最大流、线性规划、单纯形法

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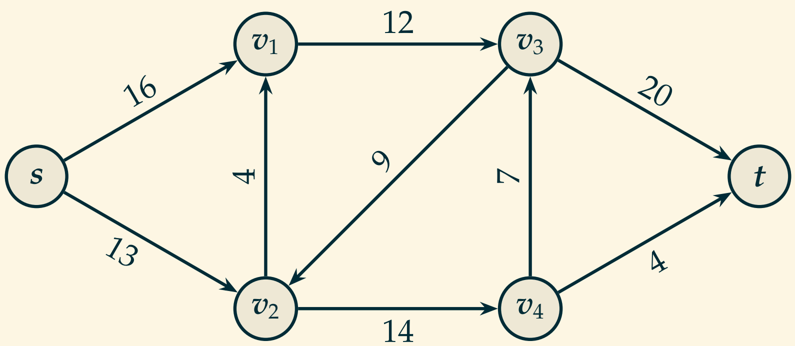
本文给出用单纯形法求解如下流网络最大流的详细过程。



先将其转化成线性规划问题，根据容量限制和流量守恒分别有

$$
\begin{cases}
0 \le x_1 \le 16 \\
0 \le x_2 \le 13 \\
0 \le x_3 \le 4 \\
0 \le x_4 \le 12 \\
0 \le x_5 \le 9 \\
0 \le x_6 \le 14 \\
0 \le x_7 \le 7 \\
0 \le x_8 \le 20 \\
0 \le x_9 \le 4
\end{cases}
\qquad
\begin{cases}
v_1: \ x_1 + x_3 - x_4 = 0 \\
v_2: \ x_2 + x_5 - x_3 - x_6 = 0 \\
v_3: \ x_4 + x_7 - x_5 - x_8 = 0 \\
v_4: \ x_6 - x_7 - x_9 = 0
\end{cases}
$$

为每个容量上限约束 $x_i \le c_i$ 引入非负松弛变量 y_i 将其转化为等式约束 $x_i + y_i = c_i$，于是最大流问题对应的标准形式的线性规划为

$$
\begin{aligned}
\max \quad & x_1 + x_2 \\
\text{s.t.} \quad & x_1 + x_3 - x_4 = 0 \\
& x_2 + x_5 - x_3 - x_6 = 0 \\
& x_4 + x_7 - x_5 - x_8 = 0
\end{aligned}
$$

$$
\begin{aligned}
&x_6 - x_7 - x_9 = 0\\
&x_1 + y_1 = 16\\
&x_2 + y_2 = 13\\
&x_3 + y_3 = 4\\
&x_4 + y_4 = 12\\
&x_5 + y_5 = 9\\
&x_6 + y_6 = 14\\
&x_7 + y_7 = 7\\
&x_8 + y_8 = 20\\
&x_9 + y_9 = 4\\
&x_i, y_i \geq 0,\ i \in [9]
\end{aligned}
$$

其中共有 18 个变量、13 个等式约束，因此基本变量有 13 个，非基本变量有 5 个。

初始不妨取 $x_{\{1,2,4,5,7\}}$ 为非基本变量，将基本变量 $x_{\{3,6,8,9\}}$ 和 $y_{\{1,\ldots,9\}}$ 由 $x_{\{1,2,4,5,7\}}$ 表出：

$$
\begin{aligned}
x_3 = -x_1 + x_4 \quad &\Rightarrow \quad x_1 + x_3 - x_4 = 0 &&\Rightarrow \quad -x_1 + x_4 + y_3 = 4\\
x_8 = x_4 - x_5 + x_7 \quad &\Rightarrow \quad -x_4 + x_5 - x_7 + x_8 = 0 &&\Rightarrow \quad x_4 - x_5 + x_7 + y_8 = 20\\
x_6 = x_2 + x_5 - x_3 \quad &\Rightarrow \quad -x_1 - x_2 + x_4 - x_5 + x_6 = 0 &&\Rightarrow \quad x_1 + x_2 - x_4 + x_5 + y_6 = 14\\
x_9 = x_6 - x_7 \quad &\Rightarrow \quad -x_1 - x_2 + x_4 - x_5 + x_7 + x_9 = 0 &&\Rightarrow \quad x_1 + x_2 - x_4 + x_5 - x_7 + y_9 = 4
\end{aligned}
$$

单纯形表为

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	
x_3	1		1	−1															0
x_6	−1	−1		1	−1	1													0
x_8				−1	1		−1	1											0
x_9	−1	−1		1	−1		1		1										0
y_1	1									1									16
y_2		1									1								13
y_3	−1			1								1							4
y_4				1									1						12
y_5					1									1					9
y_6	1	1		−1	1										1				14
y_7							1									1			7
y_8				1	−1		1										1		20
y_9	1	1		−1	1		−1											1	4
	−1	−1																	0

注意 $x_{\{3,6,8,9\}}$ 和 $y_{\{1,\ldots,9\}}$ 对应的列构成单位阵，因此令 $x_{\{1,2,4,5,7\}}=0$ 可得基本可行解

$$\left[\begin{array}{cccccccccccccccccc:c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & o \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 & 13 & 4 & 12 & 9 & 14 & 7 & 20 & 4 & 0 \end{array}\right]$$

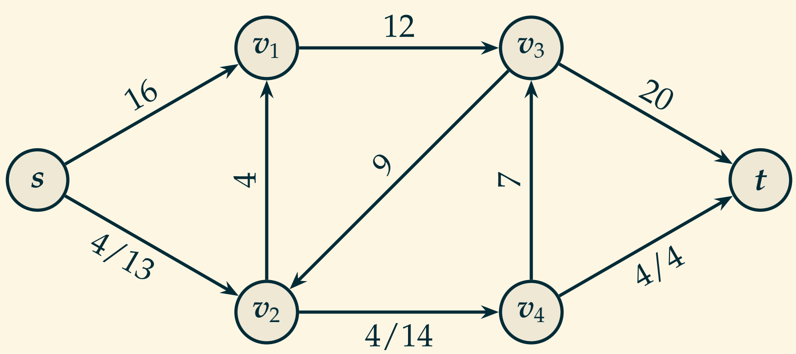
取 x_2 为输入变量，$\theta_{y_2}=13$、$\theta_{y_6}=14$、$\theta_{y_9}=4$，因此 y_9 为分离变量，做初等行变换更新单纯形表

$$\begin{array}{c|cccccccccccccccccc:c}
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & \\ \hline
x_3 & 1 & & 1 & -1 & & & & & & & & & & & & & & & 0 \\
x_6 & & & & & & 1 & -1 & & & & & & & & & & & -1 & 4 \\
x_8 & & & & -1 & 1 & & -1 & 1 & & & & & & & & & & & 0 \\
x_9 & & & & & & & & & 1 & & & & & & & & & 1 & 4 \\
y_1 & 1 & & & & & & & & & 1 & & & & & & & & & 16 \\
y_2 & -1 & & & 1 & -1 & & 1 & & & & 1 & & & & & & & -1 & 9 \\
y_3 & -1 & & & 1 & & & & & & & & 1 & & & & & & & 4 \\
y_4 & & & & 1 & & & & & & & & & 1 & & & & & & 12 \\
y_5 & & & & & 1 & & & & & & & & & 1 & & & & & 9 \\
y_6 & & & & & & & 1 & & & & & & & & 1 & & & -1 & 10 \\
y_7 & & & & & & & 1 & & & & & & & & & 1 & & & 7 \\
y_8 & & & & 1 & -1 & & 1 & & & & & & & & & & 1 & & 20 \\
x_2 & 1 & 1 & & -1 & 1 & & -1 & & & & & & & & & & & 1 & 4 \\ \hdashline
 & & & & -1 & 1 & & -1 & & & & & & & & & & & 1 & 4
\end{array}$$

当前基本可行解为

$$\left[\begin{array}{cccccccccccccccccc:c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & o \\ 0 & 4 & 0 & 0 & 0 & 4 & 0 & 0 & 4 & 16 & 9 & 4 & 12 & 9 & 10 & 7 & 20 & 0 & 4 \end{array}\right]$$

对应的流网络为



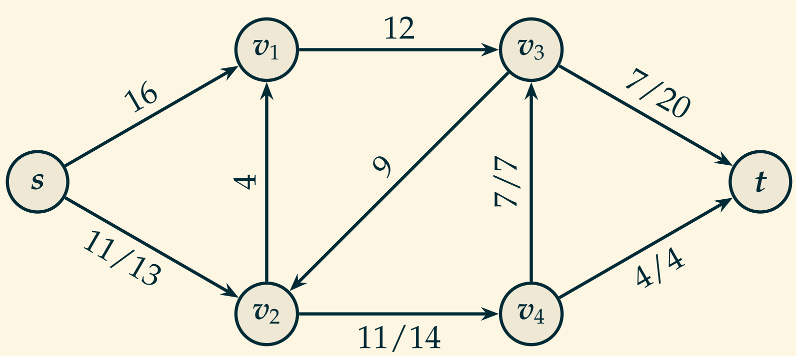
第 2 轮，取 x_7 为输入变量，$\theta_{y_2} = 9$、$\theta_{y_6} = 10$、$\theta_{y_7} = 7$、$\theta_{y_8} = 20$，因此 y_7 为分离变量，做初等行变换更新单纯形表

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	
x_3	1		1	−1															0
x_6						1										1		−1	11
x_8				−1	1			1								1			7
x_9									1									1	4
y_1	1									1									16
y_2	−1			1	−1						1					−1		−1	2
y_3	−1			1								1							4
y_4				1									1						12
y_5					1									1					9
y_6															1	−1		−1	3
x_7							1									1			7
y_8				1	−1											−1	1		13
x_2	1	1		−1	1											1		1	11
				−1	1											1		1	11

当前基本可行解为

$$\left[\begin{array}{cccccccccccccccccc:c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & o \\ 0 & 11 & 0 & 0 & 0 & 11 & 7 & 7 & 4 & 16 & 2 & 4 & 12 & 9 & 3 & 0 & 13 & 0 & 11 \end{array}\right]$$

对应的流网络为



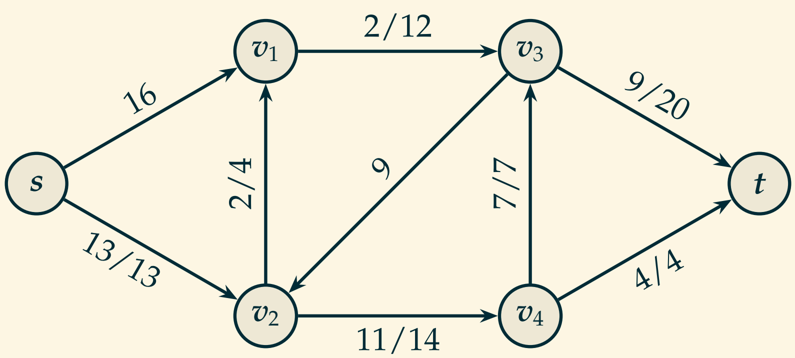
第 3 轮，取 x_4 为输入变量，$\theta_{y_2}=2$、$\theta_{y_3}=4$、$\theta_{y_4}=12$、$\theta_{y_8}=13$，因此 y_2 为分离变量，做初等行变换更新单纯形表

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	
x_3			1		-1						1					-1		-1	2
x_6						1										1		-1	11
x_8	-1							1			1							-1	9
x_9									1									1	4
y_1	1									1									16
x_4	-1			1	-1						1					-1		-1	2
y_3					1						-1	1				1		1	2
y_4	1				1						-1		1			1		1	10
y_5					1									1					9
y_6															1	-1		-1	3
x_7							1									1			7
y_8	1										-1						1	1	11
x_2		1									1								13
	-1										1								13

当前基本可行解为

$$\left[\begin{array}{cccccccccccccccccc:c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & o \\ 0 & 13 & 2 & 2 & 0 & 11 & 7 & 9 & 4 & 16 & 0 & 2 & 10 & 9 & 3 & 0 & 11 & 0 & 13 \end{array}\right]$$

对应的流网络为



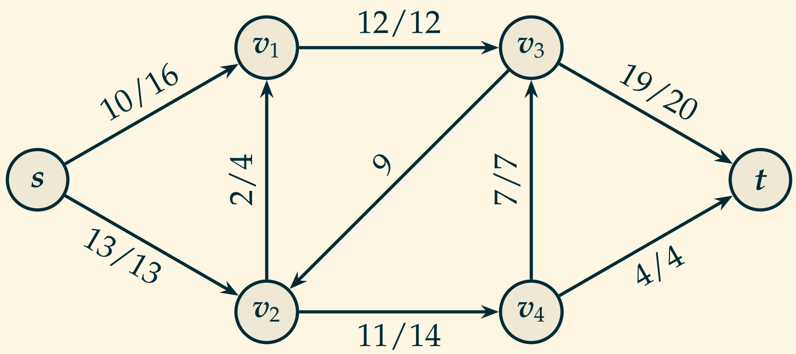
第 4 轮，取 x_1 为输入变量，$\theta_{y_1}=16$、$\theta_{y_4}=10$、$\theta_{y_8}=11$，因此 y_4 为分离变量，做初等行变换更新单纯形表

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	
x_3			1		−1						1					−1		−1	2
x_6						1										1		−1	11
x_8					1			1					1			1			19
x_9									1									1	4
y_1					−1					1	1		−1			−1		−1	6
x_4				1									1						12
y_3					1						−1	1				1		1	2
x_1	1				1						−1		1			1		1	10
y_5					1									1					9
y_6															1	−1		−1	3
x_7							1									1			7
y_8					−1								−1			−1	1		1
x_2		1									1								13
					1								1			1		1	23

当前基本可行解为

$$\left[\begin{array}{cccccccccccccccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & o \\ 10 & 13 & 2 & 12 & 0 & 11 & 7 & 19 & 4 & 6 & 0 & 2 & 0 & 9 & 3 & 0 & 1 & 0 & 23 \end{array}\right]$$

对应的流网络为



目标行所有元素均非负，因此当前解已为最优解，对应的流网络达到最大流。