Section 6-3 : Solving Exponential Equations

Now that we've seen the definitions of exponential and logarithm functions we need to start thinking about how to solve equations involving them. In this section we will look at solving exponential equations and we will look at solving logarithm equations in the next section.

There are two methods for solving exponential equations. One method is fairly simple but requires a very special form of the exponential equation. The other will work on more complicated exponential equations but can be a little messy at times.

Let's start off by looking at the simpler method. This method will use the following fact about exponential functions.

$$\text{If} \quad b^x = b^y \quad \text{then} \quad x = y$$

Note that this fact does require that the base in both exponentials to be the same. If it isn't then this fact will do us no good.

Let's take a look at a couple of examples.

Example 1 Solve each of the following.

(a) $5^{3x} = 5^{7x-2}$

(b) $4^{t^2} = 4^{6-t}$

(c) $3^z = 9^{z+5}$

(d) $4^{5-9x} = \dfrac{1}{8^{x-2}}$

Solution

(a) $5^{3x} = 5^{7x-2}$

In this first part we have the same base on both exponentials so there really isn't much to do other than to set the two exponents equal to each other and solve for *x*.

$$\begin{aligned} 3x &= 7x - 2 \\ 2 &= 4x \\ \frac{1}{2} &= x \end{aligned}$$

So, if we were to plug $x = \dfrac{1}{2}$ into the equation then we would get the same number on both sides of the equal sign.

(b) $4^{t^2} = 4^{6-t}$

Again, there really isn't much to do here other than set the exponents equal since the base is the same in both exponentials.

$$\begin{aligned} t^2 &= 6-t \\ t^2 + t - 6 &= 0 \\ (t+3)(t-2) &= 0 \qquad\qquad \Rightarrow \qquad t = -3,\ t = 2 \end{aligned}$$

In this case we get two solutions to the equation. That is perfectly acceptable so don't worry about it when it happens.

(c) $3^z = 9^{z+5}$

Now, in this case we don't have the same base so we can't just set exponents equal. However, with a little manipulation of the right side we can get the same base on both exponents. To do this all we need to notice is that $9 = 3^2$. Here's what we get when we use this fact.

$$3^z = \left(3^2\right)^{z+5}$$

Now, we still can't just set exponents equal since the right side now has two exponents. If we recall our exponent properties we can fix this however.

$$3^z = 3^{2(z+5)}$$

We now have the same base and a single exponent on each base so we can use the property and set the exponents equal. Doing this gives,

$$\begin{aligned} z &= 2(z+5) \\ z &= 2z + 10 \\ z &= -10 \end{aligned}$$

So, after all that work we get a solution of $z = -10$.

(d) $4^{5-9x} = \dfrac{1}{8^{x-2}}$

In this part we've got some issues with both sides. First the right side is a fraction and the left side isn't. That is not the problem that it might appear to be however, so for a second let's ignore that. The real issue here is that we can't write 8 as a power of 4 and we can't write 4 as a power of 8 as we did in the previous part.

The first thing to do in this problem is to get the same base on both sides and to so that we'll have to note that we can write both 4 and 8 as a power of 2. So let's do that.

$$\left(2^2\right)^{5-9x} = \frac{1}{\left(2^3\right)^{x-2}}$$

$$2^{2(5-9x)} = \frac{1}{2^{3(x-2)}}$$

It's now time to take care of the fraction on the right side. To do this we simply need to remember the following exponent property.

$$\frac{1}{a^n} = a^{-n}$$

Using this gives,

$$2^{2(5-9x)} = 2^{-3(x-2)}$$

So, we now have the same base and each base has a single exponent on it so we can set the exponents equal.

$$2(5-9x) = -3(x-2)$$
$$10-18x = -3x+6$$
$$4 = 15x$$
$$x = \frac{4}{15}$$

And there is the answer to this part.

Now, the equations in the previous set of examples all relied upon the fact that we were able to get the same base on both exponentials, but that just isn't always possible. Consider the following equation.

$$7^x = 9$$

This is a fairly simple equation however the method we used in the previous examples just won't work because we don't know how to write 9 as a power of 7. In fact, if you think about it that is exactly what this equation is asking us to find.

So, the method we used in the first set of examples won't work. The problem here is that the *x* is in the exponent. Because of that all our knowledge about solving equations won't do us any good. We need a way to get the *x* out of the exponent and luckily for us we have a way to do that. Recall the following logarithm property from the last section.

$$\log_b a^r = r\log_b a$$

Note that to avoid confusion with *x*'s we replaced the *x* in this property with an *a*. The important part of this property is that we can take an exponent and move it into the front of the term.

So, if we had,

$$\log_b 7^x$$

we could use this property as follows.

$$x \log_b 7$$

The *x* in now out of the exponent! Of course, we are now stuck with a logarithm in the problem and not only that but we haven't specified the base of the logarithm.

The reality is that we can use any logarithm to do this so we should pick one that we can deal with. This usually means that we'll work with the common logarithm or the natural logarithm.

So, let's work a set of examples to see how we actually use this idea to solve these equations.

Example 2 Solve each of the following equations.

(a) $7^x = 9$
(b) $2^{4y+1} - 3^y = 0$
(c) $\mathbf{e}^{t+6} = 2$
(d) $10^{5-x} = 8$
(e) $5\mathbf{e}^{2z+4} - 8 = 0$

Solution

(a) $7^x = 9$

Okay, so we say above that if we had a logarithm in front the left side we could get the *x* out of the exponent. That's easy enough to do. We'll just put a logarithm in front of the left side. However, if we put a logarithm there we also must put a logarithm in front of the right side. This is commonly referred to as **taking the logarithm of both sides**.

We can use any logarithm that we'd like to so let's try the natural logarithm.

$$\ln 7^x = \ln 9$$
$$x \ln 7 = \ln 9$$

Now, we need to solve for *x*. This is easier than it looks. If we had $7x = 9$ then we could all solve for *x* simply by dividing both sides by 7. It works in exactly the same manner here. Both ln7 and ln9 are just numbers. Admittedly, it would take a calculator to determine just what those numbers are, but they are numbers and so we can do the same thing here.

$$\frac{x \ln 7}{\ln 7} = \frac{\ln 9}{\ln 7}$$
$$x = \frac{\ln 9}{\ln 7}$$

Now, that is technically the exact answer. However, in this case it's usually best to get a decimal answer so let's go one step further.

$$x = \frac{\ln 9}{\ln 7} = \frac{2.19722458}{1.94591015} = 1.12915007$$

Note that the answers to these are decimal answers more often than not.

Also, be careful here to not make the following mistake.

$$1.12915007 = \frac{\ln 9}{\ln 7} \neq \ln\left(\frac{9}{7}\right) = 0.2513144283$$

The two are clearly different numbers.

Finally, let's also use the common logarithm to make sure that we get the same answer.

$$\begin{aligned} \log 7^x &= \log 9 \\ x\log 7 &= \log 9 \\ x &= \frac{\log 9}{\log 7} = \frac{0.954242509}{0.845098040} = 1.12915007 \end{aligned}$$

So, sure enough the same answer. We can use either logarithm, although there are times when it is more convenient to use one over the other.

(b) $2^{4y+1} - 3^y = 0$

In this case we can't just put a logarithm in front of both sides. There are two reasons for this. First on the right side we've got a zero and we know from the previous section that we can't take the logarithm of zero. Next, in order to move the exponent down it has to be on the whole term inside the logarithm and that just won't be the case with this equation in its present form.

So, the first step is to move on of the terms to the other side of the equal sign, then we will take the logarithm of both sides using the natural logarithm.

$$\begin{aligned} 2^{4y+1} &= 3^y \\ \ln 2^{4y+1} &= \ln 3^y \\ (4y+1)\ln 2 &= y\ln 3 \end{aligned}$$

Okay, this looks messy, but again, it's really not that bad. Let's look at the following equation first.

$$\begin{aligned} 2(4y+1) &= 3y \\ 8y+2 &= 3y \\ 5y &= -2 \\ y &= -\frac{2}{5} \end{aligned}$$

We can all solve this equation and so that means that we can solve the one that we've got. Again, the ln2 and ln3 are just numbers and so the process is exactly the same. The answer will be messier than this equation, but the process is identical. Here is the work for this one.

$$
\begin{aligned}
(4y+1)\ln 2 &= y\ln 3\\
4y\ln 2+\ln 2 &= y\ln 3\\
4y\ln 2-y\ln 3 &= -\ln 2\\
y(4\ln 2-\ln 3) &= -\ln 2\\
y &= -\frac{\ln 2}{4\ln 2-\ln 3}
\end{aligned}
$$

So, we get all the terms with *y* in them on one side and all the other terms on the other side. Once this is done we then factor out a *y* and divide by the coefficient. Again, we would prefer a decimal answer so let's get that.

$$y = -\frac{\ln 2}{4\ln 2-\ln 3} = -\frac{0.693147181}{4(0.693147181)-1.098612289} = -0.414072245$$

(c) $\mathbf{e}^{t+6} = 2$

Now, this one is a little easier than the previous one. Again, we'll take the natural logarithm of both sides.

$$\ln \mathbf{e}^{t+6} = \ln 2$$

Notice that we didn't take the exponent out of this one. That is because we want to use the following property with this one.

$$\ln \mathbf{e}^{f(x)} = f(x)$$

We saw this in the previous section (in more general form) and by using this here we will make our life significantly easier. Using this property gives,

$$
\begin{aligned}
t+6 &= \ln 2\\
t &= \ln(2)-6 = 0.69314718-6 = -5.30685202
\end{aligned}
$$

Notice the parenthesis around the 2 in the logarithm this time. They are there to make sure that we don't make the following mistake.

$$-5.30685202 = \ln(2)-6 \neq \ln(2-6) = \ln(-4) \text{ can't be done}$$

Be very careful with this mistake. It is easy to make when you aren't paying attention to what you're doing or are in a hurry.

(d) $10^{5-x} = 8$

The equation in this part is similar to the previous part except this time we've got a base of 10 and so recalling the fact that,

$$\log 10^{f(x)} = f(x)$$

it makes more sense to use common logarithms this time around.

Here is the work for this equation.

$$\begin{aligned} \log 10^{5-x} &= \log 8 \\ 5-x &= \log 8 \\ 5-\log 8 &= x \qquad \Rightarrow \qquad x = 5-0.903089987 = 4.096910013 \end{aligned}$$

This could have been done with natural logarithms but the work would have been messier.

(e) $5\mathbf{e}^{2z+4} - 8 = 0$

With this final equation we've got a couple of issues. First, we'll need to move the number over to the other side. In order to take the logarithm of both sides we need to have the exponential on one side by itself. Doing this gives,

$$5\mathbf{e}^{2z+4} = 8$$

Next, we've got to get a coefficient of 1 on the exponential. We can only use the facts to simplify this if there isn't a coefficient on the exponential. So, divide both sides by 5 to get,

$$\mathbf{e}^{2z+4} = \frac{8}{5}$$

At this point we will take the logarithm of both sides using the natural logarithm since there is an **e** in the equation.

$$\begin{aligned} \ln \mathbf{e}^{2z+4} &= \ln\left(\frac{8}{5}\right) \\ 2z+4 &= \ln\left(\frac{8}{5}\right) \\ 2z &= \ln\left(\frac{8}{5}\right) - 4 \\ z &= \frac{1}{2}\left(\ln\left(\frac{8}{5}\right) - 4\right) = \frac{1}{2}\left(0.470003629 - 4\right) = -1.76499819 \end{aligned}$$

Note that we could have used this second method on the first set of examples as well if we'd wanted to although the work would have been more complicated and prone to mistakes if we'd done that.