71-23,572

WIRT, John Gary, 1939-
OPTIMIZATION OF PRICE AND QUALITY IN SERVICE SYSTEMS.

Stanford University, Ph.D., 1971
Engineering, general

University Microfilms, A XEROX Company, Ann Arbor, Michigan

OPTIMIZATION OF PRICE AND QUALITY IN SERVICE SYSTEMS

A DISSERTATION

SUBMITTED TO THE DEPARTMENT OF ENGINEERING ECONOMIC SYSTEMS

AND THE COMMITTEE ON THE GRADUATE DIVISION

OF STANFORD UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

By

John G. Wirt

December 1970

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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(Computer Science)

Approved for the University Committee
on the Graduate Division:

Dean of the Graduate Division

OPTIMIZATION OF PRICE AND QUALITY IN SERVICE SYSTEMS

John G. Wirt, Ph.D.
Stanford University, 1970

Price and service quality are important variables in the design of optimal service systems. Price is important because of the strong consumption externality inherent in service systems. Quality is important since it directly affects consumer welfare, and is readily controlled by altering the priorities of service offered. This thesis takes a systematic view of this problem, and presents techniques for quantitative determination of the optimal prices and service quality in a wide class of systems.

A probabilistic demand model sensitive to both price and quality is derived from the microeconomic concept that a consumer chooses the service which maximizes the difference between willingness-to-pay and price. This model for the individual's demand is then aggregated to obtain a partial equilibrium macroeconomic model of demand.

After constructing the demand model's likelihood function, a Bayesian inference technique is developed for using data to reduce uncertainty about consumer's values. Records of actual consumption--rather than interview data or other indirect data--are used in the calibration procedure. Since only samples of the likelihood function are used, the inference technique is more generally applicable than just to this particular demand model.

Another phase of this work develops criteria for optimizing prices and choosing among alternative qualities of service. Due to the stochastic nature of demand and uncertainty about parameters in the demand model, system payoff is necessarily a random variable. It is argued that the optimal rule is to extremize the expected value of system payoff per unit of time. For private decision makers, expected profits are maximized, and for public decision

makers, aggregate willingness to pay is maximized. This approach emphasizes that an optimum system can be specified even though there is considerable uncertainty about demand.

A final phase of this work applies stochastic approximation to finding optimal prices in service systems. Since only a simulation of system operation is required, the method can be applied to a very wide class of service problems; the principal limitation being the amount of computer time required for optimization. Application of the method to a discrete service time, multipriority queue demonstrates that, at least in the cases tested, only a marginal increase in aggregate social welfare is obtained from establishing multiple priorities in a service system. A major shift occurs, however, in the incidence of benefits.

Approved for publication:

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ACKNOWLEDGMENTS

The helpful suggestions made during the course of this research by Dr. Richard D. Smallwood are gratefully acknowledged. His guidance and encouragement were invaluable aids to progress.

Thanks are also due to Dr. Norman Nielson and Dr. Robert C. Lind for careful reading of the manuscript and attentive remarks.

Financial support was generously provided by the Department of Engineering Economic Systems and NSF grant DK-1683.

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INTRODUCTION

1.0 OPTIMIZATION OF SERVICE SYSTEMS

1.1 General Description of the Problem

Two of the most important variables in the design of optimal service systems are price and product quality. Price is important, especially when service is performed for the public, because of the strong externality inherent in service systems.[1] This externality arises because consumption of a service by one consumer alters the quality of the product offered to subsequent consumers, thereby affecting their welfare. Unless an appropriate toll is charged to make the initial consumer aware of how his consumption affects others, he will demand more than the socially optimum amount of service. Product quality is important since it directly affects consumer welfare, and is readily controlled by varying the types of service offered.

This thesis presents a conceptual approach and a set of techniques for quantitatively determining the optimal prices and product quality in service systems. These techniques are applied to the case of priority queues, resulting in new insights about the welfare aspects of operating such systems.

Systems that provide a service are different from systems that provide a product in that the quality of output is usually a function of the level of demand. In production systems excess capacity is stored in the form of inventories, making it possible to satisfy peaks of demand while maintaining constant product quality. Just the opposite occurs in service systems, where, because of fixed capacity, excesses of demand over capacity must be eliminated by lowering service quality until a demand/supply equilibrium is achieved. An obvious example of such a feedback mechanism is the queuing system. While much work has been done on specifying optimal policies for control of inventories in production systems, comparatively little has been accomplished on specifying optimal policies in service systems.

A central feature of this thesis is that demand is a function of both price and the level of quality. If demand were assumed independent of price, price would not be an effective control for achieving an optimum. If demand were assumed independent of quality, the apparent preference of consumers for higher quality service would be denied. As a consequence of these dependence assumptions, major effort is expended on constructing a demand model that incorporates both variables.

The demand model developed here originates in the model of individual choice that has been well developed by economists, but it also incorporates the decision-theory point of view reflecting the uncertainty which is unavoidable in service systems. Two kinds of uncertainty are treated: (1) objective uncertainty corresponding to the irreducible randomness in demand, and (2) subjective uncertainty which describes the analyst's current knowledge about demand-model parameters.

1.2 Real-World Applications

Two obvious applications of the methods of this thesis are pricing computer services and regulating air traffic at airports. Currently, airports use administrative rules to regulate queuing congestion, and the optimality of these rules could be compared to pricing solutions recommended here. Data-communication services in telephone systems are an example where the institution of priorities having differential prices might improve the usefulness of service to the public.

This thesis also provides a general method for optimizing the design of any facility where quality is a controllable feature of the product. Recreation and entertainment events, discriminatory pricing in transportation service, and safe-deposit boxes are all cases where the results of this thesis could apply.

2.0 LITERATURE REVIEW

2.1 Current Practice

Current schemes for pricing service systems are either heuristic or based on average-cost principles. The Federal Government, for example, encourages the university computation centers that it supports to use average-cost pricing. This approach guarantees that facilities will yield zero profit, but produces a severe misallocation of computer time[2] because high prices are charged during low use and low prices during high use. Some facility owners have established priority rules and used differential pricing, but heuristic methods have been employed to set these policies. The Stanford University Computation Center has, for example, instituted a differential pricing scheme having several priority classes in an attempt to smooth out peak loading problems. Smidt has recently[3] proposed some heuristic pricing schemes for computation centers; however, he gives no method for determining actual prices, nor does he include the costs of delay.

2.2 Study in Operations Research

In studying queues, operations researchers have paid scant attention to optimizing queuing system design--principally because of difficult analytical problems.[4] Some effort has been made to investigate priority schemes that reduce average delay (such as "head-of-line" priority), but the effects of price-variable demand have not been extensively considered.

Kleinrock[5] has solved for the average delay in a system (assuming fixed demand) where customers are served in order according to the bribe they pay for a specific place in line. The amount that each consumer pays is a random variable that is independent of the state of delay in the queue. Kleinrock's model is optimum in the sense that he finds the probability distribution on bribes which minimizes expected cost, subject to the constraint that the average bribe is a constant. Leeman[6] has mentioned that queues can be regulated by pricing, but has published no analysis. Hillier and Lieberman[7] have taken demand as given, assigned a constant dollar-per-minute cost to delay, and then adjusted the number of priorities and system capacity to minimize total average cost. More recently, operations researchers are extending the theory of controlling Markov processes to queuing systems. McGill[8] has reported methods for minimizing expected waiting cost plus switching costs by controlling the number of exponential servers as a function of queue length. Yadin and Zacks[9] find policies for controlling the service rate in a single server, exponential queue.

Those operations researchers who endeavor to minimize expected costs must assume that demand is fixed independent of delay or price;

otherwise minimum costs are achieved by closing down the queue altogether. But if the opportunity costs of foregoing service are considered, this pathologic result is no longer true. Treatment of opportunity costs seems to imply including the effects of price on demand.

Judging by the lack of discussion in the literature, no operations researchers have attacked the problem of finding the value of time used so freely in calculating the expected cost of operation. The difficulties inherent in exogenous determination of the value of time, usually using wage rate arguments, are well known,[10] but alternative methods have yet to appear.

2.3 Study in Economics

Economists have also treated service systems in their literature, but in an entirely different way. They apply the theory of marginal cost pricing[11] (particularly the results on pricing of congestion)[12,13] to the problem of optimizing system design. To derive the marginal-cost theory they first assume that a consumer chooses his consumption set from among a fixed and finite number of products, and behaves as if he maximizes his utility function subject to a budget constraint involving prices, quantities bought, and income.

The economists assume the existence of a production sector capable of transforming inputs into consumable outputs. By bidding through a price system, an equilibrium between consumption and production is achieved. Then, invoking the normative condition that a particular distribution of resources is socially optimal if no possible rearrangement of production and distribution will yield higher utility for all individuals (Pareto condition), they show the necessary condition for optimality, that the price of the i^{th} good, p_i, should be adjusted until

$$p_i(q_1, q_2, \ldots, q_n) = \frac{\partial C(q_1 \cdot q_2, \ldots, q_n)}{\partial q_i} \qquad i = 1, \ldots, n, \tag{I-1}$$

where q_i is the "average" quantity of the i^{th} commodity purchased. In Equation I-1, $\partial C/\partial q_i$ is the marginal cost of the i^{th} commodity. This is the familiar price-equals-marginal-cost algorithm. For congestion problems, total cost C is taken as

$$C = \sum_{i=1} V q_i \Delta_i(q_1, q_2, \ldots, q_n), \tag{I-2}$$

where $\Delta_i(\cdot)$ is the expected delay in the i^{th} priority, and V is the consumer's value for time. From Equations I-1 and I-2 the optimal price to charge each consumer for the i^{th} priority equals

$$V \cdot \Delta_i(q_1, q_2, \ldots, q_n), \text{ plus } \sum_{j=1}^{n} V q_i (\partial \Delta_i / \partial q_j).$$

Simultaneous solution of Equations I-1 and I-2 yields the equilibrium levels of demand, which in turn specifies the optimal prices by Equation I-1.

Economists have raised many objections to this theory that substantially reduce the relevance of the price-equals-marginal-cost rule.[14] First, the Pareto-optimal condition is not sensitive to the distribution of income, so that by Bergson's concept of social welfare,[15] some non-Pareto-optimal allocations may be more socially optimum than some Pareto allocations. Hence, price above or below marginal cost may have desirable distribution effects. Further, it is rarely true that price equals marginal cost elsewhere in an economy. Thus moving to a price-equals-marginal-cost situation in one activity may or may not improve social welfare.

Rather than a normative analysis like the marginal cost theory, de V. Graaff argues for positive economic studies that "... contribute to our understanding of how the economic system works in practice. If positive economics can provide people with an understanding of the various far reaching indirect effects of particular policies, it will also provide them with a basis for drawing welfare conclusions. The job of the economist is not to try to reach welfare conclusions for others, but rather to make available the positive knowledge--the information and the understanding--on the basis of which laymen can pass judgment."[16]

This thesis is dedicated to the purposes outlined by de V. Graaff. It is normative in that acceptance of the Pareto condition implies a welfare judgment, but positive in that reconfiguration of the models allows comparison of various states of optimality according to their welfare implications.

Another difficulty with the classical economist's approach is that for queuing situations it does not seem reasonable to assume that the value of time, V, is constant over all consumers. Consumers placing a high value on time will be more likely to choose the priorities that have short delay, making "average value" of time higher in the short-delay channels. The obvious modification of setting $V_i = V_i(q_1, q_2, \ldots, q_n)$ for each channel does not seem promising because a satisfactory method for estimating these functions is not available.

Intuitively, it seems that including a distribution on the value of time would significantly affect system design. With a wide distribution on the value of time, it seems likely that efficiency increases will be obtained from providing high-priority service for consumers with a high value on time. Beckman, for example, has shown that if consumers have different values for time, maximum efficiency is achieved in the two-road problem when a different toll is charged on each road.[17] With consumers placing a uniform value on time, the solution is to charge the same toll on each road.

A third shortcoming of the economists' analyses of service systems is the absence of a rationale for determining the optimum number and kinds of priorities to establish. With one exception,[18] all economic models of consumer choice and welfare assume that the number of alternatives facing the consumer is fixed.

One exception to the usual treatment of service systems by economists is the recent publication by Noar[19] concerning a single-server exponential queue. Noar assigns constant values for time and service and assumes that a consumer enters service only if he values it more than the price plus the cost of time. By doing this, Noar shows that (1) net social benefit is maximized only if a toll is charged, (2) charging no toll yields an inferior solution, and (3) maximization of profits is inferior to charging no tolls. The work of this thesis is much like Noar's except that the assumptions of constant value for service and time are relaxed and multipriority queues are considered.

Neither the economist nor the operations researcher has given much attention to uncertainty in designing queuing systems--particularly uncertainty in demand-model parameters. Among operations researchers, Clark has found maximum likelihood estimators for single-server exponential systems; Benes has found a sufficient set of statistics for a similar model having an infinite number of servers, and Wolff has derived likelihood ratio tests and maximum likelihood estimators for exponential queues.[20] Economists have not treated uncertainty in congestion problems, but instead have dealt with "average" values in their models.

3.0 CONTRIBUTIONS OF THIS RESEARCH

This thesis takes a systems approach to the problem of optimizing service-system design, and treats three basic subproblems. First, a framework for evaluating the welfare aspects of system performance is developed. Then, using the model of choice developed by economists, a demand model is derived that is sensitive to both price and quality of service. Lastly, a method is presented for finding optimum prices. The method is general enough to be applied to a wide variety of service systems.

Because service-system payoff depends strongly on the demand structure, emphasis is placed on developing a demand model. Conditions are given which guarantee the existence of a willingness to pay function for an individual, and that he maximizes the difference between his willingness to pay and the price. To account for variations over a population, the willingness-to-pay function is postulated to be a functional form involving random variables which have ready interpretation as value for quality and value for service. The resulting choice model describes the probability that a consumer chooses an alternative. The model for individual choice is then aggregated as a compound Poisson process to complete the model of demand. The resulting demand model is sensitive to both prices and service quality. Demand can be predicted when the number of priorities is changed from that used for calibration.

After constructing the demand model's likelihood function, a Bayesian inference technique is developed for using data to reduce uncertainty about consumer's values. Records of actual consumption--rather than interview data or other indirect data--are used in the calibration procedure. Since only samples of the likelihood function are used, the inference technique is more generally applicable than just to this particular demand model.

Another phase of this work develops criteria for optimizing prices and choosing among alternative qualities of service. Due to the stochastic nature of demand and uncertainty about parameters in the demand model, system payoff is necessarily a random variable. It is argued that the optimal rule is to extremize the expected value of system payoff per unit of time. For private decision makers, expected profits are maximized, and for public decision makers, aggregate willingness to pay is maximized. This approach emphasizes that an optimum system can be specified even though there is considerable uncertainty about demand. The concept of value of information from decision theory is used to state when more demand data should be acquired.

A final phase of this work applies stochastic approximation to finding optimal prices in service systems. Since only a simulation of system operation is required, the method can be applied to a very wide class of service problems--the principal limitation being the amount of computer time required for optimization. Application of the method to a discrete service time, multipriority queue demonstrates the increase in social welfare obtained by changing optimal prices and using multiple priorities.

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Chapter II

FRAMEWORK FOR OPTIMIZING SERVICE SYSTEMS

1.0 DEFINITION OF A SERVICE SYSTEM

1.1 General Properties and a Definition

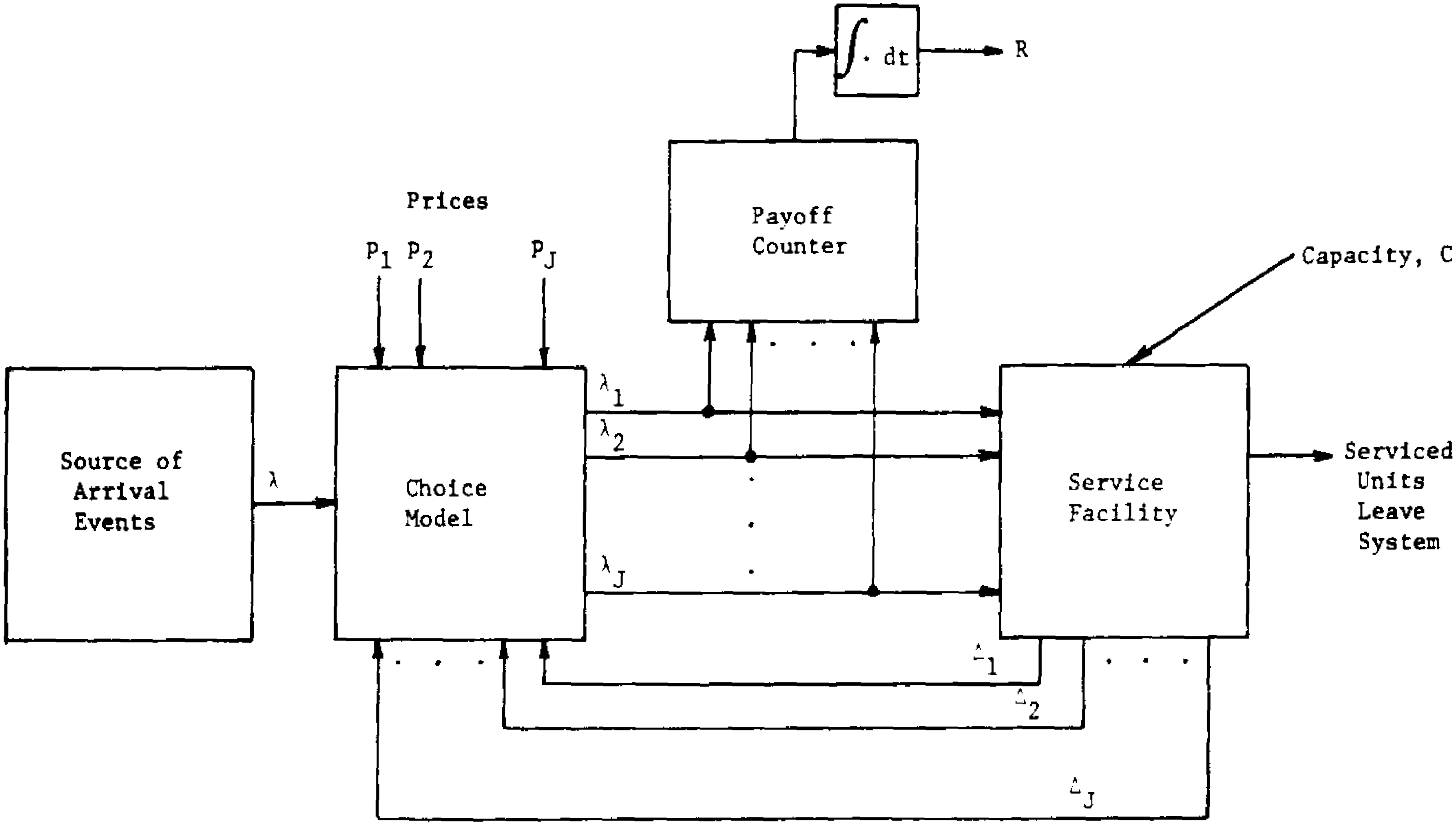
In general terms a service system is a facility in which production is a transformational process rather than manufacture of a physical good. When service is completed, the consumer is in a different state of well being, but has no additional possessions. Service is usually offered at several different qualities with a different price attached to each level of quality.

Unlike other commodities, these levels of quality are often easy to define and measure. This "measurability" property is demonstrated by the example of computer centers, where quality can be measured by expected turnaround time and the expected time-of-completion. For other commodities, such as clothing, quality is not so easily described by one or two indices. This property of service systems simplifies the relationship between quality as judged by consumers and as judged by analysts, and makes optimization of system quality a worthwhile analytical endeavor.

A second reason why optimization is a worthwhile exercise is that inexpensive alterations to the basic plant--such as changes in pricing or the classes of service--can greatly improve performance.

1.2 Basic System Configuration

A diagram of the basic service system configuration is shown in Figure II-1. Consumers arriving at the facility purchase one of the



Δ_i = current quality in i^{th} priority

λ_i = arrival rate in i^{th} priority

p_i = price in i^{th} priority

λ = arrival rate of demand events

Figure II-1--Basic Service System Configuration available priorities (or classes) of service based on price and the level of quality offered. Because the supply of service is fixed (see Section I.1.1), this consumption act lowers the level of quality offered to succeeding consumers.

It is assumed that the level of quality at time t is determined exactly for a particular system configuration once the sequence of previous arrival times, previous purchase decisions, and the consequent service loads are given. The level of quality at time t is indicated by the time dependent, quality process $\Delta_t = (\Delta_{1t}, \ldots, \Delta_{Jt})$ where Δ_{it} is the level of quality in the i^{th} class at time t.

In almost every service system, either arrival times or service loads, or both, will be modeled as stochastic processes. This forces Δ_t to be modeled as a stochastic process. Depending on the prices charged and the system configuration, the system will be in a stochastic equilibrium between the rate at which consumers arrive and the continuously fluctuating level of system quality.

The two controls in this conceptualization of a service system are price and system configuration. Price is a direct control on demand since it appears explicitly in the consumer's budget optimization. Quality, the other explicit factor in the consumer's decision, cannot be controlled directly because of the feedback effect between it and demand. Quality must be controlled indirectly through changes in the system configuration and prices.

Real-time pricing schemes are ruled out in this thesis because of the impracticality of constantly fluctuating prices. Concern here is with determining the constant price to charge over some time interval. At the end of this interval, price can be reset if desired.

2.0 SELECTION OF THE PAYOFF PROCESS

2.1 Criterion for Private Decision Makers

Each time a unit of demand enters a service system, an increment to system payoff is accumulated. Depending on the point of view, this increment equals either the price paid by the consumer or his willingness to pay.

For privately controlled service systems, price is the proper measure of payoff. This follows from the reasonable assumptions that private owners are profit maximizers, that capital investment in capacity is fixed, and that total costs are predominantly fixed costs.

Assumption 1: Privately controlled service facilities endeavor to maximize profits.

Assumption 2: Capital investment in capacity is constant.

Assumption 3: Variable operating costs are negligible compared to fixed costs.

Assumption 2 and 3 will be true only if a short-run optimization is sought since in the long run capacity is variable, and therefore all costs are variable.

Since costs are predominantly fixed, profits are maximized by maximizing revenues. The contribution of a consumption act to revenues is the price paid, and therefore the payoff random variable is a continuous time process equal to the sum of payments collected in time t:

$$R_t^p = p^1 + p^2 + \ldots + p^N. \tag{II-1}$$

The random variable p^i equals the price paid by the i^{th} consumer to arrive in the time interval $[0,t]$.

2.2 Criterion for the Social Interest

From the social viewpoint, revenue is not a reasonable choice for payoff since aggregate consumer benefit is larger than revenues. As will be shown in Chapter III, consumers endeavor to maximize the differences between what they are willing to pay over what they do pay for services consumed. Thus if the surplus of willingness to pay over cost is increased, it can be inferred that the consumer's welfare has increased. Statements about consumer welfare, though, are not sufficient to draw conclusions about social welfare, since there is no basis for interpersonal comparisons. The most common social-welfare criterion in use is Pareto optimality. It says that a particular distribution of surpluses among consumers is socially optimal if there is no rearrangement of surpluses that leaves everyone at least as much surplus as before and one person more.[1]

The Pareto principle is not a very satisfying criterion, however, since uncountably many very different patterns of surplus will usually qualify as Pareto optimal.[2]

The most convenient and commonly used criterion for partially resolving this indeterminancy is the Compensation Principle which says that society prefers Alternative A to B if the gainers under A could transfer enough of their gain to the losers so that everyone is at least as well off with A as with B. Using the willingness-to-pay surplus as a measure, each loser under Alternative A is indifferent to B if he is compensated by an income transfer equal to his decrease in surplus of willingness to pay over cost in A versus B. The gainers in A can produce this transfer payment and still remain gainers if the aggregate of surplus over all consumers is greater under Alternative A than B. Therefore society should prefer Alternative A over B if A has a higher value of willingness to pay, minus cost aggregated over all consumers. The payoff process must then be the sum of all consumer surpluses:

$$R_t = W^1 - p^1 + W^2 - p^2 + W^3 - p^3 + \dots W^N - p^n. \qquad \text{(II-2)}$$

The random variable W_i equals the willingness to pay of the i^{th} consumer for the alternative he chose.

If the service system owner is included in the social-welfare function, application of the Compensation Principle and the Pareto rule leads to the payoff process,

$$R_t = R_t + R_t^p = W^1 + W^2 + \dots W^N. \qquad \text{(II-3)}$$

It is this process which will be used as the social payoff in the following analysis.

2.3 Welfare Implications of the Compensation Principle

Many criticisms of the Compensation Principle have been made, the most penetrating being by de V. Graaff[3] and Arrow.[4] Arrow's objection is that the Compensation Principle can, if the compensation is not paid, lead to intransitivities in choice among alternatives. That is, the Compensation Principle can result in society preferring Alternative A to B, B to C, and C to A. J. de V. Graaff's objection is that the Compensation Principle does not consider the income-distribution effects of a change in social state--a sensitivity that a social-welfare function probably should exhibit. For example, suppose Plan B produces less net gain than Plan A, and that gainers under B are the poor people and under A the rich people. If compensation is not paid, choosing Plan A on the grounds of the Compensation Principle and the Pareto rule will result in the rich people being richer and the poor people being poorer. Society may not wish to make such a choice.

The only way to include income effects in the payoff process is to separate consumers by income group (or any other measure deemed appropriate) and then weight consumer contributions to willingness to pay. Assignment of weights must be a matter of judgment on the part of the decision maker. The analyst's role in such cases is to divide consumers into welfare groups identified by the administrative authorities, determine the effects of changes on each group, and then report the results to the decision makers who will use the information.

3.0 OPTIMIZATION UNDER UNCERTAINTY ON DEMAND

3.1 Sources of Uncertainty

There are two sources of uncertainty in service-system payoff. First, there is the uncertainty contributed by a lack of knowledge about parameters in the demand model. This shall be called subjective uncertainty, as it can be reduced by gathering and processing data. In systems having queuing congestion, for example, the decision maker may be unsure about the value consumers place on time, and this uncertainty will be reflected in uncertainty about payoff. The other source of uncertainty results from the impossibility of determining exactly when the next arrival will occur and which alternative will be chosen--even with perfect knowledge of demand-model parameters.

Therefore, service system payoff is a random process whose value at any time $t \in (0,\infty)$ is a random variable. The distribution of this random variable is determined by the two sources of uncertainty, and is a function of system controls, since they affect demand.

3.2 Criterion for Optimality Under Uncertainty

The service-system control problem is thus formulated as selection of an optimal control under uncertainty on system payoff. One criterion for selecting the best control is to maximize the expected value of the decision maker's utility on payoff at time T, as prescribed by statistical decision theory. There is difficulty, however, in specifying T, since most systems run for an indefinite period into the future. Making T very large is not a solution, because time preference is not easily incorporated into the utility function.

The simplest approach is to drop the specification that expected utility is to be maximized, and instead maximize the expected value of payoff at time T. The maximization of expected value amounts to assuming a risk-indifferent decision maker.

Make the further specification that it is the steady-state expected payoff per unit of time that is to be maximized. If R_t is the total payoff that accumulates the time interval $[0,t]$, the average payoff per unit of time is:

$$\overline{R} = \lim_{t\to\infty} E(R_t|\mathcal{E})/t. \qquad \text{(II-4)}$$

The expected payoff in Equation II-4 is conditioned on prior information, $\mathcal{E}$, to emphasize that all knowledge about the system and demand is to be used in calculating payoff.[6]

Assuming that the quality process (see Section 1.2) is stationary throughout the time interval [0,t], and letting R_t be the total payoff accumulated in [0,t], the expected payoff in [0,t] can be written by expansion as

$$E(R_t|E) = \sum_{N=0}^{\infty} \iint \ldots \int d\Delta_{t_1}, d\Delta_{t_2} \ldots d\Delta_{t_N} E(R_t|\Delta_{t_1}, \Delta_{t_2}, \ldots \Delta_{t_N}, N, t, E)$$

$$\cdot \Pr\{\Delta_{t_1}, \Delta_{t_2}, \ldots \Delta_{t_N}, N|t, E\}, \qquad \text{(II-5)}$$

where N is the number of arrivals in [0,t], t_i is the time of the i^{th} arrival, and Δ_{t_i} is the level of quality at t_i (Δ_{t_i} is a vector random variable with component Δ_{jt_i}, the level of quality in the j^{th} priority at time t_i). The notation $E(\cdot|\cdot)$ stands for the expected value of all random variables preceding the vertical bar, given the random variables and information following the vertical bar.

Under the assumption that the contribution to payoff by an event depends only on the value of the quality process at the time of arrival,

$$E(R_t|\Delta_{t_1}, \Delta_{t_2}, \ldots \Delta_{t_N}, N, t, E) = \sum_{c=1}^{N} E(r_{t_i}|\Delta_{t_i}), \qquad \text{(II-6)}$$

where r_{t_i} is the increment to payoff contributed by the arrival at t_i. Substituting Equation II-6 into Equation II-5,

$$E(R_t|E) = \sum_{N=0}^{\infty} \sum_{c=1}^{N} \int d\Delta_{t_i} E(r_{t_i}|\Delta_{t_i}) \Pr\{\Delta_{t_i}|N, t, \varepsilon\} \Pr\{N|t, E\}. \qquad \text{(II-7)}$$

Since the quality process is stationary, and expected payoff depends only on quality at the time of arrival, Equation II-7 reduces to

$$E(R_t|E) = \sum_{N=0}^{\infty} N \; Pr\{N|t,E\} \int d\Delta \; E(r|\Delta) Pr\{\Delta|E\}, \qquad \text{(II-8)}$$

where $Pr\{\Delta|\varepsilon\}$ is the steady state distribution on quality, and $E(r|\Delta)$ is the expected payoff at quality, Δ. If the arrivals are Poisson distributed with parameter λ,

$$\sum_{N=0}^{\infty} N \; Pr\{N|t,E\} = \int d\lambda \sum_{N=0}^{\infty} N \; Pr\{N|t,\lambda\} Pr\{\lambda|E\}$$

$$= tE(\lambda|E). \qquad \text{(II-9)}$$

Substituting Equation II-9 into Equation II-8 gives,

$$E(R_t|E) = tE(\lambda|E) \int d\Delta \; E(r|\Delta) Pr\{\Delta|E\}. \qquad \text{(II-10)}$$

Letting $t = aT$, and using Equation II-10, the average payoff per unit of time is

$$\bar{R} = \lim_{a \to \infty} E(R_{aT}|E)/aT$$

$$= \frac{1}{T} E(R_T|E). \qquad \text{(II-11)}$$

Thus to find the average payoff of a stationary process it is sufficient to find the expected value of payoff gained in a finite period, and divide that by the length of the period.* The result is independent of T and therefore the optimal controls will be independent of T. As a consequence of this, the criterion for optimality can be written

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*Assuming the quality process is stationary in this period, i.e., steady state.

as an expression involving $\bar{R}_T$. Letting C be the set of feasible controls, the criterion for optimization is stated as

$$\bar{R}^*(C) = \max_{c\in C} E(R_T|c,E) = \max_{c\in C}\left[\int R_T \Pr\{R_T|c,E\}dR_T\right]. \qquad \text{(II-12)}$$

If H is a vector of parameters in the demand model about which there is subjective uncertainty, then the probability density of R_T can be written by expansion.*

$$\Pr\{R_T|c,E\} = \int \Pr\{R_T|H,c,E\} \; \Pr\{H|c,E\}dH. \qquad \text{(II-13)}$$

The integral over H is actually a multiple integral over all the parameters in H. The condition on c can be dropped from $\Pr\{H|c,E\}$ in Equation II-13 since the prior density on A is not affected by the control.

Substituting Equation II-13 into the criterion for optimality gives an expanded form of the optimization criterion:

$$\bar{R}^*(C) = \max_{c\in C}\iint R_T \Pr\{R_T|H,c,E\} \; \Pr\{H|c,E\}dHdR_T. \qquad \text{(II-14)}$$

The significance of the criterion expressed in Equation II-14 is that an optimal system design can be selected even if only a priori information about demand is available. This is in sharp contrast to the usual approach in economics, where the philosophy of making pricing decisions under uncertainty on demand is not considered.

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*In the sequel the notation $\Pr\{\cdot\}$ will stand for both a probability density and a probability.

3.3 Value of Perfect Information About Demand

The Bayesian formulation also provides a method for calculating the value of more data.[5,6] If demand data are available, then D appears as a conditional event in each probability in Equation II-14:

$$\bar{R}^*(C,D) = \max_{c\in C} \iint R_T \Pr\{R_T | H,D,c,E\} \Pr\{H| D,c,E\}\, dH dR_T. \qquad \text{(II-15)}$$

Applying Bayes' rule to Equation II-15 and realizing that payoff will not depend on D when H is also known produces Equation II-16.

$$\bar{R}^*(C,D) = \max_{c\in C} k_0 \iint R_T \Pr\{R_T| H,c,E\} \Pr\{D| H,c,E\} \Pr\{H| c,E\}$$

$$k_0^{-1} = P_r\{D| c,E\}. \qquad \text{(II-16)}$$

If perfect information about H were available, the system payoff would be the maximum over c of $E(R_T|H,c,E)$. Taking the expectation of this maximum over the posterior distribution of H gives the expected system payoff knowing that perfect information will be available.

$$\bar{R}_I^*(C,D) = \int dH (\max_{c\in C} \int dR_T\ R_T \Pr\{R_T|H,c,E\}) \Pr\{H|D,c,E\}. \qquad \text{(II-17)}$$

The upper bound on how much should be spent reducing uncertainty on demand is then the difference between (1) the expected payoff knowing that perfect information will be available, and (2) the expected payoff.

$$V = \bar{R}_I^*(C,D) - \bar{R}^*(C,D). \qquad \text{(II-18)}$$

It follows easily that $V \geq 0$ as can be seen by substituting Equations II-16 and II-17 into Equation II-18:

$$V = \max_{c \in C}\left[\int\left[\max_{c \in C} E(R_T|H,c) - E(R_T|H,c)\right]\Pr\{H|D,c,E\}dH\right]. \qquad \text{(II-19)}$$

Since $\Pr\{H|D,c,E\}$ is non-negative and $\max_{c \in C} E(R_T|H,c) \geq E(R_T|H,c)$, the integral in Equation II-19 must be non-negative.

FOOTNOTES: CHAPTER II

1. de V. Graaff, J., op. cit., p. 8.

2. Bator, Francis, "The Simple Analytics of Welfare Maximization," American Economic Review, Vol. 4, No. 1, March 1957, p. 29.

3. de V. Graaff, J., op, cit., p. 91.

4. Arrow, K. J., Social Choices and Individual Values, John Wiley & Sons, New York, 1966, p. 34.

5. Raiffa, H., and Schlaifer, R., Applied Statistical Decision Theory, Graduate School of Business, Harvard University Press, Cambridge, 1969, p. 87.

6. Howard, R. A., "The Foundations of Decision Analysis," IEEE Trans. on Systems Science & Cybernetics, SSC-4, Vol. 3, September 1968, p. 211.

Chapter III

BEHAVIORAL MODEL OF DEMAND

1.0 DECISION THEORY APPROACH

The demand model developed in this chapter is a synthesis of the economist's and the probability analyst's points of view. Economics contributes the basic model structure, and probability theory the methods for dealing with uncertainty.

The traditional approach to modeling demand is to (1) postulate a polynomial relationship between prices (and whatever other factors are felt to affect demand) and demand; and then (2) use regression methods to estimate the polynomial's coefficients. This technique approaches the demand modeling problem from the statistician's viewpoint alone since economic reasoning is not used in either step except to guarantee the existence of a demand function.

A different approach is taken here: a behavioral model of individual choice is derived from the economist's viewpoint; and then the analyst's uncertainty about consumer values is added. The resulting individual model is then aggregated over a population to yield a macromodel for service demand. Lastly, a technique for using demand data to reduce uncertainty about consumer values is presented.

This approach yields a "structural" demand model having three useful properties:

- Calibration* can be achieved at one setting of prices if data at

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*As used here, calibration means gaining almost certain knowledge through statistical inference methods. The proviso "almost" is added, since in Bayesian inference, certain knowledge requires an infinite amount of data.

different levels of quality are available.

- o Demand can be predicted for a different number of services than that used at calibration.
- o Knowledge about consumer values is a by-product of the demand-model calibration.

The significance of the first point is that the demand model can be calibrated even if data is available at only one setting of prices (providing that data is available at different values of quality). The significance of the second point is that a demand model calibrated with data taken with J services available will also be calibrated when predicting demand for $J' \neq J$ services. The significance of the third point is that an analytic method for inferring consumers' value for quality is now available.

2.0 THEORY OF WILLINGNESS TO PAY

2.1 The Economist's Model of Choice

By summarizing a series of proofs found in Appendix A, this section proves the result that in choosing service a consumer acts as if he maximizes the surplus of willingness to pay over price. To start, we borrow the economist's notation for describing the consumer choice problem.

Definitions:

M = number of goods in the economy other than those for which a demand model is to be built.

x_i = amount of i^{th} good ($i \leq M$) consumed by the individual.

$x = (x_1, \ldots, x_M)$ = consumption bundle of an individual, other than services for which demand is being modeled.

X = subset of $\mathbb{R}^M$ to which x is constrained.

Z = set of services for which a demand model is to be built (Z is arbitrary).

Ψ = element in Z which corresponds to refusing all service.

$p(x',z) = p_x(x') + p_z(z)$ = amount paid for consumption bundle (x,z). Assume that $p_x(x')$ is linear in x'.

Y = consumer's income.

R = binary relation specifying the consumer's preferences among consumption bundles. ($(x_1,z_1)\ R(x_2,z_2)$ is equivalent to (x_2,z_2) is not preferred to (x_1,z_1)).

I = binary relation of indifference ($(x_1,z_1)\ R(x_2,z_2)$ and $(x_2,z_2)\ R(x_1,z_1)$ stands for $(x_1,z_1)I(x_2,z_2)$).

It is assumed that a consumer can choose only one element at a time from the set of choices in Z. Because of the way in which the individual demand model will be aggregated, this restriction does not preclude the same individual from choosing additional elements from Z at later times.

The set of all consumption bundles satisfying the individual's budget constraint at a price of zero for $z \in Z$ will be defined as the consumption correspondence for z.

Definition:

$$C(Y,z) = \{(x',z'):x' \in X, z' = z, p_x(x') \leq Y\}.$$

Assume that of all the elements in $C(Y,z)$, the consumer will choose one not inferior to all others in $C(Y,z)$. Define the demand correspondence $D(Y,z)$ to be the set of all such elements.

Definition:

$$D(Y,z) = \{(x',z'):(x',z') \in C(Y,z),\ (x',v') R(x,v), \text{ for all } (x,z) \in C(Y,z)\}.$$

It is obvious from the definition of $D(y,z)$ that Lemma III-1 holds.

Lemma III-1: $(x_1,z_1) \in D(Y,z)$ and $(x_2,z_2) \in D(Y,z)$ imply that $(x_1,z_1)I(x_2,z_2)$ and $z_1 = z_2 = z$.

An ordering among the sets $D(Y,z)$ for all Y and $z \in Z$ can be established by defining the relation, R, among demand correspondences.

Definition: $D(Y,z)\mathcal{R}D(Y',z)$ if and only if for some $(x,z) \in D(Y,z)$ and $(x',z) \in D(Y',z)$, $(x,z)R(x',z)$.

A relation P of strict preference follows naturally from this definition, as does a relation of indifference, I.

For future use, the following defines monotonicity for the correspondence $D(Y,z)$ with respect to the relation R.

Definition: $D(Y,z)$ is strictly monotonic as a function of Y if and only if $D(Y_1,z)PD(Y_2,z)$ whenever $Y_1 > Y_2$.

Next, seven sufficient conditions are imposed to guarantee that a consumer chooses the alternative which maximizes his surplus of willingness to pay over cost.

Condition 1: R is a complete pre-ordering of pairs of elements in $X \times Z$. A pre-ordering is a reflexive and transitive binary relation.[1]

Condition 2: R yields continuous preferences;[2] i.e., $\{(u,v):(u,v)R(x,z)$ and $(u,v) \in X \times Z\}$ and $\{(u,v):(u,v) \in X \times Z$, and $(x,z)R(u,v)\}$ are closed for all $(x,z) \in X \times Z$.

Condition 3: X is compact and convex. Z is compact.

Condition 4: For any $z \in Z$ there exists $(x,\Psi) \in C(Y,\Psi)$ such that $(x,\Psi)\ R(x',z)$, for all $(x',z) \in C(0,z)$.

Condition 5: Service can be refused at no cost to the consumer (often called free disposal).

Condition 6: Nonsatiation applies. If $x_1 = (1 + t)x_2$, and $t > 0$, then $(x_1,z)\ P(x_2,z)$.

Condition 7: Indifference curves are parallel; that is, $(x,z)I(u,v)$ implies $(x',z)I(x' - x + u,v)$ for all $u,x' \in X$ and $v,z \in Z$.

Condition 1 implies that the consumer can state a preference between any pair of bundles in $X \times Z$, and that these preferences are transitive. Condition 2 insures that for any consumption bundle (x_2,z_2), it is possible to find another consumption bundle (x_1,z_1) with a different amount of z such that $(x_1,z_1)I(x_2,z_2)$. Condition 3 requires that feasible consumptions in X are limited to closed, bounded, and connected subsets of $\mathbb{R}^M$. Condition 4 implies that Y is large enough to allow a consumption bundle (x,Ψ), which is preferred to any consumption bundle feasible with no income and a gift of z. Condition 6 requires that a proportionate increase in every good consumed increases satisfaction. The implication of Condition 7, having "parallel" indifference curves, will be seen shortly.

Lemmas III-2 and III-3 below say that if a consumer having income Y but no z is given $z \in Z$, there exists a unique amount, called the willingness to pay for z, which can be deducted from Y to make the consumer indifferent between the new income and z, and the old income and no z.

Lemma III-2: Under Conditions 1 through 5, there exists a function

$W(Y,z) \geq 0$, for all $z \in Z$, such that $D(Y - W(Y,z),z) \bar{I} D(Y,\Psi)$.

<u>Lemma III-3:</u> Under Conditions 1 through 6, $W(Y,z)$ is single valued for any Y, and $D(Y,z)$ is strictly monotonic.

If Condition 7 is included, it can be proved that $W(Y,z)$ is independent of Y, a property needed to prove the principle result in Theorem III-1.

<u>Lemma III-4:</u> Under Conditions 1 through 7 $W(Y,z)$ is constant in Y. In this case we define $W(Y,z) = W(z)$ and the following theorem can be proved.

<u>Theorem III-1:</u> Under Conditions 1 through 7, a consumer will choose $z \in Z$ only if $W(z) - p_z(z) \geq W(z') - p_z(z')$, for all $z' \in Z$.

<u>Corollary III-2:</u> Assume Conditions 1 through 7 hold. If $p_z(\Psi) = 0$, a consumer chooses $z \in Z$ only if $W(z) - p_z(z) \geq 0$.

<u>Corollary III-3:</u> Under Conditions 1 through 7 a consumer will choose $z \in Z$ if $W(z) - p_z(z) > W(z') - p_z(z')$, for all $z' \neq z$.

Theorem III-1 says that knowledge of an individual's willingness-to-pay function is sufficient for the analyst to say (except for an indeterminancy to be treated later) which alternative would be chosen, given prices for $z \in Z$. Willingness to pay incorporates just the information about a consumer's values needed to build a model of his demand for services in Z. Corollary III-2 implies that consumers only choose alternatives with a nonnegative surplus of willingness-to-pay over price. Corollary III-3 specifies a sufficient condition for choosing $z \in Z$.

The important feature of the willingness-to-pay function is its independence of the price paid. The willingness-to-pay function is not, however, independent of the prices for other goods in the economy, as can be seen from the derivation of Theorem III-1 in Appendix A. In the language of economics, willingness to pay is a partial-equilibrium concept.

2.2 Implications of the Willingness-to-Pay Model

A preference map illustrating the willingness-to-pay concept is shown in Figure III-1. For the two dimensions of Figure III-1, let the ordinate X be the single commodity money, and let the abcissa z be service of quality index $z \in \mathbb{R}$ with the null alternative Ψ corresponding to $z = 0$. The ordinal indifference curves are drawn parallel to each other, satisfying Condition 7. The consumer's set of feasible consumptions, if z is free, are all the bundles (x,z) that satisfy the budget constraint $x \leq Y$. If z_1 costs $p_z(z_1)$, then no more than $Y - p_z(z_1)$ can be spent on X.

If no z can be consumed, then the demand correspondence is $(x_0,0)$, where $x_0 = Y$. If z_1 is offered as a gift, then (x_0,z_1) would be the equilibrium consumption, since (x_0,z_1) is feasible and inferior to no other feasible consumption. If Y is reduced until the equilibrium consumption is indifferent to $(x_0,0)$, all the while consuming z_1, the distance moved along the X axis by the equilibrium consumption is the willingness to pay for z_1. It is depicted in Figure III-1 as $W(z_1)$.

Figure III-1 can be used to demonstrate that a consumer will choose z only if it maximizes $W(z) - p_z(z)$. At a price of $p_z(z_1)$ for z_1, the equilibrium consumption is (x_1,z_1), and similarly, for another service z_2, the equilibrium is (x_2,z_2). If $W(z_1) - p_z(z_1) < W(z_2) - p_z(z_2)$, then because of parallel indifference curves, the consumption (x_1,z_1)

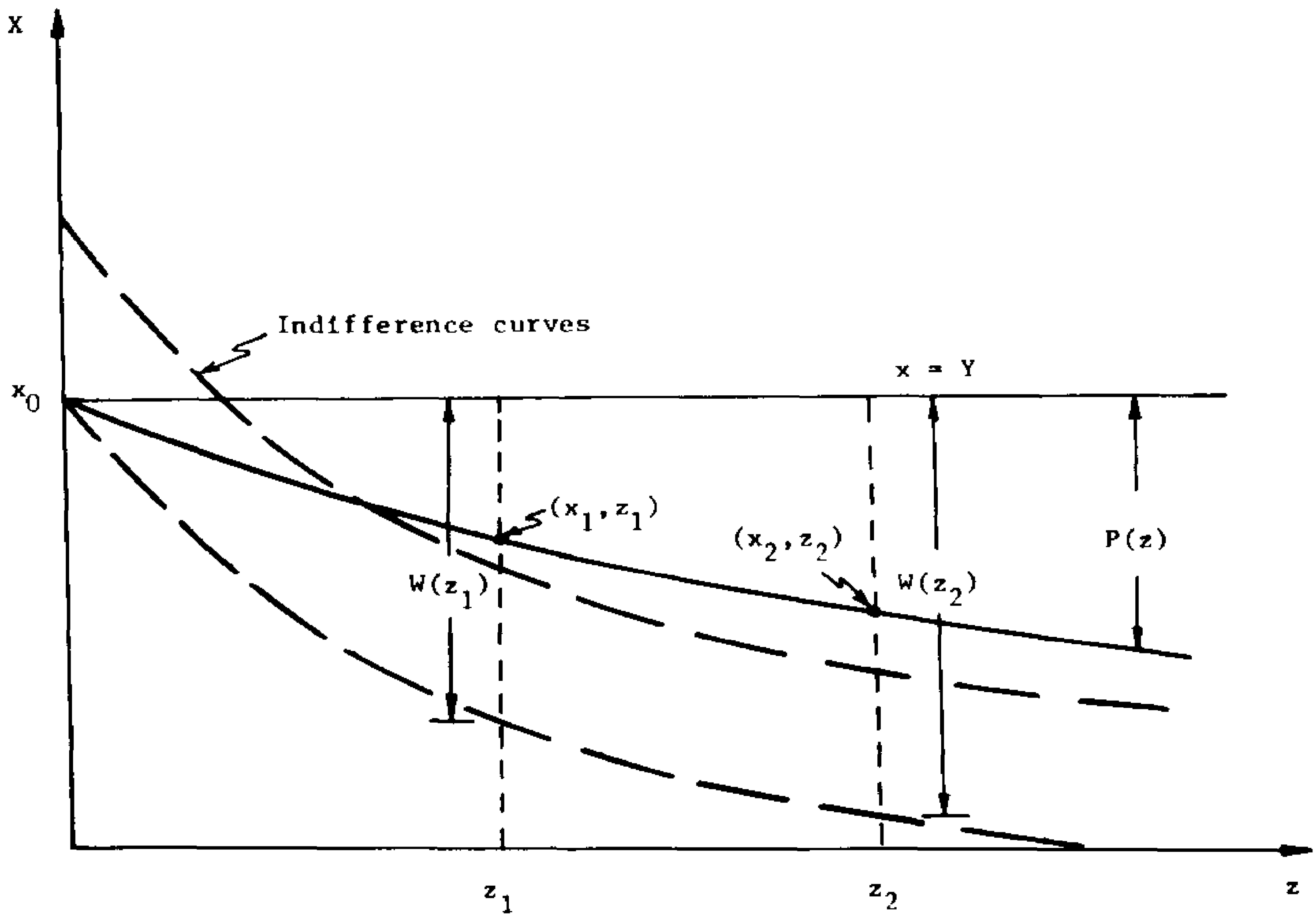


Figure III-1 -- Preference Map Illustrating Willingness-to-Pav Concept will be inferior to (x_2, z_2), and therefore not chosen.

Taken together, Theorem III-1 and Lemma III-4 prove that the z chosen by a consumer will not be a function of income. This result, which hinges on Condition 7, is not reasonable over large changes in income. If z_1, for example, was a higher-quality good than z_2, then most people would shift their choice from z_1 to z_2 at some large level of income.

However, the consumer's choice is not likely to change over small changes in income. This implies that for small changes in income, a consumer's indifference curves can be approximated with a parallel set of curves in the region of his equilibrium consumption and not change his predicted behavior. For the same reason, Condition 7 can be expected to hold when the willingness to pay is a small fraction of income. For most service systems, this is a reasonable assumption.

This formulation of willingness to pay is similar in concept to Samuelson's definition of one kind of consumer's surplus for price changes; that is, the change in income which will make trading at the old set of prices as attractive as trading at the new set of prices with the initial income.[3] Willingness to pay adds to the concept of consumer surplus the behavioral result that a consumer behaves as if he chooses the good that maximizes his surplus.

For applications, each service, z, is mapped one-to-one into a space of indices that are a measure of product quality. Let such a mapping by the function $h(\cdot)$ whose domain is Z, and range the space of quality indices, taken usually to be a subset of $\mathbb{R}^n$. Then the set of indices $\Delta = (\Delta_1, \ldots, \Delta_J)$, where J is the number of services, identifies uniquely one of the products in Z. Defining $W'(\Delta) = W(h^{-1}(\Delta))$, willingness to pay is transformed into a function on quality, indices--quantities that can be measured in the real world.

Theorem III-1 says nothing about the shape of the willingness-to-pay function because insufficient information about consumer values has been imposed. For analytical purposes it is necessary to impose a partial specification of consumer values by fixing the form of $W(\Delta)$.*

The functions $W(\Delta_i) = \xi_0 + \xi_1 \log \Delta_i$ or $W(\Delta_i) = \xi_0 + \xi_1 \Delta_i$ are typical of the broad class of functions that satisfy these assumptions.

3.0 APPLICATION IN A MODEL FOR INDIVIDUAL DEMAND

3.1 Form of the Willingness-to-Pay Function

The simplest form of an individual's willingness-to-pay function for alternative i is a linear function in two value parameters. It will be designated as willingness-to-pay Model 0.

W-P Model 0:

$$W_i(\Delta, \xi) = \xi_0 + \xi_1 \Delta_i .$$

The value parameter ξ_0 is interpreted as the value of service at zero quality index, or the intrinsic value of service. In transportation ξ_0 would be the value of getting there. The value parameter ξ_1 is interpreted as the value of quality. The deduction of $\xi_1 \Delta_i$ from ξ_0 accounts for the change in willingness to pay caused by non-ideal service. In transportation ξ_1 would be the value of time if Δ were

*The prime in $W(\Delta)$ is hereafter understood.

a measure of travel time or delay.

3.2 Incorporation of Uncertainty

Because of uncertainty, the decisionmaker can have only imperfect knowledge of a particular consumer's value parameters ξ_0 and ξ_1. This state of imperfect information is described by assuming that over a population, consumer values are distributed according to a probability density function of known form but unknown parameters. Quantitatively, the analyst assumes that ξ_0 and ξ_1 are distributed according to the form $\Pr\{\xi_0, \xi_1 | A\}$, where $A = \{a_1, a_2, \ldots, a_r\}$ is a set of unknown parameters. The analyst encodes his subjective knowledge of A with the distribution $\Pr\{A|E\}$.

In many situations, consumers will not have certain knowledge of the service quality index, and will be forced to guess the true value based on past experience. These guesses may be below or above the true value of the quality index, due to any one of a large number of factors. A principal source of consumer uncertainty in queuing systems is their lack of knowledge about how many higher-priority arrivals will occur before service can be completed in each of the lower priorities.

Sometimes consumers exhibit inconsistencies in assessing their value of quality. Instead of using the same value of quality for each alternative, consumers might use a slightly different value due to a wide variety of unpredictable factors. Biases, emotional factors, or perception errors can all contribute to random variations in ξ_0.

To the decisionmaker, these abberations make it appear that consumers are irrational--behaving as if there were an additional random component in their willingness-to-pay function. This random component will be modeled by adding another random variable, β, to the willingness-to-pay function. Depending on how β is incorporated, different sources of irrationality can be emphasized.

For simplicity, assume that each alternative has an associated irrationality random variable, β_c, and that these random variables are independent and identically distributed with probability density $\Pr\{\beta i|B\}$, where B is a set of unknown parameters.

Assumption III-1: The consumer irrationality random variables β_c, c = 1, 2, ..., J are independent and identically distributed.

If the consumer is irrational about his value for service, because of emotional factors or product qualities not captured in ξ_0, then his willingness-to-pay function would be:

W-P Model 1:

$$W_c(\Delta,\xi,\beta) = \xi_0 + \xi_1\Delta_c + \beta_c.$$

Irrationality about the value of quality is incorporated by adding an error term onto the willingness-to-pay function that is proportional to Δ_c.

W-P Model 2:

$$W_c(\Delta,\xi,\beta) = \xi_0 + \xi_1\Delta_c + \beta_c\Delta_c.$$

This model is appropriate if consumers behave as if their value for quality is not constant among alternatives, but perturbed by an amount β_c.

Irrationality in estimating the quality index is introduced by replacing Λ_c in Model 0 with $\beta_c\Lambda_c$, giving a third model.

W-P Model 3:

$$W_c(\Lambda,\xi,\beta) = \xi_0 + \xi_1\beta_c\Lambda_c.$$

This model applies if consumers estimate the *cth* quality index as $\beta_c\Lambda_c$ when the true value is Λ_c.

For queuing systems, W-P Model 3 is the most appealing, since consumers must guess the service completion time in each priority based on the current level of unserved work. The true completion time is unknown because higher priority arrivals supercede in the service order and the amount of work backlogged in waiting units is usually unknown a priori. Other sources of "irrational" behavior are less likely to dominate in queuing systems.

As a review, there are four sources of uncertainty in the decision-theoretic perception of individual demand. The decisionmaker can:

(1) know only a probability density function on the consumer's values for a product,

(2) be uncertain about the probability density function that describes the consumer's values,

(3) be uncertain about how the consumer's assessment of a product compares with his own (called consumer irrationality), and

(4) be uncertain about the degree of consumer irrationality.

Inference on demand data can be used to reduce uncertainty from sources (2) and (4) which implies that these are subjective uncertainties.

The sources (1) and (3) are, following Savage's distinction, objective uncertainties because they cannot be reduced by inference on demand data.[4]

3.3 Probability of Choice

3.3.1 *Development of a General Model*

Because willingness to pay is a random variable, information about the alternative that a consumer will choose must be stated in the form of a probability statement.

Let $Q = \{0, 1, 2, \ldots, J\}$ be the set of alternatives facing the consumer. The element zero is an index for the alternative "refuse service," which has $W(\phi) = 0$ by Lemma III-1.

Let the surplus of an alternative c be defined as

$$S_c(\Delta,p) = W_c(\Delta) - p_c. \tag{III-1}$$

The dependence of $W_c(\Delta)$ on ξ_0, ξ_1, and β_c specified by Models 0, 1, 2, and 3 is not explicitly shown in Equation 1 to minimize notational complexity.

By Theorem III-1, the probability that alternative c is chosen over all others in Q is not greater than the probability that c has maximum surplus.

$$\begin{aligned} \rho_c(\Delta,P|A,B) &= \Pr\{c \text{ is chosen when quality is } \Delta \text{ and prices are } P|A,B\} \\ &\le \Pr\{S_c(\Delta,P) \ge \max_{j\epsilon Q-c} S_j(\Delta,P)|A,B\} \\ &= \Pr\{S_c > \max_{j\epsilon Q-c} S_j|A,B\} + \Pr\{S_c = \max_{j\epsilon Q-c} S_j|A,B\}. \end{aligned} \tag{III-2}$$

Under the assumption that $\Pr\{\xi_0,\xi_1|A,B\}$ is a continuous density function, it can easily be shown that the second probability in Equation III-2 is zero.

Assumption III-2: $\Pr\{\xi_0,\xi_1|A,B\}$ is a continuous probability density.

By Corollary III-3, c is chosen if the event $S_c > \max_{j\epsilon Q-c} S_j$ occurs, therefore the probability of choosing c is not less than the probability of this event.

$$\Pr\{S_c > \max_{j\epsilon Q-c} S_j|A,B\} \leq \rho_c(\Delta,P|A,B). \qquad \text{(III-3)}$$

Together with Equation III-2, Equation III-3 and Assumption III-2 imply that

$$\rho_c(\Delta,P|A,B) = \Pr\{S_c \geq \max_{j\epsilon Q-c} S_j|A,B\}. \qquad \text{(III-4)}$$

Expanding Equation III-4 on both consumer values and the irrationality random variables gives two useful forms.

Demand Model A:

$$\begin{aligned}\rho_c(\Delta,P|A,B) &= \int d\beta \Pr\{S_c(\Delta,P) \geq \max_{j\epsilon Q-c} S_j(\Delta,P)|\beta,A\}\ \Pr\{\beta|B\} \\ &= \int d\beta I_A \Pr\{\beta|B\}.\end{aligned}$$

Demand Model B:

$$\begin{aligned}\rho_c(\Delta,P|A,B) &= \int d\xi \Pr\{S_c(\Delta,P) \geq \max_{j\epsilon Q-c} S_j(\Delta,P)|\xi,B\}\ \Pr\{\xi|A\} \\ &= \int d\xi I_B \Pr\{\xi|A\}.\end{aligned}$$

In Equations III-5a and III-5b, the indicated integrations are multiple integrals, since $\xi = (\xi_0, \xi_1)$ and $\beta = (\beta_1, \ldots, \beta_J)$. The regions of integration are implicitly the domains of the probability distributions on ξ and β. Equation III-5a is especially useful for developing qualitative insights about demand-model properties. Equation III-5b is more convenient when computing an algebraic form for some values of the parameters A and B.

The demand model represented by Equation III-4 is one of the class called random utility models by Marschak and Block.[5] One requirement for a random utility model is that there exists a random utility vector $u = (u_1, u_2, \ldots, u_J)$ with the property that $\Pr\{u_j \geq \max_i u_i\}$ equals the probability of choosing j. In our model the surplus, $S = W-p$, is the random utility having the desired property. The other requirement is that the random utility vector is unique up to a monotonic increasing function of the components. This property holds, since $(f(s_1), \ldots, f(S_J))$ will yield the same choice of alternative as $(S_1, \ldots, S_J)$ if $f(\cdot)$ is monotonic increasing.

3.3.2 *Development of Demand Model A*

For subsequent use, define $\Xi_c(\beta)$ as the set of all pairs (ξ_0, ξ_1), which, given β, produce $S_c(\Delta,p) \geq \max_{j\in Q-c} S_j(\Delta,p)$.

Defiwhich, given β, produce $S_c(\Delta,p) \geq \max_{j\in Q-c} S_j(\Delta,p)$.

Definition: $\Xi_c(\beta) = \{(\xi_0, \xi_1) : S_c(\Delta,p) \geq \max_{j\in Q-c} S_j(\Delta,p) | \beta\}$.

Expanding the integrand of Demand Model A on ξ gives,

$$I_A = \int d\xi \Pr\{S_c(\Delta,P) \geq \max_{j\in Q-c} S_j(\Delta,P) | \xi,\beta,A,B\} \Pr\{\xi|A\}. \qquad \text{(III-5)}$$

Then the integrand of Equation III-5 is easily rewritten as,

$$\text{integrand} = \begin{cases} \Pr\{\xi|A\} \text{ if } (\xi_0,\xi_1) \in \Xi_c(\beta) \\ 0 \text{ if } (\xi_0,\xi_1) \notin \Xi_c(\beta). \end{cases} \tag{III-6}$$

Substituting Equation III-6 into Equation III-5 gives a simple form for the demand model:

$$\rho_c(\Delta,P|A,B) = \int d\beta \int\limits_{\Xi_c(\beta)} d\xi \; \Pr\{\xi|A\} \; \Pr\{\beta|B\}. \tag{III-7}$$

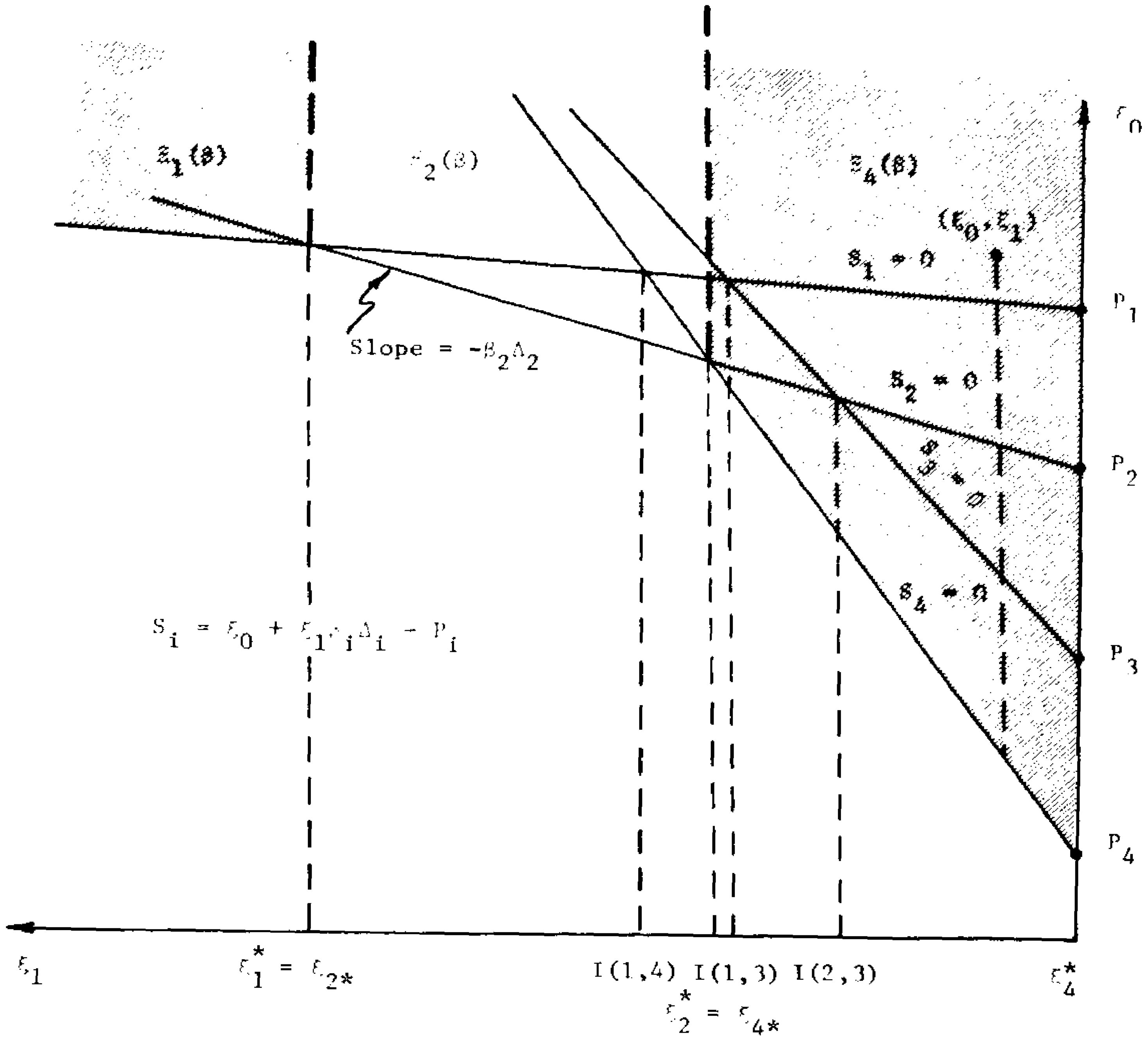
Equation III-7 cannot be further refined unless one of the willingness-to-pay models is substituted into the definition of $\Xi_c(\beta)$. As an illustration, substitute W-P Model 3 into $\Xi_c(\beta)$:

$$\Xi_c(\beta) = \{(\xi_0,\xi_1):\xi_0 + \xi_1\beta_c\Delta_c - p_c \geq 0, \xi_0 + \xi_1\beta_c\Delta_c - p_c \geq \xi_0 + \xi_1\beta_j\Delta_j - p_j \text{ for all } j = 1, 2, \ldots, J, \neq c\}. \tag{III-8}$$

A set of the $\Xi_c(\beta)$ regions for particular values of β, Δ, and prices appears in Figure III-2. The slanting lines represent points of zero surplus; i.e., $S_c = \xi_0 + \xi_1\Delta_c\beta_c - p_c = 0$, for $c = 1, 2, \ldots, J$. The slope of these lines is $\beta_c\Delta_c$, and the ξ_0 intercept is p_c. Only negative values of ξ_1 are plotted since increasing Δ usually decreases willingness to pay (assuming that $\beta_i \geq 0$). Pairs of these slanting lines intersect at a value of ξ_1 equal to

$$I(i,j) = (p_i - p_j)/(\beta_i\Delta_i - \beta_j\Delta_j); \tag{III-9}$$

a fact verified by solving $S_j = S_i$ for ξ_1.



Note: See Equation III-9 for definition of I(i,j) and page 47 for ξ_{c*} and ξ_c^*.

Figure III-2 -- Value Space Representation of Consumer Choice

Given any set of consumer values (ξ_0', ξ_1'), the surplus of alternative c will be the Euclidean distance parallel to the ξ_0 axis from (ξ_0', ξ_1') to the zero surplus line, $S_c = \xi_0 + \xi_1\Delta_c + \beta_c\Delta_c - p_c = 0$. Thus for any value pair (ξ_0', ξ_1'), the optimal alternative is immediately observable in Figure 2 as the one which intercepts a zero-surplus line at the samllest value of ξ_0. This algorithm can be used to sketch in the regions of $\Xi_c(\beta)$.

It is apparent from Figure 2 that the areas $\Xi_1(\beta)$, $\Xi_2(\beta)$, ..., etc., will be contiguous and related left to right in the same order as the prices. That is, if $p_1 \geq p_2$, then $\Xi_1(\beta)$ will be to the left of $\Xi_2(\beta)$ and connected to it.

A lower bound on ξ_1 for region $\Xi_c(\beta)$, ξ_{c*}, can be read off immediately from Figure 2 as the smallest ξ_1 intercept created by line c with all other lines of lower slope. Conversely, the upper bound, ξ_c^*, appears in Figure 2 as the largest ξ_1 intercept created by line c with all other lines of higher slope.

The effect of changes in quality on the probability of choice can be seen by visualizing $\Pr\{\xi_0, \xi_1 | A, B\}$ as a third-dimension orthogonal to the plane of (ξ_0, ξ_1). By Equation III-7, $\rho_c(\Delta, p | A, B)$ involves the integral of the function $\Pr\{\xi_0, \xi_1 | A, B\}$ over the area $\Xi_c(\beta)$. Assuming that β_c is positive, an increase in Δ_c will steepen the slope of the zero-surplus line for any value of β, thereby raising ξ_{c*} and lowering ξ_c^*. Both effects decrease the area of $\Xi_c(\beta)$, causing a decrease in $\rho_c(\Delta, p | A, B)$. Some of the reduction in probability is transferred to the alternative just superior to c and some to the alternative just inferior to c, but none to other alternatives.

Bounds on the region $\Xi_c(\beta)$ can also be evaluated mathematically, by rearranging the inequalities in Equation III-9:

$$\Xi_c(\beta) = \{(\xi_0,\xi_1):\xi_0 \geq P_c - \xi_1\beta_c\Delta_c,\ \xi_1(\beta_c\Delta_c - \beta_j\Delta_j) \geq P_c - P_j$$
$$\text{for all } j = 1,2,\ldots J \neq c\}. \qquad \text{(III-10)}$$

The definitions for the two bounds on ξ_1 implied by Equation III-10 are:

<u>Definition:</u>

$$\xi_{c*} = \max_{\substack{j \neq c,\, j\epsilon Q \\ :\beta_c\Delta_c \geq \beta_j\Delta_j}} \left(\frac{P_c - P_j}{\beta_c\Delta_c - \beta_j\Delta_j}\right)$$

$$\xi_c^* = \min_{\substack{j \neq c,\, j\epsilon Q \\ :\beta_c\Delta_c < \beta_j\Delta_j}} \left(\frac{P_c - P_j}{\beta_c\Delta_c - \beta_j\Delta_j}\right).$$

Then Equation III-10 can be simplified:

$$\Xi_c(\beta) = \{(\xi_0,\xi_1):\xi_0 \geq P_c - \xi_1\beta_c\Delta_c,\ \xi_1 \leq \xi_c^*,\ \xi_1 \geq \xi_{c*}\}. \qquad \text{(III-11)}$$

If $\xi_{c*} \leq \xi_c^*$, it follows from Equation III-8 that

$$\rho_c(\Delta,P|A,B) = \int d_\beta \int_{\xi_{c*}(\beta)}^{\xi_c^*(\beta)} d\xi_1 \int_{P_c-\xi_1\Delta_c\beta_c}^{\infty} d\xi_0 \Pr\{\xi_0,\xi_1|A\} \Pr\{\beta|B\}. \qquad \text{(III-12a)}$$

If $\xi_{c*} > \xi_c^*$, $\Xi_c(\beta)$ is void and Equation III-8 becomes

$$\rho_c(\Delta,p|A,B) = 0. \qquad \text{(III-12b)}$$

Equation III-12a applies only for the W-P Model 3, but similar forms can be derived for the other willingness-to-pay models. The next step towards achieving an algebraic form for $\rho_c(\Delta,p|A,B)$ in the set of parameters $A \cup B$ is substitution of appropriate probability distributions into Equation III-12a. While simple algebraic solutions can be found for W-P Model 0, the other W-P models lead to complex expressions. Much simpler forms can be achieved by starting with Equation III-6b.

3.3.3 *Development of Demand Model B*

Demand Model B will be developed for the particular case of W-P Model 3, rather than in full generality. Development for other cases follows virtually the same steps.

Substituting W-P Model 3 into the integrand of Demand Model B produces a probability statement about the irrationality random variables,

$$I_B = \Pr\{\beta_c \geq \frac{P_c - \xi_0}{\xi_1 \Delta_c}\ ,\ \beta_c \geq \max_{j \in Q-c} \frac{P_c - P_j + \beta_j \Delta_j}{\xi_1 \Delta_j}\ |\ \xi, B\}. \quad \text{(III-13)}$$

Since the β_i are independent random variables, Equation III-13 can be changed to Equation III-14,

$$I_B = \int_{\frac{P_c - \xi_0}{\xi_1 \Delta_c}}^{\infty} d\beta_c' \Pr\{\beta_c = \beta_c'\} \prod_{j \in Q-c} \Pr\left\{\beta_j \leq \frac{P_j - P_c + \beta_c' \Delta_c}{\xi_1 \Delta_j}\right\}. \qquad \text{(III-14)}$$

Substituting Equation III-14 back into Equation III-5b yields a form of the demand model that is useful for calculating algebraic expressions. A sample of such a calculation is carried out in Chapter IV.

The next section of this chapter discusses aggregation of individual demand into a macro-model that predicts demand by a population of consumers.

4.0 AGGREGATION OF INDIVIDUAL CHOICES

In classical microeconomics, the aggregate demand function is simply the sum of individual demand functions over all the individuals in an economy. With the decision-analysis approach taken here, the aggregate model cannot be obtained by addition, since the dimension of an economy is *a priori* unknown.

If the economy is taken to be the whole world, then the decisionmaker knows *a priori* that almost every individual will have zero demand for the services being modeled, but he will be uncertain about how many have zero demand. It seems more sensible to adopt the view from the beginning that the decisionmaker is uncertain about the number of people in an economy, and then treat this number as another in the set of unknown parameters, A.

One possibility for the aggregate model is to assert that in some time period, N individuals choose independently among the alternatives in J. The probability that any one of these individuals chooses c is $\rho_c(\Delta,p|A,B)$, making the aggregate demand for alternatives,

$$\Pr\{N_1, N_2, \ldots, N_J|A,N\} = \rho_1^{N_1}(\Delta,p|A,B) \times \ldots$$

$$\rho_J^{N_J}(\Delta,p|A,B)\left(1 - \sum_{i=1}^{J} \rho_i(\Delta,p|A,B)\right)^{N-N_1-N_2,\ldots,N_J}, \quad \text{(III-15)}$$

where N_c = number of consumers choosing c.

This is a simple multinomial model for demand. The variable N is a parameter about which the decisionmaker has subjective uncertainty.

Another possibility for the aggregate model is to assume that at exponentially distributed times t_1, t_2, t_3, ... consumers decide whether or not to accept one of J offered services--the identity of individual consumers in a sequence being unimportant. Then the aggregate model of demand is the multivariable compound Poisson process diagrammed in Figure III-3. The Poisson process has an advantage over the multinomial in queuing situations in that the distribution of arrival times in inherently available. With the multinomial process, some auxiliary specification of the distribution of arrival times would be necessary.

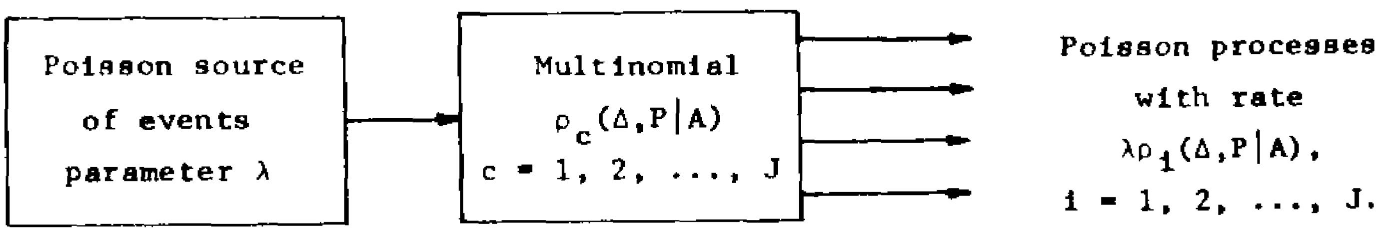


Figure III-3 -- Poisson Process Macro-Model

A further justification for using the Poisson process as a macro-model is the relationship between the order statistics of uniform distributions and the Poisson process.[6] Suppose that at time T in the interval $[0,\infty]$, N events in the Poisson macro-process have occurred. Then the random variables t_1, t_2, ..., t_N at which each Poisson event occurred are distributed as a set of order statistics of size N, taken from a uniform distribution over $[0,T]$. Or, in other words, if N consumers independently and uniformly pick a time within $[0,T]$ to appear at the service facility, and if these times are then ordered with increasing time, the distribution of times will be the same as for a Poisson process.

Hence the Poisson model is equivalent to a compound process where first the random variable N denoting the number of consumers in time T is selected (N is Poisson distributed with mean λT), and then each of these consumers chooses an arrival time uniformly in the interval $[0,T]$. This seems to be a very reasonable model for consumption processes.

FOOTNOTES--CHAPTER III

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Chapter IV

INFERENCE ON THE DEMAND MODEL

1.0 DERIVATION OF THE LIKELIHOOD FUNCTION

1.1 Theoretical Development

Often it will be useful to know the posterior moments of the demand-model parameters, especially since A and B are readily interpreted as consumer values. But the usual Bayesian technique of finding sufficient statistics for the posterior moments is impractical for the model in Section III.4.0 because of complexities in the likelihood function. As an alternative, this chapter develops a method for inferring the posterior moments when only samples of the prior distribution and the likelihood function are available. The first two steps in developing an inference procedure are to (1) define what constitutes a data event, and (2) define the likelihood function for those data events.

For the compound Poisson demand model in Section III.4.0, every purchase of service that occurs in a preselected length of time T qualifies as a data event (data events being occurrences visible to the naked eye). The times when a consumer refuses service could also be included in the data set, but will not be since it is usually difficult to distinguish between persons who have rejected service and those who are not even considering purchase at all.

Assumption IV-1: The only observable data events are purchases of service in Q.

Let a datum event be specified by the pair of numbers $E_i = (\tau_i, c_i)$, where c_i identifies the service purchased by the i^{th} consumer, and τ_i is the time elapsed between the time at which c_i is recorded, and the previous acceptance of service (or the start of data observations). If the i^{th} consumer chooses service j, then let $c_i = j$.

In the time period [0,T], N data events will occur (N a non-negative integer), but rarely will the last one occur at t = T. The time between the last observed data event (the N^{th}) and t = T describes the null event $E_{N+1} = (\tau_{N+1}, 0)$. Including the null event, a complete data set recording all the events in a time period [0,T] is:

$$D = \{E_i : i = 1,2, \ldots N + 1\}$$

$$= \{(\tau_i, c_i) : i = 1,2, \ldots N + 1\},$$

where τ_i, c_i, and N are all random variables. The likelihood function for the demand model parameters λ, A, and B will be conditioned on prices and the quality process observed in the data collection interval, since both are control variables in the compound Poisson model. From Bayes' rule the posterior probability density on demand model parameters is:

$$\Pr\{\lambda, A, B | D, \Delta_t, P, \varepsilon\} = k_o^{-1} \Pr\{D/\lambda, A, B, \Delta_t, P, \varepsilon\}$$

$$\cdot \Pr\{\lambda, A, B | \Delta_t, P, \varepsilon\}, \qquad \text{(IV-1)}$$

where $\Pr\{D|\lambda, A, B, \Delta_t, P, \varepsilon\}$ is the likelihood function and Δ_t is an abbreviation for the entire sample of the quality process observed in [0,T], and P is the vector of prices.

Derivation of the likelihood function is considerably simplified when the quality level Λ is constant over the time τ_i between data events. Unfortunately, this is rarely the case, since service performed between the times of arrival changes the level of quality.

This dilemma is avoided by defining a surrogate quality process Λ_t', where Λ_t' at time t is equal to the quality at the time of the most recent acceptance of service. Thus the surrogate quality process at time t equals the backlog of work which existed just after the previous arrival entered the service line. If quality is constant during the inter-acceptance times, then $\Lambda_t' = \Lambda_t$, and the approximation is exact. If quality is not constant, then Λ_t' approximates Λ_t. An example of Λ_t' for a single priority queueing system is shown in Figure IV-1.

<u>Assumption IV-2</u>: Between the times at which consumers accept service (acceptance times), the quality level is constant.

The data events in D will be independent under the compound Poisson demand model of Section III.4.0, since the Poisson source generates independent events and the multinomial trials are independent.

To assist in writing the likelihood function for the compound Poisson demand model, the auxiliary random variables ψ_j is defined as the time to the first acceptance of service j, starting from t = 0.

Using ψ_j, the probability density of event E_i for $i \neq N+1$ is:

$$\begin{aligned}\Pr\{E_i|\lambda,A,B,\Lambda,P,E\} &= \Pr\{\psi_1 \geq \psi_{c_i}, \ldots, \psi_{c_i}, \ldots \psi_J \geq \psi_{c_i}|\lambda,A,B,\Lambda,P,E\}\\ &= \Pr\{\psi_{c_i}^* \geq \psi_{c_i}|\psi_{c_i},\lambda,A,B,\Delta,P,E\}\\ &\quad \cdot \Pr\{\psi_{c_i}|\lambda,A,B,\Delta,P,E\}. \qquad \text{(IV-2)}\end{aligned}$$

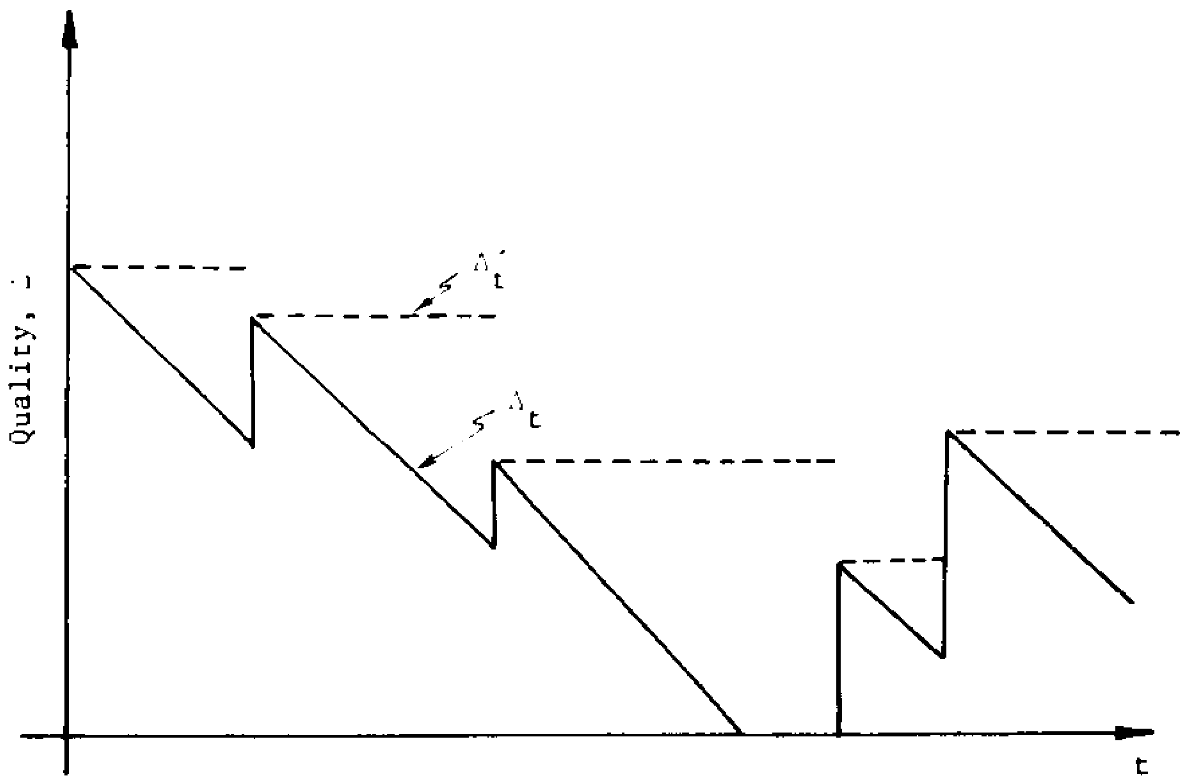

Figure IV-1--Approximation to Quality Process where $\psi_{c_i}^* = \min\{\psi_1, \ldots \psi_{c_i-1}, \psi_{c_i+1}, \ldots \psi_J\}$. For the compound Poisson demand model, $\psi_{c_i}^*$ is exponentially distributed so that

$$\Pr\{\psi_{c_i}^* \geq \psi_{c_i} | \psi_{c_i}, \lambda A, B, \Delta, P, E\}$$

$$= \exp\left[- \lambda(\rho_1 + \ldots \rho_{c_i-1} + \rho_{c_i+1} + \ldots \rho_J)\psi_{c_i}\right]. \quad \text{(IV-3)}$$

Likewise ψ_{c_i} is exponentially distributed giving

$$\Pr\{\psi_{c_i} | \lambda, A, B, \Delta, P, E\} = \lambda\rho_{c_i}(\Delta, P | A, B)$$

$$\cdot \exp\left[- \lambda\rho_{c_i}(\Delta, P | A, B)\psi_i\right]. \quad \text{(IV-4)}$$

Letting $\rho_\sigma(\Delta, P | A, B) = \sum_{j=1}^{J} \rho_j(\Delta, P | A, B)$,

$$\Pr\{E_i | \lambda, A, B, E\} = \lambda\rho_{c_i}(\Delta, P | A, B) \exp\left[- \lambda\rho_\sigma(\Delta, P | A, B)\tau_i\right]. \quad \text{(IV-5)}$$

If $i = N + 1$, then no events are observed in τ_i; therefore,

$$\Pr\{E_i | \lambda, A, B, E\} = \Pr\{\psi_i \geq \tau_i, \psi_2 \geq \tau_i, \ldots \psi_J \geq \tau_i | \lambda, A, B, \Delta, P, E\}$$

$$= \Pr\{\psi_* \geq \tau_i | \lambda, A, B, \Delta, P, E\}, \quad \text{(IV-6)}$$

where $\psi_* = \min\{\psi_1, \psi_2, \ldots \psi_J\}$. As for Equation (IV-3),

$$\Pr\{\psi_* \geq \tau_i | \lambda, A, B, \Delta, P, E\} = \exp\left[- \lambda\rho_\sigma(\Delta, P | A, B)\tau_i\right]. \quad \text{(IV-7)}$$

Letting ${}^kN_j$ = number of events in channel j at delay ${}^k\Delta$ and price ${}^kp$, and ${}^kT$ = total time over which data are observed at $({}^k\Delta, {}^kp)$, it follows from the independence of data events E_j in D that,

$$PR\{D|\lambda,A,(^k\Delta,{}^kp),E\} = \prod_{j=1}^{J}\left[\lambda\rho_\sigma(^k\Delta,{}^kp|A)\right]^{{}^kN_j} \exp\left[-\lambda\rho_\sigma(^k\Delta,{}^kp|A)^kT\right]. \quad \text{(IV-8)}$$

If data are taken at several values of delay and price, the likelihood function is clearly,

$$Pr\{D|\lambda,A,E\} = \prod_{k=1}^{L} \exp\left[-\lambda\rho_\sigma(^k\Delta,{}^kP|A)^kT\right] \prod_{j=1}^{J} (\lambda\rho_j(^k\Delta,{}^kP|A))^{{}^kN_j}. \quad \text{(IV-9)}$$

Substituting Equation (IV-9) into (Equation IV-1) yields the posterior distribution on demand model parameters.

1.2 Importance of Irrationality Random Variables

The importance of including irrationality random variables in the willingness-to-pay function can be demonstrated using Equation (IV-9). Suppose that W-P Model 0 ($W_c = \xi_0 + \xi_1\Delta_c$) holds, and that choice among two alternatives is offered with $p_1 > p_2$ and $\Delta_2 < \Delta_1$. Using the definitions on page 47, $\xi_{1*} > 0$. If ξ_1 is restricted to negative values corresponding to the implication that higher Δ lowers consumption value (i.e., $Pr\{\xi_1,\xi_0|A\} = 0$ for $\xi_1 > 0$), the condition that $\xi_{1*} > 0$ yields $\rho_1(\Delta,\rho) = 0$ (by Equation (III-12)). Then the likelihood function is zero for any data observed for which $N_1 \neq 0$. But even though $p_1 > p_2$ and $\Delta_2 < \Delta_1$ imply that the higher quality service is cheaper, out of a large number of consumers it is not impossible that a few will choose Alternative 1.

When irrationality random variables are included in the likelihood function, $\rho_1(\Delta,\rho)$ is not necessarily zero, and therefore the likelihood of some consumers choosing Alternative 1 will be nonzero.

1.3 Application to a Simple Problem

To develop an intuitive feeling for the behavior of this likelihood function as data are taken at different points, consider a simple problem. Assume W-P Model 0 holds, that $\xi_0 = s_0$, and that the complementary cumulative distribution of ξ_1 is as shown in Figure IV-2, and Equation (IV-10).

$$\Pr\{\xi_1 \geq \xi_1' | a\} = \begin{cases} 1 & \xi_1' < a \\ \xi_1'/a & a \leq \xi_1' \leq 0 \\ 0 & 0 < \xi_1' \end{cases} . \qquad \text{(IV-10)}$$

Suppose that a consumer is offered service at quality Δ and price p. The individual's demand for this service is evaluated using Equation (III-4):

$$\begin{aligned} \rho_1(\Delta,p|a) &= \Pr\{\xi_0 + \xi_1\Delta_c - P_c \geq 0\} \\ &= \Pr\left\{\xi_1 \geq \frac{P_c - s_o}{\Delta_c}\right\} \qquad \text{(IV-11)} \\ &= \begin{cases} 1 & \Delta < (P - s_o)/a \\ (P_c - s_o)/a\Delta_c & (P - s_o)/a \leq \Delta_c < \infty . \end{cases} \end{aligned}$$

Suppose ${}^{i}N$ purchases are observed during time ${}^{i}T$ at quality ${}^{i}\Delta$ and price p. Equation (IV-9) gives the likelihood function for this data:

$$\Pr\{{}^{i}N, {}^{i}T | {}^{i}\Delta, p, a, \lambda\} = (\lambda\rho_1({}^{i}\Delta, p|a))^{{}^{i}N} \exp\left(-\lambda\rho_1({}^{i}\Delta, p\ a){}^{i}T\right). \qquad \text{(IV-12)}$$

An "isolikelihood" plot of this function is shown in Figure IV-2 for two different sets of data ${}^{1}\Delta = 2$, $p = 1$, ${}^{1}N = 5$, ${}^{1}T = 10$ and

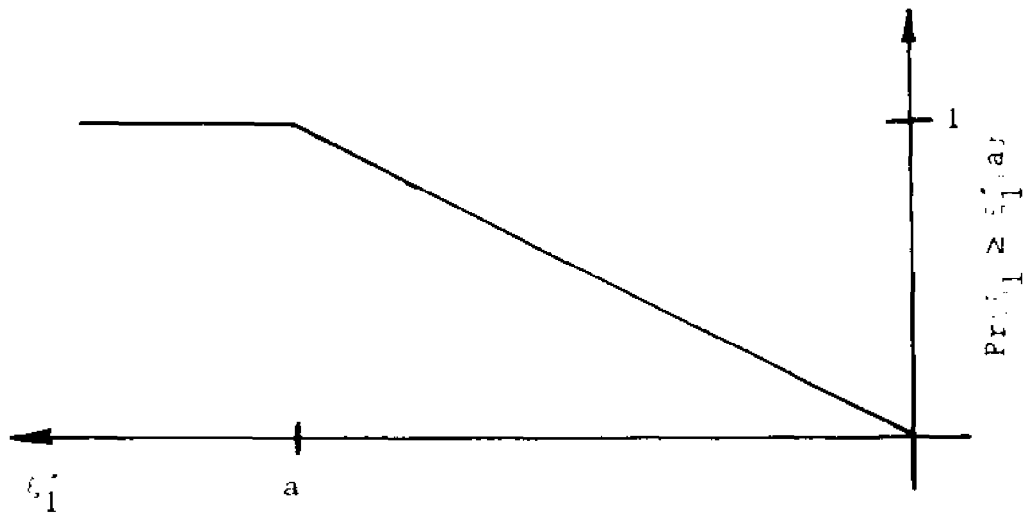

Figure IV-2--Distribution of Value of Service
for Sample Problem

${}^2\Delta = \frac{1}{2}$, $p = 1$, ${}^2N = 10$, and ${}^2T = 10$. From Equation (IV-12), $\rho_1(\Delta,p|a) = 1$ if $a > (p - S_0)/\Lambda_1$. In this region the maximum likelihood can be shown to occur at $\lambda = {}^1N/{}^1T$. When $a \le (p - S_0)^1\Lambda$, the maximum occurs at values of λ and a, for which $\lambda/a = {}^1N^1\Lambda/{}^1T(p - S_0)$.

If both sets of data are used together, the likelihood function is a product of the individual likelihood functions, and the maximum likelihood will occur at a point where the curves of maximum likelihood for the single factors intersect, if they intersect at all. In qualitative terms the likelihood function can be strongly peaked only if (1) the number of data sets is equal to or greater than the number of model parameters, and (2) a large number of events are observed. Notice that if ${}^1N/{}^1T < {}^2N/{}^2T$ and ${}^1N^1\Lambda/{}^1T(S_o - p) < {}^2N^2\Lambda/{}^2T(S_0 - p)$, the contours of maximum likelihood will not intersect. Isolikelihood contours for the composite of the likelihood functions in Figures IV-3a and b appear in Figure IV-4.

1.4 Generalizations From the Specific Example

For the general likelihood function expressed in Equation (IV-9), similar qualitative features hold. The likelihood function will be strongly peaked about some value of $\{\lambda,A,B\}$ only if the inequality

$$L\cdot J \ge \text{number of parameters in the set } \{\lambda,A,B\} \qquad \text{(IV-13)}$$

holds, and ${}^1N_j$ are all large. This inequality implies that if a narrowly peaked likelihood function is to result the number of choice probabilities multiplied together in the likelihood function must be greater than or equal to the number of unknown parameters.

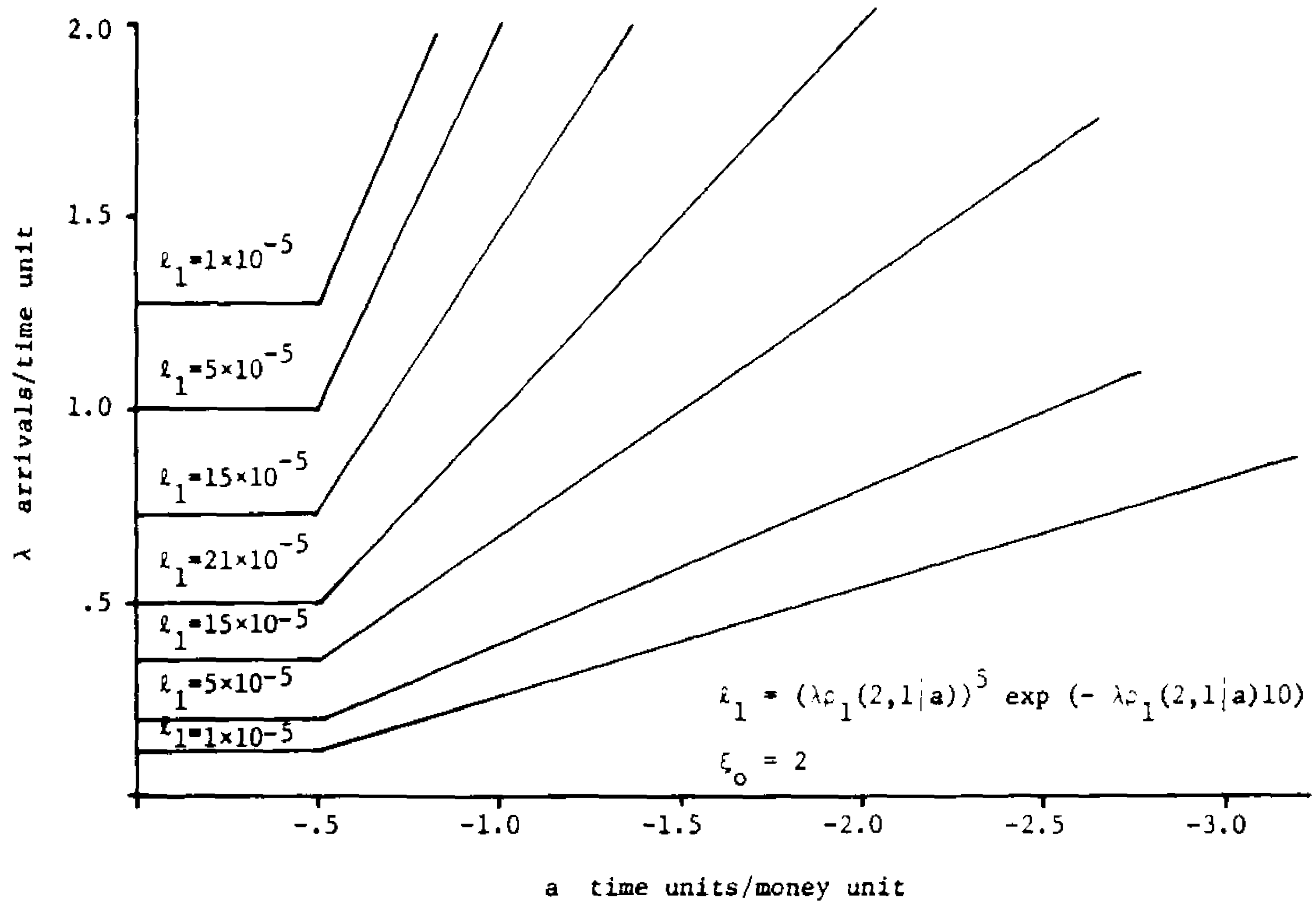

Figure IV-3a--Contours of Constant Likelihood Function Value

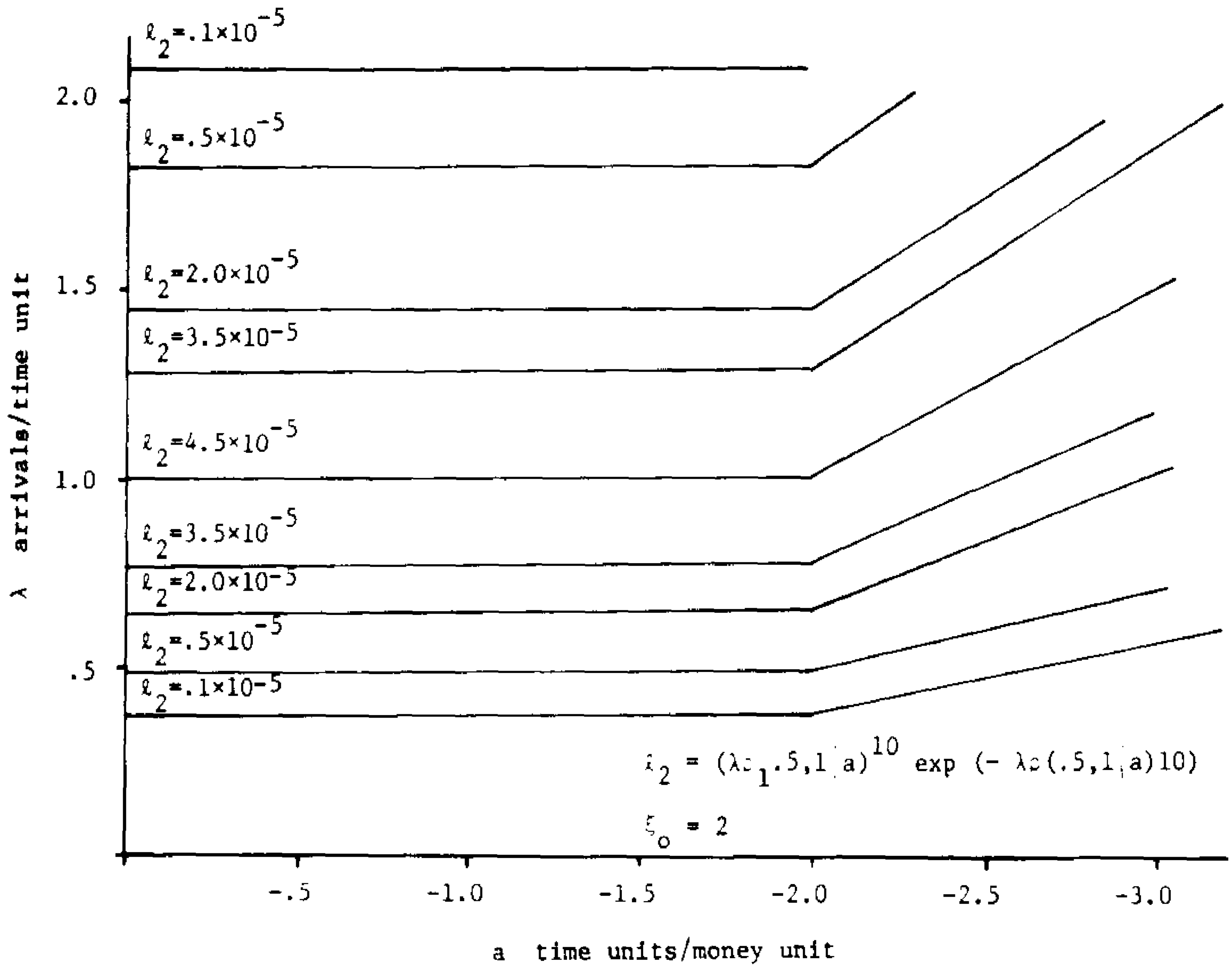

Figure IV-3b--Contours of Constant Likelihood Function Value

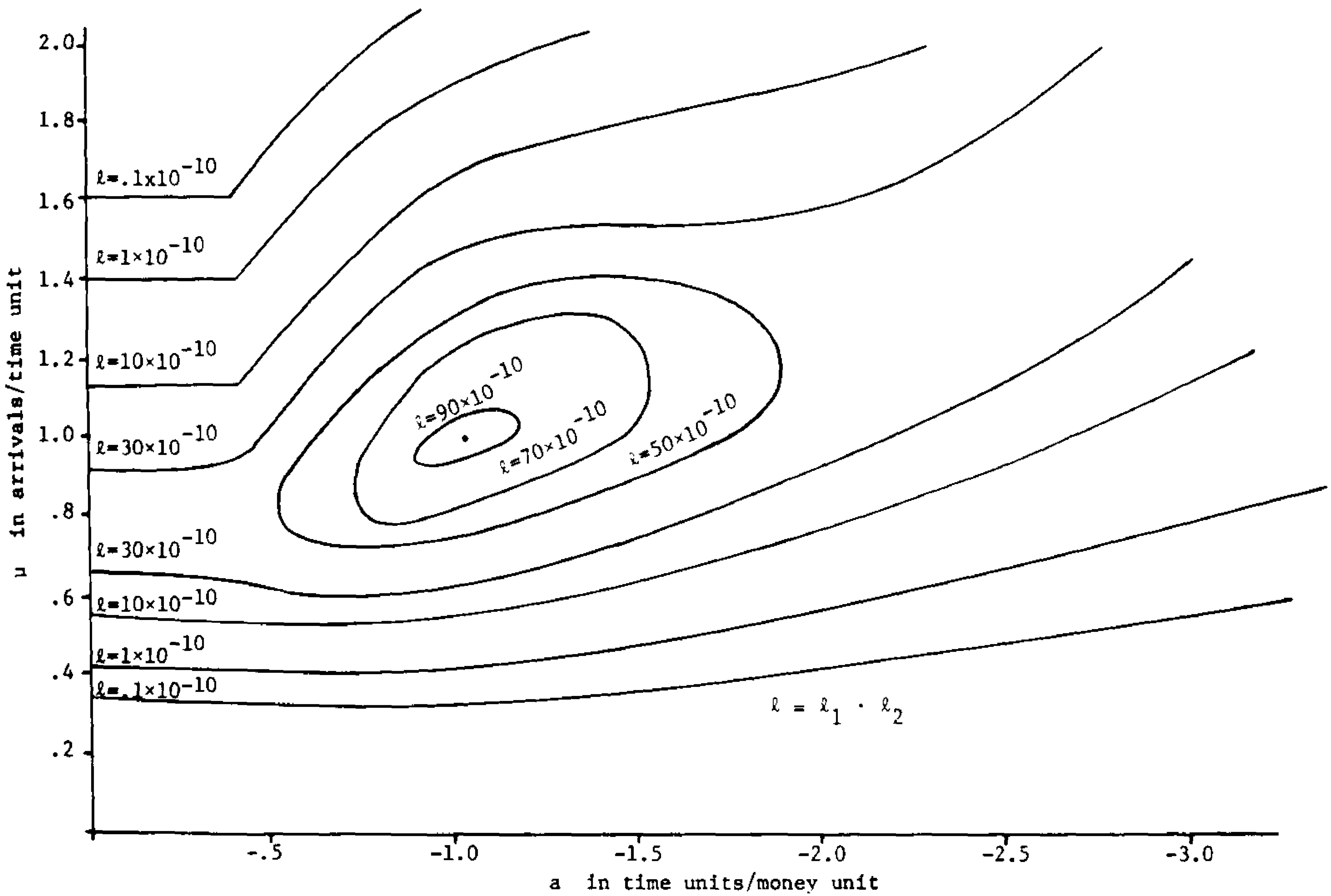

Figure IV-4--Contours of Constant Likelihood Function Value, Two Data Points

2.0 MONTE CARLO METHOD FOR CALCULATING POSTERIOR MOMENTS

2.1 Posterior Analyses and Optimization

There are two circumstances where knowledge of the posterior moments of the demand model parameters directly aids in solving the service-system optimization problem in Equation (II-12). First, when the expected payoff $E(R_T|H,C)$ is a polynomial function on the demand-model parameters, H, then integrating over H, as required by Equation (II-12), will produce an objective function that is a polynomial function of the posterior moments. Maximization of this polynomial objective function will often be possible using nonlinear programming techniques.

The other situation occurs when there are enough data to make the variance of the posterior distribution on demand-model parameters very low. With very small variance, the posterior distribution will closely approximate a delta function. There will be little error introduced into Equation (II-12) if the mean of the posterior distribution is substituted directly into the expected payoff. This will eliminate the need for integrating over the demand-model parameters, and thereby produce a simpler optimization problem.

2.2 Non-Parametric Bayesian Inference Technique for Posterior-Moment Calculation

Because of the likelihood function's complex form, it is not feasible to evaluate posterior moments analytically. It is possible, however, to evaluate them using a Monte Carlo method.

The m^{th} posterior moment of the parameter $a \in A$ (or $b \in B$) is found by performing the appropriate expectation operation on Equation (IV-1).

$$E(a^m|D) = K_D\int d\lambda\int dA\int dB\ a^m Pr\{D|A,B,\lambda\}Pr\{\lambda,A,B|E\}\ , \qquad \text{(IV-14a)}$$

where

$$K_D = \int d\lambda\int dA\int dB\ Pr\{D/A,B,\lambda\}Pr\{\lambda,A,B|E\}. \qquad \text{(IV-14b)}$$

Defining the random variables $X = a\ Pr\{D|A,B,\lambda\}$ and $Y = Pr\{D|A,B,\lambda\}$, then

$$E(a^m|D) = E(X)/E(Y), \qquad \text{(IV-15)}$$

where expectations are with respect to A,B, and λ prior to observing D.

Samples of the random variable Y are obtained by substituting into the likelihood function a sample of the random variables μ,A,B drawn from the prior distribution, $Pr\{\lambda,A,B|E\}$. Samples of the random variable X are obtained by sampling the same prior distribution, evaluating the likelihood function, and then multiplying the result by the sampled value a^m.

These samples give imperfect information about the means E(X) and E(Y); a situation described by assigning the random variables Θ whose distribution represents the state of knowledge about E(X), and Ψ whose distribution represents the state of knowledge about E(Y).

By Equation (IV-15), it follows that the state of information about the posterior m^{th} moment is described by the distribution of a random variable, Ω, related to Θ and ψ by,

$$\Omega = \Theta/\Psi. \qquad \text{(IV-16)}$$

As an estimate of the true value of $E(a^m|D)$ it is natural to consider $\overline{\Omega}$, and as a measure of how close this estimate is to $E(a^m|D)$ it is reasonable to use variance Ω. These estimates correspond respectively to the minimum-expected-cost guess and the minimum expected cost of guessing under quadratic loss.

Expanding Equation (IV-16) in a Taylor series about $\Theta = E(\Theta)$ and $\Psi = E(\Psi)$, and dropping third-order and higher terms gives a quadratic approximation for Ω.

$$\Omega \simeq \frac{\overline{\Theta}}{\overline{\Psi}} + \frac{1}{\overline{\Psi}}(\Theta - \overline{\Theta}) - \frac{\overline{\Theta}}{\overline{\Psi}^2}(\Psi - \overline{\Psi})$$

$$+ \frac{\overline{\Theta}_N}{\overline{\Psi}^3}(\overline{\Psi} - \overline{\Psi})^2 - \frac{1}{\overline{\Psi}^2}(\overline{\Psi} - \overline{\Psi})(\Theta - \overline{\Theta}). \qquad \text{(IV-17)}$$

The expected value of Ω is found by taking the expected value of Equation (IV-17)

$$\overline{\Omega} \simeq \frac{\overline{\Theta}}{\overline{\Psi}} + \frac{\overline{\Theta}}{\overline{\Psi}^3}\ \mathrm{Var}(\Psi) - \frac{1}{\overline{\Psi}^2}\ \mathrm{cov}(\Psi,\Theta). \qquad \text{(IV-18)}$$

An approximation to the variance of Ω can be found from the first three terms in Equation (IV-17).

$$\mathrm{Var}(\Omega) \simeq \frac{1}{\overline{\Psi}^2}\ \mathrm{Var}(\Theta) + \frac{\overline{\Theta}^2}{\overline{\Psi}^4}\ \mathrm{Var}(\Psi) - 2\,\frac{\overline{\Theta}}{\overline{\Psi}^3}\ \mathrm{cov}(\Theta,\Psi). \qquad \text{(IV-19)}$$

Equations (IV-18) and (IV-19) will yield approximations for the posterior mean and variance of the mean if relations can be found between $\overline{\Theta}$, $\overline{\Psi}$, Var Θ, etc., and the data samples $x_1, x_2, \ldots, x_N$ and $y_1, y_2, \ldots, y_N$. The necessary relations are derived in Appendix C, using a new technique for non-parametric Bayesian inference. The solutions are (assuming that the number of data samples are so large that prior information is negligible):

$$\bar{\Phi} = \frac{1}{N}\sum_{L=1}^{N} x_i , \quad \bar{\Psi} = \frac{1}{N}\sum_{L=1}^{N} y_i \quad , \tag{IV-20a}$$

$$\mathrm{Var}(\Phi) = \frac{1}{N^2(N+1)}\left(N\sum_{L=1}^{N} x_i^2 - \left(\sum_{L=1}^{N} x_i\right)^2\right) , \tag{IV-20b}$$

$$\mathrm{Var}(\Psi) = \frac{1}{N^2(N+1)}\left(N\sum_{L=1}^{N} y_i^2 - \left(\sum_{L=1} y_i\right)^2\right) , \tag{IV-20c}$$

$$\mathrm{cov}(\Phi,\Psi) = \frac{1}{N^2(N+1)}\left(N\sum_{L=1}^{N} x_i y_i - \sum_{L=1}^{N} x_i \sum_{L=1}^{N} y_i\right). \tag{IV-20d}$$

Since Equation (IV-15) is perfectly general, this method can be used to calculate any moment or cross-moment of the posterior distribution. For example, a cross moment can be calculated by letting $X = a_i^m a_j^n \Pr\{D|A,B\}$, and proceeding exactly as described above.

Equations (IV-19) and (IV-20) show that the posterior variance of Ω is optimized by using correlation sampling. As mentioned above, the data samples x_i and y_i are obtained by sampling $\Pr\{\mu,A,B|E\}$, evaluating the likelihood function, and then setting $x_i = a^m \Pr(D|\mu,A,B)$ and $y_i = \Pr(D|\mu,A,B)$. It is a matter of choice whether the same sample from $\Pr\{\mu,A,B|E\}$ is used in evaluating x and y. Independent samples could be used, but if the same sample is used for both, then x_i and y_i will be correlated positively, yielding a positive value for $\mathrm{cov}(\Phi,\Psi)$ and a lower value of variance. With independent samples from $\Pr\{\mu,A,B|E\}$, the term $\mathrm{cov}(\Phi_N,\Psi_N)$ would be smaller.

The question of when to stop sampling will not be addressed here, as it involves assigning a loss function. For a demand model, this loss function should be related to the payoff predicted by the demand model. Because the payoff function is likely to be complex, it is more reasonable to say that sampling should stop when the variance in the mean has been reduced to some small fraction of the mean. Just what fraction is used will be left to engineering judgment.

3.0 ASSIGNING VALUE-STRUCTURE DISTRIBUTIONS

3.1 Assignment for W-P Model 0

The likelihood function values needed in the inference procedure require algebraic expressions for the choice probabilities. To obtain algebraic expressions necessitates explicit assumptions about the willingness-to-pay functions, and the form of both the distributions $\Pr\{\xi_0, \xi_1 | A\}$ and $\Pr\{F | B\}$. This and succeeding sections present two different possibilities.

Assume first that W-P Model 0 holds ($W_c = \xi_0 + \xi_1 \Delta_c$) and that ξ_0 and ξ_1 are independent Gamma-distributed random variables.

$$\Pr\{\xi_0 | a_0, r_0\} = a_0 \frac{(a_0 \xi_0)^{r_0 - 1}}{(r_0 - 1)!} \exp(-a_0 \xi_0); \; \xi_0 \geq 0 , \quad \text{(IV-21a)}$$

$$\Pr\{\xi_1 | a_1, r_1\} = a_1 \frac{(-a_1 \xi_1)^{r_1 - 1}}{(r_1 - 1)!} \exp(-a_1 \xi_1); \; \xi_1 \leq 0 . \quad \text{(IV-21b)}$$

Equation (IV-21b) is restricted to negative values of ξ_1 under the reasonable assumption that $\partial W / \partial \Delta < 0$. The Gamma family assignment clusters most values of ξ_1 near the mean, corresponding to the intuitive notion that most consumers will place about the same value on quality.

The integrations necessary for evaluating the choice probabilities are simplified by assuming that r_0 and r_1 are positive integers. Since a_0 and a_1 are not so restricted, the integer restrictions on the r's should not significantly decrease the robustness of the Gamma family. Computational complexity is reduced even further by assuming that ξ_0 is exponentially distributed, i.e., restricting r_0 to 1. This assumption is reasonable if the probability that a consumer has a value for service between ξ_0 and $\xi_0 + \delta$ decreases as ξ_0 increases. Algebraic forms for the choice probabilities can be found even if $r_0 > 1$, but the number of computations in the final model increases in proportion to $r_0 \cdot r_1$.

Taking the case of choice among two alternatives, the choice probabilities for W-P Model 0 can be evaluated using Equation (III-7). Since the random variable, β, is not in W-P Model 0, the β and ξ integrations can be interchanged. The density $Pr\{\beta|B\}$ then integrates to one, yielding the choice probabilities shown in Equations (IV-22a) and (IV-22b). These equations apply for $p_1 \geq p_2$ and $\Delta_2 > \Delta_1$.

$$\rho_1(\Delta,p|A) = \int_{-\infty}^{d_{12}} d\xi_1 \frac{(a_1\xi_1)^{r_1-1}}{(r_1-1)!} \exp(+ a_1\xi_1)\int_{p_1-\xi_1\Delta_1}^{\infty} d\xi_0 a_0 \exp(- a_0\xi_0), \tag{IV-22a}$$

$$\rho_2(\Delta,P|A) = \int_{d_{12}}^{0} d\xi_1 \frac{(- a_1\xi)^{r_1-1}}{(r_1-1)!} \exp(a_1\xi_1) \int_{p_2-\xi_1\Delta_2}^{\infty} \exp(- a_0\xi_0), \tag{IV-22b}$$

$$A = (a_0,a_1,r_1),\ d_{12} = (p_1 - p_2)/\Delta_1 - \Delta_2. \tag{IV-22c}$$

Executing the integrations indicated by Equations (IV-22a) and (IV-22b) produces the algebraic forms in Equations (IV-23a) and (IV-23b).

$$\rho_1(\Delta,p|A) = \exp(-\, a_0 p_1 + a_2 d_{12})\left(\frac{a_1}{a_2}\right)^{r_1-1} \sum_{L=0}^{r_1-1}\left(\frac{-\, a_2 d_{12}}{L!}\right)^{i} , \qquad \text{(IV-23a)}$$

$$\rho_2(\Delta,p|A) = \exp(-\, a_0 p_2)\left(\frac{a_1}{a_3}\right)^{r_1-1} \left\{1 - \exp(a_3 d_{12}) \sum_{L=0}^{r_1-1} \frac{(a_3 d_{12})^{i}}{L!}\right\} ,$$

(IV-23b)

$$a_2 = a_1 + a_0\Delta_1 \qquad a_3 = a_1 + a_0\Delta_2 . \qquad \text{(IV-23c)}$$

3.2 Assignment for W-P Model 3

This section assumes that W-P Model 3 holds ($W_c = \xi_0 + \xi_1\beta_c\Delta_c$), that the probability density of ξ_0 conditioned on ξ_1 is exponential with $E(\xi_0|\xi_1,a_0) = \xi_1/a_0$, and that the density of ξ_1 is an inverted-gamma-1 distribution.

$$\Pr\{\xi_1/a_1,r_1\} = \frac{(a_1/\xi_1)^{r_1+1}\exp(-\, a_1/\xi_1)}{a_1(r_1 - 1)!} \qquad \xi_1 \geq 0 , \quad \text{(IV-24a)}$$

$$\Pr\{\xi_0/\xi_1,a_0\} = \frac{a_0}{\xi_1}\exp(-\, a_0\xi_0/\xi_1) \qquad \xi_0 \geq 0 . \quad \text{(IV-24b)}$$

Samples of the inverted-gamma-1 family are graphed in Figure IV-5.

The conditional dependence of ξ_0 on ξ_1 implies that consumers placing a high value on quality are more likely to have a high value on service than consumers having a low value on quality.

Assume that the irrationality random variable in W-P Model 3 is Gamma distributed with parameters b and r_2, where r_2 is a positive integer

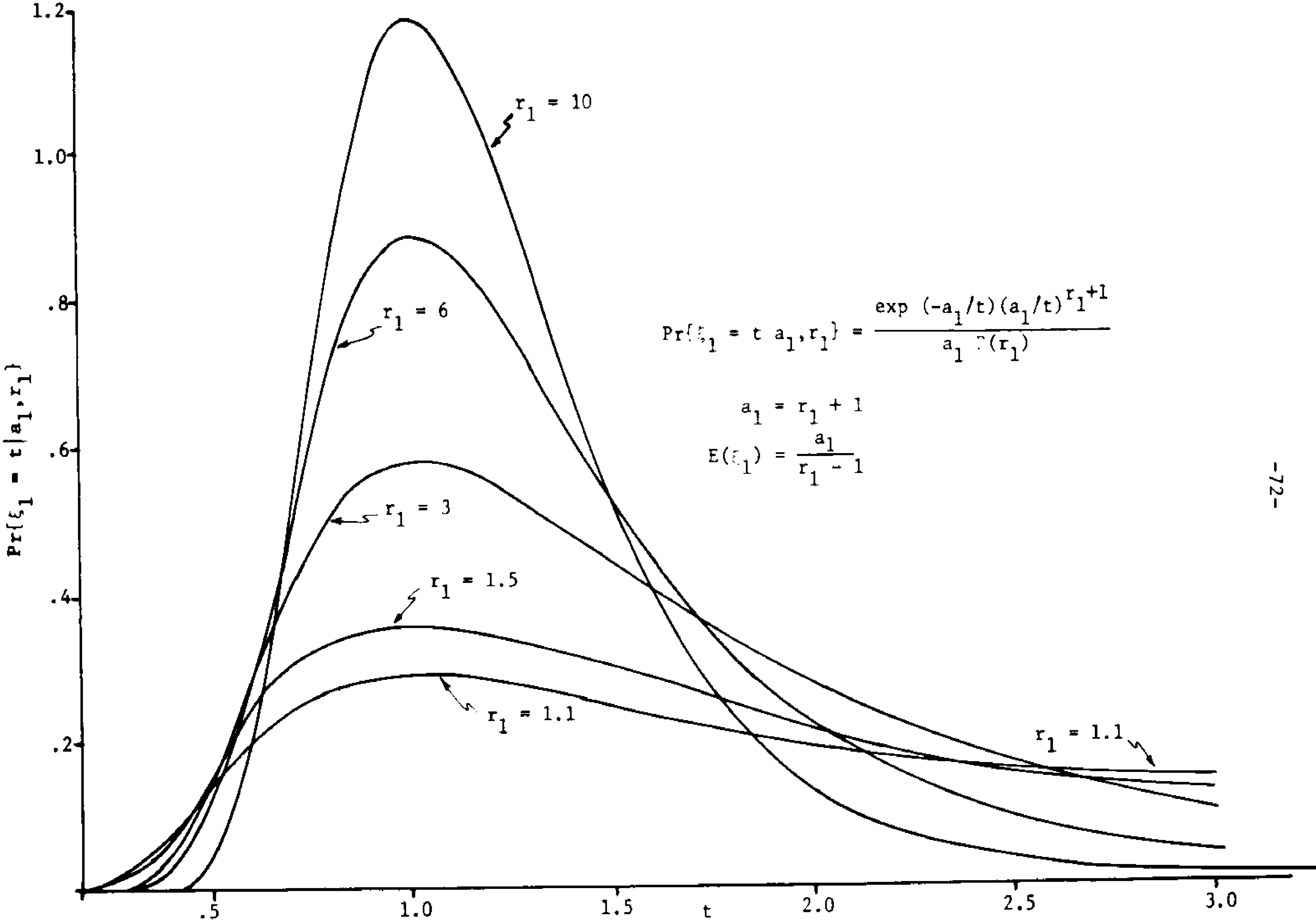

Figure IV-5--Inverted-Gamma-1 Densities

$$\Pr\{\beta_i = \beta_i' | b, r_2\} = b\,\frac{(-\,b\beta_i')^{r_2-1}\exp(b\beta_i')}{(r_2-1)!} \qquad \beta_i \le 0. \qquad \text{(IV-25)}$$

The restriction that $\beta_i \le 0$ in Equation (IV-25), and that $\delta_1 \ge 0$ in Equation (IV-24a) constrains $\partial W/\partial \Delta_i = \beta_i \delta_i$ to non-positive values. The same constraint could have been achieved by restricting $\beta_i \ge 0$, $\delta_1 \le 0$, but the former set of restrictions was used because it leads to simpler computations.

Using Demand Model B, Equation (III-14), and Equations (IV-24) and (IV-25), the choice probabilities can be calculated for the case of two alternatives:

$$p_1(\Delta,P|A,B) = C_0^{r_1} C_1^{r_2} \sum_{\ell=0}^{r_2-1} \sum_{i=0}^{r_2-1-\ell} G\left(r_1,\ell;\ C_1\,\frac{\Delta_1}{\Delta_2}\right) G\left(r_2,i;\ \frac{b_2}{a_1}\,C_0\right), \qquad \text{(IV-26a)}$$

$$p_2(\Delta,P|A,B) = C_0^{r_1} C_1^{r_2} \sum_{\ell=0}^{r_2-1} \sum_{L=0}^{r_2-1} G\left(r_1,\ell;\ \frac{b_2}{a_1}\,\frac{C_0}{C_2}\right) G\left(r_2 - \ell, i;\ \frac{\Delta_2}{\Delta_1}\,C_2\right), \qquad \text{(IV-26b)}$$

$$+\ C_2^{r_2} C_4^{r_1} \left(1 - \left(\frac{C_0}{C_4}\right)^{r_1} \sum_{\ell=0}^{r_2-1} G\left(r_1,\ell;\ \frac{b_2}{a_1}\,\frac{C_0}{C_3}\right)\right). \qquad \text{(IV-26c)}$$

where,

$A = \{a_0, a_1, r_1\}$

$B = \{b, r_2\}$

$C_0 = a_1/(a_1 + a_0 p_1 + b_2)$

$C_1 = b/(a_0\Delta_1 + b_3)$

$C_2 = b/(a_0\Delta_2 + b_4)$

$$C_3 = b/(a_0\Delta_2 + b)$$

$$C_4 = a_1/(a_1 + a_0p_2)$$

$$b_2 = b(p_1 - p_2)/\Delta_2$$

$$b_3 = b(1 + \Delta_1/\Delta_2)$$

$$b_4 = b(1 + \Delta_2/\Delta_1)$$

$$G(m,n;\ x) = \frac{\Gamma(m+n)x^n}{\Gamma(m)\Gamma(n+1)}\ .$$

Similar forms can be obtained for choice among three or more alternatives. In each case the number of summations in the first term of each choice probability will equal the number of alternatives. There will be one term in the choice probability for the first alternative and a number of terms equal to the number of alternatives in the rest.

Many other combinations of willingness-to-pay functions and distributions on consumer values can be postulated, but the cases presented in this and the previous section illustrate the basic method.

In applications, analysts may be faced with the problem of choosing the best from a set of postulated models. A theory for doing this appears in Smallwood's paper[1] on meta-modeling.

FOOTNOTES--CHAPTER IV

1. Smallwood, R. D., "A Decision Analysis of Model Selection," IEEE Transactions on Systems Science and Cybernetics, Vol. SSC-4, No. 3, September 1968, pp. 333-342.

Chapter V

APPLICATION OF STOCHASTIC APPROXIMATION TO SERVICE-SYSTEM OPTIMIZATION

1.0 DEVELOPMENT OF THE SERVICE-SYSTEM OPTIMIZATION CRITERION

1.1 Decomposition into Principal Suboptimizations

The criterion for optimizing service systems, written as Equation II-16, is repeated here for reference.*

$$\overline{R}_T^{\star}(C,D) = \max_{c\epsilon C} k_o \iint R_T \; Pr\{R_T|H,c,E\}Pr\{D|H,E\}Pr\{H|E\}dHdR_T. \qquad \text{(V-1)}$$

The parameter set H in Equation V-1 is related to the demand-model parameters in Chapters III and IV by the equality, $H = \{\lambda\} \cup A \cup B$. For later use, Equation V-1 is rewritten in a modified form:

$$\overline{R}_T^{\star}(C,D) = \max_{c\epsilon C} k_o \iint R_T \; Pr\{D|H,E\}Pr\{R_T,H|c,E\}dR_T dH. \qquad \text{(V-2)}$$

The quantity $Pr\{R_T,H|C\}$ describes the uncertainty contributed by the two sources identified in Chapter III. The normalizing constant, k_o, is independent of the controls, R_T is the payoff random variable, and $Pr\{D|H,E\}$ is the likelihood function on data outcomes.

If a separation is drawn between the optimization over price and over controls on service quality, the maximization in Equation V-2 splits into two sequential maximizations:

$$\overline{R}^{\star}(C,D) = k_o \max_{q\epsilon Q} \max_{P\epsilon P(q)} \int dR_T \int dH \; R_T \; Pr\{D|H\}Pr\{R_T H|q,P,E\}. \qquad \text{(V-3)}$$

*Since neither the likelihood function nor the prior depend on the control, c, the condition on c is dropped in these functions.

The regions Q and P(q) are related to C by $C = \bigcup_{q \epsilon Q} \{q\} \times P(q)$. The maximization over P must be done first since q will often specify the number of services available.

It is advantageous to divide the optimization problem into sequential steps when the set of elements over which q can range is limited to a finite set that is small enough to be searched exhaustively.* The global optimum is achieved by first finding the optimal price P^* for all $q \epsilon Q$, with an appropriate algorithm, and then choosing the pair (P^*, q) that maximizes expected payoff.

1.2 Finite List of Quality Alternatives

When only a finite list of services is available and two other conditions are satisfied, the optimization over quality in Equation V-3 can be done by inspection.

Let $L = \{1, 2, \ldots, J\}$ be a list of alternatives from which the quality-optimizing subset is to be drawn. Then Q is the set of all subsets of L and q is a subset of L. Assume that (1) the price of every service in q is unconstrained for every $q \epsilon Q$, and (2) the cost of providing service is independent of q.

If $q = L$, the joint distribution on payoff and demand-model parameters in Equation V-3 will be $\Pr\{R_T, H | L, P\}$--a mapping over P where P, the price vector, is an element of $\mathbb{R}^J$. If $q = L_0$, where L_0 is any subset of L containing $k < J$ elements, then the joint distribution will be $\Pr\{R_T, H | L_0, P_0\}$, where P_0 is an element in $\mathbb{R}^k$. The distribution

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*Prices are usually allowed to range over a continuum of values for any given $q \epsilon Q$, requiring a more sophisticated method of optimization.

$\Pr\{R_T, H \mid L, P\}$ can be made equal to the joint distribution $\Pr\{R_T, H \mid L_0, P_0\}$ if the price of every service in $L - L_0$ is raised until its demand is zero, and the price of every good in L_0 is set equal to its corresponding price in P_0. Under this pricing policy the expected payoff from either situation will be the same; thus any payoff attainable when q is a proper subset of L can be obtained when q equals L. In conclusion, the expected payoff in Equation V-3 is not decreased by setting q = L; therefore the optimum q is L. With J alternatives in L, Equation V-3 reduces to Equation V-4:

$$\bar{R}_T(D) = \max_{P \in \mathbb{R}^J} \iint R_T \Pr\{D \mid H\} \Pr\{R_T, H \mid P, \varepsilon\} dR_T dH. \qquad \text{(V-4)}$$

If the marginal cost of adding services is not zero, the conclusion to offer every available service may not hold. At some point the increase in payoff from more services may be more than offset by increased costs, leading to an optimum number of services that is less than J. When the cost of extra service is nonzero, a price-and-quality optimum can be found by combining a price-optimization algorithm with exhaustive search of service alternatives--a method that was described near the end of Section V.1.1.

2.0 CONVENTIONAL SOLUTION METHODS

2.1 Semi-Markov-Process Approach

The optimization problem set out in Equation V-1 is, in theory, solvable with the following method. First, the semi-Markov method is used to solve for the expected value of payoff given P and the demand-model parameters. Then the expectation over the posterior distribution of the demand-model parameters is taken, yielding the expected payoff as a function of P. The final step is maximization of an algebraic form in P.

This approach is not practical for two reasons. First, because of the likelihood function's complex form, it is unlikely that the expectation over the demand-model parameters could be calculated except for very simple demand models. Second, an inordinate amount of computation is required when using semi-Markov-processes to determine expected payoff $E(R_T|A,B,\lambda,P)$.

To see why the semi-Markov technique is impractical, consider its application to calculating $E(R_T|A,B,\lambda,P)$ for a finite-state quality process.* Let the quality process be for a J-priority, preemptive discipline queuing system having exponentially distributed service times and a single server. The quality of the i^{th} service will be the number of consumers waiting for, or in service of, the same or a higher priority. This process satisfies the requirements of a semi-Markov process, and therefore $E(R_T|A,B,\lambda,P)$ can be calculated using the semi-Markov process with rewards technique described in Howard.[2] If each priority is limited to K or less states (or places in line), the total number of states in the system is $(K + 1)^J$ (the extra state accounts zero delay). Let the state of the system be related to the quality process by the K-ary numbering scheme,

$$s = \Delta_1 + \Delta_2\ (K + 1) + \ldots\ \Delta_J(K + 1)^{J-1}. \tag{V-5}$$

*The details of this computation are carried out in Reference 2, but will be sketched here.

According to the demand model in Chapter III, the probability density on the time to the next arrival in priority j, given the demand-model parameters, is

$$\Pr\{\tau_j | \lambda, A, B, \Delta, P\} = \lambda\rho_j(\Delta, P|A,B) \exp\left(-\lambda\rho_j(\Delta,P|A,B)\tau_j\right). \qquad \text{(V-6)}$$

Likewise the probability density on the time to completion of the next service is,

$$\Pr\{\tau|\mu\} = \mu \exp(-\mu\tau), \qquad \text{(V-7)}$$

where $E(\tau) = 1/\mu$ is the expected time to complete a service.

To calculate $E(R_T|A,B,\lambda,P)$ by semi-Markov process methods, it is first necessary to obtain the probability of transition $p_{st}(P)$, between states s and t for all s and t. Any particular transition from s to t can be caused either by an arrival, or a departure; or it may be impossible to reach t from s.

If the transition, s to t, is one of those caused by an arrival, then only one alternative can be responsible. Call this alternative j_{st}. The probability density on a time to choose alternative j_{st} is,

$$\Pr\{\tau_{st}|\lambda,A,B,\Delta,P\} = \lambda\rho_{j_{st}}(\Delta,P) \exp\left(-\lambda\rho_{j_{st}}(\Delta,P)\tau_{st}\right). \qquad \text{(V-8)}$$

The joint probability-probability density function, $c_{st}(\tau,P)$, for the event that the quality process just entering state s will make its next transition to t after a holding time τ is (if the transition is caused by an arrival),

$$c_{st}(\tau,P) = \lambda\rho_{j_{st}}(\Delta,P) \exp\left(-\lambda\rho_{j_{st}}(\Delta,P)\tau\right) \exp(-\mu\tau) \cdot \exp\left(-\lambda(1-\rho_{j_{st}}(\Delta,P)\tau)\right). \qquad \text{(V-9)}$$

If the transition from s to t is caused by completion of a service, then the joint probability density function $c_{st}(\tau,P)$ is,

$$c_{st}(\tau,P) = \lambda \exp(-\lambda\tau) \exp(-\mu\tau). \tag{V-10}$$

If the transition from s to t could not have been caused by either an arrival or a departure, then

$$c_{st}(\tau,P) = 0. \tag{V-11}$$

In terms of $c_{st}(\tau,P)$ the transition probabilities for the Markov process imbedded in the quality process are[3]

$$p_{st}(P) = \int_0^\infty c_{st}(\tau,P)d\tau. \tag{V-12}$$

Howard shows that the limiting state probabilities π_s for the imbedded Markov process[4] are found by solving the linear equations,

$$\pi_s(P) = \sum_k \pi_k(P)p_{sk}(P) \qquad s = 0, 1, \ldots, J-1 \tag{V-13a}$$

and

$$\sum_s \pi_s(P) = 1. \tag{V-13b}$$

The mean holding time in state s is related to the densities $c_{st}(\tau,P)$ by[5]

$$\bar{\tau}_s(P) = \sum_t \int_0^\infty \tau c_{st}(\tau,P)d\tau. \tag{V-14}$$

The expected immediate reward for being in state s is,

$$r_s(p) = \sum_t p_{st}(P) \cdot w_{st}(P), \tag{V-15}$$

where w_{st} is the expected payoff gained in passing from state s to t. This expected payoff is the price paid if profits are being optimized; and it is the expected willingness to pay for alternative j_{st} if social welfare is being optimized (see Section II.2).

Howard shows that the expected payoff, conditional on the demand-model parameters, is the ratio of the average payoff per transition to the average number of transitions per unit of time.[6]

$$E(R_T|A,B,P) = T \sum_s \pi_s(P) r_s(P) / \sum_s \pi_s(P) \tau_s(P). \qquad (V-16)$$

Because $\pi_s(P)$ and $r_s(P)$ are complex functions of P, it is impossible in most cases to find the local maxima of Equation V-16 using the usual necessary conditions. The only alternative is an iterative procedure for finding the maximum, but this requires solving Equations 13 for $\pi_s(P)$, at each iteration. Except when the number of states is very low, the $\pi_s(P)$'s can only be found with numerical methods. But numerical methods would be very expensive, since even if the number of priorities is moderate (e.g., J = 3) and the saturation level low (e.g., K = 5), Equations 13 would require solution of $(K + 1)^J$ simultaneous equations for $(K + 1)^J$ variables.

2.2 Dynamic Programming by Policy Iteration

Another approach to optimization of service processes is Howard's policy-iteration algorithm[7] which finds the stationary, optimal control to apply given the state of a Markov process. The iteration algorithm takes the relative values, and the process gain calculated at the previous iteration, and finds the policy for each state that most improves the test quantity for that state. Then these interim best policies are used to evaluate process gain and the relative values, so that a new set of best policies can be found.

This algorithm cannot be used on our problem because prices are constrained to be the same in all states. This constraint is not compatible with the policy-iteration algorithm, since it is likely that a price policy chosen to maximize the test quantity for one state will decrease it in others. Unless a policy can be found that increases the value of the test quality in every state, there is no basis for choosing an interim best policy. Thus, the policy-improvement routine does not provide a solution to our service problem.

3.0 APPLICATION OF STOCHASTIC APPROXIMATION TO SYSTEM OPTIMIZATION

3.1 Simulation Approach to Optimization

In the absence of promising analytical techniques for finding optimal prices, it seems attractive to consider a simulation approach. A solution to the optimization problem is found simply by combining the concepts of stochastic approximation and simulation.

Stochastic approximation is an iterative method for maximizing the expected value of a controllable random variable. In our problem the random variable is $R_T \Pr\{D|H\}$, and the control variable is P, since R_T depends on P. Stochastic approximation is a modified gradient technique in that each iteration of the control requires only an approximation to the true gradient of the payoff function. This approximation is provided by sampling the random variable $R_T \Pr\{D|H\}$ at appropriately selected prices.

The sequence of events in the stochastic-approximation-simulation approach is as follows. Starting at an arbitrarily chosen set of prices for the alternatives, simulation is used to generate samples of the system payoff which indicate the direction prices should move to increase payoff. Based on this direction data, a new set of prices is chosen. The system payoff is then sampled at these new prices to find a new direction for improvement. This procedure continues until little improvement can be expected.

3.2 Kiefer-Wolfowitz Stochastic-Approximation Algorithm

To simplify notation, define the "posterior" payoff as the random variable,

$$R_T'(P) = R_T(P)\ \Pr\{D|H\}. \qquad \text{(V-17)}$$

If prices at the n^{th} iteration are P_n, the stochastic-approximation algorithm is to pick prices for the next iteration according to the relation:

$$P_{n+1} = P_n + a_n Y_n \qquad n = 0, 1, 2, \ldots \qquad \text{(V-18a)}$$

where

$$Y_n = \frac{1}{c_n} \sum_{i=1}^{J} U_i [R_T'(P_n + c_n U_i) - R_T'(P_n - c_n U_i)], \qquad \text{(V-18b)}$$

and U_i is the J-dimensional unit vector along the i^{th} coordinate of $\mathbb{R}^J$. The set of vectors $\{U_i : i = 1, 2, \ldots J\}$ will be called the basis for stochastic approximation.

More explicitly, this algorithm starts at an arbitrarily chosen set of initial prices, and obtains a sample of the payoff process R_T' at each of the prices in the base (not basis),

$$H_n = \left\{ P'_{i,n} : P'_{i,n} = P_n + c_n U_i, \qquad i = 1, 2, \ldots J \right\}. \tag{V-19}$$

Then the algorithm substitutes these samples into Equation V-18b to obtain an approximation, Y_n, of the gradient to the payoff surface. Finally, it moves in the direction of the gradient according to Equation 18a, and arrives at a new price vector. This sequence of steps can proceed indefinitely. If the payoff surface satisfies certain unimodality assumptions, and if the constants a_n and c_n are chosen correctly, the iterations will converge in mean square to the optimum prices. This technique is known in the literature as the Kiefer-Wolfowitz (K-W) stochastic-approximation procedure.

3.3 Sampling the Payoff Process

As specified by Equation V-3, samples of the random variable $R_T \Pr\{D|H\}$ are obtained by first sampling both R_T and H from the distribution $\Pr\{R_T, H|P, E\}$, and then substituting these values into $R_T \Pr\{D|H\}$. Since $\Pr\{R_T, H|P, E\}$ equals $\Pr\{R_T|H, P, E\} \Pr\{H|E\}$, a sample from $\Pr\{R_T, H|P, E\}$ is obtained by drawing a sample of H from $\Pr\{H|E\}$, and then simulating system operation using H to find the corresponding sample of R_T. With certainty on H, the sampling procedure is simplified since only simulation of the system payoff is required.

According to the demand model of Chapters III and IV and the system model in Figure I-1, a sample of the system payoff at prices $P'_{i,n}$ is obtained by the procedure described below for each value of i and + or −.

(1) Generate an exponentially distributed sequence of arrival events $\tau_1, \tau_2, \ldots, \tau_{N+1}$ such that $\sum_{i=1}^{N} \tau_i < T$, $\sum_{i=1}^{N+1} \tau_i \geq T$.

(2) Generate a set of consumer values $(\varepsilon_0^i, \varepsilon_1^i, \beta_1^i, \ldots \beta_J^i)$ for each arrival event i, and generate the service load, η^i, that each arrival event places on the system.

(3) Initialize the quality process to a given value at t = 0. Set i = 1.

(4) Evaluate the decision of the arrival at time, $t_i = \sum_{j=1}^{i} \tau_j$, using the quality state of the system at time t_{i-1}.

(5) Determine the changes in the quality process caused by the system in the time interval including the arrival at t_i.

(6) Add the contribution of this arrival to aggregate payoff, and the quality process.

(7) Repeat Steps 4 through 6 for i = 2, ... N; until all arrival events are exhausted.

For later use, let the set of random variables generated in Steps 1 through 3 be Λ.

$$\Lambda = \Big\{(\varepsilon_0^1, \varepsilon_1^1, \beta_1^1, \ldots \beta_J^1, \eta^1, \tau_1), \\ (\varepsilon_0^2, \varepsilon_1^2, \beta_1^2, \ldots \beta_J^2, \eta^2, \tau_2), \ldots \tau_N)\Big\}. \tag{V-20}$$

Recalling Section II.3.2, the random variable, R_T, is the payoff accumulated when the quality process is stationary over the whole time interval [0,T]. Since the quality process cannot be solved analytically, the stationary distribution of the quality process is a priori unknown, and the quality process cannot be properly initialized at time t = 0. As a consequence, simulation of R_T is impossible.

There are two ways to overcome this problem. One is to use the final value of the quality process at the end of the n^{th} iteration of prices as the initial value for the n + 1st iteration. If the prices at the nth iteration are sufficiently close to the prices at the n + 1st iteration, the initialization value for quality in the n + 1st iteration will be drawn from a distribution which is very close to the true distribution. Then the sample value for the payoff in the n + 1st iteration will be drawn from a distribution that is very close to the true distribution. With this method successive samples of the payoff process are dependent.

Sample Method $\mathcal{M}_s$: Initialize the quality process at the beginning of the n + 1st iteration to the final value of the quality process at the nth iteration.

Another approximate solution is achieved by initializing the quality process to the same value on each iteration. Then R_T will be a transient process, but if T is long enough, the quality process will be in the steady state during most of T, making the distribution of R_T very close to the true distribution. With this method, successive samples of the payoff process are independent.

Sample Method $\mathcal{M}_f$: Initialize the quality process to the same value on every iteration.

Aside from the ways of initializing the quality process, samples of payoff process can be obtained with correlated or uncorrelated samples of consumer values.

With uncorrelated sampling the sequence of random variables used to evaluate system payoff at each price in Π_n are identically distributed, but independently generated. Let this sequence of random variables be Λ_1^+, Λ_1^-, ..., Λ_J^-, where Λ_i^+ is the set of consumer values used to evaluate the $P_{i,n}^{\pm}$, etc.

Sample Method $\mathcal{S}_u$: The set $\Lambda_u = \{\Lambda_1^+, \Lambda_1^-, \ldots, \Lambda_J^-\}$ is a collection of mutually independent, identically distributed sets of random variables.

With uncorrelated sampling the set of payoff random variables $\{R_T'(P_{i,n}'):i = 1, \ldots J\}$ will be independent since each is a function of one and only one element in the set, Λ_u, a collection of independent sets of random variables.

Lemma V-1: With uncorrelated sampling, the set of payoff random variables is independent.

The second method is correlated sampling where the same set of consumer random variables is used to evaluate each of the payoff random variables, $R_T'(P_{1,n}^+)$, $R_T'(P_{i,n}^-)$, ..., $R_T'(P_{J,n}^-)$, and this set of random variables is drawn from the same distribution as Λ_1^+ with uncorrelated sampling.

Sample Method $\mathcal{S}_c$: The set $\Lambda_c = \{\Lambda_1^+, \Lambda_1^-, \ldots, \Lambda_J^-\}$ has $\Lambda_i' = \Lambda_1^+$ for $i = 1, 2, \ldots, J$, and $\Pr\{\Lambda_1^+|\mathcal{S}_c,E\} = \Pr\{\Lambda_1^+|\mathcal{S}_u,E\}$.

Using correlated samples is an innovation in that (1) less time is required to generate payoff samples, (2) $E(Y_n,E)$ is not changed, and (3) $\mathrm{var}\ (Y_n|\mathcal{S}_c,E) \leq \mathrm{var}\ (Y_n|\mathcal{S}_u,E)$. The reduction in $\mathrm{var}\ (Y_n|\mathcal{S}_c,E)$ improves the rate of convergence, since the gradient to the payoff surface can be measured more accurately. The equivalence of $E(Y_n|\mathcal{S}_u,E)$ and $E(Y_n|\mathcal{S}_c,E)$ guarantees that convergence to the optimum price is not lost with correlation sampling.

Lemma V-2: $E(Y_n|\mathcal{S}_c,E) = E(Y_n|\mathcal{S}_u,E)$.

Proof: By the definition of Y_n,

$$E(Y_n|\mathcal{S}_c,E) = \frac{1}{c_n} \sum_{i=1}^{J} \left[E(R_T'(P_{i,n}^+)|\mathcal{S}_c,E) - E(R_T'(P_{i,n}^-)|\mathcal{S}_c,E) \right]. \qquad \text{(V-21)}$$

By the definition of correlated sampling,

$$\Pr\{\Lambda_i^{\pm}|S_c,E\} = f_u(\Lambda_i) \qquad i = 1, 2, \ldots, J. \tag{V-22}$$

where $f_u(\cdot)$ is the probability density of Λ_i^+ with uncorrelated sampling. By expansion,

$$E(R_T'(P_{i,n}^+)|S_c,E) = \int d\Lambda_i^+ \; E(R_T'(P_{i,n}^+)|S_c,\Lambda_i^+,E)\Pr\{\Lambda_i^+|S_c,E\}, \tag{V-23}$$

and correspondingly for $P_{i,n}^-$. With knowledge of Λ_i^+, knowledge of S_c is redundant so that Equation V-23 becomes,

$$E(R_T'(P_{i,n}^+)|S_c,E) = \int d\Lambda_i^+ \; E(R_T'(P_{i,n}^+)|\Lambda_i^+,E)\Pr\{\Lambda_i^+|S_c,E\}. \tag{V-24}$$

Substituting Equation V-22 into Equation V-24,

$$\begin{aligned} E(R_T'(P_{i,n}^+)|S_c,E) &= \int d\Lambda_i^+ \; E(R_T'(P_{i,n}^+)|\Lambda_i^+,E) f_u(\Lambda_i^+) \\ &= E(R_T'(P_{i,n}^+)|S_u,E). \end{aligned} \tag{V-25}$$

Substituting Equation V-25 into Equation V-21 proves Lemma V-2.

<u>Lemma V-3</u>: If $\operatorname{cov}\,(R_T'(P_{i,n}^+),\ R_T^-(P_{i,n}^-)|S_c,E) \geq 0$, for all $i = 1, \ldots J$, then $\operatorname{var}\,(Y_n|S_c,E) \leq \operatorname{var}\,(Y_n|S_u,E)$.

<u>Proof</u>: Because the U_i's are orthogonal vectors,

$$\begin{aligned} \operatorname{var}\,(Y_n|S_c,E) &= E(\langle Y_n - E(Y_n|S_c,E)/Y_n - E(Y_n|S_c,E)\rangle) \\ &= \operatorname{var}\,(Y_n|S_u,E) - 2\sum_{j=1}^{J} \operatorname{cov}\,(R_T'(P_{j,n}^+),\ R_T'(P_{j,n})|S_c,E), \end{aligned} \tag{V-26}$$

where $\langle X/Z\rangle$ stands for the inner product of the vectors X and Z. When the covariance terms are nonnegative the conclusion of Lemma V-3 holds.

Lemma V-3 would be strengthened if the nonnegativity of the payoff covariances could be shown to hold in general for service systems, but conditions on $R_T(P)$ which insure nonnegativity could not be found.

A heuristic argument for the nonnegativity of the covariances is that if the same sequence of consumer random variables is used to evaluate payoff at $P_n + c_n U_i$ as is used at $P_n - c_n U_i$, the consumption decision, and hence the contribution to payoff, of most consumers will be the same at either price (if c_n is small enough). Therefore, when the payoff at $P^+_{i,n}$ is high, the payoff at $P^-_{i,n}$ will be high, and when the payoff at $P^+_{i,n}$ is low the payoff at $P^-_{i,n}$ will be low. This correlation produces a positive covariance.

3.4 Rotation and Dilation of Base Vectors

Two novel features have been added to the basic stochastic-approximation algorithm which improve its convergence properties.

To improve the rate of convergence on ridged payoff surfaces, the basis vectors used at the n^{th} iteration are obtained from a randomly oriented, rigid rotation of the original basis U_i, $i = 1, 2, \ldots J$. The new basis vectors at the n^{th} iteration, $U'_{i,n}$, $i = 1, 2, \ldots J$, are related to the old basis vectors by,

$$U'_{i,n} = \sum_{j=1}^{J} t^n_{j,i}\ U_{j,o}, \tag{V-27}$$

where t^n_{ji} is an element in the rotation operator T_{Θ_n}. In matrix form the rotation operation is

$$U'_n = T^t_{\Theta_n}\ U_o, \tag{V-28}$$

where U_o is a matrix of original basis vectors and U'_n is a matrix of rotated basis vectors at the n^{th} iteration.

The operator T_{Θ_n} is defined to be of the form,

$$T_{\Theta} = T_{\Theta_{1,n}} T_{\Theta_{2,n}} \cdots T_{\Theta_{m,n}} . \tag{V-29}$$

Each of the $T_{\Theta_{i,n}}$'s rotates the J-dimensional basis in the plane of two of the original basis vectors. A typical member of this family is

$$T_{\theta_{i,n}} = \begin{bmatrix} \cos\theta_{i,n} & 0 & \sin\theta_{i,n} & 0 & \\ 0 & 1 & 0 & 0 & \\ -\sin\theta_{i,n} & & \cos\theta_{i,n} & 0 & \\ 0 & & & 1 & \\ \vdots & & & & \ddots \end{bmatrix} . \tag{V-30}$$

which rotates the basis in the plane of the first and third basis vectors.

The number, m, of rotational degrees of freedom in J-dimensional Euclidean space is

$$m = \frac{J!}{2!(J-2)!} , \tag{V-31}$$

since there are $J(J - 1)/2$ pairs of axes in J-dimensional space. Or, there are $J(J - 1)/2$ ways of placing two cosine elements on the diagonal of a J-dimensional matrix.

The orientation angles $\Theta_n = \{\theta_{1,n}, \theta_{2,n}, \ldots, \theta_{m,n}\}$ are randomly selected at each iteration according to the prespecified distribution $Pr\{\theta_1, \theta_2, \ldots, \theta_m\}$. To facilitate the stochastic-approximation convergence proof, successive orientations are independent, identically distributed random variables.

$$\Pr\{\Theta n, \Theta n + 1\} = \Pr\{\Theta n\} \Pr\{\Theta n + 1\} \qquad n = 1, 2, \ldots N. \qquad \text{(V-32)}$$

Figure V-1 illustrates why rotation of the base vectors increases the convergence rate. With the base vectors oriented along the basis vectors for price, $E(Y_n)$ at $P = (p_1,p_2)$ will be smaller in magnitude and nearly orthogonal to the true gradient, grad $R_T'(P)$. Rotating the base vector by $\pi/4$ radians would substantially reduce the difference between $E(Y_n)$ and grad $R_T'(P)$.

A queuing system example, to be treated in Chapter VI, happens to have a ridged surface like the one graphed in Figure V-1. It will be demonstrated by example in Chapter VI that convergence is significantly accelerated by rotating the basis vectors.

A second feature added to the stochastic-approximation algorithm is dilation of the base vectors by the transformation T_V^t, a diagonal matrix having elements v_1, v_2, ..., v_J, which are constant over price iterations.

The reason for including T_V^t is that the number of degrees of freedom available for optimizing the convergence rate of the stochastic-approximation algorithm is increased by allowing independent adjustment of the base size in each coordinate direction. The base size (determined by c_n) is one of the parameters which must be adjusted to optimize convergence. The transformation T_V^t permits choosing a different value of c_n for each price axis.

Applying both transformations simultaneously, the new set of basis vectors is

$$U_n'' = T_V^t \, T_{\Theta_n}^t \, U, \qquad \text{(V-33)}$$

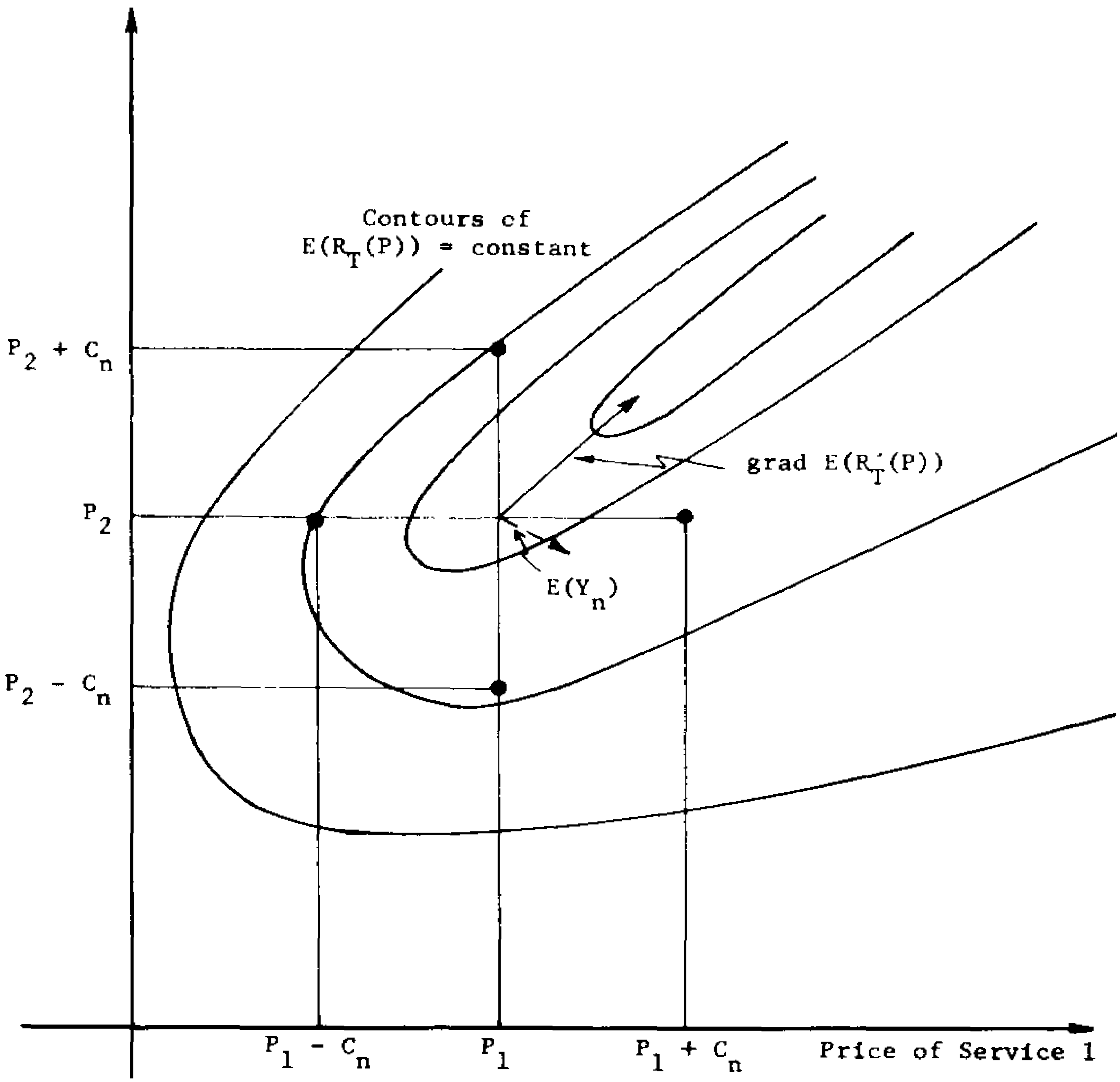

Figure V-1--Stochastic Approximation Base on a Ridge and the approximation to the gradient is

$$Y_n = \sum_{L=1}^{J} [R_T'(P_n + c_n U_{i,n}') - R_T'(P_n - c_n U_{i,n})] \frac{1}{v_i c_n} U_{i,n}'. \qquad \text{(V-34)}$$

Inclusion of these new features requires changes in the stochastic-approximation convergence proof. The modified proof given in the next section was motivated by the works of Dupac[8] and Sakrison.[9] These proofs are chosen because the assumptions required for convergence are more compatible with the service system problem than other formulations of the stochastic approximation problem.

A comprehensive review of past work in K-W stochastic approximation will not be attempted here. A review of developments from the beginning (1952) to 1960 can be found in Schmetterer.[10] Extensions since 1960 are covered or referenced in Fabian.[11]

3.5 Proof of Convergence

The purpose of this section is to show conditions on the posterior-payoff surface which guarantee convergence of the stochastic-approximation algorithm to an optimum price. Let this optimum price be P^*.

Definition: P^* satisfies

$$E(R_T'|P^*, \mathcal{S}_f, \mathcal{S}_u, \mathcal{E}) = \max_{P \in \mathbb{R}^J} E(R_T'|P, \mathcal{S}_f, \mathcal{S}_u, \mathcal{E}).$$

Actually P^* is only an approximation to the true price since the true optimum is achieved by maximizing $E(R_T'|P, \mathcal{S}_s, \mathcal{S}_u, \mathcal{E})$--the expected posterior payoff obtained when the quality process is initialized from its steady-state distribution. As T is increased, P^* approaches the true optimum.

A convergence theorem for the random variables R_T' will be proved first under assumptions about the shape of the posterior payoff surface $E(R_T' | P, \mathcal{S}_f, \mathcal{S}_u, E)$. Then this result will be modified to show that convergence can still be proved when P is constrained to a closed, bounded, convex set. Section V.3.7 shows conditions on the payoff $E(R_T' | H, P, \mathcal{S}_f, \mathcal{S}_u, E)$ which are sufficient to guarantee convergence of prices on the posterior, no matter what the form of the likelihood function.

The proof requires the following conditions on the posterior-payoff surface and the iteration parameters a_n and c_n:

Assumption V-1: $R_T'(P) \leq K_2$ for all $P \in \mathbb{R}^J$.

Assumption V-2: The sequences $\{a_n : n = 1, 2, \ldots\}$ and $\{c_n : n = 1, 2, \ldots\}$ have positive elements with

$$\sum_{n=1}^{\infty} a_n = \infty, \quad \sum_{n=1}^{\infty} a_n^2 < \infty, \quad \sum_{n=1}^{\infty} a_n c_n^4 < \infty, \text{ and } \sum_{n=1}^{\infty} (a_n / c_n)^2 < \infty.$$

Assumption V-3:

$$\frac{\partial^3 E(R_T' | P, \mathcal{S}_g, \mathcal{S}_u, E)}{\partial p_i \partial p_j p_k} \leq K_3 \qquad \text{for } i, j, k, = 1, 2, \ldots J, \text{ and } P \in \mathbb{R}^J.$$

Assumption V-4:

$$\langle \text{grad } E(R_T' | P, \mathcal{S}_f, \mathcal{S}_u, E) / P - P^* \rangle \leq - K_0 \|P - P^*\|^2 \qquad \text{for all } P \in \mathbb{R}^J.$$

Assumption V-5:

$$\| \text{grad } E(R_T' P, \mathcal{S}_f, \mathcal{S}_u, E) \| \leq K_1 \|P - P^*\|^2 \qquad \text{for all } P \in \mathbb{R}^J.$$

Theorem V-1 below proves that Assumptions V-1 through V-5 are sufficient to prove the convergence of stochastic approximation to P^* (the optimum for uncorrelated sampling) when correlated sampling is used.

Theorem V-1: Under Assumptions V-1 through V-5, the algorithm

$$P_{n+1} = P_n + \frac{a_n}{c_n} Y_n \qquad n = 0, 1, 2, \ldots$$

$$Y_n = \sum_{j=1}^{J} (R_T'(P_n + c_n U_n') - R_T'(P_n - c_n U_n')) \frac{1}{v_1} c_n U_n'$$

has

$$\lim_{n\to\infty} E(\|P_n - P^*\| | \mathcal{S}_f, \mathcal{S}_c, E) = 0.$$

Proof:

Subtracting P^* from both sides of Equation V- , and taking the inner product of each side with itself, gives

$$\| P_{n+1} - P^* \|^2 = \| P_n - P^*\|^2 + 2 a_n \langle P_n - P^*/Y_n\rangle + a_n^2 \| Y_n \|^2. \qquad \text{(V-35)}$$

The expected value of Equation V-35 given P_n, the initial prices P_0, correlated sampling, and a reset quality process is, because of linearity in an inner product,

$$\begin{aligned} E(\|P_{n+1} - P^*\|^2 | P_n, P_0, \mathcal{S}_f, \mathcal{S}_c, E) &= \| P_n - P^*\|^2 \\ &+ 2 a_n\langle P_n - P^*/E(Y_n | P_n, P_0, \mathcal{S}_f, \mathcal{S}_c, E)\rangle \\ &+ a_n^2 E(\|Y_n^2\| | P_n, P_0, \mathcal{S}_f, \mathcal{S}_c, E). \qquad \text{(V-36)} \end{aligned}$$

Because resetting the quality process to a fixed value at each iteration produces independent samples of the payoff, the expected value of

Y_n is independent of the initial price,

$$E(Y_n | P_n,\ P_0,\ \mathcal{S}_f,\ \mathcal{S}_c,\ \mathcal{E}) = E(Y_n | P_n,\ \mathcal{S}_f,\ \mathcal{S}_c,\ \mathcal{E}). \tag{V-37}$$

By Lemma V-2,

$$E(Y_n | P_n,\ \mathcal{S}_f,\ \mathcal{S}_c,\ \mathcal{E}) = E(Y_n | P_n,\ \mathcal{S}_f,\ \mathcal{S}_u,\ \mathcal{E}). \tag{V-38}$$

By Assumption V-1, $R_T' \le K_2$, and therefore,

$$\|Y_n\|^2 \le 2JK_2^2/c_n^2. \tag{V-39}$$

Under Assumptions V-3 through V-5, it is proved in Appendix C that:

$$\langle P - P^* / E(Y_n | P_n,\ \mathcal{S}_f,\ \mathcal{S}_u,\ \mathcal{E})\rangle \le -\ 2K_0\|P - P^*\|^2 + \frac{J^3}{3v} K_3\ c_n^2\|P - P^*\| \tag{V-40}$$

where $v = \min|v_i|$.

Substituting Equation V-38, and then Equations V-39 and 40 into V-36 gives,

$$E(\|P_{n+1} - P^*\|^2 | P_n,\ \mathcal{S}_f,\ \mathcal{S}_c,\ \mathcal{E}) \le \|P_n - P^*\|^2 - 4K_0 a_n\|P_n - P^*\|^2$$
$$+\ 2(J^3/3v)K_3 a_n c_n^2\|P_n - P^*\| + \left(\frac{a_n}{c_n}\right)^2 2JK_2^2. \tag{V-41}$$

Letting $b_n = E(\|P_n - P^*\|^2 | \mathcal{S}_f,\ \mathcal{S}_c,\ \mathcal{E})$, and taking the expected value over P_n given $\mathcal{S}_f$, $\mathcal{S}_c$, and $\mathcal{E}$ gives,

$$b_{n+1} \le b_n - 4K_0 a_n b_n + 2(J^3/3v)K_3 a_n c_n^2\ E(\|P_n - P^*\|\ |\mathcal{S}_f,\ \mathcal{S}_c,\ \mathcal{E})$$
$$+\ 2J\ K_2^2\left(\frac{a_n}{c_n}\right)^2. \tag{V-42}$$

Let $\epsilon_n = 2J^3K_3c_n^2/3v\epsilon\ K_0$ where $0 < \epsilon < 4$. By Lemma C-4:

$$2(J^2/3v)K_3 a_n c_n^2\ E(\|P_n - P^*\|\,|\mathcal{S}_f,\ \mathcal{S}_c,\ \mathcal{E}) \le \frac{a_n}{\epsilon K_0}\left(2(J^3/3v)K_3 c_n^2\right)^2$$
$$+\ a_n\ \epsilon\ K_0\ E(\|P_n - P^*\|^2 | \mathcal{S}_f,\ \mathcal{S}_c,\ \mathcal{E}). \tag{V-43}$$

Substituting Equation V-43 into Equation V-42,

$$b_{n+1} \le b_n(1 - (4 - \epsilon)K_0\, a_n) + a_n c_n^4\, (2(J^3/3v)K_3{}^2\, \frac{1}{\epsilon K_0} + \left(\frac{a_n}{c_n}\right)^2\, 2JK_2^2. \tag{V-44}$$

Finally, by Assumption V-2, Lemma C-2 and Lemma C-3,

$$\lim_{n\to\infty} b_n = 0, \tag{V-45}$$

proving the theorem.

The next theorem shows that satisfaction of Assumptions V-1 and V-3 to V-5 on a closed, bounded subset $\Omega \subset \mathbb{R}^J$ is sufficient for mean square convergence of prices to the optimum in Ω. This theorem enlarges the collection of surfaces for which convergence can be proved, since more surfaces satisfy Assumptions V-1 through V-5 on a compact set than on the whole of Euclidean space.

With P constrained to Ω, the definition for optimal price is slightly changed:

__Definition__: P_Ω^* is the price which satisfies

$$E(R_T'|P_\Omega^*, \mathcal{S}_f, \mathcal{S}_u, E) = \max_{P\epsilon\Omega} E(R_T'|P, \mathcal{S}_f, \mathcal{S}_u, E).$$

The Assumptions V-1 through V-5 can now be restated to reflect the constraint of P to Ω:

__Assumption V-1′__: $R_T'(P) \le K_2$ for all $P\epsilon\,\mathbb{R}^J$, and $P\epsilon\Omega$ a closed, bounded, convex subset of $\mathbb{R}^J$.

__Assumption V-2′__: The sequences of a_n and c_n are the same as in Assumption V-2.

__Assumption V-3′__: $\dfrac{\partial^3\, E(R_T'\, P,\, \mathcal{S}_f,\, \mathcal{S}_u,\, E)}{\partial p_i \partial p_j \partial p_k} \le Q$ for all $i, j, k, = 1, 2, \ldots J$ and $P\epsilon\Omega$.

Assumption V-4′: $\langle \text{grad } E(R_T'|P,\ \mathcal{S}_f,\ \mathcal{S}_u,\ \mathcal{E}/P - P^*\rangle \le -K_0\|P - P_\Omega^*\|^2$ for all $P\epsilon\Omega$.

Assumption V-5′: $\|\text{grad } E(R_T'|P,\ \mathcal{S}_f,\ \mathcal{S}_u,\ \mathcal{E})\| \le K_1\|P - P_\Omega^*\|$ for all $P\epsilon\Omega$.

Theorem V-2 can now be stated.

Theorem V-2: Under Assumptions V-1′ through V-5′, the algorithm,

$$P_{n+1}' = P_n + a_n Y_n \qquad n = 0, 1, 2, \ldots$$

$$P_{n+1} = \begin{cases} P_{n+1}' & \text{if} \quad P_{n+1}'\epsilon\Omega \\ \alpha P_{n+1}' & \text{if} \quad P_{n+1}\Omega \quad \text{where } \alpha = \max\{\alpha' : P_{n+1}'\epsilon\Omega\} \end{cases}$$

$$Y_n = \sum_{j=1}^{J} [R_T'(P_n + c_n U_j') - R_T'(P_n - c_n U_j')] \frac{1}{c_n v_j} U_j'$$

has $\lim_{n\to\infty} E(\|P_n - P_\Omega^*\| | \mathcal{S}_f,\ \mathcal{S}_c,\ \mathcal{E}) = 0.$

Proof: Almost every step in the proof of this theorem is the same as in Theorem V-1.

Without loss of generality $P_\Omega^* = 0$. Repeat the steps leading to Equations V-35 through V-41. Then, since $\alpha \le 1$,

$$E(\|P_{n+1}\|^2|P_n,\ \mathcal{S}_f,\ \mathcal{S}_c,\ \mathcal{E}) \le E(\|P_{n+1}'\|^2|P_n,\ \mathcal{S}_f,\ \mathcal{S}_u,\ \mathcal{E}). \tag{V-46}$$

After applying this inequality to Equation V-41, the proof of Theorem V-2 is the same as that of Theorem V-1.

3.6 Convergence Rate

Stochastic approximation's convergence rate is specified by the function $g(n)$ that satisfies $b_n = O(g(n))$. The notation $O(g(n))$ stands for the condition that,

$$\lim_{n\to\infty} \sup(b_n/g(n)) = G < \infty. \tag{V-47}$$

Sakrison[12] shows that the convergence rate g(n) is maximized at $g(n) = n^{2/3}$ by selecting $a_n = 1/n$, $c_n = (1/n)^{1/16}$, and $a > 1/4K_0$.

Another approach to optimizing the convergence of stochastic approximation is to minimize the bound G. The constant G depends on the shape of the posterior-payoff surface and the iteration parameters a, c, and v. Using Theorem III in Chung,[13] G is related to the coefficients of terms in Inequality V-44 by

$$G = \frac{2(J^3/3v)K_3^2 \dfrac{ac^4}{\varepsilon K_0} + \dfrac{a^2 2JK_2^2}{c^2}}{(4 - \varepsilon)\ K_0 - 2/3}, \quad \varepsilon = \varepsilon(K_1). \tag{V-48}$$

Since the reason for using stochastic approximation was lack of knowledge about the payoff surface, K_0 and K_1 are <u>a priori</u> unknown. This ignorance eliminates the possibility of choosing optimum values for a, c, and v by minimizing G over these variables.

The prospect of optimizing convergence by minimizing G is also dimmed by the weakness of Inequality V-44 (from which G is derived). Inequality V-44 is weak because it is derived from the extremely conservative inequality $E(\|Y_n\|^2 | P_n, \mathcal{S}_f, \mathcal{S}_u, \mathcal{E}) \le 2JK_2^2$. With a more accurate bound, the choice of a, c, and v would be very different from the choice following from Equation V-48. This implies that choosing a, c, and v to minimize the G in Equation V-48 could, for particular payoff surfaces, lead to a very suboptimal choice.

In place of a formal optimization criterion, the iteration parameters a and c and the v_i's must be chosen either empirically or by adaptive schemes. A rationale for making empirical choices is discussed in Chapter VI.

Kushner[12] has presented a heuristic algorithm for optimizing a, c, and the v_i's adaptively, but his methods were not adopted here because they would unduly complicate discussion without adding substance. In essence, Kushner's algorithm increases a when the price trajectory continues in a steady direction and decreases a when the trajectory is oscillating. Otherwise, a is not changed. The parameter c is set to an optimum value that is a function of surface noise and a skewness factor--both of which are learned during the iteration sequence. Application of Kushner's method to this problem is straightforward.

Hill[13] has developed a method for adjusting the parameter γ adaptively based on a heuristic criterion for measuring the rate of convergence to optimum prices. Simultaneously with the price iterations, stochastic approximation was applied to the parameter γ at a slower rate to maximize the rate of convergence criterion. Hill's technique is not adopted here for the same reason that Kushner's was not.

3.7 Satisfaction of Assumptions

This section contains a series of lemmas proving that Assumptions V-1′ through V-5′ are satisfied under certain conditions on $R_T(P)$, no matter what the form of the likelihood function. This partially answers the question of when service systems satisfy the assumptions of Theorem V-1 or Theorem V-2.

One condition assumed in the following lemmas is that the prior distribution on demand-model parameters $\Pr\{H|E\}$ has compact support; i.e., that $V = \{H:\Pr\{H|E\} > 0\}$ is compact. Since $H \subset \mathbb{R}^n$, H will be compact if it is closed and bounded. Closedness is a mild restriction on the prior shape, and boundedness is guaranteed (in any practical problem) by the largest number storable in the computer.

Under the hypotheses that $R_T(P)$ is bounded, it is easy to show that Assumption V-1′ is satisfied. Boundedness of $R_T(P)$ is reasonable, since, again, the payoff which can be accumulated in a simulation is bounded by the largest number storable in the computer.

Lemma V-3: If $R_T(P)$ is bounded for all $P\varepsilon\Omega$ and H in the domain of $\Pr\{H|E\}$, then $R_T'(P)$ is bounded on the same domain.

Proof: Since $0 \le \Pr\{D|H\} \le K$ for some $K < \infty$, $R_T(P)\Pr\{D|H\} \le K_2$ for all P, and H in V.

Assumption V-2′ is satisfied for any payoff surface, since the sequences of a_n and c_n are unrelated to the payoff surface.

Lemma V-4 guarantees that under a continuity-and-boundedness condition on the third-order partials of the conditional payoff, the third-order partials of $E(R_T'|S_f, S_u, E)$ satisfy Assumption V-3′.

Lemma V-4: If V is compact and the third-order partial derivatives of $E(R_T|H, P, S_f, S_u, E)$ are continuous on $V \times \Omega$, then the third-order partial derivatives of $E(R_T'|P, S_f, S_u, E)$ are uniformly bounded in P.

Proof: See Lemma C-5 in Appendix C.

Lemma V-5 below states that Assumption V-4′ is satisfied if the conditional payoff surface is negative definite for all $H\varepsilon V$, and has continuous first-order partial derivatives in P.

Lemma V-5: Assume that

(a) V is compact

(b) $E(R_T|H,P)$ has continuous first-order partial derivatives in P on $V \times \Omega$, and

(c) the matrix $Q(H,P) = \left[\dfrac{\partial^2 E(R_T|H,P)}{\partial p_i \partial p_k}\right]$ is negative definite for all $H \epsilon V$, and $P \epsilon \Omega$ (equivalent to concavity for all $H \epsilon V$, see Ref. 14).

Then for all $P \epsilon \Omega$, there exists $K_0 > 0$ such that

$$-\langle \text{grad } E(R_T'|P, S_f, S_u, E)/P - P^* \rangle \geq K_0 \|P - P^*\|^2.$$

Proof: See Lemma C-6 in Appendix C.

According to Lemma V-6, Assumption V-5 is satisfied if the conditional payoff surfaces satisfy a boundedness condition on the payoff-surface gradient.

Lemma V-6: Under the assumption that $E(R_T|H, P, S_f, S_u)$ has continuous second-order partial in P on $V \times \Omega$, there exists $K_1 \geq 0$ such that

$$\|\text{grad } E(R_T'|P, S_f, S_u, E)\| \leq K_1 \|P - P_\Omega^*\|.$$

Proof: See Lemma C-7 in Appendix C.

Together Lemmas V-3 through V-6 prove that if certain conditions on the conditional payoff surface are satisfied, stochastic approximation converges on the posterior payoff surface.

Theorem V-3: Assume that the following conditions on the payoff random variable $R_T(P)$ hold;

(a) Ω is a closed, bounded subset of $\mathbb{R}^J$,

(b) the prior on demand-model parameters $Pr\{H|E\}$ has compact support, V,

(c) $R_T(P) \leq K_2$ for all $H \epsilon V$ and $P \epsilon \Omega$,

(d) $E(R_T|H, P, S_f, S_u)$ has continuous third-order partial derivatives on $V \times \Omega$,

(e) the matrix $Q(H,P) = \left[\frac{\partial^2 E(R_T\ H,\ R,\ S_f,\ S_u)}{\partial p_i \partial p_k}\right]$ is negative definite in P,

(f) and the sequences of a_n and c_n satisfy Assumption V-2.

Then, the algorithm of Theorem V-2 has

$$\lim_{n\to\infty} E(\|P_n - P_\Omega^*\| | S_f, S_c, E) = 0.$$

<u>Proof</u>: Under Conditions (a) through (f), Lemmas V-3 through V-6 prove that Assumptions V-1´ through V-5´ hold.* Theorem V-2 then proves this theorem.

Ideally, conditions on the service system should be shown which guarantee satisfaction of the payoff surface Conditions (a) through (f) in Theorem V-3. This new theorem, together with Theorem V-3, would show conditions on the service system that are sufficient for mean-square convergence of the stochastic-approximation algorithm. This last step is precluded by the lack of analytical methods for calculating the payoff surface.

The regularity properties of the payoff surface $E(R_T|H, P, S_f, S_u)$ implied by Conditions (a) through (f) in Theorem V-3 should be satisfied

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*Condition d in Theorem V-3 holds only if Condition a in Lemma V-6 holds.

for many service systems. The restriction of P to Ω is important, since it requires satisfaction of the curvature Conditions (f) on only a bounded subset Ω of the payoff function's entire range. Condition (e) is a requirement for concavity of the payoff surface, and it is likely that service systems will satisfy this condition close to but not far away from the optimal price. Condition (d) is a smoothness condition. All the other conditions are easily satisfied by most service systems.

FOOTNOTES--CHAPTER V

1. The details of this computation are carried out in Reference 2, but will be sketched here.

2. Howard, Ronald A., Dynamic Probabilistic Systems, John Wiley & Sons, New York, 1971, Chapter 13.

3. Ibid., Chapter 11.

4. Ibid.

5. Ibid.

6. Ibid., Chapter 13.

7. Ibid.

8. Dupac, V., "On Kiefer-Wolfowitz Approximation Method," Selected Translations on Mathematics, Statistics, and Probability, American Mathematics Society, Providence, Rhode Island, Vol. 4, 1963, pp. 43-69.

9. Sakrison, D., Application of Stochastic Approximation Methods to System Optimization, Technical Report 391, Massachusetts Institute of Technology, 1962.

10. Schmetterer, L., "Stochastic Approximation," Proc. Fourth Berkeley Symp. Math. Stat. Prob., Vol. I, University of California Press, Berkeley, 1960, pp. 587-609.

11. Fabian, V., "Stochastic Approximation of Minima with Improved Asymptotic Speed," Annals of Math. Stat., Vol. 3, 1967, pp. 191-200.

12. Kushev, H. J., Efficient Methods for Optimizing the Performance of Multiparameter Noisy Systems, Lincoln Lab Technical Report 22G0043, October 1960.

13. Hill, J. D., "A Learning Control System Using Stochastic Approximation for Hill Climbing," Joint Automatic Computer and Control Conference, 1965, p. 243.

14. Karlin, S., Mathematical Methods in Games, Programming, and Economics, Addison Wesley, Reading, Mass., 1959, p. 405.

Chapter VI

OPTIMIZATION OF QUEUING SYSTEMS

1.0 DESCRIPTION OF THE EXAMPLE

1.1 Purpose of Solving a Sample Problem

This chapter illustrates the stochastic-approximation method by finding optimal prices for a single-server, discrete service time, non-preemptive, priority queue under a variety of assumed demand structures. These examples are presented for two purposes. First, the fact that prices converge to an optimum over a range of initial conditions will be evidence that queuing systems satisfy the stochastic-approximation assumptions V-1′ through V-5′. Second, the examples illustrate how the methods of this thesis can be used in drawing welfare conclusions about priority queues.

1.2 Demand Model for Sample Problem

Assume that consumer values are described by W-P Model 0. Then, the surplus of alternative j is

$$S_j = \xi_0 + \xi_1 \Delta_j - p_j. \tag{VI-1}$$

In all the examples it is assumed that the quality of an alternative is the total amount of higher priority work that remains unserved. If n_i is the number of waiting customers in priority i, μ is the deterministic service time, and τ the service time remaining on the unit in service, then the quality of alternative j was taken as,*

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*Quality is measured in "time units," where λ = 1 time unit.

$$\Delta_j = \sum_{i=1}^{j} n_i \lambda + \tau. \qquad \text{(VI-2)}$$

The value of service is assumed to be exponentially distributed,

$$\Pr\{\xi_0 | a_0\} = a_0 \exp(-a_0 \xi_0), \qquad \text{(VI-3)}$$

with $E(\xi_0) = 1/a_0 = 3.0$ money units. It is assumed that the value of time is Gamma distributed:

$$\Pr\{\xi_1 | a_1, r_1\} = a_1 \frac{(-a_1 \xi_1)^{r_1 - 1}}{(r_1 - 1)!} \exp(-a_1 \xi_1). \qquad \text{(VI-4)}$$

Values for the parameters a_1 and r_1 were chosen from the following list:*

(a) $E(\xi_1) = r_1/a_k = 0.3$ $\quad r_1 = 2.0$

(b) $E(\xi_1) = 0.3$ $\quad r_1 = \infty$

(c) $E(\xi_1) = 0.6$ $\quad r_1 = 2.0$

(d) $E(\xi_1) = 0.6$ $\quad r_1 = \infty$

Condition (a) states that the expected value of time is .3 money units/time unit, and so on for the other conditions. The variation from $r_1 = 2$ to $r_1 = +\infty$ allows comparison of the payoff optima achieved when the value of time is both "widely" distributed and constant. The variation from $E(\xi_1) = 0.3$ to $E(\xi_1) = 0.6$ allowed comparison of optima for both a "higher" and a "lower" value of time.

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*The parameter r_1 is unitless. The parameter $1/a_1$ is in "money units/time unit."

These values of the demand-model parameters are chosen to ensure that a long queue will build up at zero price. If the average queue were short, very little increase in efficiency could be expected from optimal prices since the queue would be empty most of the time anyway.

1.3 Service Characteristics

In every example the service distribution is assumed to be discrete with time interval $\mu = 2.90$ time units. The service discipline is assumed to be first-in-first-out within a priority and nonpreemptive between priorities. At the end of each service the next unit served is the waiting customer having the lowest priority number.

2.0 LOGICAL FLOW OF COMPUTATION

A pictorial diagram of the program's basic structure is shown in Figure VI-1. The program starts with exogenous determination of the iteration and demand-model parameters. Then the random-number generator, the quality process, and the prices are initialized to externally supplied values.

The n^{th} iteration cycle begins by calculating prices for the next iteration using the algorithm $P_{n+1} = P_n + (a_n/c_n)Y_n$. The gradient approximation Y_n is evaluated using the payoff samples collected at the previous iteration. Then the basis vectors $U'_{i,n+1}$, $i = 1, 2, \ldots J$ and the base prices, $P^{+}_{i,n} = P_n + c_n v_i\ U'_{i,n}$, $i = 1, 2, \ldots J$ are determined.

Starting with $P_n + c_n v_1 U'_{1,n}$, the queuing system is simulated for a duration of T time units. Remembering the final value of accumulated payoff, the system simulation is repeated for the other base prices.

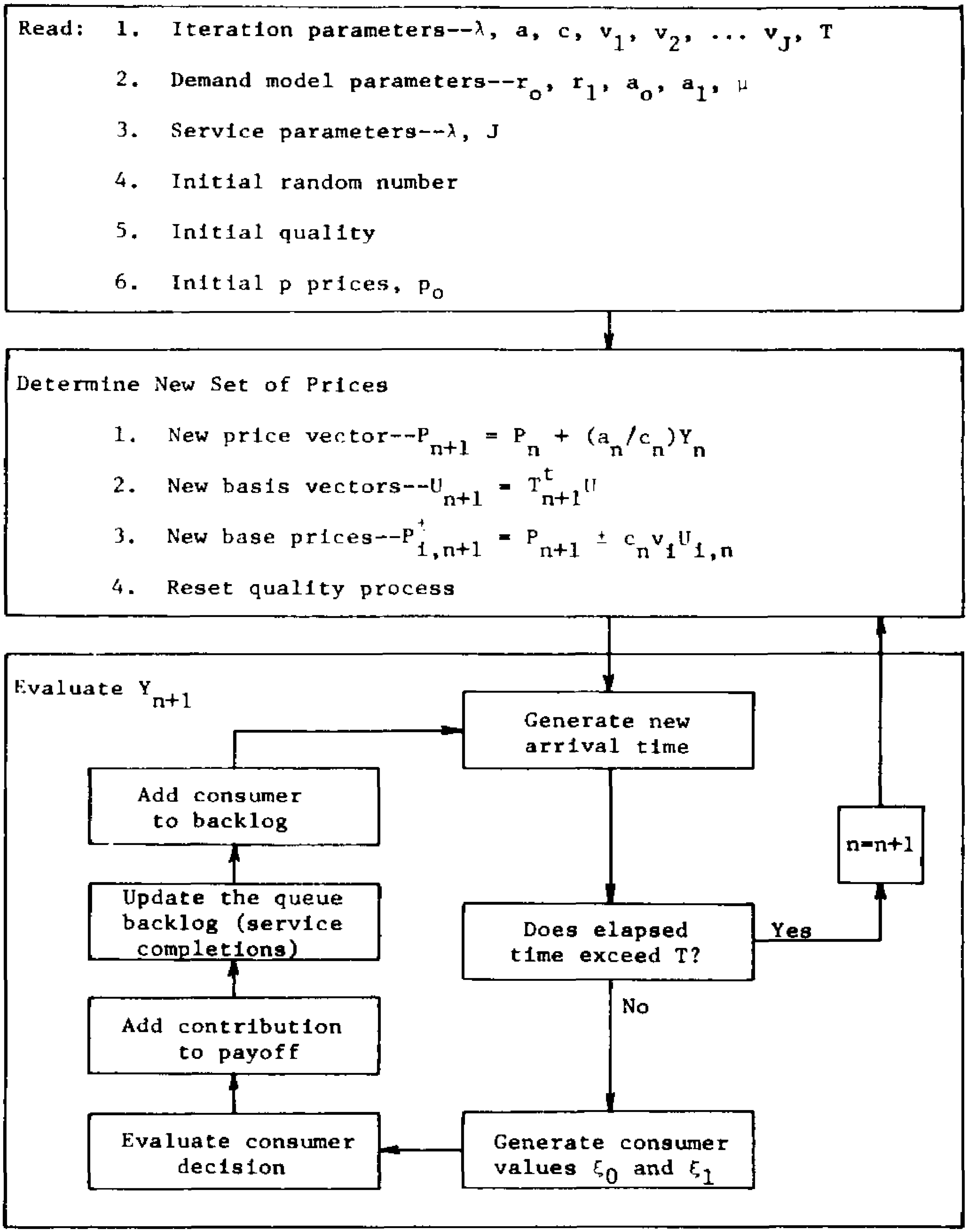

Figure VI-1--Program Flow for Stochastic Approximation

If uncorrelated sampling is used, new samples of consumer values and arrival times are used for each price in the base; if correlated sampling is used, the same values are used. The accumulated payoff samples are used to evaluate Y_n in the next iteration cycle.

3.0 EXPERIMENTAL DESIGN

3.1 Selection of Payoff Accumulation Period and Sample Plan

Except in one experiment, uncorrelated samples (Plan S_u) were used in every test run. Proof that the stochastic-approximation algorithm converges to the true optimum was not obtained until most of the computer budget had been exhausted, precluding extensive testing with correlated samples. One experiment was run to demonstrate convergence with correlated samples.

Since convergence was proved only for sample plan S_f (initialize the quality process to the same value at the beginning of each iteration), it was used throughout experimentation. The extreme choice was made of emptying the queue at the beginning of each iteration cycle. A more favorable assumption would have been to reset the queue backlog to some positive value close to the steady-state expected delay.

An accumulation time of T = 50 time units was chosen as a compromise between the need to minimize computation time and the need for a close approximation to the steady-state payoff, R_T.

The results presented in Figure VI-2 were run (using parameter values discussed in the next section) to illustrate the inaccuracies caused by choosing S_f and T = 50. These results were obtained by calculating the prices trajectories (the sequence of P_n's) for a single-priority queue in the two cases: (a) S_f and T = 50, and (b) S_s and

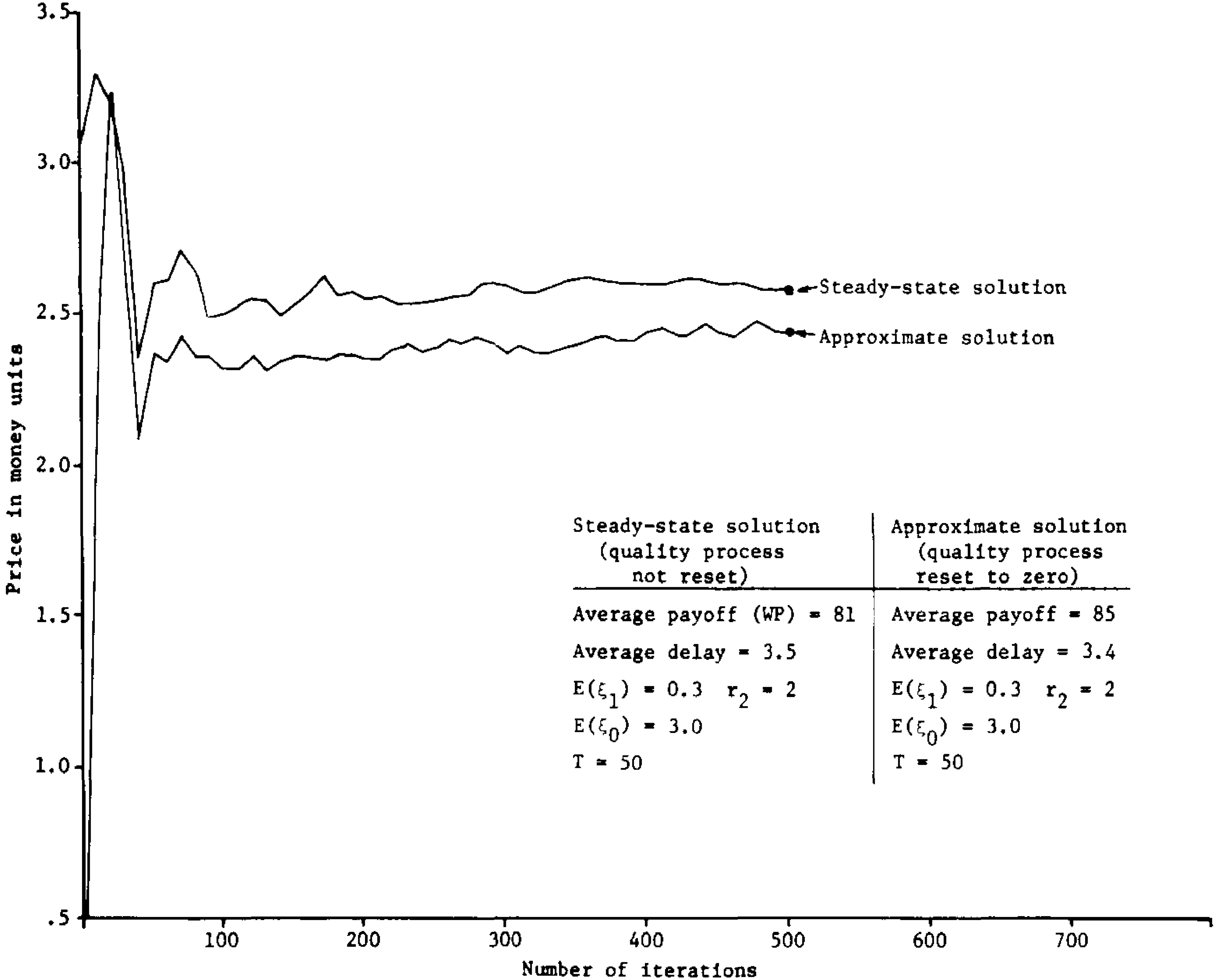

Figure VI-2--Price Trajectories for Test on Accuracy of Sample Plan

T = 50. Figure VI-1 shows that the optimal price in (a) is 92 percent of the optimal price in (b), and the optimal payoff is 5 percent more. The payoff increases because on the average more customers purchase service in time period [0,T] when the queue is initially empty than when the queue is not necessarily empty.

3.2 Selection of the Iteration Parameters

The iteration constants α and γ in the sequences of $a_n = a/_n^{\alpha}$ and $c_n = c/n^{\alpha}$ were chosen to be $\alpha = 1$ and $\gamma = 1/6$ for the reasons discussed in Section V.3.6.

The other iteration parameters which must be selected are a, c, and v_i's, but there is no formal theory for optimizing these variables (see Section V.3.6).

The effect of the step parameters on convergence can be illustrated by considering a simple problem. Assume that the payoff random variable is

$$R_T(p) = -\,p^2 + N(p), \tag{VI-5}$$

where $\{N(p):p\in\mathbb{R}\}$ is a family of independent random variables with

$$E[N(p)] = 0. \tag{VI-6}$$

With this payoff surface, the gradient approximation is

$$\begin{aligned} Y_n(p_n) &= \frac{n^{1/6}}{c}\left[-\left(p_n + \frac{c}{n^{\frac{1}{6}}}\right)^2 + \left(p_n - \frac{c}{n^{\frac{1}{6}}}\right)^2 + N(p_n^+) - N(p_n^-)\right] \\ &= -\,4\,p_n + \frac{n^{\frac{1}{6}}}{c}\left[N(p_n^+) - N(p_n^-)\right]. \end{aligned} \tag{VI-7}$$

Equation VI-7 implies that the noise present in measuring the gradient of the payoff surface is reduced by increasing the parameter c. Lower noise increases the speed of approach to an optimum since fewer steps are made in the wrong direction.

Substituting Equation VI-7 into the stochastic-approximation algorithm,

$$p_{n+1} = p_n - 4\,\frac{a}{n}\,p_n + \frac{a}{c}\,\frac{1}{n^{5/6}}\left[N(p_n^+) - N(p_n^-)\right]. \qquad \text{(VI-8)}$$

The gradient to the payoff surface is 4 p_n, so Equation VI-8 indicates that the parameter a acts like a gain determining step size. Up to a point increasing a will reduce the number of iterations needed to reach the vicinity of the optimum.

The parameter a should not be increased too far; however, for then the algorithm will overshoot the optimum. Equation VI-8 also shows that as a increases the error in determining p_{n+1} increases. The optimum a is somewhere between the two extremes. If a is too large, the algorithm will reach the vicinity of an optimum quickly, but oscillate greatly about it due to overshoot and noise. If a is too small, the algorithm will take a very long time to reach the optimum.

Equation VI-8 shows that increasing c decreases the noise in determining p_{n+1}, and does not affect the expected value of the step size, nor bias the price trajectory since $E(p_{n+1})$ is independent of c. The apparent conclusion is to make c as large as possible.

This conclusion does not follow when the payoff surface is asymetric about the optimum, for then a large bias will be introduced into the payoff trajectory. Suppose the payoff surface is:

$$R_T(p) = \begin{cases} - p^2 + N(p) & p < 0 \\ \\ - 2p^2 + N(p) & p \geq 0 \end{cases} \tag{VI-9}$$

with the noise specified as in the previous example. Then, if $- c_n < p_n < c_n$,

$$E(p_{n+1} | p_n) = p_n - 6 \frac{a}{n} p_n - \frac{a}{c_n^{5/6}} p_n^2 - \frac{ac}{n^{7/6}} . \tag{VI-10}$$

As c is increased, the bias $- ac/n^{7/6}$ increases driving $E(p_{n+1} | p_n)$ away from the optimum. As $n \to \infty$ the bias approaches zero, but convergence to the optimum is slowed by the bias.

The parameters a and c were assumed to produce similar effects in the queuing-system example. Increasing the value of a presumably accelerates the speed of approach to an optimum, at the cost of increasing the price trajectory's variance near the optimum, while increasing the value of c tends to reduce this variance (by lowering $\text{var}(Y_n)$) at the cost of biasing the approach to an optimum.

Based on these effects, the following heuristic strategy for selecting a, c, and the v_i's was adopted for the queuing-system example:

(1) Choose $c = 0.2E(\zeta_0)$, thus insuring a significant change in the expected payoff when prices are stepped around the base. Set all $v_i = 1$, $i = 1, \ldots, J$.

(2) Measure the standard deviation of the payoff at a set of prices likely to be optimum. Choose a to be 0.1 times this deviation; thus providing a moderate step in prices at each iteration.

(3) Compare two price trajectories P_1, P_2, ... and P_1', P_2', ... obtained by using different sequences of consumer values and initial prices.* Increase a by 20 percent unless $\| P_n' - P_n \|_\infty \leq 2c_{100}$ (the maximum difference in the price for any priority is less than twice the undilated base size) for 25 percent of the iterations between 75 and 100.**

(4) Compare two price trajectories P_1, P_2, ... and P_1', P_2', ... obtained by using different sequences of consumer values and prices out to the bound $n_0 = 400$. Start with $i = 1$, and if the prices for the i^{th} priority have $|p_{i,n}' - p_{i,n}| > v_i \, c_{n_0}$ on more than 25 percent of the iterations between n_0 and $n_0 - 100$, increase v_i by 20 percent. Repeat this process of comparing $p_{i,n}'$ and $p_{i,n}$, and increasing v_i if necessary for $i = 2, 3, \ldots, J$.

(5) Obtain a new pair of price trajectories, and repeat Steps (3) and (4) until $|p_{i,n}' - p_{i,n}| \leq v_i \, c_{n_0}$ on more than 75 percent of the iterations between n_0 and $n_0 - 100$.

(6) If the resulting base is so large that the base size $2v_i c_{n_0}$ is of the same order of magnitude as the price p_{i,n_0}, increase the bound n_0 keeping the parameter settings found in Steps (1) through (4) until $2v_i c_{n_0}$ is sufficiently less than p_{i,n_0} for all $i = 1, 2, \ldots, J$.

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*It is not sufficient in the above steps to obtain the trajectories for P_i' and P_i by changing only the initial price. If the same sequence of consumer values is used for P_i' and P_i, the price trajectories will converge to each other, irrespective of the initial prices and the iteration parameter values, a, c, and v_i.

**$\|X\|_\infty = \max_{1 \leq i \leq n} X_i$.

Steps (1) through (6) were found effective for choosing values of a, c, and the v_i's. After some experience with the payoff surfaces the values of a and the v_i's could be set with two or three iterations through Steps (3) and (4).

3.3 Basis Orientation

Except for one trial run all results were obtained using the base, $U_n^* = T_V U$. The realization that stochastic rotation of the approximation base would improve convergence properties came after all testing had been completed and computer funds were exhausted.

One experiment was run to demonstrate the change in convergence properties caused by basis rotation (to be discussed in Section VI.4.2).

4.0 EXPERIMENTAL RESULTS

4.1 Single-Priority Discrete-Service Time Queue

Figures VI-2 through VI-6 are the results of a series of experiments on the single-priority version of the discrete-service time queue. Each figure displays two price trajectories--each produced by a different set of starting conditions.

Figure VI-3 and Figure VI-4 show the price trajectories obtained in maximizing willingness to pay under the conditions that $E(\xi_1) = 0.3$ and $E(\xi_1) = 0.6$ respectively, while $r_1 = 2$. The optimum price turns out to be about 2.35 money units in the former case and 2.45 in the latter. Figure V-5 shows the price trajectory when all consumers have the same value of time, $\xi_1 = 0.3$. The optimal price turns out to be virtually the same as in Figure VI-3 where $r_1 = 2$.

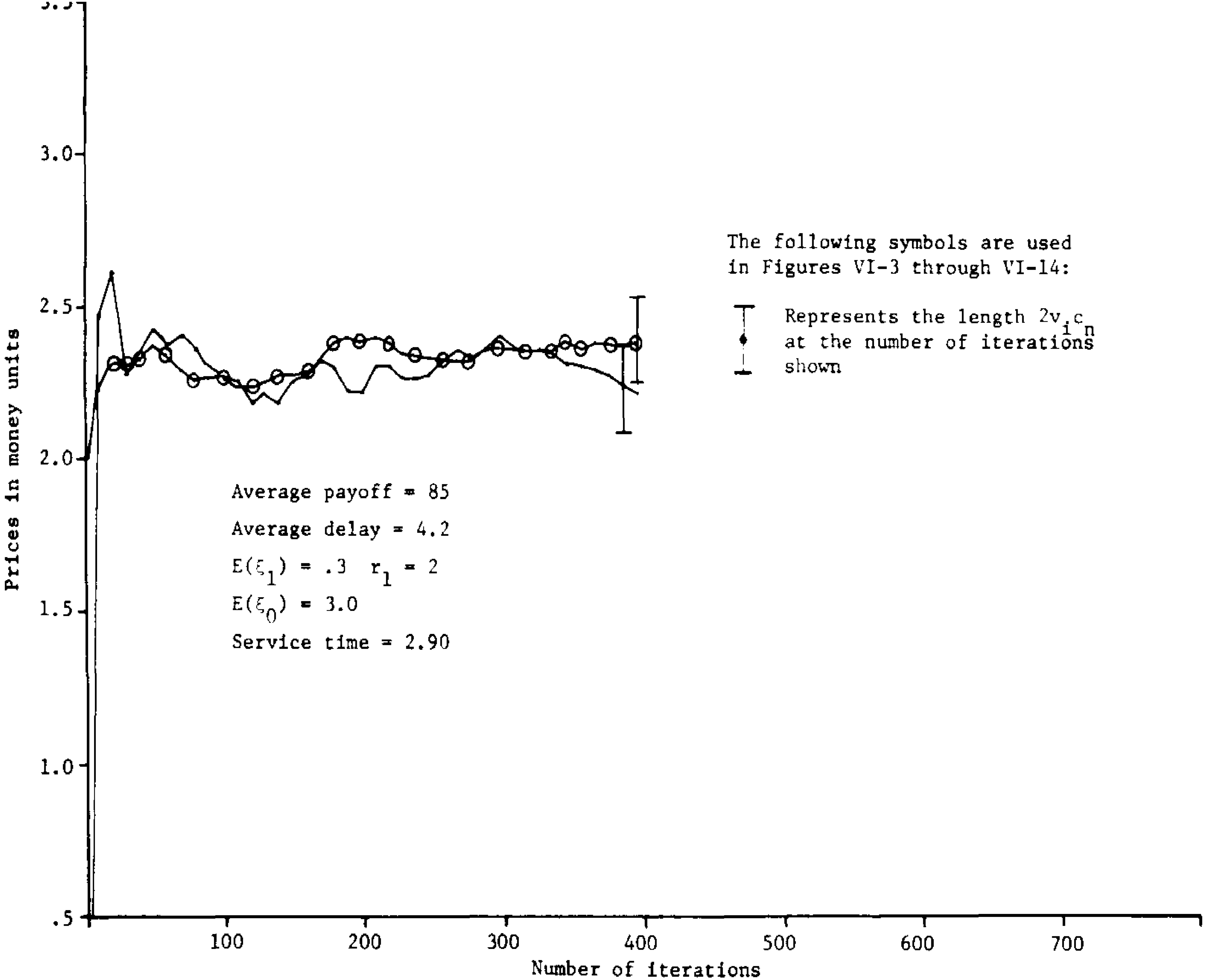


Figure VI-3--Price Trajectory for Maximum Willingness to Pay

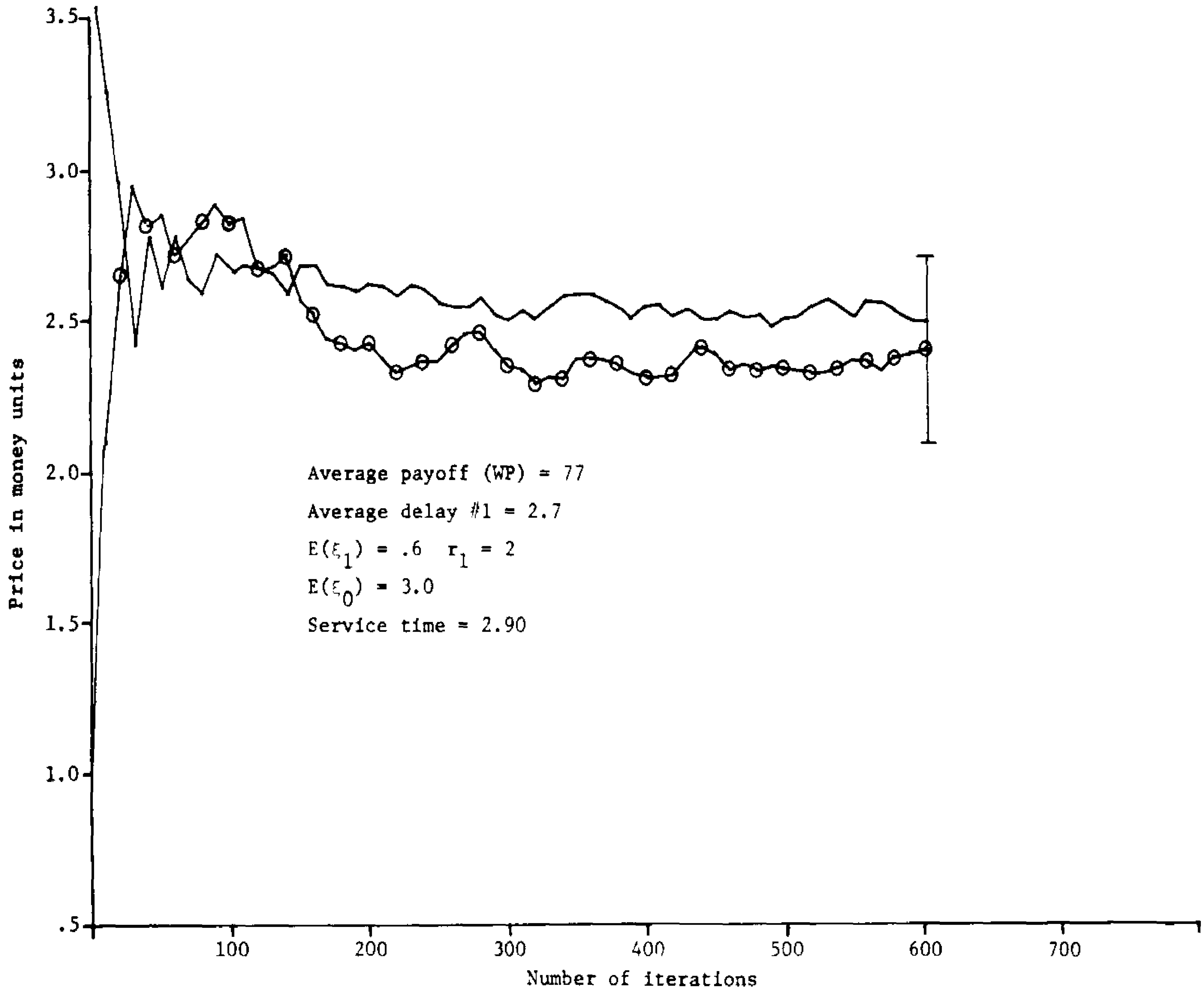


Figure VI-4--Price Trajectory for Maximum Willingness to Pay

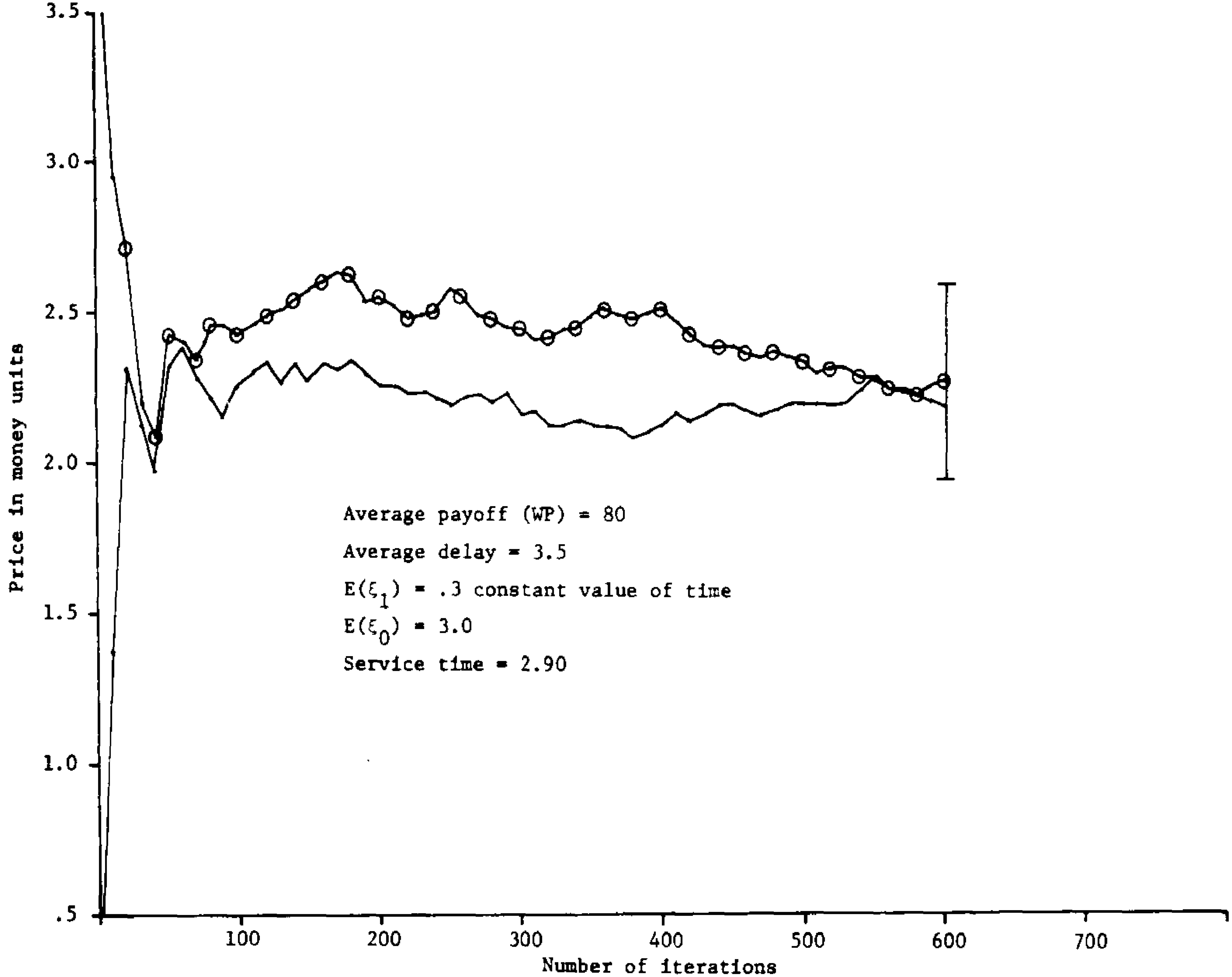


Figure VI-5--Price Trajectory for Maximum Willingness to Pay

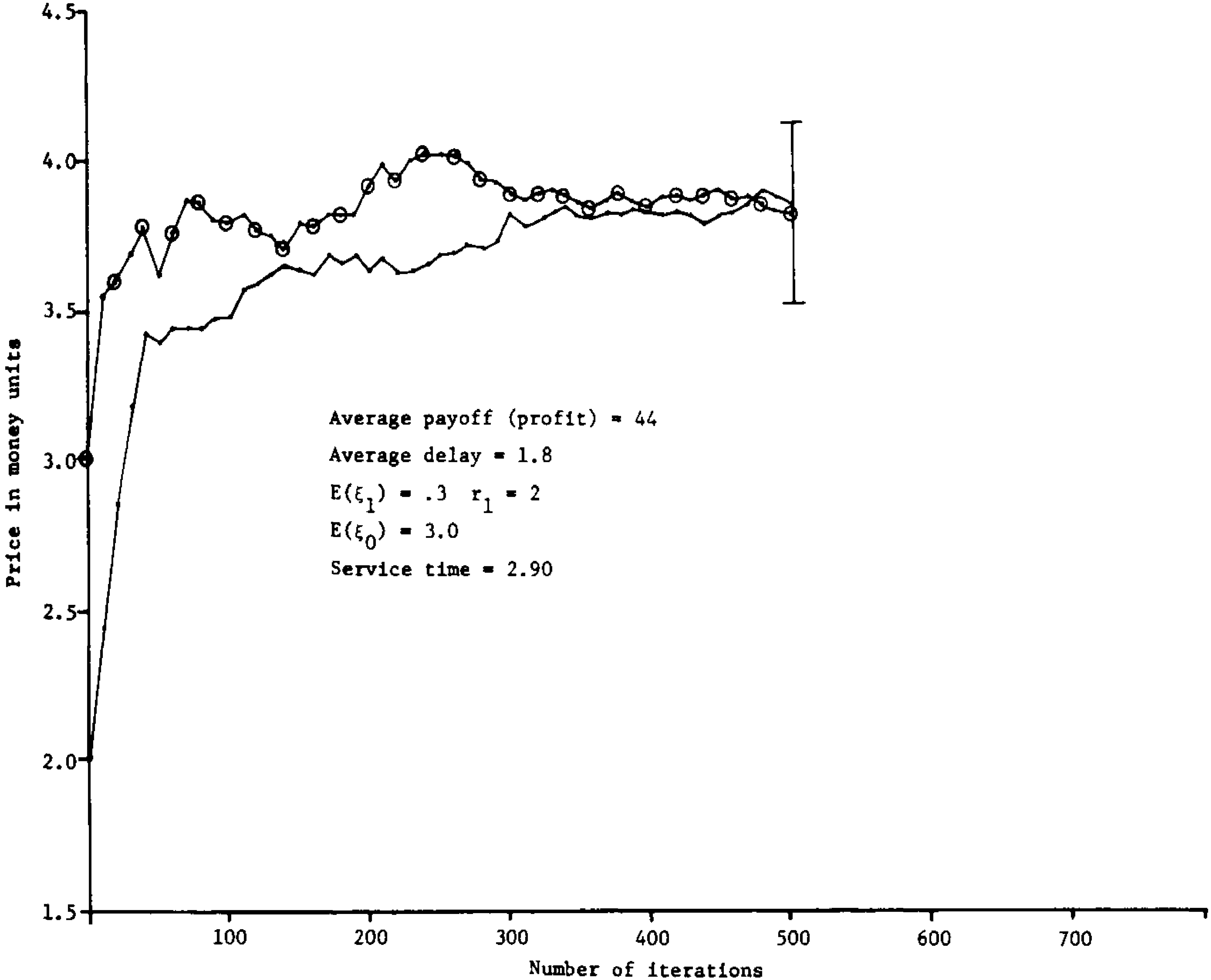


Figure VI-6--Price Trajectory for Maximum Profit

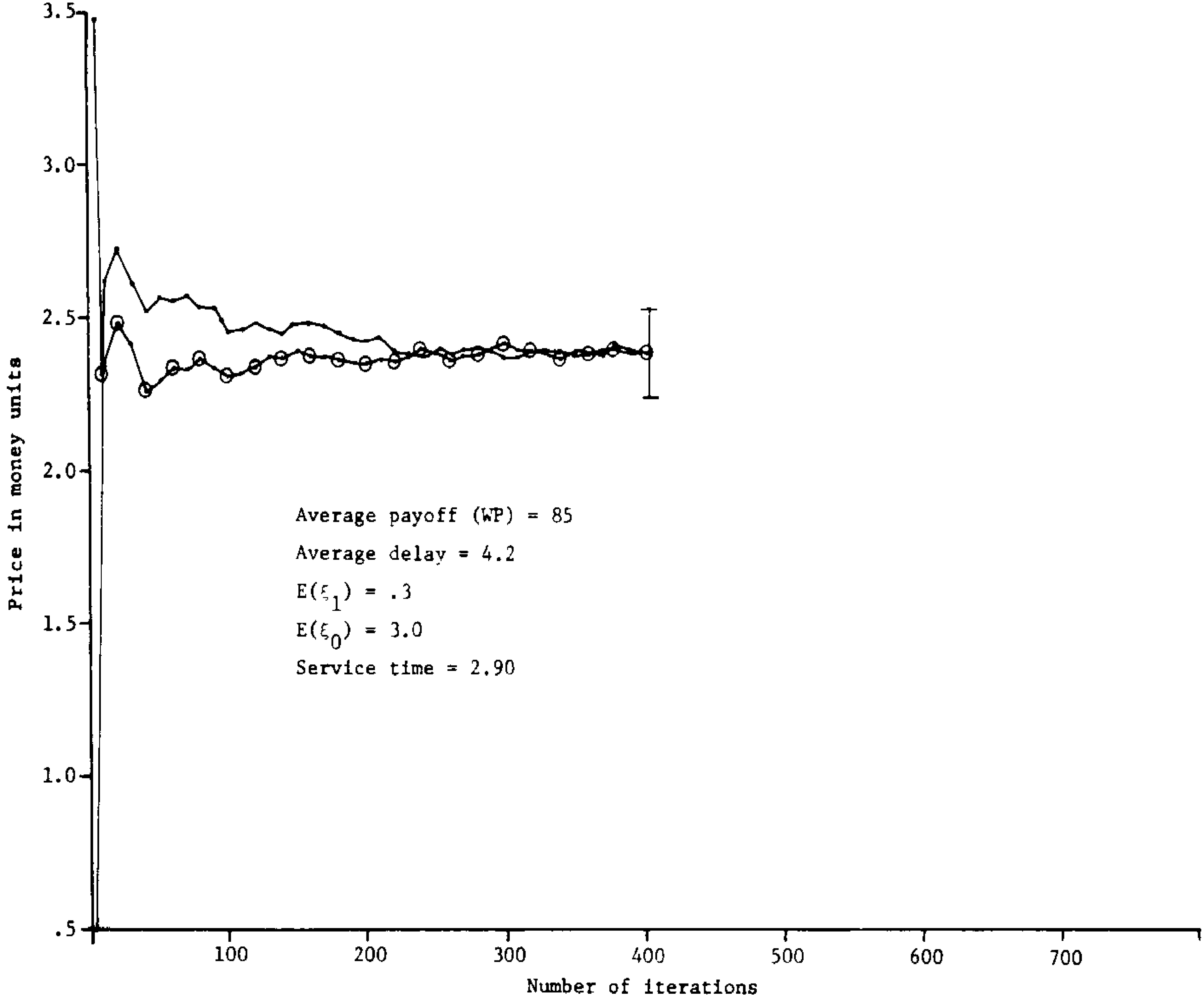


Figure VI-7--Price Trajectory for Maximum Willingness to Pay with Correlation Sampling

Figure VI-6 shows the price trajectory for maximum profit when $E(\xi_1) = 0.3$ and $r_1 = \infty$. As in Noar's[1] result for constant value of service and time, the profit maximizing price in a single-channel queue is higher than the price that optimizes social welfare.

Figure V-7 shows the price trajectory obtained with correlation sampling on the same experiment as in Figure VI-3. The speed of convergence is apparently increased with correlation sampling since the trajectory fluctuations are much smaller. The time required to complete 400 iterations was reduced from 1 msec per price iteration in Figure VI-3 to 0.6 msec per iteration in Figure VI-7. The saving in time would be more pronounced in multipriority optimizations, since the computation time increases in proportion to the number of priorities with uncorrelated sampling, but is almost constant for correlation sampling. This follows because the bulk of computer time is expended in generating random numbers.

4.2 Two-Priority Discrete-Service Time Queue

Figures VI-8 through VI-12 are the price trajectories obtained in optimizing a two-priority queue.

Figures VI-8, VI-9, and VI-10 were obtained with the same combination of demand-model parameters as were Figures VI-3, VI-4, and VI-5. With two priorities available, willingness to pay is optimized by charging a higher price (2.80 money units) in the top priority, and a lower price (1.90 money units) in the lower priority as compared to the single-priority case. There is apparently little difference in the optimal prices to charge in each case. The optimal prices for a constant value of time are slightly lower, but the uncertainty in these trajectories is higher than the trajectories in Figures VI-8 and VI-9.

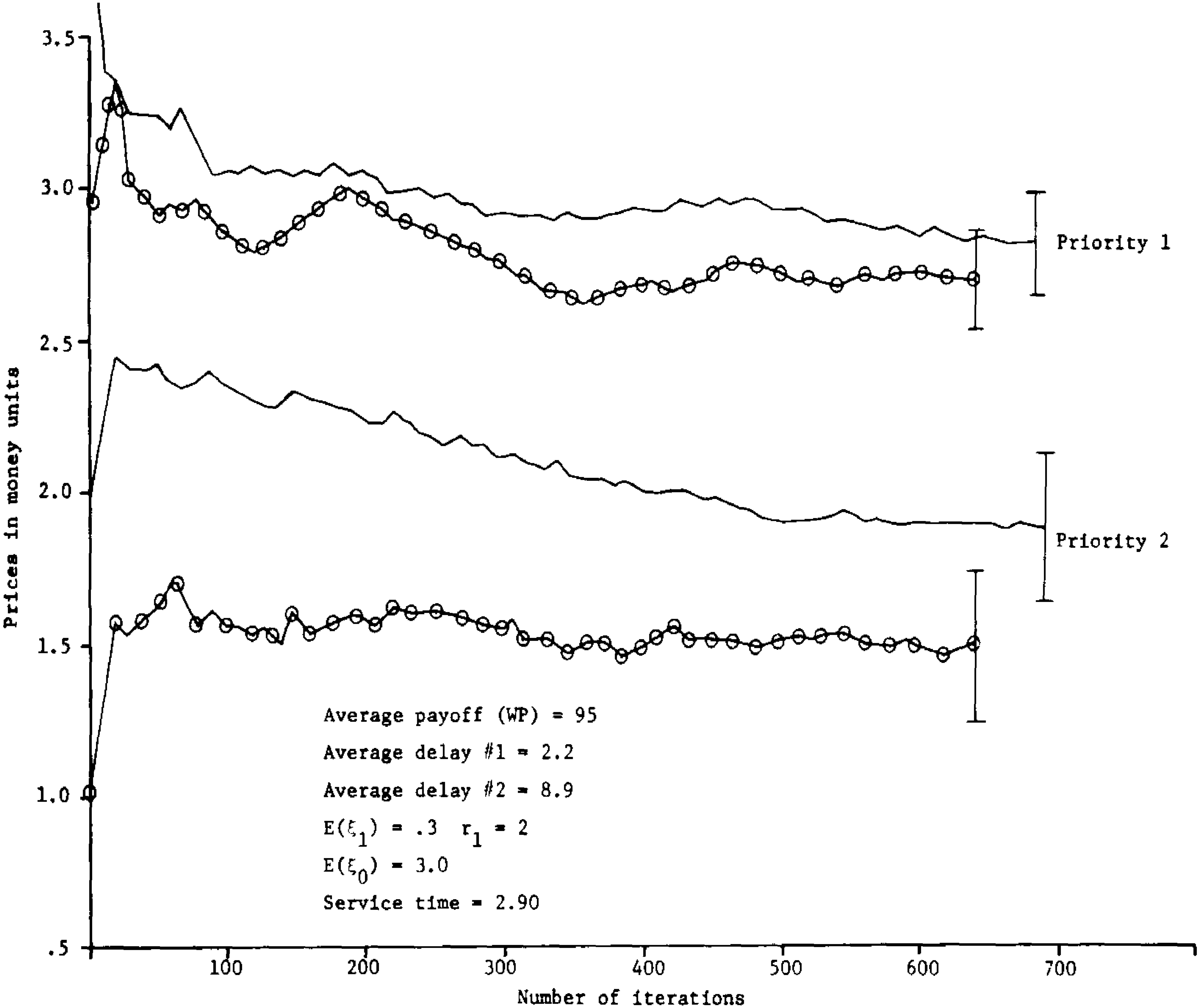


Figure VI-8--Price Trajectory for Maximum Willingness to Pay

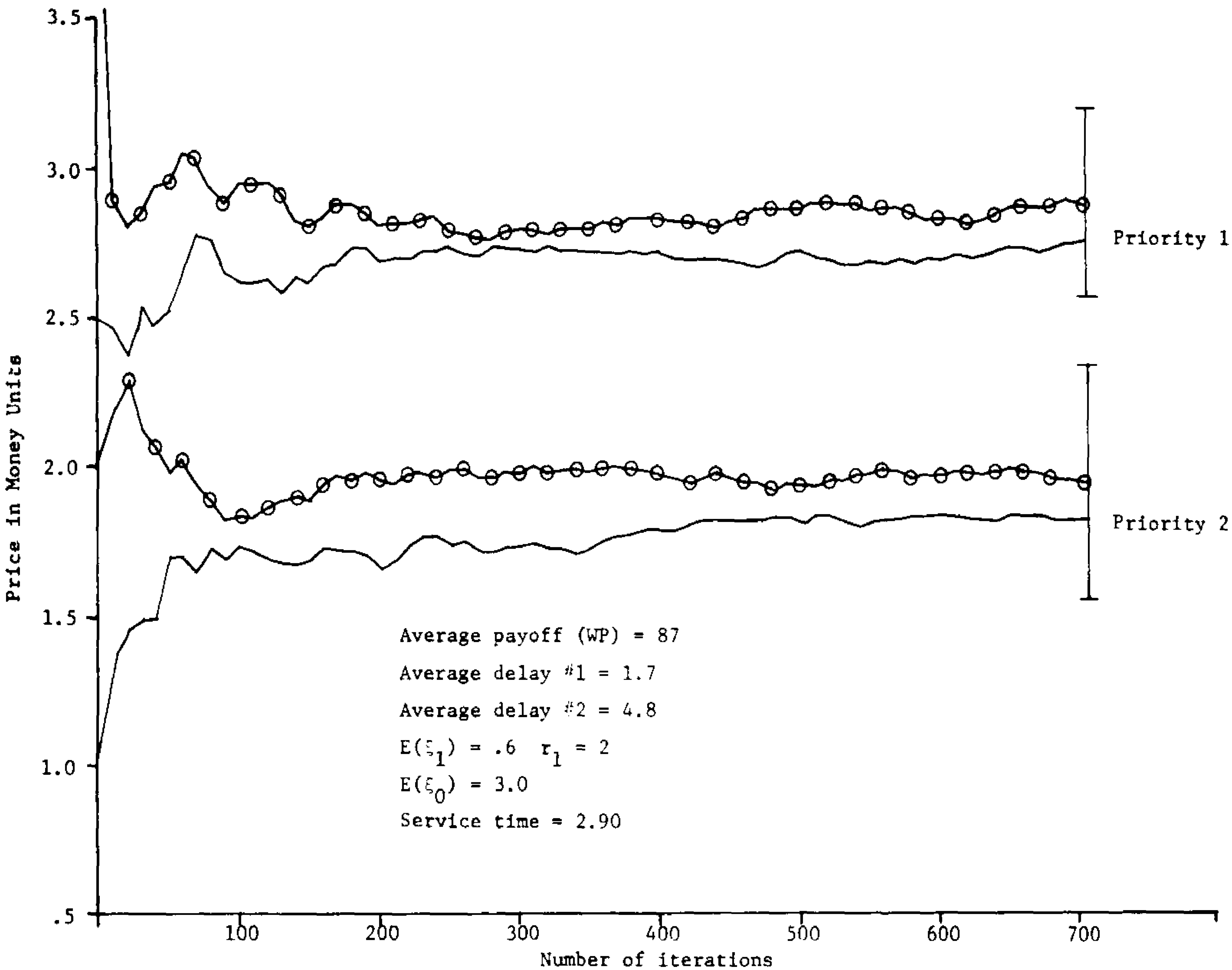


Figure VI-9--Price Trajectory for Maximum Willingness to Pay

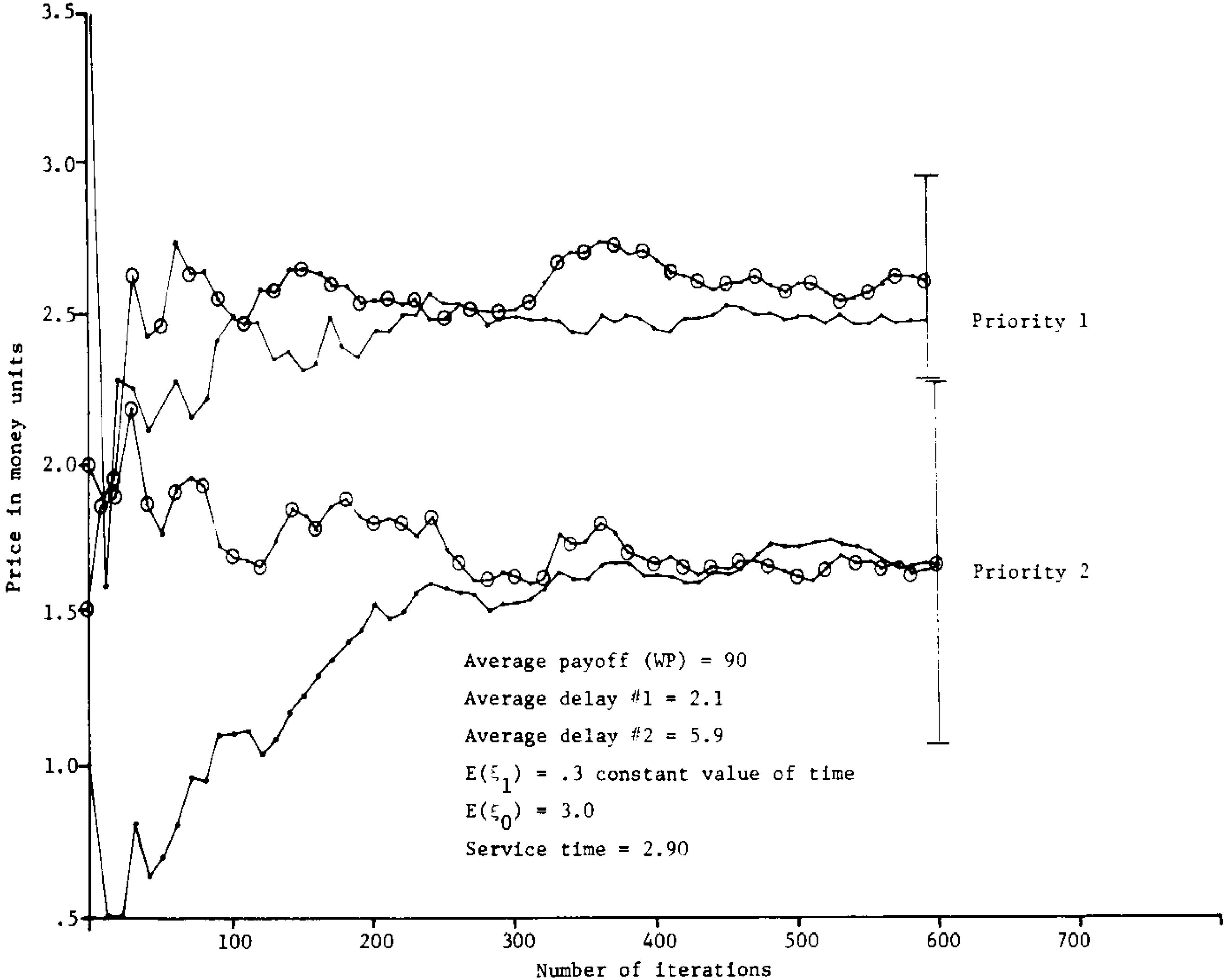


Figure VI-10--Price Trajectory for Maximum Willingness to Pay

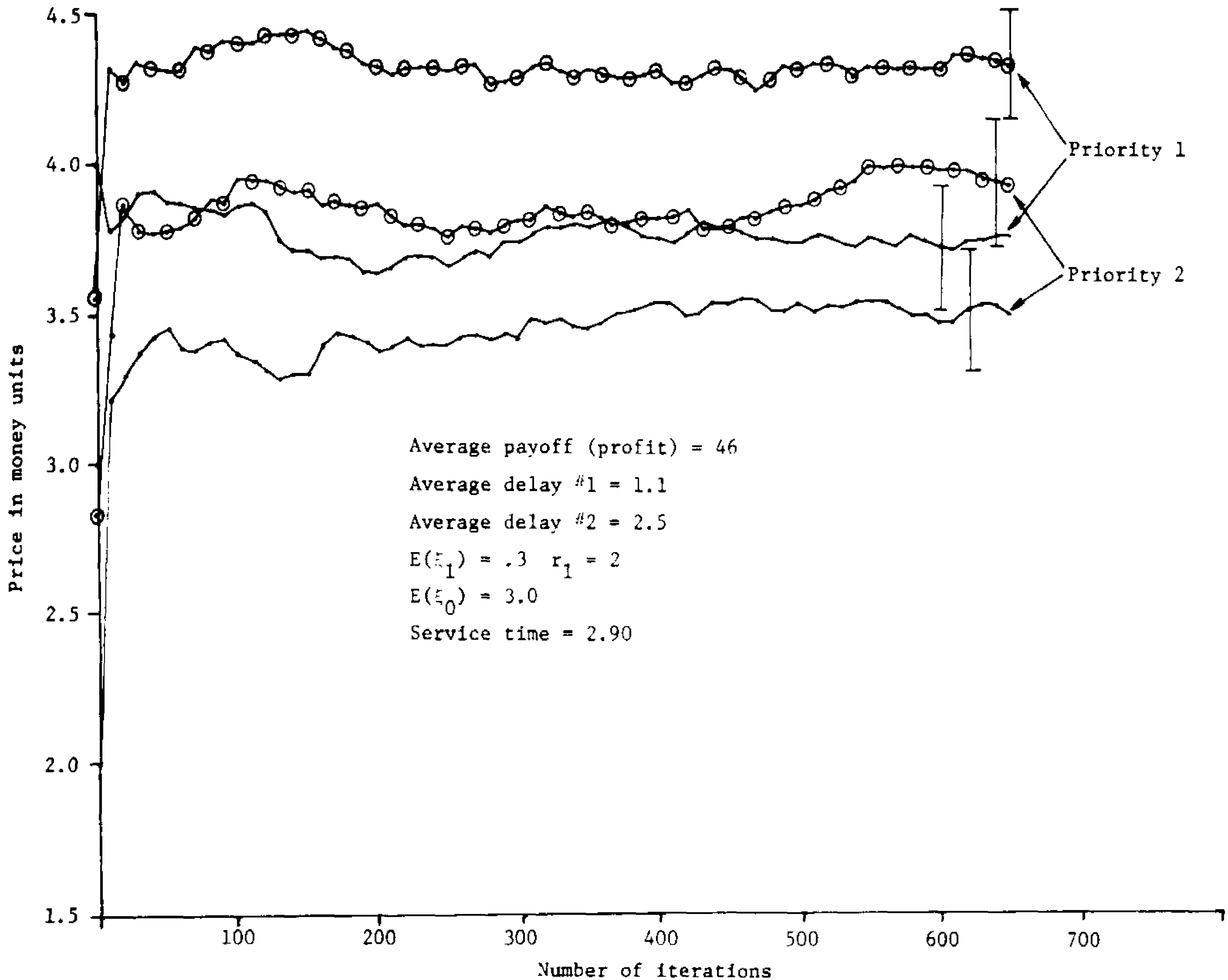


Figure VI-11--Price Trajectory for Maximum Profit

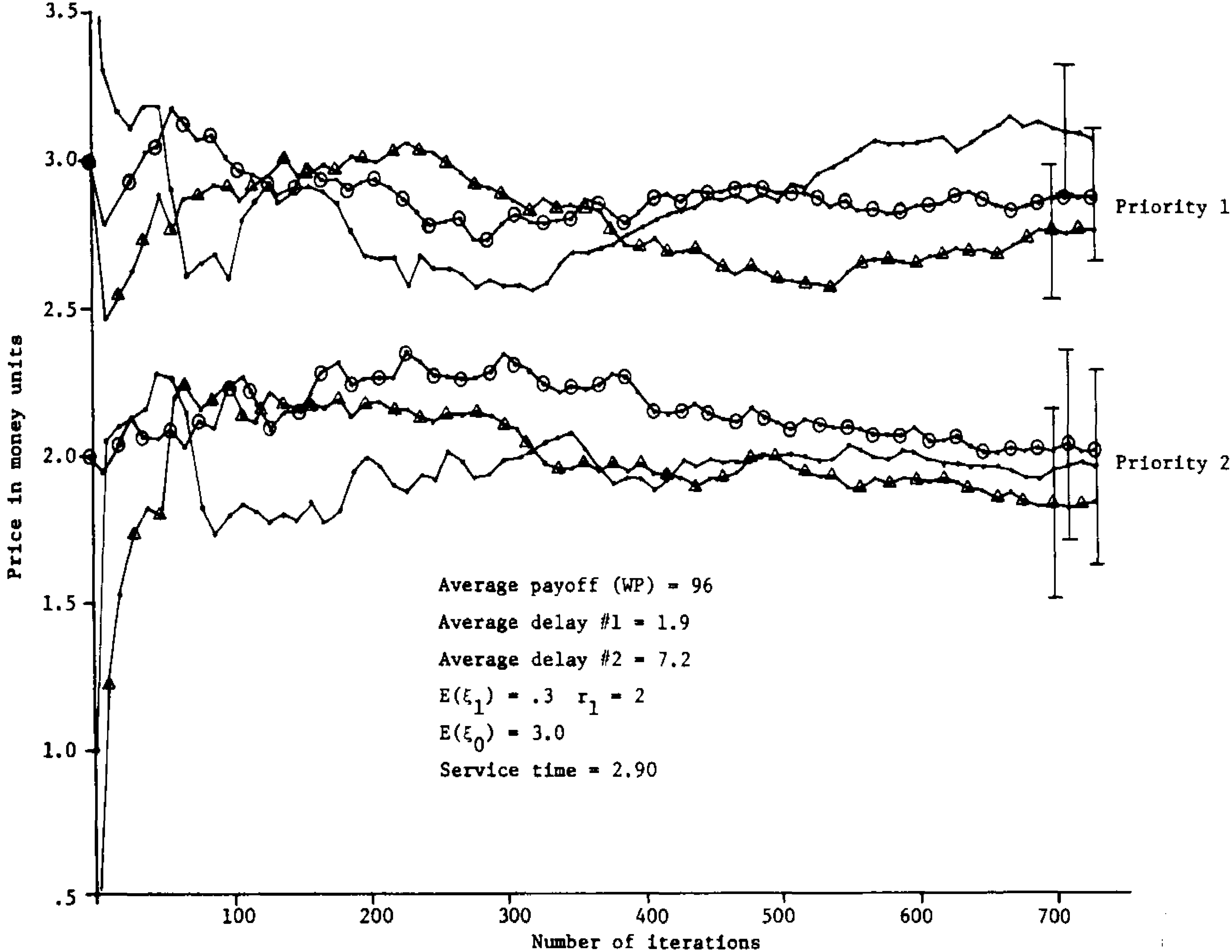


Figure VI-12--Price Trajectory for Maximum Willingness to Pay, Random Orientation of Base

The amount of uncertainty in a price trajectory is indicated by the magnitude of the base size at the last iteration. The base size in Figure VI-10 is larger than in Figures VI-8 and VI-9, leading to the assertion that the optimal price is more surely known in the former figure than in the latter figures.

Figure VI-11 shows the optimum trajectories for profit maximization when demand structure is the same as in Figures VI-3 and VI-8.

Figure VI-12 shows the price trajectories for maximizing willingness to pay under the same demand structure as in Figure VI-11, but the base vectors are randomly oriented. The rotation operator was chosen to be:

$$T^t_{\theta_n} = \begin{bmatrix} \cos\theta_n & \sin\theta_n \\ & \\ -\sin\theta_n & \cos\theta_n \end{bmatrix} . \qquad \text{(VI-14)}$$

The distribution of θ_n was taken as

$$\Pr\{\theta_n \le \theta_0\} = \begin{cases} 1 & \theta_0 > \pi/4 + 0.1 \text{ radians} \\ 5\ (\theta_0 - \frac{\pi}{4} + 0.1) & \text{otherwise} \\ 0 & \theta_0 < \pi/4 - 0.1\ . \end{cases} \qquad \text{(VI-15)}$$

The selection of ± 0.1 radians derivation was arbitrary, but the decision to center this distribution around $\pi/4$ radians is motivated by the observation that the payoff trajectories in Figures VI-8 through VI-11 follow roughly parallel paths. Such behavior is consistent with the a payoff surface that is ridged along a line at approximately a 45-degree angle to the price axis.

If the payoff surface is ridged, then orienting one vector of the base in the direction of the ridge will increase the accuracy of measuring the payoff gradient (see Figure V-2).

The trajectories produced with random orientation of the base vectors in Figure VI-11 are no longer parallel, as in Figure VI-11, but cross over each other. This indicates that the iteration algorithm is oscillating around the optimum rather than crawling up a ridge, giving increased confidence that the trajectories are in the vicinity of an optimum.

4.3 Three-Priority Discrete-Service Time Queue

Figures VI-13 and VI-14 are the price trajectories for optimizing willingness to pay in a three-priority queue.

Comparing the optimal prices for the two- and three-priority experiments, the optimal prices for the top-priority are higher while the optimal prices for the bottom priority are lower. The payoff trajectories indicate a ridged form for the payoff function as in the two-priority case, but no attempt was made to rotate the base prices.

5.0 IMPLICATIONS OF THE OPTIMA FOR ECONOMICS

The payoff data in Figures VI-3 through VI-14 are collected in Table VI-1, and converted into a more readable form in Table VI-2. Tables VI-1 and VI-2 also include payoff data from a single-priority queue offering free service.

The payoff values quoted in Table VI-1 are the mean of the expected payoff, a nonparametric Bayesian estimate first presented in

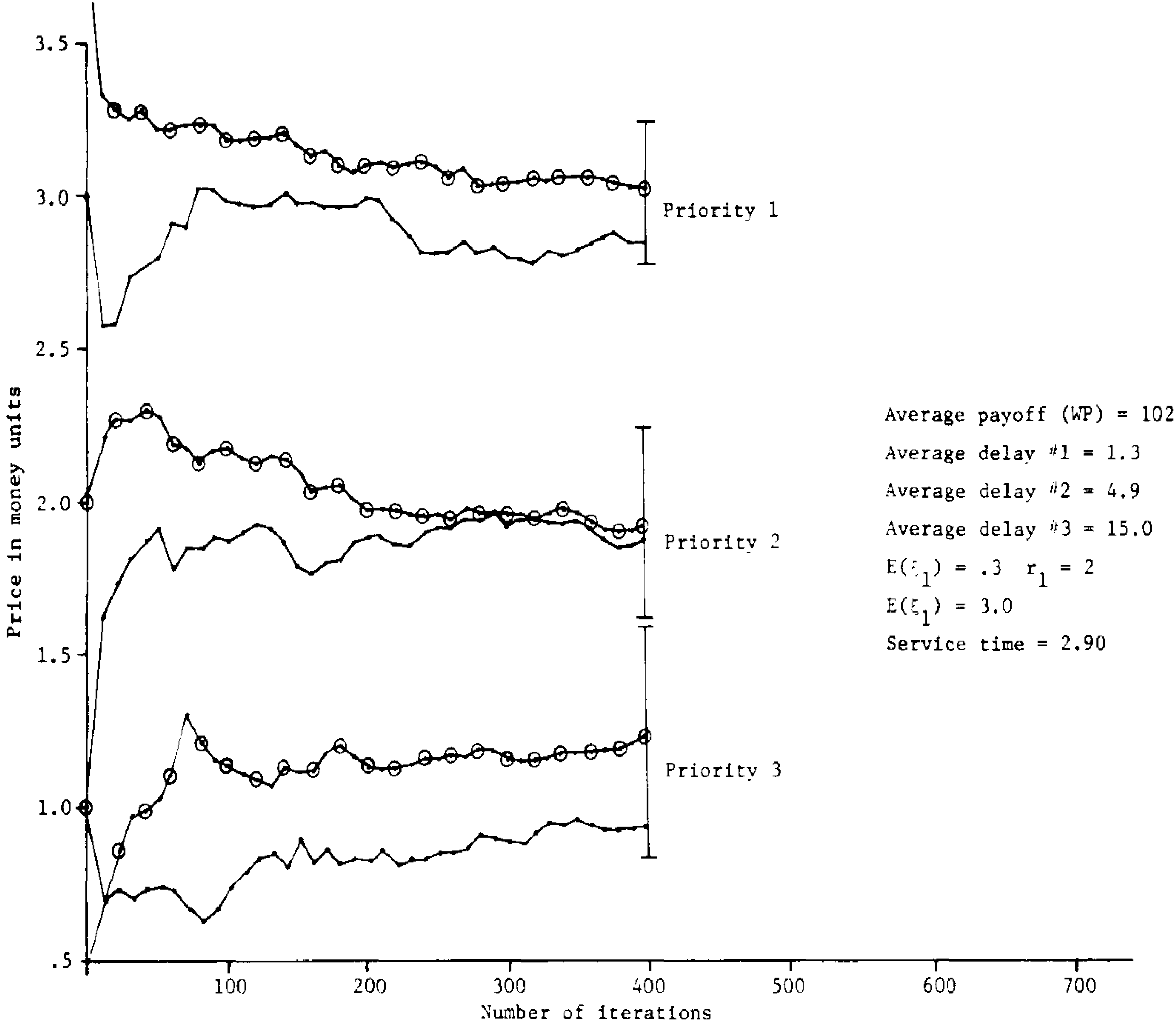
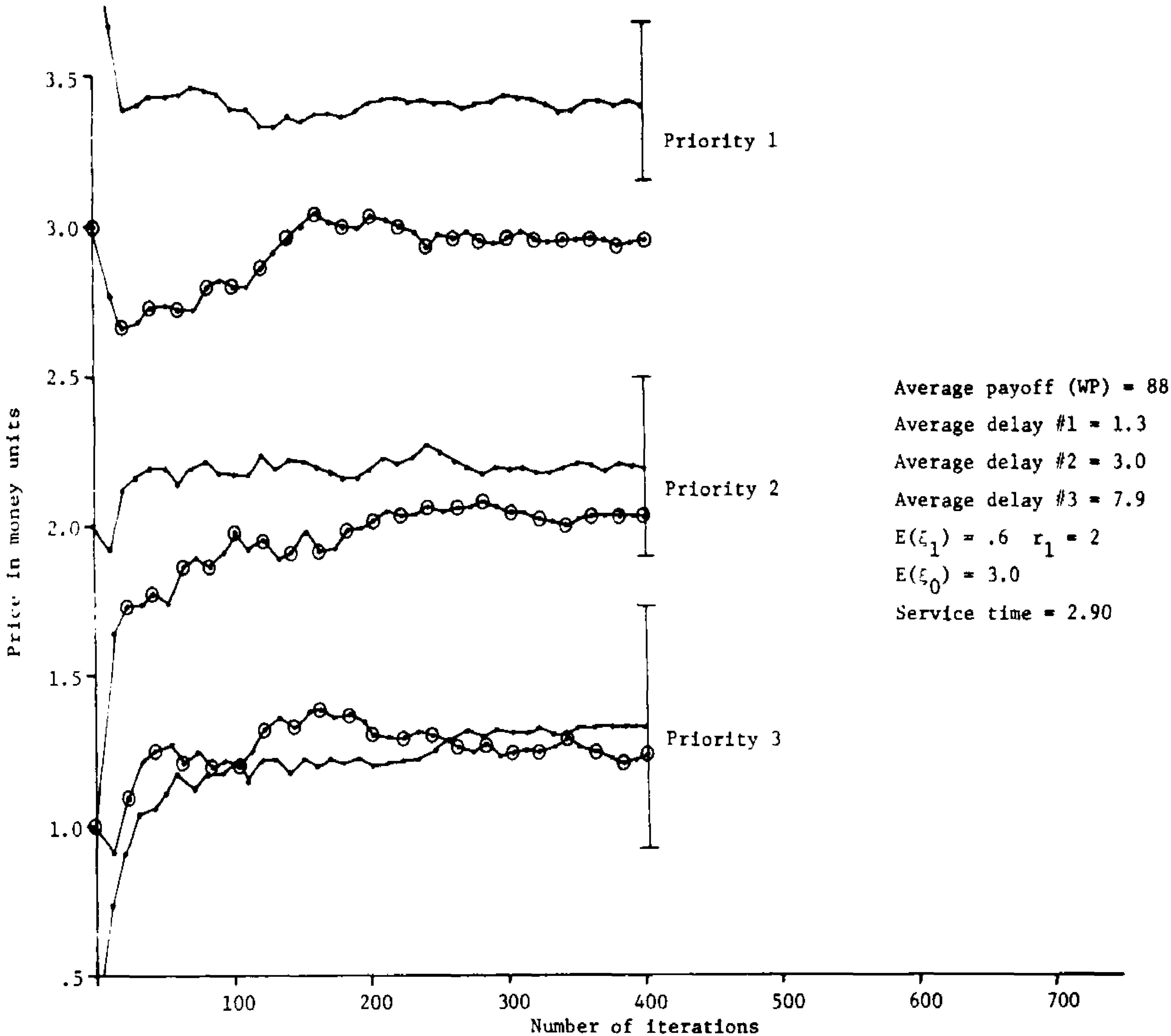


Figure VI-13--Price Trajectory for Maximum Willingness to Pay



Price in money units
3.5
3.0
2.5
2.0
1.5
1.0
.5
100
200
300
400
500
600
700
Number of iterations
Priority 1
Priority 2
Priority 3
Average payoff (WP) = 88
Average delay #1 = 1.3
Average delay #2 = 3.0
Average delay #3 = 7.9
E(ξ1) = .6 r1 = 2
E(ξ0) = 3.0
Service time = 2.90

Table VI-1

OPTIMUM PAYOFFS FOR QUEUING SYSTEM

A. Maximum Willingness to Pay

Figure	Experiment			Optimum Price			Willingness to Pay		Profit
	Number of Priorities	$E(\xi_1)$	r_1	P_1	P_2	P_3	$\overline{WP}$	$\sqrt{var(\overline{WP})}$	
V-2	1	.3	2	2.35			85	1.5	39
V-7	2	.3	2	2.80	1.75		95	1.8	37
V-12	3	.3	2	3.00	1.90	1.00	102	1.9	35
V-3	1	.6	2	2.45			77	1.4	35
V-8	2	.6	2	2.80	1.90		87	1.6	36
V-13	3	.6	2	3.20	2.10	1.20	88	1.6	33
V-4	1	.3	∞	2.25			80	1.4	--
V-9	2	.3	∞	2.55	1.70		90	1.7	--

B. Maximum Profit

V-5	1	.3	2	3.80			79	1.4	44
V-10	2	.3	2	4.10	3.60		82	1.4	46

C. Free Service

--	1	.3	2	0			65	1.3	0
--	1	.6	2	0			59	1.3	0
--	1	.3	∞	0			61	1.3	0

Table VI-2

BENEFITS FROM PRIORITY QUEUING

A. Maximum Willingness to Pay

No.	Figure	Experiment: No. of Prior- ities	Experiment: $E(\epsilon_1)$	Experiment: r_1	Increase in WP (%)	Referenced to WP in Figure	Increase in Profit (%)	Referenced to Profit in Figure
1	V-2	1	.3	2	0	V-2	-11	V-5
2	V-7	2	.3	2	11	V-2	-16	V-5
3	V-12	3	.3	2	20	V-2	-20	V-5
4	V-3	1	.6	2	0	V-3	--	--
5	V-8	2	.6	2	13	V-3	--	--
6	V-13	3	.6	2	14	V-3	--	--
7	V-4	1	.3	∞	0	V-2	--	--
8	V-9	2	.3	∞	12	V-2	--	--

B. Maximum Profit

No.	Figure	No. of Prior- ities	$E(\epsilon_1)$	r_1	Increase in WP (%)	Referenced to WP in Figure	Increase in Profit (%)	Referenced to Profit in Figure
9	V-5				-7	V-2	0	V-5
10	V-10				-4	V-2	4	V-5

C. Free Service

No.	Figure	No. of Prior- ities	$E(\epsilon_1)$	r_1	Increase in WP (%)	Referenced to WP in Figure	Increase in Profit (%)	Referenced to Profit in Figure
11	--	1	.3	2	-24	V-2	--	--
12	--	1	.6	1	-23	V-3	--	--
13	--	1	.3	∞	-24	V-4	--	--

[a] Increase in payoff = payoff from experiment/payoff a reference -1.0.

Chapter IV. If N samples of the optimum payoff, $R_T^i(P^*)$, are available, then the mean of the expected value of $R_T(P^*)$ is from Equation IV-20a,

$$\overline{wp} = \frac{1}{N} \sum_{i=1}^{N} R_T^i(P^*). \qquad \text{(VI-11)}$$

As an estimate of the error in $\overline{wp}$, the nonparametric Bayesian method provides the variance of $\overline{wp}$. After N samples of data this variance is

$$\mathrm{var}(wp) = \frac{1}{N^2(N+1)} \left[N \sum_{i=1}^{N} (R_T^i)^2 - \left(\sum_{i=1}^{N} R_T^i \right)^2 \right], \qquad \text{(VI-12)}$$

an expression derived in Appendix B.

The data in Table VI-1 were obtained by substituting 200 samples of the optimum payoff into Equations VI-11 and VI-12. Samples of the optimum payoff were obtained at a setting of

$$P^* = \frac{1}{2} \frac{1}{200} \sum_{i=n_0-N}^{n_0} (P_i + P_i'), \qquad \text{(VI-13)}$$

where P_i and P_i', $i = 1, 2, \ldots$ are the prices trajectories in Figures V-3 through V-14, and n_0 is the last iteration. Equation VI-13 is an arbitrary estimate of the optimal price, but the sensitivity of the optimum payoff to price at the optimum price was very low.

Comparing Experiments 1 with 11, 4 with 12, and 7 with 13, in Table VI-2, an optimally priced single-priority queue appears to be socially preferred to free service since willingness to pay is larger in every case.

This result is similar to the economist's prescription that a queue is optimally allocated when a toll equal to the marginal cost of service is charged.

Table VI-2 shows that establishing multiple priorities in a queue is preferred to single-class service, since at least for all the demand structures tested, the optimal willingness to pay is increased by increasing the number of priorities. Two priorities increase the payoff on the order of 12 percent; three priorities increase it somewhat more.

Comparison of Entries 2, 5, and 8 in Table VI-2 lead to the conclusion that the increase in social welfare is approximately the same for two-priority queues, no matter what the distribution on value of time. Comparison of Entries 3 and 6 in Table VI-2 shows that the same conclusion holds in the three-priority examples. Apparently, the increase in social welfare caused by multipriority queuing is to a close approximation not a function of the value of time distribution. If so, the increase in social welfare must be due to the distribution on value service.

If neither value of service or time were distributed, no social-welfare gain would have been obtained from priority queuing. But an improvement was observed that was independent of the distribution on time. Therefore, the social-welfare gain was primarily caused by the distribution on value of service.

The mechanism of this effect is the following. When the service backlog is very low, all priorities offer the same quality of service, and thus the effective price is the lowest available. The aggregate payoff is boosted because more people buy service than if single-priced service were provided. When the service backlog is very high, only the highest priority (lowest priority number) consumers will be served. These consumers will have a high value for service, since a high price must be paid to enter the top priority. The aggregate payoff is boosted because all customers served place a high value on the service they buy. It cannot be concluded that the efficiency increase from priority pricing will always be independent of the distribution on the value of time. If the variance in value of time were very much higher than produced by $r_1 = 2$, or if the values of service and time were correlated, the advantage of priority pricing might be enhanced by the distribution on value of time.

Since the efficiency gain from priority pricing appears to follow from discrimination on the value of service, equal gains could be made by adjusting the price of a single-priority queue as a function of the service backlog. The multipriority queue is an administratively convenient way to approximate setting a price schedule that is a function of delay.

The results in Figures VI-6 and VI-11 show that profits are maximized in a queuing system at a price higher than the social optimum--a result found by Noar. Evidently, the facility owner is better off (at least in these sample problems) if he keeps the system backlog low by extracting a high price from each consumer.

The most surprising profit result is that only a negligible increase in profits can be expected from establishing a priority queue, even with a distribution on the value of time (see Entries 9 and 10 in Table VI-2). Priority queuing offers the facility owner a chance to price discriminantly, but in the sample problem this does not substantially increase the potential for profit.

Entries 9 and 10 in Table VI-2 also indicate that aggregate willingness to pay at the profit optimum is only 7 percent smaller than the maximum for a single-channel queue. The average delav is much smaller (see Figures VI-3 and VI-6) at the profit maximum; however, indicating that the distribution of system benefits over the population is markedly different. The few people who can afford top-quality service are getting it to the exclusion of customers having low willingness to pay. The important result is that the (welfare) distributional effects of service are more sensitive to pricing policy than maximum attainable efficiency.

FOOTNOTES--CHAPTER VI

1. Noar, *Ibid.*, Reference 17, Chapter I.

Chapter VII

SUMMARY AND DIRECTIONS FOR FURTHER WORK

1.0 SUMMARY OF CONTRIBUTIONS

This thesis has presented methods for determining the optimal quality and prices in a service system when:

- Demand is a function of price and quality of service,
- There is uncertainty about consumer preferences, and
- The priority scheme can be varied.

Both economists and operations researchers have treated the problem of optimizing service system design, but neither have allowed uncertainty on consumer preferences (i.e., on demand). Only economists have allowed the level of demand to depend on price and quality, and only operations researchers have considered varying the priority scheme. Since all three of these conditions are allowed in this thesis, and since all three are characteristics of real systems, a more realistic set of modeling techniques is now available.

In solving the service system problem, a new method for modeling demand has been developed. Several important properties of this method are:

- Calibration (or certain knowledge of the demand model parameters) can be achieved at one setting of prices, if data at different levels of quality are available.
- Demand can be predicted for a different number of services than that used at calibration.
- Knowledge about consumer values is a by-product of the demand model calibration.

All of these properties result from the basic idea of imbedding the algorithm that a consumer acts as if he chooses the service which maximizes the difference between his willingness to pay and price in a probablistic framework. Because of these properties, the demand model concept can be used in a variety of contexts, including demand for agricultural products, entertainment events, and modal split in transportation.

Another development in this thesis is the Monte Carlo technique for estimating moments of the posterior distribution on the demand model parameters. Central to this technique is a nonparametric Bayesian method for estimating posterior moments, and calculating the variance in those estimates from a finite number of samples of the likelihood function and the prior distribution.

The application of stochastic approximation to calculation of optimal prices is original and enables determination of near optimal prices for a wide class of service systems (those for which the cost of computation is reasonable). Two innovations were included in the stochastic approximation algorithm:

- o Correlated sampling of the payoff surface, and
- o Rotation of the price basis axes at each iteration.

Correlated sampling produced a significant saving in computation time, and basis rotation improved the rate of convergence. Proof was given that neither innovation alters the optimum prices or payoff.

In running problems on an IBM 360/67, four-fifths of the time between successive price iterations with the stochastic approximation algorithm was consumed in generating exponentially distributed random numbers.* Thus, in programming the stochastic approximation algorithm, short cuts which reduce the time taken to generate random numbers should be emphasized.**

2.0 DIRECTIONS FOR FURTHER WORK

The demand model and inference procedure would be more valuable if an exact and less costly method of obtaining posterior estimates of the demand model parameters could be found. Directions to proceed are to: (1) assign more easily integrated probability distributions for the consumer values, and (2) to find algorithms for estimating maximum likelihood. One possibility under (2) is to apply Gucker's stochastic approximation technique, although this method is not exact.[1]

The proposition that the stochastic approximation algorithm converges to the true optimum when the quality process at the beginning of the $n+1^{st}$ iteration is initialized to the final value of the quality process at the n^{th} iteration is plausible, but a proof of this is needed. Use of this initialization rule would produce a closer estimate of the

---

*Exponentially distributed random numbers were generated by the familiar method of taking the logarithm of a random number uniformly distributed on the interval [0,1] and then multiplied by the appropriate scale factor. The uniform numbers were generated by the overflow method, using the Fortran IV program,

```
  IY = IY * 314159269 + 453806245
       IF (IY) 5,5,6
5 IY = IY + 2147483647 + 1
6  Y = 0.4656613 E(-9).
```

**To save computation time, the optimization routines generated for the problem in Chapter VI included a comparison test between generated values of ξ_0, the consumer's value for service, and prices. If ξ_0 was found to be less than the lowest price in the price base, the consumer was sure to reject service. Generation of the random numbers ξ_1, the consumer's value for quality, and selection of a purchase alternative could then be bypassed in the simulation.

true optimum in a finite computation time than the rule employed in this thesis where the quality process was initialized to some fixed value at the beginning of each iteration. Solving this problem requires proving convergence of the stochastic approximation algorithm when successive samples of the payoff function are dependent, an area in which little work has been done.

The sample problem treated in Chapter VI demonstrated very little change in net social advantage with changes in consumer preferences and the number of priorities, but only a restricted range of variation was considered. The question of when priority queues produce a significant advantage is raised, and could be dealt with empirically by the methods of this thesis.

FOOTNOTES--CHAPTER VII

1. Gucker, G. R., Stochastic Gradient Algorithms for Searching Multi-dimensional, Multimodal Surfaces, Technical Report 6778-7, Information Systems Laboratory, Stanford University, October 1969.

Appendix A

PROOF THAT A CONSUMER MAXIMIZES HIS SURPLUS

This appendix provides proofs of Lemmas III-2 and III-3, and Theorem III-1. The proof of Lemma III-2 is based on a lemma and some associated definitions from Debreu,[1] and Berge.[2]

Lemma A-1: Under Conditions 1 through 3 in Chapter III, $C(Y,z) \neq \varphi$ implies $D(Y,z) \neq \varphi$ (φ is the null set).

Proof: X is compact by Condition 3 and $\{x:p_x(x) \leq Y\}$ is closed; therefore $C(Y,z) = \{X \cap \{x':p_x(x') \leq Y\}\} \times \{z\}$ is compact. Let $F_x(z) = \{(x',z):(x',z)R(x,z) \cap C(Y,z)\}$. Then $F(z) = \{F_x(z):x \in C(Y,z)\}$ is a collection of closed sets with the finite intersection property. Since $F_x(z) \subset C(Y,z)$, the compactness and nonemptiness of $C(Y,z)$ implies that,[3] $\bigcap_{(x,z) \in C(Y,z)} F_x(z) \neq \varphi$. But, by definition, $D(U,z) = \bigcap_{(x,z) \in C(Y,z)} F_x(z)$.

Definition: The correspondence, $D(Y,z)$, is upper-semicontinuous at Y if for every neighborhood $\mathcal{N} \supset D(Y,z)$, there exists a neighborhood $\mathcal{V}$ of Y such that for every $Y \in \mathcal{V}$, $D(Y,z) \subset \mathcal{N}$.

Definition: The correspondence $D(Y,z)$ is upper-semicontinuous in Y on the set $\mathcal{Y}$ if $D(Y,z)$ is upper semi-continuous for every $Y \in \mathcal{Y}$.

Berge[2] proves the following lemma:

Lemma A-2: If $\mathcal{G}$ is open and $D(Y,z)$ is upper-semicontinuous on $\mathcal{Y}$, then $D^{-*}(\mathcal{G},z) = \{Y:D(Y,z) \subset \mathcal{G}\}$ is open in $\mathcal{Y}$.

Debreu[4] proves Lemma A-3 which shows that Conditions (1) to (3) are sufficient to guarantee the upper-semicontinuity of $D(Y,z)$.

Lemma A-3: Under Conditions (1) through (3), $D(Y,z)$ is upper-semicontinuous.

Lemmas A-1, A-2, and A-3 are now used to prove the main results on which the demand model of Chapter III is based.

Lemma III-2: Under Conditions (1) through (5) there exists $W(Y,x) \geq 0$, for all $z \epsilon Z$ and Y, such that $D(Y_0 - W(Y,z),z) I D(Y_0,\psi)$.

Proof: Let $(x_0,\psi) \epsilon D(Y_0,\psi)$. The continuity and connectivity of the preference sets is sufficient to guarantee the existence of an $x \epsilon X$ such that $(x,z)I(x_0,\psi)$. To see this, let $X_z = \{(x',z'):x' \epsilon X, z' = z\}$, $X_+ = \{(x,z):(x,z)R(x_0,\psi)\}$, and $X_- = \{(x,z):(x_0,\psi)R(x,z)\}$. Both of the sets $X_z \cap X_+$ and $X_- \cap X_z$ are closed, and $X \times \{z\} = (X_z \cap X_+) \cup (X_z \cap X_-)$. By hypothesis, X is connected; therefore $X \times \{z\}$ is connected and so is not the union of disjoint closed sets. Since $X \times \{z\}$ is the union of closed sets, they must have a nonvoid intersection. Hence

$$
\begin{aligned}
(X_z \cap X_+) \cap (X_z \cap X_-) &= X_z \cap (X_+ \cap X_-) \\
&= X_z \cap \{(x',z'):(x',z')I(x_0,\psi)\} \\
&= \{(x',z'):(x',z')I(x_0,\psi), z' = z\}
\end{aligned}
$$

is nonvoid.

For any point (x,z) in this nonvoid intersection, let

$$
\begin{aligned}
A^{>} &= \{(x',z'):(x,z)P(x',z')\} \\
A^{<} &= \{(x',z):(x',z')P(x,z)\} \\
A^{o} &= \{(x',z'):(x',z')I(x,z)\}.
\end{aligned}
$$

By Condition (3), $A^{>}$ and $A^{<}$ are open.

By the definition of (x,z), $D(Y_0,\psi) \subset A^{o}$. This lemma will be proved if it can be shown that $D(Y',z) \cap A^{o} \neq \varphi$ for some $Y' \epsilon [0,Y_0]$. This can be shown by contraposition.

Assume $D(Y',z) \cap A^{o} = \varphi$ for all $Y' \epsilon [0,Y_0]$. Then for any $Y \epsilon [0,Y_0]$, $D(Y,z) \subset A^{<}$ or $D(Y,z) \subset A^{>}$. Therefore

$$D^{-*}(A^{<},z) \cup D^{-*}(A^{>},z) = [0,Y_0]$$

and

$$D^{-*}(A^{<},z) \cap D^{-*}(A^{>},z) = \emptyset.$$

By Lemmas A-3 and A-2, $D^{-*}(A^{<},z)$ and $D^{-*}(A^{>},z)$ are open, and by Conditions (4) and (5) they are nonempty. The connected interval $[0,Y_0]$ cannot be the union of disjoint open sets, giving the contradiction that proves this lemma.

Adding Condition (6), $D(Y,z)$ is strictly monotonic in Y and hence Y' is unique.

Lemma III-3: Under Conditions (1) through (6), $W(Y,z)$ is a single valued function of Y and Z, and $D(Y,z)$ is strictly monotonic in Y.

Proof: Assume $Y_1 > Y_2$. Let $(x_2,z)\epsilon D(Y_2,z)$. Then $x_1 = (Y_1/Y_2)x_2$ has $(x_1,z)\epsilon C(Y_1,z)$. Therefore, by Condition (6), $(x_1,z)P(x_2,z)$. By the definition of $D(Y_1,z)$, $D(Y_1,z)P\{(x,z):(x,z)I(x_2 z)\}$, and so $D(Y_1,z)$ $PD(Y_2,z)$; proving that $D(Y,z)$ is monotonic in Y.

Since $D(Y,z)$ is strictly monotonic in $Y, D(Y - W_1(Y,z),z)$ $ID(Y - W_2(Y,z),z)$ implies that $Y - W_1(Y,z) = Y - W_2(Y,z)$ is unique.

The following simple lemma is needed in proving Lemma III-4:

Lemma A-5: Under Conditions (1) through (6), if $(x',z)\epsilon D(Y,z)$, then $p_x(x') = Y$.

Proof: $p_x(x') > Y$ is impossible, since by the definition of $D(Y,z)$, $(x'z)\epsilon C(Y,z)$.

Let $(x',z)\epsilon D(Y,z)$ and have $p_x(x') < Y$. Then $x'' = (Y/p_x(x'))x'$ satisfies $p_x(x'') = Y$ by the linearity of $p_x(\cdot)$; and thus, $(x'',z)\epsilon C(Y,z)$. Since $Y/p_x(x') > 1$, Condition (6) implies that $(x'',z)P(x',z)$. But this implies $(x',z) \not\in D(Y,z)$, a contradiction which leaves $p_x(x') = Y$ as the only possibility.

Lemma III-4: Under Conditions (1) through (7), $W(Y,z)$ is not a function of Y.

Proof: Suppose $W(Y,z)$ gives $D(Y - W(Y,z),z) I D(Y,\psi)$. Define $Y^* = Y - W(Y,z)$. By Lemma A-1, there exists

$$(x_0,\psi) \epsilon D(Y,\psi), \tag{A-1a}$$

and

$$(x^*,z) \epsilon D(Y^*,z). \tag{A-1b}$$

By the definition of Y^*,

$$(x_0,\psi) I (x^*,z). \tag{A-2}$$

By Lemma A-5,

$$p_x(x^*) = Y^*, \tag{A-3a}$$

$$p_x(x_0) = Y. \tag{A-3b}$$

For any Y', let $(x',\psi) \epsilon D(Y',\psi)$. By Condition (7),

$$(x',\psi) I (x' - x_0 + x^*,z). \tag{A-4}$$

The theorem will be proved if $(x' - x_0 + x^*, z) \epsilon D(Y' - W(Y,z),z)$, for then $D(Y',\psi) I D(Y' - W(Y,z),z)$. Since also $D(Y',\psi) I D(Y' - W(Y',z),z)$, the uniqueness of $W(Y,z)$ implies that $W(Y',z) = W(Y,z)$.

By the linearity of $p_x(\cdot)$, and Lemma A-5, $p_x(x' - x_0 + x^*) = Y' + Y^* - Y = Y' - W(Y,z)$. Thus $(x' - x_0 + x^*,z) \epsilon C(Y' - W(Y,z),z)$.

If $(x' - x_0 + x^*, z) \not\in D(Y' - W(Y,z),z)$, there exists $\bar{x}$ with

$$(\bar{x},z)P(x' - x_0 + x^*,z), \qquad \text{(A-5a)}$$

such that,

$$p_x(\bar{x}) \leq Y' - W(Y,z). \qquad \text{(A-5b)}$$

If this were not true then $D(Y' - W(Yz),z)$ would be empty. By Condition (7), and Relation A-2,

$$(\bar{x},z)I(\bar{x} - x^* + x_0, z). \qquad \text{(A-6)}$$

By Relations A-4, A-5, and A-6,

$$(\bar{x} - x^* + x_0, z)P(x' - x_0 + x^*,z)I(x',z). \qquad \text{(A-7)}$$

By the linearity of $p_x(\cdot)$, Relations A-3a, A-3b, and A-5b,

$$p_x(\bar{x} - x^* + x_0) \leq Y'. \qquad \text{(A-8)}$$

Results A-7 and A-8 together imply that $(x',z) \not\in D(Y',z)$. This contradiction proves that $(x' + x^* - x_0, z) \in D(I' - w(z),z)$.

Having Lemma III-3, it is a straightforward matter to prove Theorem III-1. Henceforth write $W(Y,z) = W(z)$, since willingness-to-pay is not a function of income.

<u>Theorem III-1</u>: A consumer will choose $z \in Z$ only if $W(z) - p(z) \geq W(z') - p_z(z')$ for all $z' \in Z$.

<u>Proof</u>: Assume z is chosen. Then paying $p_z(z)$ for z and $p_z(z')$ for z' it follows that for all $z' \in Z_1$,

$$D(Y - p_z(z),z)RD(Y - p_z(z'),z'). \qquad \text{(A-9)}$$

Adding and subtracting $W(z)$ and $W(z')$ yields for all $z' \epsilon Z$,

$$D(Y + W(z) - W(z) - p_z(z), z) \mathcal{R} D(Y + W(z') - W(z') - p_z(z'), z). \tag{A-10}$$

By Lemma III-4, $W(z)$ is not a function of income. Using Lemma III-2, the above expression reduces to,

$$D(Y + W(z) - p_z(z), \psi) \mathcal{R} D(Y + W(z') - p_z(z'), \psi), \tag{A-11}$$

for all $z' \epsilon Z_1$. By the monotonicity of $D(Y, \psi)$, this relation holds only if

$$Y + W(z) - p_z(z) \geq Y + W(z') - p_z(z'), \tag{A-12}$$

or only if for all $z' \epsilon Z$,

$$W(z) - p_z(z) \geq W(z') - p_z(z'). \tag{A-13}$$

FOOTNOTES--APPENDIX A

1. Debreu, G., *op. cit.*, Chapter 4.

2. Berge, C., *Topological Spaces*, Oliver & Boyd, London, 1963, Chapter 6.

3. Royden, H., *Real Analysis*, The MacMillan Company, New York, 1963, p. 136.

4. Debreu, G., *op. cit.*, p. 72.

Appendix B

DERIVATION OF POSTERIOR MEAN, VARIANCE OF THE MEAN, AND COVARIANCE OF TWO MEANS

This appendix presents a nonparametric Bayesian method for estimating the covariance of the means of two random variables, and outlines the derivation of estimates for the mean and variance of the mean of a random variable. The basic ideas employed in these derivations are due to Smallwood[1] except that a different approach is taken to writing the likelihood function for data events. Furthermore, the proofs presented here are more straightforward.

Suppose that N samples of two nonnegative random variables X and Y are given as $x_1,y_1,x_2,y_2,x_3,y_3, \ldots x_N,y_N$ with independence among these random variables specified by $\Pr\{x_1, \ldots y_N\} = \prod_{L=1}^{N} \Pr\{x_i,y_i\}$. Let $\{U_{xy}(\zeta_1,\zeta_2):\zeta_1 \geq 0,\ \zeta_2 \geq 0\}$ be a continuous parameter family of random variables, where $U_{xy}(\zeta_1, \zeta_2) = \Pr\{X > \zeta_1,\ Y > \zeta_2\}$. The joint density functions over the random variable $U_{xy}(\zeta_1,\zeta_2)$ describe the analyst's state of knowledge about the joint distribution on X and Y. These random variables are related to the means by

$$\Lambda = \int_0^\infty U_{xy}(\zeta_1,0)\,d\zeta_1\ , \qquad \text{(B-1a)}$$

$$\Gamma = \int_0^\infty U_{xy}(0,\zeta_2)\,d\zeta_2\ . \qquad \text{(B-1b)}$$

Since U_{xy} is a random variable, Λ and Γ are random variables.

One statistic of interest is the covariance of the random variables Λ and Γ:

$$\text{cov}\ (\Lambda,\Gamma) = E(\Lambda\Gamma) - E(\Lambda)\ E(\Gamma). \qquad \text{(B-2)}$$

We desire a relation between the data samples $x_1, y_1, \ldots x_N, y_N$ and the estimate, cov (Λ, T). Substituting Equations B-1 into the terms of Equation C-2,

$$E(\Lambda) = E\left(\int_0^\infty U_{xy}(\zeta_1, 0)\, d\zeta_1\right), \tag{B-3a}$$

$$E(T) = E\left(\int_0^\infty U_{xy}(0, \zeta_2)\, d\zeta_2\right), \tag{B-3b}$$

$$E(\Lambda, T) = E\left(\int_0^\infty\int_0^\infty U_{xy}(\zeta_1, 0) U_{xy}(0, \zeta_2)\, d\zeta_1 \zeta_2\right). \tag{B-3c}$$

If it is assumed that $U_{xy}(\zeta_1, \zeta_2)$ has a bounded mean for both x and y, the Fubini Theorem[2] allows expectation and integration to be interchanged. Defining $E(U_{xy}(\zeta_1, \zeta_2)) = \overline{U_{xy}(\zeta_1, \zeta_2)}$, gives

$$E(\Lambda) = \int_0^\infty \overline{U_{xy}(\zeta_1, 0)}\, d\zeta_1, E(T) = \int_0^\infty \overline{U_{xy}(0, \zeta_2)}\, d\zeta_2, \tag{B-4a}$$

$$E(\Lambda, T) = \int_0^\infty\int_0^\infty \overline{U_{xy}(\zeta_1, 0) U_{xy}(0, \zeta_2)}\, d\zeta_1, d\zeta_2 \ . \tag{B-4b}$$

If the auxiliary random variables,

$$V(\zeta_1, \zeta_2) = \Pr\{X > \zeta_1,\ Y \leq \zeta_2\}, \tag{B-5a}$$

$$W(\zeta_1, \zeta_2) = \Pr\{X \leq \zeta_1,\ Y > \zeta_2\}, \tag{B-5b}$$

$$Z(\zeta_1, \zeta_2) = \Pr\{X > \zeta_1,\ Y > \zeta_2\}, \tag{B-5c}$$

are defined, then

$$U_{xy}(\zeta_1,)) = V(\zeta_1, \zeta_2) + Z(\zeta_1, \zeta_2), \tag{B-6a}$$

$$U_{xy}(0, \zeta_2) = W(\zeta_1, \zeta_2) + Z(\zeta_1, \zeta_2). \tag{B-6b}$$

The regions $G_i(\zeta_1,\zeta_2)$ are defined as,

$$G_1(\zeta_1,\zeta_2) = \{(x,y): 0 \le x < \zeta_1,\ y \ge \zeta_2\}$$

$$G_2(\zeta_1,\zeta_2) = \{(x,y): x \ge \zeta_1,\ y \ge \zeta_2\}$$

$$G_3(\zeta_1,\zeta_2) = \{(x,y): 0 \le x < \zeta_1,\ 0 \le y < \zeta_2\}$$

$$G_4(\zeta_1,\zeta_2) = \{(x,y): x \ge \zeta_1,\ 0 \le y < \zeta_2\}. \tag{B-9}$$

Then for the data event (x_i,y_i), the following continuous parameter discrete-valued random variable is defined:

$$D_{x_iy_i}(\zeta_1,\zeta_2) = \begin{cases} E_1 & \text{if} \quad (x_i,y_i)\in G_1(\zeta_1,\zeta_2) \\ E_2 & \text{if} \quad (x_i,y_i)\in G_2(\zeta_1,\zeta_2) \\ E_3 & \text{if} \quad (x_i,y_i)\in G_3(\zeta_1,\zeta_2) \\ E_4 & \text{if} \quad (x_i,y_i)\in G_4(\zeta_1,\zeta_2), \end{cases} \tag{B-10}$$

where the E_i's are mutually exclusive events since the regions G_i are disjoint. It is necessary to assume that the posterior distribution of the random variables $V(\zeta_1,\zeta_2)$, $W(\zeta_1,\zeta_2)$, and $Z(\zeta_1,\zeta_2)$ given the data point (x_i,y_i) is equal to the posterior distribution given $D_{x_iy_i}(\zeta_1,\zeta_2)$.

<u>Assumption B-1</u>:

$$\Pr\{V(\zeta_1,\zeta_2),\ W(\zeta_1,\zeta_2),\ Z(\zeta_1,\zeta_2)\,|\,x_i,y_i,\ \mathcal{E}\}$$

$$= \Pr\{V(\zeta_1,\zeta_2),\ W(\zeta_1,\zeta_2),\ Z(\zeta_1,\zeta_2)\,|\,D_{x_iy_i}(\zeta_1,\zeta_2),\ \mathcal{E}\}.$$

This amounts to asserting that the posterior probability that the joint probabilities V, W, and Z at (ζ_1,ζ_2) are greater than the values

(v_0, w_0, z_0) is dependent on only the region in which the data occurred, and not on the precise value of the data outcome.

Using Bayes' rule and Assumption B-1, the posterior distribution on V, W, and Z is (dropping the (ζ_1, ζ_2) notation),

$$\Pr\{V,W,Z|x_i,y_i,\mathcal{E}\} = \Pr\{D_{x_i y_i}|V,W,Z,\mathcal{E}\}\Pr\{V,W,Z|\mathcal{E}\}/\Pr\{D_{x_i y_i}|\mathcal{E}\}. \quad \text{(B-11)}$$

The likelihood function in Equation B-11 is easily written as

$$\Pr\{D_{x_i y_i}|V,W,Z\} = \begin{cases} V & \text{if} \quad (x_i,y_i) \in G_1 \\ W & \text{if} \quad (x_i,y_i) \in G_2 \\ Z & \text{if} \quad (x_i,y_i) \in G_3 \\ 1-V-W-Z & \text{if} \quad (x_i,y_i) \in G_4. \end{cases} \quad \text{(B-12)}$$

For N data events it follows from independence of the data samples that

$$\Pr\{D_{x_1 y_1}, D_{x_2 y_2}, \ldots, D_{x_N y_N}\} = V^n W^m Z^r (1-V-W-Z)^{N-n-m-r}, \quad \text{(B-13)}$$

where

$n = n(\zeta_1, \zeta_2)$ = number of data points in set with $x > \zeta_1$ and $y \le \zeta_2$

$m = M(\zeta_1, \zeta_2)$ = number of data points in set with $x \le \zeta_1$ and $y > \zeta_2$

$r = r(\zeta_1, \zeta_2)$ = number of data points in set with $x > \zeta_1$ and $y > \zeta_2$.

A conjugate prior for Equation B-13 is obviously

$$\Pr\{V,W,Z/\mathcal{E}\} = B(\alpha,\beta,\gamma,\delta) V^{\alpha-1} W^{\beta-1} Z^{\gamma-1} (1-V-W-Z)^{\delta-1}, \quad \text{(B-14)}$$

where $B(\cdot)$ is a complete Beta function in four arguments.

$$B(\cdot) = \Gamma(\alpha + \beta + \gamma + \delta)/\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\delta). \quad \text{(B-15)}$$

In the usual method for conjugate distributions,[3] the posterior parameters are related to the prior parameters by:

$$
\begin{aligned}
\alpha' &= n + \alpha \\
\beta' &= m + \beta \\
\gamma' &= r + \gamma \\
\delta' &= N - n - m - r + \delta. \qquad \text{(B-16)}
\end{aligned}
$$

The constants α, β, γ, and δ of the prior distribution cannot be chosen arbitrarily as there is a consistency condition on the prior that must be imposed. The random variables $U_{xy}(\zeta_1,0)$ and $U_{xy}(0,\zeta_2)$ are related to the random variables V, W, and Z in two different ways: as specified in Equations B-6a and B-6b, and by

$$U_{xy}(\zeta_1,0) = Z(\zeta_1,0), \qquad \text{(B-17a)}$$

$$U_{xy}(0,\zeta_2) = Z(0,\zeta_2). \qquad \text{(B-17b)}$$

These change of variables expressions specify two different ways of obtaining the distributions of $U_{xy}(\zeta_1,0)$ and $U_{xy}(0,\zeta_2)$.

Performing the change of variables operation on the prior density (Equation B-14) indicated by Equation B-6a gives the probability density:

$$\Pr\{U_{xy}(\zeta_1,0)|E\} = B\{\alpha + \gamma, \beta + \delta\} U_{xy}^{(\alpha+\gamma)(\zeta_1,\zeta_2)-1} (1 - U_{xy})^{(\beta+\delta)(\zeta_1,\zeta_2)-1} \qquad \text{(B-18)}$$

where $(\alpha + \gamma)(\zeta_1,\zeta_2) = \alpha(\zeta_1,\zeta_2) + \gamma(\zeta_1,\zeta_2)$, etc.

Performing the change of variables operation on the prior density (Equation B-14) indicated by Equation B-17a gives the probability density,

$$\Pr\{U_{xy}(\zeta_1,0)|E\} = B(\gamma,\alpha + \beta + \delta)U_{xy}^{\gamma(\zeta_2,0)-1}(1 - U_{xy})^{(\alpha+\beta+\delta)(\zeta_2,0)}. \tag{B-19}$$

Similarly, Equation B-6b and B-17b require that:

$$\Pr\{U_{xy}(0,\zeta_2)|E\} = B(\beta + \gamma, \alpha + \delta)U_{xy}^{(\beta+\gamma)(\zeta_1,\zeta_2)-1}(1 - U_{xy})^{(\alpha+\delta)(\zeta_1,\zeta_2)}. \tag{B-20a}$$

$$\Pr\{U_{xy}(0,\zeta_2)|E\} = B(\gamma,\alpha + \beta + \delta)U_{xy}^{\gamma(0,\zeta_2)-1}(1 - U_{xy})^{(\alpha+\beta+\delta)(0,\zeta_2)-1}. \tag{B-20b}$$

Equations B-18 and B-19 will be equivalent expressions for $\Pr\{U_{xy}(\zeta_1,0)|E\}$ for all ζ_1 and ζ_2 only if:

$$(\alpha + \gamma)(\zeta_1,\zeta_2) = \gamma(\zeta_1,0) \tag{B-21a}$$

$$(\beta + \delta)(\zeta_1,\zeta_2) = (\alpha + \beta + \delta)(\zeta_1,0). \tag{B-21b}$$

Likewise, Equations B-20 will be equivalent expressions for $\Pr\{U_{xy}(0,\zeta_2)|E\}$ only if:

$$(\beta + \gamma)(\zeta_1,\zeta_2) = \gamma(0,\zeta_2) \tag{B-21c}$$

$$(\alpha + \delta)(\zeta_1,\zeta_2) = (\alpha + \beta + \delta)(0,\zeta_2). \tag{B-21d}$$

Equations B-21a and B-21b imply that $\alpha + \beta + \gamma + \delta$ is not a function of ζ_2; and Equations B-21c and B-21d imply that $\alpha + \beta + \gamma + \delta$ is not a function of ζ_1. Therefore

$$\alpha + \beta + \gamma + \delta = M, \tag{B-22}$$

where M is an as yet unspecified constant. From Equations B-21a and B-21c, respectively:

$$\alpha + \alpha = f(\zeta_1), \tag{B-23a}$$

$$\beta + \gamma = g(\zeta_2). \tag{B-23b}$$

For later use, it is desirable to relate $f(\zeta_1)$, $g(\zeta_2)$, and M to the prior distributions on x and y. From Equation B-18 it can be calculated that,

$$E(U_{x,y}(\zeta_1,0)|E) = \frac{(\alpha + \gamma)(\zeta_1,0)}{(\alpha + \beta + \gamma + \delta)(\zeta_1,0)} . \tag{B-24}$$

Expanding the prior probability that $x > \zeta$ on $U_{xy}(\zeta,0)$ gives:

$$\Pr\{x > \zeta|E\} = \int_0^\infty \Pr\{x > \zeta|U_{xy}(\zeta,0),E\} \Pr\{U_{xy}(\zeta,0)|E\} d\, U_{xy}(\zeta,0). \tag{B-25}$$

By the definition of $U_{xy}(\zeta_1,0)$, $\Pr\{x > \zeta|U_{xy}(\zeta,0)E\} = U_{xy}(\zeta,0)$. Therefore, Equation B-25 can be reduced to,

$$\Pr\{x > \zeta|E\} = E(U_{xy}(\zeta,0)|E). \tag{B-26}$$

By Equations B-24, B-22 and B-23a,

$$\Pr\{x > \zeta|E\} = \frac{f(\zeta)}{M} . \tag{B-27a}$$

Thus, the function, $f(\zeta)$, is the prior complementary cumulative distribution for x, modified by the constant M. Similarly,

$$\Pr\{y > \zeta|E\} = g(\zeta)/M, \tag{B-27b}$$

$$\Pr\{x > \zeta_1,\ y > \zeta_2|E\} = \gamma(\zeta_1,\zeta_2)/M. \tag{B-27c}$$

There are now enough results to find an expression relating the estimate cov (Λ,T) to the data samples $x_1,y_1, \ldots x_N,y_N$. Substituting B-6a and B-6b into B-4b and rearranging terms,

$$E(\Lambda T) = \int_0^\infty\int_0^\infty \frac{(\alpha + \gamma)(\beta + \gamma) + \gamma}{(\alpha + \beta + \gamma + \delta + 1)(\alpha + \beta + \gamma + \delta)}\, d\zeta_1, d\zeta_2. \tag{B-28}$$

A posteriori, $E(\Lambda T)$, is from Equations B-16,

$$E(\Lambda T) = \int_0^\infty\int_0^\infty \frac{(n + r + f(\zeta_1))(m + r + g(\zeta_2)) + (r + \gamma)}{(M + N + 1)(M + N)}\, d\zeta_1, d\zeta_2. \tag{B-29}$$

Define the function $u_{xy}(\zeta_1,\zeta_2)$ by,

$$u_{xy}(\zeta_1,\zeta_2) = \begin{cases} 1 & \text{if} \quad x > \zeta_1,\ y > \zeta_1 \\ 0 & \text{otherwise}\ . \end{cases} \tag{B-30}$$

Then,

$$n + r = \sum_{j=1}^{N} \sum_{i=1}^{N} u_{x_i y_j}(\zeta_1, 0), \tag{B-31a}$$

$$m + r = \sum_{j=1}^{N} \sum_{L=1}^{N} u_{x_i y_j}(0, \zeta_2), \tag{B-31b}$$

$$r = \sum_{j=1}^{N} \sum_{L=1}^{N} u_{x_i y_j}(\zeta_1, \zeta_2). \tag{B-31c}$$

Putting Equations B-31 and B-30 into B-29 and integrating,

$$E(\Lambda T) = (M + N + 1)^{-1}(M + N)^{-1}\left[\left(M\nu_1 + \sum_{i=1}^{N} x_i\right)\left(M\nu_1 + \sum_{i=1}^{N} y_i\right) + \sum_{i=1}^{N} x_i y_i + M(\nu\mu)_1\right], \tag{B-32}$$

where

$$\nu_1 = \int_0^\infty \Pr\{x > \zeta | E\} d\zeta , \tag{B-33a}$$

$$\mu_1 = \int_0^\infty \Pr\{y > \zeta | E\} d\zeta , \tag{B-33b}$$

$$(\nu\mu)_1 = \int_0^\infty\int_0^\infty d\zeta_1 d\zeta_2 \Pr\{x > \zeta_1 , y > \zeta_2 | E\} . \tag{B-33c}$$

It is obvious from Equations B-32 that ν_1, μ_1, and $(\nu,\mu)_1$ are the prior means and the prior cross moment, respectively. The constant M is interpreted as the weight given to these prior estimates by the analyst. High M implies that the analyst is very sure of the prior parameters and lower values for M imply more uncertainty.

In a similar manner it can be shown that

$$E(\Lambda) = (M + N)^{-1}\left(M\nu_1 + \sum_{i=1}^{N} x_i\right), \tag{B-34a}$$

$$E(\Gamma) = (M + N)^{-1}\left(M\mu_1 + \sum_{i=1}^{N} y_i\right). \tag{B-34b}$$

Equations B-32 and B-34 together with B-2 give

$$\mathrm{cov}\ (\Lambda, \Gamma) = (M + N + 1)^{-1}(M + N)^{-2}\left[(M + N)\left(M(\nu\mu)_1 + \sum_{i=1}^{N} x_i y_i\right)\right.$$
$$\left. - \sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i - M\nu_1 \sum_{i=1}^{N} y_i - M\mu_1 \sum_{i=1}^{N} x_i - M^2\nu_1\mu_1\right] . \tag{B-35}$$

Letting $x_i = y_i$, the variance of the state of information for the mean is

$$\text{var}\ (\Lambda) = (M + N + 1)^{-1}(M + N)^{-2}\left[(M + N)\left(2M\nu_2 + \sum_{i=1}^{N} x_i^2\right)\right.$$

$$\left. - \left(\sum_{i=1}^{N} x_i\right)^2 - 2M\nu_1 \sum_{i=1}^{N} x_i - M^2\nu_1^2\right], \tag{B-36a}$$

where

$$\nu_2 = \int_0^\infty\int_0^\infty d\zeta_1 d\zeta_2\ \Pr\{x > \zeta_1,\ x > \zeta_2 | E\}$$

$$= \int_0^\infty d\zeta_1 \int_0^{\zeta_1} d\zeta_2\ \Pr\{x > \zeta_1 | E\} + \int_0^\infty d\zeta_2 \int_0^{\zeta_2} d\zeta_1\ \Pr\{x > \zeta_2 | E\}$$

$$= 2\int_0^\infty \zeta_1\ \Pr\{x > \zeta_1 | E\} d\zeta_1, \tag{B-36b}$$

or ν_2 is the prior variance of X.

If M is very small, or if N is very large then to a close approximation,

$$E(\Lambda) = N^{-1}\left(\sum_{L=1}^{N} x_i\right)$$

$$E(T) = N^{-1}\left(\sum_{L=1}^{N} y_i\right)$$

$$\text{cov}\ (\Lambda, T) = (N + 1)^{-1}\ N^{-2}\left(\sum_{L=1}^{N} x_i y_i - \sum_{L=1}^{N} x_i \sum_{L=1}^{N} y_i\right)$$

$$\text{var}\ (\Lambda) = (N + 1)^{-1}\ N^{-2}\left(\sum_{L=1}^{N} x_i^2 - \left(\sum_{L=1}^{N}\right) x_i\right)^2 . \tag{B-37}$$

FOOTNOTES--APPENDIX B

1. Smallwood, R. D., "A Theory of Non-parametric Bayesian Inference," unpublished work, Engineering Economic Systems Department, Stanford University.

2. Royden, H., op. cit., p. 233.

3. Raiffa, H., and R. Schlaifer, Applied Statistical Decision Theory, Harvard University, 1961.

Appendix C

MISCELLANEOUS LEMMAS FOR STOCHASTIC APPROXIMATION

Lemma C-1 proves an equality about the inner product

$$\langle P - P^* / E(Y_n | P, \mathcal{S}_f, \mathcal{S}_u, E)\rangle .$$

The notation and definitions adopted in Chapter IV are used here without change. The distinction between a vector and its representation will be indicated by a bar over the vector's symbol. Thus, the price vector, P, has representation $\overline{P} = (p_1, p_2, \ldots, p_J)$ in the basis U. In the following, let $v = \min_i v_i$.

Lemma C-1: Under Assumption V-3 and V-4,

$$\langle P - P^*/E(Y_n | P, \mathcal{S}_c, \mathcal{S}_u, E)\rangle \leq -2K_0 \|P - P^*\|^2$$
$$+ \left(\frac{J}{\sqrt{3V}}\right)^3 K_3 c_n^2 \|P - P^*\| .$$

Proof: Suppressing the condition on $\mathcal{S}_f$, $\mathcal{S}_u$, and E, the definition of Y_n produces,

$$E(Y_n | P) = \sum_{i=1}^{J} [E(R_T' | P + c_n U^*_{i,n}) - E(R_T' | P - c_n U^*_{i,n})] \frac{1}{v_i c_n} U'_{i,n} . \tag{C-1}$$

The representation of this vector in the basis $U'_{i,n}$ is,

$$\overline{E(Y_n | P)} = c_n^{-1} T_V^{-1} [E(R_T' | P + c_n U^*_{1,n}) - E(R_T' | P - c_n U^*_{1,n})$$
$$\ldots, E(R_T' | P + c_n U^*_{J,n}) - E(R_T' | P - c_n U^*_{J,n})] . \tag{C-2}$$

The Taylor's expansion of $E(R_T'|P \pm c_n U^{\#}_{i,n})$ at P is,

$$E(R_T'|P \pm c_n U^{\#}_{i,n}) = E(R_T'|P)$$
$$\pm c_n \langle \text{grad } E(R_T'|P)/U^{\#}_{i,n} \rangle$$
$$+ \frac{1}{2!} c_n^2 \langle U^{\#}_{i,n}/HE(R_T'|P)/U^{\#}_{i,n} \rangle$$
$$+ \frac{1}{3!} T^{\pm}_{3,i} \; c_n^3, \tag{C-3}$$

where $HE(R_T'|P)$ is the Hessian of payoff at P, and $T_{3,i}$ is the third-order error term,

$$T^{\pm}_{3,i} = \sum_{j=1}^{J} \sum_{k=1}^{J} \sum_{\ell=1}^{J} \frac{\partial^3 E(R_T'|\bar{P}^{\pm})}{\partial p_j \partial p_k \partial p_\ell} u^{\#n}_{j,i} \; u^{\#n}_{k,i} \; u^{\#n}_{\ell,i} \;, \tag{C-4}$$

where $\bar{P}^{\pm} = P + \alpha U^{\#}_{i,n}$, $0 < \alpha < 1$; and where $U^{\#}_{i,n} = (u^{\#n}_{1,i}, u^{\#n}_{2,i}, \ldots, u^{\#n}_{J,i})$ is the representation of $U^{\#}_{i,n}$, a vector in the orthogonal basis $U^{\#}_n$. The second-order term in Equation C-3 is,

$$\langle U^{\#}_{i,n}/HE(R_T'|P)/U^{\#}_{i,n} \rangle = \sum_{j=1}^{J} \sum_{k=1}^{J} u^{\#n}_{j,i} \; u^{\#n}_{k,i} \; \frac{\partial^2 E(R_T'(P))}{\partial p_j \partial p_k} \; . \tag{C-5}$$

Let $\bar{G} = (g_1, g_2, \ldots, g_J)$ be the representation of grad $E(R_T'(P))$ in the basis U. Then the representation of grad $E(R_T'(P))$ in the basis $U^{\#}_n$ is,[1]

$$\bar{G}^{\#}_n = (g^{\#}_1, g^{\#}_2, \ldots, g^{\#}_J)$$
$$= (T_{\Theta_n} T_V)^{-1} \; \bar{G} \; . \tag{C-6}$$

The representation of $U^{\#}_{i,n}$ in the same basis is obviously a unit vector with a one in the i^{th} entry. As a consequence,

$$\langle \text{grad } E(R_T^-|P)/U^{\#}_{i,n}\rangle = g^{\#}_i \; . \tag{C-7}$$

Let P be the representation of P in U. Then the representations of P and P* in U'_n are:

$$P' = T^{-1}_{\oplus_n} P \tag{C-8a}$$

$$P^{*\prime} = T^{-1}_{\oplus_n} P^* \; . \tag{C-8b}$$

Taking the inner product of Equation C-1 with P − P*, and representing the result in the basis $U'_{i,n}$ gives,

$$\langle P - P^*/E(Y_n|P)\rangle = \left(E(Y_n^{T}P)'\right)^t (P' - P^{*\prime}) \; . \tag{C-9}$$

Substituting Equation C-7 into Equation C-3, and then Equation C-3 into Equation C-9 gives,

$$\langle P - P^*/E(Y_n|P)\rangle = 2G^{*t} T_V^{-1} (\bar{P}' - \bar{P}^{*\prime})$$

$$+ \frac{c_n}{3!} T_3(\bar{P}' - P^{*\prime}) , \tag{C-10}$$

where $T_3 = (T^+_{3,1} + T^-_{3,1}, \ldots, T^+_{3,J} + T^-_{3,J})$.

Substituting Equations C-8a and C-8b, and C-6 into C-10,

$$\langle P - P^*/E(Y_n|P)\rangle = 2\bar{G}^t \left(T_V^{-1} \; T^{-1}_{\oplus_n}\right)^t T_V^{-1} \; T^{-1}_{\oplus_n} (P - \bar{P}^*)$$

$$+ \frac{2c_n^{\;2}}{3!} T_3 T^{-1}_{\oplus_n} (\bar{P} - \bar{P}^*) \; . \tag{C-11}$$

Since T_V is diagonal, and T_n is unitary (it is a rotation operator),

$$\langle P - P^*/E(Y_n|P)\rangle = 2\bar{G}^t(\bar{P} - \bar{P}^*) + \frac{2c_n^2}{3!} T_3 T_{\oplus_n}^{-1}(\bar{P} - \bar{P}^*). \tag{C-12}$$

From Equation C-4, Assumption V-3, and the definition $v = \min_i v_i$,

$$|T_{3,i}^{+}| \leq J^3 K_3 \frac{1}{v^3}. \tag{C-13}$$

Since each element in $T_{\oplus_n}$ has magnitude less than 1,

$$|T_3 T_{\oplus_n}^{-1}(\bar{P} - P^*)| \leq 2\left(\frac{J}{v}\right)^3 K_3 \|P - P^*\|. \tag{C-14}$$

By the definition of $\bar{G}$,

$$\bar{G}^t(P - P^*) = \langle \text{grad } E(R_T'|P)/P - P^*\rangle . \tag{C-15}$$

Applying Assumption V-4 to Equation C-15, and then substituting Equations C-14 and C-15 into Equation C-12 proves the lemma.

Lemma C-2:

If $1 > d_n > 0$, then $\sum_{n=1}^{\infty} d_n < \infty$ if and only if $\prod_{i=1}^{n} (1 - d_i) \to a > 0$.

Proof: $\text{Log} \prod_{i=1}^{n} (1 - d_i) = \sum_{i=1}^{n} \log (1 - d_i) = \sum_{i=1}^{n} y_i$. By the limit comparison test $\sum_{i=1}^{n} y_i$ converges if any only if $\sum_{i=1}^{n} d_i$ converges whenever $\lim_{n\to\infty} (y_n/d_n) \neq 0$. But $\lim_{n\to\infty} (y_n/d_n) = -1$, proving the lemma.

<u>Lemma C-3:</u> Assume that $b_{n+1} \leq b_n d_n + W_n$, where $d_n = 1 - 2k_0 a_n$, and $W_n = W_0 r_n$. Also assume that $\sum_{n=1}^{\infty} a_n \to \infty$, $1 > a_n > 0$, $\sum_{n=1}^{\infty} r_n < \infty$. Then $b_n \to 0$.

<u>Proof:</u> Iterating the hypothesis on b_n $n - n_0$ times gives

$$b_{n+1} \leq b_{n_0} \prod_{k=n_0}^{n} dk + \sum_{k=n_0}^{n-1} W_k \prod_{j=k+1}^{n} d_j. \tag{C-16}$$

Choose n_0 so large that $0 < d_n < 1$ for $n > n_0$. Then

$$\sum_{k=n_0}^{n-1} W_k \prod_{j=k+1}^{n} d_j < \sum_{k=n_0}^{n-1} W_k. \tag{C-17}$$

Since W_n converges, n_0' can be chosen large enough that

$$\sum_{k=n_0}^{n-1} W_k < \varepsilon/2. \tag{C-18}$$

Taking the contrapositive of Lemma C-2 there exists an n_0'' large enough that for all $n > n_0''$,

$$\prod_{k=n_0''}^{n} d_k < \varepsilon/2\, b_{n_0}. \tag{C-19}$$

Then for all $n \geq \max\{n_0, n_0', n_0''\} = N_0$,

$$b_{n+1} \leq b_{n_0}\, \varepsilon/2b_{n_0} + \varepsilon/2 = \varepsilon. \tag{C-20}$$

<u>Lemma C-4 (Sakrison,[2] Inequality A-18):</u> For arbitrary $\varepsilon > 0$

$$E(\|P - P^*\|) \leq \varepsilon + \frac{E(\|P - P^*\|^2)}{\varepsilon} .$$

<u>Proof:</u>

$$E(\|P - P^*\|) = \int_B \|P - P^*\| d\omega + \int_{\bar{B}} \|P - P^*\| d\omega, \tag{C-21}$$

where $B = \{\omega : \|P - P^*\| \leq \varepsilon\}$

$$\leq \varepsilon \ \Pr\{B\} + \frac{1}{\varepsilon} \int \|P - P^*\|^2 d\omega$$

$$\leq \varepsilon + \frac{E(\|P - P^*\|^2)}{\varepsilon} . \tag{C-22}$$

To simplify notation, the conditions on $\mathcal{S}_n$, $\mathcal{S}_f$, and E in $E(R_T'|P,\mathcal{S}_n,\mathcal{S}_f,E)$ are suppressed in the next two lemmas.

<u>Lemma C-5:</u> If V is compact, and the third-order partial derivatives of $E(R_T|H,P)$ are continuous on $V \times \Omega$, then the third-order partial derivatives of $E(R_T'|P,E)$ are bounded over $P \in \Omega$.

<u>Proof:</u> By expansion,

$$E(R_T'|P) = \int dH \ E(R_T|P,H) \ \Pr\{H|D\} . \tag{C-23}$$

Since the third-order partial derivatives of $E(R_T|P,H)$ are continuous, Theorem XIV in Taylor[3] allows interchange of integration and differentiation giving,

$$\frac{\partial^3 E(R_T'|P)}{\partial p_i \partial p_j \partial p_k} = \int dH \ \frac{\partial^3 E(R_T)H,P)}{\partial p_i \partial p_j \partial p_k} \Pr\{H|D\}; \ i,\ j,\ k = 1,\ 2,\ \ldots,\ J. \tag{C-24}$$

Since $\dfrac{\partial^3 E(R_T|P,H)}{\partial p_i \partial p_j \partial p_k}$ is continuous over $V \times \Omega$, a compact space, it is bounded on $V \times \Omega$. If this bound is $K_3(H)$, $K_3(H)$ is also bounded because $V \times \Omega$ is compact. Let this bound be K_3. Then Equation C-24 yields,

$$\frac{\partial^3 E(R_T'|P)}{\partial p_i \partial p_j \partial p_k} \le K_3 \int dH \; Pr\{H|D\} = K_3 , \tag{C-25}$$

proving this lemma.

The next lemma shows that Assumption V-4 is satisfied if the second partial derivatives of the conditional payoff surfaces are negative definite over Ω.

Lemma C-6: If $Pr\{H\}$ has compact support V, and

(a) $E(R|H,P)$ has continuous first-order partial derivatives in P on $V \times \Omega$ and

(b) the matrix $Q(H,P) = \left[\dfrac{\partial^2 E(R_T|H,P)}{\partial p_i \partial p_k}\right]$ is negative definite in P for all $H \in V$, then there exists $K_0 > 0$ such that for all $P \in \Omega$, $-\langle \text{grad } E(R_T'|P)/P - P^* \rangle \ge K_0 \|P - P^*\|^2$.

Proof: The Taylor expansion of $E(R_T|H,P)$ about P^*_Ω is,

$$E(R_T|H,P) = E(R_T|H,P^*_\Omega) + \langle \text{grad } E(R_T/H,P^*_\Omega)/P - P^*_\Omega \rangle$$
$$+ \frac{1}{2} \langle P - P^*/Q(H,P^*_\Omega + \alpha(P - P^*_\Omega))/P - P^*_\Omega \rangle, \tag{C-26}$$

for some $\alpha \in [0,1]$. The matrix Q can be replaced with the real symmetric matrix $S = (1/2)(Q + Q^T)$ without changing the value of Equation C-26.[4]

Since S is real and symmetric, there is a unitary transformation, T, that produces,

$$S = T\Phi(H, P^*_\Omega + \alpha(P - P^*_\Omega))T^t, \tag{C-27}$$

where Φ is a diagonal matrix of eigenvalues of S. The matrix S is negative definite if and only if it has negative eigenvalues,[5] $\phi_i(H, P^*_\Omega + \alpha(P - P^*_\Omega))$, $i = 1, \ldots, J$. Letting,

$$- K_0(H,P) = \max_i \phi_i, \tag{C-28}$$

it follows that $K_0(H,P) > 0$. Since V and Ω are compact,

$$\inf\{K_0(H,P) : H\epsilon V, P\epsilon\Omega\} = K_0 > 0.$$

Substituting Equation C-27 into the quadratic form in Equation C-26 (the third term) gives,

$$\text{Quad }(H,P) = \frac{1}{2}\langle P - P^*/T\Phi T^t(P - P^*)\rangle$$

$$= \frac{1}{2}\langle T^t(P - P^*)/\Phi/T^t(P - P^*)\rangle. \tag{C-29}$$

Letting $P' = T^t(P - P^*)$, Equation C-29 becomes,

$$\text{Quad }(H,P) = \frac{1}{2}\sum_{i=1}^{J}\phi_i(p_i')^2$$

$$\leq - K_0\frac{1}{2}\sum_{i=1}^{J}(p_i')^2$$

$$= - K_0\langle T^t(P - P^*)/T^t(P - P^*)\rangle$$

$$= - K_0\langle(P - P^*)/TT^t(P - P^*)\rangle. \tag{C-30}$$

Since T is unitary,

$$\text{Quad }(H,P) \leq - K_0\|P - P^*\|^2. \tag{C-31}$$

Substituting Inequality C-31 into Equation C-26,

$$E(R_T|H,P) - E(R_T|H,P^*) \leq \langle \text{grad } E(R_T|H,P^*)/P - P^*\rangle - K_0\|P - P^*\|^2. \tag{C-32}$$

Taking the expected values of Inequality C-32 over the posterior distribution on H gives,

$$E(R_T'|P) - E(R_T'|P^*) \leq - K_0\|P - P^*\|^2. \tag{C-33}$$

Condition b implies that $E(R_T|H,P)$ is concave in P.[6] Thus for all H and P_1, P_2,

$$\alpha E(R_T|H,P_1) + (1 - \alpha)E(R_T|H,P_2) \leq E(R_T|H,\alpha P_1 + (1 - \alpha)P_2). \tag{C-34}$$

Multiplying Inequality C-34 by the likelihood, $Pr\{D|H\}$, and taking the expected value of Inequality C-34 over the prior distribution on H, proves that $E(R_T'|P)$ is concave in P for all H.

A necessary and sufficient condition for the concavity of $E(R_T'|P)$[7] is that for P_1 and P_2,

$$E(R_T'|P_1) - E(R_T'|P_2) \geq \langle \text{grad } E(R_T'|P_1)/P_1 - P_2\rangle. \tag{C-35}$$

Letting $P_2 = P^*$ and $P_1 = P$ in Inequality C-35, and applying it in conjunction with Inequality C-33 completes the proof of this lemma.

<u>Lemma C-7:</u> Let Pr{H} have compact support, V. If $E(R_T|H,P)$ has continuous second-order partial derivatives in P on $V \times \Omega$, then there exists $K_1 > 0$ such that $\|\text{grad } E(R_T|P)\| \leq K_1\|P - P^*_\Omega\|$.

<u>Proof:</u> Since $E(R_T|H,P)$ has continuous second-order partial derivatives in P on $V \times \Omega$ a compact space, there exists K_1 such that for all $i,j = 1,2,\ldots J$, $\left|\frac{\partial^2 E(R_T|H,P)}{\partial p_i \partial p_j}\right| \leq K_1$. Then letting $p_{j,i}$ be the j^{th} component of P_i for some $0 < \alpha < 1$,

$$\frac{\partial E(R_T|H,P_1)}{\partial p_i} = \frac{\partial E(R_T|H,P_2)}{\partial p_i} + \sum_{j=1}^{J} \frac{\partial^2 E(R_T|H,\alpha P_2 + (1-\alpha)P_1)}{\partial p_i \partial p_j} \cdot (p_{j,1} - p_{j,2}). \tag{C-36}$$

From the boundedness of the second it follows from C-36 that,

$$\|\text{grad } E(R_T|H,P_1) - \text{grad } E(R_T|H,P_2)\| \leq K_1\|P_1 - P_2\|. \tag{C-37}$$

Setting $P_2 = P^*_\Omega$ and $P_1 = P$, Inequality C-37 becomes,

$$\|\text{grad } E(R_T|H,P) - \text{grad } E(R_T|H,P^*_\Omega)\| \leq K_1\|P - P^*_\Omega\|. \tag{C-38}$$

Taking the expected value of Inequality C-38 with respect to the posterior distribution on H, and using the relation $E(\|x\|) \geq \|E(x)\|$ gives,

$$\|E(\text{grad } E(R_T|H,P)) - E(\text{grad } E(R_T|H,P^*))\| \leq K_1\|P - P^*\|. \tag{C-39}$$

By the continuity of the first-order partial derivatives, expectation and differentiation can be interchanged in Inequality C-39:

$$\|\text{grad } E(R_T'|P) - \text{grad } E(R_T'|P_\Omega^*)\| \leq K_1\|P - P^*\|. \qquad \text{(C-40)}$$

By the definition of P_Ω^*, grad $E(R_T'|P_\Omega^*) = 0$, proving the lemma.

FOOTNOTES--APPENDIX C

1. Block, A. D., Introduction to Tensor Analysis, Merrill Books, Columbus, Ohio, 1962, p. 8.

2. Sakrison, D. J., op. cit., p. 64.

3. Taylor, A. E., Advanced Calculus, Ginn, Boston, 1955.

4. Karlin, S., Mathematical Methods in Games, Programming, and Economics, Addison Wesley, Reading, Mass., 1959, p. 381.

5. Ibid., p. 382.

6. Ibid., p. 406.

7. Ibid., p. 404.