Grade 11 Quad 4 Math Notes

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Contents

Chapter 1

Course 1 - MCR3U7

1.1 Review Unit

1.1.1 Quadratics Review

Sum and Product of Roots of Quadratic

For a quadratic equation, $ax^2 + bx + c = 0$ (where $a \neq 0$), let r_1, r_2 be the roots. Then,

$$r_1 + r_2 = -\frac{b}{a} \tag{1.1.1}$$

and

$$r_1 r_2 = \frac{c}{a} \tag{1.1.2}$$

These can be proven by writing r_1 and r_2 in terms of a,b,c by using the quadratic formula. Relation to equation:

$$\begin{aligned} ax^2 + bx + c &= 0 \\ \frac{ax^2 + bx + c}{a} &= 0 \\ x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\ x^2 - \left(-\frac{b}{a}\right)x + \frac{c}{a} &= 0 \\ x^2 - \mathrm{S}x + \mathrm{P} &= 0 \end{aligned}$$

where S is the sum of roots and P is the product of roots.

Sum and Product of Polynomial Roots

In the general case, for a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$,

$$S = -\frac{a_{n-1}}{a_n} \tag{1.1.3}$$

$$P = (-1)^n \frac{a_0}{a_n} \tag{1.1.4}$$

where S is the sum and P is the product of the roots.

These can be derived by comparing the coefficients of the following two forms of $P(x)$:

$$\begin{aligned} P(x) &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \\ P(x) &= a(x - r_1)(x - r_2) \ \cdots (x - r_n) \end{aligned}$$

1.1.2 Complex Numbers Review

For a complex number z,

- Rectangular form: $z = a + bi$, where $a, b \in \mathbb{R}$
- Conjuagate: z^* or $\bar{z} = a - bi$
- Modulus: $|z| = \sqrt{a^2 + b^2}$

Also, $z \times \bar{z} = |z|^2 = a^2 + b^2$

1.1.3 Factor Theorem & Remainder Theorem Review

Notation:

$$\begin{aligned} P(x) &\leftarrow \text{polynomial} \\ D(x) &\leftarrow \text{divisor} \\ Q(x) &\leftarrow \text{quotient} \\ R(x) &\leftarrow \text{remainder} \end{aligned}$$

Division statement (2 ways):

$$P(x) = D(x)Q(x) + R(x) \tag{1.1.5}$$

or

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)} \tag{1.1.6}$$

Note that in both forms, $D(x) \neq 0$.

Factor Theorem

- When $P(x)$ is divided by $x - b$ the remainder is $P(b)$. If $P(b) = 0$, then $(x - b)|P(x)$.
- When $P(x)$ is divided by $ax - b$ the remainder is $P(\frac{b}{a})$. If $P(\frac{b}{a}) = 0$, then $(ax - b)|P(x)$.

1.1.4 Transformation of a Function Review

General form of a transformed function:

$$f(x) = a\,[\,b\,(x - c)\,] + d \tag{1.1.7}$$

$a > 0$	no reflection
$a < 0$	reflection in x-axis
$a > 1$	vertical stretch by a factor of a
$0 < a < 1$	vertical compression by a factor of $\frac{1}{a}$
$b < 0$	reflection in y-axis
$b > 1$	horizontal compression by a factor of b
$0 < b < 1$	horizontal stretch by a factor of $\frac{1}{b}$
$c > 0$	horizontal translation c units to the right
$c < 0$	horizontal translation c units to the left
$d > 0$	vertical translation d units up
$d < 0$	vertical translation d units down

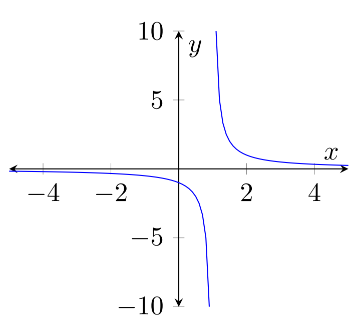
1.2 Rational Functions

- v.a. means vertical asymptote
- h.a. means horizontal asymptote

1.2.1 Reciprocal of a Linear Function

$$y = \frac{a}{kx - c}$$

v.a.	$x = \frac{c}{k}$
h.a.	$y = 0$
y-int	$(0, \frac{a}{c})$

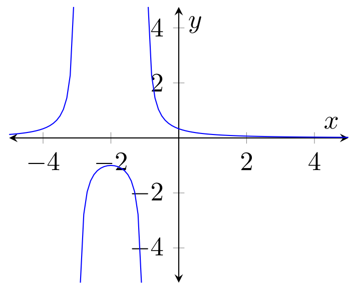


1.2.2 Reciprocal of a Quadratic

Two real roots

$$y = \frac{a}{(x - r)(x - s)}$$

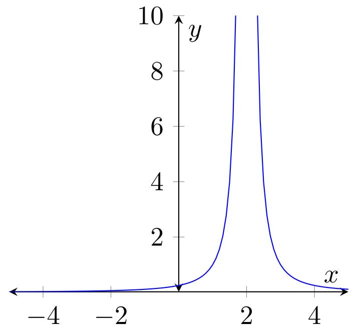
v.a.:	$x = r$ and $x = s$
h.a.:	$y = 0$
local max at	$x = \frac{r+s}{2}$ (same as the parabola's vertex)



One real root

$$y = \frac{a}{(x-r)^2}$$

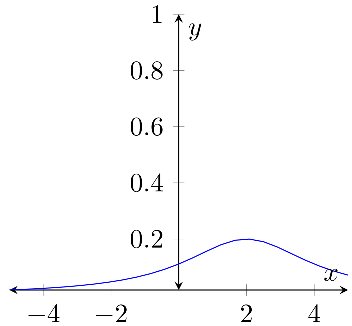
v.a.:	$x = r$
h.a.:	$y = 0$



No real roots

$$y = \frac{a}{(x-r)^2 + b}$$

h.a.:	$y = 0$
local max at	$x = \frac{r+s}{2}$ (same as the parabola's vertex)



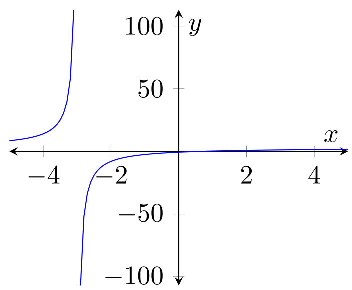
1.2.3 Linear divided by Linear

$$y = \frac{ax+b}{cx+d}$$

v.a.: $x = -\frac{d}{c}$
h.a.: $y = \frac{a}{c}$

y-int: $\left(0, \frac{b}{d}\right)$

x-int: $\left(-\frac{b}{a}, 0\right)$



Remember to label h.a. and v.a. when graphing.
Also label x-axis and y-axis.

1.2.4 General Rational Functions

Given two ploynomial functions, $P(x)$ of degree p, and $Q(x)$ of degree q, the function $\frac{P(x)}{Q(x)}$:

- has a horizontal asymptote $y = 0$ if $q > p$
- has a horizontal asymptote $y = k$ if $q = p$
  - where k is found by dividing the leading cofficients
- has an oblique asymptote if $p > q$ and that asymptote has order $p - q$.

1.2.5 Rational Equations and Inequalities

Equations

Method:

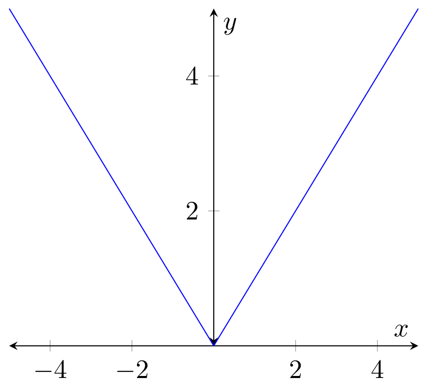
1. Note the restrictions on x (values that make the denominator are 0.
2. Make both LHS and RHS into 1 fraction each.
3. Cross-multiply, expand, simplify, and solve.
4. Check that the solutions are valid from step 1.

Inequalities

Method:

1. Bring all the terms to the left side, forming a rational function.
2. Use a factor table to find where the rational function is positive or negative.
3. Account for vertical asymptotes. The LHS can be zero at x-intercepts but not at vertical asymptotes.

1.3 Absolute Value Function



Note that:

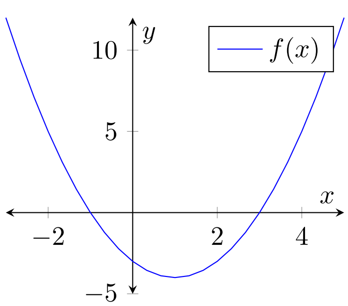
$$|x| = \sqrt{x^2} \tag{1.3.1}$$

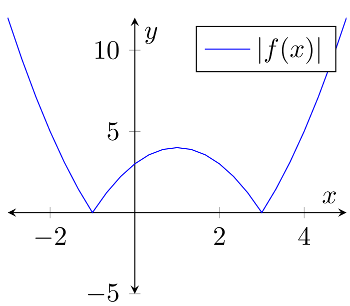
Properties of the Absolute Value Function

- $|ab| = |a| \times |b|$
- $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ (if $b \neq 0$)

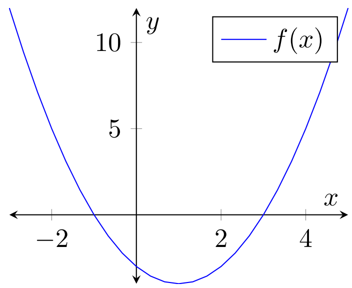
1.3.1 $|f(x)|$ and $f(|x|)$

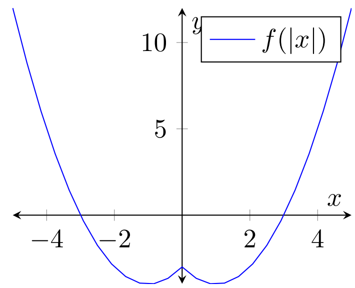
For $|f(x)|$, simply fold the parts of $f(x)$ that are below the x-axis over the x-axis. Note that this may create cusps in the graph.





For $f(|x|)$, erase the part of the graph to the left of the y-axis, and reflect the remaining part across the y-axis. For example,





1.3.2 Absolute Value Equations and Inequalities

Equations Examine each interval seperately (use cases). For example, if the equation is

$$|3x - 4| + 4x^2 - 2 = |5x + 1|$$

the cases would be $x \in (-\infty, -\frac{1}{5}), [-\frac{1}{5}, \frac{4}{3}), [\frac{4}{3}, \infty)$. Remember to include the boundary values of x only once ($-\frac{1}{5}$ and $\frac{4}{3}$).

Each term in absolute value brackets will be positive or negative depending on the interval.

Remember to reject values of x which fall outside the domain being considered in that case.

Note: Another method is to square both sides to remove absolute values by using this identity (1.3.1). However, that method may introduce extraneous solutions because squaring is a non-reversible operation. If using that method, remember to verify all solutions.

Inequalities For simple inequalities, this rule may suffice

If $|f(x)| < a$, where $a > 0$, then $-a < x < a$
If $|f(x)| > a$, where $a > 0$, then $x < -a$ or $x > a$

For more complex ones, examine each interval seperately like you do when solving absolute value equalities. Remember to verify that the solution is inside the domain of that case. If the solution partially overlaps with the domain, take the intersection of the two intervals.

1.4 Inverse and Composite Functions

$f \circ g(x)$ is read as "f compose g of x". It is synonymous to $f(g(x))$.

Note that $f \circ g(x) \neq g \circ f(x)$. (composition is not always commutative)

But $f \circ f^{-1} = f^{-1} \circ f(x) = x$.

Self inverse function: $f(x) = f^{-1}(x)$.

Given $f(x)$, to find the intersection of $f(x)$ and $f^{-1}(x)$ without solving for $f^{-1}(x)$, you simply find the intersection of $f(x)$ and $y = x$. This works because $f^{-1}(x)$ is a reflection of $f(x)$ across the line $y = x$.

1.4.1 Techniques to Solve Function Composition Problems

There are 2 common types of function composition problems:

1. Given $(f \circ g)(x) = \dots$ and $f(x) = \dots$, find $g(x)$
2. Given $(f \circ g)(x) = \dots$ and $g(x) = \dots$, find $f(x)$

If you are given the outer function (type 1), plug in $g(x)$ into $(f \circ g)(x)$, and solve for $g(x)$.

If you are given the inner function (type 2), first find $g^{-1}(x)$ and then plug in $g^{-1}(x)$ into $(f \circ g)(x)$. This will give,

$$f(g(g^{-1}(x))) = g^{-1}(x)^2 + 4g^{-1}(x) + \cdots = f(x)$$

and you will have found an expression for $f(x)$.

1.5 Reciprocal and Square of a Function

Think of squaring and reciprocating $f(x)$ as applying a transformation onto $f(x)$.

1.5.1 Square of a function

Given $f(x)$, graph $[f(x)]^2$.

1. Invariant points have $y = 0, 1$. Also, points which have a y-coordinate of -1 are reflected across the x-axis.
2. The horizontal asymptote is squared.
3. For, $f(x) > 0$ it seems like a vertical stretch. For $f(x) < 0$, reflect across y-axis, then stretch.
4. Sharp points in the graph of $f(x)$, such as those in $y = |x|$, may become cusps.

The horizontal asymptote is transformed, but the vertical asymptote remains the same.

If $f(x)$ is approximately a line at its x-intercept, $[f(x)]^2$ has a curve which touches the x-axis at that point. This is because the square of a linear expression is a quadratic (parabola).

In general, any part of $f(x)$ that is a line becomes a parabola.

1.5.2 Reciprocal of a function

1. Invariant points have $y = -1, 1$.
2. Vertical asymptotes become x-intercepts.
3. x-intercepts become vertical asymptotes.
4. Horizontal asymptotes are reciprocated.
5. If $f(x) > 0$, $\frac{1}{f(x)} > 0$
6. If $f(x) < 0$, $\frac{1}{f(x)} < 0$
7. If $f(x) > 1$, $\frac{1}{f(x)} < 1$
8. If $f(x) < 1$, $\frac{1}{f(x)} > 1$

Examine each branch individually and reason out how the reciprocal would look like.

1.6 Exponential and Logarithmic Functions

Exponent terminology

$$\sqrt[n]{x}$$

$$\begin{array}{ll} n & \text{is called the index} \\ x & \text{is called the radicand} \end{array}$$

Also in the exponential form:

$$a^{\frac{b}{c}}$$

$$\begin{array}{ll} b & \text{is the exponent} \\ c & \text{is the index} \end{array}$$

1.6.1 Exponential Functions

$$f(x) = a^x$$

Where $a > 0$, $a \neq 1$.

1.6.2 Logarithmic Functions

$$f(x) = \log_b x$$

Where $x > 0$, $b > 0$, $b \neq 1$.

$y = \log_a x$ is the inverse of $y = a^x$.

Power Law

$$\log_{b^m}(x^n) = \frac{n}{m} \times \log_b x \tag{1.6.1}$$

1.6.3 Expontential Growth and Decay

Growth

$$N(t) = N_0(2)^{\frac{t}{h}} \tag{1.6.2}$$

Decay

$$N(t) = N_0\left(\frac{1}{2}\right)^{\frac{t}{h}} \tag{1.6.3}$$

$$= N_0(2)^{-\frac{t}{h}} \tag{1.6.4}$$

$N(t)$	is final amount
N_0	is initial amount
t	is time
h	is half-life (same units as t)

Both (1.6.3) and (1.6.4) can be used to solve problems.

General Form

$$N(t) = N_0 e^{\frac{k}{t}} \tag{1.6.5}$$

where k is the exponential growth/decay constant of the specific situation. k can be calculated if you are given the half-life or doubling time (or even the time it takes to reach some fraction/multiple of the original amount.

If $k > 0$	it's exponential growth
If $k < 0$	it's exponential decay

1.6.4 Compound Interest

Non-continuous

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \tag{1.6.6}$$

A	is the final amount
P	is the initial amount
r	is the interest rate (in decimal form)
n	is the compounding period
t	is the time (years)

n refers to how frequently the interest is paid. For example:

monthly	means $n = 12$
quarterly	means $n = 4$
semi-annually	means $n = 2$
annually	means $n = 1$

Continous Compounding

$$A = Pe^{rt} \tag{1.6.7}$$

Use (1.6.7) when the interest compounds continually.

1.7 Sequences and Series

1.7.1 Sequences

A sequence is a function whose domain is a subset of the set of natural numbers ($\mathbb{N}$). The values in the range are called the terms of the sequence.

Ex. $20, 15, 10, 5, \dots, -30$

x	y
1	20
2	15
3	10
4	10

Arithmetic Sequences

A sequence is arithmetic if the difference between the terms is constant. The general term of an arithmetic sequence is

$$t_n = a + (n-1)d \tag{1.7.1}$$

where a is the first term
d is the common difference ($d = t_n - t_{n-1}$)
n is the number of terms

Geometric Sequences

A sequence is geometric if the *ratio* between the terms is constant. The general term of a geometric sequence is

$$t_n = ar^{n-1} \tag{1.7.2}$$

where a is the first term
r is the common ratio $\left(r = \frac{t_n}{t_{n-1}}\right)$
n is the number of terms

Arithmetic Means

If a question says "insert n arithmetic means between a and b", where n,a,b are given, you need to find n numbers between a and b which form an arithmetic sequence along with a and b.

Geometric Means

If a question says "insert n geometric means between a and b", where n,a,b are given, you need to find n numbers between a and b which form a geometric sequence along with a and b.

Simple Interest

Unlike compound interest (1.6.4), *simple* interest means you only receive interest on the original amount. For example, a simple interest of 6% per annum on an investment of \$200, means the amount will be

$$\$200,\ \$212,\ \$224,\ \$236,\ \ldots$$

which makes it an arithmetic series.

1.7.2 Series

A series is the sum of the terms of a sequence. Given a sequence $t_1, t_2, t_2, \ldots, t_n$, S_n denotes the sum of the first n terms. This is called a partial sum.

Arithmetic Series

$$S_n = \frac{n}{2}(a + t_n) \tag{1.7.3}$$

$$S_n = \frac{n}{2}(2a + (n-1)d) \tag{1.7.4}$$

Proof.

$$\begin{aligned} S_n &= t_1 + t_2 \quad + t_3 \quad + \cdots + t_{n-1} + t_n \\ S_n &= t_n + t_{n-1} + t_{n-2} + \cdots + t_2 \quad + t_1 \end{aligned}$$

Writing both of these in terms of a and d gives

$$\begin{aligned} S_n &= a \qquad\qquad + a + d \qquad\quad + \cdots + a + (n-2)d + a + (n-1)d \\ S_n &= a + (n-1)d + a + (n-2)d + \cdots + a + d \qquad\quad + a \end{aligned}$$

Adding the two,

$$2S_n = 2a + (n-1)d + 2a + (n-1)d + \cdots + 2a + (n-1)d + 2a + (n-1)d \tag{1.7.5}$$

$$\begin{aligned} 2S_n &= n(2a + (n-1)d) \\ S_n &= \frac{n}{2}(2a + (n-1)d) \end{aligned}$$

which gives (1.7.4). To get (1.7.3) we can continue,

$$\begin{aligned} S_n &= \frac{n}{2}(a + (a + (n-1)d)) \\ S_n &= \frac{n}{2}(a + t_n) \end{aligned}$$

Geometric Series

$$S_n = \frac{a(1-r^n)}{1-r}, \qquad r \neq 1 \tag{1.7.6}$$

$$S_n = \frac{a(r^n-1)}{r-1}, \qquad r \neq 1 \tag{1.7.7}$$

Proof.

$$\begin{aligned}
S_n &= a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} \\
rS_n &= ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} + ar^n \\
S_n - rS_n &= a - ar^n \\
S_n(1-r) &= a(1-r^n) \\
S_n &= \frac{a(1-r^n)}{1-r}
\end{aligned}$$

which gives (1.7.6).

Sum of infinite geometric series:

$$S_\infty = \frac{a}{1-r} \tag{1.7.8}$$

A useful equation to solve problems is

$$t_n = S_n - S_{n-1} \tag{1.7.9}$$

(1.7.9) is useful when you are given an expression for S_n and asked to find one of the terms of t.

Sigma notation

Properties:

$$\sum_{i=1}^{n} c = cn \tag{1.7.10}$$

$$\sum_{i=1}^{n} c\,t_i = c\sum_{i=1}^{n} t_i \tag{1.7.11}$$

$$\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i \tag{1.7.12}$$

$$\sum_{i=1}^{n} t_i = \sum_{i=1}^{c} t_i + \sum_{i=c+1}^{n} t_i, \quad 1 < c < n \tag{1.7.13}$$

1.8 Trig Review

$$\sin^{-1} \neq csc$$

Because $\sin^{-1}$ denotes the inverse function of sin, which is arcsin. This is similar to how $f^{-1}(x)$ denotes the inverse function of f.

But note that $\sin^2(x) = [\sin(x)]^2$, or $\sin^3(x) = [\sin(x)]^3$, etc.

1.8.1 Terminology

Standard form An angle in standard form has one arm on the positive x-axis, called the initial arm, and the other arm anywhere else, called the terminal arm.

Principal angle The counter clockwise angle between the initial and terminal arm of an angle in standard position.

Co-terminal angles are angles whose terminal arms have the same standard position. If α and θ are co-terminal angles,

$$\alpha - \theta = k \times 360° \quad \text{where } k \in \mathbb{Z} \tag{1.8.1}$$

or in other words

$$\alpha = k \times 360° + \theta \quad \text{where } k \in \mathbb{Z} \tag{1.8.2}$$

So to find a co-terminal angle of θ you can simply add or subtract multiples of 360°.

Related Acute Angle (RAA) a.k.a. **Reference Angle (RA)** is the positive acute angle between the terminal arm and x-axis of an angle in standard position.

Angle of Elevation/Depression in word problems, the angle of elevation or depression is the angle made from the horizontal to the line of sight.

1.8.2 Method to Evaluate a Trig Ratio

First find the RAA of θ. Let's call it α. Then calculate $\sin\alpha$. Then use the CAST rule to determine the sign of $\sin\theta$ and apply it to $\sin\alpha$.

1.8.3 Circle Stuff

Radians

$$1 \text{ rad } = \frac{180°}{\pi} \tag{1.8.3}$$

$$1° = \frac{\pi}{180°} \text{ radians} \tag{1.8.4}$$

Sectors and Segments

Arc length, a

$$a = r\theta \tag{1.8.5}$$

Area of a sector, A

$$A = \frac{r^2}{2}\theta \tag{1.8.6}$$

Area of a segment, A

$$A = \frac{r^2}{2}(\theta - \sin\theta) \tag{1.8.7}$$

Note that θ is in radians.

1.8.4 Compass Bearings

Compass Bearing an angle measured from a specified direction. For example, South 30° East (or S30°E) means an angle 30° East from the South direction.

True Bearing like a compass bearing but always measured from North. For example, the true bearing 150°T is the same as the compass bearing S30°E. Also, if the angle is less than 100°, prepend with a zero. For example: 040°T.

1.8.5 Graphing Trig Functions

General form

$$y = A\sin[k(x-a)] + c \tag{1.8.8}$$

A	is amplitude (vertical stretch) $\left(\frac{\text{max}-\text{min}}{2}\right)$
$y = c$	is the equation of the sinusoidal axis (the midline) $\left(\frac{\text{max}+\text{min}}{2}\right)$
a	is phase shift
$\frac{2\pi}{k}$	is the period

Important Increment (II): $\frac{\text{period}}{4}$

Num of II	1	2	3	4
sin	axis	max	axis	min
cos	max	axis	min	axis

Scaling Increments (SI) When graphing, the smallest tick on the x-axis should divide the phase shift and II. Let's call the smallest tick length the Scaling Increment, or SI. The SI is used as the smallest tick mark when graphing.

$$\text{An easy way to find the SI for: } \left(\frac{\pi}{a}, \frac{\pi}{b}\right) \text{ is } \frac{\pi}{\text{lcm}(a,b)}$$

1.8.6 Steps to Graph

1. Find max $= c + a$ and min $= c - a$
2. Determine period $= \frac{2\pi}{k}$
3. Calculate II $= \frac{\text{period}}{4}$
4. Calculate Scaling Increment
5. Graph using the pattern for sin or cos. Remember to account for reflections and phase shift.

Finding the Equation of a Trigonometric Function from its Graph

Steps for determining the equation of a sinusoidal function from its graph

1. Find the maximum and minimum values

2. Use the formula $a = \frac{max-min}{2}$ and $c = \frac{max+min}{2}$ to determine the amplitude and the vertical displacement (which is also the equation of the axis of the curve). On the graph, draw a **dashed line** to represent the equation of the axis.

3. Highlight one complete cycle that is closest to the y-axis.
   - To write your equation in terms of **cosine**: Use two consecutive maximum values of the function as the starting point and end point for your cycle
   - To write your equation in terms of **sine**: Use two values on the axis of the curve of the function as a starting point and end point for your cycle

4. Using your highlighted cycle, find the period of your function and use the period to find the k value
   - $New\ period = \frac{Original\ period}{|k|}$ so $|k| = \frac{2\pi}{p}$ for $sin(x)$ and $cos(x)$, and $|k| = \frac{\pi}{k}$ for tan(x)

5. Find the phase shift of your function by determining the horizontal distance of the beginning of your highlighted cycle from the y-axis.

6. Incorporate all of the transformations into the equation $y = a\mathbf{sin}[k(x-d)] + c$ or $y = a\mathbf{cos}[k(x-d)] + c$

7. To convert from sine to cosine: phase shift by $-\frac{\pi}{2k}$ in the (x-d) bracket (i.e. add ~~subtract~~ a quarter of the period from x).

   To convert from cosine to sine phase shift by $+\frac{\pi}{2k}$ in the (x-d) bracket (i.e. subtract ~~add~~ a quarter of the period to x).

1.9 General Solutions to Trig Problems

Solutions come in multiples of the period, so the final answer would look like:

$$\theta = \frac{7\pi}{6} + k\pi \quad \text{or} \quad \theta = \frac{\pi}{6} + k\pi, \qquad k \in \mathbb{Z}$$

If a domain is given, find all the solutions in the domain, otherwise, write down the general solution.

1.9.1 Inverse Trig Functions

$\sin^{-1} x$ Domain: $-1 \leq x \leq 1$
Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$\cos^{-1} x$ Domain: $-1 \leq x \leq 1$
Range: $0 \leq y \leq \pi$

$\tan^{-1} x$ Domain: $-1 \leq x \leq 1$
Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

The following equations hold only when x is in the range of $\sin^{-1}$ or $\cos^{-1}$.

$$\sin^{-1}(\sin x) = x$$
$$\cos^{-1}(\cos x) = x$$
$$\tan^{-1}(\tan x) = x$$

Note that if the inverse function is on the inside it always equals x. For example $\sin(\sin^{-1} x)$ for all x such that $-1 \leq x \leq 1$.

1.9.2 Identities

These are also called Pythagorean Identities

$$1 + \tan^2 x = \sec^2 x \tag{1.9.1}$$
$$1 + \cot^2 x = \csc^2 x \tag{1.9.2}$$

1.10 Complex Numbers

Cartesian (or Rectangular Form)

$$z = a + bi$$

Polar form

$$z = r \operatorname{cis}(\theta + 2\pi k)$$

r is the modulus $\left(r = \sqrt{a^2 + b^2}\right)$
θ is the argument

1.10.1 Adding and Subtracting in Polar Form

Polar form gives you nothing special for adding/subtracting. Convert to cartesian form then add/subtract.

1.10.2 Multiplying in Polar Form

$$\begin{aligned}(r_1 \operatorname{cis}\theta_1)(r_2 \operatorname{cis}\theta_2) &= [r_1(\cos\theta_1 + i\sin\theta_1)][r_2(\cos\theta_2 + i\sin\theta_2)] \\ &= r_1r_2[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)] \\ &= r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)] \\ &= r_1r_2 \operatorname{cis}(\theta_1+\theta_2)\end{aligned}$$

When mulitplying complex numbers in polar form, *magnitudes multiply and angles add.*

1.10.3 Conjugate

If $z = a + bi = r\operatorname{cis}\theta$, then the conjugate, $\overline{z}$ or z^* is:

$$z^* = r\operatorname{cis}(-\theta) \tag{1.10.1}$$

You can see this graphically.

Recall:

$$\begin{aligned}zz^* &= (a+bi)(a-bi) \\ &= a^2 + b^2 \\ &= |z|^2\end{aligned}$$

In polar form, this becomes,

$$\begin{aligned}zz^* &= (r\operatorname{cis}\theta)(r\operatorname{cis}(-\theta)) \\ &= r^2\operatorname{cis}(\theta-\theta) \\ &= r^2(\cos 0 + i\sin 0) \\ &= r^2 \\ &= |z|^2\end{aligned}$$

In the above equations, you can skip a step if you see that since the argument $\theta - \theta = 0$, the number lies on the Real-axis.

1.10.4 Division

Recall: to divide complex numbers, we multiply the numerator and denominator by the conjugate.

$$\begin{aligned}\frac{r_1 \operatorname{cis}\theta_1}{r_2 \operatorname{cis}\theta_2} = \frac{z_1}{z_2} &= \frac{z_1\overline{z}_2}{z_2\overline{z}_2}\\ &= \frac{(r_1 \operatorname{cis}\theta_1)(r_2 \operatorname{cis}(-\theta_2))}{r^2}\\ &= \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2\end{aligned}$$

Chapter 2

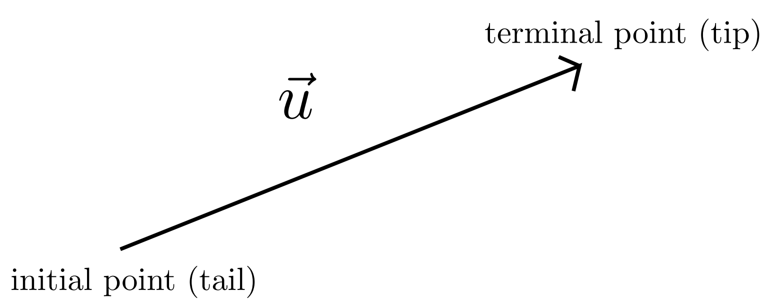
Course 2 - MDM4U7

2.1 Vectors

2.1.1 Basic Definitions

Vectors have:

- initial point (tail)
- terminal point (tip)



The **magnitude** of a vector $\vec{u}$ is denoted by $|\vec{u}|$.

Free vector: a vector whose position is not fixed.

Vector Equality (definition)

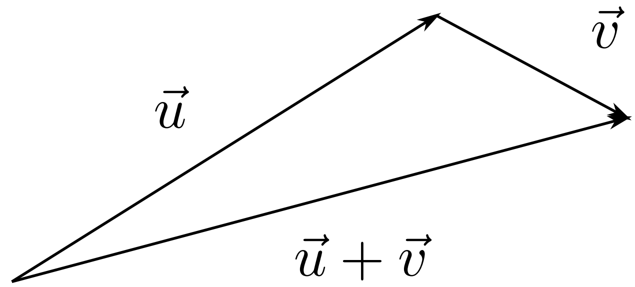
Two vector are equal if and only if:

- they have the same magnitude
- they have the same direction

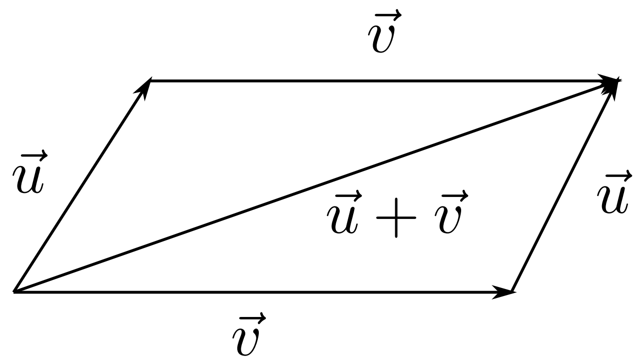
we write $\vec{u} = \vec{v}$ if they are equal.

Vector Addition (definition)

Place the tail of $\vec{v}$ on the tip of $\vec{u}$. $\vec{u}+\vec{v}$ is defined as the vector from the tail of $\vec{u}$ to the tip of $\vec{v}$.



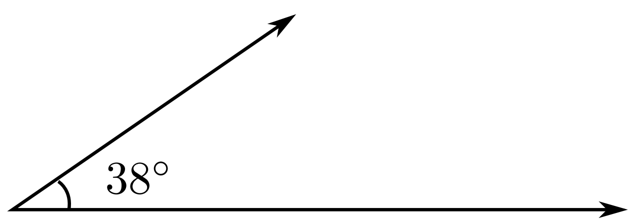
Or, you can use the parallelogram law.



$\vec{u}+\vec{v}$ is called the **sum** or **resultant** of $\vec{u}$ and $\vec{v}$.

Angles between Vectors

The angle between two vectors is the angle formed when the vectors are placed *tail to tail*.



Opposite

The *opposite* of $\vec{u}$ is a vector with the same magnitude but exactly opposite direction. The opposite of $\vec{u}$ is denoted by $-\vec{u}$. The sum of a vector and its opposite is a vector with zero magnitude. This vector is called a *zero vector* or $\vec{0}$.

$$\vec{u}+(-\vec{u})=\vec{0} \tag{2.1.1}$$

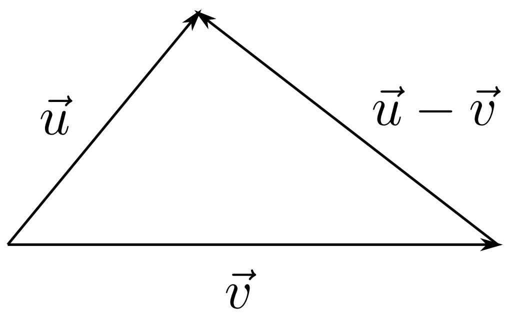
A zero vector has indeterminate direction and magnitude 0.

Vector Subtraction

Vector subtraction is defined as "adding the opposite".

$$\vec{u} - \vec{v} = \vec{u} + (-\vec{v}), \quad \text{by definition} \tag{2.1.2}$$

It can also be illustrated as:



Unit Vector

A unit vector is a vector with magnitude 1. A unit vector in the direction of any vector $\vec{u}$ can be found by dividing $\vec{u}$ by its magnitude.

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|} \tag{2.1.3}$$

where $\hat{u}$ is a unit vector.

Properties of Vector Addition

$$\vec{u} + \vec{v} = \vec{v} + \vec{u} \tag{2.1.4}$$

$$\vec{u} - \vec{v} = \vec{u} + (-\vec{v}) \tag{2.1.5}$$

$$\vec{u} + (-\vec{u}) = \vec{0} \tag{2.1.6}$$

$$\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w} \tag{2.1.7}$$

Multiplying Vectors by Scalars

For vector $\vec{u}$ and real number c, the **scalar multiple of $\vec{u}$ by c** is a vector with the following characteristics:

magnitude:	$\lvert c \rvert \times \lvert \vec{u} \rvert$
direction:	if $c > 0$, same as that of $\vec{u}$
	if $c < 0$, opposite to that of $\vec{u}$

This vector is denoted by $c\vec{u}$.

Properties of Scalar Multiplication

For vectors $\vec{u}$ and $\vec{v}$, and scalars a and b,

$$a(\vec{u}+\vec{v}) = a\vec{u} + a\vec{v} \tag{2.1.8}$$

$$(a+b)\vec{u} = a\vec{u} + b\vec{u} \tag{2.1.9}$$

$$(ab)\vec{u} = a(b\vec{u}) \tag{2.1.10}$$

2.1.2 Forces

To describe a force, it is necessary to state its:

- direction
- magnitude
- the point at which it is applied

Note that a force is *not* a free vector.

The **resultant** is the sum of the vectors of the forces that are applied at the same point. The **equilibrant** is the opposite force to the resultant. It is the force that would exactly counterbalance the resultant.

Relative Velocities

All velocites are relative to something else. We call this reference point a **frame of reference**.

The velocity of an object A relative to object B is

$$\vec{v}_A - \vec{v}_B \tag{2.1.11}$$

where $\vec{v}_A$	is the velocity of A and
$\vec{v}_B$	is the velocity of B.

2.1.3 Theorems

Definition 2.1.1. A **linear combination** of vectors $\vec{u}$ and $\vec{v}$ is a vector $\vec{w}$ of the form

$$\vec{w} = c\vec{u} + d\vec{v} \tag{2.1.12}$$

where c and d are scalars

Definition 2.1.2. Vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3, \ldots, \vec{u}_n$ are **linearly dependent** if there exist scalars $a_1, a_2, a_3, \ldots, a_n$ not all zero such that

$$a_1\vec{u}_1 + a_2\vec{u}_2 + a_3\vec{u}_3 + \cdots + a_n\vec{u}_n = \vec{0} \tag{2.1.13}$$

Theorem 2.1.1. For two vectors this means

$$\vec{u}_1 = k\vec{u}_2 \tag{2.1.14}$$

for some scalar k.

Theorem 2.1.2. For three vectors, they are linearly dependent if and only if at least one can be written as a linear combination of the other two.

Definition 2.1.3. Two vectors are **co-linear** if they lie on a straight line when placed tip to tail.

The following are all equivalent statements:

- $\vec{u}$ and $\vec{v}$ are parallel.
- $\vec{u}$ and $\vec{v}$ are linearly dependent.
- $\vec{u}$ and $\vec{v}$ are co-linear
- $\vec{u}$ is a scalar multiple of $\vec{v}$.

Definition 2.1.4. Vectors are **coplanar** if they lie on a plane when they are arranged tail-to-tail.

Theorem 2.1.3. If three vectors are linearly dependent, then they are coplanar.

The following statements are all equivalent:

- $\vec{u}$, $\vec{v}$ and $\vec{w}$ are linearly dependent.
- at least one of teh vectors can be written as a linear combination of the other two.
- $\vec{u}$, $\vec{v}$ and $\vec{w}$ are coplanar.

Definition 2.1.5. Vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3, \ldots, \vec{u}_n$ are **linearly independent** if the only linear combination of those vectors that produces the zero vector is

$$0\vec{u} + 0\vec{u}_2 + 0\vec{u}_3 + \cdots + 0\vec{u}_n = \vec{0} \tag{2.1.15}$$

Definition 2.1.6. Vectors that lie in the same plane are called **planar vectors**.

Definition 2.1.7. Two planar vectors form a **basis for a plane** if every vector in the plane can be written as a linear combination of the two vectors.

Theorem 2.1.4. Any two linearly independent vectors form a basis for the plane in which they lie.

The following are all equivalent statements:

- $\vec{u}$ and $\vec{v}$ form a basis for the plane.
- $\vec{u}$ and $\vec{v}$ are linearly independent.

Definition 2.1.8. Three vectors form a **basis for a space** if every vector in that space can be written as a linear combination of the three vectors.

Theorem 2.1.5. Any three linearly independent vectors form a basis for the space in which they exist.

2.1.4 Algebraic Vectors

Definition 2.1.9. A **position vector** for a point P, with respect to the origin O, is the fixed vector $\overrightarrow{OP}$.

Vectors in a Plane

Every vector in the plane can be written in **component form**:

$$\overrightarrow{OP} = (a, b)$$

The **unit vectors** are defined as

$$\hat{\imath} = (1, 0) \qquad \hat{\jmath} = (0, 1) \tag{2.1.16}$$

Every vector in a plane can also be written in **vector form**:

$$\overrightarrow{OP} = a\hat{\imath} + b\hat{\jmath}$$

Thus

$$(a, b) = a\hat{\imath} + b\hat{\jmath} \tag{2.1.17}$$

a and b are called the **components** or scalar components of $\overrightarrow{OP}$.

Vectors in Space

The convention for vectors in space is to use a **right-handed system**. To construct x-,y- and z-axes for space:

1. Choose three mutually perpendicular lines intersecting at a point and call them the x-axis, y-axis and z-axis.
2. Choose a positive x direction and positive y direction. Then define the positive z direction by curling the fingers of your right hand in the direction of a rotation from the positive x-axis to the positive Y-axis. The direction in which your thumb points is the positive z-direction.

Every vector in space can be written in **component form**:

$$\overrightarrow{OP} = (a, b, c)$$

The **unit vectors** are defined as

$$\hat{\imath} = (1, 0, 0) \qquad \hat{\jmath} = (0, 1, 0) \qquad \hat{k} = (0, 0, 1) \tag{2.1.18}$$

Every vector in space can also be written in **vector form**

$$\overrightarrow{OP} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$$

Thus

$$(a, b, c) = a\hat{\imath} + b\hat{\jmath} + c\hat{k} \tag{2.1.19}$$

a,b and c are called the **components** or scalar components of $\overrightarrow{OP}$.

Vector Operations in Component Form

Vector Equality

Given $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$, $\vec{u} = \vec{v}$ if and only if

$$u_1 = v_1 \quad \text{and} \quad u_2 = v_2$$

Given $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$, $\vec{u} = \vec{v}$ if and only if

$$u_1 = v_1 \quad \text{and} \quad u_2 = v_2 \quad \text{and} \quad u_3 = v_3$$

Vector Addition

Given $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$

$$\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2)$$

Given $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$

$$\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

Scalar Multiplication

Given $\vec{u} = (u_1, u_2)$ and $k \in \mathbb{R}$

$$k\vec{u} = (ku_1, ku_2)$$

Given $\vec{u} = (u_1, u_2, u_3)$ and $k \in \mathbb{R}$

$$k\vec{u} = (ku_1, ku_2, ku_3)$$

Vector Subtraction

Given $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$

$$\vec{u} - \vec{v} = (u_1 - v_1, u_2 - v_2)$$

Given $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$

$$\vec{u} - \vec{v} = (u_1 - v_1, u_2 - v_2, u_3 - v_3)$$

Length of a Vector

Given $\vec{u} = (u_1, u_2)$, then $|\vec{u}| = \sqrt{u_1^2 + u_2^2}$.
Given $\vec{u} = (u_1, u_2, u_3)$, then $|\vec{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$.

Vectors between two points

Given $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, then

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1)$$

Given $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$, then

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

2.1.5 Dot Product (scalar product)

For non-zero vectors $\vec{u}$ and $\vec{v}$ we define the dot product between $\vec{u}$ and $\vec{v}$ as

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta \tag{2.1.20}$$

where θ is the angle between $\vec{u}$ and $\vec{v}$ and $0 \leq \theta \leq \pi$. i.e. θ is the smaller angle.

If $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$, then it is defined as $\vec{u} \cdot \vec{v} = 0$.

Dot Product in Component Form

For $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$,

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 \tag{2.1.21}$$

For $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$,

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \tag{2.1.22}$$

Properties of the Dot Product

1. For non-zero vectors $\vec{u}$ and $\vec{v}$, $\vec{u}$ and $\vec{v}$ are perpendicular if and only if $\vec{u} \cdot \vec{v} = 0$.
2. For any vector $\vec{u}$,
$$\vec{u} \cdot \vec{u} = |\vec{u}|^2$$
3. For any vectors $\vec{u}$ and $\vec{v}$,
$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$
In other words, the dot product is **Commutative**.
4. For any vectors $\vec{u}$ and $\vec{v}$, and scalar $k \in \mathbb{R}$,
$$(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v})$$
In other words, the dot product is **Associative**.
5. For any vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$,
$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$
In other words, the dot product is **Distributive**

Applications of the Dot Product

Work

We define work done in moving an object through a displacement $\vec{s}$, under an applied force $\vec{f}$ acting at an angle θ as

$$w = |\vec{f}||\vec{s}| \cos\theta \tag{2.1.23}$$

$$= \vec{f} \cdot \vec{s} \tag{2.1.24}$$

Note that $\vec{s}$ must be in metres and $\vec{f}$ in Newtons. *Don't forget to convert to metres.*

2.1.6 Cross Product (vector product)

For non-zero vectors $\vec{u}$ and $\vec{v}$ we define the cross product between $\vec{u}$ and $\vec{v}$ as

$$\vec{u} \times \vec{v} = |\vec{u}||\vec{v}| \sin\theta \hat{n} \tag{2.1.25}$$

where θ is the angle between $\vec{u}$ and $\vec{v}$ such that $0 \leq \theta \leq \pi$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{u}$ and $\vec{v}$ such that $\vec{u}$, $\vec{v}$ and $\hat{n}$ form a right-handed system.

The magnitude of the cross product is not affected by $\hat{n}$, since it is a unit vector. So, the magnitude is

$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta \tag{2.1.26}$$

which looks similar to the dot product.

To find the direction of $\hat{n}$:

1. Wrist at tail, fingers at tip of $\vec{u}$.
2. Curl fingers towards $\vec{v}$ (through the shorter angle). Thumb points in direction of $\vec{u} \times \vec{v}$.

Properties of Cross Product

1. For non-zero vectors $\vec{u}$ and $\vec{v}$, they are collinear if and only if $\vec{u} \times \vec{v} = \vec{0}$.
2. For any vectors $\vec{u}$ and $\vec{v}$
$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$
which means the cross product is **not Commutative**. But note that $|\vec{u} \times \vec{v}| = |\vec{v} \times \vec{u}|$, which means their magnitudes are commutative, which makes sense because they are just pointing in opposite directions.
3. For any vectors $\vec{u}$ and $\vec{v}$, and scalar $k \in \mathbb{R}$,
$$(k\vec{u}) \times \vec{v} = k(\vec{u} \times \vec{v}) = \vec{u} \times (k\vec{v})$$
which means the cross product is **Associative**.
4. For any vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$,
$$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$$
which means the cross product is **Distributive**.

Cross Product in Component Form

$$\begin{aligned}
\hat{\imath} \times \hat{\imath} &= \vec{0} & \hat{\imath} \times \hat{\jmath} &= \hat{k} & \hat{\jmath} \times \hat{\imath} &= -\hat{k} \\
\hat{\jmath} \times \hat{\jmath} &= \vec{0} & \hat{\jmath} \times \hat{k} &= \hat{\imath} & \hat{k} \times \hat{\jmath} &= -\hat{\imath} \\
\hat{k} \times \hat{k} &= \vec{0} & \hat{k} \times \hat{\imath} &= \hat{\jmath} & \hat{\imath} \times \hat{k} &= -\hat{\jmath}
\end{aligned}$$

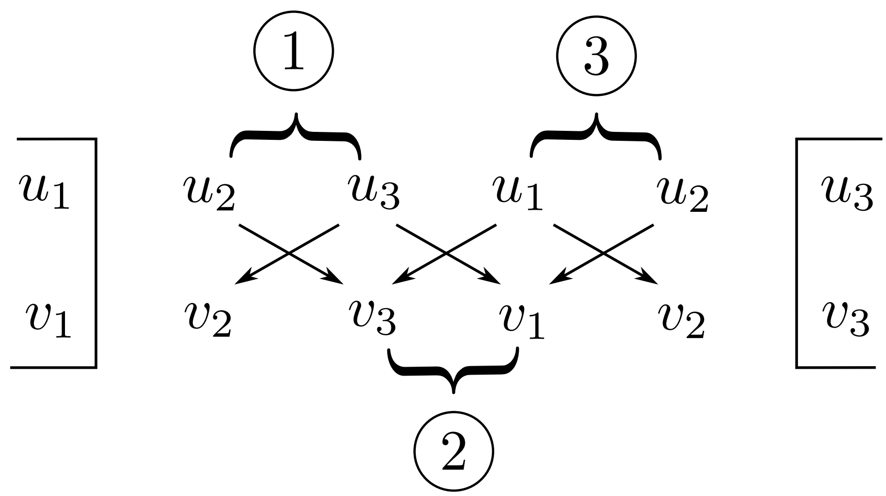
For two vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$,

$$\vec{u} \times \vec{v} = \underset{1}{(u_2v_3 - u_3v_2)}\hat{\imath} + \underset{2}{(u_3v_1 - u_1v_3)}\hat{\jmath} + \underset{3}{(u_1v_2 - u_2v_1)}\hat{k} \tag{2.1.27}$$

Trick for finding cross product in component form

For two vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$,

1. Write out the first vector twice, then write the second vector twice underneath it.
2. Chop off the ends.
3. Take diagonal products as shown below and subtract them. To obtain each component of $\vec{u} \times \vec{v}$, move one column to the right.



where the circled numbers correspond to the circled numbers in (2.1.27).

Triple Scalar Product is defined as

$$(\vec{u} \times \vec{v}) \cdot \vec{w} \tag{2.1.28}$$

Triple Vector Product is defined as

$$(\vec{u} \times \vec{v}) \times \vec{w} \tag{2.1.29}$$

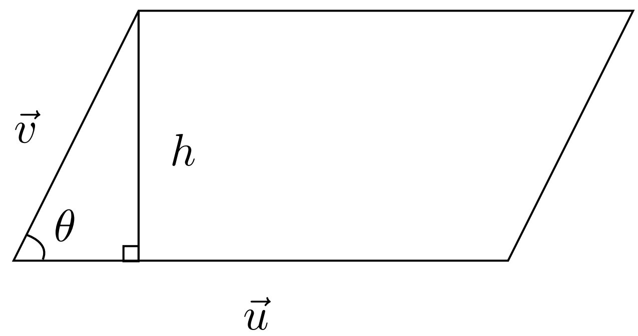
Cross Product Theorems

Theorem 2.1.6. For any non-zero vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, $(\vec{u} \times \vec{v}) \cdot \vec{w} = 0$ if and only if $\vec{u}$, $\vec{v}$ and $\vec{w}$ are **coplanar**.

Corollary 2.1.6.1. for any non-zero vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, if $(\vec{u} \times \vec{v}) \cdot \vec{w} \neq 0$, then $\vec{u}$, $\vec{v}$ and $\vec{w}$ are not coplanar and they form a basis for space.

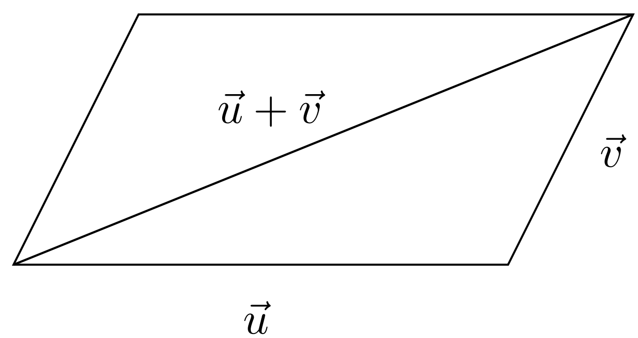
Applications of Cross Product

Area of a Parallelogram



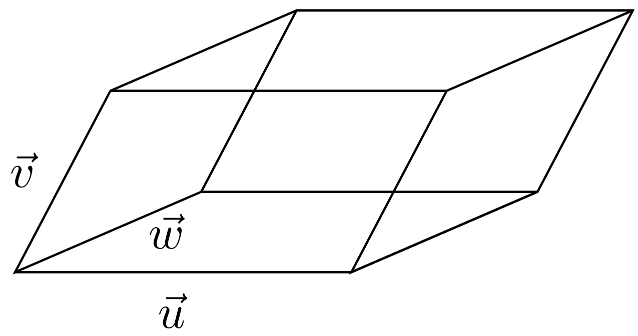
$$
\begin{aligned}
\text{Area} &= \text{base} \times \text{height} \\
&= |\vec{u}| \times |\vec{v}| \sin\theta \\
&= |\vec{u}||\vec{v}| \sin\theta \\
\text{Area} &= |\vec{u} \times \vec{v}| \qquad (2.1.30)
\end{aligned}
$$

This method can be extended to find the area of a triangle.



$$
\begin{aligned}
\text{Area}_{\Delta} &= \frac{1}{2}(\text{Area of the parallelogram}) \\
\text{Area}_{\Delta} &= \frac{1}{2}\left|\vec{u} \times \vec{v}\right| \qquad (2.1.31)
\end{aligned}
$$

Volume of a Parallelepiped



$$\text{Volume} = |(\vec{u} \times \vec{v}) \cdot \vec{w}| \tag{2.1.32}$$

Note that the volume doesn't depend on which vectors are chosen as the ones forming the base, which makes sense because (2.1.32) has an absolute value.

Volume of a Tetrahedron

$$\text{Volume} = \frac{1}{6} |(\vec{u} \times \vec{v}) \cdot \vec{w}| \tag{2.1.33}$$

This is since the volume of a tetrahedron, V is $\frac{1}{3}$ of the volume of the triangular prism enclosing it. Also, the volume fo a triangular prism is $\frac{1}{2}$ of the volume of the enclosing parallelepiped (you can see this by cutting the parallelpiped in half in a particular way).

In general, the volume, V, of a pyramid is $\frac{1}{3}$ of the volume of the prism enclosing it.

$$\text{V} = \frac{\text{Area of Base} \times \text{height}}{3}$$

Torque (or Moment)

We define the torque (or moment) $\vec{T}$ of an applied force as

$$\vec{T} = \vec{r} \times \vec{f} \tag{2.1.34}$$

$$= |\vec{r}||\vec{f}| \sin\theta \hat{n} \tag{2.1.35}$$

where $\vec{f}$ is the applied force.
$\vec{r}$ is the vector determined by the lever arm acting from the axis of rotation.
θ is the angle between the force and the lever arm.
$\hat{n}$ is a unit vector perpendicular to both $\vec{r}$ and $\vec{f}$ such that they form a right-handed system.

2.1.7 Equations of Lines in Planes and Space

Vector Equation of a Line

$$\vec{r} = \vec{p} + t\vec{d}, \quad t \in \mathbb{R} \tag{2.1.36}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}, t \in \mathbb{R} \tag{2.1.37}$$

where $\vec{r}$ is the position vector of any point on the line, $\vec{p}$ is the position vector of a particular point on the line and $\vec{d}$ is the **direction vector** for the line.

Parametric Equation of a Line

$$x = x_0 + td_1 \tag{2.1.38}$$

$$y = y_0 + td_2 \tag{2.1.39}$$

$$z = z_0 + td_3 \tag{2.1.40}$$

This system of equations are found by equating the components of the vectors in the vector equation of a line. d_1, d_2 and d_3 are called the **direction numbers** of the line.

Symmetric Equation of a Line (or Cartesian Equation)

$$\frac{x - x_1}{d_1} = \frac{y - y_0}{d_2} = \frac{z - z_0}{d_3}, \quad d_1, d_2, d_3 \neq 0 \tag{2.1.41}$$

This equation is found by rearranging the parametric equation of a line to solve for t and equating the two expressions for t. If any of the direction numbers are 0, it is not possible to write a symmetric equation of the line.

Scalar Equation of a Line (in a Plane)

A line in space has no scalar equation. Only lines in a plane can be described by a scalar equation.

$$Ax + By + C = 0 \tag{2.1.42}$$

where $A \in \mathbb{Z}^+$ and $B, C \in \mathbb{Z}$.

A scalar equation of the line through $P_0(x_0, y_0)$ with normal $\vec{n} = (n_1, n_2)$ is

$$n_1 x + n_2 y + c = 0 \tag{2.1.43}$$

where c can be determined by plugging in values of x_0 and y_0.

To get the normal from a direction vector, or the direction vector from a normal, swap the components switch the sign of one of them:

$$\vec{n} = (n_1, n_2) = (-d_2, d_1) = (d_2, -d_1)$$
$$\vec{d} = (d_1, d_2) = (-n_2, n_1) = (n_2, -n_1)$$

Also the relationship between slope m and normal vector $\vec{n} = (n_1, n_2)$:

$$m = -\frac{n_1}{n_2}$$

Meaning of Cartesian Equation

Line in a plane,	Cartesian means symmetric.
Line in space,	Cartesian means symmetric (it doesn't even have a scalar equation, so no confusion).
Plane,	Cartesian means scalar (a plane doesn't have a symmetric equation).

2.1.8 Intersection of Lines

2-space

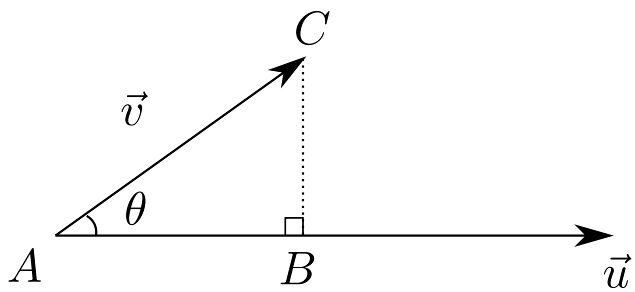
- Parallel lines - distinct (inconsistent linear system - no solution)
- Parallel lines - coincident (consistent dependent system - infinite solutions)
- Non-parallel lines - intersect (consistent independent system - one solution)

3-space

- Parallel lines - distinct (inconsistent linear system - no solution)
- Parallel lines - coincident (consistent dependent system - infinite solutions)
- Non-parallel lines - intersect (consistent independent system - one solution)
- Non-parallel lines - Skew lines that do not intersect (lie in different planes)

When finding a point of intersection of two lines $l_1 = (x_1, y_1) + t(d_{1,d}2)$ and $l_2 = (x_2, y_2) + s(d_3, d_4)$ in space, you must verify that the values of s and t found satisfy all three of the parametric equations for each line. This is since the lines could be skew lines.

Projections



$$|\text{proj}_{\vec{u}}\vec{v}| = |\overrightarrow{AB}| = |\vec{v}|\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|} \tag{2.1.44}$$

Distance

2D In 2D you can find the distance from:

- a point to a line
- a line to a line (parallel lines)

3D In 3D you can find the distance from:

- a point to a line
- a line to a line (parallel lines)
- a line to a line (skew lines)

You cannot find the distant between two lines if they are coincident or intersecting. Thus we can only find the distance between lines if they are

- parallel, or
- skew lines

In a Plane

Line to a Point (in a Plane)

Formula for the distance from line $Ax + By + C = 0$ and point (x_1, y_1). Or in vector form, line with normal $\vec{n}$ and point P_2 (on the line) to the point P_1 (not on the line).

$$D = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \text{ or} \tag{2.1.45}$$

$$D = \frac{|\overrightarrow{P_1P_2}\cdot\vec{n}|}{|\vec{n}|} \tag{2.1.46}$$

Line to a Parallel Line (in a Plane)

To find the distance between two parallel lines in a plane, you can use the formulas for the distance between a line to a point. Simply pick a point on one line and find the distance between that point and the other line.

There is also a shortcut formula if the lines are given in scalar form.

$$D = \frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}} \tag{2.1.47}$$

where the lines are $Ax + By + C_1 = 0$ and $Ax + By + C_2 = 0$. Notice that the coefficents of x and y, i.e. A and B are the same because the lines share a normal because the lines are parallel.

In Space

Line to a Point (in Space)

Formula for the distance from line with direction vector $\vec{d}$ and a point P_2 (on the line) to a point P_1 not on the line.

$$D = \frac{|\overrightarrow{P_1P_2} \times \vec{d}|}{|\vec{d}|}$$

Line to a Parallel Line (in Space)

Use the formula for the distance between a line to a point in space. Simply pick a point P_1 on one line and find the distance between P_1 to the other line. This works since both lines are parallel.

Line to a Line (Skew lines)

$$\frac{|\overrightarrow{P_1P_2} \cdot \vec{n}|}{|\vec{n}|} \tag{2.1.48}$$

where $\vec{n}$ is a vector perpendicular to both lines found by taking the cross product of the direction vectors of the two lines. i.e.

$$\vec{n} = \vec{d_1} \times \vec{d_2}$$

2.1.9 Equation of Planes

Vector Equation of a Plane

$$(x, y, z) = (x_0, y_0, z_0) + s(a_1, a_2, a_3) + t(b_1, b_2, b_3), \quad s, t \in R \tag{2.1.49}$$

Parametric Equation of a Plane

$$x = x_0 + sa_1 + tb_1 \tag{2.1.50}$$
$$y = y_0 + sa_2 + tb_2 \tag{2.1.51}$$
$$z = z_0 + sa_3 + tb_3 \tag{2.1.52}$$

Note that vector and parametric equations of a plane are not unique.

Scalar Equation of a Plane (or Cartesian Equation)

$$Ax + By + Cz + D = 0 \tag{2.1.53}$$

where the normal to the plane $\vec{n} = (A, B, C)$. The constant D can be found by plugging in values of a point $P = (x, y, z)$ that lies on the plane. Note that this form of the scalar equation is unique, unlike others. Another two versions of the scalar equation are

$$\vec{n} \cdot (\vec{r} - \vec{a}) = 0 \tag{2.1.54}$$
$$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{a} \tag{2.1.55}$$

where $\vec{r}$ and $\vec{a}$ are the position vectors of two distinct points on the plane.

Normal to a Plane To find the normal vector $\vec{n}$ to the plane, take the cross product of two direction vectors of the plane.

$$\vec{n} = \vec{d_1} \times \vec{d_2} \tag{2.1.56}$$

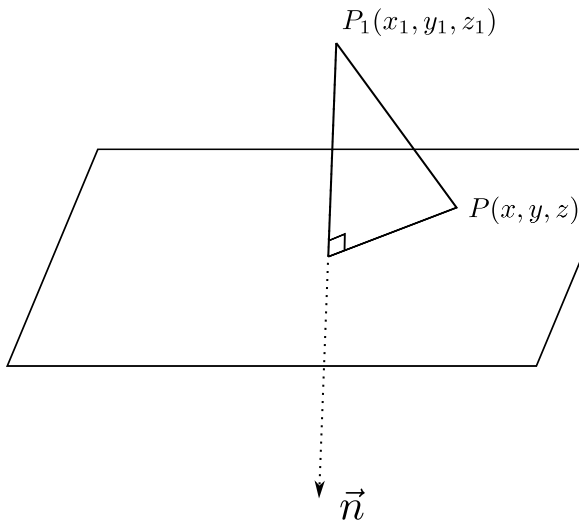
where $\vec{d_1}$ and $\vec{d_2}$ are the two direction vectors of the plane. This vector $\vec{n}$ is perpendicular to every vector in the plane.

Useful Facts about Planes

- Two planes are parallel if and only if their normals are collinear.
- Two planes are parallel and coincident if and only if they are parallel and they share a point.
- A plane can be formed from two parallel distinct lines.
- A plane can be formed from two non-parallel intersecting lines

2.1.10 Distance for Planes

Point to a Plane



Formula for the distance between plane $Ax + By + Cz + D = 0$ and point $P_1(x_1, y_1, z_1)$, or in vector form, the distance between a plane which contains point P (on the plane) and has normal $\vec{n}$ to point P_1 (not on the plane).

$$D = \frac{|\overrightarrow{P_1P} \cdot \vec{n}|}{|\vec{n}|} \tag{2.1.57}$$

$$D = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}} \tag{2.1.58}$$

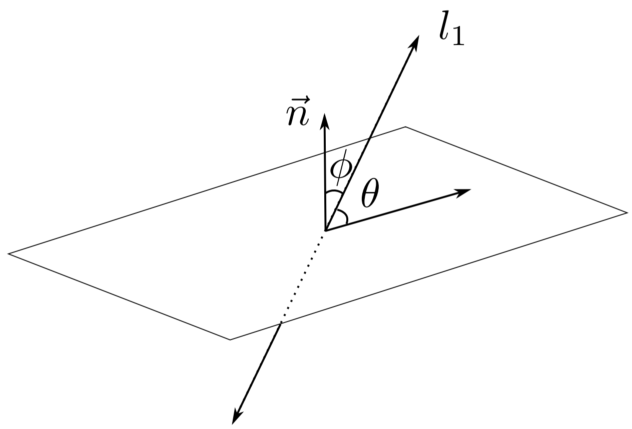
Plane to a Parallel Plane

Just pick a point P on one plane and use the formula for the distance between a point and a plane to find the distance between the planes. If you are given the scalar equation, you can use the shortcut formula:

$$D = \frac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}} \tag{2.1.59}$$

where D_1 and D_2 are the constants in the scalar equations of the planes.

Angle between a Line and a Plane



$$\cos\phi = \frac{\vec{n}\cdot\vec{d}}{|\vec{n}||\vec{d}|} \tag{2.1.60}$$

To find the angle between the line and the plane (i.e. θ), solve for ϕ and then subtract it from $90°$. Or, another method is to use the fact that

$$\cos\phi = \sin\theta \tag{2.1.61}$$

since $\theta = 90° - \phi$. Using this fact, we obtain

$$\theta = \sin^{-1}\left(\frac{\vec{n}\cdot\vec{d}}{|\vec{n}||\vec{d}|}\right) \tag{2.1.62}$$

2.1.11 Intersection of Planes

Intersection of a Line and a Plane

Given a line in parametric form

$$\begin{aligned} x &= 5 + t \\ y &= 2 - 2t \\ z &= 6t \end{aligned}$$

and a plane in scalar form, $4x + 2y + z - 6 = 0$, to find the point of intersection of the line and the plane, simply substitute the expressions for x, y, z from the parametric equations of the line into the scalar equation of the plane and solve for t. Then sub the value of t back in the parametric equations of the line to find the point of intersection. For ex.

$$4(5+t) + 2(2-2t) + (6t) - 6 = 0t{=}\ 3$$

so the point of intersection is

$$\begin{gathered} (5+t, 2-2t, 6t) \\ (5+3, 2-2(3), 6(3)) \\ (8, -4, 18) \end{gathered}$$

Intersection of two Planes

Two planes can be:

- parallel and distinct
- parallel and coincident
- intersecting in a line

There are many methods to find the line of intersection, but I will show the best one. Suppose you want to find the line of intersection between these planes

$$\pi_1 : 2x - 2y + 5z + 10 = 0$$
$$\pi_2 : 2x + y - 4z + 7 = 0$$

1. Eliminate a variable, say x.

$$\pi_1 - \pi_2 : -3y + 9z + 3 = 0$$

2. Let $z = t$ and solve for y in terms of t.

$$y = 1 + 3t$$

3. Substitute the expression for y and $z = t$ into either of the original equations of the planes.

$$\pi_1 : 2x - 2(1 + 3t) + 5(t) + 10 = 0$$

4. Solve for x in terms of t.

$$x = -4 + \frac{1}{2}t$$

5. Now you can write the parametric equation of the line of intersection.

$$\begin{aligned} x &= -4 + \frac{1}{2}t \\ y &= 1 + 3t \\ z &= t \end{aligned}$$

6. You can scale t by any value since it just corresponds to scaling the direction vector of the line. Let's scale it by 2 in order to eliminate the fraction.

$$\begin{aligned} x &= -4 + t \\ y &= 1 + 6t \\ z &= 2t \end{aligned}$$

And now you have a parametric equation of the line of intersection.

You can also find the line of intersection by solving the system of equations using matrices. You will end up with a free variable which you should set to t which will similarly lead to parametric equations for the line of intersection.

A Useful Fact The direction vector for the line of intersection of two planes is perpendicular to the normals of both of the two planes. This follows from the fact that the direction vector of a line which lies on a plane is perpendicular to that plane's normal. Since the line of intersection lies on both planes, it is perpendicular to both the normals. This means the direction vector can be found by calculating the cross product of the two normal vectors

$$\vec{d} = \vec{n}_1 \times \vec{n}_2$$

where $\vec{d}$ is the direction vector of the line and $\vec{n}_1, \vec{n}_2$ are the normals of the two planes.

Angle between two Planes

Formula for the angle between two planes

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} \tag{2.1.63}$$

where $\vec{n}_1$ and $\vec{n}_2$ are the normal vectors of the two planes.

Note that planes make 2 angles. The other angle is simply $180° - \theta$. If a question asks for the acute, find the acute one.

2.1.12 Operations on Matrices

There are 3 valid manipulations on matrices. These will be used to convert matrices to reduced row echelon form.

1. Interchange two rows
2. Multiply a row by a non-zero constant
3. Add a scalar multiple of one row to any other row

2.2 Counting

2.2.1 Factorial

Note that $0! = 1$, by definition.

2.2.2 Permutations

nPr or $P(n, r)$ is read as "n perm r".

$$nPr = \frac{n!}{(n-r)!} \tag{2.2.1}$$

Permutations with Repetition

The number of permutations of n objects of which a objects are alike and another b objects are alike and another c objects are alike, etc is

$$\frac{n!}{a!b!c!\dots}$$

Circular Permutations

The formula for the number of ways to arrange n people around a circular table is

$$(n-1)!$$

Form of Answer If the answer is greater than or equal to 1 billion (it is a 10 digit number), you may leave it in the form nPr. However, if the number has 9 or fewer digits, you must evaluate it to get a numerical value.

2.2.3 Sets

Principle of Inclusion and Exclusion

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) \tag{2.2.2}$$

Note that

$$n(A \cup B) \leq n(A) + n(B) \tag{2.2.3}$$

This can be extended to more than two sets. For three sets, A, B, C,

$$\begin{aligned} n(A \cup B \cup C) \quad = \quad & n(A) + n(B) + n(C) \\ & - n(A \cap B) - n(B \cap C) - n(A \cap C) \\ & + n(A \cap B \cap C) \end{aligned} \tag{2.2.4}$$

As before,

$$n(A \cup B \cup C) \leq n(A) + n(B) + n(C) \tag{2.2.5}$$

2.2.4 A Counting Trick

The total number of selelctions you can make from p items of one kind, q items of another kind, r items of another kind is

$$(p+1)(q+1)(r+1)\cdots - 1 \tag{2.2.6}$$

You subtract 1 if you don't want to count the case where you selelct none. But if do you want to count this empty set, don't subtract one.

2.2.5 Binomial Theorem

The Binomial Theorem can be expanded to rational exponents. $n \in \mathbb{Q}$.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \tag{2.2.7}$$

$$\tag{2.2.8}$$

where the general term t_{r+1} is given by

$$t_{r+1} = \frac{(n)(n-1)(n-2)\cdots(n-r+1)}{r!}x^r \tag{2.2.9}$$

There is a restriction for this expansion. $|x| < 1$ must be true in order for the series to converge and thus for the expansion to be valid.

2.3 Probability

2.3.1 Basic Definitions and Theorems

Experiment:	a well-defined process from which observations can be made
Outcome:	one possible result of an experiment
Event:	a subset of all outcomes

For any event A, $0 \le P(A) \le 1$.

Also,

$$P(A) = \frac{n(A)}{n(S)} \tag{2.3.1}$$

where $n(A)$ is the number of outcomes related to event A and $n(S)$ is the total number of outcomes assuming all outcomes are equally likely.

It is clear that

$$P(A') = 1 - P(A) \tag{2.3.2}$$

2.3.2 Mutually Exclusive

If two sets have no outcomes in common, they are said to be mutually exclusive.

For mutually exclusive events A and B,

$$P(A \cup B) = P(A) + P(B) \tag{2.3.3}$$

This follows from the principle of inclusion-exclusion, $P(A\cup B) = P(A) + P(B) - P(A\cap B)$ and the fact that $P(A \cap B) = 0$ for mutually exclusive events.

2.3.3 Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{2.3.4}$$

which can also be written as

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{n(A \cap B)}{n(S)}}{\frac{n(B)}{n(S)}} = \frac{n(A \cap B)}{n(B)} \tag{2.3.5}$$

2.3.4 Independent Events

Events where the occurrence of one event does not affect the occurrence of the other event are called independent events.

For independent events A and B,

$$P(A|B) = P(A) \tag{2.3.6}$$

because event B occurring doesn't affect the probability of event A occurring.

Substituting (2.3.6) into (2.3.4) gives this fact

$$P(A \cap B) = P(A) \times P(B) \tag{2.3.7}$$

2.3.5 Bayes's Theorem

Law of Total Probability

$$p(B) = p(A) \times p(B|A) + p(A') \times p(B|A')$$

This equation is much easier to follow if you look at a tree diagram.

2.4 IB Conventions

1. The set of Natural numbers includes 0. i.e. $\mathbb{N} = \{0, 1, 2, \dots\}$.
2. All polar answers in cis notation must be expressed with argument θ such that $\pi \leq \theta \leq \pi$. But answers in Euler's form must have argument θ such that $0 \leq \theta \leq 2\pi$. For ex. $2e^{-\pi i}$ is not allowed, but $2\operatorname{cis}(-\pi)$ is allowed.