最大流、线性规划、单纯形法

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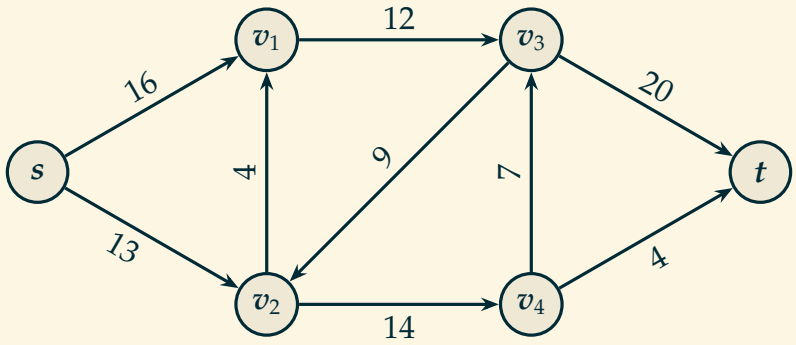
本文给出用单纯形法求解如下流网络最大流的详细过程。



先将其转化成线性规划问题，根据容量限制和流量守恒分别有

$$\begin{cases} 0 \le x_1 \le 16 \\ 0 \le x_2 \le 13 \\ 0 \le x_3 \le 4 \\ 0 \le x_4 \le 12 \\ 0 \le x_5 \le 9 \\ 0 \le x_6 \le 14 \\ 0 \le x_7 \le 7 \\ 0 \le x_8 \le 20 \\ 0 \le x_9 \le 4 \end{cases} \qquad \begin{cases} v_1: \ x_1 + x_3 - x_4 = 0 \\ v_2: \ x_2 + x_5 - x_3 - x_6 = 0 \\ v_3: \ x_4 + x_7 - x_5 - x_8 = 0 \\ v_4: \ x_6 - x_7 - x_9 = 0 \end{cases}$$

其标准形式的线性规划为

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_3 - x_4 = 0 \\ & x_2 + x_5 - x_3 - x_6 = 0 \\ & x_4 + x_7 - x_5 - x_8 = 0 \end{aligned}$$

$$
\begin{aligned}
x_6 - x_7 - x_9 &= 0 \\
x_1 + y_1 &= 16 \\
x_2 + y_2 &= 13 \\
x_3 + y_3 &= 4 \\
x_4 + y_4 &= 12 \\
x_5 + y_5 &= 9 \\
x_6 + y_6 &= 14 \\
x_7 + y_7 &= 7 \\
x_8 + y_8 &= 20 \\
x_9 + y_9 &= 4 \\
x_i, y_i &\geq 0,\ i \in [9]
\end{aligned}
$$

共有 18 个变量、13 个等式约束，因此基本变量有 13 个，非基本变量有 5 个。

不妨取 $x_{\{1,2,4,5,7\}}$ 为非基本变量，将基本变量由非基本变量表出：

$$
\begin{aligned}
x_1 + x_3 - x_4 = 0 &\Longrightarrow x_3 = -x_1 + x_4 \\
x_4 + x_7 - x_5 - x_8 = 0 &\Longrightarrow x_8 = x_4 - x_5 + x_7 \\
x_2 + x_5 - x_3 - x_6 = 0 &\Longrightarrow x_6 = x_2 + x_5 - x_3 = x_1 + x_2 - x_4 + x_5 \\
x_6 - x_7 - x_9 = 0 &\Longrightarrow x_9 = x_6 - x_7 = x_1 + x_2 - x_4 + x_5 - x_7
\end{aligned}
$$

易知初始单纯形表为

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	
x_3	1		1	−1															0
x_6	−1	−1		1	−1	1													0
x_8				−1	1		−1	1											0
x_9	−1	−1		1	−1		1		1										0
y_1	1									1									16
y_2		1									1								13
y_3	−1			1								1							4
y_4				1									1						12
y_5					1									1					9
y_6	1	1		−1	1										1				14
y_7							1									1			7
y_8				1	−1		1										1		20
y_9	1	1		−1	1		−1											1	4
	−1	−1																	0

注意基本变量对应的列构成单位阵，因此令 $x_{\{1,2,4,5,7\}}=0$ 可得基本可行解

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	o
0	0	0	0	0	0	0	0	0	16	13	4	12	9	14	7	20	4	0

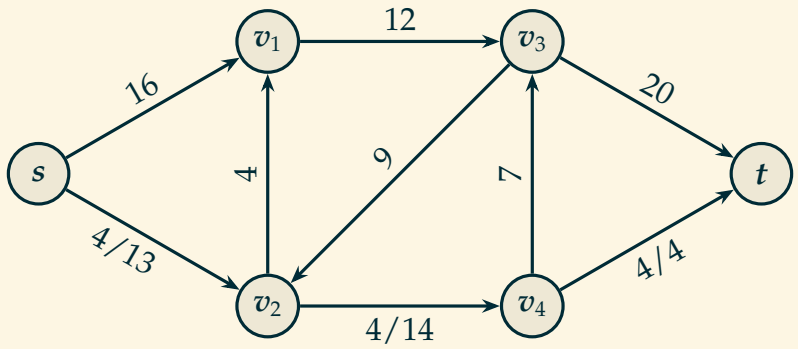
取 x_2 为输入变量，$\theta_{y_2}=13$、$\theta_{y_6}=14$、$\theta_{y_9}=4$，因此 y_9 为分离变量，做初等行变换更新单纯形表

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	
x_3	1		1	−1															0
x_6						1	−1												4
x_8				−1	1		−1	1											0
x_9									1									1	4
y_1	1									1									16
y_2	−1			1	−1		1				1							−1	9
y_3	−1			1								1							4
y_4				1									1						12
y_5					1									1					9
y_6							1								1			−1	10
y_7							1									1			7
y_8				1	−1		1										1		20
x_2	1	1		−1	1		−1											1	4
				−1	1		−1											1	4

当前基本可行解为

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	o
0	4	0	0	0	4	0	0	4	16	9	4	12	9	10	7	20	0	4

对应的流网络为



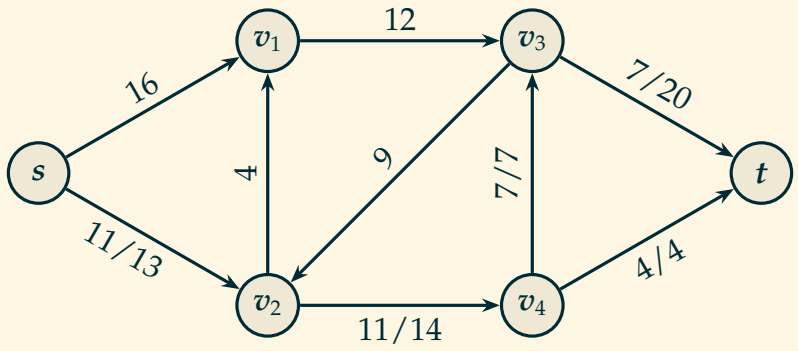
第 2 轮，取 x_7 为输入变量，$\theta_{y_2}=9$、$\theta_{y_6}=10$、$\theta_{y_7}=7$、$\theta_{y_8}=20$，因此 y_7 为分离变量，做初等行变换更新单纯形表

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	
x_3	1		1	−1															0
x_6						1										1			11
x_8				−1	1			1								1			7
x_9									1									1	4
y_1	1									1									16
y_2	−1			1	−1						1					−1		−1	2
y_3	−1			1								1							4
y_4				1									1						12
y_5					1									1					9
y_6															1	−1		−1	3
x_7							1									1			7
y_8				1	−1											−1	1		13
x_2	1	1		−1	1											1		1	11
				−1	1											1		1	11

当前基本可行解为

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	o
0	11	0	0	0	11	7	7	4	16	2	4	12	9	3	0	13	0	11

对应的流网络为



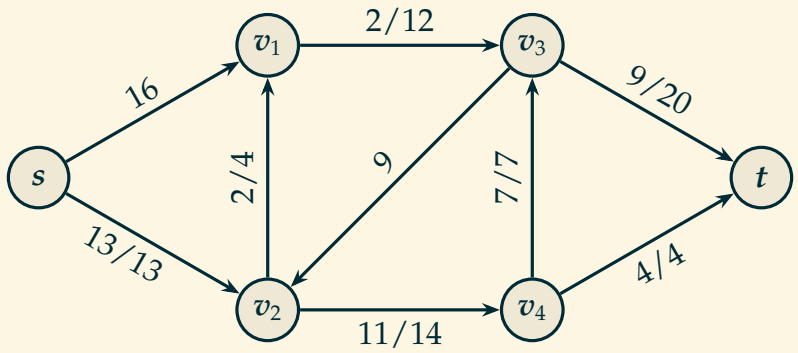
第 3 轮，取 x_4 为输入变量，$\theta_{y_2}=2$、$\theta_{y_3}=4$、$\theta_{y_4}=12$、$\theta_{y_8}=13$，因此 y_2 为分离变量，做初等行变换更新单纯形表

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	
x_3			1		−1						1					−1		−1	2
x_6						1										1			11
x_8	−1							1			1							−1	9
x_9									1									1	4
y_1	1									1									16
x_4	−1			1	−1						1					−1		−1	2
y_3					1						−1	1				1		1	2
y_4	1				1						−1		1			1		1	10
y_5					1									1					9
y_6															1	−1		−1	3
x_7							1									1			7
y_8	1										−1						1	1	11
x_2		1									1								13
	−1										1								13

当前基本可行解为

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	o
0	13	2	2	0	11	7	9	4	16	0	2	10	9	3	0	11	0	13

对应的流网络为



第 4 轮，取 x_1 为输入变量，$\theta_{y_1}=16$、$\theta_{y_4}=10$、$\theta_{y_8}=11$，因此 y_4 为分离变量，做初等行变换更新单纯形表

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	
x_3			1		−1						1					−1		−1	2
x_6						1										1			11
x_8					1			1					1			1			19
x_9									1									1	4
y_1					−1					1	1		−1			−1		−1	6
x_4				1									1						12
y_3					1						−1	1				1		1	2
x_1	1				1						−1		1			1		1	10
y_5					1									1					9
y_6															1	−1		−1	3
x_7							1									1			7
y_8					−1								−1			−1	1		1
x_2		1									1								13
					1								1			1		1	23

当前基本可行解为

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	o
10	13	2	12	0	11	7	19	4	6	0	2	0	9	3	0	1	0	23

目标行所有元素均非负，因此这就是最优解，对应的流网络达到最大流