Section 6-5 : Applications

In this final section of this chapter we need to look at some applications of exponential and logarithm functions.

Compound Interest

This first application is compounding interest and there are actually two separate formulas that we'll be looking at here. Let's first get those out of the way.

If we were to put *P* dollars into an account that earns interest at a rate of *r* (written as a decimal) for *t* years (yes, it must be years) then,

1. if interest is compounded *m* times per year we will have
$$A = P\left(1 + \frac{r}{m}\right)^{tm}$$
dollars after *t* years.

2. if interest is compounded continuously then we will have
$$A = P\mathbf{e}^{rt}$$
dollars after *t* years.

Let's take a look at a couple of examples.

Example 1 We are going to invest \$100,000 in an account that earns interest at a rate of 7.5% for 54 months. Determine how much money will be in the account if,

(a) interest is compounded quarterly.
(b) interest is compounded monthly.
(c) interest is compounded continuously.

Solution

Before getting into each part let's identify the quantities that we will need in all the parts and won't change.

$$P = 100,000 \qquad r = \frac{7.5}{100} = 0.075 \qquad t = \frac{54}{12} = 4.5$$

Remember that interest rates must be decimals for these computations and *t* must be in years! Now, let's work the problems.

(a) Interest is compounded quarterly.

In this part the interest is compounded quarterly and that means it is compounded 4 times a year. After 54 months we then have,

$$
\begin{aligned}
A &= 100000\left(1+\frac{0.075}{4}\right)^{(4)(4.5)} \\
&= 100000(1.01875)^{18} \\
&= 100000(1.39706686207) \\
&= 139706.686207 = \$139,706.69
\end{aligned}
$$

Notice the amount of decimal places used here. We didn't do any rounding until the very last step. It is important to not do too much rounding in intermediate steps with these problems.

(b) Interest is compounded monthly.
Here we are compounding monthly and so that means we are compounding 12 times a year. Here is how much we'll have after 54 months.

$$
\begin{aligned}
A &= 100000\left(1+\frac{0.075}{12}\right)^{(12)(4.5)} \\
&= 100000(1.00625)^{54} \\
&= 100000(1.39996843023) \\
&= 139996.843023 = \$139,996.84
\end{aligned}
$$

So, compounding more times per year will yield more money.

(c) Interest is compounded continuously.
Finally, if we compound continuously then after 54 months we will have,

$$
\begin{aligned}
A &= 100000\mathbf{e}^{(0.075)(4.5)} \\
&= 100000(1.40143960839) \\
&= 140143.960839 = \$140,143.96
\end{aligned}
$$

Now, as pointed out in the first part of this example it is important to not round too much before the final answer. Let's go back and work the first part again and this time let's round to three decimal places at each step.

$$
\begin{aligned}
A &= 100000\left(1+\frac{0.075}{4}\right)^{(4)(4.5)} \\
&= 100000(1.019)^{18} \\
&= 100000(1.403) \\
&= \$140,300.00
\end{aligned}
$$

This answer is off from the correct answer by \$593.31 and that's a fairly large difference. So, how many decimal places should we keep in these? Well, unfortunately the answer is that it depends. The larger the initial amount the more decimal places we will need to keep around. As a general rule of thumb, set your calculator to the maximum number of decimal places it can handle and take all of them until the final answer and then round at that point.

Let's now look at a different kind of example with compounding interest.

Example 2 We are going to put \$2500 into an account that earns interest at a rate of 12%. If we want to have \$4000 in the account when we close it how long should we keep the money in the account if,

(a) we compound interest continuously.

(b) we compound interest 6 times a year.

Solution

Again, let's identify the quantities that won't change with each part.

$$A = 4000 \qquad P = 2500 \qquad r = \frac{12}{100} = 0.12$$

Notice that this time we've been given *A* and are asking to find *t*. This means that we are going to have to solve an exponential equation to get at the answer.

(a) Compound interest continuously.

Let's first set up the equation that we'll need to solve.

$$4000 = 2500\mathbf{e}^{0.12t}$$

Now, we saw how to solve these kinds of equations a couple of sections ago. In that section we saw that we need to get the exponential on one side by itself with a coefficient of 1 and then take the natural logarithm of both sides. Let's do that.

$$\begin{aligned}\frac{4000}{2500} &= \mathbf{e}^{0.12t}\\ 1.6 &= \mathbf{e}^{0.12t}\\ \ln 1.6 &= \ln \mathbf{e}^{0.12t}\\ \ln 1.6 &= 0.12t \qquad \Rightarrow \qquad t = \frac{\ln 1.6}{0.12} = 3.917\end{aligned}$$

We need to keep the amount in the account for 3.917 years to get \$4000.

(b) Compound interest 6 times a year.

Again, let's first set up the equation that we need to solve.

$$4000 = 2500\left(1 + \frac{0.12}{6}\right)^{6t}$$

$$4000 = 2500\left(1.02\right)^{6t}$$

We will solve this the same way that we solved the previous part. The work will be a little messier, but for the most part it will be the same.

$$\begin{aligned}
\frac{4000}{2500} &= (1.02)^{6t} \\
1.6 &= (1.02)^{6t} \\
\ln 1.6 &= \ln (1.02)^{6t} \\
\ln 1.6 &= 6t \ln (1.02) \\
t = \frac{\ln 1.6}{6 \ln (1.02)} &= \frac{0.470003629246}{6(0.019802627296)} = 3.956
\end{aligned}$$

In this case we need to keep the amount slightly longer to reach \$4000.

Exponential Growth and Decay

There are many quantities out there in the world that are governed (at least for a short time period) by the equation,

$$Q = Q_0 \mathbf{e}^{kt}$$

where Q_0 is positive and is the amount initially present at $t = 0$ and *k* is a non-zero constant. If *k* is positive then the equation will grow without bound and is called the **exponential growth** equation. Likewise, if *k* is negative the equation will die down to zero and is called the **exponential decay** equation.

Short term population growth is often modeled by the exponential growth equation and the decay of a radioactive element is governed the exponential decay equation.

Example 3 The growth of a colony of bacteria is given by the equation,

$$Q = Q_0 \mathbf{e}^{0.195\,t}$$

If there are initially 500 bacteria present and *t* is given in hours determine each of the following.

(a) How many bacteria are there after a half of a day?

(b) How long will it take before there are 10000 bacteria in the colony?

Solution

Here is the equation for this starting amount of bacteria.

$$Q = 500\,\mathbf{e}^{0.195\,t}$$

(a) How many bacteria are there after a half of a day?

In this case if we want the number of bacteria after half of a day we will need to use $t = 12$ since *t* is in hours. So, to get the answer to this part we just need to plug *t* into the equation.

$$Q = 500\,\mathbf{e}^{0.195(12)} = 500(10.3812365627) = 5190.618$$

So, since a fractional population doesn't make much sense we'll say that after half of a day there are 5190 of the bacteria present.

(b) How long will it take before there are 10000 bacteria in the colony?

Do NOT make the mistake of assuming that it will be approximately 1 day for this answer based on the answer to the previous part. With exponential growth things just don't work that way as we'll see. In order to answer this part we will need to solve the following exponential equation.

$$10000 = 500\,\mathbf{e}^{0.195\,t}$$

Let's do that.

$$\begin{aligned} \frac{10000}{500} &= \mathbf{e}^{0.195\,t} \\ 20 &= \mathbf{e}^{0.195\,t} \\ \ln 20 &= \ln \mathbf{e}^{0.195\,t} \\ \ln 20 &= 0.195t \qquad \Rightarrow \qquad t = \frac{\ln 20}{0.195} = 15.3627 \end{aligned}$$

So, it only takes approximately 15.4 hours to reach 10000 bacteria and NOT 24 hours if we just double the time from the first part. In other words, be careful!

Example 4 Carbon 14 dating works by measuring the amount of Carbon 14 (a radioactive element) that is in a fossil. All living things have a constant level of Carbon 14 in them and once they die it starts to decay according to the formula,

$$Q = Q_0\mathbf{e}^{-0.000124t}$$

where *t* is in years and Q_0 is the amount of Carbon 14 present at death and for this example let's assume that there will be 100 milligrams present at death.

(a) How much Carbon 14 will there be after 1000 years?

(b) How long will it take for half of the Carbon 14 to decay?

Solution

(a) How much Carbon 14 will there be after 1000 years?

In this case all we need to do is plug in *t=1000* into the equation.

$$Q = 100\mathbf{e}^{-0.000124(1000)} = 100\left(0.883379840883\right) = 88.338 \text{ milligrams}$$

So, it looks like we will have around 88.338 milligrams left after 1000 years.

(b) How long will it take for half of the Carbon 14 to decay?

So, we want to know how long it will take until there is 50 milligrams of the Carbon 14 left. That means we will have to solve the following equation,

$$50 = 100\mathbf{e}^{-0.000124t}$$

Here is that work.

$$\frac{50}{100} = \mathbf{e}^{-0.000124t}$$

$$\frac{1}{2} = \mathbf{e}^{-0.000124t}$$

$$\ln\frac{1}{2} = \ln\mathbf{e}^{-0.000124t}$$

$$\ln\frac{1}{2} = -0.000124t$$

$$t = \frac{\ln\frac{1}{2}}{-0.000124} = \frac{-0.69314718056}{-0.000124} = 5589.89661742$$

So, it looks like it will take about 5589.897 years for half of the Carbon 14 to decay. This number is called the **half-life** of Carbon 14.

We've now looked at a couple of applications of exponential equations and we should now look at a quick application of a logarithm.

Earthquake Intensity

The **Richter scale** is commonly used to measure the intensity of an earthquake. There are many different ways of computing this based on a variety of different quantities. We are going to take a quick look at the formula that uses the energy released during an earthquake.

If E is the energy released, measured in joules, during an earthquake then the magnitude of the earthquake is given by,

$$M = \frac{2}{3}\log\left(\frac{E}{E_0}\right)$$

where $E_0 = 10^{4.4}$ joules.

Example 5 If 8×10^{14} joules of energy is released during an earthquake what was the magnitude of the earthquake?

Solution

There really isn't much to do here other than to plug into the formula.

$$M = \frac{2}{3}\log\left(\frac{8\times10^{14}}{10^{4.4}}\right) = \frac{2}{3}\log\left(8\times10^{9.6}\right) = \frac{2}{3}\left(10.50308999\right) = 7.002$$

So, it looks like we'll have a magnitude of about 7.

Example 6 How much energy will be released in an earthquake with a magnitude of 5.9?

Solution

In this case we will need to solve the following equation.

$$5.9 = \frac{2}{3}\log\left(\frac{E}{10^{4.4}}\right)$$

We saw how solve these kinds of equations in the previous section. First we need the logarithm on one side by itself with a coefficient of one. Once we have it in that form we convert to exponential form and solve.

$$\begin{aligned}
5.9 &= \frac{2}{3}\log\left(\frac{E}{10^{4.4}}\right)\\
8.85 &= \log\left(\frac{E}{10^{4.4}}\right)\\
10^{8.85} &= \frac{E}{10^{4.4}}\\
E &= 10^{8.85}\left(10^{4.4}\right) = 10^{13.25}
\end{aligned}$$

So, it looks like there would be a release of $10^{13.25}$ joules of energy in an earthquake with a magnitude of 5.9.