Least Squares

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1 Direct measurements

1.1 Theory

We want to estimate $\bar{x}$. The measurements are y_i ($i \in \{1, 2, \ldots, k\}$), with uncertainties R_i. So y_i is normally distributed with mean $\bar{x}$, covariance R_i.

$$\begin{aligned} y_i &= \bar{x} + v_i \\ v_i &\sim N(0, R_i) \end{aligned} \tag{1}$$

v_i are the error vectors. The energy term to minimize:

$$E(x) = \sum_{i=1}^{k} (x - y_i)^T R_i^{-1} (x - y_i) \tag{2}$$

Let's calculate the derivative of $E(x)$ w.r.t. x:

$$\frac{\partial E}{\partial x} = \sum_{i=1}^{k} 2(x - y_i)^T R_i^{-1} \tag{3}$$

Setting the derivative to zero, and solving for x:

$$\boxed{\hat{x} = \left(\sum_{i=1}^{k} R_i^{-1} \right)^{-1} \cdot \sum_{i=1}^{k} R_i^{-1} y_i} \tag{4}$$

The variance of the estimation (assuming that the measurements are independent) can be easily calculated, the result is:

$$\boxed{\text{Var}(\hat{x}) = \left(\sum_{i=1}^{k} R_i^{-1} \right)^{-1}} \tag{5}$$

$$\hat{x} \tag{6}$$

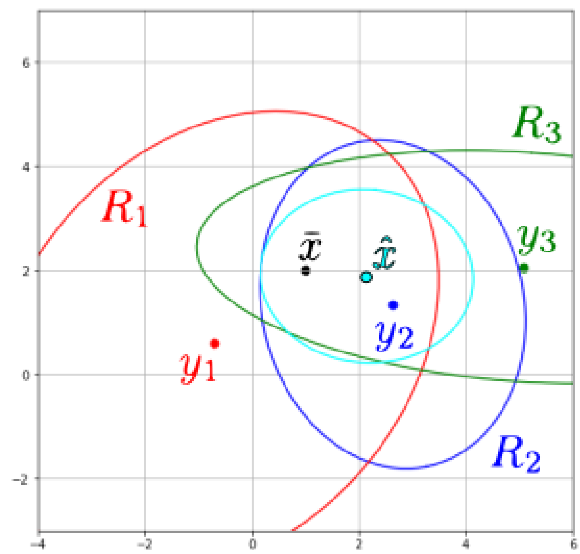

Figure 1: Example. We have 3 measurements y_1, y_2 and y_3 with the corresponding covariance matrices R_i. $\hat{x}$ is the maximum likelihood solution. Note that the uncertainty of $\hat{x}$ is less than any of the measurements.

1.2 Example

Figure 1 shows an example where we have 3 measurements for estimating $\bar{x}$.