<u>TECHNISCHE UNIVERSITEIT EINDHOVEN</u>
Faculteit Wiskunde en Informatica

Final examination Logic & Set Theory (2IT61/2IT07)
(correction model)

Thursday October 31, 2013, 9:00–12:00 hrs.

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(1) 1. Determine $\mathcal{P}(\{0,1\}) \times \mathcal{P}(\emptyset)$.

Solution:

$$\begin{aligned}\mathcal{P}(\{0,1\}) \times \mathcal{P}(\emptyset) &= \{\emptyset, \{0\}, \{1\}, \{0,1\}\} \times \{\emptyset\} \\ &= \{(\emptyset,\emptyset), (\{0\},\emptyset), (\{1\},\emptyset), (\{0,1\},\emptyset)\}\ .\end{aligned}$$

Correction suggestions: A correct answer (1 point) specifies the set of pairs of sets as above. I propose to also allow $\{\}$ as notation for the empty set.

An answer that demonstrates that the candidate knows how to compute either power sets or Cartesian products, but not both, can be awarded with 0.5 point.

(2) 2. Prove that the formulas

$$P \Rightarrow ((Q \Rightarrow R) \wedge (Q \vee R)) \quad \text{and} \quad (\neg P \Rightarrow Q) \Rightarrow R$$

are comparable (i.e., the left-hand side formula is stronger than the right-hand side formula, or vice versa).

Solution: We have the following calculation, proving that

$$(\neg P \Rightarrow Q) \Rightarrow R \ \stackrel{val}{\models\!\!=}\ P \Rightarrow ((Q \Rightarrow R) \wedge (Q \vee R))\ :$$

$(\neg P \Rightarrow Q) \Rightarrow$

$\stackrel{val}{=\!=}$ { Implication (2×) }

$\neg(\neg\neg P \vee Q) \vee R$

$\stackrel{val}{=\!=}$ { Double Negation }

$\neg(P \vee Q) \vee R$

$\stackrel{val}{=\!=}$ { De Morgan }

$(\neg P \wedge \neg Q) \vee R$

$\stackrel{val}{\models}$ { $\wedge$-$\vee$-weakening + Monotonicity }

$\neg P \vee R$

$\stackrel{val}{=\!=}$ { `True`/`False`-elimination }

$\neg P \vee (\texttt{False} \vee R)$

$\stackrel{val}{=\!=}$ { Contradiction }

$\neg P \vee ((\neg Q \wedge Q) \vee R)$

$\stackrel{val}{=\!=}$ { Distributivity }

$\neg P \vee ((\neg Q \vee R) \wedge (Q \vee R))$

$\stackrel{val}{=\!=}$ { Implication (2×) }

$P \Rightarrow ((Q \Rightarrow R) \wedge (Q \vee R))$

Hence, the two formulas are comparable.

Correction suggestions: A correct answer (2 points) consists of a proof showing that the right-hand side formula is stronger than the left-hand side formula. The proof need not be a calculation; it may also be an argument using truth tables.

If a candidate provides a calculation, I propose not to be too strict on the use of hints, but, clearly, the argument should not contain logically incorrect steps. Do subtract 0.5 point if the $\stackrel{val}{=\!=}$ -signs are missing.

If the candidate mentions that the right-hand side formula is stronger than the left-hand side formula, but the proof is flawed or absent, then award 0.5 point. If there is an argument that contains some reasonable steps, but is incomplete or incorrect, you may award 1 or 1.5 points, depending on the seriousness of the mistakes.

3. Let P be the set of all people, let Anna denote a particular person in P, and let B be the set of all books. Furthermore, let O be a predicate on $P \times B$, and let L be a predicate on $P \times B \times P$, with the following interpretations:

   $O(p, b)$ means 'p owns b', and

   $L(p, b, q)$ means 'p has lent out b to q'.

Give formulas of predicate logic that express the following statements:

(1) (a) Everybody owns a book.

(1) (b) Anna has only lent out books she owns.

<u>*Solution:*</u>

(a) $\forall_p[p \in P : \exists_b[b \in B : O(p, b)]]$;

(b) $\forall_b[b \in B : \exists_p[p \in P : L(\text{Anna}, b, p)] \Rightarrow O(\text{Anna}, b)]$.

<u>*Correction suggestions:*</u>

(a) A correct answer (1 point) consists of a correct formula. It is hard to predict what kind of variations you will encounter; award 0.5 for an 'almost' correct formula.

(b) Idem.

(2) 4. Prove with a *derivation* (i.e., using the methods described in *Part II* of the book) that the formula

$$\exists_x \forall_y[\neg P(y) : Q(y, x)] \Rightarrow \forall_z[P(z) \vee \exists_u[Q(z, u)]]$$

is a tautology.

<u>*Solution:*</u>

{ Assume: }

(1) $\boxed{\exists_x \forall_y[\neg P(y) : Q(y, x)]}$

{ Assume: }

(2) $\boxed{\textbf{var } z;\ \texttt{True}}$

{ Assume: }

(3) $\boxed{\neg P(z)}$

{ $\exists^*$-elim on (1): }

(4) Pick an x with $\forall_y[\neg P(y) : Q(y, x)]$

{ $\forall$-elim on (4): }

(5) $Q(z, x)$

{ $\exists^*$-intro on (5): }

(6) $\exists_u[Q(z, u)]$

{ $\vee$-intro on (3) and (6): }

(7) $P(z) \vee \exists_u[Q(z, u)]$

$$\begin{array}{ll|l} & & \{ \forall\text{-intro on (2) and (7): } \} \\ (8) & & \forall_z[P(z) \vee \exists_u[Q(z,u)]] \end{array}$$

$\{ \Rightarrow\text{-intro on (1) and (8): } \}$

(9) $\exists_x \forall_y[\neg P(y) : Q(y,x)] \Rightarrow \forall_z[P(z) \vee \exists_u[Q(z,u)]]$

Correction suggestions: A correct answer (2 points) gives a derivation showing that the formula is a tautology. It is especially important in this exercise that free variables are not renamed, and that the $\forall$-elim is applied correctly (e.g., don't conclude from $\forall_y[\neg P(y) : Q(y,x)]$ that $Q(y,x)$); derivations including such mistakes should not be awarded with more than 1 point.

Tentative suggestion for awarding a fraction of the points for partially correct derivations:

Two outermosts flag ($\Rightarrow$-intro and $\forall$-intro): 0.5 point.
Also correct $\vee$-intro: 1 point.
Also correct $\exists^*$-elim and $\forall$-elim: $1\frac{1}{2}$ point.

A deriviation with logically incorrect steps should not be awarded with more than 1 point.

I propose not to be too strict on omission of hints. If all hints are missing, you could subtract (at most) 0.5 points.

A complete and correct proof that is *not* presented in the form of a derivation (e.g., a proof presented as a calculation, or a proof that uses truth tables), you may award with (at most) 1 point.

(3) 5. Prove that the following formula holds for all sets A, B and C:

$$B \subseteq A \ \Rightarrow \ B \subseteq (C \backslash A)^{\mathrm{C}} \ .$$

Solution: Suppose that $B \subseteq A$. To prove that $B \subseteq (C\backslash A)^{\mathrm{C}}$ it suffices, by the *property of* $\subseteq$, to establish that for all $x \in B$ we have $x \in (C\backslash A)^{\mathrm{C}}$. So let $x \in B$; to prove that $x \in (C\backslash A)^{\mathrm{C}}$, by the property of $\_^{\mathrm{C}}$ it is enough to derive a contradiction from the assumption that $x \in C\backslash A$. Note that, on the one hand, from $x \in C\backslash A$, it follows by the property of $\backslash$ that $x \in C$ and $x \notin A$. On the other hand, from $x \in B$ and $B \subseteq A$ it follows that $x \in A$ by the property of $\subseteq$. Thus, we have now established $x \in A$ and $x \notin A$; a contradiction.

Correction suggestions: A correct answer (3 points) proves the implication.

The proof should mention (or at least clearly apply) the four set-theoretic properties ($\subseteq$, $\_^{\mathrm{C}}$, $\backslash$, and again $\subseteq$); I propose to award 0.5 point per correctly applied property.

The logical part of the argument is then worth 1 point. It should be clear from the argument that the candidate understands how implications, universal quantifications and negations are proved, and how a conjunction can be used in a proof.

6. Consider the mapping $f : \mathbb{R} \to \mathbb{R}$ that is defined, for all $x \in \mathbb{R}$, by

$$f(x) = x^2 - 4x + 4 \ .$$

(2) (a) Determine $f^{\leftarrow}(\{1\})$.

(1) (b) Give the formula that expresses 'f is an injection' and show with a counterexample that f is not an injection.

<u>*Solution:*</u>

(a) $f^{\leftarrow}(\{1\}) = \{1, 3\}$.

(b) The formula that expresses 'f is an injection' is $\forall_{x_1,x_2}[x_1, x_2 \in \mathbb{R} : f(x_1) = f(x_2) \Rightarrow x_1 = x_2]$. To see that f is *not* an injection, note that $f(1) = 1 = f(3)$, but $1 \neq 3$.

<u>*Correction suggestions:*</u>

(a) A correct answer (2 points) mentions that $f^{\leftarrow}(\{1\})$ is the set $\{1, 3\}$.

If the candidate shows some understanding of the concept 'source of a mapping', but does not correctly specify or describe the set $\{1, 3\}$ (e.g., writes $f^{\leftarrow}(\{1\}) = 1, 3$), then I think he/she deserves 1 point.

For the answers $f^{\leftarrow}(\{1\}) = \{1\}$ and $f^{\leftarrow}(\{1\}) = \{3\}$, also award 1 point.

(b) A correct answer (1 point) gives the formula and a counterexample. If only the formula is there, but the counterexample is lacking, or vice versa, award 0.5 point.

(3) 7. Prove by induction that $n^3 - n$ is divisible by 3 for all $n \in \mathbb{N}$.

<u>*Solution:*</u> We prove by induction on n that $n^3 - n$ is divisible by 3 for all $n \in \mathbb{N}$.

If $n = 0$, then $n^3 - n = 0 = 3 \cdot 0$, so $n^3 - n$ is divisible by 3.

Let $n \in \mathbb{N}$, and suppose that $n^3 - n$ is divisible by 3 (IH). Then

$$(n+1)^3 - (n+1) = n^3 + 3n^2 + 2n = (n^3 - n) + 3n^2 + 3n$$

and since $n^3 - n$ is divisible by 3 by the induction hypothesis, and $3n^2 + 3n = 3(n^2 + n)$ is also divisible by 3, it follows that $(n+1)^3 - (n+1)$ is divisible by 3.

<u>*Correction suggestions:*</u> A correct answer (3 points) proves with induction that $n^3 - n$ is divisible by 3 for all $n \in \mathbb{N}$. The proof should then clearly

distinguish a basis case ($n = 0$) and a step case. It should be clear from the step case that the candidate understands that he should prove a universally quantified implication; that is, there should be a declaration of $n \in \mathbb{N}$ and an explicit assumption of the induction hypothesis. The step case should then proceed to establish the property for $n + 1$.

Although the question explicitly asks for a proof by induction, I propose to still award 3 points for a complete and correct proof that does not use induction (or does not use the induction hypothesis). Do pay attention to the logical details, then (declaration of $n \in \mathbb{N}$, appropriate case distinction, etc.). In hindsight, I now feel that I should've listened better to Jaap's warnings; this is not a good exam question.

For incorrect, or incomplete, proofs, award points as follows:

- correct basis case: 0.5 point;
- declaration of $n \in \mathbb{N}$ in step case: 0.5 point;
- clear statement of the induction hypothesis: 1 point;
- argument that $(n+1)^3 - (n+1)$ is divisible by 3: 1 point.

If the proof mentions that $n^3 - n = 3 \cdot k$ without a declaration of k, then subtract 0.5 point.

8. Define the relation R on $\mathbb{N} \times \mathbb{N}$ by

$$(x_1, y_1)\ R\ (x_2, y_2) \text{ if, and only if, } x_2 - x_1 < y_2 - y_1 \ .$$

(2) (a) Prove that $\langle \mathbb{N} \times \mathbb{N}, R\rangle$ is an *irreflexive ordering.*

(1) (b) Draw a Hasse diagram of $\langle \{0,1\} \times \{0,1\}, R\rangle$.

(1) (c) Give the maximal and minimal elements of the set $\{0,1\} \times \{0,1\}$ in the irreflexive ordering $\langle \mathbb{N} \times \mathbb{N}, R\rangle$.

Solution:

(a) It suffices to establish that R is irreflexive and transitive.

To see that R is irreflexive, let $(x, y) \in \mathbb{N} \times \mathbb{N}$ and suppose that $(x, y)\ R\ (x, y)$; we derive a contradiction. Note that, from $(x, y)\ R\ (x, y)$, on the one hand, it follows from the definition of R that $x - x < y - y$, while, on the other hand, $x - x = 0 = y - y$; a contradiction.

To see that R is transitive, let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{N} \times \mathbb{N}$, and suppose that $(x_1, y_1)\ R\ (x_2, y_2)$ and $(x_2, y_2)\ R\ (x_3, y_3)$; we establish that $(x_1, y_1)\ R\ (x_3, y_3)$.
Note that from $(x_1, y_1)\ R\ (x_2, y_2)$ it follows according to the definition of R that $x_2 - x_1 < y_2 - y_1$, and from $(x_2, y_2)\ R\ (x_3, y_3)$ it follows according to the definition of R that $x_3 - x_2 < y_3 - y_2$. Then

$$x_3 - x_1 = (x_3 - x_2) + (x_2 - x_1) < (y_3 - y_2) + (y_2 - y_1) = y_3 - y_1 \ ,$$

and hence, according to the definition of R, we have $(x_1, y_1)\ R\ (x_3, y_3)$.

(b) A Hasse diagram of $\langle \{0,1\} \times \{0,1\}, R\rangle$:

$$
\begin{tikzcd}
 & (0,1) \arrow[dl, dash] \arrow[dr, dash] & \\
(0,0) \arrow[dr, dash] & & (1,1) \arrow[dl, dash] \\
 & (1,0) &
\end{tikzcd}
$$

(c) The maximal element of $\{0,1\} \times \{0,1\}$ in $\langle \mathbb{N} \times \mathbb{N}, R\rangle$ is $(0,1)$; the minimal element of $\{0,1\} \times \{0,1\}$ in $\langle \mathbb{N} \times \mathbb{N}, R\rangle$ is $(1,0)$.

Correction suggestions:

(a) A correct answer (2 points) shows that R is irreflexive and transitive.

Award 1 point for a complete and correct proof of each property. Award 0.5 if the candidate shows that he/she knows the defining formula for the property, and award 0.5 for the actual proof.

(b) A correct answer (1 point) gives a Hasse diagram. If there is only one superfluous or one missing connection, then you can still award 0.5 point.

(c) A correct answer (1 point) mentions the maximal and minimal elements. Award 0.5 point for the correct maximal element, and 0.5 point for the correct minimal element.