Physica A 539 (2020) 122928

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

A quantum model of supply and demand

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ARTICLE INFO

Article history:
Received 16 May 2019
Received in revised form 16 September 2019
Available online 4 October 2019

JEL classification:
D00
D80
G10

Keywords:
Quantum economics
Quantum finance
Quantum cognition
Entropic forces
Harmonic oscillator
Quantum agent-based model

ABSTRACT

One of the most iconic and influential graphics in economics is the figure showing supply and demand as two lines sloping in opposite directions, with the point at which they intersect representing the equilibrium price which perfectly balances supply and demand. The figure, which dates back to the nineteenth century, can be seen as a graphical representation of Adam Smith's invisible hand, which is said to guide prices to their optimal level, and features in nearly every introductory textbook. However this figure suffers from a number of basic drawbacks. One is that it does not express a dynamical view of market forces, so it is not clear how prices converge on an equilibrium. Another is that it views supply and demand as deterministic, when in fact they are intrinsically uncertain in nature. This paper addresses these issues by using a quantum framework to model supply and demand as, not a cross, but a probabilistic wave, with an associated entropic force. The approach is used to derive from first principles a technique for modeling asset price changes using a quantum harmonic oscillator, that has been previously used and empirically tested in quantum finance. The method is demonstrated for a simple system, and applications in other areas of economics are discussed.

1. Introduction

A basic question in economics is that of the relationship between supply and demand. The standard interpretation, known as the law of supply and demand, is traditionally illustrated using versions of a graph first published in an 1870 essay by Fleeming Jenkin. It has since become the most famous figure in economics, and is taught at every undergraduate economics class. The figure shows two intersecting curves or lines, which describe how supply and demand are related to price. When price is low, supply is low as well, because producers have little incentive to enter the market; but when price is high, supply also increases. Conversely, demand is lower at high prices because fewer consumers are willing to pay that much. The point where the two lines cross gives the unique price at which supply and demand are in perfect balance, and is therefore a pictorial representation of Adam Smith's invisible hand.

The law of supply and demand not only plays an important role in many economic models, but also justifies the widespread assumption in economics that prices are drawn to a stable equilibrium. However there are a number of basic problems with it. One is that it is generally impossible to measure supply or demand curves, because all we have is transactions which involve both quantities. The parameters are therefore underdetermined. Another problem is that the law is deterministic, while economic interactions are intrinsically probabilistic (or indeterminate). The law by itself also gives little sense of underlying dynamics (according to the efficient market hypothesis, equilibrium is achieved instantly). Finally, the law assumes continuity, but goods are sold in discrete amounts, and financial transactions are inherently discontinuous.

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https://doi.org/10.1016/j.physa.2019.122928
0378-4371/

These issues can be addressed via the adoption of a quantum formalism, which is explicitly designed to handle systems that are discrete, indeterminate, and dynamic. In recent years the quantum methodology has been applied to a number of areas in social science, from cognitive psychology to international relations (see [1,2] for an overview). One of the first areas to explore the quantum approach was finance [3,4]. The basic idea of quantum finance is that asset prices are indeterminate until measured through transactions, so can be modeled using wave functions that collapse to a certain price when measured. Quantum cognition, meanwhile, treats mental states as indeterminate until measured through decisions. These theories come together naturally in the question of supply and demand, which involves decisions about financial transactions.

This paper applies the quantum methodology to a simple but illustrative case of supply and demand, that can be extended to a variety of situations. The probabilistic approach is similar to that of Kondratenko [5], who also argues for a quantum link, however the paper derives dynamic equations that are interpreted in terms of entropic forces; uses these to generate equations for an oscillator model; draws an explicit connection with recent research in quantum finance, where a similar quantum oscillator model has been used to model asset price changes; and motivates the technique's application in other areas of economics.

Before proceeding it should be noted that the results, as presented here, can also be reproduced using classical models, and indeed much of the paper is devoted to exploring the relationship between the classical and quantum approaches. The motivation for adopting a quantum formalism is that prices, being based on information flows, do not behave in a classical fashion — they are fundamentally indeterminate and are only known through a measurement procedure, which in turn affects the system. As argued in [6] this implies that the quantum formalism is a natural and appropriate framework for analysis. In particular, a quantum model for supply and demand supplies an interface for considering interference effects of the sort studied in quantum cognition, or financial entanglements through debt. For example, the model assumes a certain mean price and price sensitivity on the part of buyers and sellers; however in a complete model, their formation and dynamics would be the result of cognitive processes which as argued elsewhere elude a classical treatment (see e.g. [7,8]). More generally, we will see that the quantum oscillator offers a simple, parsimonious, and effective way to model a range of phenomena.

The outline is as follows. Section 2 begins with a simplified case where there is only one buyer and one seller, but where the desire to buy or sell at a particular price is modeled probabilistically. Section 3 extends this model to the general case with multiple agents. Section 4 uses the concept of entropic forces to analyze the dynamics of the system, and Section 5 exploits a particular feature of the ground state of a quantum harmonic oscillator to show how a quantum approach can be used to account for the indeterminate and dynamic nature of the system. Section 6 demonstrates the methods for simple examples, discusses the results in the context of previous works, and proposes applications in economic modeling. Section 7 summarizes the results.

2. Case with single buyer and seller

As a starting point, first consider the case of a single buyer and seller, who are negotiating a transaction involving a certain good (say a stock, or a house). The buyer might have a certain offer price μ_o in mind, while the seller has a bid price μ_b. Because price is a relative quantity, we will treat it as a logarithmic variable. Since it generally holds that $\mu_o < \mu_b$ there will be no transaction unless at least one party shows some flexibility. It is therefore necessary to broaden the constraints, so instead of having a central fixed price, each participant is willing to consider a range of prices, with the propensity to sell or purchase at each price described by a function. The use of the term "propensity" here is similar to that in stochastic chemical kinetics, where it refers to the probability of a molecular reaction occurring in a certain time [9]. The situation is shown graphically in Fig. 1, where $P_o(x)$ is the offer propensity function, and $P_b(x)$ the bid propensity function. Both functions are assumed to be normal (Gaussian), with standard deviations σ_o and σ_b. The case for the common scenario where the sales price is fixed over a trading period would be modeled by setting $\sigma_b = 0$ so P_b is a delta function.

This assumption of normally distributed prices may seem a little strange, since it implies that buyers will not purchase items that seem too cheap, and sellers will be reluctant to sell above a certain price. One way to think of these curves is as a kind of schedule, where the buyer and seller mentally partition their offers and bids, with a peak at a central price which they consider to be ideal but not too unrealistic, and in a manner that is constrained by the condition that the integral of the function equals 1. Viewed this way, it would not make sense for a buyer to commit to buy only at a very low price, since they would then have to decline any reasonable offer outside that range. Note also that transactions take place in the middle ground between the mean bid and offer prices, so what matters is the behavior of the propensity functions over this range.

If we assume independence, then the joint propensity function, which describes the joint probability of a transaction actually occurring at a particular price, is the product $P_t(x) = P_o(x) P_b(x)$, shown by the blue line in the figure. The area of this graph measures the propensity for trade. It is easily shown [10] that the product of two normal distribution curves is a scaled normal curve, with mean and standard deviation

$$\mu = \frac{\sigma_b^2 \mu_o + \sigma_o^2 \mu_b}{\sigma_o^2 + \sigma_b^2}$$

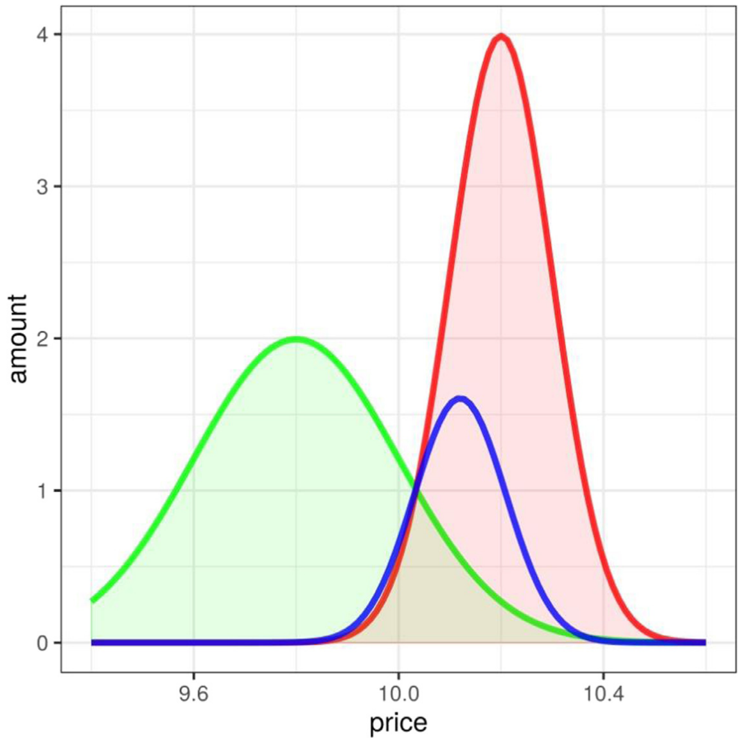

Fig. 1. Plot showing the buyer's propensity function (green) and the seller's propensity function (red). The joint propensity for a transaction occurring is the product of these functions, shown by the blue line. Price is treated as a logarithmic variable.

$$\sigma = \frac{\sigma_o \sigma_b}{\sqrt{\sigma_o^2 + \sigma_b^2}}.$$

The scaling factor α is itself a normal distribution of the form

$$\alpha = \frac{1}{\sqrt{2\pi\sigma_t^2}} e^{-\frac{\mu_t^2}{2\sigma_t^2}}$$

where $\mu_t = \mu_o - \mu_b$ is the spread, and the standard deviation $\sigma_t = \sqrt{\sigma_o^2 + \sigma_b^2}$ is a measure of price flexibility [10]. The main parameters and equations are also summarized in Appendix.

For the case of a financial market, price quotes in the order book often come from market makers. The expected profit over a trading cycle for a market maker will depend on the operating spread, which represents the profit per transaction, multiplied by the amount traded. The propensity for trade scales with α so if we assume the operating spread scales with μ_t, then the profit scales with the product $\mu_t \alpha$ which has a maximum when the spread satisfies $\mu_t = \sigma_t$. If we further assume that market makers adjust the spread in this way in order to maximize profit, then setting this value for μ_t in the expression for the propensity for trade α gives $\alpha \propto \exp(-H)$ where $H = \log\left(2\pi e \sigma_t^2\right)/2$ is the differential entropy of the normal distribution [11]. Market negotiations which align the expectations of the buyer and seller will also tend to reduce σ_t and therefore minimize the entropy, which in information theoretic terms is equivalent to minimizing the missing information about the system [12].

3. Multiple agents

So far we have only considered the case of a single buyer and seller, who are negotiating the price of a single item, but the same methodology carries over easily to the case with multiple units and agents. The bid functions and offer functions are aggregated to give a total bid function over all buyers, and a total offer function over all sellers, measured in numbers of units. In an agent-based model this would be performed by summing the propensity functions directly. If for simplicity we assume that the bid propensity functions all share the same mean and standard deviation, and likewise for the offer functions, then the effect is to simply scale the propensity functions by the numbers of sellers and buyers respectively.

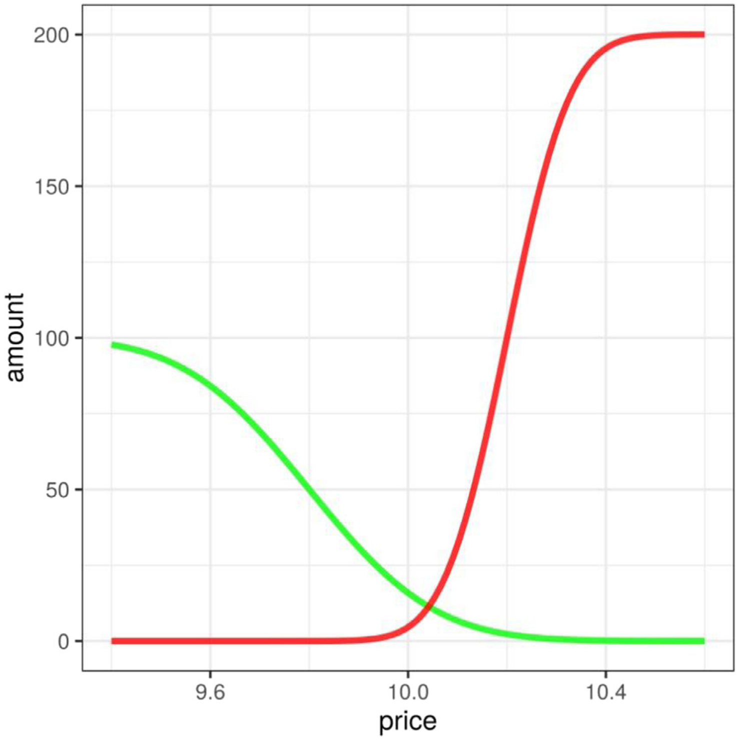

Fig. 2. Cumulative bid/offer propensity curves, scaled by the number of participants, for the case with 100 buyers and 200 sellers. The red line shows the total number of units sold $C_b(x)$, assuming that an individual buyer starts with the lowest available price and works their way up to the price x. The maximum for this curve is 200 which corresponds to the number of units available, assuming vendors sell a maximum of 1 unit each. The green line shows the number bought $C_o(x)$, assuming that an individual seller starts with the highest available price and works their way down to x.

The expected trading volume, expressed as a rate, is then given by $V = N_o N_b r \alpha$. This is the same as mass action kinetics in chemistry, where a reaction between two chemical species in solution occurs at a rate which is proportional to the concentrations of the reactants, and their chemical affinity, but depends also on factors such as the temperature. Here the propensity for trade α is adjusted by a rate parameter r which accounts for the exact structure of the market and the degree and nature of the interactions between buyers and sellers. Note that as in chemistry [13], this equation serves as a useful first-order model but may need to be modified under certain conditions (or agents can be modeled individually). In a stochastic model the number of transactions over a trading cycle of duration τ follows a Poisson distribution with mean $\lambda = V\tau$.

This population model also gives a different perspective on the propensity functions. The red line in Fig. 2 shows the cumulative number of units $C_b(x)$ sold to an individual buyer, assuming that the buyer starts with the lowest available bid price and works their way up to the price x. This curve is given by the seller cumulative propensity function multiplied by the number of sellers:

$$C_b(x) = N_b \int_{-\infty}^{x} P_b(x)\, dx.$$

The green line shows the number of units $C_o(x)$ sold to an individual seller, assuming that the seller starts with the highest available offer price and works their way down to the price x, which is given by

$$C_o(x) = N_o - N_o \int_{-\infty}^{x} P_o(x)\, dx.$$

These cumulative propensity curves are very unrealistic because they assume that single large buy and sell orders are broken into infinitesimal chunks and processed in order. In reality, transaction charges would mean that the orders would be handled in a small number of large transactions. The curves also remove any probabilistic uncertainty, because buyers and sellers are assumed to have perfect information. However they are interesting because they resemble the traditional plots of supply and demand, with the difference that the independent variable price is on the horizontal, rather than vertical, axis. If we consider cash as carrying momentum [14], then a large purchase (or sale) can be viewed as a transfer of momentum which will perturb the price point. Far from being an inert medium of exchange, money is the basis of a measurement procedure which affects the system being measured ([15]: 20).

We here notice a clear difference between the deterministic and probabilistic interpretations. In the former, the equilibrium price is the intersection point at which supply and demand are equal, obtained with a probability 1, while in the latter, the expected price is normally distributed. What counts in the probabilistic picture is not just the number of buyers or sellers, but their flexibility in negotiating prices, as expressed by the inverse of the variance. As discussed further below, the model can be generalized to simulate group influences where suppliers collectively decide to change their price ranges.

This simple model assumes that buyers and sellers in the population are homogeneous in the sense that they share the same offer and bid functions. Even without this assumption, it should often be possible to approximate the total offer function using a normal distribution. Also, while we have considered normal distributions here because of their mathematical convenience, one could consider different shapes for the propensity functions. The main thing is that the product of these functions, in the region around the price-point of interest, should be approximated by a normal curve, which is the case if the buyer and seller forces defined below are locally linear. In general, it seems reasonable to suppose that transactions will occur over a limited range and can be approximated by the kind of model described here.

4. The entropic oscillator

The bid and offer propensity functions in Fig. 1 can be viewed as representing the mental state of the buyer/seller. As shown by cognitive psychology, decisions contain a random component, so should be modeled as probabilistic processes [16]. However we can also think of these curves as describing a kind of force. To motivate the treatment, suppose that the current price x is higher than the buyer's central price μ_o. The probability to purchase is then given by the propensity function $P_o(x)$. The resistance to changing to some nearby price $x + \Delta x$ will depend on the change in propensity conditional on (or relative to) the current propensity. This is equal to the slope of the propensity, divided by the current propensity, or $P_o{}'(x)/P_o(x)$. We therefore define the supply and demand forces as

$$F_o(x) = \gamma \frac{P_o{}'(x)}{P_o(x)} = \frac{-\gamma(x-\mu_o)}{\sigma_o^2} = -k_o(x-\mu_o)$$

$$F_b(x) = -\gamma \frac{P_b{}'(x)}{P_b(x)} = \frac{\gamma(x-\mu_b)}{\sigma_b^2} = k_b(x-\mu_b)$$

where $k_o = \gamma/\sigma_o^2$ and $k_b = \gamma/\sigma_b^2$ are force constants, and γ is a scaling parameter with units of energy.[1] The demand force slopes downwards, because there is resistance to increasing price, while the supply force slopes upwards. The forces therefore represent the mental desire for the buyer or seller to adjust the price to their own preferred level. Note that, because the propensity functions are chosen to be normal curves, the corresponding forces are linear in price. They can therefore be viewed as a first-order approximation to the dynamics in the region of the central equilibrium point. As seen in the Appendix, these forces are the cognitive version of entropic forces which reflect the tendency of a thermodynamic system to maximize entropy by evolving to states that are statistically more probable (with the difference that they act in the opposite direction, so decrease the entropy).

We can similarly define the transaction force as the entropic force generated by the joint probability, which is just the sum of the buyer and seller forces:

$$F_t(x) = \gamma \frac{P_t{}'(x)}{P_t(x)} = \gamma \frac{P_o(x)P_b{}'(x) + P_o{}'(x)P_b(x)}{P_o(x)P_b(x)} = F_o(x) + F_b(x).$$

The point at which the probability of a transaction is highest can be found by setting the derivative of the joint propensity function to zero, so

$$P_t'(x) = P_o{}'(x)P_b(x) + P_o(x)P_b{}'(x) = 0$$

or

$$F_o(x) = -F_b(x)$$

which occurs at the price

$$\mu = \frac{k_o\mu_o + k_b\mu_b}{k_o + k_b}.$$

The equilibrium price is therefore the point where the supply and demand forces are in balance and $F_t(x) = 0$, as expected.

The existence of a restoring force is consistent with the idea that market sentiment tends to oscillate over time, alternating between periods of e.g. greed and fear. To picture the dynamics, we can imagine the force $F_t(x)$ acting on a mass $m = m_o + m_b$, where m_o and m_b represent the resistance to change of the buyer and seller respectively, and these masses are joined together as shown in Fig. 3. The equation of motion for this coupled system (diagram C in the figure) is then $m\ddot{x} = -k(x-\mu)$ where $k = k_o + k_b$. This has the oscillatory solution $x = \mu + A\cos(\omega t + \varphi)$ where A is the amplitude, $\omega = \sqrt{k/m}$ is the frequency of oscillation, and the phase φ depends on the starting point.

[1] ([5]: 137–138) proposes similar force terms, without a corresponding expression for mass, on the basis that these terms cancel at equilibrium.

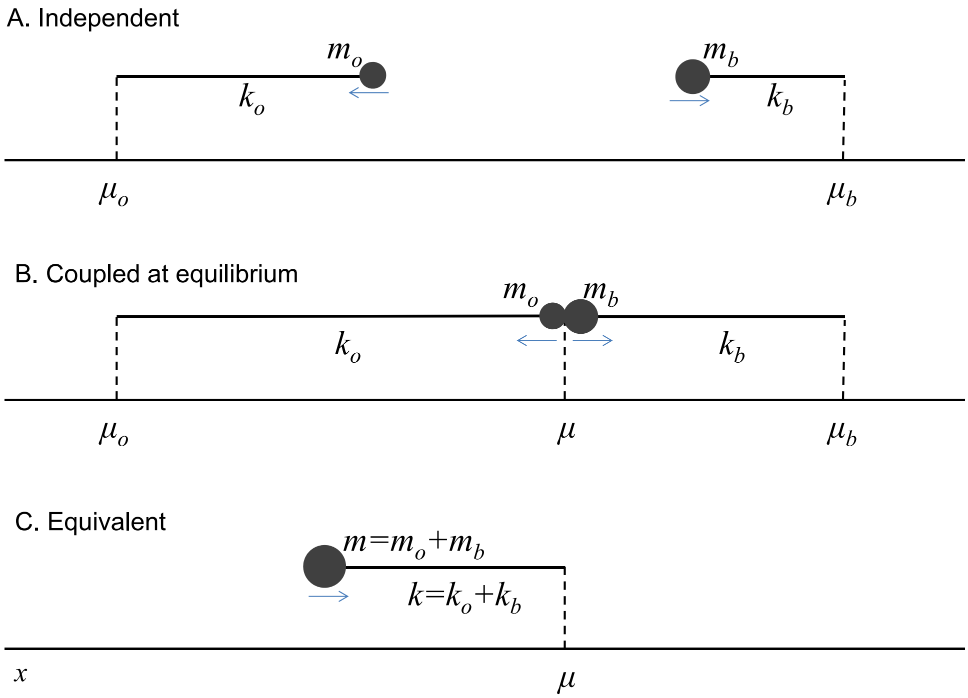

Fig. 3. In diagram A the buyer and seller are represented by the masses m_o and m_b which oscillate independently around their central points with spring constants k_o and k_b. The arrows show the direction of force. Diagram B shows the coupled system where the two masses are attached, and are at their equilibrium point μ. This is equivalent to the oscillator in diagram C with mass $m = m_o + m_b$ and spring constant $k = k_o + k_b$.

While such a force would express the restoring tendency towards a central price, it again is deterministic rather than probabilistic. Also, unless additional damping terms are added, the price would simply bounce back and forth between two extremes that would depend on the initial conditions. The probability distribution of prices is given by the equation

$$P(x, A) = \frac{1}{\pi\sqrt{A^2 - (x-\mu)^2}}$$

which as shown in Fig. 4 below is highest at the extremes (where the rate of change is slowest) and lowest in the midpoint (where it is fastest), which is inconsistent with the probabilistic picture in Fig. 1.

A more realistic approach would therefore be to assume the oscillator is driven by random noise. This results in an Ornstein–Uhlenbeck process which is a mean-reverting random walk given by the stochastic differential equation

$$dx = -\theta kx\, dt + \sqrt{2D}\, dW.$$

Here dW is a Wiener process, D is the diffusion coefficient, and θ is the mobility which measures the drift velocity induced by a given force [17]. The probability density function P then satisfies the Fokker–Planck equation

$$\frac{\partial P}{\partial t} = \theta k \frac{\partial}{\partial x}(xP) + D\frac{\partial^2 P}{\partial x^2}.$$

The steady state solution $P(x)$ is a Gaussian with standard deviation $\sigma = \sqrt{D/k}$. Excited states relax back to this steady state due to dissipation. In physics, if we assume that the system is perturbed by thermal noise, then according to the Einstein relation we have $D = \theta k_B T$, where k_B is the Boltzmann constant and T is temperature. In the quantum model developed below the force constant satisfies $k = \gamma/\sigma^2$ from which it follows that $\gamma = D = \theta k_B T$. For $\theta = 1$ this is the same relationship arrived at by interpreting the supply and demand forces as entropic forces, see Appendix.

5. The quantum harmonic oscillator

While it is certainly reasonable to model the system as a stochastic differential equation in this way, an alternative approach is to shift to a quantum framework, which offers a natural way to handle its indeterminate, dynamic properties. For example, while the stochastic approach assumes that price has a well-defined value at each time, the quantum model acknowledges that prices, and indeed the mental states of buyers and seller, are indeterminate until measured through a transaction, and this measurement process affects the price. As mentioned in ([18]: 4072), an obvious feature

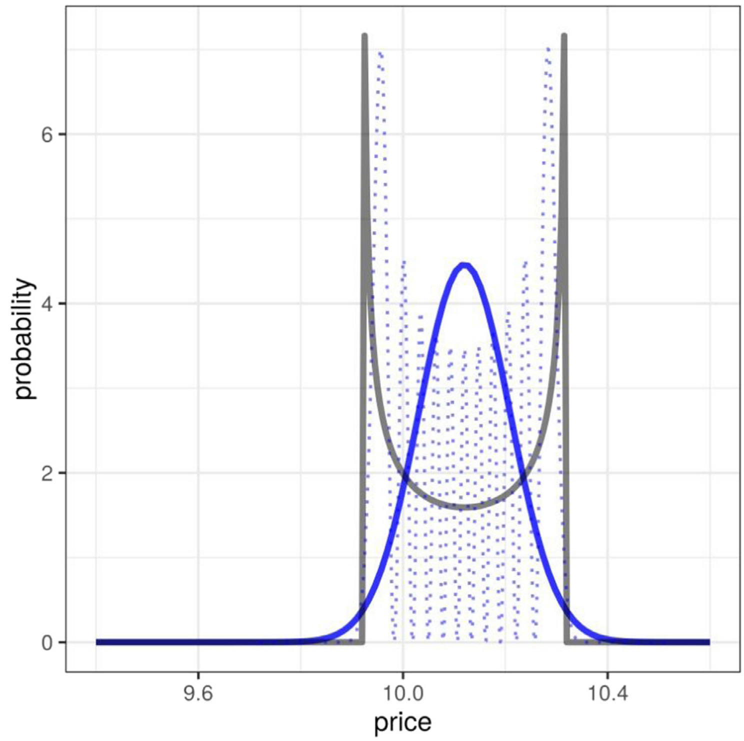

Fig. 4. Probability distributions for position for the classical harmonic oscillator (gray line) at high energy, and the quantum model in its ground state (blue line) and in the tenth excited eigenstate (dotted). In the classical oscillator model, the probability is highest at the extremes rather than at the center, and the range – which here equals the bid/offer spread – depends on the initial conditions. The quantum oscillator matches the probability distribution of price for transactions at low energy, and converges when smoothed to the classical distribution as energy increases. The tenth eigenstate is shown for illustrative purposes, only the lower-energy states are typically used.

of financial markets is that it is impossible to observe both an asset's price, and its instantaneous rate of change: this "lack of simultaneous observability appears capable of precise mathematical formulation only in quantum terms". More generally, as mentioned in the introduction, the information flows that make up economic transactions do not behave in a classical fashion and are often better suited to a quantum approach.

In the quantum formalism, the price of an asset is represented by a wave function that collapses to a particular value when measured through a transaction, just as the wave function for a particle's position collapses down to a single number when measured. We can move to the quantum framework by "quantizing" the classical equations. The quantum version of the spring equation is the quantum harmonic oscillator, which is restricted to a discrete set of energy levels. The ground state is described by the wave function

$$\psi_E(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}(x-\mu)^2\right).$$

The corresponding probability distribution for x is a normal distribution with mean μ and standard deviation

$$\sigma = \sqrt{\frac{\hbar}{2m\omega}}.$$

From the Schrödinger equation, the time evolution of an oscillator in the ground state is given by the complex wave function

$$\psi_E(x,t) = \exp\left(-\frac{i\omega t}{2}\right)\psi_E(x)$$

which rotates around the imaginary axis with an angular frequency $\omega/2$. As the energy increases, the probability distribution peaks at the extremes, as with the classical case in Fig. 4, in accordance with Bohr's correspondence principle ([19]: 40). In practice only low energy states will be used here, as discussed in Section 6.

To see how this applies to supply and demand, we identify the probability distributions as the ground states of quantum oscillators. For the buyer or seller, the oscillator can represent a kind of mental oscillation over prices, while for the transaction price it represents an oscillation between the buyer's preferred price and that of the seller. As discussed below the parameter $\hbar$ in this model will determine the transition between energy levels, while ω represents a characteristic frequency. It is then easily shown (see Appendix) that the equations for the mean and standard deviation of the quantum oscillator for the coupled supply/demand system are the same as the equations for the joint propensity function from Section 2. Mass terms for the buyer and seller scale inversely with the variance of the propensity functions, so are a measure of price flexibility. The corresponding price distribution is shown in Fig. 4, where it is assumed that a transaction has taken place (so the total probability is 1).

This equivalence between the models does not rely on the fitting of any parameters. The sole additional assumption is that demand and supply forces scale in a consistent way with variance of the propensity functions. A consequence is that the frequencies of the buyer and seller in the transaction process are the same. In the classical model (Fig. 3) this was enforced by physically joining their corresponding masses, as is necessary if they are to represent a single price.

As discussed in the final section, the parameters $\hbar$ and ω serve as a parsimonious way to fit the higher energy level states that are characteristic of observed data. Comparison with the entropic version (see Appendix) shows that $\gamma = \hbar\omega/2 = \theta k_B T$ where the mobility factor θ was set to 1. In physics, both $\hbar$ and k_B are well-defined constants, with one quantizing mechanical action and the other quantizing entropy [20]. In economics they have no such set value, however the presence of the mobility factor, which depends on the details of the system, is a reminder that in economics these parameters are context-dependent, and need to be fit for a particular model. Note also that quantizing the system changes the meaning of the parameters (instead of a scaled temperature, there is a scaled frequency) but does not increase their number, which is important from a modeling perspective.

In the classical picture, with price modeled by a classical oscillator (which can be viewed as representing a kind of dynamic bargaining process), one would represent a price negotiation by adding energy to the system in order to induce an oscillation. In the quantum picture, we can similarly add an amount of energy E_d using a displacement operator (for example, if the system is initially in the ground state, then a displacement of 2σ raises the energy by $E_d = \hbar\omega$). Again, this can be viewed as the result of a negotiation process, where the forces exerted by the buyer and seller shift and adjust in response to each other. Interactions between groups of buyers or sellers could have similar effects. The probability of observing the system in a particular state then follows a Poisson distribution with mean $\lambda = E_d/(\hbar\omega)$.

A number of authors have developed models of financial trading based on the operator approach, as in quantum field theory where creation and annihilation operators are used to model the behavior of particles [3,21–23]. For example in the model of Gonçalves and Gonçalves [24] the number of buyers and sellers is represented by a number operator which counts the participants. At the start of each trading cycle, the system is in a ground state which is then perturbed by a displacement operator. This puts the system into a so-called coherent state where the number of participants follows a Poisson distribution. While a discussion of the operator approach is beyond the scope of this paper, it does suggest the following possible interpretation, which is to treat the supply/demand system at the start of a trading cycle as a quantum oscillator in its ground state. When a negotiation begins, the effect is to perturb the system with an increase in energy of $E_d = \hbar\omega\lambda$ units. In this picture, the bringing of money to the table therefore acts as a kind of financial kick to the system. The result is a coherent state, which can be viewed as a quantum version of a classical oscillating state. The probability density is Gaussian but oscillates around the mean, and the energy level when measured follows a Poisson distribution with mean λ, so corresponds to the number of transactions in the probabilistic model.

To summarize, the state of the system is being modeled as a quantum harmonic oscillator whose properties can all be derived from the probability distributions for the buyer and seller, as measured in a trading context. The energy of the oscillator, and therefore the probability of transactions occurring during a trading cycle, reflects both price spread and price flexibility. The quantum model can be seen as mediating between two classical models: the ground state corresponds to the normal-shaped static probabilistic model of supply and demand, while as energy increases (i.e. for excited states) the model converges to the dynamic spring model where prices oscillate around the mean, and have the highest probability of being observed at the extremes. The statistical behavior is essentially the same as that derived from a classical stochastic model, however there a number of key differences. The system state is modeled by a complex wave function, and variables such as prices or the mental states of the buyers and sellers are treated as indeterminate until measured through a transaction, which as discussed further below has implications for things like interference effects and entanglement. The quantum model has a non-trivial ground state which reflects uncertainty rather than random noise. Also, while excited states damp out in the stochastic model, in the quantum model they persist until the wave function is collapsed through measurement. The main operating assumption is that the buyer and seller forces $F_o(x)$ and $F_b(x)$ are linear in the region of the equilibrium price. The entropic nature of these forces makes clear the connection between information exchanges, quantum behavior, and economic transactions.

6. Discussion

The probabilistic quantum model can be applied in a number of ways to model financial transactions. The most basic is to use it as a way to generate stochastic models of supply and demand. For example, Fig. 5 shows two simulations for a simple system where the price of some good is adjusted by the seller so as to maintain a certain level of inventory (see Appendix for details). The dashed line shows the equilibrium demand level using a classical systems dynamic approach. The solid line shows a scenario where the price is set by the seller as before, but now the number of units purchased at that price follows a Poisson distribution as described in Section 3. The effect is to create stochastic noise in the price

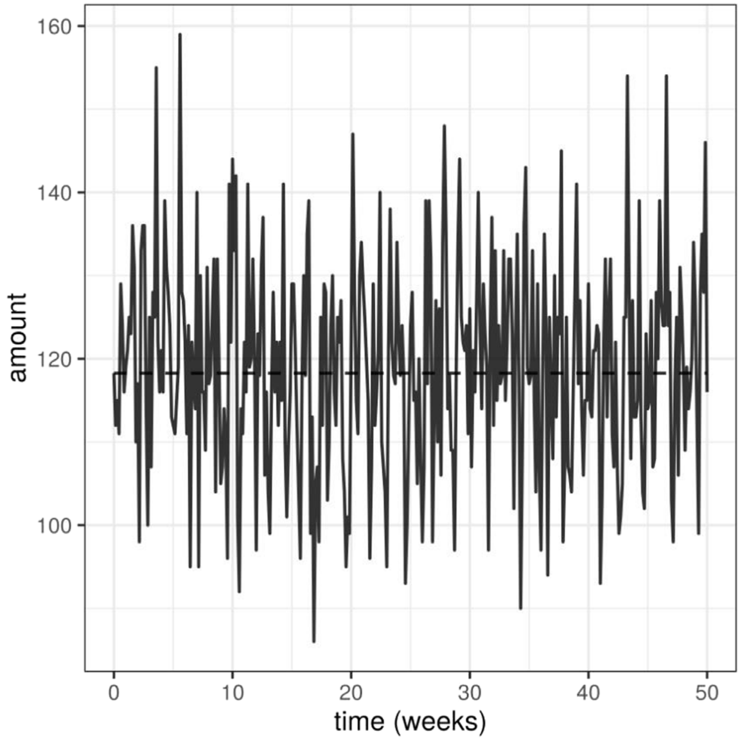

Fig. 5. Straight dashed line shows a simulation of demand in a systems dynamics model where price is set dynamically by the seller in order to maintain inventory at four times the level of demand. The system starts at equilibrium so demand is stable. Solid line shows a simulation where price is again set dynamically by the seller, but the demand at that price is stochastic.

level, even when the system is unperturbed. In other words, random changes are here caused not by external events, as assumed in conventional theories such as the efficient market hypothesis [25], but are due to the innate uncertainty of the system. Again, this is because the quantum oscillator has a ground state with non-zero energy.

Such stochastic models have been widely used in areas such as systems biology, where it has been shown that certain system properties such as negative feedback actively damp stochastic variations due to a small number of molecules [26,27], but in economics their use is usually limited to assessing the effects of random external shocks rather than internal dynamics. A first step therefore would be to replace deterministic supply/demand equations in conventional models with dynamic probabilistic versions. Larger models could take advantage of the computational techniques developed for systems biology models [28].

This type of application would only exploit the probabilistic aspect of the approach; however the most interesting features of the quantum oscillator are its quantized energy structure, and the possibility for phenomena such as interference and entanglement between multiple oscillators, which are very relevant for economics. As a simple illustrative example, Fig. 6 shows a prototype quantum agent-based model where 100 buyers and 200 sellers perform transactions. The mass term of the buyers has an oscillatory component which gives a seasonal variation. In terms of quantum decision theory [8], this variation could be attributed to a subjective attraction factor which reflects seasonal attitudes and interferes with objective calculations of utility on the part of the buyer. Seasonal behavior can also of course be produced by classical models, but the advantage of the quantum approach is that it provides a general framework for handling such effects.

As mentioned in the introduction, a number of authors have previously used the quantum oscillator to model asset price changes in financial markets as oscillations in a potential well [29–31], with the restoring force representing reversion to the mean (as opposed to the bottom-up interpretation here in terms of probabilistic interactions between buyer and seller). The quantum Hamiltonian can be viewed as an expression of a stock's risk: the kinetic term captures the degree of price momentum, while the potential term reflects deviation from equilibrium. The mass m is seen as reflecting properties such as market capitalization that affect the rate of price adjustment, while ω is a characteristic oscillating frequency. Piotrowski et al. [32] also derived a model of asset price changes, based on quantum game theory, that followed a Ornstein–Uhlenbeck process, and used it to obtain a formula for the price of a European call option.

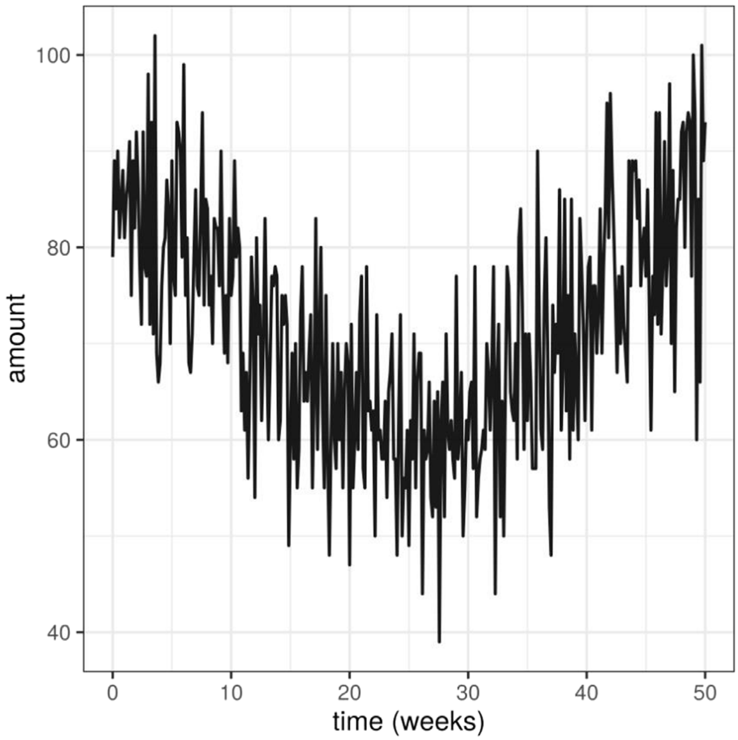

Fig. 6. Simulations of the daily number of transactions for a model where the mass term for the buyer has a seasonal component. When buyers are less flexible (higher mass) there are fewer transactions. Price follows a similar pattern.

As seen above, energy levels in the oscillator are quantized, with a normal ground state and higher energy levels that show more complicated distributions. Ahn et al. [31] showed that the quantum oscillator model outperformed traditional stochastic process models for fitting historical price changes in the Financial Times Stock Exchange (FTSE) All Share Index. The system was found to be in the ground state nearly all (about 98 percent) of the time, with the next two levels contributing the skewness and kurtosis that characterized the data. Higher levels had negligible effect. The frequency ω was interpreted as a measure of the speed of mean reversion of stock returns. This number will of course depend on the particular market and asset; for example Balvers et al. [33] analyzed a number of stock markets and estimated a reversion half-life of three to three and a half years.

The quantum approach is also compatible with Ising-type models from statistical mechanics which have been used to simulate stock market dynamics [34]. In physics, the Ising model was initially developed to simulate ferromagnetic materials, where the magnetic dipole moments of atomic spins can be in one of two states ($+1$ or -1). When an external magnetic field is applied, interactions between atoms lead to phase transitions between a random state and ones in which spins are aligned. The same idea can be applied to simulate contagion in the stock market, where market participants collectively change their stance towards asset valuation. For example Gusev et al. [35] created an empirically-fitted model where prices oscillate in a potential well which is determined in part by the propagation of news and opinions. While they used a classical version of the Ising model, a quantum version would give similar results, though again with the feature of a ground state where fluctuations occur even in the absence of new information.

Finally, financial markets are characterized by entanglement of two sorts. The first is through social factors such as culture or news, the second (and more direct) is through the use of financial instruments such as loans and derivatives. As discussed in [6,36] a loan agreement can be modeled as an entangled system, where the borrower's mental state to pay or default is a quantum state which is entangled via the loan with that of the creditor (so default immediately affects the status of the loan even if the creditor does not find out immediately). Entangled oscillators are a staple of quantum physics and some of the techniques could carry over into economics. The oscillator model of supply and demand could for example be incorporated in quantum agent-based models where decisions to buy or sell are viewed as the outcome of quantum dynamic processes, that are susceptible to entanglement through social influences but also through the financial system itself. The development of such a model is a project for future work.

7. Conclusions

The quantum approach provides a natural framework for modeling supply and demand. The main parameters are measures of preferred prices and flexibility, which comprise a minimal description of buyer/seller behavior. To summarize, the main conclusions are:

- The propensity for buyers and sellers to take part in a transaction can be modeled as a joint propensity curve that represents a probability distribution.
- The entropic force corresponding to this curve describes an oscillator, whose mass is given by an inverse variance term that measures resistance to change.
- The quantized version of this entropic oscillator has a complex wave function whose squared amplitude gives the probability distribution for price.
- The ground state corresponds to the original propensity curve, which shows the connection between information flows and quantum dynamics.
- The uncertainty of the quantum ground state represents the indeterminate nature of the system, so price changes may reflect not new information, as in the classical model, but rather the *absence* of information.
- While the classical model assumes that market exchanges drive the system to a state of equilibrium, the quantum model suggests they drive it to a state of higher information (lower entropy).
- Excited energy states of the oscillator contribute the skew and kurtosis that characterize financial statistics.
- The model has a number of applications, including as a tool to perform stochastic simulations, or as the basis for a quantum agent-based model.

While as shown above a somewhat similar model can be produced using stochastic differential equations, a distinguishing feature of the quantum version is that price is modeled by a wave function which only collapses to a set value when measured during a transaction. This correctly reflects the indeterminacy of financial systems; incorporates the fact that the measurement procedure affects the system being measured; and forms a natural interface to explore interference effects in cognition, entanglement through social and financial bonds, and the dynamics of excited energy states.

Acknowledgments

The author would like to thank Michael Schnabel and four anonymous reviewers for their comments on earlier drafts which led to a significantly improved manuscript.

Appendix

This appendix gives extra mathematical background on the equations for the propensity force from Section 4, the harmonic oscillator (Section 5), and the inventory model (Section 6), as well as tables listing the key parameters and equations.

Entropic forces

In statistical mechanics, a force $f(x)$ with potential $U(x)$ acting on a single particle yields a probability distribution $P(x)$ for the position of the particle of

$$P(x) = C \exp\left(-\frac{U(x)}{k_B T}\right)$$

where T is temperature, k_B is the Boltzmann constant, and C is a normalization constant. It therefore follows that

$$f(x) = -k_B T \frac{d}{dx} \log P(x) = -k_B T \frac{P'(x)}{P(x)}.$$

These forces are also called entropic forces because they reflect the tendency of the system to achieve maximum entropy [37]. The propensity force can therefore be viewed as an entropic force acting on the mental state of the buyer/seller, with constant $\gamma = k_B T$. In physics, this term is the amount of heat required to increase the thermodynamic entropy of a system measured in natural units. As seen next, the quantum model gives $\gamma = \hbar\omega/2$ which is the lowest energy state of the oscillator, and can be interpreted as a heat level for the system. This result is in agreement with the derivation of Hooke's law using entropic forces and the principle of minimal information in [38]. The propensity forces are therefore the cognitive version of entropic forces, with the difference that they counteract the tendency towards randomness and therefore act in the opposite direction. From $k_B T = \hbar\omega/2$ we can also write the period as $\tau = 2\pi/\omega = \pi\tau_B$ where the Boltzmann time $\tau_B = \hbar/(k_B T)$ is the theoretical order of time needed for an arbitrary (so not necessarily realistic) nonstationary state to reach thermal equilibrium [39].

Equations for the coupled supply/demand oscillator

Using the equation $\sigma = \sqrt{\frac{\hbar}{2m\omega}}$ for the standard deviation of the ground state of a quantum harmonic oscillator, we can write the corresponding masses of the buyer and seller as

$$m_o = \frac{\hbar}{2\omega_o\sigma_o^2}$$
$$m_b = \frac{\hbar}{2\omega_b\sigma_b^2}.$$

We will assume that the force constants k_o and k_b scale in the same way with mass, which implies that the oscillating frequencies for the buyer and seller are the same, so $\omega_o = \omega_b = \omega$. Using the expressions for frequency and mass, these are

$$k_o = m_o\omega^2 = \frac{\hbar\omega}{2\sigma_o^2}$$
$$k_b = m_b\omega^2 = \frac{\hbar\omega}{2\sigma_b^2}.$$

Note these are the same as the force constants for the demand and supply forces in Section 2, where the scaling factor there is set to $\gamma = \hbar\omega/2$. Since $k = k_o + k_b$ for the joint supply/demand system, we can write

$$\frac{\hbar\omega}{2\sigma^2} = \frac{\hbar\omega}{2\sigma_b^2} + \frac{\hbar\omega}{2\sigma_b^2}$$

and solving for the standard deviation σ gives

$$\sigma = \frac{\sigma_o\sigma_b}{\sqrt{\sigma_o^2 + \sigma_b^2}}.$$

The corresponding mass is

$$m = m_o + m_b = \frac{\hbar}{2\omega\sigma^2}$$

and the center of mass is

$$\mu = \frac{k_o\mu_o + k_b\mu_b}{k_o + k_b} = \frac{\sigma_b^2\mu_o + \sigma_o^2\mu_b}{\sigma_o^2 + \sigma_b^2}.$$

These parameters are the same as for the product of the normal probability distributions for supply and demand. It is interesting to note that the inverse relationship between mass m and volatility σ^2 is derived through completely independent quantum finance analyses in ([3,4]: 52).

Inventory model

The inventory model was adapted from the systems dynamics model presented in [40]. At each time step, the price is adjusted in order to maintain a level of inventory equal to four times the current demand. In the classical systems dynamics model, the inventory equation is

$$i_{t+1} = i_t + s_t - d_t$$

where supply s_t and demand d_t are determined by their respective schedules. The price for the next time step is given by

$$p_{t+1} = \left(1 + \frac{(1 - r_t)}{\gamma}\right) p_t$$

where the inventory ratio is

$$r_t = \frac{i_t}{4d_t}$$

and $\gamma = 30$ reflects inertia in adjusting the price. The system is therefore at equilibrium when inventory is four times the demand.

In the quantum model, the equations are the same except that demand at the price set by the seller is probabilistic, so the number of units purchased follows a Poisson distribution. The effect is to introduce a level of stochastic noise.

Table A.1
List of main parameters.

Parameter	Symbol	Default value	Dimensions
Mean bid price	μ_b	10.2	L
Mean offer price	μ_o	9.8	L
Standard deviation seller	σ_b	0.1	L
Standard deviation buyer	σ_o	0.2	L
Seller quantity	N_b	200	–
Buyer quantity	N_o	100	–
Planck constant	$\hbar$	See text	$ML^2\ T^{-1}$
Frequency	ω	See text	T^{-1}
Rate parameter	r	0.01	LT^{-1}

Table A.2
List of main equations.

Variable	Symbol	Equation	Dimensions
Offer propensity	$P_o(x)$	$\frac{1}{\sqrt{2\pi\sigma_o^2}}\exp\left(-\frac{\mu_o^2}{2\sigma_o^2}\right)$	L^{-1}
Bid propensity	$P_b(x)$	$\frac{1}{\sqrt{2\pi\sigma_b^2}}\exp\left(-\frac{\mu_b^2}{2\sigma_b^2}\right)$	L^{-1}
Demand force	$F_o(x)$	$-k_o(x-\mu_o)$	MLT^{-2}
Supply force	$F_b(x)$	$k_b(x-\mu_b)$	MLT^{-2}
Demand force constant	k_o	$\frac{\hbar\omega}{2\sigma_o^2}$	MT^{-2}
Supply force constant	k_b	$\frac{\hbar\omega}{2\sigma_b^2}$	MT^{-2}
Joint mean	μ	$\frac{\sigma_b^2\mu_o+\sigma_o^2\mu_b}{\sigma_o^2+\sigma_b^2}$	L
Joint standard deviation	σ	$\frac{\sigma_o\sigma_b}{\sqrt{\sigma_o^2+\sigma_b^2}}.$	L
Mass of coupled system	m	$\frac{\hbar}{2\omega\sigma^2}$	M
Bid-offer spread	μ_t	$\mu_b-\mu_o$	L
Standard deviation for α	σ_t	$\sqrt{\sigma_o^2+\sigma_b^2}$	L
Scaling factor for propensity of trade	α	$\frac{1}{\sqrt{2\pi\sigma_t^2}}\exp\left(-\frac{\mu_t^2}{2\sigma_t^2}\right)$	L^{-1}
Joint propensity	$P_t(x)$	$\frac{\alpha}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{\mu^2}{2\sigma^2}\right)$	L^{-2}
Trading volume	$V(x)$	$N_oN_br\alpha$	T^{-1}

Key parameters and equations

All simulations in the paper were performed using the default parameters of the QuantumSD app available at: https://david-systemsforecasting.shinyapps.io/supplydemand/. The key parameters are listed in Table A.1. Dimensions are M for mass, T for time, and L for the natural logarithm of price.

The Table A.2 lists the main equations used.

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