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A unified model for price return distributions used in econophysics

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ABSTRACT

For a decade, a new theoretical movement called "econophysics" has been initiated by some physicists who began to publish articles devoted to the study of economic and financial phenomena. Since then, econophysicists have written a very prolific literature about the way of characterizing the evolution of financial prices. Today, there is an "extreme diversity" of models recently developed by econophysicists whose research is sometimes presented as an ill-defined field. The objective of this paper is precisely to provide a unified framework in order to contribute to unify econophysics and to base this new field on shared scientific standards.

0. Introduction

For the 1970s, a new theoretical movement has been initiated by some physicists who began publishing articles devoted to the study of social phenomena, such as the formation of social groups [1] or social mimetism [2].[1] The next decade confirmed this new theoretical trend (labelled *sociophysics*[2]), as the number of physicists publishing papers devoted to the explanation of social phenomena and the number of themes analysed continued to increase.[3] During the 1990s, physicists[4] turned their attention to economics, and particularly financial economics, giving rise to econophysics.[5] Although the movement's official birth for example announcement came in a 1996 article by Stanley et al. [11],[6] econophysics was at that time still a young and ill-defined field. Econophysics can be defined as "a quantitative approach using ideas, models,

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1 Regarding the emergence and history of sociophysics, see Ref. [3].

2 This term was proposed by Serge Galam in a 1982 article. In his view, one of the reasons why physicists attempt to explain social phenomena stems from a kind of mismatch between the theoretical power of physics and the inert nature of its subject matter: "During my research, I started to advocate the use of modern theory phase transitions to describe social, psychological, political and economical phenomena. My claim was motivated by an analysis of some epistemological contradictions within physics. On the one hand, the power of concepts and tools of statistical physics were enormous, and on the other hand, I was expecting that physics would soon reach the limits of investigating inert matter" [2, p. 50].

3 Let us mention, for example industrial strikes [4], democratic structures [5], and elections [3,6].

4 The influence of physics on the study of financial markets is not new, as witnessed by the work of Bachelier [7] and Black and Scholes [8]. Nevertheless, we cannot yet refer to Black & Scholes' model as econophysics in the term's current meaning, since it was completely integrated into the dominant theoretical current of economics and finance [9]. Econophysics is not an "adapted import" of the methodology used in physics; rather, it is closer to a "methodological invasion". We return to this point in the next section.

5 For a historical analysis, see Ref. [10].

6 This article is also the origin of the term econophysics.

We would point out, however, that Kutner and Grech [12] trace the informal birth of the approach to the paper by Mategna [13] that studied the evolution of returns on financial markets in terms of Lévy processes. This definition seemed to gain ground as a compromise, and is found in a number of books and articles produced by the current, for example by Wang et al. [14, p. 1] or Rickles [15].

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conceptual and computational methods of statistical physics''.[7] Today, econophysics is an institutionalized field, [17] with different journals proposing a prolific literature about the way of characterizing the evolution of financial prices. There is an ''extreme diversity'' of models recently developed by econophysicists [18] and many theoretical frameworks still emerge.

In this paper, our objective is precisely to provide a such unified framework. Indeed, the standardization of knowledge through a common scientific culture is a necessary condition to become a strong discipline [19]. We propose a generic formula characterizing the statistical distributions usually used by econophysicists (Levy, Truncated Levy or no stable Levy distributions). Such formula will contribute to unify econophysics and to base this new field on shared scientific standards since the possibility to find a generalized formula is derived from the common conceptual tools shared by econophysicists. This will enable econophysics be no longer an ill-defined field. Moreover, such generalized formula allows a systematic comparison between the different models used by econophysicists.

1. Econophysics: a new field of research

According to Kutner and Grech [12], econophysics as a field of research dates back to 1991 when Mantegna published a paper about Levy process in finance. However, one can trace the roots of the basic ideas of econophysics to papers by Benoît Mandelbrot [20,21] who saw an analogy between the evolution of financial markets and the phenomenon of turbulence. It is only about thirty years later that these discussions re-emerged under the label ''econophysics''. As the name suggests, econophysics presents itself as a hybrid discipline which can be defined in methodological terms as ''a quantitative approach using ideas, models, conceptual and computational methods of statistical physics'' applied to economic and financial phenomena [22, p. 1].

Econophysics presents itself as a new way of thinking about the economic and financial systems through the ''lenses'' of physics [23]. As much as neoclassical economics imported models from classical physics as formulated by Lagrange [24], and financial economics built on the model of Brownian motion also imported from physics, econophysics tries to model economic phenomena using analogies taken from modern condensed matter physics and its associated mathematical tools and concepts. Using the standard tools of statistical mechanics including microscopic models like Ising model and scaling laws, econophysicists aim at explaining how complex economic systems behave. Broadly speaking, econophysics is founded on general statistical properties that reappear across many and diverse phenomena [25]. This statistical regularity can be characterized by scaling laws that are considered as the heart of econophysics[8] [26] or [27, p. 288]. These scaling laws can take a variety of forms. The objective of the next section is to offer a generic formula characterizing the main distributions usually used by econophysicists.

2. Generalized formula for price return distributions

For describing the probability distributions of stock market price changes, many models using different types of probability functions are proposed in the econophysics literature. However, Gringras and Schinckus [17] showed that *Physica A* appears to be the leading journal and that Mantegna, Bouchaud, Mandelbrot, Sornette and Lux are the most cited authors in econophysics. Our analysis is based on these results. We also add other important authors such as Stanley, Gopikrishnan or Plerou who are also very cited authors in econophysics (*Web of Science*). Among the authors identified, we have selected econophysics papers dedicated to distribution of price returns (see Tables 1 and 2). From articles listed in Tables 1 and 2, we propose the following generalized formula:

$$P(x) = Cf(x)\mathrm{e}^{-g[h(x)]+d} \tag{1}$$

where, C and d are constants that might have temporal variation.

The analytical form of $f(x)$ is not always known for all the values of x, but it has a power law variation in the limit of large $x(x \to \infty)$:

$$f(x) \sim \frac{1}{x^{b_1+a_1\alpha}} \quad \text{for } |x| \to \infty \tag{2}$$

a_1 and b_1 are two parameters (usually equal to 1) that define the shape of the distribution at large x, and α is the principal exponent of the power law. The function g introduced in Eq. (1) has the form:

$$g(x) = (a_2 h(x) + b_2)^{c_2} \tag{3}$$

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[7] [11, p. 2] To present econophysics as an extension of statistical mechanics necessitates a better definition of this approach in physics. Statistical mechanics attempts mainly to explain in statistical terms the behaviour and macroscopic evolution of a complex system on the basis of interactions of a large number of microscopic constituents (atoms, electrons, ions, etc.) that make it up [16, p. 155]. Applied to finance, this type of reasoning allows one to consider the market as the statistical and macroscopic results of a very large number of heterogeneous interactions at the microscopic level.

[8] These scaling laws can then be viewed as a macroresult of the behaviour of a large number of interacting components from lower levels. As Rickles [18] explains, ''The idea is that in statistical physics, systems that consist of a large number of interacting parts often are found to obey 'universal laws' – laws independent causally of microscopic details and dependent on just a few macroscopic parameters''.

Table 1
Lévy stable – Gaussian distributions.

Reference	Author formula	Generic formula
[7] — The first model for the stochastic process of returns		
[8] — An option pricing technique — the Gaussian distribution is one of the principal assumptions	$P(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/(2\sigma^2)}$	$P(x) \sim C \frac{1}{x^{b_1+a_1\alpha}} e^{-(a_2x+b_2)^{c_2}+d}$ for $x \to \infty$ $C = \frac{1}{\sqrt{2\pi\sigma^2}}$, $h(x) = x$ $a_1 = b_1 = 0$, $a_2 = 1/(2\sigma^2)^{0.5}$, $b_2 = 0$, $c_2 = 2$, $d = 0$
[28] — Mixture of Gaussian distributions		
[29]	$P(s', t_f; s, t_i) = (4\pi\sigma^2 Ts's)^{-1} \exp\left[-\frac{(\log(s'/s)+BT)^2}{2\sigma^2 T}\right]$ s' — price at time t_f; s — price at time t; $T = t_f - t_i$	$P(x) \sim C \frac{1}{x^{b_1+a_1\alpha}} e^{-(a_2 \log(x)+b_2)^{c_2}+d}$ for $x \to \infty$ $h(x) = \log(x)$ $a_1 = 0$, $b_1 = -1$, $a_2 = 1/(2\sigma 2T)^{0.5}$, $b_2 = [BT - \log(s)]/(2\sigma^2 T)^{0.5}$, $c_2 = 2$, $d = 0$

with two possible forms for $h(x)$:x or $\log(x)$. The use of lognormal law in finance was introduced by Osbone [53] in order to avoid the theoretical possibility to have negative prices. Moreover, this use is also based on the assumption that the rate of returns rather than the change of prices, are independent random variables. In Eq. (3) a_2, b_2 and c_2 are parameters that are different from one model to another, defining the final shape of the distribution function. Finally our generalized formula:

$$P(x) \sim C \frac{1}{x^{b_1+a_1\alpha}} e^{-(a_2x+b_2)^{c_2}+a}. \tag{4}$$

3. Application

Our formula allows to rewrite and to compare the distribution of price changes used in the main econophysics models. This section, for providing a classification of the main econophysics models uses this formula. Three classes of econophysics models are considered depending the distribution used (Levy, Truncated Levy or no stable Levy distribution). However, we will apply Eq. (4) to the Gaussian distribution first.

3.1. Gaussian distribution

The Gaussian distribution, which is the most simple particular case of Eq. (4), can be obtained when $c = \frac{1}{\sqrt{2\pi\sigma^2}}$, $a_1 = b_1 = 0$, $b_2 = d = 0$, $c_2 = 2$ and $a_2 = \frac{1}{\sqrt{2\sigma^2}}$:

$$P(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/(2\sigma^2)}.$$

Gaussian distribution is intensively used in neoclassical finance for describing price variations. Unfortunately this distribution does not describe correctly the empirical data especially for high price variations. With a Gaussian distribution, the probability of having very high price variations is much lower than what is observed in real data, thus the appearance of financial crashes is highly underestimated [25].

Because econophysicists adopt an empiricist perspective [54], they are looking for the distribution functions $P(x)$ that must fit empirical financial data without a priorism. Sometimes, distribution functions are directly derived from physics models used for describing stochastic dynamic process. A common characteristic of these distribution functions used by econophysicists refers to a power law variation for large values of x (Pareto law). However, the exponent of the power law differs from one authors to another.[9] Also the functional form of $P(x)$ for small and medium values of x is also quite different depending from the author. We will now present the three categories of models used by econophysicists that we have identified from our survey.

3.2. Lévy stable distributions

These distribution functions were first proposed by Mandelbrot [20] and used afterwards by the pioneers of econophysics since they describe better the tail of the distribution of financial data then a Gaussian distribution. Most important, for large

[9] See Ref. [55] for a survey about the potential value of this exponent.

Table 2
Lévy stable – Paretian distributions.

Reference	Author formula	Generic formula
[30] They use the conditionally exponential decay model – power law distribution – Apply this distribution function for modelling daily returns of the DJIA and S&P500 financial indices as well as returns of the USD/DEM exchange rate	$f(r)=\begin{cases}C_1(\lambda r)^{\alpha-1}, & 0\le\lambda r\le 1\\ C_2(\lambda r)^{-(\alpha/k)-1}, & \lambda r\ge 1\end{cases}$ $\alpha\cong 1.09$ C_1, C_2 depends upon parameters α, λ and k	$P(x)\sim C\frac{1}{x^{b_1+a_1\alpha}}e^{-(a_2x+b_2)^{c_2}+d}$ for $x\to\infty$ $a_1=-1, b_1=1, a_2=b_2=c_2=0$ $\alpha\cong 1.09$ $h(x)=x, d=0$
[31] They use the Generalized Lotka–Volterra (GLV) model to explain power law distributions in individual wealth (Pareto laws) and in financial market returns	$P(x_i)=x_i^{-1-\alpha}e^{-2a/(Dx_i)}$ $\alpha=1.5$	$P(x)\sim C\frac{1}{x^{b_1+a_1\alpha}}e^{-(a_2x+b_2)^{c_2}+d}$ for $x\to\infty$ $a_1=1, b_1=1,\ a_2=2a/D, b_2=0, c_2=-1$ $\alpha\cong 1.5$ $h(x)=x, d=0$
[32] They model the distribution of population in cities and that of the wealth Power law $\alpha\approx 1.4$	$P(w)\propto w^{-1-\alpha}$ $\alpha\approx 1.4$	$P(x)\sim C\frac{1}{x^{b_1+a_1\alpha}}e^{-(a_2x+b_2)^{c_2}+d}$ for $x\to\infty$ $a_1=1, b_1=1, a_2=b_2=c_2=0$ $h(x)=x, d=0$ $\alpha\cong 1.4$
[33] Power law $\alpha=1.5$	Power law $\alpha=1.5$ Idem to 3 Obs. Articles 2, 3, 4 are connected	$P(x)\sim C\frac{1}{x^{b_1+a_1\alpha}}e^{-(a_2x+b_2)^{c_2}+d}$ for $x\to\infty$ $a_1=1,\ b_1=1,\ a_2=b_2=c_2=0,$ $h(x)=x, d=0$ $\alpha\cong 1.5$
[34] Lévy α stable distribution Ex: $\alpha=1.57, \beta=0.159, \gamma=6.76\times10^{-3}$, $\delta=3.5\times10^{-4}$ General forms for Lévy distributions – The paper illustrates a procedure for fitting financial data with Lévy α-stable distributions	The characteristic function for symmetric distributions: $L_\alpha(k)=\exp(-a\lvert k\rvert^\alpha)$ $L_\alpha(x)\sim\frac{1}{\lvert x\rvert^{1+\alpha}}$ for $\lvert x\rvert\to\infty$	$P(x)\sim C\frac{1}{x^{b_1+a_1\alpha}}e^{-(a_2x+b_2)^{c_2}+d}$ for $x\to\infty$ $a_1=1, b_1=1, a_2=b_2=c_2=0$ $h(x)=x, d=0$ $\alpha\cong 1.4$
[35] Lévy stable distributions	Idem to 5	
[36] —They analyse variations on the Norwegian and USA stock markets - measure the local Hurst exponent H (related to α; $\alpha=1/H$) — Reach the conclusion that Lévy stable distributions describe empirical data much better than Gaussian. — H is obtained from log–log plot of P(Dt) when $D_x=0$	Lévy stable-$\alpha=1.64$	$P(x)\sim C\frac{1}{x^{b_1+a_1\alpha}}e^{-(a_2x+b_2)^{c_2}+d}$ for $x\to\infty$ $a_1=1, b_1=1, a_2=b_2=c_2=0$ $h(x)=x, d=0$ $\alpha\cong 1.64$

(*continued on next page*)

x, the Lévy stable distributions are well approximated by a power law as described in Eq. (2) with the exponent α having values between 1 and 2, generally around 1.5. When compared to Eq. (5) most of the authors use $a_1=b_1=1$ [31–37,56] with one special case where $a_1=-1$ and $b_1=1$ [30]. The parameter a_2 is non-zero only in three cases [31,37,56] and $c_2=-1$ in Ref. [31]. The others parameters of Eq. (5) are always taken to be zero in this case. The observed non-zero a_2 involves the presence of an exponential term in the distribution function $P(x)$ which is derived by using models to explain the empirical data (for example, generalized Lotka–Volterra model [31] and the Percolation model [37,56]). But most often the authors focus to calculate the power law exponent of the distribution tail in some specific situations.

Table 2 (*continued*)

Reference	Author formula	Generic formula
[37] Percolation theory – one gets the distribution for the cluster (of financial operators) size – identical to the distribution of price changes according to **percolation theory** — The probability that financial operators interact between them is **c/N**; N is the total number of operators	$P(S) \sim \frac{1}{S^{5/2}} \exp\left(-\varepsilon^2 S\right)$ S – cluster size; $\varepsilon = 1 - c$ –for $c = 1$ one has a pure power law, Lévy symmetrical, distribution with $\alpha = 3/2$.	$P(x) \sim C \frac{1}{x^{b_1 + a_1 \alpha}} e^{-(a_2 x + b_2)^{c_2} + d}$ for $x \to \infty$ $a_1 = 1, b_1 = 1, a_2 = \varepsilon^2, b_2 = 0, c_2 = 1$ $\alpha \cong 1.5$ $h(x) = x, d = 0$ $\varepsilon = 1 - c$ $h(x) = x, d = 0$
[26] Discussion of results of 8	$P(S) \sim \frac{1}{S^{5/2}} \exp\left(-\varepsilon^2 S\right)$ $\alpha = 3/2$	$P(x) \sim C \frac{1}{x^{b_1 + a_1 \alpha}} e^{-(a_2 x + b_2)^{c_2} + d}$ for $x \to \infty$ $a_1 = 1, b_1 = 1, a_2 = \varepsilon^2, b_2 = 0, c_2 = 1$ $h(x) = x, d = 0$ $\alpha \cong 1.5$

Table 3
Lévy non-stable distributions.

Reference	Author formula	Generic formula
[38] Power law (asymptotic) distribution $\alpha = 3$	$P(g_i(t) > g) \propto g^{-\alpha}$ $\alpha \approx 3$	$P(x) \sim C \frac{1}{x^{b_1 + a_1 \alpha}} e^{-(a_2 x + b_2)^{c_2} + d}$ for $x \to \infty$ $a_1 = 1, b_1 = 1, a_2 = b_2 = c_2 = 0$ $h(x) = x, d = 0$ $\alpha \cong 3$
[39] Examination of German stocks with stable Levy distributions – A detailed analysis leads to conclusion that Lévy stable is not such a good fit – it is assumed that the real distribution is not stable	See comments – Power law – Lévy non-stable	$P(x) \sim C \frac{1}{x^{b_1 + a_1 \alpha}} e^{-(a_2 x + b_2)^{c_2} + d}$ for $x \to \infty$ $a_1 = 1, b_1 = 1, a_2 = b_2 = c_2 = 0$ $h(x) = x, d = 0$ $\alpha > 2$
[40] They analyse the values of the S&P500 index	$P(g > x) \propto \frac{1}{x^\alpha}$ g is a normalized return: $g = \frac{G - \langle G \rangle_T}{v}$ $v^2 = \langle G^2 \rangle_T - \langle G \rangle_T^2$ volatility $\alpha \approx 3$ for $3 < g < 50$; $\alpha \approx 1.6$ for $3 < g < 50$	$P(x) \sim C \frac{1}{x^{b_1 + a_1 \alpha}} e^{-(a_2 x + b_2)^{c_2} + d}$ for $x \to \infty$ $a_1 = 1, b_1 = 1, a_2 = b_2 = c_2 = 0$ $h(x) = x, d = 0$ $\alpha \cong 3$
[41] Theoretical model that uses Fokker–Planck equation (describes anomalous diffusion) to derive the probability function Non-constant diffusion coefficient — one approaches a power law for high u and a quadratic variation for D	$F(u) = C \exp\left(\frac{-\lvert u \rvert}{D_0 \varepsilon}\right) (\varepsilon \lvert u \rvert + 1)^{\alpha - 1}$ for $D = D_0(1 + \varepsilon \lvert u \rvert)$ $u = \frac{x}{\sqrt{t}}$, x price return $F(u) = \frac{C}{(1 + \varepsilon u^2)^{1+\beta}}$, $D = D_0(1 + \varepsilon u^2)$	$P(x) \sim C \frac{1}{x^{b_1 + a_1 \alpha}} e^{-(a_2 x + b_2)^{c_2} + d}$ for $x \to \infty$ $a_1 = -1, b_1 = 1, a_2 = 1/(D_0 \varepsilon), b_2 = 0, c_2 = 1$ $h(x) = x, d = 0$
[42] — They estimate the power law tail exponent (α) using Hill estimator, for the personal income in Australia and Italy	$P(i > x) \propto \frac{1}{x^\alpha}$ $\alpha \sim 2.3$ — Australia $\alpha \sim 2.5$ — Italy	$P(x) \sim C \frac{1}{x^{b_1 + a_1 \alpha}} e^{-(a_2 x + b_2)^{c_2} + d}$ for $x \to \infty$ $a_1 = 1, b_1 = 1, a_2 = b_2 = c_2 = 0$ $h(x) = x, d = 0$ $\alpha \cong 2.5$

The main drawback of Lévy stable distributions is that they have infinite variance, a situation that in physics cannot be accepted. As Gupta and Campanha [38, p. 32] point out, "Lévy flight have mathematical properties that discourage a physical approach because they have infinite variance". Physicists have chosen to characterize financial phenomena through Lévy processes but they explicitly reject the idea of infinite variance. In this perspective, some physicists have developed statistical methods in order to truncate the Levy stable distribution (see Tables 3–5).

Table 4
Lévy truncated distributions.

Reference	Author formula	Generic formula
[43] Truncated Lévy distribution, $\alpha = 1.5$ — Lévy distribution normalized by a constant. — Not stable, has finite variance.	$P(x) = \begin{cases} 0, & x > l \\ cP_l(x), & -l < x < l \\ 0, & x < l \end{cases}$ $P_l(x)$ — Lévy function	$P(x) \sim C \frac{1}{x^{b_1+a_1\alpha}} e^{-(a_2x+b_2)^{c_2}+d}$ for $x \to \infty$ $a_1 = 1, b_1 = 1, a_2 = b_2 = c_2 = 0$ $\alpha \cong 1.5$ for $-l < x < l$. $h(x) = x, d = 0$ $P(x) = 0$ for all other values of x, $a_2 \sim \infty$
[44] – Study of market indices from Brazil, Mexico, Argentine Exponentially truncated Lévy distribution $\alpha < .2$	They use the characteristic function in simulations – Probably same kind of equation as at 3	$P(x) \sim C \frac{1}{x^{b_1+a_1\alpha}} e^{-(a_2x+b_2)^{c_2}+d}$ for $x \to \infty$ $a_1 = 1, b_1 = 1, a_2 = 1/k, b_2 = -1/k, c_2 = 1$ $h(x) = x, d = 0$ $\alpha < 2$ for $\|x\| > 1$
[45] – They propose a gradually truncated Lévy distribution	$P(x) = \begin{cases} cL_\alpha(x, \Delta t), -l_C \le x \le l_C \\ CL_\alpha(x, \Delta t) \exp\left\{-\left(\frac{\|x\| - l_C}{k}\right)^\beta\right\}, \|x\| > l_C \end{cases}$ k, β are constants $\alpha \cong 1.2, \beta \approx 0.6$ pour S&P500	$P(x) \sim C \frac{1}{x^{b_1+a_1\alpha}} e^{-(a_2x+b_2)^{c_2}+d}$ for $x \to \infty$ $a_1 = 1, b_1 = 1, a_2 = 1/k, b_2 = -1/k, c_2 = \beta \sim 0.6$ $h(x) = x, d = 0$ $\alpha = 1.2$ for $\|x\| > 1$
[37] – Percolation theory – one gets the distribution for the cluster (of financial operators) size – identical to the distribution of price changes –The probability that operators interact between them is c/N; N is the total number of operators	$P(S) \sim \frac{1}{S^{5/2}} \exp\left(-\varepsilon^2 S\right)$ S – cluster size; $\varepsilon = 1 - c$ – for $c < 1$ one has an exponentially truncated Lévy distribution	$P(x) \sim C \frac{1}{x^{b_1+a_1\alpha}} e^{-(a_2x+b_2)^{c_2}+d}$ for $x \to \infty$ $a_1 = 1, b_1 = 1, a_2 = -\varepsilon^2, b_2 = 0, c_2 = 1$ $h(x) = x, d = 0$ $\alpha \cong 1.5$ $\varepsilon = 1 - c$
[26] — Discussion of results of 4	$P(S) \sim \frac{1}{S^{5/2}} \exp\left(-\varepsilon^2 S\right)$	$P(x) \sim C \frac{1}{x^{b_1+a_1\alpha}} e^{-(a_2x+b_2)^{c_2}+d}$ for $x \to \infty$ $a_1 = 1, b_1 = 1, a_2 = -\varepsilon^2, b_2 = c_2 = 0$ $h(x) = x, d = 0$ $\alpha \cong 1.5$

3.3. *Truncated Lévy distributions*

Some econophysicists suggested that it would be preferable to use distributions with finite variance for describing the stock price variations. Two reasons can be evoked for this: on one hand, the fact that a finite variance is more in line with a physical approach and, on the other hand, this notion of variance usually refers to the idea of risk in finance.

These econophysicists developed truncated Lévy distributions in order to solve the problem of infinite variance. For these cases, stable Lévy distributions are used with the specific condition that there is a cut-off length for the price variations above which the distribution function is set to zero in the simplest case [44], or decreases exponentially [45, p. 52–55]. These functions are chosen in order to obtain the best fit with the empirical data [44,46] or in other situations are derived from models like percolation theory [47,48] or the generalized Fokker–Planck equation [52].

We can find these truncated distributions in our generalized formula when $a_1 = b_1 = 1$ and at least a_2 and c_2 different of zero (b_2 is non-zero in Refs. [45,46,52]). Only the Ref. [52] gives a distribution function with the d constant non-zero. For the simply truncated distribution from Ref. [44] one can consider a_2 very large (going to ∞) beyond the cut-off length.

3.4. *Other non-stable Lévy distributions*

Some empirical studies about financial markets suggested that Lévy stable could overestimate the presence of large price variations even though they are much closer of data than a Gaussian [55].

Table 5
Other distributions.

Reference	Author formula	Generic formula
[46] Tsallis power law distribution — the value of q smaller than 5/3 allows a **finite variance** — Can be classified as Lévy stable	$P(x,t) = \frac{1}{Z(t)}\left\{1+\beta(t)(q-1)\left[x-\bar{x}(t)\right]^2\right\}^{\frac{1}{1-q}}$ $P(x,t) = x^{\frac{-2}{q-1}} = x^{-(\alpha+1)}$ for x large q Tsallis parameter, (1.64 for S&P500) – good fit on the empirical data for $q = 5/3$ one gets Gaussian; for $1 < q \leq 5/3$ – Lévy non stable; for $5/3 < q < 3$ — Lévy stable;	$P(x) \sim C\frac{1}{x^{b_1+a_1\alpha}}\mathrm{e}^{-(a_2x+b_2)^{c_2}+d}$ for $x \to \infty$ $a_1 = 1, b_1 = 1, a_2 = b_2 = c_2 = 0$ $h(x) = x, d = 0$ $\alpha \sim 2.15$
[47] Student's t-distribution	$St_\alpha(x) = \frac{C(\alpha)}{\left(A^2+x^2\right)^{(1+\alpha)/2}}$ $St_\alpha(x) \sim x^{-(1+\alpha)}$ for $x \to \infty$ behaves like Lévy distribution for large x	$P(x) \sim C\frac{1}{x^{b_1+a_1\alpha}}\mathrm{e}^{-(a_2x+b_2)^{c_2}+d}$ for $x \to \infty$ $a_1 = 1, b_1 = 1, a_2 = b_2 = c_2 = 0$ $h(x) = x, d = 0$ α- parameter
[48] Power law distribution	$P(\varphi) = \frac{1}{Z\lvert\varphi\rvert^{1-\frac{J}{D}}}\exp\left\{\left(2b\varphi - c\varphi^2\right)/D\right\}$ φ is related to price return b, c, D, J parameters	$P(x) \sim C\frac{1}{x^{b_1+a_1\alpha}}\mathrm{e}^{-(a_2x+b_2)^{c_2}+d}$ for $x \to \infty$ $h(x) = x, d = -b^2/(cD)$ $a_2 = c^{1/2}, b_2 = -b/c^{1/2}, c_2 = 2, a_1, b_1$, and α-parameters
[49] — empirical Exponential Distribution function for intraday trading of bonds and foreign exchange, written in terms of returns $x = \ln(p(t)/p(t0))$	$f(x,t) = \frac{\nu^2}{\gamma+\nu}\mathrm{e}^{-\nu(x-R\Delta t)}, x > R\Delta t$ $f(x,t) = \frac{\gamma^2}{\gamma+\nu}\mathrm{e}^{\gamma(x-R\Delta t)}, x < R\Delta t$ $\nu, \gamma \propto \Delta t$	$P(x) \sim C\frac{1}{x^{b_1+a_1\alpha}}\mathrm{e}^{-(a_2x+b_2)^{c_2}+d}$ for $x \to \infty$ $a_1 \neq 0, b_1 = 0, a_2 = 0, b_2 = 0, c_2 = 0$ $h(x) = \log(x), d = 0$
[50] — empirical Exponential Distribution function for intraday trading of bonds and foreign exchange, written in terms of returns $x = \ln(p(t)/p(t0))$	$f(x,t) = \frac{\nu^2}{\gamma+\nu}\mathrm{e}^{-\nu(x-R\Delta t)}, x > R\Delta t$ $f(x,t) = \frac{\gamma^2}{\gamma+\nu}\mathrm{e}^{\gamma(x-R\Delta t)}, x < R\Delta t$ $\nu, \gamma \propto \Delta t$	$P(x) \sim C\frac{1}{x^{b_1+a_1\alpha}}\mathrm{e}^{-(a_2x+b_2)^{c_2}+d}$ for $x \to \infty$ $a_1 \neq 0, b_1 = 0, a_2 = 0, b_2 = 0, c_2 = 0$ $h(x) = \log(x), d = 0$
[52] — empirical Exponential Distribution function for intraday trading of bonds and foreign exchange, written in terms of returns $x = \ln(p(t)/p(t0))$	$f(x,t) = \frac{\nu^2}{\gamma+\nu}\mathrm{e}^{-\nu(x-R\Delta t)}, x > R\Delta t$ $f(x,t) = \frac{\gamma^2}{\gamma+\nu}\mathrm{e}^{\gamma(x-R\Delta t)}, x < R\Delta t$ $\nu, \gamma \propto \Delta t$	$P(x) \sim C\frac{1}{x^{b_1+a_1\alpha}}\mathrm{e}^{-(a_2x+b_2)^{c_2}+d}$ for $x \to \infty$ $a_1 \neq 0, b_1 = 0, a_2 = 0, b_2 = 0, c_2 = 0$ $h(x) = \log(x), d = 0$

In order to solve this point, some authors have developed a power law variation of $P(x)$ for large x but with the values of the exponent α greater than 2. The parameters of Eq. (4) are in this case $a_1 = b_1 = 1$ [39–41,43] with one special case where $a_1 = -1$ and $b_1 = 1$ [42]; $a_2 = b_2 = c_2 = 0$ except for Ref. [42] where a_2 is non-zero and $c_2 = 1$. In this last case one used the Fokker–Planck equation for anomalous diffusion to derive the probability function. Derived distributions like that obtained in Ref. [42] are in general not Lévy like distribution but they approach a Lévy distribution in the limit of large x. A Lévy non-stable distribution for describing price variations is also obtained as a special case of a Tsallis distribution derived in Ref. [49]. One should also note that for large x a student distribution used in Ref. [50] approaches Lévy distributions.

Finally in Refs. [49–51] authors proposed exponential distribution functions in terms of logarithmic price differences, for the intraday trading of bonds and foreign exchange. In terms of price differences this distribution would be a power law as described in Eq. (2); thus in this case one has $h(x) = \log(x)$.

When we compare the three categories described in this section, we can see that the main difference refers between Levy stable distribution and Levy non-stable distribution refers to the value of α which is between 1 and 2 for the two first cases and above 2 for the third category. In these situations a_2 and b_2 are always zero. Concerning truncated Lévy distributions we have non-zero a_2 and b_2 parameters that assure the stability of the distributions and the flintiness of the variance. The value of a_2 must be quite high in order to describe the financial data.

4. Conclusion and implications

We acknowledge that no simple function can perfectly uniquely describe the financial data. The generalized distribution given by Eq. (4) is a "meta-equation" derived from the main models used in econophysics and which describes well the empirical data at largest values of x (with the mention that there are few non-determined parameters). Econophysicists want to describe the financial phenomena as they are and not as they should be Ref. [25]. In this empiricist perspective, they want to go beyond the Gaussian framework because financial data cannot be empirically described by a Gaussian distribution. In order to describe the complexity of financial data in a more realistic way, econophysicists had then to develop more sophisticated tools. Therefore, they developed different Levy processes for which this paper provides an unified framework.

Our generalized formula contributes to structure econophysics such as a scientific discipline with a clear method and a common scientific culture. This conclusion is directly in line with the bibliometric and sociological conclusions given by Gingras and Schinckus [17] concerning the strong institutionalization of econophysics. In this perspective, econophysics appears more and more as specific field independent from economics with a "lack of awareness of work that has been done within economics" [59, p. 1]. Our formula also helps to overcome some limitations of econophysics to become the next dominant paradigm in financial theory, such as identified in Ref. [10].

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