Section 6-4 : Solving Logarithm Equations

In this section we will now take a look at solving logarithmic equations, or equations with logarithms in them. We will be looking at two specific types of equations here. In particular we will look at equations in which every term is a logarithm and we also look at equations in which all but one term in the equation is a logarithm and the term without the logarithm will be a constant. Also, we will be assuming that the logarithms in each equation will have the same base. If there is more than one base in the logarithms in the equation the solution process becomes much more difficult.

Before we get into the solution process we will need to remember that we can only plug positive numbers into a logarithm. This will be important down the road and so we can't forget that.

Now, let's start off by looking at equations in which each term is a logarithm and all the bases on the logarithms are the same. In this case we will use the fact that,

$$\text{If } \log_b x = \log_b y \text{ then } x = y$$

In other words, if we've got two logs in the problem, one on either side of an equal sign and both with a coefficient of one, then we can just drop the logarithms.

Let's take a look at a couple of examples.

Example 1 Solve each of the following equations.

(a) $2\log_9\left(\sqrt{x}\right) - \log_9\left(6x-1\right) = 0$

(b) $\log x + \log\left(x-1\right) = \log\left(3x+12\right)$

(c) $\ln 10 - \ln\left(7-x\right) = \ln x$

Solution

(a) $2\log_9\left(\sqrt{x}\right) - \log_9\left(6x-1\right) = 0$

With this equation there are only two logarithms in the equation so it's easy to get on one either side of the equal sign. We will also need to deal with the coefficient in front of the first term.

$$\log_9\left(\sqrt{x}\right)^2 = \log_9\left(6x-1\right)$$
$$\log_9 x = \log_9\left(6x-1\right)$$

Now that we've got two logarithms with the same base and coefficients of 1 on either side of the equal sign we can drop the logs and solve.

$$x = 6x - 1$$
$$1 = 5x \qquad\qquad \Rightarrow \qquad\qquad x = \frac{1}{5}$$

Now, we do need to worry if this solution will produce any negative numbers or zeroes in the logarithms so the next step is to plug this into the **original** equation and see if it does.

$$2\log_9\left(\sqrt{\frac{1}{5}}\right)-\log_9\left(6\left(\frac{1}{5}\right)-1\right)=2\log_9\left(\sqrt{\frac{1}{5}}\right)-\log_9\left(\frac{1}{5}\right)=0$$

Note that we don't need to go all the way out with the check here. We just need to make sure that once we plug in the *x* we don't have any negative numbers or zeroes in the logarithms. Since we don't in this case we have the solution, it is $x=\dfrac{1}{5}$.

(b) $\log x+\log\left(x-1\right)=\log\left(3x+12\right)$

Okay, in this equation we've got three logarithms and we can only have two. So, we saw how to do this kind of work in a set of examples in the previous so we just need to do the same thing here. It doesn't really matter how we do this, but since one side already has one logarithm on it we might as well combine the logs on the other side.

$$\log\left(x\left(x-1\right)\right)=\log\left(3x+12\right)$$

Now we've got one logarithm on either side of the equal sign, they are the same base and have coefficients of one so we can drop the logarithms and solve.

$$x\left(x-1\right)=3x+12$$
$$x^2-x-3x-12=0$$
$$x^2-4x-12=0$$
$$\left(x-6\right)\left(x+2\right)=0 \qquad\Rightarrow\qquad x=-2,\ x=6$$

Now, before we declare these to be solutions we MUST check them in the original equation.

$x=6$:

$$\log 6+\log\left(6-1\right)=\log\left(3\left(6\right)+12\right)$$
$$\log 6+\log 5=\log 30$$

No logarithms of negative numbers and no logarithms of zero so this is a solution.

$x=-2$:

$$\log\left(-2\right)+\log\left(-2-1\right)=\log\left(3\left(-2\right)+12\right)$$

We don't need to go any farther, there is a logarithm of a negative number in the first term (the others are also negative) and that's all we need in order to exclude this as a solution.

Be careful here. We are not excluding $x = -2$ because it is negative, that's not the problem. We are excluding it because once we plug it into the original equation we end up with logarithms of negative numbers. It is possible to have negative values of x be solutions to these problems, so don't mistake the reason for excluding this value.

Also, along those lines we didn't take $x = 6$ as a solution because it was positive, but because it didn't produce any negative numbers or zero in the logarithms upon substitution. It is possible for positive numbers to not be solutions.

So, with all that out of the way, we've got a single solution to this equation, $x = 6$.

(c) $\ln 10 - \ln(7 - x) = \ln x$

We will work this equation in the same manner that we worked the previous one. We've got two logarithms on one side so we'll combine those, drop the logarithms and then solve.

$$\ln\left(\frac{10}{7-x}\right) = \ln x$$

$$\frac{10}{7-x} = x$$

$$10 = x(7-x)$$

$$10 = 7x - x^2$$

$$x^2 - 7x + 10 = 0$$

$$(x-5)(x-2) = 0 \qquad \Rightarrow \qquad x = 2, \; x = 5$$

We've got two possible solutions to check here.

$x = 2$:

$$\ln 10 - \ln(7 - 2) = \ln 2$$

$$\ln 10 - \ln 5 = \ln 2$$

This one is okay.

$x = 5$:

$$\ln 10 - \ln(7 - 5) = \ln 5$$

$$\ln 10 - \ln 2 = \ln 5$$

This one is also okay.

In this case both possible solutions, $x = 2$ and $x = 5$, end up actually being solutions. There is no reason to expect to always have to throw one of the two out as a solution.

Now we need to take a look at the second kind of logarithmic equation that we'll be solving here. This equation will have all the terms but one be a logarithm and the one term that doesn't have a logarithm will be a constant.

In order to solve these kinds of equations we will need to remember the exponential form of the logarithm. Here it is if you don't remember.

$$y = \log_b x \qquad\qquad \Rightarrow \qquad\qquad b^y = x$$

We will be using this conversion to exponential form in all of these equations so it's important that you can do it. Let's work some examples so we can see how these kinds of equations can be solved.

Example 2 Solve each of the following equations.

      (a) $\log_5(2x+4) = 2$

      (b) $\log x = 1 - \log(x-3)$

      (c) $\log_2(x^2 - 6x) = 3 + \log_2(1-x)$

Solution

(a) $\log_5(2x+4) = 2$

To solve these we need to get the equation into exactly the form that this one is in. We need a single log in the equation with a coefficient of one and a constant on the other side of the equal sign. Once we have the equation in this form we simply convert to exponential form.

So, let's do that with this equation. The exponential form of this equation is,
$$2x+4 = 5^2 = 25$$

Notice that this is an equation that we can easily solve.
$$2x = 21 \qquad \Rightarrow \qquad x = \frac{21}{2}$$

Now, just as with the first set of examples we need to plug this back into the **original** equation and see if it will produce negative numbers or zeroes in the logarithms. If it does it can't be a solution and if it doesn't then it is a solution.

$$\log_5\left(2\left(\frac{21}{2}\right)+4\right) = 2$$
$$\log_5(25) = 2$$

Only positive numbers in the logarithm and so $x = \dfrac{21}{2}$ is in fact a solution.

(b) $\log x = 1 - \log(x-3)$

In this case we've got two logarithms in the problem so we are going to have to combine them into a single logarithm as we did in the first set of examples. Doing this for this equation gives,

$$\log x + \log(x-3) = 1$$
$$\log(x(x-3)) = 1$$

Now, that we've got the equation into the proper form we convert to exponential form. Recall as well that we're dealing with the common logarithm here and so the base is 10.

Here is the exponential form of this equation.

$$x(x-3) = 10^1$$
$$x^2 - 3x - 10 = 0$$
$$(x-5)(x+2) = 0 \qquad \Rightarrow \qquad x = -2, \ x = 5$$

So, we've got two potential solutions. Let's check them both.

$x = -2$:
$$\log(-2) = 1 - \log(-2-3)$$
We've got negative numbers in the logarithms and so this can't be a solution.

$x = 5$:
$$\log 5 = 1 - \log(5-3)$$
$$\log 5 = 1 - \log 2$$
No negative numbers or zeroes in the logarithms and so this is a solution.

Therefore, we have a single solution to this equation, $x = 5$.

Again, remember that we don't exclude a potential solution because it's negative or include a potential solution because it's positive. We exclude a potential solution if it produces negative numbers or zeroes in the logarithms upon substituting it into the equation and we include a potential solution if it doesn't.

(c) $\log_2(x^2 - 6x) = 3 + \log_2(1-x)$

Again, let's get the logarithms onto one side and combined into a single logarithm.

$$\log_2(x^2 - 6x) - \log_2(1-x) = 3$$
$$\log_2\left(\frac{x^2 - 6x}{1-x}\right) = 3$$

Now, convert it to exponential form.

$$\frac{x^2 - 6x}{1 - x} = 2^3 = 8$$

Now, let's solve this equation.

$$x^2 - 6x = 8(1 - x)$$
$$x^2 - 6x = 8 - 8x$$
$$x^2 + 2x - 8 = 0$$
$$(x + 4)(x - 2) = 0 \qquad \Rightarrow \qquad x = -4, \; x = 2$$

Now, let's check both of these solutions in the original equation.

$x = -4$:

$$\log_2\left((-4)^2 - 6(-4)\right) = 3 + \log_2\left(1 - (-4)\right)$$
$$\log_2\left(16 + 24\right) = 3 + \log_2(5)$$

So, upon substituting this solution in we see that all the numbers in the logarithms are positive and so this IS a solution. Note again that it doesn't matter that the solution is negative, it just can't produce negative numbers or zeroes in the logarithms.

$x = 2$:

$$\log_2\left(2^2 - 6(2)\right) = 3 + \log_2(1 - 2)$$
$$\log_2(4 - 12) = 3 + \log_2(-1)$$

In this case, despite the fact that the potential solution is positive we get negative numbers in the logarithms and so it can't possibly be a solution.

Therefore, we get a single solution for this equation, $x = -4$.