TECHNISCHE UNIVERSITEIT EINDHOVEN
Faculteit Wiskunde en Informatica

Solutions to the Examination Logic & Set Theory (2IT61)

Monday January 21, 2013, 14:00–17:00 hrs.

You are **not** allowed to use any books, notes, or other course material. Your solutions to the problems have to be formulated and written down in a clear and precise manner.

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(1) 1. Show that the following abstract proposition is a contingency (i.e., not a tautology and not a contradiction):

$$((a \wedge b) \Leftrightarrow (\neg c \vee b)) \wedge \neg(a \Rightarrow c) \ .$$

Solution: Let us denote by φ the formula in the exercise.

To see that φ is not a tautology, note that if $a = 0$, then $a \Rightarrow c = 1$, so $\neg(a \Rightarrow c) = 0$, and hence $\varphi = 0$.

To see that φ is not a contradiction, note that if $a = 1$, $b = 1$, and $c = 0$, then $(a \wedge b) = 1$ and $(\neg c \vee b) = 1$, so $(a \wedge b) \Leftrightarrow (\neg c \vee b) = 1$, and $a \Rightarrow c = 0$, so $\neg(a \Rightarrow c) = 1$, and hence $\varphi = 1$.

We conclude that φ is a contingency.

(2) 2. Prove with a *calculation* (i.e., using the formal system based on standard equivalences and weakenings described *Part I* of the book) that the abstract propositions

$$P \Rightarrow ((Q \Rightarrow R) \wedge (Q \vee R)) \text{ and } (\neg P \Rightarrow Q) \Rightarrow R$$

are *comparable* (i.e., the left-hand side formula is stronger than the right-hand side formula, or the right-hand side formula is stronger than the left-hand side formula).

Solution: On the one hand, the calculation

$$\begin{array}{ll}
 & P \Rightarrow ((Q \Rightarrow R) \wedge (Q \vee R)) \\
\stackrel{val}{==} & \{ \text{ Implication } (2\times) \ \} \\
 & \neg P \vee ((\neg Q \vee R) \wedge (Q \vee R)) \\
\stackrel{val}{==} & \{ \text{ Distributivity } \} \\
 & \neg P \vee ((\neg Q \wedge Q) \vee R) \\
\stackrel{val}{==} & \{ \text{ Contradiction } \} \\
 & \neg P \vee (\texttt{False} \vee R) \\
\stackrel{val}{==} & \{ \ \texttt{True}/\texttt{False}\text{-elimination } \} \\
 & \neg P \vee R
\end{array}$$

proves that $P \Rightarrow ((Q \Rightarrow R) \wedge (Q \vee R)) \stackrel{val}{=\!=} \neg P \vee R$.

On the other hand, the calculation

$$
\begin{array}{ll}
 & (\neg P \Rightarrow Q) \Rightarrow R \\
\stackrel{val}{=\!=} & \{ \text{ Implication } (2\times) \ \} \\
 & \neg(\neg\neg P \vee Q) \vee R \\
\stackrel{val}{=\!=} & \{ \text{ Double Negation } \} \\
 & \neg(P \vee Q) \vee R \\
\stackrel{val}{=\!=} & \{ \text{ De Morgan } \} \\
 & (\neg P \wedge \neg Q) \vee R
\end{array}
$$

proves that $(\neg P \Rightarrow Q) \Rightarrow R \stackrel{val}{=\!=} (\neg P \wedge \neg Q) \vee R$.

Since

$$
\begin{array}{ll}
 & (\neg P \wedge \neg Q) \vee R \\
\stackrel{val}{\models} & \{ \ \wedge\text{-}\vee\text{-weakening + Monotonicity } \} \\
 & \neg P \vee R \ ,
\end{array}
$$

it follows that

$$
\begin{array}{ll}
 & (\neg P \Rightarrow Q) \Rightarrow R \\
\stackrel{val}{=\!=} & \\
 & (\neg P \wedge \neg Q) \vee R \\
\stackrel{val}{\models} & \\
 & \neg P \vee R \\
\stackrel{val}{=\!=} & \\
 & P \Rightarrow ((Q \Rightarrow R) \wedge (Q \vee R)) \ .
\end{array}
$$

Hence, the formulas $P \Rightarrow ((Q \Rightarrow R) \wedge (Q \vee R))$ and $(\neg P \Rightarrow Q) \Rightarrow R$ are comparable.

(2) 3. Determine whether the formula

$$\forall_x [x \in \mathbb{Z} : \exists_y [y \in \mathbb{Z} : 2x - y = 3]]$$

is true or false, and give arguments for your answer.

Solution: Let $x \in \mathbb{Z}$, and define y by $y = 2x - 3$; then, clearly, $y \in \mathbb{Z}$ and $2x - y = 2x - (2x - 3) = 3$. This proves that for all $x \in \mathbb{Z}$ there exists $y \in \mathbb{Z}$ such that $2x - y = 3$, and hence the formula is true.

(2) 4. Prove with a *derivation* (i.e., using the methods described in *Part II* of the book) that the formula

$$\exists_x \forall_y [P(x) \Rightarrow Q(y)] \;\Rightarrow\; (\neg\forall_u [P(u)] \vee \exists_v [Q(v)])$$

is a tautology.

<u>*Solution:*</u>

{ Assume: }

(1) $\boxed{\exists_x \forall_y [P(x) \Rightarrow Q(y)]}$

{ Assume: }

(2) $\boxed{\neg\neg\forall_u [P(u)]}$

{ $\neg\neg$-elim on (2): }

(3) $\forall_u [P(u)]$

{ $\exists^*$-elim on (1): }

(4) Pick an x with $\forall_y [P(x) \Rightarrow Q(y)]$

{ $\forall$-elim on (4): }

(5) $P(x) \Rightarrow Q(x)$

{ $\forall$-elim on (3) and (4): }

(6) $P(x)$

{ $\Rightarrow$-elim on (5) and (6): }

(7) $Q(x)$

{ $\exists^*$-intro on (7): }

(8) $\exists_v [Q(v)]$

{ $\vee$-intro on (2) and (8): }

(9) $\neg\forall_u [P(u)] \vee \exists_v [Q(v)]$

{ $\Rightarrow$-intro on (1) and (10): }

(10) $\exists_x \forall_y [P(x) \Rightarrow Q(y)] \Rightarrow (\neg\forall_u [P(u)] \vee \exists_v [Q(v)])$

(2) 5. Check whether the formula

$$A \cap B \subseteq C \;\Rightarrow\; A \backslash C \subseteq B^{\mathrm{C}}$$

holds for all sets A, B and C. If so, then give a proof; if not, then give a counterexample.

Solution: We prove that the formula

$$A \cap B \subseteq C \;\Rightarrow\; A \backslash C \subseteq B^{\mathrm{C}}$$

holds for all sets A, B and C.

Let A, B and C be sets such that $A \cap B \subseteq C$. To prove that $A \backslash C \subseteq B^{\mathrm{C}}$, by the definition of $\subseteq$ it suffice to prove that all $x \in A \backslash C$ satisfy $x \in B^{\mathrm{C}}$. So, let $x \in A \backslash C$; we establish that $x \in B^{\mathrm{C}}$. By the property of ${}^{\mathrm{C}}$, to establish that $x \in B^{\mathrm{C}}$, it suffices to show that $x \notin B$. To this end, we assume that $x \in B$ and derive a contradiction. Note that from $x \in A \backslash C$ it follows that $x \in A$ and $x \notin C$. From $x \in A$ and $x \in B$ it follows, by the property of $\cap$, that $x \in A \cap B$, and hence, since $A \cap B \subseteq C$, by the property of $\subseteq$, that $x \in C$. We now have that both $x \in C$ and $x \notin C$: a contradiction.

(2) 6. (a) Show with a counterexample that the formula

$$\forall_x[x \in A : G_1(F(x)) = G_2(F(x))] \Rightarrow \forall_y[y \in B : G_1(y) = G_2(y)]$$

does not hold for all sets A, B and C, and for all mappings $F : A \to B$, $G_1 : B \to C$ and $G_2 : B \to C$.

(2) (b) Let A, B and C be sets and let $F : A \to B$, $G_1 : B \to C$ and $G_2 : B \to C$ be mappings.

Prove: if F is a *surjection*, then the formula

$$\forall_x[x \in A : G_1(F(x)) = G_2(F(x))] \Rightarrow \forall_y[y \in B : G_1(y) = G_2(y)]$$

does hold.

Solution:

(a) Let $A = \{0\}$, $B = \{0, 1\}$, and $C = \{0, 1, 2\}$, define $F : A \to B$ for all $x \in A$ by $F(x) = x$, define $G_1 : B \to C$ for all $y \in B$ by $G_1(y) = y$, and define $G_2 : B \to C$ for all $y \in B$ by $G_2(y) = 2 \cdot y$. Then $G_1(F(0)) = G_1(0) = 0 = 2 \cdot 0 = G_2(0) = G_2(F(0))$, so $G_1(F(x)) = G_2(F(x))$ for all $x \in A$, but $G_1(1) = 1 \neq 2 = 2 \cdot 1 = G_2(1)$ refuting that $G_1(y) = G_2(y)$ for all $y \in B$.

(b) Let $F : A \to B$, $G_1 : B \to C$ and $G_2 : B \to C$ be mappings, and suppose that F is a surjection. To prove that the formula

$$\forall_x[x \in A : G_1(F(x)) = G_2(F(x))] \Rightarrow \forall_y[y \in B : G_1(y) = G_2(y)]$$

holds, we assume that $G_1(F(x)) = G_2(F(x))$ holds for all $x \in A$ and establish that then also $G_1(y) = G_2(y)$ for all $y \in B$. Let $y \in B$. Then, since F is a surjection, there exists $x \in A$ such that $F(x) = y$. From $G_1(F(x)) = G_2(F(x))$ it then immediately follows that $G_1(y) = G_1(F(x)) = G_2(F(x)) = G_2(y)$.

(4) 7. Prove that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 for every natural number n.

Solution: We prove with induction on $n \in \mathbb{N}$ that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

If $n = 0$, then $n^3 + (n+1)^3 + (n+2)^3 = 0^3 + 1^3 + 2^3 = 0 + 1 + 8 = 9$, which is clearly divisible by 9.

Let $n \in \mathbb{N}$ and suppose that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 (the induction hypothesis); we should establish that $(n+1)^3 + (n+2)^3 + (n+3)^3$ is divisible by 9. To this end, note that

$$
\begin{aligned}
&(n+1)^3 + (n+2)^3 + (n+3)^3 \\
&\qquad = (n+1)^3 + (n+2)^3 + (n+3)(n+3)(n+3) \\
&\qquad = (n+1)^3 + (n+2)^3 + (n+3)(n^2 + 6n + 9) \\
&\qquad = (n+1)^3 + (n+2)^3 + (n^3 + 3n^2 + 6n^2 + 18n + 9n + 27)) \\
&\qquad = (n+1)^3 + (n+2)^3 + n^3 + 9n^2 + 27n + 27 \\
&\qquad = n^3 + (n+1)^2 + (n+2)^3 + 9(n^2 + 3n + 3) \ .
\end{aligned}
$$

Since $n^3 + (n+1)^2 + (n+2)^3$ is divisible by 9 by the induction hypothesis and $9(n^2 + 3n + 3)$ is clearly also divisible by 9, it follows that $(n+1)^3 + (n+2)^3 + (n+3)^3$ is divisible by 9.

8. Define the binary relation R on $\mathbb{N} \times \mathbb{N}$ for all $k, \ell, m, n \in \mathbb{N}$ by

$$(k, \ell) \ R \ (m, n) \text{ if, and only if, } k \le m \wedge (k = m \Rightarrow \ell \le n) \ .$$

(2) (a) List the three properties the relation R should satisfy for being a *reflexive ordering* together with their defining formulas, and prove one of these properties. (You may choose yourself which of the three properties you prefer to prove.)

(1) (b) Draw a Hasse diagram of $\langle \{0,1,2\} \times \{0,1,2\}, R \rangle$.

Solution:

(a) R is an equivalence relation if, and only if,

i. R is *reflexive*, i.e., $(k, \ell) \ R \ (k, \ell)$ for all $(k, \ell) \in \mathbb{N} \times \mathbb{N}$;

ii. R is *transitive*, i.e., for all $(k, \ell), (m, n), (o, p) \in \mathbb{N} \times \mathbb{N}$, if $(k, \ell) \ R \ (m, n)$ and $(m, n) \ R \ (o, p)$, then $(k, \ell) \ R \ (o, p)$; and

iii. R is *anti-symmetric*, i.e., for all $(k, \ell), (m, n) \in \mathbb{N} \times \mathbb{N}$, if $(k, \ell) \ R \ (m, n)$ and $(m, n) \ R \ (k, \ell)$, then $(k, \ell) = (m, n)$.

We prove each of these properties (although, according to the exercise, it is enough to establish one of them):

i. To prove that R is reflexive, let $(k, \ell) \in \mathbb{N} \times \mathbb{N}$. Since $k \le k$ and $\ell \le \ell$, it is immediate from the definition of R, that $(k, \ell) \ R \ (k, \ell)$.

ii. To prove that R is transitive, let $(k,\ell),(m,n),(o,p) \in \mathbb{N}\times\mathbb{N}$ and suppose that $(k,\ell)\ R\ (m,n)$ and $(m,n)\ R\ (o,p)$. Then, according to the definition of R, $k \leq m$ and $m \leq o$ so $k \leq o$. Moreover, if $k = o$, then from $k \leq m$ and $m \leq o$ it follows that $k = m$ and $m = o$, so, again according to the definition of R, $\ell \leq n$ and $n \leq p$, and hence $\ell \leq p$. We have established that $k \leq o$ and $k = o$ implies $\ell \leq p$, so by the definition of R it follows that $(k,\ell)\ R\ (o,p)$.

iii. To prove that R is anti-symmetric, let $(k,\ell),(m,n) \in \mathbb{N}\times\mathbb{N}$ and suppose that $(k,\ell)\ R\ (m,n)$ and $(m,n)\ R\ (k,\ell)$. Then, according to the definition of R, $k \leq m$ and $m \leq k$, and hence $k = m$. Moreover, from $k = m$ it follows from the definition of R that $\ell \leq n$, and, since $k = m$ implies $m = n$, also $n \leq \ell$, and hence $\ell = n$. Thus, we have now established that $(k,\ell) = (m,n)$.

(b) The Hasse diagram of $\langle \{0,1,2\}\times\{0,1,2\}, R\rangle$:

$$\begin{tikzcd}
(2,2) \arrow[d, dash] \\
(2,1) \arrow[d, dash] \\
(2,0) \arrow[d, dash] \\
(1,2) \arrow[d, dash] \\
(1,1) \arrow[d, dash] \\
(1,0) \arrow[d, dash] \\
(0,2) \arrow[d, dash] \\
(0,1) \arrow[d, dash] \\
(0,0)
\end{tikzcd}$$

The number between parentheses in front of a problem indicates how many points you score with a correct answer to it. A partially correct answer is sometimes awarded with a fraction of those points. The grade for this examination will be determined by dividing the total number of scored points by 2.