Matrix Inversion Lemma

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Contents

1 $(I + UV)^{-1}$

$I \in \mathbb{R}^{n\times n}$, $U \in \mathbb{R}^{n\times m}$, $V \in \mathbb{R}^{m\times n}$.

$$\boxed{(I_n + UV)^{-1} = I_n - U(I_m + VU)^{-1}V} \tag{1}$$

1.1 Proof.

$$A = I_n + UV \tag{2}$$

$$VA = V + VUV \tag{3}$$

$$VA = (I_m + VU)V \tag{4}$$

$$(I_m + VU)^{-1}VA = V \tag{5}$$

$$U(I_m + VU)^{-1}VA = UV \tag{6}$$

$$U(I_m + VU)^{-1}VA = A - I_n \tag{7}$$

$$U(I_m + VU)^{-1}V = I_n - A^{-1} \tag{8}$$

$$A^{-1} = I_n - U(I_m + VU)^{-1}V \tag{9}$$

2 $(R + PQP^T)^{-1}$

$R \in \mathbb{R}^{n\times n}$ invertible, $P \in \mathbb{R}^{n\times m}$, $Q \in \mathbb{R}^{m\times m}$ invertible.

$$\boxed{(R + PQP^T)^{-1} = R^{-1} - R^{-1}P(Q^{-1} + P^TR^{-1}P)^{-1}P^TR^{-1}} \tag{10}$$

2.1 Proof 1

$$A = R + PQP^T \tag{11}$$

We want to find a formulae for A^{-1}.

$$R^{-1}A = I + R^{-1}PQP^T \tag{12}$$

$$P^T R^{-1}A = P^T + P^T R^{-1}PQP^T \tag{13}$$

$$P^T R^{-1}A = (I + P^T R^{-1}PQ)P^T \tag{14}$$

$$P^T R^{-1}A = (Q^{-1} + P^T R^{-1}P)QP^T \tag{15}$$

$$(Q^{-1} + P^T R^{-1}P)^{-1}P^T R^{-1}A = QP^T \tag{16}$$

$$P(Q^{-1} + P^T R^{-1}P)^{-1}P^T R^{-1}A = PQP^T \equiv A - R \tag{17}$$

$$P(Q^{-1} + P^T R^{-1}P)^{-1}P^T R^{-1}A = A - R \tag{18}$$

$$R = A - P(Q^{-1} + P^T R^{-1}P)^{-1}P^T R^{-1}A \tag{19}$$

$$R = (I - P(Q^{-1} + P^T R^{-1}P)^{-1}P^T R^{-1})A \tag{20}$$

$$RA^{-1} = I - P(Q^{-1} + P^T R^{-1}P)^{-1}P^T R^{-1} \tag{21}$$

$$A^{-1} = R^{-1} - R^{-1}P(Q^{-1} + P^T R^{-1}P)^{-1}P^T R^{-1} \tag{22}$$

2.2 Proof 2

Here we use (1).

$$A = R + PQP^T = R(I + R^{-1}PQP^T) \tag{23}$$

$$A^{-1} = (I + R^{-1}PQP^T)^{-1}R^{-1} \tag{24}$$

With $U = R^{-1}P$ and $V = QP^T$, we can write:

$$A^{-1} = (I - R^{-1}P(I + QP^TR^{-1}P)^{-1}QP^T)R^{-1} \tag{25}$$

$$A^{-1} = (I - R^{-1}P(Q^{-1} + P^TR^{-1}P)^{-1}P^T)R^{-1} \tag{26}$$

$$A^{-1} = R^{-1} - R^{-1}P(Q^{-1} + P^TR^{-1}P)^{-1}P^TR^{-1} \tag{27}$$