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TWO-DIMENSIONAL ISING MODEL WITH ANNEALED RANDOM FIELDS*

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We investigate the influence of annealed random fields on the phase diagram of the ferromagnetic Ising model on the square lattice. We find that the gaussian random field makes the ferromagnetic ground state unstable and we have a super-antiferromagnetic state at low temperatures. For the binary random field, in which the random field takes $+h$ with probability p and $-h$ with probability $1-p$, there is a critical field h_c, and the ferromagnetic ground state is stable for $h<h_c$ but unstable for $h>h_c$. When the ferromagnetic ground state is unstable, we have a re-entrant phase transition.

1. Introduction

Since the work of Imry and Ma[1]), the influence of quenched random fields on the phase diagram and on the critical phenomena for Ising models has been investigated by many authors. The big controversy has been on the value of the lower critical dimension, i.e. the dimension below which long-range ferromagnetic order cannot exist for the ferromagnetic Ising models. The results from the experimental and the theoretical investigations are not yet conclusive for this problem. See, for example, the paper by Birgeneau et al.[2,3]) for the experimental investigations, and the paper by Imry[4]) and by Imbrie[5]) for the theoretical aspects. Also see references given in these papers.

The exact results were obtained so far for the one-dimensional Ising model[6–8]), for the Ising model on a Cayley tree[9,10]), for the Husimi–Temperly model[11,12]) and for the spherical model[13]). In the Hushimi–Temperley model, the ferromagnetic phase is unstable at all temperatures against local random fields if their standard deviation is large enough.

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Recently an interesting approach to the quenched–random field problem was given by Schwartz[14]. He constructed an equivalent annealed system with complicated field distribution to a quenched system. Quite recently, Gonçalves and Stinchcombe[15] have studied the Ising model with an annealed random field. In that model, the random field which is allowed to assume the values $\pm h$, 0 with probabilities $p_{\pm}$, p_0 is generated by introducing decorated bonds along only one of the lattice directions. They also required a pseudo-chemical potential which is determined by specifying the probability p_0. They found that the ferromagnetic state is stable and is favoured by the random field.

In the present paper, we consider the ferromagnetic Ising model with annealed random fields on the square lattice. A random field is also generated by introducing decorated bonds along one of the lattice directions in a similar procedure as the one adopted by Gonçalves and Stinchcombe[15]. The different, but important, point is that each decorating spin couples to two spins with opposite signs. We find that the random field makes the ferromagnetic phase unstable at low temperatures.

In section 2, we describe the system considered in the present paper. In section 3, we investigate the phase diagram for the random field with a binary density on the Gaussian density. Concluding remarks are given in section 4.

2. The model

We consider the square lattice $\mathbb{Z}^2$. We express a part of $\mathbb{Z}^2$ by Λ which is equal to set $\{(i, j) \mid i, j \in \mathbb{I}_N\}$. Here we denote the set of integers from 1 to N by $\mathbb{I}_N$. We consider the Ising model with an annealed random field on Λ. The Hamiltonian is given by

$$H = -J \sum_{(i,j)} (\sigma_{ij}\sigma_{i\,j+1} + \sigma_{ij}\sigma_{i+1\,j}) - \sum_{(i,j)} h_{ij}\sigma_{ij} - \mu \sum_{(i,j)} h_{ij}\,, \tag{1}$$

where σ_{ij} is the Ising spin variable taking on values ± 1 and h_{ij} is assumed to be an annealed random field applied to spin σ_{ij} on a lattice site (i,j). μ is the chemical potential and J is the ferromagnetic exchange integral. We impose periodic boundary conditions to the system on Λ, and then the summations in eq. (1) are taken over the set Λ.

The distributions of the random fields (h_{ij}) are not independent from each other in annealed random fields. Here we further assume that those along the one of two lattice directions, say the y-direction, have a correlation. More explicitly, we assume that the sum of these fields along the y-direction with a fixed coordinate in the x-direction is zero,

$$\sum_{j=1}^{N} h_{ij} = 0 . \tag{2}$$

For example, we may consider a discrete distribution of these fields, whose probability density is given as follows:

$$P\{h_{ij} \mid j \in \mathbb{I}_N\} = \sum_{k=0}^{N} p^{N-k}(1-p)^k \sum_{\Phi_k} F\{h_{in} \mid n \in \Phi_k\} \prod_{l \in \mathbb{I}_N \setminus \Phi_k} \delta(h_{il}) , \tag{3}$$

where Φ_k is a set given by

$$\Phi_k = \{ j_1, j_1 - 1, j_2, j_2 - 1, \ldots, j_k, j_k - 1 \mid j_n \in \mathbb{I}_N, j_1 \neq j_2 \neq \cdots \neq j_k \} . \tag{4}$$

The function $F\{h_{in} \mid n \in \Phi_k\}$ is given as follows:

$$F\{h_{in} \mid n \in \Phi_k\} = \prod_{n=1}^{k}{}' \delta(h_{ij_n} + h)\delta(h_{i\, j_n - 1} - h) , \tag{5}$$

where the product with a prime means that a product of two factors, for example $\delta(h_{ij} + h)$ and $\delta(h_{i\, j'-1} - h)$ is equal to $\delta(h_{ij})$ whenever $j = j' - 1$. The summation over the set Φ_k in the above equation is taken over all the possible combinations of k integers out of $\mathbb{I}_N$. Then for fixed k, we have ${}_N C_k$ terms and the relation

$$\sum_{k=0}^{N} p^{N-k}(1-p)^k {}_N C_k = 1 . \tag{6}$$

The probability is determined usually through the chemical potential.

We look at this system from another viewpoint. When we introduce a set of auxiliary fields $\{\tau_{ij}\}$, we may express our system in a slightly different way. Namely, we assume that

$$h_{ij} = (-\tau_{i\, j-1} + \tau_{ij})/2 . \tag{7}$$

Then condition (2) is automatically satisfied under the periodic conditions, no matter what kind of distribution for $\{\tau_{ij}\}$ we have. Hamiltonian (1) is now rewritten as follows:

$$H = -J \sum_{(i,j)} (\sigma_{ij}\sigma_{i\, j+1} + \sigma_{ij}\sigma_{i+1\, j}) - \tfrac{1}{2} \sum_{(i,j)} (\tau_{ij} - \tau_{i\, j-1})\sigma_{ij} . \tag{8}$$

We notice that the chemical potential does not appear in the above equation.

This is a different point from the usual annealed system. We then investigate the present system under such a condition that we have a rigid probability density for τ_{ij}.

The probability density for $\{h_{ij} \mid j \in \mathbb{I}_N\}$ given by eq. (3) is obtained by assuming that the probability density for τ_{ij} is a binary density:

$$P(\tau_{ij}) = \begin{cases} p\,, & \text{for } \tau_{ij} = h\,, \\ 1-p\,, & \text{for } \tau_{ij} = -h\,. \end{cases} \tag{9}$$

Actually, $P\{h_{ij} \mid j \in \mathbb{I}_N\}$ is calculated from the following equation:

$$P\{h_{ij} \mid j \in \mathbb{I}_N\} = \frac{1}{(2\pi)^N} \int \prod_{j=1} \mathrm{d}q_j \left\langle \prod_{j=1} \exp[\mathrm{i}q_j(h_{ij} - \tfrac{1}{2}\tau_{ij} + \tfrac{1}{2}\tau_{i\ j-1})] \right\rangle, \tag{10}$$

where the angular brackets mean the average over distributions for $\{\tau_{ij}\}$.

In the next section, we will calculate the partition function

$$Z = \left\langle \sum_{\{\sigma\}} \exp\{-\beta H\} \right\rangle, \tag{11}$$

where $\beta = 1/k_{\mathrm{B}}T$ as usual. We will consider two types of probability density for τ_{ij} and will discuss the phase diagrams there. Before that, we give here a comment on the system. Our system given by eq. (8) is very similar to the one which has recently been investigated by Gonçalves and Stinchcombe[15]). In their system, the random field does not satisfy the condition shown in eq. (2), and their approach is restricted to a discrete distribution of random fields with components $\pm h$, 0 in which only the probability for the component zero of the field is specified.

3. Phase diagram

We investigate in detail the system described by eq. (8) for two types of distribution for the random field, namely for a binary density and for the Gaussian density.

First we consider the case of a binary density. We set $\tau_{ij} = ht_{ij}$ where t_{ij} takes on ± 1. Hamiltonian (8) is rewritten as

$$H = -J \sum_{(i,j)} (\sigma_{ij}\sigma_{i\ j+1} + \sigma_{ij}\sigma_{i+1\ j}) - \frac{h}{2} \sum_{(i,j)} (t_{ij} - t_{i\ j-1})\sigma_{ij}\,, \tag{12}$$

and eq. (9) is interpreted as

$$P(t_{ij}) = (pq)^{1/2} \exp\left\{ \frac{1}{2} \ln\left(\frac{p}{q}\right) t_{ij} \right\}, \tag{13}$$

where $q = 1 - p$. We assume that p is not equal to 0 nor 1. The partition function is now expressed as follows:

$$Z = (pq)^{N^2/2} \sum_{\{\sigma\}} \sum_{\{t\}} \exp\left\{ K \sum_{(i,j)} (\sigma_{ij}\sigma_{i\,j+1} + \sigma_{ij}\sigma_{i+1\,j}) + \sum_{(i,j)} t_{ij} \left(\frac{\beta h}{2} (\sigma_{ij} - \sigma_{i\,j+1}) + \frac{1}{2} \ln\left(\frac{p}{q}\right) \right) \right\}, \tag{14}$$

where $K = \beta J$. Taking the summation over the variables $\{t_{ij}\}$, we obtain

$$Z = (1 + 4pq \sinh^2 \beta h)^{N^2/4} Z_{\mathrm{sq}}(K - C, K), \tag{15}$$

where

$$C = \tfrac{1}{4} \ln\{1 + 4pq \sinh^2 \beta h\}. \tag{16}$$

Here $Z_{\mathrm{sq}}(K_1, K_2)$ is the partition function of the Ising model on the square lattice with nearest-neighbour coupling constants $k_{\mathrm{B}}TK_1$ and $k_{\mathrm{B}}TK_2$ [16]).

The critical temperature is determined by the following equation[16]):

$$\sinh 2|K - C| \sinh 2K = 1. \tag{17}$$

The phase diagram is obtained by solving this equation numerically. The results are given in fig. 1. Due to the symmetry at $p = \frac{1}{2}$, we present the results only for $p \geqslant \frac{1}{2}$. The behavior of C in the limit of $T = 0$ determines the nature of the ground state. The critical field is $2J$, and thus we have three cases. For $h < 2J$, the ground state is ferromagnetic as in a two-dimensional Ising model. An example of this case is shown in fig. 1 for $h = J$. We have the ferromagnetic state at temperatures below the critical temperature, and this state is stable for the random field. For $h = 2J$, the ground state is ferromagnetic as in a one-dimensional Ising model. Thus in fig. 1, we have the two-dimensional ferromagnetic state at temperatures below the critical temperature except at $T = 0$ and this state is unstable at $T = 0$ for the random field. For $h > 2J$, the ground state is super-antiferromagnetic[17]). Two examples are shown in fig. 1, namely for $h = 2.5J$ and $h = 3J$. We have in this case a critical concentration $p_{\mathrm{c}}(h)$, which depends on the field h. For $p_{\mathrm{c}}(h) < p < 1 - p_{\mathrm{c}}(h)$, the ferromagnetic state is totally unstable for the random field. For $p < p_{\mathrm{c}}(h)$ or $p > 1 - p_{\mathrm{c}}(h)$, we have the paramagnetic state, the ferromagnetic state, again the

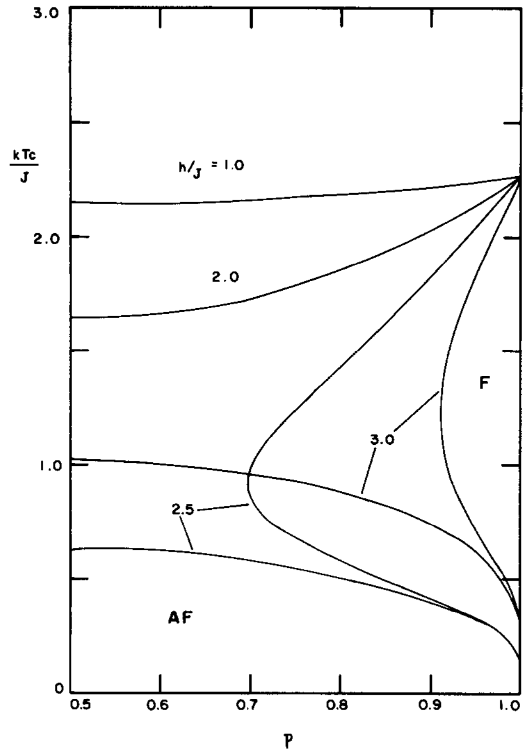

Fig. 1. The critical temperature for an annealed random field with a binary density. The abscissa denotes the probability and the ordinate the temperature. We notice that the graphs are symmetric at $p = \frac{1}{2}$, so only half of them are given.

paramagnetic state and the super-antiferromagnetic state as the temperature decreases. Thus the ferromagnetic state is unstable for the random field at low temperatures and a re-entrant phase transition is observed.

We have a comment here. When we consider the case in which the distribution of the random field τ_{ij} is probability p for h_1 and $1-p$ for h_2, we have the same results obtained above. In order to see this, we may write τ_{ij} as follows:

$$\tau_{ij} = \tfrac{1}{2}(h_1 + h_2) + \tfrac{1}{2}(h_1 - h_2)t_{ij} \,. \tag{18}$$

Substitute this into eq. (8); Hamiltonian (8) is equal to Hamiltonian (12) with $(h_1 - h_2)/2$ instead of h.

Next we consider the case in which the probability density of the annealed random field is the Gaussian density. Namely, we have

$$P(\tau_{ij}) = \frac{1}{(2\pi)^{1/2}\rho} \exp\left\{-\frac{1}{2\rho^2}(\tau_{ij} - h)^2\right\}, \tag{19}$$

where ρ is the standard deviation. The partition function (11) is evaluated as follows:

$$Z = \exp\{\tfrac{1}{4}N^2\beta^2\rho^2\} Z_{\mathrm{sq}}(K - \tfrac{1}{4}\beta^2\rho^2, K). \tag{20}$$

The critical temperature is determined by the following equation[16]:

$$\sinh|2K - \tfrac{1}{2}\beta^2\rho^2| \sinh 2K = 1. \tag{21}$$

Solving this equation numerically, we have the phase diagram as shown in fig. 2. For $\rho < 1.95J$, we have the paramagnetic state, the ferromagnetic state, again the paramagnetic state and the super-antiferromagnetic state as the temperature decreases. We observe a re-rentrant phase transition in this case

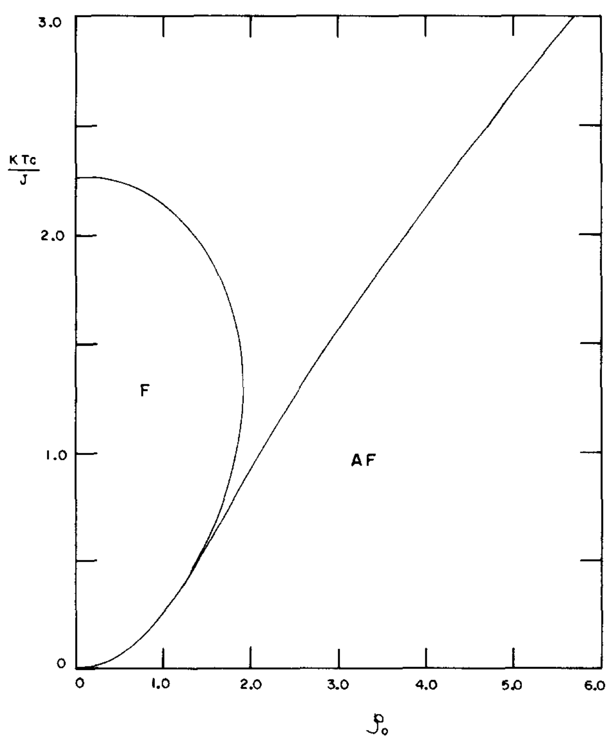

Fig. 2. The critical temperatures for an annealed random field with the Gaussian density. The abscissa denotes the reduced standard deviation $\rho_0 = \rho/J$ and the ordinate the temperature.

and the ferromagnetic state is unstable for the random field at low temperatures. For $\rho > 1.95J$, the ferromagnetic state is totally unstable for the random field and we have only the super-antiferromagnetic state as for the ordered state. This phase diagram does not depend on the value of h. This situation is similar to the one which occurs in the case of the binary density, namely the phase diagram does not depend on the value of $(h_1 + h_2)/2$.

4. Concluding remarks

In this paper, we investigated the influence of an annealed random field on the phase transition of the ferromagnetic Ising model on the square lattice. The annealed random field considered in the present paper has the rather artificial condition that the sum of them along one of two lattice directions is zero. Nevertheless, it is interesting to have found that this sort of annealed random field makes the ferromagnetic state unstable at low temperatures and that there exists a critical field for the annealed random field with the binary density. The existence of such a critical field for the binary density has been suggested by several authors for the quenched random field problems. See for example the paper by de Queiroz and dos Santos[18]).

As a related system to the one studied in the present paper, we consider an Ising model on the square lattice whose Hamiltonian is given by

$$H = -J \sum_{(i,j)} (-\sigma_{ij}\sigma_{i\,j+1} + \sigma_{ij}\sigma_{i+1\,j}) . \tag{22}$$

When we apply the following annealed random field to the above system:

$$\Delta H = -\tfrac{1}{2} \sum_{(i,n)} \{\tau_{i2n+1}(\sigma_{i2n+1} + \sigma_{i2n+2}) - \tau_{i2n}(\sigma_{i2n} + \sigma_{i2n+1})\} , \tag{23}$$

the random field makes the super-antiferromagnetic state unstable at low temperatures. This is easily confirmed: if we change σ_{i2n} for $-\sigma_{i2n}$, then system (22) with (23) becomes the system described in section 2.

Finally, we have a comment. In the calculation of the partition function, the averages of random fields $\{h_{ij}\}$ and also the spins $\{\sigma_{ij}\}$ have been carried out with the Gibbs factor. However, due to condition (2), the chemical potential term disappears in the present system. Then we have investigated the system under the fixed distribution of the random fields. When a distribution of random fields is not fixed, these could adjust their distribution through the spin variables in the thermal equilibrium at each temperature. By using moments of fields, we can obtain their distribution at each temperature. We are now proceeding an investigation in this direction.

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