Discrete Structures

Topic : Set Theory

Content

- Definition of a Set
- Properties of a Set
- Representation of a Set
- Cardinality of a Set
- Types of Sets
- Venn Diagrams
- Set Operations

Definition of a Set

- Set are the fundamental discrete structures on which all the discrete structures are built.
- Sets are used to group objects together, formally speaking.
- “A well-defined, unordered collection of distinct objects (Called elements or members of a set) of same type”.

Cont…

For example:

A= {1,2,3,3,4,4}

B= {a,b,c,d}

C= {1,2,3,4,}

D= {2,1,4,3}

E={a,1,#,&}

Specifying a set

- Sets are usually represented by a capital letter (A, B, S, etc.).
- Elements are usually represented by an
- lower-case letter (*a*, *x*, *y*, etc.).
- Easiest way to specify a set is to list all the elements: A = {1, 2, 3, 4, 5}

 (Not always feasible for large or infinite sets)

Specifying a set

A set is said to "contain" the various "members" or "elements" that make up the set

- If an element *a* is a member of (or an element of) a set S, we use then notation $a \in S$
  - $4 \in \{1, 2, 3, 4\}$
- If an element is not a member of (or an element of) a set S, we use the notation $a \notin S$
  - $7 \notin \{1, 2, 3, 4\}$
  - Virginia $\notin \{1, 2, 3, 4\}$

Some Important Sets

- **N** − the set of all natural numbers
  = {1,2,3,4,.....}
- **Z** − the set of all integers
  = {.....,−3,−2,−1,0,1,2,3,.....}
- $\mathbf{Z}^+$ − the set of all positive integers
- **Q** − the set of all rational numbers {Numbers that can be written in the form of p/q, where q≠0.}
- **W** − the set of all whole numbers{0,1,2,3….}

Representation of a Set

Sets can be represented in two ways −

1. Roster or Tabular Form
2. Set Builder Notation

Roster or Tabular Form

The set is represented by listing all the elements. The elements are enclosed within curly braces and separated by commas.

Example 1 − Set of vowels in English alphabet, A={a,e,i,o,u}

Example 2 − Set of odd numbers less than 10, B={1,3,5,7,9}

Note: Used with simple and small sets.

Cont…

Set Builder Notation

The set is defined by specifying a property that elements of the set have in common. The set is described as $A=\{x:p(x)\}$

Example 1 − The set {a,e,i,o,u} is written as −

$A=\{x:x \text{ is a vowel in English alphabet}\}$

Example 2 − The set {1,3,5,7,9} is written as −

$B=\{x:1\leq x<10 \text{ and } (x\%2)\neq 0\}$

Note: Used in complex analysis with large sets.

Cardinality of a Set

Cardinality of a set S, denoted by |S|, is the number of elements of the set. The number is also referred as the cardinal number. If a set has an infinite number of elements, its cardinality is ∞.

Cardinal number: Number of distinct element in group. It is represented by n(A) or | A |.

Cont..

The cardinality of a set is the number of elements in a set

Written as $|A|$

Examples

Let $R = \{1, 2, 3, 4, 5\}$. Then $|R| = 5$

$|\varnothing| = 0$

Let $S = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$. Then $|S| = 4$

A set with one element is sometimes called a singleton set

Cont…

For Example

A= {1,2,3,5} ,   cardinality n(A) = 4

B= {1,2,2,2} ,   cardinality n(B) = 2

C= {1,{2,3},5}, cardinality n(C) = 3

Types of Sets

Sets can be classified into many types. Some of which are finite, infinite, subset, universal, proper, singleton set, etc.

1. Singleton Set or Unit Set:

Singleton set or unit set contains only one element. A singleton set is denoted by {s}.

Example − $S=\{x \mid x \in N, 7<x<9\} = \{8\}$

Cont…

2. Empty Set or Null Set

An empty set contains no elements. It is denoted by ∅. As the number of elements in an empty set is finite, empty set is a finite set. The cardinality of empty set or null set is zero.

Example − $S=\{x \mid x \in N \text{ and } 7<x<8\}=\emptyset$ or { }.

If a set has zero elements, it is called the empty (or null) set

Written using the symbol $\varnothing$

Thus, $\varnothing = \{\ \}$ ← **VERY IMPORTANT**

Cont..

- If you get confused about the empty set in a problem, try replacing $\varnothing$ by { }
- As the empty set is a set, it can be a element of other sets

  { $\varnothing$, 1, 2, 3, x } is a valid set

Note that $\varnothing \neq \{ \varnothing \}$

- The first is a set of zero elements
- The second is a set of 1 element (that one element being the empty set)

Replace $\varnothing$ by { }, and you get: $\{ \} \neq \{ \{ \} \}$

Cont…

3. Equal Set

If two sets contain the same elements they are said to be equal.

Example − If A={1,2,6} and B={6,1,2}, they are equal as every element of set A is an element of set B and every element of set B is an element of set A.

Cont..

- Two sets are equal if they have the same elements
  - {1, 2, 3, 4, 5} = {5, 4, 3, 2, 1}
    - Remember that order does not matter!
  - {1, 2, 3, 2, 4, 3, 2, 1} = {4, 3, 2, 1}
    - Remember that duplicate elements do not matter!
- Two sets are not equal if they do not have the same elements
  - {1, 2, 3, 4, 5} ≠ {1, 2, 3, 4}

Cont…

4. Equivalent Set

If the cardinalities of two sets are same, they are called equivalent sets.

Example − If A={1,2,6} and B={16,17,22}, they are equivalent as cardinality of A is equal to the cardinality of B. i.e. $|A|=|B|=3$.

Cont..

4. Universal Set (U)

It is a collection of all of elements (or the "universe") from which given any set is drawn.

Universal sets are represented as U

For example : The set of the students in this class, ***U*** would be all the students in the University (or perhaps all the people in the world).

Cont..

Finite Set

A set which contains a definite number of elements is called a finite set.

Example − $S=\{x \mid x \in N \text{ and } 70>x>50\}$

Infinite Set

A set which contains infinite number of elements is called an infinite set.

Example − $S=\{x \mid x \in N \text{ and } x>10\}$

Subset

- If all the elements of a set S are also elements of a set T, then S is a subset of T

  For example, if S = {2, 4, 6} and T = {1, 2, 3, 4, 5, 6, 7}, then S is a subset of T

  This is specified by $S \subseteq T$

  Or by $\{2, 4, 6\} \subseteq \{1, 2, 3, 4, 5, 6, 7\}$

  If S is not a subset of T, it is written as such: $S \subseteq T$

  For example, $\{1, 2, 8\} \subseteq \{1, 2, 3, 4, 5, 6, 7\}$

Cont..

Note that any set is a subset of itself!

- Given set S = {2, 4, 6}, since all the elements of S are elements of S, S is a subset of itself
- This is kind of like saying 5 is less than or equal to 5
- Thus, for any set S, $S \subseteq S$

Cont..

- The empty set is a subset of *all* sets (including itself!)
  - Recall that all sets are subsets of themselves
- *All* sets are subsets of the universal set
- A horrible way to define a subset:
  - $\forall x\ (x \in A \rightarrow x \in B)$
  - English translation: for all possible values of x, (meaning for all possible elements of a set), if x is an element of A, then x is an element of B
  - This type of notation will be gone over later

Proper Subset

if S is a subset of T, and S is not equal to T, then S is a proper subset of T

- Let T = {0, 1, 2, 3, 4, 5}
- If S = {1, 2, 3}, S is not equal to T, and S is a subset of T
- A proper subset is written as $S \subset T$
- Let R = {0, 1, 2, 3, 4, 5}. R is equal to T, and thus is a subset (but not a proper subset) of T Can be written as: $R \subseteq T$ and $R \not\subset T$.
- Let Q = {4, 5, 6}. Q is neither a subset of T nor a proper subset of T

Cont..

- The difference between “subset” and “proper subset” is like the difference between “less than or equal to” and “less than” for numbers.

- The empty set is a proper subset of all sets other than the empty set (as it is equal to the empty set).

Power Set

- Power set of a set S is the set of all subsets of S including the empty set.
- The cardinality of a power set of a set S of cardinality n is 2n.
- Power set is denoted as P(S).

Cont..

Given the set S = {0, 1}. What are all the possible subsets of S?

- They are: ∅ (as it is a subset of all sets), {0}, {1}, and {0, 1}
- The power set of S (written as P(S)) is the set of all the subsets of S
- P(S) = { ∅, {0}, {1}, {0,1} }

  Note that |S| = 2 and |P(S)| = 4

Cont..

- Let T = {0, 1, 2}. The P(T) = { ∅, {0}, {1}, {2}, {0,1}, {0,2}, {1,2}, {0,1,2} }

  Note that |T| = 3 and |P(T)| = 8

- P(∅) = { ∅ }

  Note that |∅| = 0 and |P(∅)| = 1

- If a set has *n* elements, then the power set will have 2^n elements

Example

For a set S={a,b,c,d}

let us calculate the subsets

I. Subsets with 0 elements: {ø} (the empty set)

II. Subsets with 1 element: {a},{b},{c},{d}

III. Subsets with 2 elements:

{a,b},{a,c},{a,d},{b,c},{b,d},{c,d}

iv. Subsets with 3 elements:

{a,b,c},{a,b,d},{a,c,d},{b,c,d}

v . Subsets with 4 elements − {a,b,c,d}

Cont..

Hence, $P(S)=$

$\{\{\emptyset\},\{a\},\{b\},\{c\},\{d\},\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},$

$\{c,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\},\{a,b,c,d\}\}$

$|P(S)|=16$

Cartesian product

The **Cartesian product** of two sets A and B, denoted $A \times B$, is the set of all possible ordered pairs where the elements of A are first and the elements of B are second. In set-builder notation, $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$.

Cont..

Example − If we take two

sets $A=\{a,b\}$ and $B=\{1,2\}$,

The Cartesian product of A and B is written as

− $A \times B=\{(a,1),(a,2),(b,1),(b,2)\}$

The Cartesian product of B and A is written as

− $B \times A=\{(1,a),(1,b),(2,a),(2,b)\}$

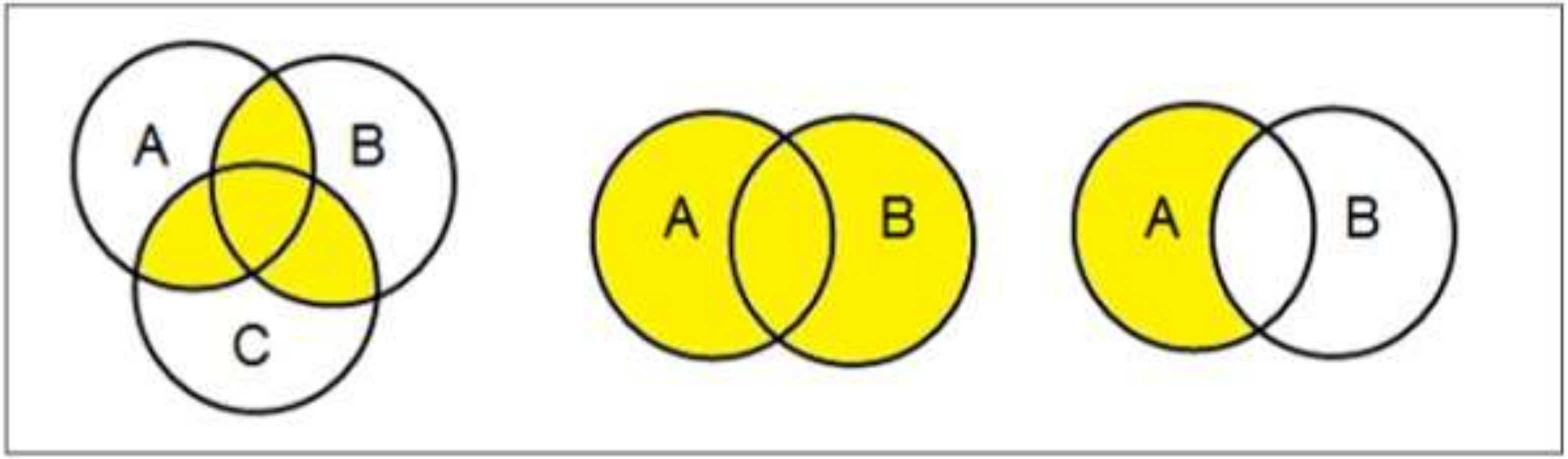
Venn Diagrams

Venn Diagrams

Venn diagram, invented in 1880 by John Venn, is a schematic diagram that shows all possible logical relations between different mathematical sets.

Cont..

Example



Set Operations

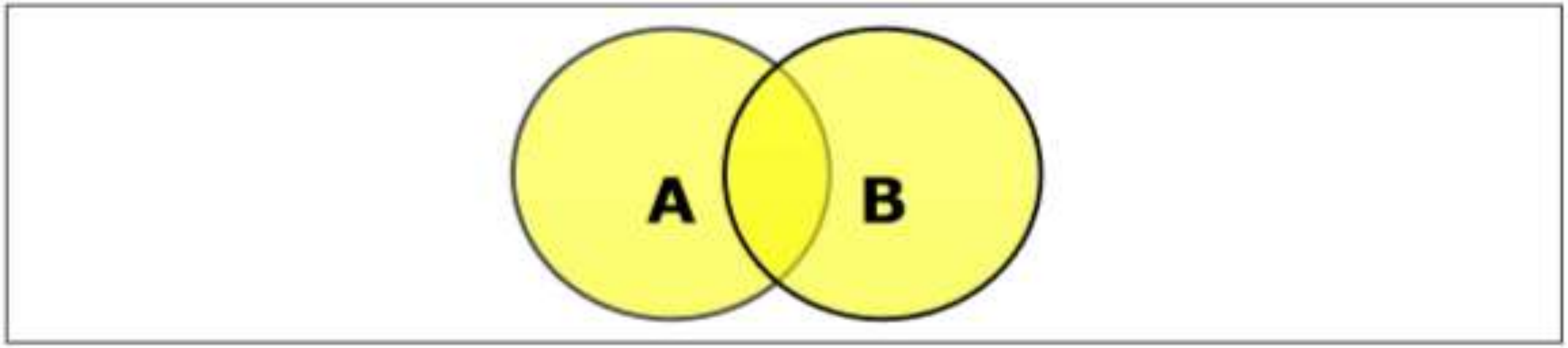
Set Operations include Set Union, Set Intersection, Set Difference, Complement of Set.

Set Union Operation

The union of sets A and B (denoted by $A \cup B$) is the set of elements which are in A, in B, or in both A and B. Hence, $A \cup B = \{x \mid x \in A \text{ OR } x \in B\}$

Example − If $A=\{10,11,12,13\}$ and $B= \{13,14,15\}$ then $A \cup B= \{10,11,12,13,14,15\}$

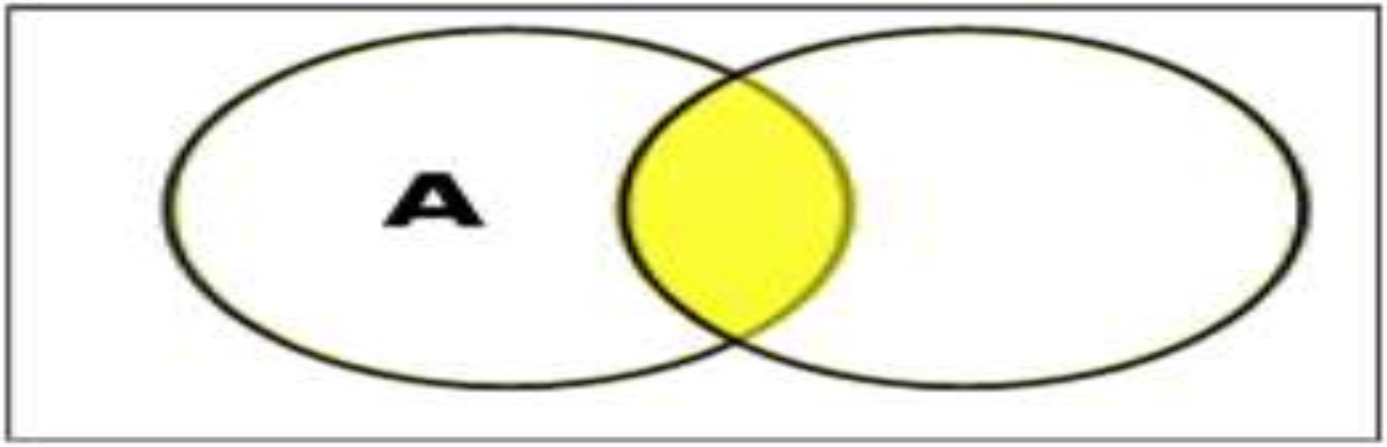
(Note: The common element occurs only once)



Set Intersection Operation

The intersection of sets A and B (denoted by $A \cap B$) is the set of elements which are in both A and B. Hence, $A \cap B = \{x \mid x \in A \text{ AND } x \in B\}$.

Example − If $A=\{11,12,13\}$ and $B=\{13,14,15\}$, then $A \cap B=\{13\}$.



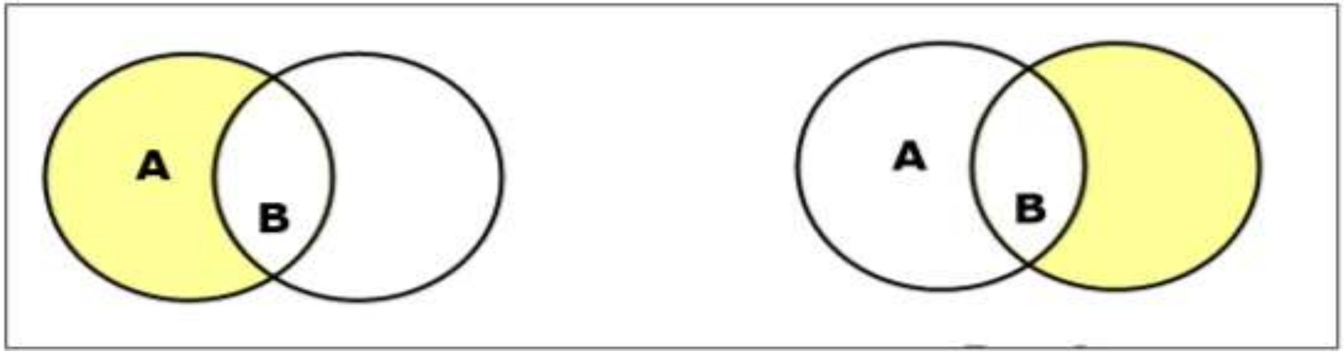
Set Difference Operation

The set difference of sets A and B (denoted by A–B) is the set of elements which are only in A but not in B. Hence, $A-B=\{x \mid x \in A \text{ AND } x \notin B\}$.

Example:

If $A=\{10,11,12,13\}$ and $B=\{13,14,15\}$then $(A-B)=\{10, 11,12\}$ and $(B-A)=\{14,15\}$.

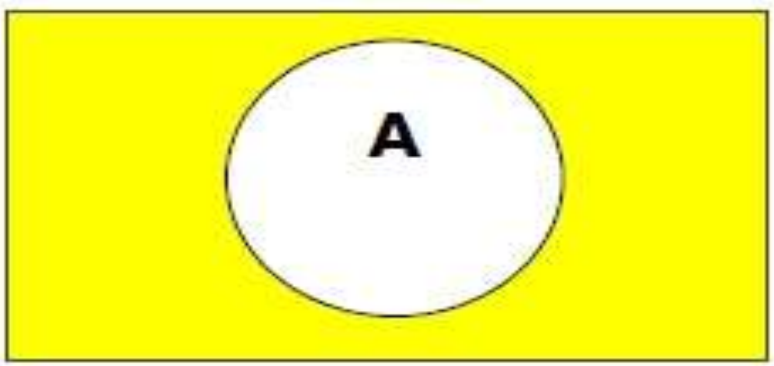
Here, we can see $(A-B)\neq(B-A)$



Set Complement Operation

The complement of a set A (denoted by A′) is the set of elements which are not in set A. Hence, $A' = \{x | x \notin A\}$. More specifically, $A' = (U - A)$ where U is a universal set which contains all objects.

Example If A= {x | x belongs to set of odd integers} then A′={y |y does not belong to set of odd integers}.



Disjoint Set

- Two sets are disjoint if the have NO elements in common.
- Formally, two sets are disjoint if their intersection is the empty set.
- Example: the set of the even numbers and the set of the odd numbers.

Disjoint Set

Formal definition for disjoint sets: Two sets are disjoint if their intersection is the empty set.

Further examples

- {1, 2, 3} and {3, 4, 5} are not disjoint
- {Indore , Ujjain} and {3, 4} are disjoint
- {1, 2} and ∅ are disjoint

  Their intersection is the empty set
- ∅ and ∅ are disjoint!

  Their intersection is the empty set

Reference Books

1. Trembley, J.P & Manohar; “Discrete Mathematical Structure with Application CS”, Tata McGraw Hill.
2. C.L.Liu, “Elements of Discrete Mathematics”,4th Edition, Tata McGraw-Hill, 2012.
3. Bisht, “Discrete Mathematics”, Oxford University Press, 2015.
4. Biswal,” Discrete Mathematics & Graph Theory”,3rd Edition, PHI, 2011.