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Modeling of the financial market using the two-dimensional anisotropic Ising model

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HIGHLIGHTS

- The two-dimensional Ising model in an external field and an ion single anisotropy term has been used as a mathematical model for the price dynamics of the financial market.
- The free energy of the model and the mean price $\langle S\rangle$ have been calculated using a mean field approximation.
- The influence of the anisotropy Δ on the behavior of the mean price has been gotten.

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ABSTRACT

We have used the two-dimensional classical anisotropic Ising model in an external field and with an ion single anisotropy term as a mathematical model for the price dynamics of the financial market. The model presented allows us to test within the same framework the comparative explanatory power of rational agents versus irrational agents with respect to the facts of financial markets. We have obtained the mean price in terms of the strong of the site anisotropy term Δ which reinforces the sensitivity of the agent's sentiment to external news.

1. Introduction

The study of complex systems in a unified framework has become recognized in recent years as a new scientific discipline, the ultimate of interdisciplinary fields. Among many things, the dynamics of prices of derivative securities has been studied in the literature since 1960s, where the celebrated Black Scholes formula [1] for the price of a European option was one of the first fundamental results in this direction. The development of the derivatives pricing theory has resulted in that, nowadays the volume of derivatives traded is much higher than the volume of basic assets [2].

Is well known that the modeling of a financial system with a large number of decision makers is analogous to modeling a physical system consisting of many degrees of freedom [3]. Since, the economical and sociological systems have been a great field for the application of concepts and mathematical methods of theoretical physics used to tackle complex systems [4–7]. One important model to treat the financial markets is the Ising model and its extensions [8–12]. This is a model many employed in statistical mechanics and which is very simple. It presents a binary variable S_i that makes it appealing to researchers from other branches of science including the economy [9]. However, there are different families of Econophysics models. For instance, the Mike and Farmer model [13] and the modified Mike and Former model [14]. Mike and Farmer have constructed a very powerful and realistic behavioral model to minimize the dynamic process of stock price formation based on the empirical regularities of order placement and cancellation in a purely order-driven market, which can successfully

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reproduce the whole distribution of returns, not only the well-known power-law tails, together with several other important stylized facts. The three key ingredients in the Mike–Farmer (MF) model: the long memory of order signs characterized by the Hurst index H_s, the distribution of relative order prices x in reference to the best price is described by a distribution such as the Student distribution and the dynamics of order cancellation. Gao-Feng Gu et al. [14] have improved the Mike–Farmer model for order-driven markets by introducing long memory in the order aggressiveness, which is a new stylized fact identified using the ultra-high-frequency data of 23 liquid Chinese stocks traded on the Shenzhen stock exchange in 2003.

From a general way, it is not much clear how to define the price of a market [9]. The only obvious requirement is that the price should go up when there is more demand than supply and vice versa. We can define the price x_i as a stochastic variable and as the normalized difference between demand. This behaves as the magnetization $x_t = 1/N\sum_{i=1}^{N} S_i(t)$ of a magnetic system, where N (the sites number) is the size of the system. Many studies have shown that the stock market fluctuations are inversely proportional to the frequency on some power that points to self-similarity in time for processes underlying the market [15,16,47].

One important thing in finance is the observation related to scaling laws in financial markets that is the widespread power-law behavior exhibited by large price changes. This is corroborated for practically all types of financial data and markets. The quantity of interest is the relative price change or return, defined as $r_t = (p_t - p_{t-1})/p_{t-1}$ where p_t denotes the price of an asset at time t [17–19]. The distribution of returns has been studied in detail and is well known present the inverse cubic-law $P(> v) \sim v^{-\beta}$, where $\beta \sim 3$ is the tail index and v is the volatility that can be defined as the modulus of the return, $v = |r|$ [14,20–26]. In contrast, empirical analyses for other stock markets have unveiled power-law tail exponents other than the Lévy regime and the inverse cubic-law. Makowiec and Gnaciński have studied the main index of Warsaw stock exchange in Poland for five years and found that the distribution of return follows power-law behaviors in three parts with $\beta = 0.76$, 2.03 and 3.88 for the positive tail and $\beta = 0.69$, 1.83 and 3.06 for the negative tail [27]. Bertram [28] focuses on the high-frequency dates of 200 most actively traded stocks in the Australian stock exchange in the period from 1993 to 2002, and reported that the distribution of returns has power-law tails with $\beta > 3$, which varies with different time interval Δt from 10 to 60 min. Coronel-Brizio et al. [29] analyzed the daily data (1990–2004) of the Mexican stock market index (IPC) and find that the distribution of the daily returns followed a power-law distribution with the exponent $\beta^+ = 3.33$ (positive tail) and $\beta^- = 3.12$ (negative tail) by selecting a suitable cutoff value [25]. Yan et al. [30] investigated the daily returns of 104 stocks (76 from the Shanghai stock exchange and 28 from Shenzhen stock exchange) in the Chines stock markets in the period from 1994 to 2001 and argue that the tail exponent is $\beta^+ = 2.44$ for the positive part and $\beta^- = 4.29$ for the negative part. After removing the opening and close returns of high-frequency data for the Shanghai stock exchange composite index, the tail exponents are much closer to $\beta = 3$. There are also controversial results for some markets. An example comes from the Indian stock market. Matia et al. [31] analyzed the daily returns of 49 largest stocks in the National stock exchange over 8 years (1994–2002) and find that the distribution of daily returns significantly deviates from the power-law form but decays exponentially in the form of $P(r) = e^{-\beta r}$ with the decay coefficient $\beta = 1.34$ for the positive tail and $\beta = 1.51$ for the negative tail. In contrast, Pan and Sinha [32] have studied the daily data of two stock indices (Nifty, 1990–2006 and Sensex, 1991–2006) and found the daily returns are exponentially distributed followed by power-law decay in the tails ($\beta^+ = 3.10$ and $\beta^- = 3.18$ for Nifty and $\beta^+ = 3.33$ and $\beta^- = 3.45$ for Sensex). They also analyze the high-frequency data of 489 stocks containing the information about all the transactions carried out in the national stock exchange (NSE) for two-year period (2003–2004) and observed power-law tails with $\alpha^+ = 2.87$ and $\alpha^- = 2.52$ for $\Delta t = 5$ and $\beta \sim 3$ for Δt ranging from 10 to 60 min.

The simple Ising spin model can be employed to describe the mechanism of price formation in financial markets. Its simplicity makes it appealing to researchers from other branches of science such as biology, sociology and economy [33–40]. In spite of simple rules, the model exhibits a complicated dynamics in one and more dimensions. In contrast to usual majority rules, in this model the influence was spreading outward from the center. This idea seemed appealing and we adapted it to model financial markets. New dynamic rules describing the behavior of two types of market players such as the trend followers and fundamentalists were obtained with the properties of simulated price trajectories duplicated those of analyzed historic data sets. Hence this simple and parameter free model is a good first approximation of a number of real financial markets [9,41].

In the present work we study the behavior of the mean price or return of the financial market using the Ising model in an external field with an anisotropy site term in the mean field approximation (MF). The plan of this paper is the following. In Section 2 we describe the model, in Section 3 we talk about the method employed and in Section 4, we present our conclusions and final remarks.

2. The model

The model is described by the following Hamiltonian

$$\mathcal{H} = \sum_{\langle ij \rangle} J_{ij} S_i S_j - B \sum_i S_i + \Delta \sum_i S_i^2 \tag{1}$$

where we make $J_{ij} = 1$, if i and j are neighboring sites, and $J_i j = 0$, otherwise. J_{ij} is interpreted here as the coefficient of influence of the agent j on agent i according to the following rule [8]: $J_{ij} = b_i + \alpha J_i(t-1) + \beta r(t-1) B(t-1)$, where b_i is the idiosyncratic imitation tendency of the agent i and α and β are arbitrary constants. The spins are interpreted as market participants attitude. An up-spin ($S_i = 1$) represents a trader who is bullish and places buy orders, whereas a down-spin ($S_i = -1$) represents a trader who is bearish and places sell orders. The pairs are connected according to some graph topology reflecting the physics of social nature of the problem. The use of Ising models of a similar type to describe opinion formation has a long history [42–44]. The model Eq. (1) represents a model of interacting investors, J_{ij} represents also the relative propensity of the trader i to be contaminated by the sentiment of her friend j, B describes the impact of external news and S_i is the relative sensitivity of the agent's sentiment to the news being uniformly distributed in the interval $[0, S_{max}]$ and frozen to represent the heterogeneity of the agents. The Δ term comes reinforce the sensitivity of the agent's sentiment.

We can also replace the third term in the model above by the condition

$$\sum_i S_i^2 = N. \tag{2}$$

If this condition is satisfied by the spins of the Ising model, the partition function is given by

$$Z = \int_{\sum_i S_i^2 = N} dS_1 \cdot \dots \cdot dS_N \exp\left[\beta\left(B\sum_i S_i - \sum J_{ij} S_i S_j\right)\right], \tag{3}$$

where $\beta = 1/T$ and the model becomes the spherical model [45].

A geometrical picture may help to clarify the relation between the Gaussian and Ising models. The variable S_i may be thought of as the components of an N-dimensional vector. Then each configuration of the system may be represented by a point in an N-dimensional space. The partition function of the spherical model is obtained by integrating a slightly simpler Boltzmann factor over a spherical shell of radius $\sqrt{N}$. The partition function of the Ising model is obtained by adding the values taken by the same, simple Boltzmann factor at the corners of an unit hypercube. At these corners the hypercube touches the spherical model's sphere, so the spherical model's partition function includes that of the Ising model [45].

3. Method

In order to reduce the Hamiltonian to the diagonal form, we choose the unperturbed model [46] for $\mathcal{H}$ to be

$$\mathcal{H} = -\sum_i B' S_i - B\sum_i S_i \tag{4}$$

where B' is the "collective field" representing the effect of all the other spins with labels $i \neq j$ on the spin at the lattice site i. Sometimes it is convenient to lump the two magnetic fields together as an "effective field" B_E: $B_E = B' + B$. Now we can work out our upper bound for the system free energy F, using the statistic of the model system. First we obtain the partition function Z_0 as

$$Z_0 = [2\cosh(\beta B_E)]^N. \tag{5}$$

The free energy of the model system F_0 follows from the bridge equation, as

$$F_0 = -\frac{N}{\beta}\ln[2\cosh(\beta B_E)]. \tag{6}$$

We use the Bogoliubov inequality written in the form

$$F \le F_0 + \langle\mathcal{H}_I\rangle_0 \le F_0 + \langle\mathcal{H} - \mathcal{H}_0\rangle_0, \tag{7}$$

where we have reexpressed the correction term as the difference between the exact Hamiltonian and the model system Hamiltonian. This condition reflects on the free energy as

$$F \le F_0 - \sum_{\langle ij\rangle} J_{ij}\langle S_i S_j\rangle_0 + B'\sum_i\langle S_j\rangle_0 + \Delta\sum_j\langle S_j^2\rangle_0. \tag{8}$$

Using the mean field decoupling

$$\sum_{\langle ij\rangle} J_{ij}\langle S_i S_j\rangle_0 = \sum_{\langle ij\rangle} J_{ij}\langle S_i\rangle_0\langle S_j\rangle_0 = \frac{JNz}{2}\langle S\rangle_0^2 \tag{9}$$

where $Nz/2$ is the number of nearest-neighbor pairs, we have

$$F \le F_0 - \frac{JNz + N\Delta}{2}\langle S\rangle_0^2 + N(B_E - B)\langle S\rangle_0. \tag{10}$$

We employ the variational method differentiating F with respect to B' and set the result equal to zero. Note that B' always occurs as part of B_E, the condition for an extremum that can be written as

$$\frac{\partial F}{\partial B_E} = 0. \tag{11}$$

From the condition of the extremum, we obtain:

$$B_E - B = zJ\langle S\rangle_0, \tag{12}$$

which is the condition for minimum free energy. In this model, the magnetization is the mean value of the spin:

$$\langle S\rangle_0 = \tanh(\beta B + zJ\beta\langle S\rangle_0). \tag{13}$$

In order to identify a phase transition, we put $B = 0$ and the equation above becomes

$$\langle S\rangle_0 = \tanh(zJ\beta\langle S\rangle_0). \tag{14}$$

From the equations above, we obtain the optimal free energy

$$F = -\frac{N}{\beta}\ln[2\cosh(\beta B_E)] - \frac{(JNz + N\Delta)}{2}\langle S\rangle_0^2 + N(B_E - B)\langle S\rangle_0, \tag{15}$$

and

$$F = -\frac{N}{\beta}\ln[2\cosh(\beta B_E)] - \frac{(JNz + N\Delta)}{2}\frac{(B_E - B)^2}{z^2J^2} + N\frac{(B_E - B)^2}{zJ}. \tag{16}$$

With the cancellation as appropriate, we arrive at the neat form:

$$F = -\frac{N}{\beta}\ln[2\cosh(\beta B_E)] + \frac{N}{zJ}(B_E - B)^2 - \frac{N\Delta}{2z^2J^2}(B_E - B)^2. \tag{17}$$

Bearing in mind that the free energy is always our main objective in statistical physics, we can use this result to obtain the exact value of the mean spin using the procedures of the linear response theory, where we write

$$\langle S\rangle = -\frac{1}{N}\frac{\partial F}{\partial B} \tag{18}$$

and hence

$$\langle S\rangle = -\frac{1}{z^2J^2}(zJ + \Delta)(B_E - B) = \langle S\rangle_0. \tag{19}$$

Thus we have established the fact that the exact solution is the same as the zero-order result. This implies that we can also write the equation above in terms of exact values of the mean spin, thus:

$$\langle S\rangle = \tanh(\beta B + zJ\beta\langle S\rangle). \tag{20}$$

The market price is updated according to $p(t) = p(t-1)\exp[r(t)]$, where $r(t)$ is the return is given by

$$r(t) = \frac{1}{\lambda}\langle S(t)\rangle \tag{21}$$

λ gives a measure the market depth or liquidity. The return is thus proportional to the "magnetization" or aggregated decisions of the agents [8].

We solve numerically the Eq. (20) for different values of B and the result of the behavior of $\langle S\rangle$ with Δ is showed in the Fig. 1. We obtain the "magnetization" that here is interpreted as the mean price is ever negative. There is a decrease of $\langle S\rangle$ with Δ and thus, we have that the decrease of $r(t)$ and $\langle S\rangle$ with Δ is lesser when the value of B increases. Hence we must have a lesser decrease of the mean price $\langle p(t)\rangle$ with the decrease of B and therefore we have an larger increase of the impact of external news on the mean price. Hence there is an increase of the relative sensitivity S_i of the agent's sentiment to external news.

We also have that the price x_t of the market can be defined as the normalized difference between the demand and supply as $x_t = (1/N)\sum_{i=1}^{N} S_i(t)$, where N is the system size and $x_t \in [-1, 1]$, where $|x_t|$ can be treated as the probability distribution.

In Fig. 2 we show the distribution of returns which is well known to obey the law of power $P(|r|) = 1/|r|^{\alpha}$ with $\alpha \sim 3$. We obtain an increase of α with Δ parameter, where we obtain $\alpha > 3$ for $\Delta > 0$. The straight line are the best power-law fits to the data. The regression fits were performed in the region of large $|r|$ (tail distribution). The asymptotic behavior of the functional form of the cumulative distribution is consistent with the power law, $P(G_j > |r|) \sim 1/|r|^{\alpha}$ where α is the exponent characterizing the power-law decay. How we can see, for $\Delta > 0$, the behavior of tail distribution tends to deviates of the inverse cubic law. The return G_j can be defined as $G_j \equiv G_i(t, \Delta t) \equiv \ln S_i(t + \Delta t) - \ln S_i(t)$. Where for small changes in $S_i(t)$, the return G_j is the approximately the forward relative change

$$G_j \approx \frac{S_i(t + \Delta t) - S_i(t)}{S_i(t)}. \tag{22}$$

The correlation function of returns of volatilities at time scale τ is $C(|r|, |r|) \equiv \langle |r_\tau(t)||r_\tau(t + l)|\rangle$, where l is the time lag. It has similar amplitudes and decays with the same characteristic time scale as a function of time lag. This is very

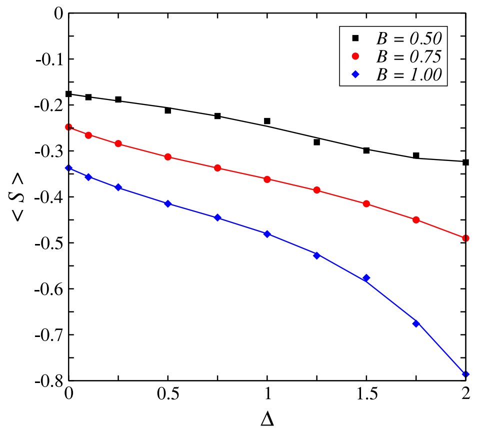

Fig. 1. Behavior of the mean value $\langle S \rangle$ with Δ for different values of constant B as $B = 0.5$ $B = 1.0$ and $B = 0.75$. The return is given as $r(t) = \frac{1}{\lambda}\langle S(t) \rangle$ that relate with the market price $p(t) = p(t-1)\exp[r(t)]$.

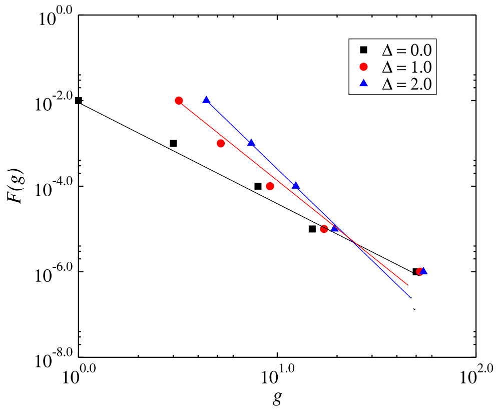

Fig. 2. Distribution of return $F(g)$ where $g = |r|$. The least squares fits of power laws varies with Δ as: $\alpha = 2.42215$ for $\Delta = 0.0$ and for $\alpha > 3$ when $\Delta > 0$: $\alpha = 3.29536$ for $\Delta = 1.0$ and $\alpha = 3.73099$ for $\Delta = 2.0$.

different from the observed correlations of financial markets, with very short memory for returns and long-memory for the volatility [8]. One of the stylized facts is the long-memory of volatility, which is usually characterized by a decay of the volatility autocorrelation as a power-law with a exponent k around 0.2–0.3. For the case $\Delta = 0.0$ is well-known that $C(l)$ can be approximated by $C(l) = 1 - k\ln(l/T)$, which is undistinguishable from the power law form $C(l) \sim 1/(l/T)^k$, if the exponent k is typically smaller that 0.3, for range of time lags l usually available. We have by expanding the exponential and for $k\ln(l/T) < 1$, that the two expressions are approximately the same and the financial dates do not allow one to distinguish between the two formulations. For $\Delta \neq 0.0$, we must have the same behavior for the correlation function of returns of volatilities into the mean field approximation employed here, expressed by the Eq. (9), where the last term is replaced by the condition given by Eq. (2) where the partition function obeys the Eq. (3).

In Fig. 3, we present the Log–Log graphic with objective to determine the Hurst exponent. We employ a rescaled range analysis on the volatilities $g = |r|$ for the model Eq. (1). The Hurst exponent is used as a measure of the long-term memory of time series. It relates with the autocorrelations of the time series and the rate at which these decrease as the lag between pairs of values increases. The rescaled range (R/S) analysis is the oldest and best-known method to estimate the Hurst indexes. An alternative method is the detrended fluctuation analysis (DFA) which in general, shows a excellent power-law dependence of the fluctuation function $F(l)$ with respect to the timescale l. We obtain the standard deviation using the mean field approach as $s = \sqrt{\langle S^2 \rangle - \langle S \rangle^2}$ and the range R as

$$R = \left[\sum_i^N S_i - N\langle S \rangle\right]_{max} - \left[\sum_i^N S_i - N\langle S \rangle\right]_{min} = \left(\sum_i^N S_i\right)_{max} - \left(\sum_i^N S_i\right)_{min} = 2N, \tag{23}$$

where H is the Hurst exponent. In the rescaled range analysis, R/s is given as

$$\frac{R}{s} = 2N\cosh(\beta B_E) = \left(\frac{N}{2}\right)^H, \tag{24}$$

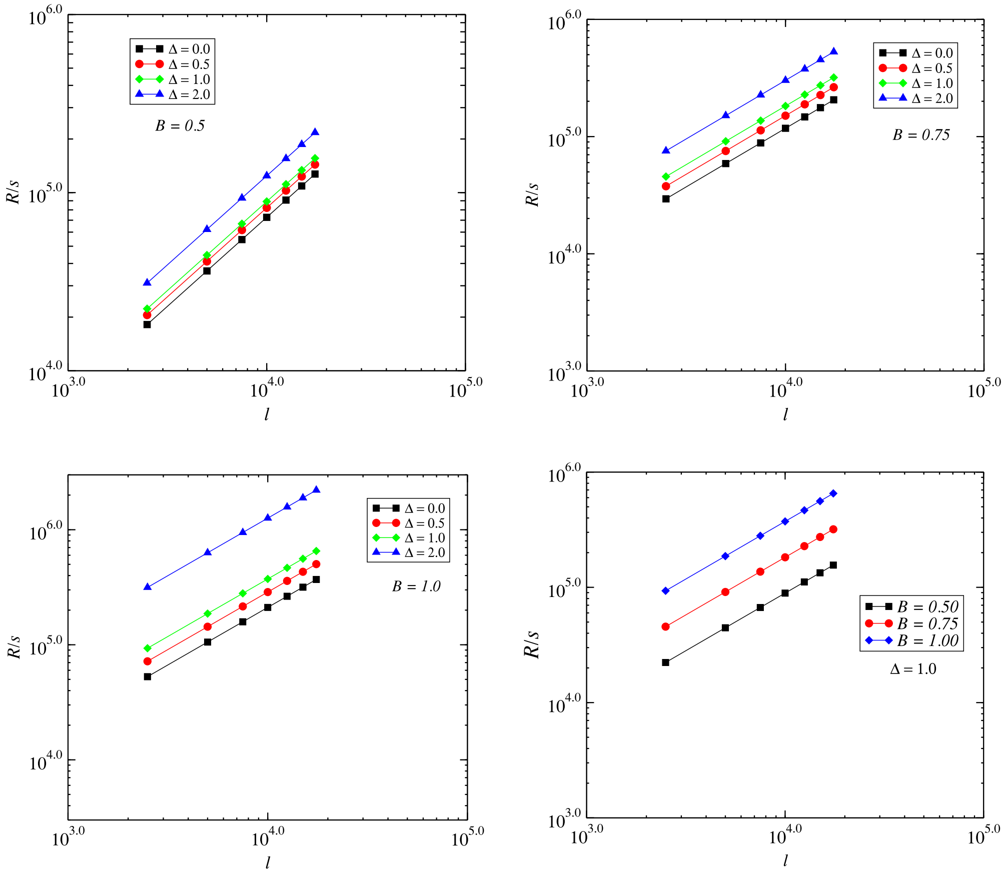

Fig. 3. Log–Log graphic of the rescaled range (RS) of the variability vs. $l = \log N$ for a time series, for different values of B and Δ. We have gotten the Hurst indexes H as $H \sim 1$ and none variation of the Hurst index with B.

and hence

$$\log\left(\frac{R}{s}\right) = \log 2N + \log\left[\cosh\left(\beta B_E\right)\right] = H \log\left(\frac{N}{2}\right), \tag{25}$$

where $B_E = B + zJ\langle S\rangle$.

According of the results obtained, we do not have obtained a variation of the Hurst index with B and Δ parameters for the model Eq. (1). We obtain the Hurst exponent for the volatilities in all cases analyzed as $H \sim 1$, and hence, we have a strong correlation in this system or a long memory in the volatilities. The cause of the large value of the Hurst exponent is the fact of the spin variable of the Ising model with external field and an anisotropy parameter to force all the spins to align in one direction and consequently we must have a total correlation in this system. The valuer obtained is inconsistent with empirical results. Usually, $H = 0.7 - 0.85$. It means that the model cannot be a model for the financial markets or it must be improved to be one. This is subject of a future work. In Fig. 4, we present the behavior of the return time series for the model Eq. (1).

4. Conclusions and final remarks

In summary, we have employed the Ising model in an external field together with a self-energy term as a model for the behavior of the financial market. We have obtained the variation of the mean price, that relate with the average $\langle S\rangle$, with the Δ and B parameters into the mean field approximation for this generalized Ising model. How the last term of the Eq. (1), in the mean field approximation, obeys the condition given by Eq. (2), the effect of this term is only generate a contribution additive to "energy" as $\epsilon' \to \epsilon + \Delta$. Consequently, the system analyzed here has the same properties of the model considered in Ref. [8] with the correlation function of returns of volatilities $C(l)$ scaling in the same way, i.e. the long-memory of volatility characterized by a decay of the volatility autocorrelation $C(l)$ well-approximated as $C(l) = 1 - k\ln(l/T)$, and which can be approximately equal to power-law behavior $C(l) \sim 1/(l/T)^k$ for $k\ln(l/T) < 1$. From a general way, is well known that the modeling of a financial system is analogous to modeling of a physical system of many degrees of freedom [3].

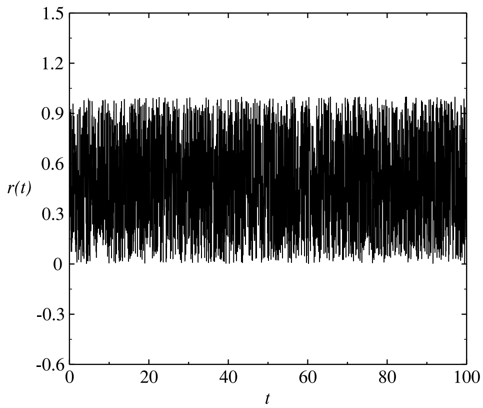

Fig. 4. A simulated return times series, $r(t)$, obtained for the model Eq. (1).

Our mathematical model comprises some theoretical analog of these features. Preliminary computations indicate that the mathematical model produces over-reactions, fluctuations, and oscillations similar to the laboratory experiments and financial markets.

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References

[1] F. Black, M. Scholes, J. Polit. Econ. 81 (1973) 637.
[2] M.V. Zhitlukhin, A.N. Shiryaev, Theory Probab. Appl. 57 (2013) 497.
[3] G. Caginalp, C.B. Ermentrout, Appl. Math. Lett. 3 (1990) 17.
[4] Tiziana Assenzaa, Domenico Delli Gattia, Jakob Grazzinia, J. Econom. Dynam. Control 50 (2015) 5.
[5] L.J.L. Cirto, L.S. Lima, F.D. Nobre, J. Stat. Mech. (2015) P04012.
[6] Cees Diks, Juanxi Wang, J. Econom. Dynam. Control 69 (2016) 68.
[7] Daan in t Veld, J. Econom. Dynam. Control 69 (2016) 45.
[8] W.-X. Zhou, D. Sornette, Eur. Phys. J. B 55 (2007) 175.
[9] K. Sznajd-Weron, R. Weron, Internat. J. Modern Phys. C 13 (2002) 115.
[10] E. Callen, D. Shapero, Phys. Today 27 (1974) 23.
[11] E.W. Montroll, W.W. Badger, Introduction to Quantitative Aspects of Social Phenomena, Gordon and Breach, New York, 1974.
[12] A. Orléan, J. Econ. Behav. Org. 28 (1995) 274.
[13] S. Mike, J.D. Farmer, J. Econom. Dynam. Control 32 (2008) 200.
[14] G.-F. Gu, W.-X. Zhou, Europhys. Lett. 86 (2009) 48002.
[15] E.E. Peters, Chaos and Order in the Capital Markets: A New View of Cycles, Prices, and Market Volatility, J Wiley, New York, 1994.
[16] R. Mantegna, H.E. Stanley, An Introduction to Econophysics, Cambridge Univ. Press, Cambridge, 2000.
[17] B. Mandelbrot, J. Bus. 35 (1963) 394.
[18] E. Fama, E. Mandelbrot, J. Bus. 35 (1963) 420.
[19] Thomas Lux, Michele Marchesi, Nature 397 (1999) 498.
[20] P. Gopikrishnan, M. Meyer, L.A.N. Amaral, H.E. Stanley, Eur. Phys. J. B 3 (1998) 139.
[21] Parameswaran Gopikrishnan, Vasiliki Plerou, Lus A. Nunes Amaral, Martin Meyer, H. Eugene Stanley, Phys. Rev. E 60 (1999) 5305.
[22] Vasiliki Plerou, Parameswaran Gopikrishnan, Lus A. Nunes Amaral, Martin Meyer, H. Eugene Stanley, Phys. Rev. E 60 (1999) 6519.
[23] Federico Botta, Helen Susannah Moat, H. Eugene Stanley, Tobias Preis, PLoS ONE 10 (2015) e0135600. http://dx.doi.org/10.1371/journal.pone.0135600.
[24] G.-F. Gu, W.-X. Zhou, Eur. Phys. J. B 67 (4) (2009) 585–592.
[25] H. Meng, F. Ren, G.-F. Gu, X. Xiong, Y.-J. Zhang, W.-X. Zhou, W. Zhang, Europhys. Lett. 98 (2012) 38003.
[26] J. Zhou, G.-F. Gu, Z.-Q. Jiang, X. Xiong, W. Zhang, W.-X. Zhou, Comput. Econ. (2016). http://dx.doi.org/10.1007/s10614-016-9612-1.
[27] D. Makowiec, P. Gnaciński, Acta Phys. Polon. B 32 (2001) 1487.
[28] W.K. Bertram, Physica A 341 (2004) 533.
[29] H.F. Coronel-Brizio, A.R. Hernandez-Montoya, Physica A 354 (2005) 437.
[30] C. Yan, J.-W. Zhang, Y. Zhang, Y.-N. Tang, Physica A 353 (2005) 425.
[31] K. Matia, M. Pal, H. Salunkay, H.E. Stanley, Europhys. Lett. 66 (2004) 909.
[32] R.K. Pan, S. Sinha, Europhys. Lett. 5 (2007) 58004.
[33] D. Derrida, P.G. Higgs, J. Phys. A 24 (1991) 1985.
[34] F. Schweitzer, J.A. Holyst, Eur. Phys. J. B 15 (2000) 723.
[35] R. Savit, R. Manuca, R. Riolo, Phys. Rev. Lett. 82 (1999) 2203.
[36] A. Cavagna, J.P. Garrahan, I. Giardina, D. Sherrington, Phys. Rev. Lett. 83 (1999) 4429.
[37] D. Chowdhury, D. Stauffer, Eur. Phys. J. B 8 (1999) 477.
[38] R. Cont, J.P. Bouchaud, Macroeconomic Dyn. 4 (2000) 170.
[39] D. Challet, M. Marsili, R. Zecchina, Phys. Rev. Lett. 84 (2000) 1824.

[40] V.M. Eguiluz, M.G. Zimmermann, Phys. Rev. Lett. 85 (2000) 5659.
[41] Seungsik Min, Kyuseong Lim, Ki-Ho Chang, Il-Hwan Park, Kyungsik Kim, Fractals 24 (2016) 1650016.
[42] S. Galan, S. Moscovici, Eur. J. Soc. Psychol. 21 (1991) 49.
[43] S. Galan, Physica A 238 (1997) 66.
[44] Q. Michard, J.P. Bouchaud, Eur. Phys. J. B 47 (2005) 151.
[45] J.J. Binney, N.J. Dowrick, A.J. Fisher, M.E.J. Newman, The Theory of Critical Phenomena, an Introduction to The Renormalization Group, Oxford University Press, New York, 1992.
[46] W.D. McCommb, Renormalization Methods a Guide for Beginners, Oxford claredon press, Oxford, 2004.
[47] K. Ivanova, M. Ausloos, H. Takayasu, arXiv:cond-mat/0301268v1, 2003.

Further reading

[1] J.-W. Zhang, Y. Zhang, H. Kleinert, Physica A 377 (2007) 166.
[2] Yue Zhang, Yan Zheng, Xi Liu, Qingling Zhang, Aihua Li, Physica A 462 (2016) 222.