A note on the derivation of $\mathbb{E}[\lambda_t]$ for Hawkes Processes

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Abstract

We give a simple proof of this quantity using Martingale Theory.

1 A result from Martingale Theory

Let θ be a random variable such that $\theta \in L^2$, i.e.

$$\mathbb{E}\left[\theta^2\right] < \infty,$$

and let M be a martingale. Then the following holds:

$$\mathbb{E}\left[\int \theta dM\right] = 0. \tag{1}$$

1.1 Some martingales in point processes

Given the point process N with intensity λ, the process

$$M_t := N_t - \int_0^t \lambda_u du \tag{2}$$

is a martingale for all $t \in [0, T_{hor}]$ with T_{hor} being a fixed horizon, which is a number. The proof can be found in Daley and Vere-Jones [2003].

2 A general formula for $x(t) := \mathbb{E}[\lambda_t]$

We have a Hawkes process N with kernel ϕ and jump times T_i with intensity λ taking the form

$$\begin{aligned}\lambda_t &= \mu + \sum_{T_i<t} \phi(t-T_i) \\ &= \mu + \int_0^t \phi(t-s)dN_s \\ &= \mu + \int_0^t \phi(t-s)dM_s + \int_0^t \phi(t-s)\lambda_s ds, \quad \text{using equation (2).}\end{aligned}$$

Taking expectation both sides, we get

$$\begin{aligned}\mathbb{E}[\lambda_t] &= \mu + \mathbb{E}\left[\int_0^t \phi(t-s)dM_s\right] + \mathbb{E}\left[\int_0^t \phi(t-s)\lambda_s\right] ds \\ &= \mu + 0 + \int_0^t \phi(t-s)\mathbb{E}\left[\lambda_s\right] ds,\end{aligned}$$

where we have used equation (1). Also note that the interchanging of integral and expectation is valid due to Fubini-Tonelli. Performing a change of variable $y = t - s$ and let $x(t) := \mathbb{E}[\lambda(t)]$ yields

$$x(t) = \mu + \int_0^t \phi(y)x(t-y)dy. \tag{3}$$

3 Some remarks of this result

Relation to your paper. This equation (3) coincides with your equation (11). Obvious extension when ϕ include *iid* marks - the same representation holds, but with a factor multiplying the integral. This factor is the mean of the mark of the distribution.

Uses of integration w.r.t. martingale in machine learning. The continuous version of this result has been exploited by [Sato and Nakagawa, 2014], see equation (18). In fact they use and exploit this successively, see equations (55) and (85), all of which is referenced to Sato and Nakagawa [2014].

Bibliography

Daryl J. Daley and David Vere-Jones. *An Introduction to the Theory of Point Processes*, volume I: Elementary Theory and Methods. Springer, 2nd edition, 2003.

Issei Sato and Hiroshi Nakagawa. Approximation analysis of stochastic gradient langevin dynamics by using fokker-planck equation and ito process. In *Proceedings of the 31st International Conference on Machine Learning (ICML-14)*, pages 982–990, 2014.