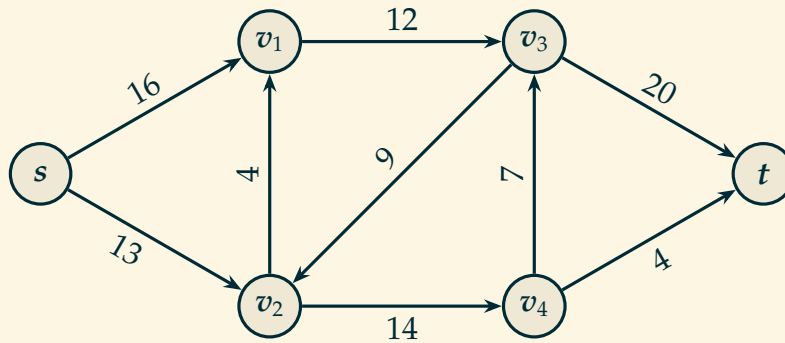


最大流、线性规划、单纯形法

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本文给出用单纯形法求解如下流网络最大流的详细过程。



先将其转化成线性规划问题，根据容量限制和流量守恒分别有

$$\left\{ \begin{array}{l} 0 \leq x_1 \leq 16 \\ 0 \leq x_2 \leq 13 \\ 0 \leq x_3 \leq 4 \\ 0 \leq x_4 \leq 12 \\ 0 \leq x_5 \leq 9 \\ 0 \leq x_6 \leq 14 \\ 0 \leq x_7 \leq 7 \\ 0 \leq x_8 \leq 20 \\ 0 \leq x_9 \leq 4 \end{array} \right. \quad \left\{ \begin{array}{l} v_1 : x_1 + x_3 - x_4 = 0 \\ v_2 : x_2 + x_5 - x_3 - x_6 = 0 \\ v_3 : x_4 + x_7 - x_5 - x_8 = 0 \\ v_4 : x_6 - x_7 - x_9 = 0 \end{array} \right.$$

为每个容量上限约束 $x_i \leq c_i$ 引入非负松弛变量 y_i 将其转化为等式约束 $x_i + y_i = c_i$ ，于是最大流问题对应的标准形式的线性规划为

$$\begin{array}{ll} \max & x_1 + x_2 \\ \text{s.t.} & x_1 + x_3 - x_4 = 0 \\ & x_2 + x_5 - x_3 - x_6 = 0 \\ & x_4 + x_7 - x_5 - x_8 = 0 \end{array}$$

$$x_6 - x_7 - x_9 = 0$$

$$x_1 + y_1 = 16$$

$$x_2 + y_2 = 13$$

$$x_3 + y_3 = 4$$

$$x_4 + y_4 = 12$$

$$x_5 + y_5 = 9$$

$$x_6 + y_6 = 14$$

$$x_7 + y_7 = 7$$

$$x_8 + y_8 = 20$$

$$x_9 + y_9 = 4$$

$$x_i, y_i \geq 0, i \in [9]$$

其中共有 18 个变量、13 个等式约束，因此基本变量有 13 个，非基本变量有 5 个。

初始不妨取 $x_{\{1,2,4,5,7\}}$ 为非基本变量，将基本变量 $x_{\{3,6,8,9\}}$ 和 $y_{\{1,\dots,9\}}$ 由 $x_{\{1,2,4,5,7\}}$ 表出：

$$\begin{aligned} x_3 = -x_1 + x_4 &\Rightarrow x_1 + x_3 - x_4 = 0 &\Rightarrow -x_1 + x_4 + y_3 = 4 \\ x_8 = x_4 - x_5 + x_7 &\Rightarrow -x_4 + x_5 - x_7 + x_8 = 0 &\Rightarrow x_4 - x_5 + x_7 + y_8 = 20 \\ x_6 = x_2 + x_5 - x_3 &\Rightarrow -x_1 - x_2 + x_4 - x_5 + x_6 = 0 &\Rightarrow x_1 + x_2 - x_4 + x_5 + y_6 = 14 \\ x_9 = x_6 - x_7 &\Rightarrow -x_1 - x_2 + x_4 - x_5 + x_7 + x_9 = 0 &\Rightarrow x_1 + x_2 - x_4 + x_5 - x_7 + y_9 = 4 \end{aligned}$$

单纯形表为

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	
x_3	1		1	-1															0
x_6	-1	-1		1	-1	1													0
x_8				-1	1		-1	1											0
x_9	-1	-1		1	-1		1		1										0
y_1	1									1									16
y_2		1									1								13
y_3	-1			1								1							4
y_4				1									1						12
y_5					1									1					9
y_6	1	1		-1	1										1				14
y_7							1									1			7
y_8				1	-1		1										1		20
y_9	1	1		-1	1		-1											1	4
	-1	-1																	0

注意 $x_{\{3,6,8,9\}}$ 和 $y_{\{1,\dots,9\}}$ 对应的列构成单位阵, 因此令 $x_{\{1,2,4,5,7\}} = 0$ 可得基本可行解

$$\left[\begin{array}{cccccccccccccccccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 13 & 4 & 12 & 9 & 14 & 7 & 20 & 4 & 0 \end{array} \right]$$

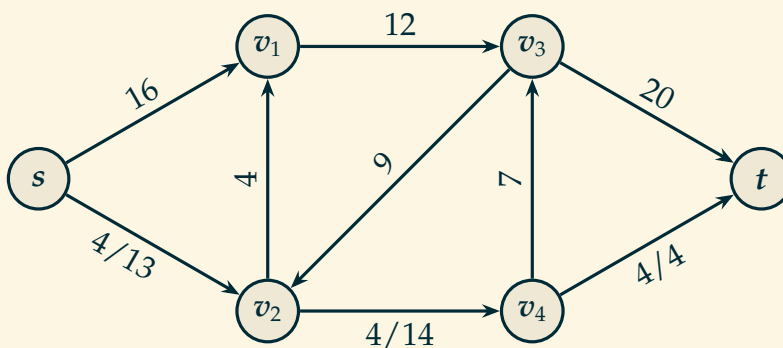
取 x_2 为输入变量, $\theta_{y_2} = 13$ 、 $\theta_{y_6} = 14$ 、 $\theta_{y_9} = 4$, 因此 y_9 为分离变量, 做初等行变换更新单纯形表

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	
x_3	1		1	-1															0
x_6						1	-1											-1	4
x_8				-1	1		-1	1											0
x_9									1									1	4
y_1	1									1									16
y_2	-1			1	-1		1				1							-1	9
y_3	-1			1								1							4
y_4				1									1						12
y_5					1									1					9
y_6							1								1			-1	10
y_7							1									1			7
y_8				1	-1		1										1		20
x_2	1	1		-1	1		-1											1	4
				-1	1		-1											1	4

当前基本可行解为

$$\left[\begin{array}{cccccccccccccccccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & 0 \\ 0 & 4 & 0 & 0 & 0 & 4 & 0 & 0 & 4 & 16 & 9 & 4 & 12 & 9 & 10 & 7 & 20 & 0 & 4 \end{array} \right]$$

对应的流网络为



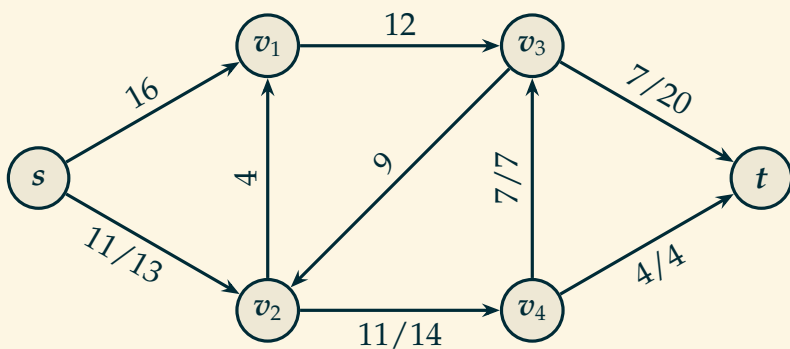
第 2 轮, 取 x_7 为输入变量, $\theta_{y_2} = 9$ 、 $\theta_{y_6} = 10$ 、 $\theta_{y_7} = 7$ 、 $\theta_{y_8} = 20$, 因此 y_7 为分离变量, 做初等行变换更新单纯形表

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	
x_3	1		1	-1															0
x_6						1										1	-1		11
x_8				-1	1			1								1			7
x_9									1									1	4
y_1	1									1									16
y_2	-1			1	-1						1					-1	-1		2
y_3	-1			1								1							4
y_4				1									1						12
y_5					1									1					9
y_6															1	-1	-1		3
x_7							1									1			7
y_8				1	-1											-1	1		13
x_2	1	1		-1	1											1		1	11
				-1	1											1		1	11

当前基本可行解为

$$\left[\begin{array}{cccccccccccccccccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & o \\ 0 & 11 & 0 & 0 & 0 & 11 & 7 & 7 & 4 & 16 & 2 & 4 & 12 & 9 & 3 & 0 & 13 & 0 & 11 \end{array} \right]$$

对应的流网络为



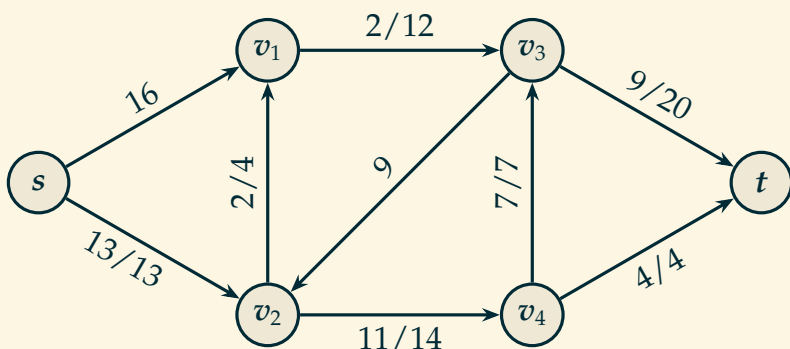
第 3 轮, 取 x_4 为输入变量, $\theta_{y_2} = 2$ 、 $\theta_{y_3} = 4$ 、 $\theta_{y_4} = 12$ 、 $\theta_{y_8} = 13$, 因此 y_2 为分离变量, 做初等行变换更新单纯形表

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	
x_3			1	-1							1					-1	-1		2
x_6						1										1	-1		11
x_8	-1							1			1							-1	9
x_9									1									1	4
y_1	1									1									16
x_4	-1			1	-1						1					-1	-1		2
y_3					1						-1	1				1	1		2
y_4	1				1						-1		1			1	1		10
y_5					1									1					9
y_6															1	-1	-1		3
x_7							1									1			7
y_8	1										-1						1	1	11
x_2		1									1								13
	-1										1								13

当前基本可行解为

$$\left[\begin{array}{cccccccccccccccccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & o \\ 0 & 13 & 2 & 2 & 0 & 11 & 7 & 9 & 4 & 16 & 0 & 2 & 10 & 9 & 3 & 0 & 11 & 0 & 13 \end{array} \right]$$

对应的流网络为



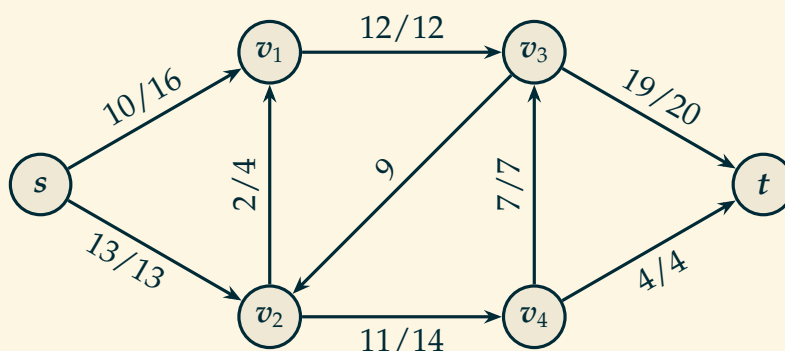
第 4 轮, 取 x_1 为输入变量, $\theta_{y_1} = 16$ 、 $\theta_{y_4} = 10$ 、 $\theta_{y_8} = 11$, 因此 y_4 为分离变量, 做初等行变换更新单纯形表

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	
x_3			1		-1						1					-1	-1		2
x_6						1										1	-1		11
x_8					1			1					1			1			19
x_9									1									1	4
y_1					-1					1	1		-1			-1	-1		6
x_4				1									1						12
y_3					1						-1	1				1	1		2
x_1	1				1						-1		1			1	1		10
y_5					1									1					9
y_6															1	-1	-1		3
x_7							1									1			7
y_8					-1								-1			-1	1		1
x_2		1									1								13
					1								1			1	1		23

当前基本可行解为

$$\left[\begin{array}{cccccccccccccccccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & o \\ 10 & 13 & 2 & 12 & 0 & 11 & 7 & 19 & 4 & 6 & 0 & 2 & 0 & 9 & 3 & 0 & 1 & 0 & 23 \end{array} \right]$$

对应的流网络为



目标行所有元素均非负, 因此当前解已为最优解, 对应的流网络达到最大流。