t)$ if $j < i$ when considering agent i. In other words, we have

$$E[s_j](t) = H(j-i)s_j(t-1) + H(i-j)s_j(t), \tag{9}$$

where H is the Heaviside function. In the literature on adaptive learning, this procedure is said to be backward-looking, because it does not model the other agents' future behavior given all past information, for instance, because agents are not able to form beliefs about the other agents' future choices. These rather naive procedures have been suggested by psychologists and animal behaviorists in the 1950s and have been tested in laboratory settings.

2.3. Restriction to model [27] with $\alpha_i = 0$ corresponding to the absence of memory of the coefficients K_{ij}'s of imitation

Our model is a straightforward generalization of the artificial stock market model formulated by Gonçalves [27], in which the coefficients K_{ij} of the influence of j on i are taken identical for all j's and are updated as follows:

$$K_{ij}(t) = K_i(t) = b_i + \beta r(t-1)G(t-1). \tag{10}$$

The coefficients b_i's capture the "natural" propensity of humans for imitation, which may vary from agent to agent. These are drawn at the beginning of the simulation from a uniform distribution between 0 and some maximum positive value and remain fixed thereafter during the dynamics. The main difference between (10) and our specification (5) is the absence of the persistence or memory of $K_{ij}(t)$ on its past values in (10).

Based on numerical simulations and synthesis, Gonçalves [27] argues that the model (1) using the implementation (9) corresponding to an information cascade along a specific path and with (10) reproduce all the important stylized facts of the stock market only for $\beta > 0$. This is interesting but actually not quite correct. Consider, for instance, the parameters $b_{\max} = 0.22-0.24$, $\sigma_{\max} = 0.14$–0.15 and $\mathrm{CV} = 0.8$–0.9, which are recommended by Gonçalves [27] to reproduce the main stylized facts (fat tailed distributions of returns, clustering of volatility, bubbles, crashes). The problem is that the time series of returns generated with these parameters have unrealistic bimodal distributions of returns, as shown in Fig. 1. The origin of this bimodal structure results from the fact that the model explores the ordered regime of the Ising model too often, corresponding roughly speaking to an average coupling constant $\langle K_i \rangle$ larger that the critical Ising value K_c.

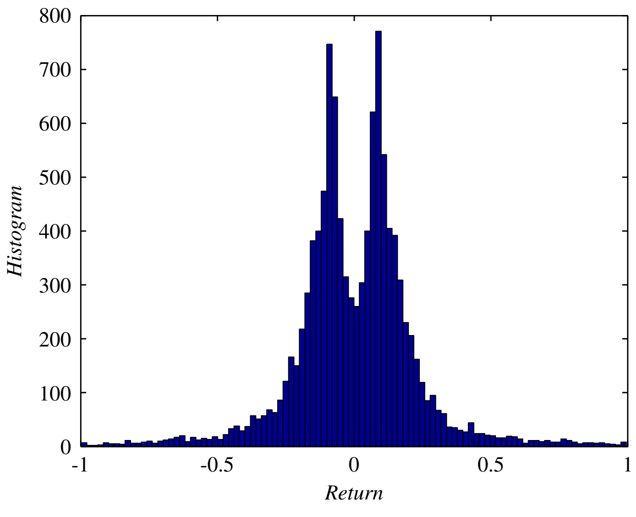

Fig. 1. Density distribution of returns r_1 for a realization of the artificial stock market model formulated by Gonçalves [27] generated using $b_{\max} = 0.22$–0.24, $\sigma_{\max} = 0.14$–0.15 and $\mathrm{CV} = 0.8$–0.9 as recommended by this author. The time series of returns have been kindly provided by Gonçalves. Our own simulations reproduce the same results.

In this case, agents form a crowd with a majority opinion which can flip with time between ± 1 due to the feedback process of the news and return dynamics on the coupling coefficient. In other words, the two bumps of the distribution of returns are direct signatures of the existence of the two spontaneous ordered states of the Ising model when the coupling coefficient is above its critical value. The dynamics of decisions is thus characterized by more or less random flips of crowds of agents between the two opinions. The resulting stylized facts cannot therefore be taken as genuine signatures of a realistic dynamics of agents' opinions.

3. Simulations and results of the model with memory in the dynamics of influence coefficients

The origin of the unrealistic bimodal distribution of returns shown in Fig. 1 has to be found in the rather artificial property that the influence coefficients $K_{ij}(t)$'s, which are updated according to (10), lose instantaneously from one time to the next the memory of their past values. As a consequence, the coefficients $K_{ij}(t)$ fluctuate with time approximately as a white noise with an amplitude controlled by that of the returns and this gives rise to abrupt shifts of the majority opinion as explained in the previous section. This property is unrealistic to model the real persistence of interactions between agents. Indeed, in the real world, people are connected through social networks that do not instantaneously reshuffle in response to external effects as expression (10) describe. Rather, social connections evolve slowly and exhibit significant persistence, as documented in numerous studies (see, for instance, Refs. [57,58] and references therein). Networks of investors communicating their opinions and sentiment on the stock market are similarly persistent.

We thus turn to model (1) implemented with (6) or along a specific information cascade (9), as described in Section 2.2. We assume the existence of a memory in the dynamics of the influence coefficients, which are updated according to the simplified version of (5) given by

$$K_i(t) = b_i + \alpha K_i(t-1) + \beta r(t-1)G(t-1), \tag{11}$$

where the b_i's are uniformly distributed in $(0, b_{\max})$ at the beginning of the simulation and remain frozen thereafter. The value b_i of agent i represents her idiosyncratic imitation tendency. Compared with (5), the memory parameter α is taken the same for all agents. We use a 50×50 lattice as the geometrical implementation of the social network, in which the agents are located at the nodes and each agent interacts only with her four nearest neighbors. The sensitivity σ_i of agent i to the global news $G(t)$ is uniformly distributed in the interval $(0, \sigma_{\max})$. The value σ_i of agent i is again specific to her and quantifies her susceptibility to be influenced by external news. The idiosyncratic or private information term $\varepsilon_i(t)$ of agent i is drawn at each time step from a normal distribution with zero mean and a standard deviation $s_{\varepsilon,i}$ which is also different from one agent to another: $s_{\varepsilon,i}$ is chosen at the beginning of the simulation (like b_i and σ_i) to characterize agent i according to a value equal to the sum of a common constant CV and of a uniform random variable in the interval $[0, 0.1]$.

In our simulations, we fix $\lambda = 40$ (which determines the scale of the returns to a value comparable to that of empirical observations) and $\alpha = 0.2$. We have also investigated other values $\alpha = 0.4$, 0.6, and 0.8 and obtain similar results. We explore the properties of the model in the parameter space of $b_{\max}$, $\sigma_{\max}$ and CV. There is no loss of generality in fixing $|\beta| = 1$ to explore the relative importance of the term $\beta r(t-1)G(t-1)$, since the typical scale of the K_i's is set by $b_{\max}$ whose amplitude is varied in our numerical exploration. However, the sign of β is fundamental as explained in Section 2.1. We thus consider in turn the two cases $\beta = -1$ and $\beta = +1$ and explore in each case a large sample of triplets $(b_{\max}, \sigma_{\max}, \mathrm{CV})$.

We ask whether the model can account for the most often reported stylized facts of financial markets. In other words, we would like to validate the model. For this, we consider the following metrics: (i) the distribution of returns at different time scales; (ii) the correlation function of returns and of the absolute value of the returns (taken as a proxy for the financial volatility); (iii) the scaling of the moments of increasing orders of the absolute values of the returns (testing multifractality); (iv) the existence of a hierarchy of exponents controlling the relaxation of the volatility after an endogenous shock (another hallmark of multifractality); (v) the existence of bubbles and crashes and their properties. Our strategy is to search for a robust set of model parameters for which all these stylized facts are reproduced not only qualitatively, but also quantitatively. The main result of our analysis is that it is impossible to validate the model for $\beta < 0$ while we find sets of parameters for $\beta > 0$ which nicely fit the stylized facts of real financial markets.

3.1. News predicts the next return → decrease of imitation: $\beta = -1$

In this case, when the news G happens to correctly predict the return ($rG > 0$), the agents reduce their mutual imitation. On the contrary, when $rG < 0$, imitation among agents strengthens. Thus, $\beta < 0$ corresponds to agents behaving according to standard rational expectations with respect to the flow of external news: a rational agent would concluded that, if the stock market is in agreement with the news then, the impact of imitation is minor and the news are the dominating factor, as standard economic textbooks describe, since the strategy $s_i(t) = \text{sign}[G(t)]$ consisting in following the news is found as good as or better than the more elaborate dynamics of the agents' actions $s_i(t)$ incorporating the imitation process. For $\beta < 0$, agents decrease their propensity to imitation as long as the news are in agreement with the stock market moves.

The following argument shows that the attractor of the dynamics is characterized by negligible imitation and only the news and private information terms are important for the dynamics. Consider a population of traders at time t with their propensity $K_i(t)$ to imitate on average above the critical Ising value K_c such that imitation initially dominates the dynamics. The news $G(t)$ becomes known and the decisions of the agents given by (1) decide collectively the return $r(t+1)$ from time t to $t+1$ through expression (4). Since the $K_i(t)$'s are overall above K_c, it is well-known from many past studies of the Ising model (see, for instance, Ref. [10] and references therein) that the corresponding "ferromagnetic" phase of the system is characterized by a strong slaving to external fields such as $G(t)$. Hence, the collective opinion $\sum_i s_i$ takes the sign of $G(t)$ with a high probability. As a result, with a large probability, $r(t+1)G(t)$ is positive, which entails a downgrading of K_i by an amount $r(t+1)$ since $\beta = -1$ (in addition to the other terms which tend to reverse $K_i(t)$ to the value $b_i/(1-\alpha)$). This behavior continues as long as the K_i's are above or close to K_c (since the K_i's are heterogeneous, the effective critical value is modified compared with the homogeneous case and our argument remains valid when using this modified value). Alternatively, if the K_i's are on average smaller than K_c, the collective decision $\sum_i s_i$, and therefore the market return have little or no relationship with the external news. Hence, the term $\beta r(t+1)G(t)$ takes random signs from one time step to the next, leading to an effective random forcing added to the autoregressive equation $K_i(t) = b_i + \alpha K_i(t-1)$. The coefficients $K_i(t)$ evolve to fluctuate around the asymptotic value $b_i/(1-\alpha)$. We thus expect Gaussian distributions of returns when $b_i/(1-\alpha)$ is smaller than K_c and bimodal distributions when $b_i/(1-\alpha) > K_c$ reflecting the slaving of the global opinion to the sign of the news.

In our simulations, we have scanned $b_{\max}$ from 0.1 to 0.5 with spacing 0.1, $\sigma_{\max}$ from 0.005 to 0.08 with spacing 0.005, and CV from 0.1 to 1.1 with spacing 0.2. This corresponds to a total of 480 different models. We use (1) implemented according to the information cascade of sentiment formation explained in the item associated with Eq. (9). This choice is made to minimize the computational cost. Tests using the other updates over limited time span suggest that the results we are interested in are not sensitive to the details of the updating rules. We run each model over 10^4 time steps and then analyze the time series of returns.

The first metric we analyze is the distribution of returns at different time scales τ, defined according to

$$r_\tau(t) = \ln[p(t)/p(t-\tau)], \tag{12}$$

where τ is a multiple of the time step. We observe two classes of shapes. For large idiosyncratic noise (large CV) and not too large $b_{\max}$, the distribution of returns is Gaussian for all time scales τ. For smaller CV's and larger $b_{\max}$, we observe multimodal return distributions, as illustrated by the typical example shown in Fig. 2 obtained for $b_{\max} = 0.2$, $\sigma_{\max} = 0.045$, and CV $= 0.1$. The number of peaks in the distribution of r_τ is $\tau + 1$. These multimodal distributions correspond to the regime where the news $G(t)$ controls the collective opinion of the traders which tend to coordinate their decisions as explained above when $b_i/(1-\alpha)$ is larger than K_c. For given $b_{\max}$ and CV, the bimodal structure of the distribution of r_1 becomes more and more significant as $\sigma_{\max}$ increases. Alternatively, for fixed $\sigma_{\max}$ and CV (say, $\sigma_{\max} = 0.02$ and CV $= 0.1$), a bimodal distribution is obtained for sufficiently small $b_{\max}$ and we observe a crossover from bimodal ($b_{\max} = 0.1$) to unimodal ($b_{\max} = 0.5$) with a crossover with a plateau ($b_{\max} = 0.3$). This corresponds to the regime in which a majority of agents react to the global news in the same manner and buy or sell simultaneously. Since the news are $G(t) = \pm 1$ with equal probability, the decisions of the agents, and therefore the returns most often jump between two values of equal amplitude and opposite signs. The multimodal structure of the distributions of

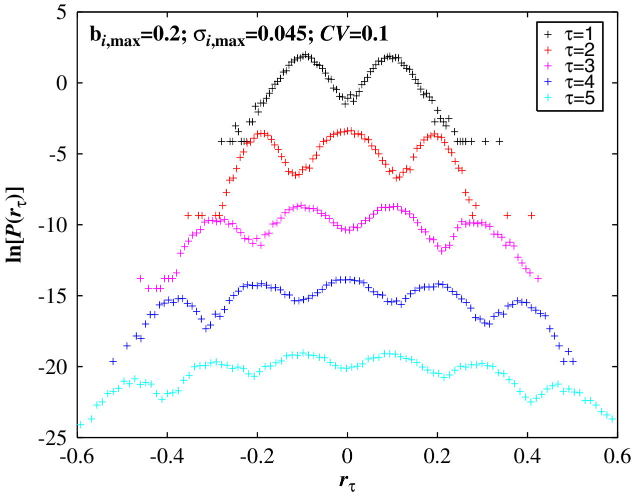

Fig. 2. A typical example of the multimodal distribution for $b_{\max} = 0.2$, $\sigma_{\max} = 0.045$, and CV = 0.1.

returns at time scale τ then results from the properties of the convolution of the bimodal distribution of the returns r_1.

In the parameter space that we have explored and notwithstanding our best attempts, we have not been able to find a set of parameters leading to distributions of returns exhibiting a monomodal shape with fat tails for small time scales, evolving slowly towards Gaussian distributions at large time scales, as can be observed in empirical data [59,60].

In addition, we observe that the correlation function of returns ($C_\tau(r, r)$) and of volatilities ($C_\tau(|r|, |r|)$) have similar amplitudes and decay with the same characteristic time scale as a function of time lag. This is very different from the observed correlations of financial markets, with very short memory for returns and long-memory for the volatility.

We have also investigated the impact of the updating rule of the agents' sentiments. If instead of the information cascade (third item of Section 2.2), we use the parallel update (6), for the same range of parameters $b_{\max}$, $\sigma_{\max}$, and CV, we find that most of the returns are between two values proportional to ± 1 and the distribution of returns is close to $P(r_1) = \delta(r_1^2 - 1)$. In other words, the bimodality of the distribution of returns is much more pronounced than shown in Fig. 2 that was obtained for the information cascade updating scheme. For large idiosyncratic noise (large CV), the bimodality disappears and is replaced by pdf's which are approximately Gaussian. With CV and $b_{\max}$ (resp. $\sigma_{\max}$) fixed, the volatility increases with increasing $\sigma_{\max}$ (resp. decreasing $b_{\max}$) and the distribution is bimodal when $\sigma_{\max}$ (resp. $b_{\max}$) is large (resp. small) enough. The time evolution exhibits in addition long transients with returns fluctuating around zero before bifurcating to the bimodal state. The duration of the transient decreases with increasing $\sigma_{\max}$ or $b_{\max}$. Scanning the parameters, we never obtained realizations with realistic distributions of returns at different time scales. We conclude that the updating scheme (11) with $\beta = -1$ corresponding to boundedly rational agents cannot explain the stylized facts of empirical finance.

3.2. *News predicts the next return → increase of imitation: $\beta = 1$*

In our simulations, we fix $\alpha = 0.2$, $b_{\max}$ varies from 0.1 to 0.5 with spacing 0.1, $\sigma_{\max}$ from 0.01 to 0.08 with spacing 0.01, and CV from 0.1 to 0.7 with spacing 0.2. This gives 160 models that we explore by generating time series of length equal to 10 000 time steps. We use (1) implemented according to the information cascade of sentiment formation explained in the item associated with Eq. (9).

When CV is very large, the distribution of return r_τ is Gaussian, simply because the returns are dominated by the idiosyncratic noise modeling private information, which is chosen Gaussian. This regime is not interesting since it erases both the effect of news and of imitation. For smaller CV's, we also observe bimodal distributions of the returns r_1 for certain ranges of parameters, as in the case $\beta = -1$. This occurs for large $\sigma_{\max}$ and small $b_{\max}$. Similar bimodal distributions of returns are obtained for $\alpha = 0$, as described in Section 2.3.

We have found several parameter combinations which lead to realistic stylized facts. For instance, for the following sets of $(b_{\max}, \sigma_{\max}, \mathrm{CV})$, (0.3, 0.03, 0.1), (0.4, 0.04, 0.1), (0.4, 0.05, 0.1), (0.5, 0.06, 0.1), (0.1, 0.01, 0.3), (0.1, 0.02, 0.3), (0.2, 0.02, 0.3), (0.2, 0.03, 0.3), (0.3, 0.04, 0.3), (0.5, 0.05, 0.3), (0.5, 0.07, 0.3), (0.3, 0.03, 0.5), and (0.5, 0.05, 0.5), the distribution of returns r_τ is a stretched exponential (or close to a power law) [61] for small τ, exponential for intermediate τ, and Gaussian for large τ, as in real financial data.

In the following, we exemplify the obtained stylized facts with $\alpha = 0.2$ and with the parameter combination $(b_{\max} = 0.3, \sigma_{\max} = 0.03, \mathrm{CV} = 0.1)$ which is typical.

3.2.1. *Probability density functions of log-returns at different time scales*

Fig. 3 shows a realization of the logarithm of the price over a time interval of 10^5 time steps. Fig. 4 shows the corresponding time series of the log-returns defined by (3,4), where $\lambda = 40$ is set to scale the returns to realistic values. Note the existence of clusters of volatility which are qualitatively similar to those observed in real financial data. The solid lines in Fig. 5 show the logarithm of the probability distribution densities (pdf) of the log-returns at different time scales defined by (12) as a function of the returns r_τ scaled by their standard deviation σ_τ. The pdf curves have been translated vertically for clarity. In the semi-log representation of Fig. 5, a straight line qualifies an exponential law. We observe stretched exponential laws at short time scales that cross over smoothly to a Gaussian law at the largest shown time scale. This evolution of pdf's with time scales comply with the well-known stylized fact of financial markets [59]. The model thus obtains price series with the correct monomodal shape with fat tails, and the correct progressive transition to a Gaussian distribution at large time scales. We stress that this comparison improves on those involving only one time scale.

It is interesting to compare the obtained pdf's at time scales τ larger than the elementary time step 1 with those which would derive by τ-fold convolution of the pdf at the unit time scale. In the absence of time dependence, the two should be asymptotically identical. For instance, the probability density function $\mathrm{pdf}(r_4)$ of returns r_4 over four time steps would be given by $\mathrm{pdf}(r_4) = \mathrm{pdf}(r_1) \otimes \mathrm{pdf}(r_1) \otimes \mathrm{pdf}(r_1) \otimes \mathrm{pdf}(r_1)$. Upon convolution in the absence of dependence, the variance is additive, which gives a prediction for the standard deviation of the pdf of returns at τ time steps, $\sigma_\tau = \sigma_1\sqrt{\tau}$, which can be used to normalize the pdfs in terms of the reduced variable r_τ/σ_τ. The corresponding pdf's of r_4, r_{16}, r_{64}, and r_{256} obtained by convolution of the pdf of r_1 are drawn in Fig. 5 with dashed lines. We see clearly that the true pdf's for $\tau = 4, 16, 64$ have a much

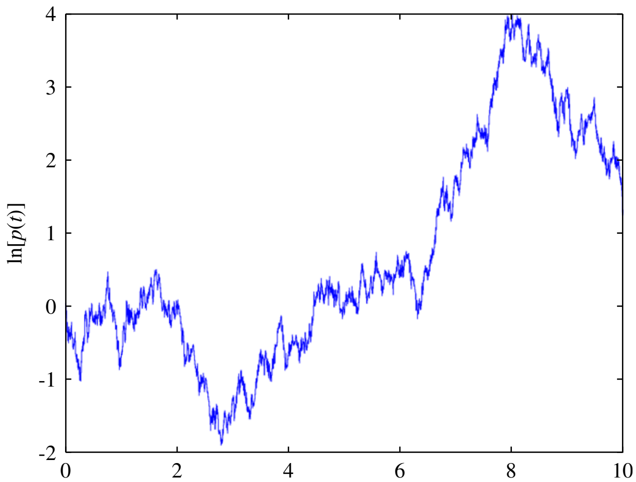

Fig. 3. A realization of the logarithm of the price over 10^5 time steps generated using $\alpha = 0.2$, $b_{\max} = 0.3$, $\sigma_{\max} = 0.03$ and $\mathrm{CV} = 0.1$ of the generalized artificial stock market model defined by (1), (4) and (10).

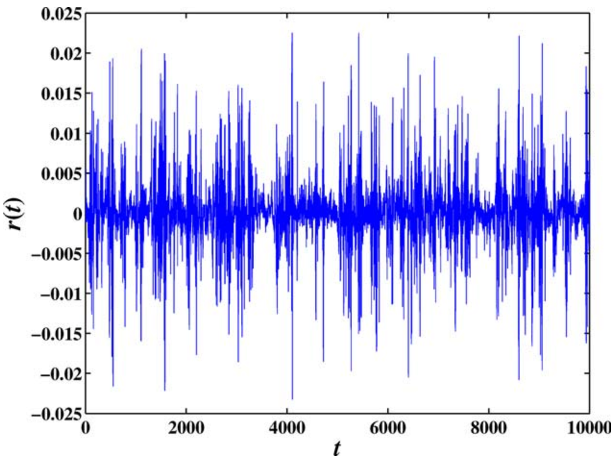

Fig. 4. Time series of the log-returns of the price shown in Fig. 3.

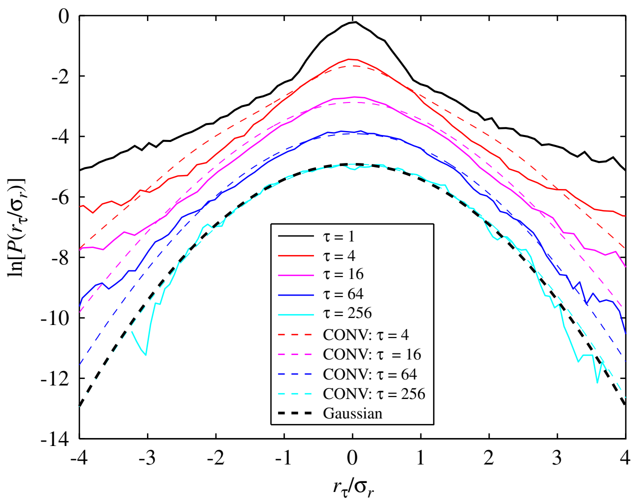

Fig. 5. Empirical (solid lines) and theoretical (dashed thin lines) probability distribution density (in logarithmic scales) of log-returns at different time scales τ of the price time series shown in Fig. 3. The log-returns r_τ are normalized by their corresponding standard deviations σ_τ. The pdf curves are translated vertically for clarity. The thick dashed line is the Gaussian pdf (color online).

fatter tail compared with the theoretical pdf's based on the absence of dependence. Such behavior is very similar to what is observed in real data (see, for instance, Ref. [62, Fig. 2.2]).

3.2.2. Autocorrelations of log-returns and volatility

Fig. 6 shows the temporal correlation of the log-returns r_1 as a function of the time lag ℓ. One can observe a very short correlation time, of duration smaller than one time step. Fig. 7 presents the temporal correlation of the absolute value of log-returns r_1, taken as a proxy for the volatility, both in linear–linear and in linear–log scales. It is apparent that the volatility exhibits a strong correlation with memory lasting approximately 100 time steps for this set of parameter. The bottom panel of Fig. 7 represents the correlation of the volatility as a function of the logarithm of the time lag. This representation is suggested by the multifractal random walk (MRW) model, which is constructed by definition with a correlation decaying linearly with the logarithm of the time lag, up to a so-called integral time scale T [63–66]. The right panel of Fig. 7 shows that this dependence suggested by the MRW provides a reasonable approximation of the numerical data. The integral

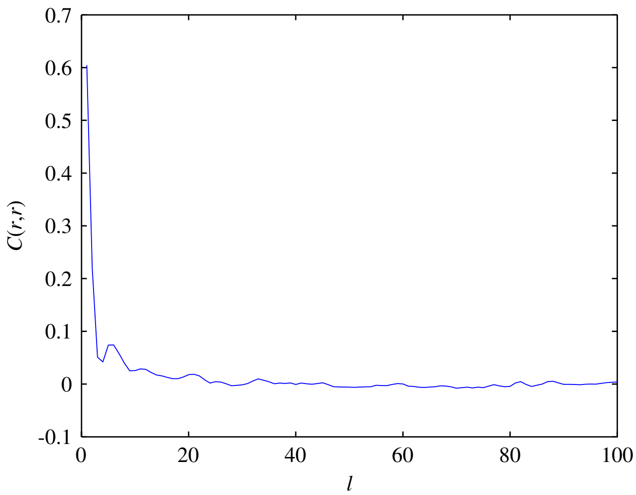

Fig. 6. Autocorrelation function of the log-returns of the realization shown in Fig. 3.

time scale is here estimated around 100 time steps. These observations are in good agreement with the stylized facts on the correlation of returns and of volatility of real financial markets. Indeed, one of the key stylized facts observed empirically is that there are only very short-range correlations in price changes and the time memory is less than one trading day and as small as minutes for the most liquid markets [60,67,68]. In contrast, volatility exhibits a memory over up to the order of one year.

If we combine the information on the time scales from the pdf's of returns shown in Fig. 5, the correlation of returns shown in Fig. 6 and the correlation of the volatility shown in Fig. 7, we can obtain a rough idea of the correspondence between the time step of the model and real trading time. From Fig. 5, we see that the pdf of returns at time scale $\tau = 4$ and 16 are similar to the empirical one at the daily scale for major stocks and indices. This suggests that one day corresponds to roughly 4–16 time steps of the model. This correspondence is compatible with Fig. 6 for the absence of correlation at the daily time scale observed empirically. This correspondence gives with Fig. 7 an integral time scale for the volatility correlation of about 100/4 to 100/16 days, i.e., 6–25 days. This is about a factor of ten shorter than observed on real markets, if we believe the relevance of the MRW and its calibration of T for real markets.

The memory of the autocorrelation of the volatility (as well as the correlation of the returns) is sensitive to the value taken by the parameter α introduced in our model, which embodies the dependence of the coefficient K_{ij} of imitation on its past values. Fig. 8 shows that much longer ranges for the correlation of the absolute value of returns are found for larger values of α, however at the cost of introducing an unrealistic correlation of the returns. This figure suggests that α cannot be larger than 0.2–0.3 without producing unrealistic correlation in returns.

3.2.3. Multifractal properties

The MRW also predicts (and this is well-verified by empirical data) that the autocorrelation functions of $|r_\tau(t)|$ for different τ should superimpose for time lags larger than their respective τ [63,64]. Fig. 9 shows that this is approximately the case.

Another important stylized facts is the multifractal structure of the absolute values of log-returns [65,69–76], which led to the proposition that the MRW might be a good model for financial price time series [63–66]. Fig. 10 shows in log–log scale the structure function

$$M_q(\tau) \equiv \langle |r_\tau|^q \rangle \tag{13}$$

as a function of the time scale τ. The power law dependence of the structure functions $M_q(\tau)$ as a function of τ is found to be reasonable. The slopes of the lines in log–log plots give the exponents ξ_q

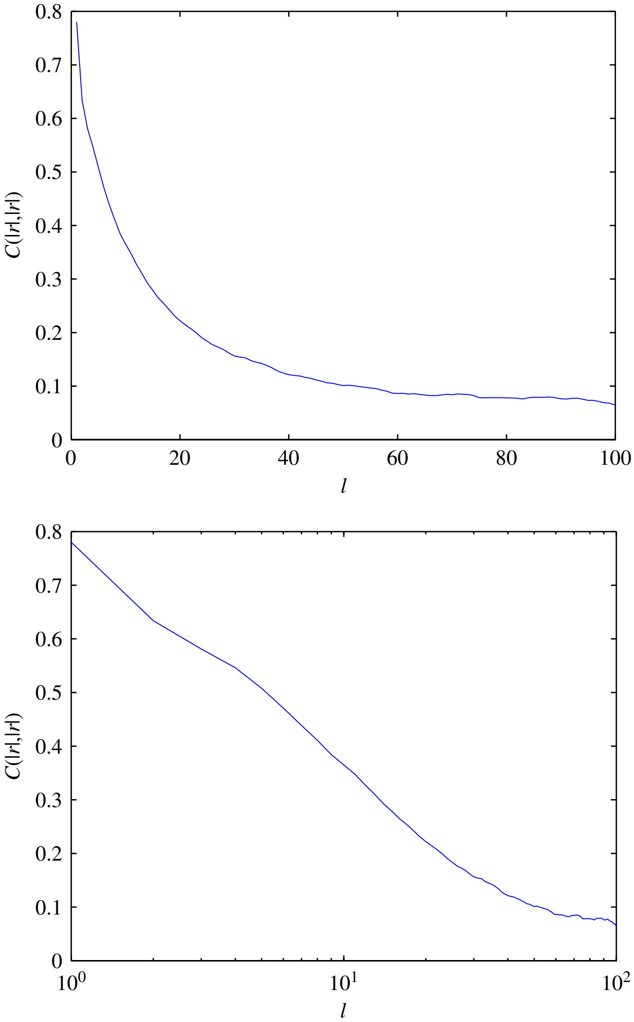

Fig. 7. Autocorrelation function of the absolute value of log-returns of the realization shown in Fig. 3. The top panel show the correlation in linear–linear scale. The bottom panel plots the correlation function as a function of the logarithm of the time lag, as suggested by the multifractal random walk model (see text).

defined by

$$M_q(\tau) \sim \tau^{\xi_q}. \tag{14}$$

Multifractality is qualified by a nonlinear dependence of ξ_q as a function of the order q of the structure function [59,77], as reported in Fig. 11.

3.2.4. Endogenous versus exogenous shocks

The dynamical process described by (5) together with (6) and (4) describes a flux of external news $G(t)$ which are "digested" by the collective behavior of the population of traders to create a time series of returns presenting long-range memory in the volatility and multifractal properties, similar to the MRW model [63–66]. In addition, Sornette et al. [66] have discovered a new consequence of multifractality in the form of a continuous dependence of the exponent of the power law relaxation of the volatility after a spontaneous peak as a function of the amplitude of this peak (see also other applications in Refs. [78,79]). We proceed to test if such an effect is found in our model.

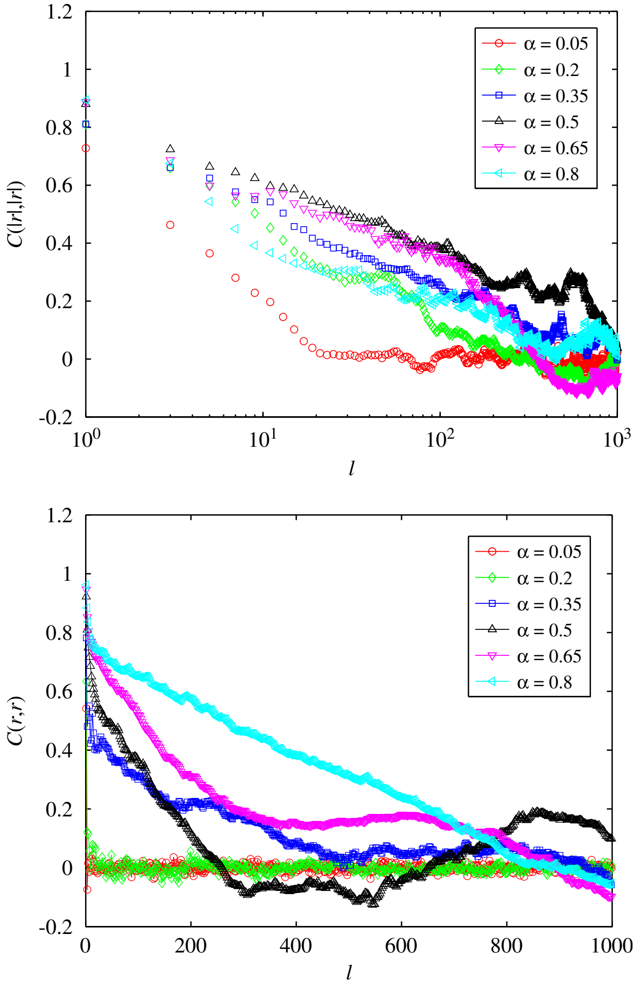

Fig. 8. The impact of α on the autocorrelation of the absolute values of the returns and of the returns.

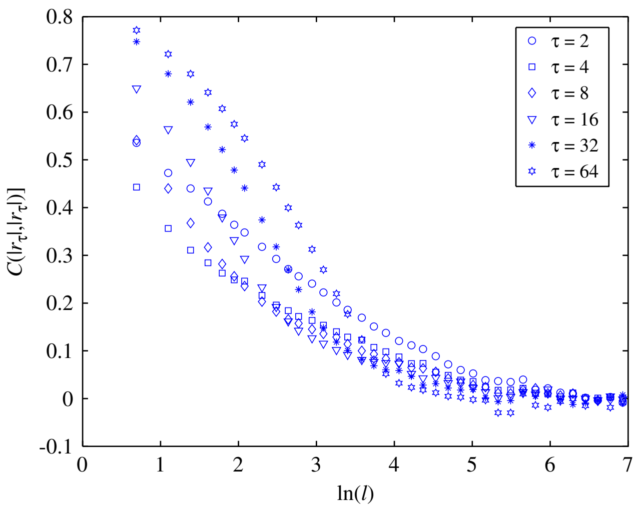

Fig. 9. Scaling of the autocorrelation functions of $|r_\tau(t)|$ for different time scales τ of the realization shown in Fig. 3.

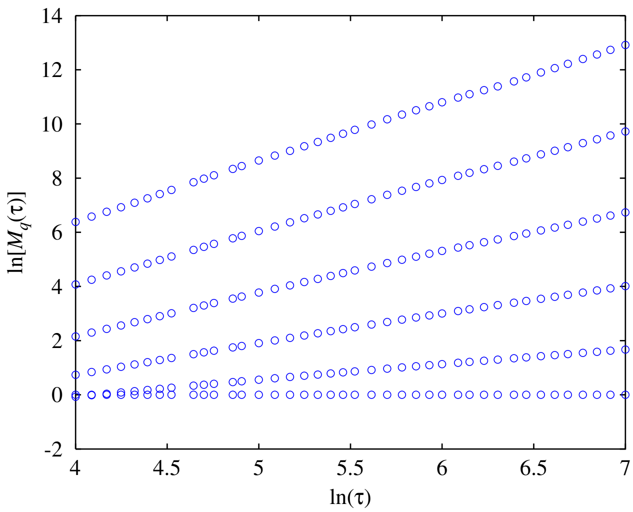

Fig. 10. Scaling of the structure functions $M_q(\tau)$ of log-returns shown in Fig. 4 at different scales τ for different orders q.

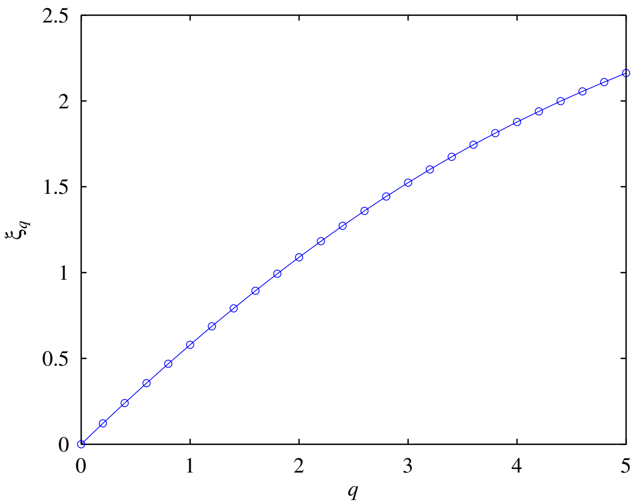

Fig. 11. Dependence of the scaling exponents ξ_q defined in (14) as a function of the order q of the structure functions $M_q(\tau) \sim \tau^{\xi_q}$. The concavity of ξ_q as a function q is the hallmark of multifractality.

Consider a realization giving a time series of returns $r(t)$. Let us define the local volatility within a window $[t+1, t+\Delta t]$ of size Δt by

$$\sigma^2_{\Delta t}(t) = \sum_{i=1}^{\Delta t} |r(t+i)|^2. \tag{15}$$

Using this definition, we construct a time series for the volatility $\sigma^2_{\Delta t}(t)$ in moving windows of size Δt. The (unconditional) average volatility $E[\sigma^2]$ is nothing but the average of the time series of $\sigma^2_{\Delta t}(t)$. Let us follow Sornette et al. [66] and consider local burst of volatility $\sigma^2_{\Delta t}(t)$ with amplitude scaled to the average volatility $E[\sigma^2]$:

$$\sigma^2_{\Delta t}(t) = e^{2s(t)} E[\sigma^2]. \tag{16}$$

The parameter s thus quantifies the relative amplitude of a local burst of volatility in units of the average volatility. Following Sornette et al. [66], for a given s, we identify all times t_s whose volatility $\sigma^2_{\Delta t}(t_s)$ is close to $e^{2s}E[\sigma^2]$, that is,

$$e^{2(s-ds)}E[\sigma^2] \le \sigma^2_{\Delta t}(t_s) \le e^{2(s+ds)}E[\sigma^2], \tag{17}$$

where $ds \ll 1$. For a given relative log-amplitude s, we translate and superimpose all time series starting at all the previously found times t_s. Averaging over these time series of volatility obtains the average conditional relaxation function of the volatility $E[\sigma(t|s)^2]$ following a local burst of volatility of amplitude (16). The MRW model predicts a power dependence

$$E[\sigma(t|s)^2] \sim t^{-\alpha(s)}, \tag{18}$$

with

$$\alpha(s) = \frac{2s}{3/2 + \ln(T/\Delta t)}, \tag{19}$$

when $\Delta t < t \ll \Delta t e^{|s|/\lambda^2}$, where $\lambda^2 \approx 0.02$. We keep the symbol $\alpha(s)$ for the exponent in (18) in line with the notation of Sornette et al. [66], but this should not be confused with the parameter α in (5) which controls the memory of the imitation coefficients K_{ij}.

Fig. 12 shows the average normalized conditional volatility $E[\sigma(t|s)^2]/E[\sigma^2]$ as a function of the time $t - t_s$ to the local burst of volatility at time t_s for different log-amplitudes s in double logarithmic coordinates. As expected from predictions (18) and (19) of the MRW, the average volatility after a burst decays (respectively, increases) when $s > 0$ (respectively, $s < 0$). Similar to real data analyzed by Sornette et al. [66], we observe approximate power laws. The power law exponents for different values of Δt are plotted in Fig. 13. The exponent $\alpha(s)$ depends linearly on s, with a slope increasing with Δt, in agreement with the prediction of the MRW and the finding of Section 3.2.3.

We can actually obtain a direct estimation of the integral time scale T by studying how the slope $1/k$ of $\alpha(s)$ as a function of s depends upon Δt. Expression (19) predicts that $k(\Delta t) = -\frac{1}{2}\ln(\Delta t) + \frac{1}{2}\ln(T) + \frac{3}{4}$. A linear regression of k as a function of $\ln(\Delta t)$ gives $k(\Delta t) = -0.48\ln(\Delta t) + 2.08$. The first coefficient -0.48 is nicely close to the exact value $-1/2$ predicted by the multifractal theory, which provides an independent check on the validity of multifractality. Identifying $\frac{1}{2}\ln(T) + \frac{3}{4}$ with 2.08 yields $T = 14.3$.

Fig. 14 shows the average relaxation of the volatility after an exogenous shock, created by imposing a very large news impact $G(t_s)$ at a single time t_s and then letting the system evolve according to its normal dynamics thereafter. To gather sufficient statistics, we impose such large shocks with a periodicity of several hundred

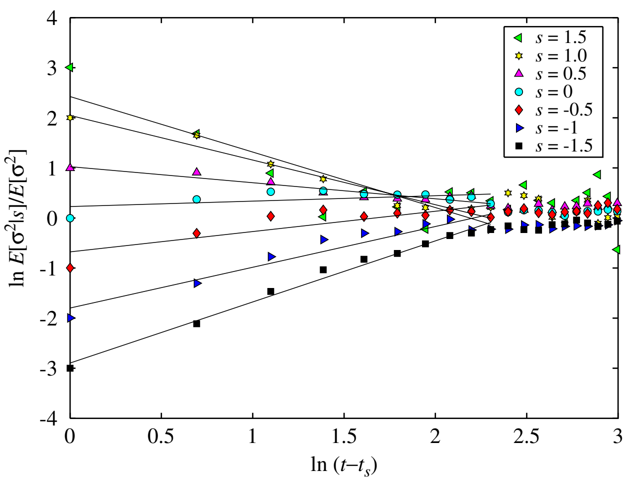

Fig. 12. Average normalized conditional volatility $\sigma^2_{\Delta t}(t)/E[\sigma^2]$ as a function of the time $t - t_s$ from the local burst of volatility at time t_s for different log-amplitudes s in double logarithmic coordinates.

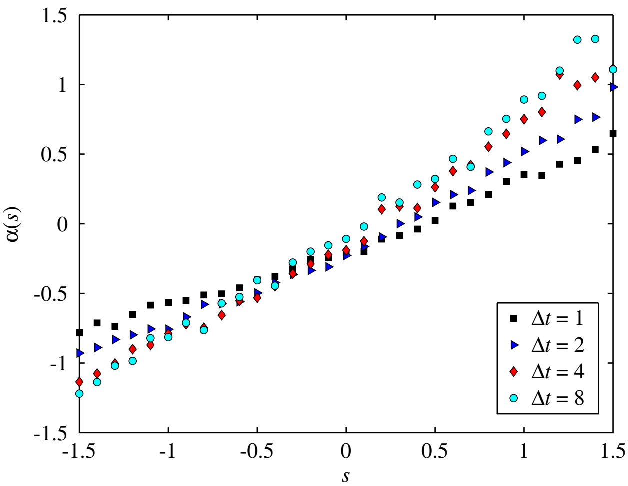

Fig. 13. Exponent $\alpha(s)$ of the conditional volatility response as a function of the endogenous shock amplitude S for $\Delta t = 1$, 2, 4, and 8.

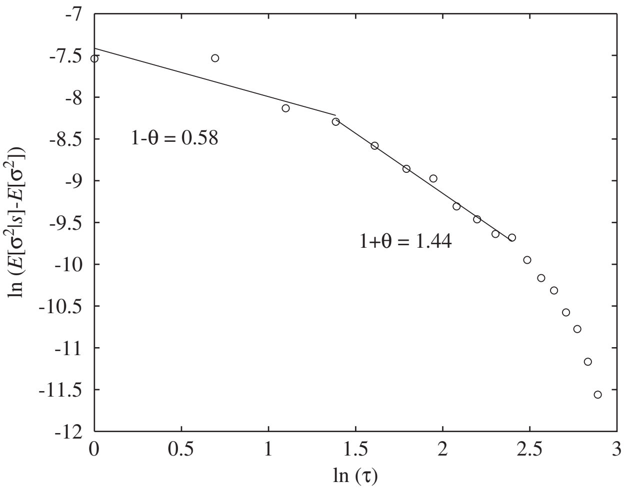

Fig. 14. Relaxation of superposed excess volatility after exogenous shocks obtained by imposing a very large news $G(t_s)$ for $\Delta t = 1$.

time steps, which is sufficiently long to allow the system to relax back to its normal fluctuating volatility. We have checked that the relaxation shown in Fig. 14 is independent of the amplitude of the shock when sufficiently large. Note that, immediately after the news impact, the volatility first increases over a few time step before relaxing, which reflects the strong increase of the coupling coefficient K_{ij} and the resulting stronger cooperativity of the agents. In order to mimic the previous analysis of the data by Sornette et al. [66], the origin of time for the relaxation of the volatility is taken at the peak time, rather than from the incipient exogenous news shock at t_s. This shifts the origin of time by approximately 2–3 time steps, which is compatible with the correspondence that one trading day corresponds to 4–16 time steps, as discussed in Section 3.2.2.

According to the theory relating the endogenous relaxation to the exogenous response function of the MRW developed by Sornette et al. [66], the relaxation shown in Fig. 14 should be characterized by an exponent close to $1/2$ over approximately the same range of times $t - t_s$ as found for the power law dependence of the relaxations shown in Fig. 12. We indeed observe a first decay regime which is compatible with a power law with an exponent close to 0.5 (but of course the range is too short to provide anything other than an indication). We also observe a crossover to a faster decay, compatible with a faster decaying power with exponent close to 1.5. This behavior is actually expected if the system is not exactly critical but close to critical, as shown by Sornette and Helmstetter [78] and Sornette et al. [79]: the response function to an exogenous shock should in this case cross over from a dependence proportional to $1/t^{1-\theta}$ to $1/t^{1+\theta}$, with $\theta = 1/2$ for a multifractal system. Fig. 14 thus suggests that the multifractal properties of our system hold only up to a finite time scale T beyond which a crossover to a non-critical behavior dominates.

4. Discussion

In this paper, we have extended the artificial stock market model introduced by Gonçalves [27] to include a memory in the dynamics of the influence coefficient on its past realization. This additional memory turns out to be a key ingredient to reproduce the major stylized facts of financial stock markets. In the previous specification $\alpha = 0$ of Gonçlaves [27], the influence coefficients K_i adjust instantaneously to previous news and returns realization. With a non-zero α as proposed here, the influence coefficients exhibit an inertia. We believe that this is a crucial property of the interactions between social agents who only relatively slowly update their tendency to imitate their colleagues or friends. That this parameter provides, together with the competition between imitation and news, the main stylized facts of financial stock markets is an encouraging sign that we have correctly captured some of the most important ingredients at the origin of the organization of financial stock markets.

These ingredients comprise imitation between agents, their influence by external news and the impact of their private information. The imitation plus the idiosyncratic part of the decision process together give the dynamics of the Ising model. The news act then as an time-dependent external field. The addition of an evolution in the influence coefficients (which can also be called ‘‘coupling coefficients’’) make the strength of the imitations between agents a function of the past realization of the news and returns.

The empirical stylized facts of financial stock markets have been found only for $\beta > 0$ and α neither too small nor too large. The condition $\beta > 0$ means that agents increase the propensity to imitate if the external news have been predictive of the returns in the past. This behavior corresponds to agents who misinterpret, or misattribute the source of the prediction of returns. Alternatively, this behavior corresponds to overconfident agents. Technically, the stylized facts in this regime result from the fact that the model operates around the critical point of the corresponding Ising model, with coupling coefficients which are time-dependent and endowed with a memory of past realizations. The critical point of the Ising model is associated with a critical value K_c for the average coupling coefficient. Close to this value, agents organize spontaneously within clusters of similar opinions, which become very susceptible to small external influences, such as a change of news. This may explain the occurrence of crashes, as argued previously (see, Ref. [80, Chapter 5]). The present model exhibits bubbles and crashes as shown in Fig. 15, whose detailed study will be reported elsewhere.

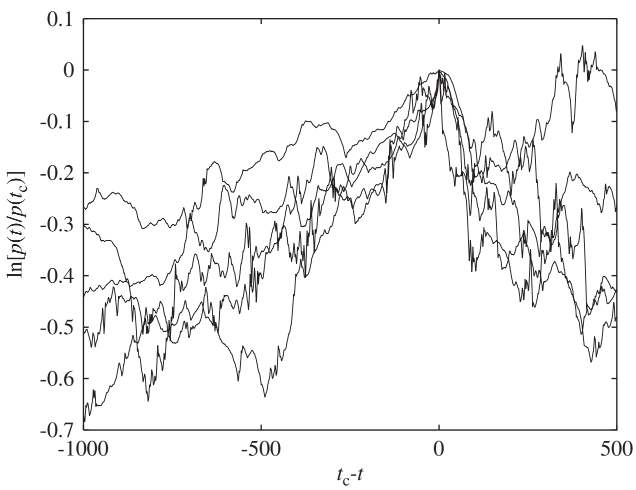

Fig. 15. Five price trajectories showing bubbles preceding crashes that occur at the shifted time 0. The five time series have been translated so that the time of their crash is placed at the origin $t = 0$.

Intuitively, the critical slowing down well-known to characterize the proximity to the critical Ising point can explain the long-term memory of the volatility while the almost absent correlation of the returns themselves is ensured by the impact of the news and the random idiosyncratic decisions.

The fact that the most basic stylized facts (monomodal shape with fat tails, short-time return correlations and long-memory of the volatility) cannot be obtained for $\beta < 0$ suggests the importance of the imitation behavior captured by (10) with $\beta > 0$. In this model, the creation of anomalous volatility, of its persistence, and of multifractality result from the tendency of agents to misinterpret the combined information of the news and of the stock market as resulting from the influence of the agents and their imitation. Or seen from a different view point, conditioned on their role of reflecting the stock market, the news serve as the substrate for fostering social interactions and reinforcing herding. By the mechanism of intermittent reinforcing of social interaction in (10), the coupling coefficient K_{ij} will vary and sometimes increase close to or cross a critical value at which critical fluctuations occur and beyond which global cooperativity dominates.

As a bonus, we have discovered that this simple model exhibits a rich multifractal structure, diagnosed not only by the standard convexity of the exponents of the structure functions, but also by distinct power law response functions to endogenous compared with exogenous volatility shocks [66,78,79]. To our knowledge, this is the first nonlinear model in which such clear distinction is documented quantitatively, based on a bottom-up self-organization. In contrast, the multifractal random walk which has provided the theoretical predictions used here is a descriptive phenomenological model.

Let us end by noting the connection with the model of Wyart and Bouchaud [81], which can be embedded in our model by putting all K_{ij} to 0 (no imitation) and adding a dependence of σ_i on past correlations between the realized returns $r(t)$ and the available information $G(t)$, similar to the dynamics of K_{ij} in our model. Wyart and Bouchaud [81] have studied the limit of self-fulfilling conventions created by the belief of agents on the existence of correlations between information and returns. By this belief, traders try to estimate this correlation from past time series and act on it, thus creating it. Our model emphasizes the other class of conventions based on imitation and moods. For the future, it would be interesting to combine both mechanisms as they are arguably present together in real markets, in order to clarify their relative importance and interplay.

Acknowledgements

We are grateful to Carlos Pedro Gonçalves, Jean-François Muzy and Matthieu Wyart for stimulating discussions. The research by W.-X. Zhou was supported by NSFC (No. 70501011) and by NSFC/PetroChina jointly through a major project on multiscale methodology (No. 20490200).

Appendix: Justification that the imitation term is rational in the absence of reliable information

Form (6) of the formulation of the decision of agent i (without the term $\sigma_i(t)G(t)$) derives naturally from an argument of bounded rationality, as follows.

Let us denote $N(i)$ the number of traders directly connected to i on the graph of acquaintance ($N_i = 4$ for the 2D-square topology). The traders buy or sell one asset at price $p(t)$ which evolves as a function of time assumed to be discrete and measured in units of the time step Δt. In the simplest version of the model, each agent can either buy or sell only one unit of the asset. This is quantified by the buy state $s_i = +1$ or the sell state $s_i = -1$. Each agent can trade at time $t-1$ at the price $p(t-1)$ based on all previous information including that at $t-1$. We assume that the asset price variation is determined by the following equation:

$$\frac{p(t) - p(t-1)}{p(t-1)} = F\left(\frac{\sum_{i=1}^{N} s_i(t-1)}{N}\right) + \sigma\eta(t), \tag{20}$$

where σ is the price volatility per unit time and $\eta(t)$ is a white Gaussian noise with unit variance. The first term in the r.h.s. of (20) is the systematic price drift resulting from the possible imbalance between buyers and sellers. The impact function $F(x)$ is such that $F(0) = 0$ and is monotonically increasing with its argument as shown by recent empirical studies [82–85]: perfect balance between buyers and sellers does not move the price; a larger (resp. smaller) number of buyers than sellers drive the price up (resp. down). An often used dependence is simply a linear relationship $F(x) = \mu x$ [86–88]. The second stochastic term of the r.h.s. of (20) accounts for noisy sources of price fluctuations. Taken alone, it would give the usual log-normal random walk process.

At time $t-1$, just when the price $p(t-1)$ has been announced, the trader i defines her strategy $s_i(t-1)$ that she will hold from $t-1$ to t, thus realizing the profit/loss $(p(t) - p(t-1))s_i(t-1)$. To define $s_i(t-1)$, the trader calculates her expected profit P_E, given the past information and her position, and then chooses $s_i(t-1)$ such that P_E is maximum. Within the rational expectation model, all traders have full knowledge of the fundamental equation (20) of their financial world. However, they cannot poll the positions s_j of all other traders which will determine the price drift according to (20). The next best thing that trader i can do is to poll her $N(i)$ "neighbors" and construct her prediction for the price drift from this information. Note that, in this approach, the "neighbors" are by definition those who are polled by the trader according to her network of acquaintance. The trader needs an additional information, namely the a priori probability P_+ and P_- for each trader to buy or sell. The probabilities P_+ and P_- are the only information that she can use for all the traders who are not polled directly. From this, she can form her expectation of the price change. The simplest case corresponds to a neutral market where $P_+ = P_- = 1/2$. The trader i expects the following relative price change:

$$\mu\left(\frac{\sum_{j=1}^{*N(i)} s_j(t-1)}{N}\right) + \sigma\eta(t), \tag{21}$$

where the index j runs over the neighborhood of agent i. Notice that the sum is now restricted to the $N(i)$ neighbors of trader i. The contribution of all other traders, whom she cannot poll directly, is one contribution to the stochastic term $\sigma\eta(t)$. The restricted sum over the neighbors is represented by the star symbol. Her expected profit is thus

$$\left(\mu\left(\frac{\sum_{j=1}^{*N(i)} s_j(t-1)}{N}\right) + \sigma\eta(t)\right)p(t-1)s_i(t-1). \tag{22}$$

The strategy that maximizes her profit is

$$s_i(t-1) = \mathrm{sign}\left(\frac{\mu}{N}\sum_{j=1}^{N(i)*} s_j(t-1) + \sigma\eta(t)\right). \tag{23}$$

The equation recovers (6) (without the term $\sigma_i(t)G(t)$) by identifying $\mu/\sigma N$ with $K_{ij} = K$ taken uniform. The evolution of opinions given by (23) is nothing but the dynamical version of the Ising model in which the sentiments $\{s_i\}$ are called "magnetic spins." Recall that the Ising model exhibits a phase transition between two phases:

(1) For weak coupling between spins (or large idiosyncratic noise σ), the spins take random signs and there is no majority opinion. The average opinion is zero.
(2) Above a threshold K_c for the average coupling coefficient (or below a threshold σ_c), the spins align spontaneously along a preferred direction; there is a non-zero majority opinion, which can be either $+1$ or -1 depending on the history. The properties of this transition and the spontaneous symmetry breaking in the context of social agents is described in details by Sornette [80, see Chapter 5].

In the extension (1) with (5), the coupling coefficients K_{ij} between "spins" is allowed to change with time. During their dynamical evolution, one can expect that the coupling coefficients K_{ij} between agents' sentiments explore a large set of possible values according to a process similar to the increments of a random walk-like trajectory. At some times, a significant fraction of the K_{ij} may approach or pass above the critical value K_c: this will lead, according to the mechanism of the phase transition in the Ising model, to the occurrence of a majority of opinion, and thus to strong herding behavior. The introduction of the dynamics of the coupling coefficient $K_{ij}(t)$ is reminiscent of the mechanism discussed by Stauffer and Sornette [89] in the similar context of a percolation model of cluster of agents impacting the price evolution Cont and Bouchaud [11], in which the connectivity parameter is varied randomly in time. It is also providing a specific mechanism for the occurrence of critical times as explained by Sornette [80, see Chapter 5]. The additional existence of external news corresponds in the language of the Ising model to an external forcing ''field'' which help the opinion of the majority in the herding phase to bifurcate towards one or the other values ± 1. However, the dynamics of (1) with (5) is clearly more complicated than for the Ising model due to the feedback of the external news and the collective decision process captured by the term $\beta r(t)G(t)$ on the coefficients of influences K_{ij}.

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