Spintronics Note

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1 The Quantum Mechanics of Spin

1.1 Pauli Matrices

We have the following relations

$$\boxed{\begin{aligned}&[S_i, S_j] = \varepsilon_{ijk} i\hbar S_k\\ &S^2 |s,m\rangle = \hbar^2 s(s+1)|s,m\rangle\\ &S_z|s,m\rangle = \hbar m|s,m\rangle\\ &S_\pm |s,m\rangle = (S_x \pm iS_y)|s,m\rangle = \hbar\sqrt{s(s+1)-m(m\pm 1)}\,|s,m\pm 1\rangle\end{aligned}} \tag{1.1a}$$

We have two states for spin 1/2

$$\alpha = \begin{pmatrix}1\\0\end{pmatrix}; \beta = \begin{pmatrix}0\\1\end{pmatrix}$$

And we can know

$$S^2\alpha = \frac{3}{4}\hbar^2\alpha; S^2\beta = \frac{3}{4}\hbar^2\beta$$

Let $S^2 = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$, then we can get

$$S^2 = \frac{3}{4}\hbar^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Similarly, we can get

$$S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}; S_+ = \hbar\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}; S_- = \hbar\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

Or

$$S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}; S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}; S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix};$$

And we let $S = \frac{\hbar}{2}\sigma$, where σ is what we call Pauli Matrix

$$\boxed{\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}; \sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}; \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix};}$$

Or to make it easer to remember, we get a generally form

$$\sigma_a = \begin{pmatrix}\delta_{a3} & \delta_{a1} - i\delta_{a2}\\ \delta_{a1} + i\delta a2 & -\delta_{a3}\end{pmatrix}$$

where δ_{ai} is Kronecker delta.

After we get the Pauli matrices, we can correspondingly get their eigenstates which are:

$$\psi_{z+} = \begin{pmatrix}1\\0\end{pmatrix}; \psi_{z-} = \begin{pmatrix}0\\1\end{pmatrix}$$
$$\psi_{x+} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}; \psi_{x-} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$
$$\psi_{y+} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}; \psi_{y-} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$

1.2 The Pauli Equation and Spinors

We can set the electron' wavefunction as

$$[\psi(\boldsymbol{x})] = \begin{bmatrix} \phi_1(\boldsymbol{x}) \\ \phi_2(\boldsymbol{x}) \end{bmatrix} \tag{1.2}$$

Where $\boldsymbol{x} = (x, y, z, t)$. Then we can recast the Schrodinger Equation

$$\left(\hat{H} - i\hbar\frac{\partial}{\partial t}[I]\right)[\psi(\boldsymbol{x})] = [0] \tag{1.3}$$

This is a set of two simultaneous differential equations for the two components of the spinor wavefunction–ϕ_1 and ϕ_2. And this is referred to as the *Pauli Equation*.

Right now, we can calculate the expected value of the spin components with special expressions.

$$\langle S_n \rangle = \frac{\hbar}{2}[\psi(\boldsymbol{r}, t)]^\dagger \sigma_n [\psi(\boldsymbol{r}, t)] \tag{1.4}$$

Specifically,

$$\langle S_x \rangle = \frac{\hbar}{2}\begin{bmatrix} \phi_1^\dagger(\boldsymbol{x}) & \phi_2^\dagger(\boldsymbol{x}) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \phi_1(\boldsymbol{x}) \\ \phi_2(\boldsymbol{x}) \end{bmatrix} = \hbar Re(\phi_1^\dagger(\boldsymbol{x})\phi_2(\boldsymbol{x})) \tag{1.5}$$

$$\langle S_y \rangle = \hbar Im(\phi_1^\dagger(\boldsymbol{x})\phi_2(\boldsymbol{x})) \tag{1.6}$$

$$\langle S_z \rangle = \hbar(|\phi_1(\boldsymbol{x})|^2 - |\phi_2(\boldsymbol{x})|^2) \tag{1.7}$$