Question 2

Avadhoot Jadhav - 210050027
Hrishikesh Jedhe Deshmukh - 210050073

October 2022

1 Analytical form of transformed data 'y'

We know, if random variable X has PDF f(x) then PDF of a random variable Y = g(X) is given by:

$$h(y) = f(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right|$$

$$\text{Here g(x) } = \frac{-log(x)}{\lambda}$$

$$g^{-1}(y) = e^{-\lambda y}$$

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & otherwise \end{cases}$$

$$\boldsymbol{h(y) = \begin{cases} 0 & y < 0 \\ \lambda e^{-\lambda y} & y \geq 0 \end{cases}}$$

So the transform data is of exponential form

2 Maximum Likelihood Estimate

If $y_1, y_2...y_n$ are the values obtained as data then Likelihood function will be given as:

$$L(y_1, y_2...y_n|\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda y_i}$$

$$\therefore L(y_1, y_2...y_n|\lambda) = \lambda^n e^{-\lambda \Sigma y_i}$$

Maximum likelihood estimate $\hat{\lambda}^{ML}$ will be 'λ' at which likelihood function have maximum value. We can get it by differentiation.

$$\frac{d}{d\lambda} L(y_1, y_2...y_n|\lambda) = 0$$

$$\frac{d}{d\lambda} \lambda^n e^{-\lambda \Sigma y_i} = 0$$

$$n\lambda^{n-1} e^{-\lambda \Sigma y_i} - \lambda^n e^{-\lambda \Sigma y_i} \Sigma y_i = 0$$

$$\boldsymbol{\hat{\lambda}^{ML} = \frac{n}{\Sigma y_i}}$$

So to get ML estimate of mean we will generate random numbers from uniform distribution and transform it with $\boldsymbol{\frac{-log(x)}{\lambda}}$ and then divide number of values generated by sum of values.

3 Posterior Mean

- **(5 points) Derive a formula for the posterior mean.**

We know,

$$Posterior \propto Prior \times Likelihood$$
$$\therefore Posterior \propto \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} \times \lambda^n e^{-\lambda \Sigma y_i}$$
$$\therefore Posterior \propto \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{n+\alpha-1} e^{-\lambda(\beta+\Sigma y_i)}$$

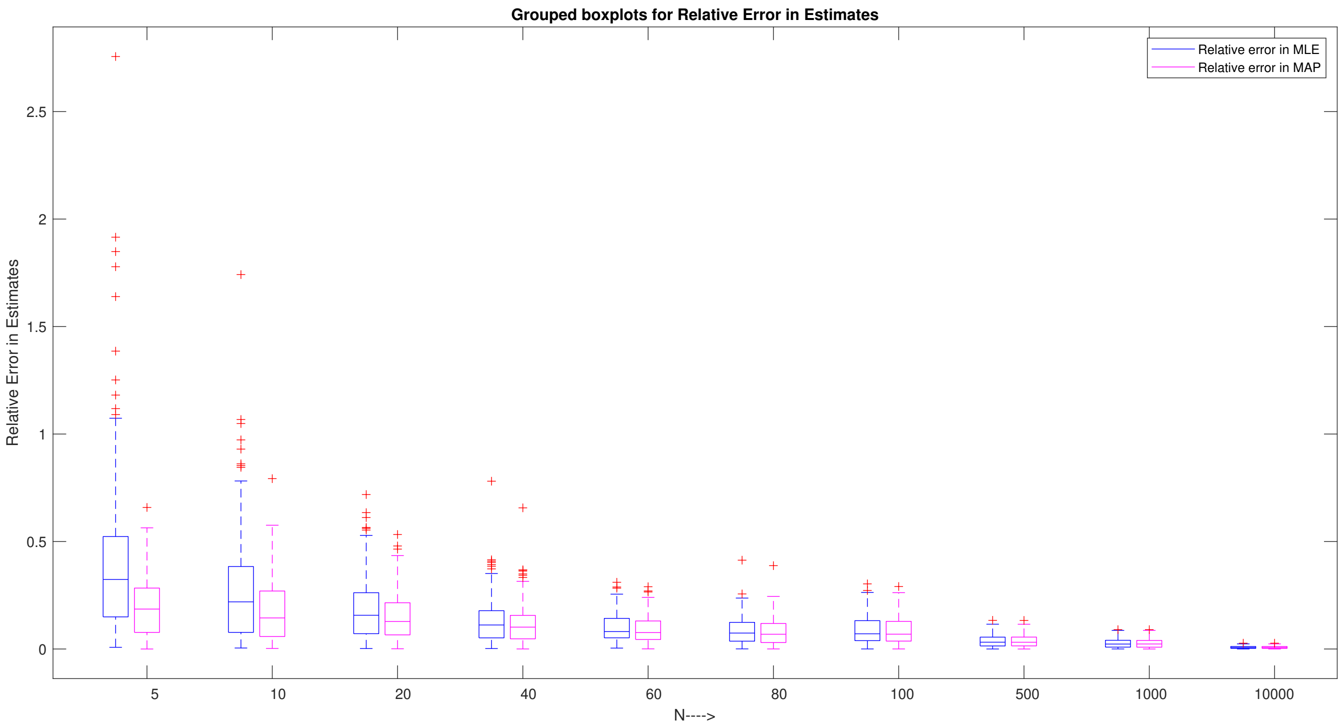
$$\boldsymbol{Posterior \sim Gamma(\lambda; n+\alpha, \beta+\Sigma y_i)}$$

It means posterior is also gamma distribution with parameters $\boldsymbol{n+\alpha}$ and $\boldsymbol{\beta+\Sigma y_i}$.
So the posterior mean:

$$\boldsymbol{\hat{\lambda}^{PosteriorMean} = \frac{n+\alpha}{\beta+\Sigma y_i}}$$

4 Interpretation from graph

Following boxplots are obtained for relative errors in $\hat{\lambda}^{MLE}$ and $\hat{\lambda}^{PosteriorMean}$



From above boxplots it is clear that for small values of N maximum likelihood estimator have higher relative errors compared to posterior mean estimator. This difference in relative errors decreases as N increases and as N tends to ∞ both estimation converges to same value.

$$\lim_{x\to\infty} \hat{\lambda}^{PosteriorMean} = \frac{N}{\Sigma y_i} = \hat{\lambda}^{MLE}$$

Since at large values of N both estimators converge to true value, but at small values of N posterior mean estimator have less relative errors; We would prefer posterior mean estimator.