Consistency of a Recurrent Language Model
With Respect to Incomplete Decoding

Sean Welleck [1] [*]   **Ilia Kulikov** [1] [*]   **Jaedeok Kim** [1] [2]   **Richard Yuanzhe Pang** [1]   **Kyunghyun Cho** [1] [3] [4]

Abstract

Despite strong performance on a variety of tasks, neural sequence models trained with maximum likelihood have been shown to exhibit issues such as length bias and degenerate repetition. We study the related issue of receiving infinite-length sequences from a recurrent language model when using common decoding algorithms. To analyze this issue, we first define inconsistency of a decoding algorithm, meaning that the algorithm can yield an infinite-length sequence that has zero probability under the model. We prove that commonly used incomplete decoding algorithms – greedy search, beam search, top-k sampling, and nucleus sampling – are inconsistent, despite the fact that recurrent language models are trained to produce sequences of finite length. Based on these insights, we propose two remedies which address inconsistency: consistent variants of top-k and nucleus sampling, and a self-terminating recurrent language model. Empirical results show that inconsistency occurs in practice, and that the proposed methods prevent inconsistency.

1. Introduction

Neural sequence models trained with maximum likelihood estimation (MLE) have become a standard approach to modeling sequences in a variety of natural language applications such as machine translation (Bahdanau et al., 2015), dialogue modeling (Vinyals et al., 2015), and language modeling (Radford et al., 2018). Despite this success, MLE-trained neural sequence models have been shown to exhibit issues such as length bias (Sountsov & Sarawagi, 2016; Stahlberg & Byrne, 2019) and degenerate repetition (Holtzman et al., 2019). These issues are suspected to be related to the maximum likelihood objective's local normalization, which results in a discrepancy between the learned model's

distribution and the distribution induced by the decoding algorithm used to generate sequences (Lafferty et al., 2001; Andor et al., 2016). This has prompted the development of alternative decoding methods (Wu et al., 2016; Holtzman et al., 2019) and training objectives (Murray & Chiang, 2018; Welleck et al., 2019). In this paper, we formalize and study this discrepancy between the model and the decoding algorithm.

We begin by formally defining *recurrent neural language models*, a family that encompasses neural models used in practice, such as recurrent neural networks (Elman, 1990; Cho et al., 2014; Hochreiter & Schmidhuber, 1997), and transformers (Vaswani et al., 2017). Next, we formally define a decoding algorithm – a function that induces a distribution over sequences given a recurrent language model and a context distribution – which is used to obtain probable sequences from a model. In this paper, we show that the distribution induced by a decoding algorithm can contradict this intended use; instead, the decoding algorithm may return improbable, infinite-length sequences.

Our main finding is that a sequence which receives zero probability under a recurrent language model's distribution can receive nonzero probability under the distribution induced by a decoding algorithm. This occurs when the recurrent language model always ranks the sequence termination token outside of the set of tokens considered at each decoding step, yielding an infinite-length, zero probability sequence. This holds whenever the decoding algorithm is *incomplete*, in the sense that the algorithm excludes tokens from consideration at each step of decoding, which is the case for common methods such as greedy search, beam search, top-k sampling (Fan et al., 2018), and nucleus sampling (Holtzman et al., 2019). We formalize our main finding using the notion of *consistency* (Chen et al., 2017) – whether a distribution assigns probability mass only to finite sequences – and prove that a consistent recurrent language model paired with an incomplete decoding algorithm can induce an inconsistent sequence distribution.

Based on the insight that inconsistency occurs due to the behavior of the termination token under incomplete decoding, we develop two methods for addressing inconsistency. First, we propose *consistent sampling methods* which guarantee

---

[*]Equal contribution [1]New York University [2]Samsung Research [3]Facebook AI Research [4]CIFAR Associate Fellow. Correspondence to: Sean Welleck <wellecks@nyu.edu>.

that the termination token is not excluded from selection during decoding. Second, we introduce a *self-terminating recurrent language model* which ensures that the termination token is eventually ranked above all others, guaranteeing consistency under incomplete decoding.

To empirically measure inconsistency, we decode sequences from trained recurrent language models and measure the proportion of sequences with lengths far exceeding the maximum training sequence length. Our experiments on the Wikitext2 dataset (Merity et al., 2016) suggest that inconsistency occurs in practice when using incomplete decoding methods, while the proposed consistent sampling methods and self-terminating model parameterization prevent inconsistency and maintain language modeling quality.

The theoretical analysis reveals defects of existing decoding algorithms, providing a way to develop future models, inference procedures, and learning algorithms. We present methods related to sampling and model parameterization, but there are more directions which we leave to the future; we close with directions related to sequence-level learning.

2. Background

We begin our discussion by establishing background definitions. First, we define a sequence which is the main object of our investigation.

Definition 2.1 (Sequence). *A sequence Y is an ordered collection of items from a predefined finite vocabulary V. A sequence of finite length always ends with a special token $\langle eos \rangle \in V$ that only appears at the end of a sequence.*

Each model we consider generates a sequence conditioned on context information, such as a prefix in sentence completion. To consider this, we define a context distribution.

Definition 2.2 (Context distribution). *A context distribution $p(C)$ is a probability distribution defined over a set $\mathcal{C}$. An element $C \in \mathcal{C}$ is called a context.*

2.1. Recurrent Language Models

A recurrent language model is an autoregressive model of a sequence distribution, where each conditional probability is parameterized with a neural network. Importantly, we assume that all tokens in a sequence are dependent on each other under a recurrent language model. This allows us to avoid cases in which the model degenerates to a Markovian language model, such as an n-gram model with a finite n.

Definition 2.3 (Recurrent language model). *A recurrent language model p_θ is a neural network that computes the following conditional probability at each time step*

$$p_\theta(y_t = v \,|\, y_{<t}, C) = \frac{\exp(u_v^\top h_t + c_v)}{\sum_{v' \in V} \exp(u_{v'}^\top h_t + c_{v'})},$$

where $h_t = f_\theta(y_t, h_{t-1})$ and $h_0 = g_\theta(C)$, and u, c, θ are parameters. A recurrent language model thereby computes the probability of a sequence $Y = (y_1, \ldots, y_T)$ by

$$p_\theta(Y \,|\, C) = \prod_{t=1}^{T} p_\theta(y_t \,|\, y_{<t}, C),$$

where $y_{<t} = (y_1, \ldots, y_{t-1})$. This distribution satisfies

$$y_i \not\perp y_j \,|\, C, \;\; \forall i < j.$$

Practical variants of the recurrent language model differ by the choice of transition function f_θ (Elman, 1990; Hochreiter & Schmidhuber, 1997; Cho et al., 2014; Vaswani et al., 2017). The use of softmax (Bridle, 1990) implies that every unique token in the vocabulary is considered at every location of a sequence.

Remark 2.1. *Under the conditional distribution of a recurrent language model, every token $v \in V$ is assigned a positive probability. This implies that $0 < p_\theta(v \,|\, y_{<t}, C) < 1$. In addition, it follows that any finite sequence is probable by a recurrent language model under any context, i.e., $p_\theta(Y \,|\, C) > 0$ for any sequence Y of finite length.*

2.2. Decoding Algorithms

Because it is intractable to decode the most probable sequence, it is necessary in practice to use an approximate decoding algorithm.

Definition 2.4 (Decoding algorithm). *A decoding algorithm $\mathcal{F}(p_\theta, C)$ is a function that generates a sequence $\tilde{Y}$ given a recurrent language model p_θ and context C. Let $q_\mathcal{F}$ denote the distribution induced by the decoding algorithm $\mathcal{F}$.*

We consider two families of decoding algorithms. In our analysis we only consider decoding algorithms that decode in a single pass, forward in time, without modifying previously selected tokens.

Stochastic decoding. The first family consists of stochastic algorithms. Among them, ancestral sampling is asymptotically unbiased and can be used for finding the most probable sequence, although it requires a substantial number of samples to achieve a low-variance estimate.

Definition 2.5 (Ancestral sampling). *Ancestral sampling $\mathcal{F}_{anc}$ generates a sequence from a recurrent language model p_θ given context C by recursively sampling from $p_\theta(y_t \,|\, \tilde{y}_{<t}, C)$ until $\tilde{y}_t = \langle eos \rangle$:*

$$\tilde{y}_t \sim p_\theta(y_t \,|\, \tilde{y}_{<t}, C).$$

In order to avoid the high variance, two approximate stochastic decoding algorithms have recently been proposed and

tested with recurrent language models. Top-k sampling considers only a subset of the k most probable tokens from the vocabulary at a time, while nucleus sampling considers only the minimal subset of most probable tokens whose total probability is higher than a predefined threshold.

Definition 2.6 (Top-k sampling (Fan et al., 2018)). *Top-k sampling $\mathcal{F}_{top\text{-}k}$ generates a sequence from a recurrent language model p_θ given context C by recursively sampling from the following proposal distribution:*

$$q(v) \propto \begin{cases} p_\theta(v \mid y_{<t}, C), & \text{if } v \in \arg \, top\text{-}k \, p_\theta(v' \mid y_{<t}, C), \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.7 (Nucleus sampling (Holtzman et al., 2019)). *Nucleus sampling $\mathcal{F}_{nuc\text{-}\mu}$ generates a sequence from a recurrent language model p_θ given context C by recursively sampling from the following proposal distribution. Let $v_1, \ldots, v_{|V|}$ denote tokens in V such that $p_\theta(v_i \mid y_{<t}, C) \geq p_\theta(v_j \mid y_{<t}, C)$ for all $i < j$, and define*

$$q(v) \propto \begin{cases} p_\theta(v \mid y_{<t}, C), & \text{if } v \in V_\mu, \\ 0, & \text{otherwise,} \end{cases}$$

where $V_\mu = \{v_1, \cdots, v_{k_\mu}\}$ with

$$k_\mu = \min \left\{ k \, \middle| \, \sum_{i=1}^{k} p_\theta(v_i \mid y_{<t}, C) > \mu \right\}.$$

Deterministic decoding. The other family consists of deterministic decoding algorithms, where a token is selected deterministically according to a rule at each decoding step. The most naive algorithm, called greedy decoding, simply takes the most probable token at each step.

Definition 2.8 (Greedy decoding). *Greedy decoding $\mathcal{F}_{greedy}$ generates a sequence from a recurrent language model p_θ given context C by recursively selecting the most likely token from $p_\theta(y_t | \tilde{y}_{<t}, C)$ until $\tilde{y}_t = \langle eos \rangle$:*

$$\tilde{y}_t = \arg \max_{v \in V} \log p_\theta(y_t = v \mid \tilde{y}_{<t}, C).$$

In contrast to greedy decoding, beam search operates on the level of partial sequences or prefixes.

Definition 2.9 (Prefix). *A prefix ρ_t is an ordered collection of items from V. The score of a prefix is*

$$s(\rho_t) = \sum_{\tau=1}^{t} \log p_\theta(y_\tau = \rho_t[\tau] \mid \rho_t[< \tau], C),$$

where $\rho_t[\tau]$ is a token at time τ from ρ_t.

Starting from a set of empty prefixes, at each iteration a new prefix set is formed by expanding each prefix, then choosing the highest scoring expanded prefixes.

Definition 2.10 (Beam search). *Beam search with width k, $\mathcal{F}_{beam-k}$, generates a sequence from a recurrent language model p_θ by maintaining a size-k prefix set P_t^{top}. Starting with $P_0^{top} = \varnothing$, at each iteration $t \in \{1, 2, \ldots\}$ beam search forms a new prefix set P_t^{top} by expanding the current set, $\mathrm{P}_t = \bigcup_{\rho \in \mathrm{P}_{t-1}^{top}} \{\rho \circ v \mid v \in V\}$ (where $\rho \circ v$ is concatenation), then choosing the k highest scoring elements,*

$$\mathrm{P}_t^{top} = \arg \, top\text{-}k \, s(\rho).$$
$$\scriptstyle \rho \in \mathrm{P}_t$$

Any $\rho \in \mathrm{P}_t^{top}$ ending with $\langle eos \rangle$ is restricted from being expanded further, and is added to a set S. Beam search ends when S contains k sequences, and returns the highest scoring sequence in S.

Incompleteness. Other than ancestral sampling, the decoding algorithms above are *incomplete* in that they only consider a strict subset of the the full vocabulary V at each time step, aside from the trivial case of $k = |V|$.[1]

Definition 2.11 (Incomplete Decoding). *A decoding algorithm $\mathcal{F}$ is incomplete when for each context C and prefix $y_{<t}$, there is a strict subset $V_t' \subsetneq V$ such that*

$$\sum_{v \in V_t'} q_{\mathcal{F}}(y_t = v \mid y_{<t}, C) = 1.$$

3. Consistency of a Decoding Algorithm

Definition of consistency. A recurrent language model p_θ may assign a positive probability to an infinitely long sequence, in which case we call the model inconsistent. This notion of consistency was raised and analyzed earlier, for instance by Booth & Thompson (1973) and Chen et al. (2017), in terms of whether the distribution induced by p_θ is concentrated on finite sequences. We extend their definition to account for the context C.

Definition 3.1 (Consistency of a recurrent language model). *A recurrent language model is consistent under a context distribution $p(C)$ if $p_\theta(|Y| = \infty) = 0$. Otherwise, the recurrent language model is said to be inconsistent.*

Any sequence decoded from a consistent model for a given probable context is guaranteed to terminate.

Lemma 3.1. *If a recurrent language model p_θ is consistent, $p_\theta(|Y| = \infty \mid C) = 0$ for any probable context C.*[2]

Next, we establish a practical condition under which a recurrent language model is consistent.

Lemma 3.2. *A recurrent language model p_θ is consistent if $\|h_t\|_p$ is uniformly bounded for some $p \geq 1$.*

---

[1] Nucleus sampling is incomplete when for every context C and prefix $y_{<t}$, $\min_{v \in V} p_\theta(v|y_{<t}, C) < 1 - \mu$.

[2] Proofs of Lemmas 3.1-3.3 are in Appendix A.

Proof sketch. If $\|h_t\|_p$ is bounded, then each $u_v^\top h_t$ is bounded, hence $p_\theta(\langle\text{eos}\rangle \mid y_{<t}, C) > \xi > 0$ for a constant ξ. Thus $p_\theta(|Y| = \infty) \leq \lim_{t\to\infty}(1 - \xi)^t = 0$, meaning that p_θ is consistent. $\square$

Although this condition is practical because layer normalization or bounded activation functions (Elman, 1990; Cho et al., 2014; Vaswani et al., 2017) result in bounded h_t, we show that even if a recurrent language model is consistent, a decoding algorithm may produce an infinite-length sequence. We formalize this discrepancy using the consistency of a decoding algorithm.

Definition 3.2 (Consistency of a decoding algorithm). *A decoding algorithm $\mathcal{F}$ is consistent with respect to a consistent recurrent language model p_θ under a context distribution $p(C)$ if the decoding algorithm $\mathcal{F}$ preserves the consistency of the model p_θ, that is, $q_{\mathcal{F}}(|Y| = \infty) = 0$.*

When a consistent recurrent language model p_θ and a decoding algorithm $\mathcal{F}$ induce a consistent distribution $q_{\mathcal{F}}$, we say that p_θ paired with $\mathcal{F}$ is consistent. For instance, any consistent recurrent language model paired with ancestral sampling is consistent, because the induced distribution $q_{\mathcal{F}_{\text{anc}}}$ is the same as the distribution of the original model. We also have an analogue of Lemma 3.1.

Lemma 3.3. *A consistent decoding algorithm with respect to a consistent recurrent language model decodes only probable sequences. That is, if $q_{\mathcal{F}}(Y \mid C) > 0$, then $p_\theta(Y \mid C) > 0$ for any probable context C.*

Inconsistency of incomplete decoding. Any incomplete decoding algorithm (Definition 2.11) can be inconsistent regardless of the context distribution, because there is a recurrent language model that places $\langle\text{eos}\rangle$ outside of V_t' at every step of decoding. To show this, we construct a consistent recurrent language model whose distribution induced by an incomplete decoding algorithm is inconsistent.

Theorem 3.4 (Inconsistency of an incomplete decoding algorithm). *There exists a consistent recurrent language model p_θ from which an incomplete decoding algorithm $\mathcal{F}$, that considers only up to $(|V| - 1)$-most likely tokens according to $p_\theta(y_t \mid y_{<t}, C)$ at each step t, finds a sequence $\tilde{Y}$ whose probability under p_θ is 0 for any context distribution.*

Proof. We prove this theorem by constructing a $\tanh$ recurrent network. We define the recurrent function f_θ as

$$h_t = f_\theta(y_t, h_{t-1})$$
$$= \tanh\left(\left[\begin{array}{c|c} W_h & \mathbf{0} \\ \hline \mathbf{0} & I \end{array}\right] h_{t-1} + \left[\begin{array}{c} \mathbf{0} \\ e(y_t) \end{array}\right]\right),$$

where $e(y_t) \in \mathbb{R}^{|V|}$ is a one-hot representation of y_t, $W_h \in \mathbb{R}^{d \times d}$ where every entry is positive, and I is an identity

matrix of size $|V| \times |V|$. $h_0 = g_\theta(C)$ is constructed to consist of positive values only. Because each element of $|h_t|$ is bounded by 1, the constructed recurrent language model p_θ is consistent by Lemma 3.2.

For $v \neq \langle\text{eos}\rangle$, we set u_v (see Definition 2.3) to be

$$u_v = \left[\begin{array}{c} \bar{u}_v \\ e(v) \end{array}\right],$$

where all elements of $\bar{u}_v$ are positive and $e(v)$ is a one-hot representation of v. c_v is set to zero. Next, let

$$u_{\langle\text{eos}\rangle} = \left[\begin{array}{c} \bar{u}_{\langle\text{eos}\rangle} \\ e(\langle\text{eos}\rangle) \end{array}\right],$$

where all elements of $\bar{u}_{\langle\text{eos}\rangle}$ are negative.

This defines a valid recurrent language model (Definition 2.3), since the conditional distribution at each time t is influenced by all the previous tokens. More specifically, the logit of a token v depends on $\sum_{t'=1}^{t} \mathbb{1}(y_{t'} = v)$, where $\mathbb{1}$ is an indicator function.

This recurrent language model always outputs positive logits for non-$\langle\text{eos}\rangle$ tokens, and outputs negative logits for the $\langle\text{eos}\rangle$ token. This implies $p(\langle\text{eos}\rangle \mid y_{<t}, C) < p(v \mid y_{<t}, C)$ for all $v \in V \setminus \{\langle\text{eos}\rangle\}$. This means that $\langle\text{eos}\rangle$ is always ranked last at each time step, so an incomplete decoding algorithm that considers at most $(|V| - 1)$ most probable tokens at each time step from $p_\theta(y_t \mid y_{<t}, C)$ cannot decode $\langle\text{eos}\rangle$ and thus always decodes an infinitely long sequence.

The log-probability of this infinitely long sequence $\hat{Y}$ is

$$\log p_\theta(\hat{Y} \mid C) = \sum_{t=1}^{\infty} \log p_\theta(\hat{y}_t \mid \hat{y}_{<t}, C).$$

For any $v \in V$,

$$p_\theta(v \mid \hat{y}_{<t}, C) = \frac{\exp(u_v^\top h_t)}{\exp(u_v^\top h_t) + \sum_{v' \neq v} \exp(u_{v'}^\top h_t)}$$
$$\leq \frac{\exp(u_v^\top h_t)}{\exp(u_v^\top h_t) + b_v} \leq \frac{\exp(\|u_v\|_1)}{\exp(\|u_v\|_1) + b_v} < 1,$$

where $b_v = \sum_{v' \neq v} \exp(-\|u_{v'}\|_1)$. The last inequality holds because $x/(x + b_v)$ is increasing in $x > 0$. Therefore, the log-probability $\log p_\theta(\hat{Y} \mid C)$ diverges as $|\hat{Y}| \to \infty$, and thus $p_\theta(\hat{Y} \mid C) = 0$, which implies the decoding algorithm $\mathcal{F}$ is inconsistent by Lemma 3.3. $\square$

Greedy decoding, beam search, top-k sampling, and nucleus sampling are all inconsistent according to this theorem; there are consistent models p_θ that induce inconsistent distributions when paired with these decoding algorithms.

4. Fixing the inconsistency

In this section, we consider two ways to prevent inconsistency arising from incomplete decoding algorithms. First, we introduce consistent versions of top-k and nucleus sampling. Second, we introduce the *self-terminating recurrent language model*, which is consistent when paired with any of the decoding algorithms considered in this paper.

4.1. Consistent Sampling Algorithms

The proof of Theorem 3.4 suggests that inconsistency of incomplete decoding algorithms arises from the fact that $\langle eos \rangle$ may be excluded indefinitely from the set of top-ranked tokens. We propose a simple modification to top-k and nucleus sampling that forces $\langle eos \rangle$ to be included at each step of decoding. First, we give a condition for when a particular model p_θ paired with a decoding algorithm $\mathcal{F}$ is consistent.

Theorem 4.1. *Let p_θ be a consistent recurrent language model. If a decoding algorithm $\mathcal{F}$ satisfies $q_\mathcal{F}(\langle eos \rangle \,|\, y_{<t}, C) \geq p_\theta(\langle eos \rangle \,|\, y_{<t}, C)$ for every prefix $y_{<t}$ and context C, then the decoding algorithm $\mathcal{F}$ is consistent with respect to the model p_θ.*

Proof. Let P'_{t-1} denote a set of all prefixes $y_{<t}$ of length $t - 1$. For $t \geq 1$,

$$q_\mathcal{F}(|Y| > t \,|\, C) = \sum_{y_{<t} \in P'_{t-1}} (1 - q_\mathcal{F}(\langle eos \rangle \,|\, y_{<t}, C))$$

$$\leq \sum_{y_{<t} \in P'_{t-1}} (1 - p_\theta(\langle eos \rangle \,|\, y_{<t}, C)) = p_\theta(|Y| > t \,|\, C).$$

Taking the limit $t \to \infty$ and expectation over C on both sides, we have

$$q_\mathcal{F}(|Y| = \infty) = \mathbb{E}\left[q_\mathcal{F}(|Y| = \infty \,|\, C)\right]$$
$$\leq \mathbb{E}\left[p_\theta(|Y| = \infty \,|\, C)\right] = 0,$$

from which the decoding algorithm is consistent. $\square$

We define consistent variants of top-k and nucleus sampling which satisfy this condition.

Definition 4.1 (Consistent top-k sampling). *Consistent top-k sampling is top-k sampling with the following modified proposal distribution:*

$$q(v) \propto \begin{cases} p_\theta(v|y_{<t}, C), & \text{if } v \in V', \\ 0, & \text{otherwise}, \end{cases}$$

where $V' = \{\langle eos \rangle\} \cup \arg \underset{v'}{top\text{-}k}\, p_\theta(v' \,|\, y_{<t}, C)$.

Definition 4.2 (Consistent nucleus sampling). *Consistent nucleus sampling is nucleus sampling with the following modified proposal distribution:*

$$q(v) \propto \begin{cases} p_\theta(v \,|\, y_{<t}, C), & \text{if } v \in V_\mu \cup \{\langle eos \rangle\}, \\ 0, & \text{otherwise}. \end{cases}$$

The induced probability of $\langle eos \rangle$ under these two algorithms is always equal to or larger than the model's probability. By Theorem 4.1, these algorithms are consistent with respect to any consistent recurrent language model.

4.2. A Self-Terminating Recurrent Language Model

Although these consistent sampling algorithms can be used with any recurrent language model, their stochastic nature may not be suitable for finding a single, highly probable sequence. To avoid this limitation, we propose the *self-terminating recurrent language model* (STRLM).

Definition 4.3 (Self-terminating recurrent language model). *A self-terminating recurrent language model computes the following conditional probability at each time step:*

$$p_\theta(v \,|\, y_{<t}, C) = \begin{cases} 1 - \alpha(h_t), & \text{if } v = \langle eos \rangle, \\ \frac{\alpha(h_t) \exp(u_v^\top h_t + c_v)}{\sum_{v' \in V'} \exp(u_{v'}^\top h_t + c_{v'})}, & \text{otherwise}, \end{cases}$$

where

$$\alpha(h_0) = \sigma(u_{\langle eos \rangle}^\top h_0 + c_{\langle eos \rangle}),$$
$$\alpha(h_t) = \sigma(u_{\langle eos \rangle}^\top h_t + c_{\langle eos \rangle}) \left[1 - p_\theta(\langle eos \rangle \,|\, y_{<t-1}, C)\right],$$

with $\sigma : \mathbb{R} \to [0, 1 - \epsilon]$ and $\epsilon \in (0, 1)$. h_t is computed as in the original recurrent language model.

The underlying idea is that the probability of $\langle eos \rangle$ increases monotonically. The model is consistent when paired with greedy decoding.

Theorem 4.2. *Greedy decoding is consistent with respect to any self-terminating recurrent language model.*

Proof. Let $p_t^{\langle eos \rangle}$ denote $p_\theta(\langle eos \rangle \,|\, y_{<t}, C)$ and $a_t^{\langle eos \rangle}$ denote $u_{\langle eos \rangle}^\top h_t + c_{\langle eos \rangle}$. By Definition 4.3 we have

$$p_t^{\langle eos \rangle} = 1 - \sigma(a_t^{\langle eos \rangle})(1 - p_{t-1}^{\langle eos \rangle})$$

$$= 1 - \prod_{t'=0}^{t} \sigma(a_{t'}^{\langle eos \rangle}) \geq 1 - (1 - \epsilon)^{t+1}.$$

Take $B = -\log 2 / \log(1 - \epsilon)$. We then have $p_t^{\langle eos \rangle} > 1/2$ for all $t > B$, which implies that $\langle eos \rangle$ is always the most probable token after time step B. Hence, the sequence length is less than B with probability 1. $\square$

Beam search is also consistent with respect to any self-terminating recurrent language model according to a similar argument; see Appendix B for the proof.

5. Empirical Validation

The theoretical results rely on the existence of a model that results in inconsistency; it remains to be shown that inconsistency with respect to incomplete decoding occurs with

recurrent language models encountered in practice. Moreover, while the proposed consistent sampling methods and self-terminating recurrent language model carry theoretical guarantees in terms of consistency, we must check whether they retain language modeling quality. To do so, we perform two experiments using a sequence completion task. In each experiment, we use the beginning of a sequence as context, then decode continuations from a trained recurrent language model and measure the proportion of non-terminated sequences in order to approximately measure inconsistency. The first experiment (§5.1) shows that inconsistency occurs in practice, and the second experiment (§5.2) shows the effectiveness of the proposed approaches.

Sequence completion. We evaluate recurrent language models on a sequence completion task, which has previously been used to evaluate the effectiveness of sequence models, e.g. Sutskever et al. (2011); Graves (2013); Radford et al. (2018); Holtzman et al. (2019); Welleck et al. (2019). Sequence completion is a general setting for studying the behavior of language models, encompassing machine translation (Bahdanau et al., 2015), story generation (Fan et al., 2018), and dialogue modeling (Vinyals et al., 2015). The task consists of decoding a continuation $\hat{Y} \sim \mathcal{F}(p_\theta, C)$ given a length-k prefix $C = (c_1, \ldots, c_k)$, resulting in a completion $(c_1, \ldots, c_k, \hat{y}_1 \ldots, \hat{y}_T)$.

Dataset. We use the Wikitext2 dataset (Merity et al., 2016) consisting of paragraphs from Wikipedia, since it has frequently been used to evaluate language models (Grave et al., 2017; Melis et al., 2018; Merity et al., 2018). We split each paragraph into sentences using Spacy,[3] resulting in roughly 100k sequences (78,274 train, 8,464 valid, 9,708 test). We split each sequence, using the first k tokens as a context and the remaining tokens as a continuation. To ensure that each sequence contains a prefix, we prepend padding tokens to make it length k. Special ⟨bos⟩ and ⟨eos⟩ tokens are then inserted at the beginning and end of every sequence. Our experiments use $k = 10$. We model sequences at the word level with a vocabulary size of 33,182. The average training sequence length is 24 tokens, with a maximum of 137.

Context distribution. We define empirical context distributions with prefixes from the train, valid, and test sets,

$$p_{\text{emp}}(C; \mathcal{D}) = \frac{1}{|\mathcal{D}|} \sum_{n=1}^{|\mathcal{D}|} \mathbb{1}(C = C^{(n)}),$$

where $\mathcal{D} = \{(C^{(n)}, Y^{(n)})\}_{n=1}^N$ is a dataset split.

Evaluation metrics. We use finite sequences to approximately measure the consistency of a model paired with a

decoding algorithm, since decoding an infinite-length sequence is impossible. We use the proportion of decoded continuations that are longer than a predefined limit,

$$r_L = \frac{1}{|\mathcal{D}|} \sum_{n=1}^{|\mathcal{D}|} \mathbb{1}(|\hat{Y}^{(n)}| \geq L),$$

where $\hat{Y}^{(n)} \sim \mathcal{F}(p_\theta, C^{(n)})$ for each context $C^{(n)}$ in $\mathcal{D}$. We call r_L the *non-termination ratio* of the decoding algorithm $\mathcal{F}$ for an underlying model and context distribution. A value of r_L greater than zero means that some sequences did not terminate within L steps. When L is infinity, this implies that the model paired with the decoding algorithm is inconsistent. In practice, we use a finite L that is substantially larger than the maximum training sequence length, and we interpret a non-zero r_L as evidence that the model paired with the decoding algorithm is inconsistent. We use $L = 1500$, which is more than 10 times the maximum training sequence length.

In each experiment, we report the mean and standard deviation of metrics across 10 independent initializations. Unless specified otherwise, we report metrics using the test context distribution, since the train, valid, and randomly generated context distributions had similar results.

Training. We train recurrent language models for sequence completion with maximum likelihood, using the following loss on each sequence $Y = (c_1, \ldots, c_k, y_1, \ldots, y_T)$:

$$\mathcal{L}_{\text{MLE}}(p_\theta, Y) = - \sum_{t=1}^{T} \log p_\theta(y_t \mid y_{<t}, c_1, \ldots, c_k).$$

This amounts to running the full training sequence through a recurrent model and zeroing the loss for the first k tokens, so that the first k steps correspond to learning a g_θ that encodes the context. Each model is trained on a single Nvidia P40 GPU for up to 100 epochs, stopping early when validation perplexity does not decrease for 10 consecutive epochs.

Models. We consider recurrent neural networks with hyperbolic tangent activations ($\tanh$-RNN) (Elman, 1990) and LSTM units (LSTM-RNN) (Hochreiter & Schmidhuber, 1997). We perform an initial hyper-parameter sweep and select the best set of hyper-parameters for each of $\tanh$-RNN and LSTM-RNN based on the validation perplexities.[4] With this best set of hyperparameters, we train each of these models with 10 different initializations. The choice of $\tanh$ and LSTM RNNs implies that all of the recurrent language models that we train are consistent according to Lemma 3.2. Our LSTM models achieve similar test perplexity (91.86 ± 0.4) to those reported in previous work (Merity et al., 2018); see Appendix C for further details.

---

[3] https://spacy.io/

[4] Refer to Appendix C for the hyper-parameter ranges.

Table 1. Non-termination ratio (r_L (%)) of decoded sequences using ancestral sampling and incomplete decoding methods.

	$\tanh$-**RNN**	**LSTM-RNN**
ancestral	0.00 ± 0.0	0.00 ± 0.0
greedy	6.07 ± 5.6	1.03 ± 0.3
beam-2	1.21 ± 0.3	0.07 ± 0.1
beam-4	0.29 ± 0.1	0.00 ± 0.0
topk-2	0.84 ± 0.8	0.00 ± 0.0
topk-4	0.02 ± 0.0	0.00 ± 0.0
nucleus-0.2	2.49 ± 0.2	0.76 ± 0.3
nucleus-0.4	0.32 ± 0.1	0.22 ± 0.1

Table 2. Non-termination ratio (r_L (%)) of decoded sequences using consistent sampling methods.

	$\tanh$-**RNN**	**LSTM-RNN**
consistent topk-2	0.00 ± 0.0	0.00 ± 0.0
consistent topk-4	0.00 ± 0.0	0.00 ± 0.0
consistent nucleus-0.2	0.00 ± 0.0	0.01 ± 0.0
consistent nucleus-0.4	0.00 ± 0.0	0.01 ± 0.0

Additionally, we train self-terminating $\tanh$-RNN and LSTM-RNN variants (Definition 4.3) at various values of ϵ, which controls a lower bound on the termination probability at each step. We use $\sigma(x) = (1 - \epsilon)\text{sigmoid}(x)$. We use the hyper-parameters selected in the preceding grid search.

5.1. Inconsistency of Recurrent Language Models

In this experiment, we demonstrate evidence of inconsistency with incomplete decoding methods (Theorem 3.4).

Table 1 shows non-termination ratios for the recurrent language models using the incomplete decoding algorithms considered in this work, along with ancestral sampling. Decoding with ancestral sampling always resulted in sequences that terminated within L steps, since the induced distribution is the same as that of the consistent model. On the other hand, the non-zero non-termination ratios for the incomplete decoding algorithms suggest inconsistency with respect to each algorithm, providing evidence for Theorem 3.4.

In particular, greedy search, beam search, and nucleus sampling yielded non-terminating sequences with both the $\tanh$ and LSTM RNNs. Using greedy decoding, roughly 6% of all contexts resulted in a non-terminating continuation with the $\tanh$-RNN, and roughly 1% with the LSTM-RNN. Nucleus sampling also produced non-terminating sequences with the $\tanh$-RNN (2.49%, nuc-0.2) and LSTM-RNN (0.76%, nuc-0.2), with the amount of non-termination decreasing as μ increased (see Definition 2.7), likely due to $\langle\text{eos}\rangle$ having a higher chance of being included in V_μ. Top-k sampling resulted in non-terminating sequences with the $\tanh$-RNN, but not with the LSTM, implying that $\langle\text{eos}\rangle$ was

ranked within the top k positions on at least one timestep during each decoding. Beam search produced non-terminating sequences with both the $\tanh$-RNN (beam-2,4) and LSTM-RNN (beam-2) models. This means that $\langle\text{eos}\rangle$ was outside of the top tokens (determined by the beam width) considered at each step, since in our experiments we terminated the beam search when a single beam prefix contained $\langle\text{eos}\rangle$. With the LSTM-RNN, a larger beam width (beam-4) prevented non-termination.

5.2. Consistency of the Proposed Methods

In this experiment, we evaluate the consistent variants of top-k and nucleus sampling (§4.1) as well as the self-terminating recurrent language model (§4.2) in terms of consistency and language modeling quality.

Consistent sampling. Table 2 shows that consistent nucleus and top-k sampling (§4.1) resulted in only terminating sequences, except for a few cases that we attribute to the finite limit L used to measure the non-termination ratio. The example continuations in Table 3 show that the sampling tends to preserve language modeling quality on prefixes that led to termination with the baseline (first row). On prefixes that led to non-termination with the baseline (second & third rows), the quality tends to improve since the continuation now terminates. Since the model's non-$\langle\text{eos}\rangle$ token probabilities at each step are only modified by a multiplicative constant, the sampling process can still enter a repetitive cycle (e.g. when the constant is close to 1), though the cycle is guaranteed to eventually terminate.

Self-terminating RNN. As seen in Table 5, the self-terminating recurrent language models with $\epsilon \in \{10^{-2}, 10^{-3}\}$ are consistent with respect to greedy decoding, at the expense of perplexity compared to the vanilla model. The value of ϵ from Definition 4.3, which controls a lower-bound on termination probability at each step, influences both r_L and perplexity. When ϵ is too large ($\epsilon = 10^{-2}$), perplexity degrades. When ϵ is too small ($\epsilon = 10^{-4}$), the lower-bound grows slowly, so $\langle\text{eos}\rangle$ is not guaranteed to be top-ranked within L steps, and the metrics resemble the baseline's. An ϵ of 10^{-3} balanced consistency and language modeling quality, with a zero non-termination ratio and perplexity within 3 points of the baseline.

For the example decoded sequences in Table 4, generation quality is similar when both the self-terminating and baseline models terminate (first row). For prefixes that led to non-termination with the baseline, the self-terminating variant can yield a finite sequence with reasonable quality (second row). This suggests that some cases of degenerate repetition (Holtzman et al., 2019; Welleck et al., 2019) may be attributed to inconsistency. However, in other cases the self-terminating model enters a repetitive (but finite) cycle

Table 3. Example continuations using nucleus and consistent nucleus ($\mu = 0.4$) sampling with the LSTM-RNN.

Prefix	*He had a guest @-@ starring role on the television*
nucleus	film the website , with whom he wrote to the title of The Englishwoman 's Domestic Magazine . ⟨eos⟩
c-nucleus	film the website , but he did not be a new sequel . ⟨eos⟩
Prefix	*Somewhere between 29 and 67 species are recognised in the*
nucleus	⟨unk⟩ , ⟨unk⟩ , ⟨unk⟩ , ⟨unk⟩ , ⟨unk⟩ , ⟨unk⟩ , ⟨unk⟩ , ⟨unk⟩ , ⟨unk⟩ , ⟨unk⟩ , ⟨unk⟩ , ⟨unk⟩ , ⟨unk⟩ , ⟨unk⟩ , ⟨unk⟩ , ⟨unk⟩ . . .
c-nucleus	⟨unk⟩ , with the exception of an average of 6 @.@ 4 in (1 @.@ 6 mm) . ⟨eos⟩
Prefix	*The Civil War saw more ironclads built by both sides*
nucleus	and towns , including ⟨unk⟩ , the British Empire , ⟨unk⟩ , ⟨unk⟩ , ⟨unk⟩ , ⟨unk⟩ , ⟨unk⟩ , ⟨unk⟩ , ⟨unk⟩ , ⟨unk⟩ , ⟨unk⟩ . . .
c-nucleus	and towns , including ⟨unk⟩ , the British Empire , ⟨unk⟩ , ⟨unk⟩ , ⟨unk⟩ , ⟨unk⟩ , ⟨unk⟩ ⟨eos⟩

Table 4. Example continuations with the LSTM-RNN and a self-terminating LSTM-RNN ($\epsilon = 10^{-3}$).

Prefix	*With 2 : 45 to go in the game ,*
Baseline	the team was able to gain a first down . ⟨eos⟩
STRLM	the Wolfpack was unable to gain a first down . ⟨eos⟩
Prefix	*As of 2012 , she is a horse riding teacher*
Baseline	, and a ⟨unk⟩ , and a ⟨unk⟩ , and a ⟨unk⟩ , and a ⟨unk⟩ , and a ⟨unk⟩ , and a ⟨unk⟩ , and a ⟨unk⟩ , and a ⟨unk⟩ , and a ⟨unk⟩ , . . .
STRLM	, and a member of the ⟨unk⟩ ⟨unk⟩ . ⟨eos⟩
Prefix	*Nintendo Power said they enjoyed Block Ball and its number*
Baseline	of songs , including the " ⟨unk⟩ " , " ⟨unk⟩ " , " ⟨unk⟩ " , . . . , " Ode to a Nightingale " , " Ode to a Nightingale " , " Ode to a . . .
STRLM	of songs , including the " ⟨unk⟩ " , " ⟨unk⟩ " , . . . {" ⟨unk⟩ " ,}^{45} . . . " ⟨unk⟩ " , " ⟨eos⟩

Table 5. Non-termination ratio (r_L (%)) of greedy-decoded sequences and test perplexity for self-terminating recurrent models.

	ST	ϵ	r_L (%)	**perplexity**
tanh-RNN	✓	10^{-2}	0.00 ± 0.0	150.07 ± 2.7
	✓	10^{-3}	0.00 ± 0.0	138.01 ± 0.6
	✓	10^{-4}	1.04 ± 0.6	138.67 ± 1.8
	✗	–	6.07 ± 5.6	136.57 ± 1.8
LSTM	✓	10^{-2}	0.00 ± 0.0	101.24 ± 0.3
	✓	10^{-3}	0.00 ± 0.0	94.33 ± 0.6
	✓	10^{-4}	0.94 ± 0.5	94.15 ± 0.8
	✗	–	1.03 ± 0.3	91.86 ± 0.4

that resembles the baseline (third row), showing that consistency does not necessarily eliminate degenerate repetition.

6. Future Directions

The methods we proposed in this paper have focused on how to resolve inconsistency from the viewpoint of decoding algorithms or model parameterization. Another approach is to address the issue of inconsistency in the *learning* phase.

One interesting direction is to investigate whether maximum likelihood learning is a cause of inconsistency. Given a training set $\left\{ (C^{(n)}, Y^{(n)}) \right\}_{n=1}^{N}$ drawn from a data distribution, maximum likelihood learning solves:

$$\max_{\theta} \sum_{n=1}^{N} \log p_{\theta}(Y^{(n)} \mid C^{(n)}) + \lambda \Omega(\theta),$$

where $\Omega(\theta)$ is a regularizer and λ is a regularization weight.

Inconsistency may arise from the lack of decoding in solving this optimization problem. Maximum likelihood learning fits the model p_{θ} using the data distribution, whereas a decoded sequence from the trained model follows the distribution $q_{\mathcal{F}}$ induced by a decoding algorithm. Based on this discrepancy, we make a strong conjecture: we cannot be guaranteed to obtain a good consistent sequence generator using maximum likelihood learning and greedy decoding. Sequence-level learning, however, uses a decoding algorithm during training (Ranzato et al., 2016; Bahdanau et al., 2016). We hypothesize that sequence-level learning can result in a good sequence generator that is consistent with respect to incomplete decoding.

7. Conclusion

We extended the notion of consistency of a recurrent language model put forward by Chen et al. (2017) to incorporate a decoding algorithm, and used it to analyze the discrepancy between a model and the distribution induced by a decoding algorithm. We proved that incomplete decoding is inconsistent, and proposed two methods to prevent this: consistent decoding and the self-terminating recurrent language model. Using a sequence completion task, we confirmed that empirical inconsistency occurs in practice, and that each method prevents inconsistency while maintaining the quality of generated sequences. We suspect the absence of decoding in maximum likelihood estimation as a cause behind this inconsistency, and suggest investigating sequence-level learning as an alternative in the future.

Acknowledgements

References

Andor, D., Alberti, C., Weiss, D., Severyn, A., Presta, A., Ganchev, K., Petrov, S., and Collins, M. Globally normalized transition-based neural networks. In *54th Annual Meeting of the Association for Computational Linguistics, ACL 2016 - Long Papers*, volume 4, pp. 2442–2452, 2016. ISBN 9781510827585. doi: 10.18653/v1/p16-1231. URL http://www.tensorflow.org.

Bahdanau, D., Cho, K., and Bengio, Y. Neural machine translation by jointly learning to align and translate. In *3rd International Conference on Learning Representations, ICLR 2015, San Diego, CA, USA, May 7-9, 2015, Conference Track Proceedings*, 2015. URL http://arxiv.org/abs/1409.0473.

Bahdanau, D., Brakel, P., Xu, K., Goyal, A., Lowe, R., Pineau, J., Courville, A., and Bengio, Y. An actor-critic algorithm for sequence prediction, 2016.

Booth, T. L. and Thompson, R. A. Applying probability measures to abstract languages. *IEEE Transactions on Computers*, C-22(5):442–450, May 1973. ISSN 2326-3814. doi: 10.1109/T-C.1973.223746.

Bridle, J. S. Probabilistic interpretation of feedforward classification network outputs, with relationships to statistical pattern recognition. In *Neurocomputing*, pp. 227–236. Springer, 1990.

Chen, Y., Gilroy, S., Maletti, A., May, J., and Knight, K. Recurrent neural networks as weighted language recognizers. *arXiv preprint arXiv:1711.05408*, 2017.

Cho, K., van Merriënboer, B., Bahdanau, D., and Bengio, Y. On the properties of neural machine translation: Encoder–decoder approaches. In *Proceedings of SSST-8, Eighth Workshop on Syntax, Semantics and Structure in Statistical Translation*, pp. 103–111, Doha, Qatar, October 2014. Association for Computational Linguistics. doi: 10.3115/v1/W14-4012. URL https://www.aclweb.org/anthology/W14-4012.

Elman, J. L. Finding structure in time. *Cognitive science*, 14(2):179–211, 1990.

Fan, A., Lewis, M., and Dauphin, Y. Hierarchical neural story generation. *arXiv preprint arXiv:1805.04833*, 2018.

Grave, E., Joulin, A., and Usunier, N. Improving neural language models with a continuous cache. In *5th International Conference on Learning Representations, ICLR 2017 - Conference Track Proceedings*, 2017.

Graves, A. Generating Sequences With Recurrent Neural Networks. 2013. URL http://arxiv.org/abs/1308.0850.

Hochreiter, S. and Schmidhuber, J. Long short-term memory. *Neural Computation*, 9(8):1735–1780, 1997.

Holtzman, A., Buys, J., Forbes, M., and Choi, Y. The curious case of neural text degeneration. *arXiv preprint arXiv:1904.09751*, 2019.

Lafferty, J., McCallum, A., and Pereira, F. C. N. Conditional random fields: Probabilistic models for segmenting and labeling sequence data. *ICML '01 Proceedings of the Eighteenth International Conference on Machine Learning*, 2001. ISSN 1750-2799. doi: 10.1038/nprot.2006.61.

Melis, G., Dyer, C., and Blunsom, P. On the state of the art of evaluation in neural language models. In *6th International Conference on Learning Representations, ICLR 2018 - Conference Track Proceedings*, 2018.

Merity, S., Xiong, C., Bradbury, J., and Socher, R. Pointer sentinel mixture models. *ArXiv*, abs/1609.07843, 2016.

Merity, S., Keskar, N. S., and Socher, R. Regularizing and optimizing LSTM language models. In *6th International Conference on Learning Representations, ICLR 2018 - Conference Track Proceedings*, 2018.

Murray, K. and Chiang, D. Correcting length bias in neural machine translation. In *Proceedings of the Third Conference on Machine Translation: Research Papers*, pp. 212–223, Brussels, Belgium, October 2018. Association for Computational Linguistics. doi: 10.18653/v1/W18-6322. URL https://www.aclweb.org/anthology/W18-6322.

Radford, A., Wu, J., Child, R., Luan, D., Amodei, D., and Sutskever, I. Language Models are Unsupervised Multitask Learners. 2018.

Ranzato, M., Chopra, S., Auli, M., and Zaremba, W. Sequence level training with recurrent neural networks. In *4th International Conference on Learning Representations, ICLR 2016, San Juan, Puerto Rico, May 2-4, 2016, Conference Track Proceedings*, 2016. URL http://arxiv.org/abs/1511.06732.

Sountsov, P. and Sarawagi, S. Length bias in encoder decoder models and a case for global conditioning. In *Proceedings of the 2016 Conference on Empirical Methods in Natural Language Processing*, pp. 1516–1525, Austin,

Texas, November 2016. Association for Computational Linguistics. doi: 10.18653/v1/D16-1158. URL https://www.aclweb.org/anthology/D16-1158.

Stahlberg, F. and Byrne, B. On NMT search errors and model errors: Cat got your tongue? In *Proceedings of the 2019 Conference on Empirical Methods in Natural Language Processing and the 9th International Joint Conference on Natural Language Processing (EMNLP-IJCNLP)*, pp. 3354–3360, Hong Kong, China, November 2019. Association for Computational Linguistics. doi: 10.18653/v1/D19-1331. URL https://www.aclweb.org/anthology/D19-1331.

Sutskever, I., Martens, J., and Hinton, G. Generating text with recurrent neural networks. In *Proceedings of the 28th International Conference on Machine Learning, ICML 2011*, 2011. ISBN 9781450306195.

Vaswani, A., Shazeer, N., Parmar, N., Uszkoreit, J., Jones, L., Gomez, A. N., Kaiser, Ł., and Polosukhin, I. Attention is all you need. In *Advances in Neural Information Processing Systems*, 2017.

Vinyals, O., Quoc, G., and Le, V. A Neural Conversational Model. In *ICML Deep Learning Workshop*, 2015.

Welleck, S., Kulikov, I., Roller, S., Dinan, E., Cho, K., and Weston, J. Neural text generation with unlikelihood training. *arXiv preprint arXiv:1908.04319*, 2019.

Wu, Y., Schuster, M., Chen, Z., Le, Q. V., Norouzi, M., Macherey, W., Krikun, M., Cao, Y., Gao, Q., Macherey, K., Klingner, J., Shah, A., Johnson, M., Liu, X., Kaiser, Ł., Gouws, S., Kato, Y., Kudo, T., Kazawa, H., Stevens, K., Kurian, G., Patil, N., Wang, W., Young, C., Smith, J., Riesa, J., Rudnick, A., Vinyals, O., Corrado, G., Hughes, M., and Dean, J. Google's Neural Machine Translation System: Bridging the Gap between Human and Machine Translation. 2016. URL http://arxiv.org/abs/1609.08144.

A. Proof of Lemmas in Section 3

Lemma 3.1. *If a recurrent language model p_θ is consistent, $p_\theta(|Y| = \infty \,|\, C) = 0$ for any probable context C.*

Proof. Suppose there exists a probable context $\tilde{C}$ such that $p_\theta(|Y| = \infty \,|\, \tilde{C}) > 0$. Then

$$p_\theta(|Y| = \infty) = \mathbb{E}\left[p_\theta(|Y| = \infty \,|\, C)\right]$$
$$\geq p(\tilde{C})p_\theta(|Y| = \infty \,|\, \tilde{C}) > 0,$$

which contradicts the consistency of the model p_θ. $\quad\square$

Lemma 3.2. *A recurrent language model p_θ is consistent if $\|h_t\|_p$ is uniformly bounded for some $p \geq 1$.*

Proof. Let $B > 0$ be an upper bound such that $\|h_t\|_p < B$ for all t. Let q be the conjugate of p satisfying $1/p + 1/q = 1$. Then we have from Hölder's inequality, for all $v \in V$ and t,

$$u_v^\top h_t \leq \|u_v^\top h_t\|_1 \leq \|h_t\|_p \|u_v\|_q < Bu^+,$$

where $u^+ = \max_{v \in V} \|u_v\|_q$. Note that

$$\log \sum_{v \in V} e^{u_v^\top h_t + c_v} \leq \log \left(\max_{v \in V} e^{u_v^\top h_t + c_v} \times |V|\right)$$
$$\leq \max_{v \in V}\{u_v^\top h_t + c_v\} + \log|V|$$
$$< Bu^+ + c^+ + \log|V|,$$

where $c^+ = \max_{v \in V} c_v$. For a given $y_{<t}$ and context C,

$$\log p_\theta(\langle \text{eos} \rangle \,|\, y_{<t}, C)$$
$$= (u_{\langle \text{eos} \rangle}^\top h_t + c_{\langle \text{eos} \rangle}) - \log \sum_{v \in V} e^{u_v^\top h_t + c_v}$$
$$> (-Bu^+ + c_{\langle \text{eos} \rangle}) - (Bu^+ + c^+ + \log|V|) > -\infty,$$

and it follows that $p_\theta(\langle \text{eos} \rangle \,|\, y_{<t}, C) > \xi > 0$ for some strictly positive constant ξ. Then

$$p_\theta(|Y| = \infty) = \lim_{t \to \infty} p_\theta(|Y| > t)$$
$$= \lim_{t \to \infty} \mathbb{E}\left[p_\theta(|Y| > t \,|\, C)\right]$$
$$= \mathbb{E}\left[\lim_{t \to \infty} p_\theta(|Y| > t \,|\, C)\right]$$
$$\leq \mathbb{E}\left[\lim_{t \to \infty} (1 - \xi)^t\right] = 0,$$

and hence p_θ is consistent. $\quad\square$

Lemma 3.3. *A consistent decoding algorithm with respect to a consistent recurrent language model decodes only probable sequences. That is, if $q_{\mathcal{F}}(Y \,|\, C) > 0$, then $p_\theta(Y \,|\, C) > 0$ for any probable context C.*

Proof. Suppose there exists a decoded sequence $\tilde{Y}$ by $\mathcal{F}$ and probable context $\tilde{C}$ such that $q_{\mathcal{F}}(\tilde{Y} \,|\, \tilde{C}) > 0$ but $p_\theta(\tilde{Y} \,|\, \tilde{C}) = 0$. By Remark 2.1, the sequence $\tilde{Y}$ is of infinite length and thus $q_{\mathcal{F}}(|Y| = \infty \,|\, \tilde{C}) \geq q_{\mathcal{F}}(\tilde{Y} \,|\, \tilde{C}) > 0$, which contradicts the consistency of $q_{\mathcal{F}}$ by Lemma 3.1. $\quad\square$

B. Consistency of STRLM

Theorem 4.3. *Beam search with width k, $\mathcal{F}_{beam-k}$, is consistent with respect to any STRLM.*

Proof. Let $S(\rho)$ be the size-k set of sequences kept by $\mathcal{F}_{\text{beam}-k}$ that start with a prefix ρ.

Take $B = -\log 2/\log(1-\epsilon)$ as in the proof of Theorem 4.2. Suppose that there exists at least one prefix $\hat{\rho} \in P_B^{\text{top}}$ which does not end with $\langle \text{eos} \rangle$.

We first want to show that $\hat{\rho}$ induces at most k more steps in beam search with width k, that is, $Y \in S(\hat{\rho})$ implies $|Y| \leq B + k$.

We know from the proof of Theorem 4.2 that an STRLM p_θ satisfies: for any context C and $v \in V \setminus \{\langle \text{eos} \rangle\}$,

$$p_\theta(\langle \text{eos} \rangle \,|\, \hat{\rho}, C) > p_\theta(v \,|\, \hat{\rho}, C).$$

For any subsequence $y = (y_1, \ldots, y_l)$ with $y_1 \neq \langle \text{eos} \rangle$,

$$p_\theta(\hat{\rho} \circ y \,|\, \hat{\rho}, C) = \prod_{i=1}^{l} p_\theta(y_i \,|\, \hat{\rho} \circ y_{<i}, C)$$
$$\leq p_\theta(y_1 \,|\, \hat{\rho}, C)$$
$$< p_\theta(\langle \text{eos} \rangle \,|\, \hat{\rho}, C).$$

Thus, $\hat{\rho} \circ \langle \text{eos} \rangle$ is the most probable sequence among sequences starting with the prefix $\hat{\rho}$, and it follows that $\hat{\rho} \circ \langle \text{eos} \rangle \in S(\hat{\rho})$.

Thus, in $S(\hat{\rho})$, there are $(k-1)$ sequences starting with $\hat{\rho} \circ v$ for $v \in V \setminus \{\langle \text{eos} \rangle\}$. By the same argument, at each step at least one sequence ending with $\langle \text{eos} \rangle$ is added to $S(\hat{\rho})$, and therefore at time step $(B + k)$, k sequences ending with $\langle \text{eos} \rangle$ are in $S(\hat{\rho})$.

Note that the result set S by $\mathcal{F}_{\text{beam}-k}$ (Definition 2.10) satisfies

$$S \subseteq \bigcup_{\rho \in P_B^{\text{top}}} S(\rho).$$

Since each $\rho \in P_B^{\text{top}}$ induces sequences of length at most $B + k$, we have

$$p_\theta(|Y| > B + k \,|\, C) = 0.$$

Taking the expectation over C yields the consistency of the model p_θ. $\quad\square$

C. Additional Details and Results

Additional example continuations. Table 6 shows additional greedy-decoded continuations using a self-terminating LSTM-RNN and the baseline LSTM-RNN.

Table 6. More example continuations from the LSTM-RNN and a self-terminating LSTM-RNN ($\epsilon = 10^{-3}$).

Prefix	*In his " ⟨unk⟩ with Francis ⟨unk⟩ " series of*
Baseline	the song , " The ⟨unk⟩ " , " The ⟨unk⟩ " , " The ⟨unk⟩ " , " The ⟨unk⟩ " , " The Tallest Man " , " The Tallest Man " , " The Tallest Man " , " The Tallest Man " , " The Tallest Man " , " The Tallest Man " , " The Avalanche " , " The Tallest Man " , " The Avalanche " , " The Tallest Man " , " The Avalanche " ...
STRLM	the same name , the song was released in the United States on November 18 , 2003 .
Prefix	*T " as one of the most important songs on*
Baseline	the album , and " The ⟨unk⟩ " , " The ⟨unk⟩ " , " The ⟨unk⟩ " , " The ⟨unk⟩ " , " The Tallest Man " , " The Tallest Man " , " The Tallest Man " , " The Tallest Man " , " The Tallest Man " , " The Tallest Man " , " The Avalanche " , " The Tallest Man " , " The Avalanche " , " The Tallest Man " , " The Avalanche " , " The Tallest Man " ...
STRLM	the album , the song was released on the Billboard Hot 100 , and was certified gold by the Recording Industry Association of America (RIAA) .
Prefix	*Pound , who believed that the " Hellenic hardness "*
Baseline	of the poem was " ⟨unk⟩ " , and " ⟨unk⟩ " , " ⟨unk⟩ " , " ⟨unk⟩ " , " ⟨unk⟩ " , " ⟨unk⟩ " , " ⟨unk⟩ " , " ⟨unk⟩ " , " ⟨unk⟩ " , " ⟨unk⟩ " , " ⟨unk⟩ " , " ⟨unk⟩ " , " ⟨unk⟩ " , " ⟨unk⟩ " , " ⟨unk⟩ " , " ⟨unk⟩ " , " ⟨unk⟩ " , " ⟨unk⟩ " , " ⟨unk⟩ " , " " ⟨unk⟩ " , " ⟨unk⟩ " , " " ⟨unk⟩ " , " " ⟨unk⟩ " , " " ⟨unk⟩ " , " " ⟨unk⟩ " , ...
STRLM	was " a ⟨unk⟩ of the ⟨unk⟩ of the ⟨unk⟩ " .
Prefix	*Due to industrial waste and automobiles , Manila suffers from*
Baseline	the ⟨unk⟩ ⟨unk⟩ , ...
STRLM	the ⟨unk⟩ of the ⟨unk⟩ , and the ⟨unk⟩ of the ⟨unk⟩ ⟨unk⟩ , the ⟨unk⟩ of the ⟨unk⟩ , and the ⟨unk⟩ of the ⟨unk⟩ .

Table 7. Grid search specification. The values selected for the **LSTM-RNN** and tanh-*RNN* models are shown in bold and italics, respectively.

Parameter	Values
Hidden Size	{*256*, **512**, 1024}
Dropout	{0.1, *0.3*, **0.5**}
Embedding Weight Tying	{***True***, False}

Table 8. Perplexities of trained recurrent language models.

model	context	perplexity
tanh-RNN	train	91.54 ± 7.9
tanh-RNN	test	136.57 ± 1.8
LSTM-RNN	train	45.80 ± 2.5
LSTM-RNN	test	91.86 ± 0.4

Hyper-parameters. Table 7 shows the grid search specification. All models were 2 layers and were trained with the Adam optimizer.

Model perplexities. Table 8 shows train and test perplexities for the tanh-RNN and LSTM-RNN models.