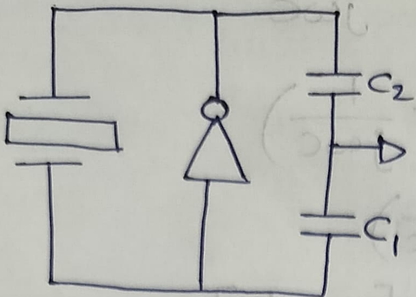
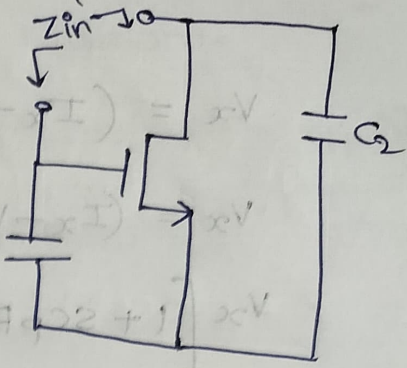
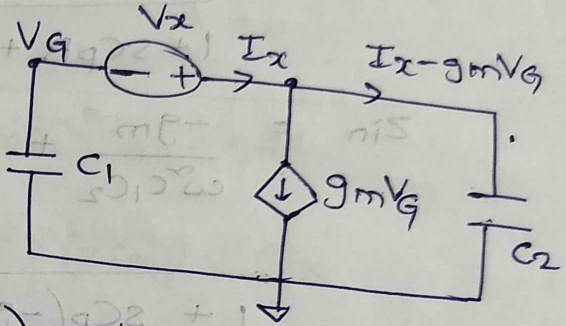
Numerical Derivations



Equivalent circuit



small-signal model



Derivation :-

$$V_G = \frac{-I_x}{sC_1}$$

$$\frac{I_x}{sC_1} - V_x + \left(\frac{I_x - g_m V_G}{sC_2}\right) = 0$$

$$\frac{I_x}{s}\left[\frac{1}{C_1} + \frac{1}{C_2}\right] + \frac{g_m}{sC_1}\left(\frac{I_x}{sC_2}\right) = V_x$$

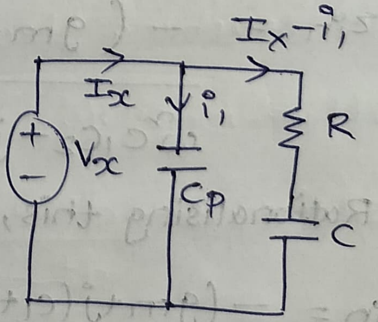
$$\frac{V_x}{I_x} = Z'_{in} = \frac{1}{j\omega\left(\frac{C_1C_2}{C_1+C_2}\right)} + \left(\frac{-g_m}{\omega^2 C_1 C_2}\right)$$

$$\boxed{Re(Z'_{in}) = \frac{-g_m}{\omega^2 C_1 C_2}}, \quad \boxed{Im(Z'_{in}) = \frac{1}{j\omega\left(\frac{C_1C_2}{C_1+C_2}\right)}}$$

Now, For calculating negative resistance :-

Here, consider,

$$R = \frac{-g_m}{\omega^2 C_1 C_2}, \quad C = \frac{C_1 C_2}{C_1 + C_2}$$



$$-V_x + \frac{i_1}{sC_p} = 0$$

$$i_1 = V_x(sC_p)$$

$$-V_x + (I_x - V_x sC_p)R + \frac{(I_x - V_x sC_p)}{j\omega C} = 0$$

$$V_x = (I_x - V_x sC_p)\left(R + \frac{1}{j\omega C}\right)$$

$$V_x = (I_x - V_x sC_p)\left(R + \frac{1}{sC}\right)$$

$$V_x\left[1 + sC_pR + \frac{C_p}{C}\right] = I_x\left[R + \frac{1}{sC}\right]$$

$$Z_{in} = \frac{R + \frac{1}{sC}}{1 + sC_pR + C_p/C}$$

$$Z_{in} = \frac{\frac{-g_m}{\omega^2 C_1 C_2} + \frac{1}{s\left(\frac{C_1C_2}{C_1+C_2}\right)}}{1 + sC_p\left(\frac{-g_m}{\omega^2 C_1 C_2}\right) + \frac{C_p}{\left(\frac{C_1C_2}{C_1+C_2}\right)}}$$

$$Z_{in} = \frac{\frac{-g_m}{\omega^2 C_1C_2} - \frac{j(C_1+C_2)\omega}{\omega^2 C_1C_2}}{1 - \frac{j\omega C_p g_m}{\omega^2 C_1 C_2} + \frac{C_p(C_1+C_2)}{C_1C_2}}$$

$$Z_{in} = \frac{-(g_m + j(C_1+C_2)\omega)}{\omega^2 C_1C_2 - j\omega C_p g_m + \omega^2 C_p(C_1+C_2)}$$

On Rationalising this, we get;

$$Z_{in} = \frac{-(g_m + j(C_1+C_2)\omega)(\omega^2(C_1C_2 + C_1C_p + C_2C_p) + j\omega C_p g_m)}{[\omega^2(C_1C_2 + C_pC_1 + C_pC_2)]^2 - (j\omega C_p g_m)^2}$$

$Z_{in} =$

Solving Real part from this Equation:-

$$Re(Z_{in}) = \frac{-[g_m \omega^2 (C_1C_2 + C_1C_p + C_2C_p) - \omega^2 g_m (C_pC_1 + C_pC_2)]}{[\omega^2 (C_1C_2 + C_pC_1 + C_pC_2)]^2 + \omega^2 C_p^2 g_m^2}$$

$$Re(Z_{in}) = \frac{-g_m \cancel{\omega^2} (C_1C_2)}{\omega^2 \cdot \cancel{\omega^2} (C_1C_2 + C_pC_1 + C_pC_2)^2 + \cancel{\omega^2} C_p^2 g_m^2}$$

$$Re(Z_{in}) = \frac{-g_m (C_1C_2)}{\omega^2 (C_1C_2 + C_pC_1 + C_pC_2)^2 + C_p^2 g_m^2}$$

$\therefore$ Negative Resistance $= Re(Z_{in}) = R_N$

$$\boxed{|R_N| = \frac{g_m (C_1C_2)}{\omega^2 (C_1C_2 + C_pC_1 + C_pC_2)^2 + C_p^2 g_m^2}}$$

Now, For Start-up time ;-

$$E = \frac{1}{2} L_m I^2$$

$$\frac{dE}{dt} = L_m I \left(\frac{dI}{dt}\right) \longrightarrow (1)$$

$$P = R_{net} (I_{rms})^2 = (|R_N| - R_m) \frac{I^2}{2} \longrightarrow (2)$$

Equating (1) and (2) ;-

$$L_m \cancel{I} \left(\frac{dI}{dt}\right) = (|R_N| - R_m) \frac{I^{\cancel{2}}}{2}$$

$$L_m \left(\frac{dI}{dt}\right) = (|R_N| - R_m) \left(\frac{I}{2}\right)$$

$$\frac{dI}{I} = \left(\frac{|R_N| - R_m}{2 L_m}\right) dt$$

Integrating on Both sides :-

$$\ln(I(t)) = \left(\frac{|R_N| - R_m}{2 L_m}\right) t + c$$

$$I(t) = I_m(0)\, e^{\left(\frac{|R_N| - R_m}{2 L_m}\right) t}$$

$$\left\{ \begin{array}{l} t \to T_s \\ \because e^{c} \to I_m(0) \end{array} \right\}$$

$$T_s = \frac{2 L_m}{|R_N| - R_m} \ln\left[\frac{I(t)}{I_m(0)}\right]$$

On reaching steady-state oscillation, we get,

$$V(t) = V_{DD} \sin(\omega t)$$

$$I(t) = C_T \frac{d\,V(t)}{dt} \Rightarrow C_T \omega V_{DD} \cos(\omega t)$$

So, $I_{peak} = C_T \omega V_{DD}$

To declare oscillations, we want to achieve $0.9 I_{max}$:

$$I_{Target} = 0.9 C_T \omega V_{DD}$$

Here, $C_T = \dfrac{C_1 C_2}{C_1 + C_2} + C_p$ → (2)

Hence,

$$\boxed{T_s = \frac{2L_m}{|R_N| - R_m}\left[\ln\left(\frac{0.9\omega C_T V_{DD}}{|I_m(0)|}\right)\right]}$$

→ We know that,

$$R_N = \frac{g_m C_1 C_2}{(g_m C_p)^2 + \omega^2 (C_1 C_2 + C_2 C_p + C_1 C_p)^2}$$

For getting maximum value of R_N;

Derivative of R_N wrt $g_m = 0$ [By u/v rule]

$$C_1 C_2 \left[(g_m C_p)^2 + \omega^2 (C_1 C_2 + C_2 C_p + C_1 C_p)^2\right] - g_m C_1 C_2 \left[2 g_m C_p^2\right] = 0 = \frac{dR_N}{dg_m}$$

$$\omega^2 (C_1 C_2 + C_2 C_p + C_1 C_p)^2 = g_m^2 C_p^2$$

This g_m is optimised value of g_m;

$$\boxed{\therefore\ g_{m,optimised} = \omega\left(\frac{C_1 C_2}{C_p} + C_2 + C_1\right)}$$

→ Now, substitute this $g_{m,optimised}$ value in place of g_m in R_N, to get maximum value of R_N

$$R_{N,max} = \frac{\omega\left(\frac{C_1C_2}{C_p}+C_1+C_2\right)C_1C_2}{2\omega^2(C_1C_2+C_1C_p+C_2C_p)^2}$$

$$R_{N,max} = \frac{\cancel{\omega(C_1C_2+C_1C_p+C_2C_p)}\left(\frac{C_1C_2}{C_p}\right)}{2\omega^{\cancel{2}}(C_1C_2+C_1C_p+C_2C_p)^{\cancel{2}}}$$

$$R_{N,max} = \frac{1}{\frac{2\omega C_p}{C_1C_2}(C_1C_2+C_1C_p+C_2C_p)}$$

$$R_{N,max} = \frac{1}{2\omega C_p\left(1+\frac{C_p}{C_2}+\frac{C_p}{C_1}\right)}$$

$$\boxed{R_{N,max} = \frac{1}{2\omega C_p\left(1+C_p\left(\frac{C_1+C_2}{C_1C_2}\right)\right)}}$$

→ The start-up time at $g_{m,optimised}$:-

$$T_s = \frac{2L_m}{1/2\omega C_p\left(1+C_p\left(\frac{C_1+C_2}{C_1C_2}\right)\right)-R_m}\left[\ln\left(\frac{0.9\omega C_T V_{DD}}{|I_m(0)|}\right)\right]$$

→ The Total-Energy during the start-up is given by :-

$$E = P \times t$$

$$E = V_{DD} \times I_D \times T_s$$

where, I_D is the current consumption of the crystal oscillator.

$$I_D = \frac{\mu C_{ox} W}{2L} (V_{GS} - V_{Th})^2$$

$$g_m = \frac{\partial I_D}{\partial V_{GS}} = \frac{\mu C_{ox} W}{L} (V_{GS} - V_{Th})$$

$V_{ov} = V_{GS} - V_{Th}$ is the overdrive voltage.

$$I_D = \frac{1}{2}\left[\frac{\mu C_{ox} W}{L}(V_{GS} - V_{Th})\right](V_{GS} - V_{Th})$$

$$\boxed{I_D = \frac{1}{2} g_m V_{ov}}$$

→ The Voltage variation due to switching of capacitors :-

Before switching the charge is ;

$$Q_0 = C_0 V_0$$

Now, capacitor C_u is added (switched ON)

∴ Here, charge remains constant.

$$C_{new} = C_0 + C_u$$

$$Q_0 = C_0 V_0 = (C_0 + C_u) V$$

$$V = \frac{C_0 V_0}{C_0 + C_u} = V_0 \left[\frac{1}{1 + \frac{C_u}{C_0}} \right]$$

Voltage Drop (ΔV) = $V_0 - V$

$$\Delta V = V_0 - V_0 \left[\frac{1}{1 + \frac{C_u}{C_0}} \right]$$

$$\Delta V = \frac{V_0 (C_u / C_0)}{(1 + C_u / C_0)} = V_0 \left(\frac{1}{1 + C_0 / C_u} \right)$$

$$\boxed{\Delta V = V_0 \left(\frac{1}{1 + \frac{C_0}{C_u}} \right)}$$