Section 3-4 : The Definition of a Function

We now need to move into the second topic of this chapter. Before we do that however we need a quick definition taken care of.

Definition of Relation

A **relation** is a set of ordered pairs.

This seems like an odd definition but we'll need it for the definition of a function (which is the main topic of this section). However, before we actually give the definition of a function let's see if we can get a handle on just what a relation is.

Think back to Example 1 in the Graphing section of this chapter. In that example we constructed a set of ordered pairs we used to sketch the graph of $y = (x-1)^2 - 4$. Here are the ordered pairs that we used.

$$(-2,5) \quad (-1,0) \quad (0,-3) \quad (1,-4) \quad (2,-3) \quad (3,0) \quad (4,5)$$

Any of the following are then relations because they consist of a set of ordered pairs.

$$\{(-2,5) \quad (-1,0) \quad (2,-3)\}$$
$$\{(-1,0) \quad (0,-3) \quad (2,-3) \quad (3,0) \quad (4,5)\}$$
$$\{(3,0) \quad (4,5)\}$$
$$\{(-2,5) \quad (-1,0) \quad (0,-3) \quad (1,-4) \quad (2,-3) \quad (3,0) \quad (4,5)\}$$

There are of course many more relations that we could form from the list of ordered pairs above, but we just wanted to list a few possible relations to give some examples. Note as well that we could also get other ordered pairs from the equation and add those into any of the relations above if we wanted to.

Now, at this point you are probably asking just why we care about relations and that is a good question. Some relations are very special and are used at almost all levels of mathematics. The following definition tells us just which relations are these special relations.

Definition of a Function

A **function** is a relation for which each value from the set the first components of the ordered pairs is associated with exactly one value from the set of second components of the ordered pair.

Okay, that is a mouth full. Let's see if we can figure out just what it means. Let's take a look at the following example that will hopefully help us figure all this out.

Example 1 The following relation is a function.

$$\{(-1,0)\ \ (0,-3)\ \ (2,-3)\ \ (3,0)\ \ (4,5)\}$$

Solution

From these ordered pairs we have the following sets of first components (*i.e.* the first number from each ordered pair) and second components (*i.e.* the second number from each ordered pair).

$$1^{\text{st}} \text{ components : } \{-1,0,2,3,4\} \qquad\qquad 2^{\text{nd}} \text{ components : } \{0,-3,0,5\}$$

For the set of second components notice that the "-3" occurred in two ordered pairs but we only listed it once.

To see why this relation is a function simply pick any value from the set of first components. Now, go back up to the relation and find every ordered pair in which this number is the first component and list all the second components from those ordered pairs. The list of second components will consist of exactly one value.

For example, let's choose 2 from the set of first components. From the relation we see that there is exactly one ordered pair with 2 as a first component, $(2,-3)$. Therefore, the list of second components (*i.e.* the list of values from the set of second components) associated with 2 is exactly one number, -3.

Note that we don't care that -3 is the second component of a second ordered par in the relation. That is perfectly acceptable. We just don't want there to be any more than one ordered pair with 2 as a first component.

We looked at a single value from the set of first components for our quick example here but the result will be the same for all the other choices. Regardless of the choice of first components there will be exactly one second component associated with it.

Therefore, this relation is a function.

In order to really get a feel for what the definition of a function is telling us we should probably also check out an example of a relation that is not a function.

Example 2 The following relation is not a function.

$$\{(6,10)\ \ (-7,3)\ \ (0,4)\ \ (6,-4)\}$$

Solution

Don't worry about where this relation came from. It is just one that we made up for this example.

Here is the list of first and second components

$$1^{\text{st}} \text{ components : } \{6,-7,0\} \qquad\qquad 2^{\text{nd}} \text{ components : } \{10,3,4,-4\}$$

From the set of first components let's choose 6. Now, if we go up to the relation we see that there are two ordered pairs with 6 as a first component : $(6,10)$ and $(6,-4)$. The list of second components associated with 6 is then : 10, -4.

The list of second components associated with 6 has two values and so this relation is not a function.

Note that the fact that if we'd chosen -7 or 0 from the set of first components there is only one number in the list of second components associated with each. This doesn't matter. The fact that we found even a single value in the set of first components with more than one second component associated with it is enough to say that this relation is not a function.

As a final comment about this example let's note that if we removed the first and/or the fourth ordered pair from the relation we would have a function!

So, hopefully you have at least a feeling for what the definition of a function is telling us.

Now that we've forced you to go through the actual definition of a function let's give another "working" definition of a function that will be much more useful to what we are doing here.

The actual definition works on a relation. However, as we saw with the four relations we gave prior to the definition of a function and the relation we used in Example 1 we often get the relations from some equation.

It is important to note that not all relations come from equations! The relation from the second example for instance was just a set of ordered pairs we wrote down for the example and didn't come from any equation. This can also be true with relations that are functions. They do not have to come from equations.

However, having said that, the functions that we are going to be using in this course do all come from equations. Therefore, let's write down a definition of a function that acknowledges this fact.

Before we give the "working" definition of a function we need to point out that this is NOT the actual definition of a function, that is given above. This is simply a good "working definition" of a function that ties things to the kinds of functions that we will be working with in this course.

"Working Definition" of Function

A **function** is an equation for which any *x* that can be plugged into the equation will yield exactly one *y* out of the equation.

There it is. That is the definition of functions that we're going to use and will probably be easier to decipher just what it means.

Before we examine this a little more note that we used the phrase "*x* that can be plugged into" in the definition. This tends to imply that not all *x*'s can be plugged into an equation and this is in fact correct. We will come back and discuss this in more detail towards the end of this section, however at this point just remember that we can't divide by zero and if we want real numbers out of the equation we can't take the square root of a negative number. So, with these two examples it is clear that we will not always be able to plug in every *x* into any equation.

Further, when dealing with functions we are always going to assume that both *x* and *y* will be real numbers. In other words, we are going to forget that we know anything about complex numbers for a little bit while we deal with this section.

Okay, with that out of the way let's get back to the definition of a function and let's look at some examples of equations that are functions and equations that aren't functions.

Example 3 Determine which of the following equations are functions and which are not functions.

(a) $y = 5x + 1$

(b) $y = x^2 + 1$

(c) $y^2 = x + 1$

(d) $x^2 + y^2 = 4$

Solution

The "working" definition of function is saying is that if we take all possible values of *x* and plug them into the equation and solve for *y* we will get exactly one value for each value of *x*. At this stage of the game it can be pretty difficult to actually show that an equation is a function so we'll mostly talk our way through it. On the other hand, it's often quite easy to show that an equation isn't a function.

(a) $y = 5x + 1$

So, we need to show that no matter what *x* we plug into the equation and solve for *y* we will only get a single value of *y*. Note as well that the value of *y* will probably be different for each value of *x*, although it doesn't have to be.

Let's start this off by plugging in some values of *x* and see what happens.

$$\begin{array}{ll} x = -4: & y = 5(-4) + 1 = -20 + 1 = -19 \\ x = 0: & y = 5(0) + 1 = 0 + 1 = 1 \\ x = 10: & y = 5(10) + 1 = 50 + 1 = 51 \end{array}$$

So, for each of these values of *x* we got a single value of *y* out of the equation. Now, this isn't sufficient to claim that this is a function. In order to officially prove that this is a function we need to show that this will work no matter which value of *x* we plug into the equation.

Of course, we can't plug all possible value of *x* into the equation. That just isn't physically possible. However, let's go back and look at the ones that we did plug in. For each *x*, upon plugging in, we first multiplied the *x* by 5 and then added 1 onto it. Now, if we multiply a number by 5 we will get a single value from the multiplication. Likewise, we will only get a single value if we add 1 onto a number. Therefore, it seems plausible that based on the operations involved with plugging *x* into the equation that we will only get a single value of *y* out of the equation.

So, this equation is a function.

(b) $y = x^2 + 1$

Again, let's plug in a couple of values of *x* and solve for *y* to see what happens.

$$\begin{aligned} x &= -1: \qquad & y &= (-1)^2 + 1 = 1 + 1 = 2 \\ x &= 3: \qquad & y &= (3)^2 + 1 = 9 + 1 = 10 \end{aligned}$$

Now, let's think a little bit about what we were doing with the evaluations. First, we squared the value of *x* that we plugged in. When we square a number there will only be one possible value. We then add 1 onto this, but again, this will yield a single value.

So, it seems like this equation is also a function.

Note that it is okay to get the same *y* value for different *x*'s. For example,

$$x = -3: \qquad y = (-3)^2 + 1 = 9 + 1 = 10$$

We just can't get more than one *y* out of the equation after we plug in the *x*.

(c) $y^2 = x + 1$

As we've done with the previous two equations let's plug in a couple of value of *x*, solve for *y* and see what we get.

$$\begin{aligned} x &= 3: \qquad & y^2 &= 3 + 1 = 4 \qquad & \Rightarrow \qquad & y = \pm 2 \\ x &= -1: \qquad & y^2 &= -1 + 1 = 0 \qquad & \Rightarrow \qquad & y = 0 \\ x &= 10: \qquad & y^2 &= 10 + 1 = 11 \qquad & \Rightarrow \qquad & y = \pm\sqrt{11} \end{aligned}$$

Now, remember that we're solving for *y* and so that means that in the first and last case above we will actually get two different *y* values out of the *x* and so this equation is NOT a function.

Note that we can have values of *x* that will yield a single *y* as we've seen above, but that doesn't matter. If even one value of *x* yields more than one value of *y* upon solving the equation will not be a function.

What this really means is that we didn't need to go any farther than the first evaluation, since that gave multiple values of *y*.

(d) $x^2 + y^2 = 4$

With this case we'll use the lesson learned in the previous part and see if we can find a value of *x* that will give more than one value of *y* upon solving. Because we've got a y^2 in the problem this shouldn't be too hard to do since solving will eventually mean using the square root property which will give more than one value of *y*.

$$x = 0: \qquad 0^2 + y^2 = 4 \quad \Rightarrow \quad y^2 = 4 \quad \Rightarrow \quad y = \pm 2$$

So, this equation is not a function. Recall, that from the previous section this is the equation of a circle. Circles are never functions.

Hopefully these examples have given you a better feel for what a function actually is.

We now need to move onto something called **function notation**. Function notation will be used heavily throughout most of the remaining chapters in this course and so it is important to understand it.

Let's start off with the following quadratic equation.

$$y = x^2 - 5x + 3$$

We can use a process similar to what we used in the previous set of examples to convince ourselves that this is a function. Since this is a function we will denote it as follows,

$$f\left(x\right) = x^2 - 5x + 3$$

So, we replaced the *y* with the notation $f\left(x\right)$. This is read as "f of *x*". Note that there is nothing special about the *f* we used here. We could just have easily used any of the following,

$$g\left(x\right) = x^2 - 5x + 3 \qquad h\left(x\right) = x^2 - 5x + 3 \qquad R\left(x\right) = x^2 - 5x + 3$$

The letter we use does not matter. What is important is the "$\left(x\right)$" part. The letter in the parenthesis must match the variable used on the right side of the equal sign.

It is very important to note that $f\left(x\right)$ is really nothing more than a really fancy way of writing *y*. If you keep that in mind you may find that dealing with function notation becomes a little easier.

Also, this is **NOT** a multiplication of *f* by *x*! This is one of the more common mistakes people make when they first deal with functions. This is just a notation used to denote functions.

Next we need to talk about **evaluating functions**. Evaluating a function is really nothing more than asking what its value is for specific values of *x*. Another way of looking at it is that we are asking what the *y* value for a given *x* is.

Evaluation is really quite simple. Let's take the function we were looking at above

$$f\left(x\right)=x^2-5x+3$$

and ask what its value is for $x=4$. In terms of function notation we will "ask" this using the notation $f\left(4\right)$. So, when there is something other than the variable inside the parenthesis we are really asking what the value of the function is for that particular quantity.

Now, when we say the value of the function we are really asking what the value of the equation is for that particular value of *x*. Here is $f\left(4\right)$.

$$f\left(4\right)=\left(4\right)^2-5\left(4\right)+3=16-20+3=-1$$

Notice that evaluating a function is done in exactly the same way in which we evaluate equations. All we do is plug in for *x* whatever is on the inside of the parenthesis on the left. Here's another evaluation for this function.

$$f\left(-6\right)=\left(-6\right)^2-5\left(-6\right)+3=36+30+3=69$$

So, again, whatever is on the inside of the parenthesis on the left is plugged in for *x* in the equation on the right. Let's take a look at some more examples.

Example 4 Given $f\left(x\right)=x^2-2x+8$ and $g\left(x\right)=\sqrt{x+6}$ evaluate each of the following.

(a) $f\left(3\right)$ and $g\left(3\right)$

(b) $f\left(-10\right)$ and $g\left(-10\right)$

(c) $f\left(0\right)$

(d) $f\left(t\right)$

(e) $f\left(t+1\right)$ and $f\left(x+1\right)$

(f) $f\left(x^3\right)$

(g) $g\left(x^2-5\right)$

Solution

(a) $f\left(3\right)$ **and** $g\left(3\right)$

Okay we've got two function evaluations to do here and we've also got two functions so we're going to need to decide which function to use for the evaluations. The key here is to notice the letter that is in front of the parenthesis. For $f(3)$ we will use the function $f(x)$ and for $g(3)$ we will use $g(x)$. In other words, we just need to make sure that the variables match up.

Here are the evaluations for this part.

$$f(3) = (3)^2 - 2(3) + 8 = 9 - 6 + 8 = 11$$
$$g(3) = \sqrt{3+6} = \sqrt{9} = 3$$

(b) $f(-10)$ **and** $g(-10)$

This one is pretty much the same as the previous part with one exception that we'll touch on when we reach that point. Here are the evaluations.

$$f(-10) = (-10)^2 - 2(-10) + 8 = 100 + 20 + 8 = 128$$

Make sure that you deal with the negative signs properly here. Now the second one.

$$g(-10) = \sqrt{-10+6} = \sqrt{-4}$$

We've now reached the difference. Recall that when we first started talking about the definition of functions we stated that we were only going to deal with real numbers. In other words, we only plug in real numbers and we only want real numbers back out as answers. So, since we would get a complex number out of this we can't plug -10 into this function.

(c) $f(0)$

Not much to this one.

$$f(0) = (0)^2 - 2(0) + 8 = 8$$

Again, don't forget that this isn't multiplication! For some reason students like to think of this one as multiplication and get an answer of zero. Be careful.

(d) $f(t)$

The rest of these evaluations are now going to be a little different. As this one shows we don't need to just have numbers in the parenthesis. However, evaluation works in exactly the same way. We plug into the *x*'s on the right side of the equal sign whatever is in the parenthesis. In this case that means that we plug in *t* for all the *x*'s.

Here is this evaluation.

$$f(t) = t^2 - 2t + 8$$

Note that in this case this is pretty much the same thing as our original function, except this time we're using *t* as a variable.

(e) $f(t+1)$ **and** $f(x+1)$

Now, let's get a little more complicated, or at least they appear to be more complicated. Things aren't as bad as they may appear however. We'll evaluate $f\left(t+1\right)$ first. This one works exactly the same as the previous part did. All the *x*'s on the left will get replaced with $t+1$. We will have some simplification to do as well after the substitution.

$$\begin{aligned} f\left(t+1\right) &= \left(t+1\right)^2 - 2\left(t+1\right)+8 \\ &= t^2+2t+1-2t-2+8 \\ &= t^2+7 \end{aligned}$$

Be careful with parenthesis in these kinds of evaluations. It is easy to mess up with them.

Now, let's take a look at $f\left(x+1\right)$. With the exception of the *x* this is identical to $f\left(t+1\right)$ and so it works exactly the same way.

$$\begin{aligned} f\left(x+1\right) &= \left(x+1\right)^2 - 2\left(x+1\right)+8 \\ &= x^2+2x+1-2x-2+8 \\ &= x^2+7 \end{aligned}$$

Do not get excited about the fact that we reused *x*'s in the evaluation here. In many places where we will be doing this in later sections there will be *x*'s here and so you will need to get used to seeing that.

(f) $f\left(x^3\right)$

Again, don't get excited about the *x*'s in the parenthesis here. Just evaluate it as if it were a number.

$$\begin{aligned} f\left(x^3\right) &= \left(x^3\right)^2 - 2\left(x^3\right)+8 \\ &= x^6-2x^3+8 \end{aligned}$$

(g) $g\left(x^2-5\right)$

One more evaluation and this time we'll use the other function.

$$\begin{aligned} g\left(x^2-5\right) &= \sqrt{x^2-5+6} \\ &= \sqrt{x^2+1} \end{aligned}$$

Function evaluation is something that we'll be doing a lot of in later sections and chapters so make sure that you can do it. You will find several later sections very difficult to understand and/or do the work in if you do not have a good grasp on how function evaluation works.

While we are on the subject of function evaluation we should now talk about **piecewise functions**. We've actually already seen an example of a piecewise function even if we didn't call it a function (or a piecewise function) at the time. Recall the mathematical definition of absolute value.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

This is a function and if we use function notation we can write it as follows,

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

This is also an example of a piecewise function. A piecewise function is nothing more than a function that is broken into pieces and which piece you use depends upon value of *x*. So, in the absolute value example we will use the top piece if *x* is positive or zero and we will use the bottom piece if *x* is negative.

Let's take a look at evaluating a more complicated piecewise function.

Example 5 Given,

$$g(t) = \begin{cases} 3t^2 + 4 & \text{if } t \leq -4 \\ 10 & \text{if } -4 < t \leq 15 \\ 1 - 6t & \text{if } t > 15 \end{cases}$$

evaluate each of the following.

(a) $g(-6)$

(b) $g(-4)$

(c) $g(1)$

(d) $g(15)$

(e) $g(21)$

Solution

Before starting the evaluations here let's notice that we're using different letters for the function and variable than the ones that we've used to this point. That won't change how the evaluation works.

Do not get so locked into seeing *f* for the function and *x* for the variable that you can't do any problem that doesn't have those letters.

Now, to do each of these evaluations the first thing that we need to do is determine which inequality the number satisfies, and it will only satisfy a single inequality. When we determine which inequality the number satisfies we use the equation associated with that inequality.

So, let's do some evaluations.

(a) $g(-6)$

In this case -6 satisfies the top inequality and so we'll use the top equation for this evaluation.

$$g(-6) = 3(-6)^2 + 4 = 112$$

(b) $g(-4)$

Now we'll need to be a little careful with this one since -4 shows up in two of the inequalities. However, it only satisfies the top inequality and so we will once again use the top function for the evaluation.

$$g(-4) = 3(-4)^2 + 4 = 52$$

(c) $g(1)$

In this case the number, 1, satisfies the middle inequality and so we'll use the middle equation for the evaluation. This evaluation often causes problems for students despite the fact that it's actually one of the easiest evaluations we'll ever do. We know that we evaluate functions/equations by plugging in the number for the variable. In this case there are no variables. That isn't a problem. Since there aren't any variables it just means that we don't actually plug in anything and we get the following,

$$g(1) = 10$$

(d) $g(15)$

Again, like with the second part we need to be a little careful with this one. In this case the number satisfies the middle inequality since that is the one with the equal sign in it. Then like the previous part we just get,

$$g(15) = 10$$

Don't get excited about the fact that the previous two evaluations were the same value. This will happen on occasion.

(e) $g(21)$

For the final evaluation in this example the number satisfies the bottom inequality and so we'll use the bottom equation for the evaluation.

$$g(21) = 1 - 6(21) = -125$$

Piecewise functions do not arise all that often in an Algebra class however, they do arise in several places in later classes and so it is important for you to understand them if you are going to be moving on to more math classes.

As a final topic we need to come back and touch on the fact that we can't always plug every *x* into every function. We talked briefly about this when we gave the definition of the function and we saw an example of this when we were evaluating functions. We now need to look at this in a little more detail.

First, we need to get a couple of definitions out of the way.

Domain and Range

The **domain** of an equation is the set of all *x*'s that we can plug into the equation and get back a real number for *y*. The **range** of an equation is the set of all *y*'s that we can ever get out of the equation.

Note that we did mean to use equation in the definitions above instead of functions. These are really definitions for equations. However, since functions are also equations we can use the definitions for functions as well.

Determining the range of an equation/function can be pretty difficult to do for many functions and so we aren't going to really get into that. We are much more interested here in determining the domains of functions. From the definition the domain is the set of all *x*'s that we can plug into a function and get back a real number. At this point, that means that we need to avoid division by zero and taking square roots of negative numbers.

Let's do a couple of quick examples of finding domains.

Example 6 Determine the domain of each of the following functions.

(a) $g(x) = \dfrac{x+3}{x^2+3x-10}$

(b) $f(x) = \sqrt{5-3x}$

(c) $h(x) = \dfrac{\sqrt{7x+8}}{x^2+4}$

(d) $R(x) = \dfrac{\sqrt{10x-5}}{x^2-16}$

Solution

The domains for these functions are all the values of *x* for which we don't have division by zero or the square root of a negative number. If we remember these two ideas finding the domains will be pretty easy.

(a) $g\left(x\right)=\dfrac{x+3}{x^2+3x-10}$

So, in this case there are no square roots so we don't need to worry about the square root of a negative number. There is however a possibility that we'll have a division by zero error. To determine if we will we'll need to set the denominator equal to zero and solve.

$$x^2+3x-10=\left(x+5\right)\left(x-2\right)=0 \qquad x=-5,\ x=2$$

So, we will get division by zero if we plug in $x=-5$ or $x=2$. That means that we'll need to avoid those two numbers. However, all the other values of *x* will work since they don't give division by zero. The domain is then,

$$\text{Domain : All real numbers except } x=-5 \text{ and } x=2$$

(b) $f\left(x\right)=\sqrt{5-3x}$

In this case we won't have division by zero problems since we don't have any fractions. We do have a square root in the problem and so we'll need to worry about taking the square root of a negative numbers.

This one is going to work a little differently from the previous part. In that part we determined the value(s) of *x* to avoid. In this case it will be just as easy to directly get the domain. To avoid square roots of negative numbers all that we need to do is require that

$$5-3x\geq 0$$

This is a fairly simple linear inequality that we should be able to solve at this point.

$$5\geq 3x \qquad \Rightarrow \qquad x\leq\frac{5}{3}$$

The domain of this function is then,

$$\text{Domain : } x\leq\frac{5}{3}$$

(c) $h\left(x\right)=\dfrac{\sqrt{7x+8}}{x^2+4}$

In this case we've got a fraction, but notice that the denominator will never be zero for any real number since x^2 is guaranteed to be positive or zero and adding 4 onto this will mean that the denominator is always at least 4. In other words, the denominator won't ever be zero. So, all we need to do then is worry about the square root in the numerator.

To do this we'll require,

$$7x+8\geq 0$$
$$7x\geq -8$$
$$x\geq -\frac{8}{7}$$

Now, we can actually plug in any value of *x* into the denominator, however, since we've got the square root in the numerator we'll have to make sure that all *x*'s satisfy the inequality above to avoid problems. Therefore, the domain of this function is

$$\text{Domain}: x\geq -\frac{8}{7}$$

(d) $R(x)=\frac{\sqrt{10x-5}}{x^2-16}$

In this final part we've got both a square root and division by zero to worry about. Let's take care of the square root first since this will probably put the largest restriction on the values of *x*. So, to keep the square root happy (*i.e.* no square root of negative numbers) we'll need to require that,

$$10x-5\geq 0$$
$$10x\geq 5$$
$$x\geq \frac{1}{2}$$

So, at the least we'll need to require that $x\geq \frac{1}{2}$ in order to avoid problems with the square root.

Now, let's see if we have any division by zero problems. Again, to do this simply set the denominator equal to zero and solve.

$$x^2-16=(x-4)(x+4)=0 \qquad \Rightarrow \qquad x=-4,\ x=4$$

Now, notice that $x=-4$ doesn't satisfy the inequality we need for the square root and so that value of *x* has already been excluded by the square root. On the other hand, $x=4$ does satisfy the inequality. This means that it is okay to plug $x=4$ into the square root, however, since it would give division by zero we will need to avoid it.

The domain for this function is then,

$$\text{Domain}: x\geq \frac{1}{2} \ \text{ except } \ x=4$$