TECHNISCHE UNIVERSITEIT EINDHOVEN
Faculteit Wiskunde en Informatica

Solutions to the Final examination Logic & Set Theory (2IT61)

Monday November 5, 2012, 9:00–12:00 hrs.

You are **not** allowed to use any books, notes, or other course material. Your solutions to the problems have to be formulated and written down in a clear and precise manner.

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(2) 1. Show that the following abstract proposition is a contingency (i.e., not a tautology and not a contradiction):

$$(a \Rightarrow (b \wedge c)) \wedge (\neg d \Leftrightarrow (b \vee c)) \wedge (a \wedge \neg e) \ .$$

<u>*Solution:*</u> Let us denote the formula of the exercise by φ.

If $a = 0$, then $(a \wedge \neg e) = 0$, so $\varphi = 0$, so φ is not a tautology.

If $a = 1$, $b = 1$, $c = 1$, $d = 0$ and $e = 0$, then $(b \wedge c) = 1$ so $a \Rightarrow (b \wedge c) = 1$, both $\neg d$ and $(b \vee c) = 1$ so $(\neg d \Leftrightarrow (b \vee c)) = 1$, and $(a \wedge \neg e) = 1$. It follows that $\varphi = 1$, and hence φ is not a contradiction.

2. Determine for each of the following propositions whether it is true or false. Give arguments for your answers.

(1) (a) $\forall_x[x \in \mathbb{Z} : \exists_y[y \in \mathbb{Z} : 2x - y = 3]]$;

(1) (b) $\exists_y[y \in \mathbb{Z} : \forall_x[x \in \mathbb{Z} : 2x - y = 3]]$.

<u>*Solution:*</u>

(a) The formula is true. To see this, let $x \in \mathbb{Z}$; then also $2x - 3 \in \mathbb{Z}$ and $2x - (2x - 3) = 3$, so there exists $y \in \mathbb{Z}$ such that $2x - y = 3$.

(b) The formula is false. To see this, note that, for every $y \in \mathbb{Z}$, it holds, e.g., for $x = \lceil (y + 3)/2 \rceil + 1$ that $2x - y > 3$.

(3) 3. Prove that the abstract propositions

$$(P \Leftrightarrow \neg Q) \quad \text{and} \quad (P \vee \neg(Q \Rightarrow P))$$

are *comparable* (i.e., the left-hand side formula is stronger than the right-hand side formula, or the right-hand side formula is stronger than the left-hand side formula).

Solution: On the one hand, the calculation

$$
\begin{aligned}
& P \Leftrightarrow \neg Q \\
\stackrel{val}{==} \; & \{ \text{ Bi-implication } \} \\
& (P \Rightarrow \neg Q) \wedge (\neg Q \Rightarrow P) \\
\stackrel{val}{==} \; & \{ \text{ Implication } (2\times) \} \\
& (\neg P \vee \neg Q) \wedge (\neg\neg Q \vee P) \\
\stackrel{val}{==} \; & \{ \text{ Implication } (2\times) \} \\
& (\neg P \vee \neg Q) \wedge (\neg\neg Q \vee P) \ ,
\end{aligned}
$$

proves that $P \Leftrightarrow \neg Q \stackrel{val}{==} (\neg P \vee \neg Q) \wedge (P \vee \neg\neg Q)$, and on the other hand, the calculation

$$
\begin{aligned}
& P \vee \neg(Q \Rightarrow P) \\
\stackrel{val}{==} \; & \{ \text{ Implication } \} \\
& P \vee \neg(\neg Q \vee P) \\
\stackrel{val}{==} \; & \{ \text{ De Morgan } \} \\
& P \vee (\neg\neg Q \wedge \neg P) \\
\stackrel{val}{==} \; & \{ \text{ Distributivity } \} \\
& (P \vee \neg\neg Q) \wedge (P \vee \neg P) \\
\stackrel{val}{==} \; & \{ \text{ Excluded Middle } \} \\
& (P \vee \neg\neg Q) \wedge \texttt{True} \\
\stackrel{val}{==} \; & \{ \texttt{True}/\texttt{False}\text{-elimination } \} \\
& P \vee \neg\neg Q
\end{aligned}
$$

proves that $P \vee \neg(Q \Rightarrow P) \stackrel{val}{==} P \vee \neg\neg Q$. Since

$$
\begin{aligned}
& (P \vee \neg\neg Q) \wedge (\neg P \vee \neg Q) \\
\stackrel{val}{\models} \; & \{ \wedge\text{-}\vee\text{-weakening } \} \\
& P \vee \neg\neg Q \ ,
\end{aligned}
$$

It follows that

$$P \Leftrightarrow \neg Q \stackrel{val}{\models} P \vee \neg(Q \Rightarrow P) \ ,$$

so $P \Leftrightarrow \neg Q$ and $P \vee \neg(Q \Rightarrow P)$ are comparable.

(2) 4. Prove with a *derivation* (i.e., using the methods described in *Part II* of the book) that the formula

$$(\forall_x[P(x)] \wedge \neg\exists_y[P(y) \wedge Q(y)]) \Rightarrow \neg\exists_x[Q(x)]$$

is a tautology.

Solution:

{ Assume: }

(1) $\boxed{\forall_x[P(x)] \wedge \neg\exists_y[P(y) \wedge Q(y)]}$

{ Assume: }

(2) $\boxed{\exists_z[Q(z)]}$

{ $\exists^*$-elim on (2) }

(3) Pick a z with (`True` and) $Q(z)$

{ $\wedge$-elim on (1) }

(4) $\forall_x[P(x)]$

{ $\forall$-elim on (3) and (4): }

(5) $P(z)$

{ $\wedge$-intro on (3) and (5): }

(6) $P(z) \wedge Q(z)$

{ $\exists^*$-intro on (6): }

(7) $\exists_y[P(y) \wedge Q(y)]$

{ $\wedge$-elim on (1): }

(8) $\neg\exists_y[P(y) \wedge Q(y)]$

{ $\neg$-elim on (7) and (8): }

(9) `False`

{ $\neg$-intro on (2) and (9): }

(10) $\neg\exists_x[Q(z)]$

{ $\Rightarrow$-intro on (1) and (10): }

(11) $(\forall_x[P(x)] \wedge \neg\exists_y[P(y) \wedge Q(y)]) \Rightarrow \neg\exists_z[Q(z)]$

5. Check for each of the following formulas whether it holds for all sets A, B and C. If so, then give a proof; if not, then give a counterexample.

(1) (a) $((A \cap (B \backslash C)) = \emptyset) \;\Rightarrow\; ((A \cap B) = (A \cap C))$;

(1) (b) $((A \cap B) = (A \cap C)) \;\Rightarrow\; ((A \cap (B \backslash C)) = \emptyset)$.

<u>*Solution:*</u>

(a) The formula is not true for all sets A, B and C, for if $A = C = \{0\}$ and $B = \emptyset$, then $A \cap (B \backslash C) = \{0\} \cap (\emptyset \backslash \{0\}) = \{0\} \cap \emptyset = \emptyset$, while $(A \cap B) = \{0\} \cap \emptyset = \emptyset \neq \{0\} = \{0\} \cap \{0\}$.

(b) The formula is true for all sets A, B and C. For suppose that $(A \cap B) = (A \cap C)$; we need to establish that $(A \cap (B \backslash C)) = \emptyset$. To this end, by the Property of $\emptyset$ it suffices to derive a contradiction from the assumption that $x \in A \cap (B \backslash C)$ for some $x \in \mathcal{U}$. Then, by the Property of $\cap$, $x \in A$ and $x \in B \backslash C$, and hence, by the Property of $\backslash$, $x \in B$ and $\neg(x \in C)$. From $x \in A$ and $x \in B$, it follows, by the Property of $\cap$, that $x \in A \cap B$, and hence, since $(A \cap B) = (A \cap C)$, it follows that $x \in A \cap C$. We then get by the Property of $\cap$ that $x \in A$ and $x \in C$, and the latter contradicts $\neg(x \in C)$.

6. Consider the mapping $F : \mathbb{Z} \to \mathbb{Z}$ defined by

$$F(x) = 2x^2 + 3 \ .$$

(1) (a) Determine $F(\{-1, 0, 1\})$.

(1) (b) Is F a bijection? (Motivate your answer!)

<u>*Solution:*</u>

(a) $F(\{-1, 0, 1\}) = \{3, 5\}$.

(b) No, F is not a bijection, for $F(-1) = 5 = F(1)$, but $-1 \neq 1$.

(3) 7. Let R be an equivalence relation on V, and let $a, b, c \in V$. Prove that

$$(c \in K(a) \wedge \neg(c \in K(b))) \Rightarrow \neg(b \in K(a)) \ .$$

(As usual, $K(x)$ denotes the equivalence class of x.)

<u>*Solution:*</u> Suppose that $c \in K(a)$ and $\neg(c \in K(b))$; to prove that $\neg(b \in K(a))$ we assume that $b \in K(a)$ and derive a contradiction. From the definitions of $K(a)$ and by the Property of $\in$, it follows that $a \ R \ c$ and $a \ R \ b$. Hence, since R is symmetric, $b \ R \ a$, and, since R is transitive, $b \ R \ c$. It follows that $c \in K(b)$ which is in contradiction with $\neg(c \in K(b))$.

(4) 8. Prove that $4n^3 - 4n$ is divisible by 3 for every natural number n.

<u>*Solution:*</u> We prove that $4n^3 - 4n$ is divisible by 3 by induction on n.

If $n = 0$, then $4n^3 - 4n = 0$, and 0 is clearly divisible by 3.

Let $n \geq 0$, and suppose that $4n^3 - 4n$ is divisible by 3 (induction hypothesis). Then

$$\begin{aligned} 4(n+1)^3 - 4(n+1) &= 4(n^2 + 2n + 1)(n+1) - 4(n+1) \\ &= 4(n^3 + 2n^2 + n + n^2 + 2n + 1) - 4(n+1) \\ &= 4(n^3 + 3n^2 + 3n + 1) - 4(n+1) \\ &= (4n^3 - 4n) + 12n^2 + 12n \ . \end{aligned}$$

Since $4n^3 - 4n$ is divisible by 3 by the induction hypothesis, and $12n^2 + 12n = 3(4n^2 + 4n)$ is clearly also divisible by 3, it follows that $(4n^3 - 4n) + 12n^2 + 12n$ is divisible by 4, and hence $4(n+1)^3 - 4(n+1)$ is divisible by 3.