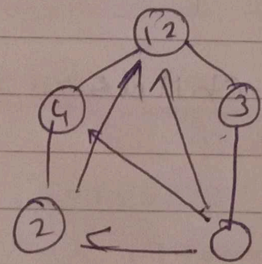
Unit

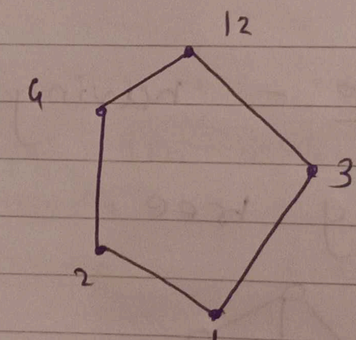
# * Lattices & Boolean Algebra *

- **Partially ordered sets** -
  - A relation R on a set A is called partial order if R is reflexive, anti-symmetric and transitive.
  - A set A together with the partial order R is called a partially ordered set, or simply a poset, denoted by (A, R).

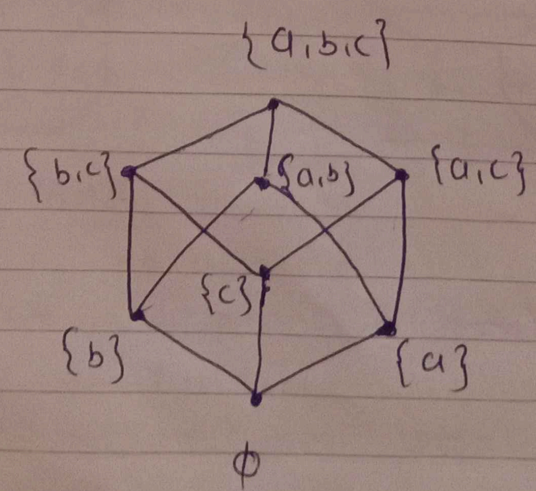
eg.

① let A = {1, 2, 3, 4, 12}. Consider the partial order of divisibility on A. Draw the corresponding Hasse diagram.




② S = {a, b, c} & A = P(S). Draw the Hasse diagram of the poset A with the partial order '⊆'.

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- <u>Comparable</u> -

If $(A, \leq)$ is a poset, elements $a$ & $b$ are comparable if $a \leq b$ or $b \leq a$.

<u>Note</u> - if every pair of elements in a poset A is comparable, we say that A is linear ordered set, and the partial order is called linear order. we also say that A is a chain or totally ordered set.

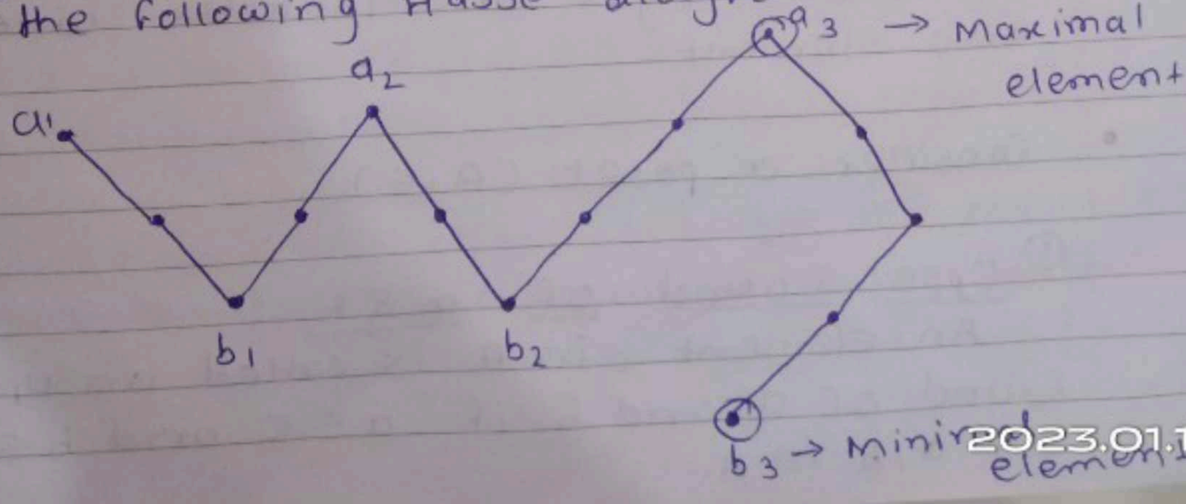
- ~~Extrem.~~ <u>Extremal Elements of Partially ordered sets</u>.

consider a poset $(A, \leq)$

(1) <u>Maximal Element</u> - An element $a$ in A is called a maximal element of A if there is no element $c$ in A such that $a \leq c$.

(2) <u>Minimal element</u> - An element $b$ in A is called a minimal element of A if there is no element $c$ in A such that $c \leq b$

eg. Find the maximal & minimal elements in the following Hasse diagram.



eg. ② A be the poset of nonnegative real number with the usual partial order $\leq$. Then 0 is a minimal element of A & there is no maximal elements of A.

③ The poset $\mathbb{Z}$ with the usual partial order $\leq$ has no maximal element & has no minimal elements.

- <u>Greatest element</u> :
  An element a in A is called a greatest element of A if $x \leq a$ for all x in A.

- <u>Least element</u> :
  An element a in A is called a least element of A if $a \leq x$ for all x in A.

<u>Note</u> - an element a of $(A, \leq)$ is greatest (or least) element iff it is a least (or greatest) element of $(A, \geq)$.

- <u>Unit element</u> -
  The greatest element of a poset, if it exists, it denoted by 1 and is often called the unit element.

- <u>zero element</u> -
  The least element of a poset if it exists is denoted by 0 and is often called the zero element.

- Consider a poset $(A, \leq)$

① <u>Upper bound of a & b</u> :
An element c in A is called an upper bound of a and b if $a \leq c$ and $b \leq c$ all a, b in A.

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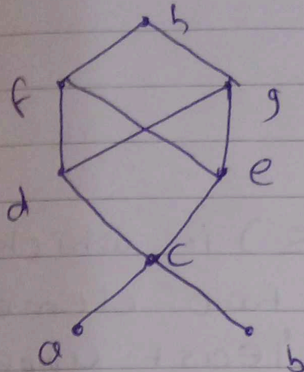
② Lower bound of a and b:

An element d in A is called a lower bound of a and b if $d \le a$ & $d \le b$ for all a, b in A.

eg. find the upper and lower bound of the following subset of A:

(a) $B_1 = \{a, b\}$ ; $B_2 = \{c, d, e\}$.




- $B_1$ has no lower bound, The upper bounds of $B_1$ are c, d, e, f, g and h.

- $B_2$ having lower bound c, a & b & The upper bounds of $B_2$ are f, g & h.

- Consider a poset $(A, \le)$ and a, b in A.

① Least upper bound -

An element c in A is called a least upper bound of a and b, if ⇒ c

① c is an upper bound ~~then c ≤ c'~~ of a and b; i.e. $a \le c$ & $b \le c$

② if c' is another upper bound then $c \le c'$.

② Greatest lower bound.

An element g in A is called a greatest lower bound of a and b, if (i) g

① g is a lower bound of a & b; i.e $g \le a$ & $g \le b$

② if g' is another lower bound then $g' \le g$.

eg. let $A = \{1, 2, 3 - - - 11\}$ be the ~~m~~ poset whose Hasse diagram is shown below. find the LUB and GLB of $B = \{6, 7, 10\}$,

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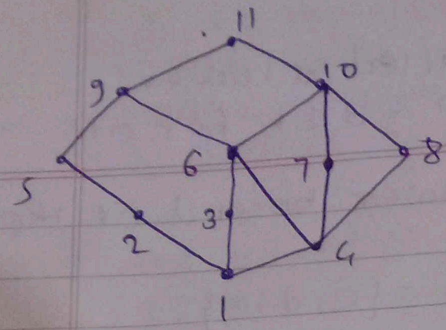



- The upperbounds of B ar 10 & 11
& LUB (B) is 10

- The lower bounds of B are 1, 4 and GLB (B) is 4

★ Lattices :

- Lattices as a poset,

A lattices is a poset $(L, \leq)$ in which every subset $\{a, b\}$ consisting of two elements has a ~~least low~~ Greatest. a 'least upper bound' and a 'greatest lower bound'.

we denote

- $LUB(\{a,b\})$ by $a \oplus b$ (the join or sum of a and b).
- $GLB(\{a, b\})$ by $a * b$ (the meet or product of a and b).

Eg.

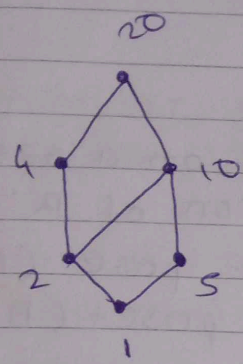
let S be a set and let $L = P(S)$.
$\subseteq$ is a partial order relation on L.
let A and B belong to the poset $(L, \subseteq)$.
Then,
$a \oplus b = A \cup B$ & $a * b = A \cap B$.

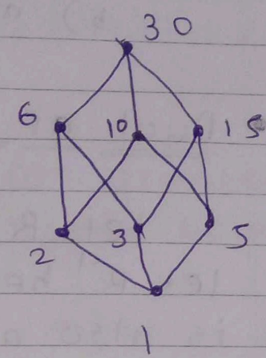
→ Assuming c is a lower bound of $\{a, b\}$
then $A \subseteq C$ and $B \subseteq C$ thus $A \cup B \subseteq C$

Assuming c is a lower bound of $\{a, b\}$
then $C \subseteq A$ and $C \subseteq B$ thus
$C \subseteq A \cap B$.

eg. let $n$ be a positive integer and $D_n$ be the set of all positive divisors of $n$. Then $D_n$ is a lattice under the relation of divisibility. For instance.

e- $D_{20} = \{1, 2, 4, 5, 10, 20\}$ $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 20\}$




- **Properties of lattices :**

① $(a \oplus b) \oplus c = a \oplus (b \oplus c) = a \oplus b \oplus c$

② $(a * b) * c = a * (b * c) = a * b * c$.

③ $LUB(\{a_1, a_2, \ldots, a_n\}) = a_1 \oplus a_2 \oplus \ldots \oplus a_n$

④ $GLB(\{a_1, a_2, \ldots, a_n\}) = a_1 * a_2 * \ldots * a_n$

- **Important Result :**

let $L$ be a lattice. Then for every $a$ and $b$ in $L$.

a) $a \oplus b = b$ if and only if $a \leq b$

b) $a * b = a$ if and only if $a \leq b$

c) $a * b = a$ if and only if $a \oplus b = b$

- **Important Result : isotonicity.**

let $L$ be a lattice. Then, for every $a, b$ & $c$ in $L$.

1. If $b \leq c$, then

   a) $a \oplus b \leq a \oplus c$

   b) $a * b \leq a * c$.

2) $a \le b$ and $a \le c$ if and only if $a \le b \oplus c$

3) $a \le b$ and $a \le c$ if and only if $a \le b * c$

4) If $a \le b$ and $c \le d$, then

a) $a \oplus c \le b \oplus d$

b) $a * c \le b * d$

- **Dual of a Lattice :**

Let R be a partial order on a set A, and let $R^{-1}$ be the inverse relation of R. Then $R^{-1}$ is also a partial order. The poset $(A, R^{-1})$ is called the dual of the poset $(A, R)$.

eg.

Let $(L, \le)$ be a lattice, then the $(L, \ge)$ is called dual lattice of $(L, \le)$.

- In $(L, \le)$, if $a \oplus b = c$ ; $a * b = d$, then in dual lattice $(L, \ge)$, $a \oplus b = d$ ; $a * b = c$

- **Principle of duality :**

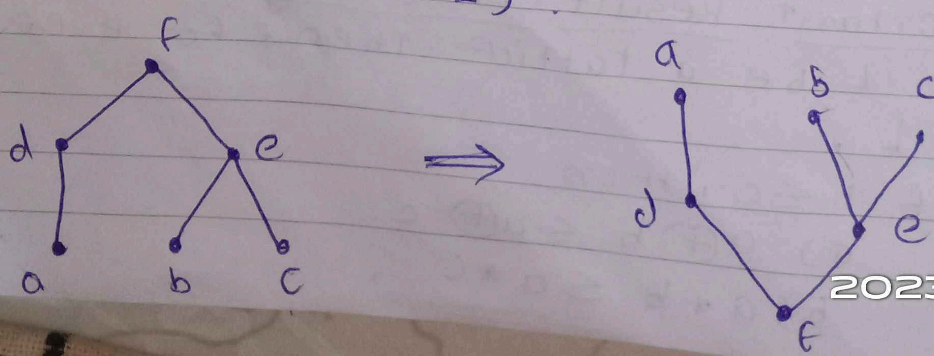
If P is a valid statement in a lattice then the statement obtained by interchanging meet and join everywhere and replacing $\le$ by $\ge$ is also a valid statement

eg.

a) Show the Hasse diagram of a poset $(A, \le)$, where,

$$A = \{a, b, c, d, e, f\}$$

b) Show the Hasse diagram of the dual poset $(A, \ge)$.



## Some properties of dual of poset :

1) The upper bounds in $(A, \leq)$ correspond to lower bounds in $(A, \geq)$ (for the same set of element).

2) The lower bounds in $(A, \leq)$ correspond to upper bounds in $(A, \geq)$ (for the same set of elements).

3) Similar statements hold for greatest lower bounds and least upper bounds.

Note – An element a of $(A, \leq)$ is a greatest (or least) element if and only if it is a least (or greatest) element of $(A, \geq)$.

## Lattices as algebraic structures :

- An algebric structure $(L, \vee, \wedge)$, consisting of a set L and two binary operations $\vee$ and $\wedge$, on L is a lattice if the following axiomatic identities hold for all elements a, b, c of L

① Commutative laws –

$a \oplus b = b \oplus a$ , $a * b = b * a$ .

② Associative laws ,

$a \oplus (b \oplus c) = (a \oplus b) \oplus c$

$a * (b * c) = (a * b) * c$ .

③ Absorption laws

$a \oplus (a * b) = a$

$a * (a \oplus b) = a$

Idempotent law.

④ $a \oplus a = a$

$a * a = a$

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- **Sub-Lattices :**

- Consider a non-empty subset $L_1$ of a lattice $<L, *, \oplus>$. Then $L_1$ is called a sub-lattice of $L$ if $L_1$ itself is a lattice i.e, the operation of $L$ i.e. $a \oplus b \in L_1$ and $a * b \in L_1$ whenever $a \in L_1$ & $b \in L_1$. $<L_1, *, \oplus>$

eg.

① consider the lattice of all +ve integers $I_+$ under the operation of divisibility. The lattice $D_n$ of all divisors of $n > 1$ is a sublattice of $I_+$

- Determine all the sub lattices of $D_{30}$ that contain at least four elements,

$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$.

→ The sub lattices of $D_{30}$ that contain at least four elements are as follows.

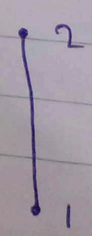
1) $\{1, 2, 6, 30\}$ 2) $\{1, 2, 3, 30\}$
3) $\{1, 5, 15, 30\}$ 4) $\{1, 3, 6, 30\}$
5) $\{1, 5, 10, 30\}$ 6) $\{1, 3, 15, 30\}$.
7) $\{2, 6, 10, 30\}$

- **Direct product of lattices :**

Let $(L_1, \oplus_1, *_1)$ and $(L_2, \oplus_2, *_2)$ be two lattices. Then $(L, *, \oplus)$ is the direct product of lattices, where $L = L_1 \times L_2$ in which the binary operation $\oplus$ (Join) & $*$ (meet) on $L$ are such that for any $(a_1, b_1)$ and $(a_2, b_2)$ in $L$.

- $(a_1, b_1) \oplus (a_2, b_2) = (a_1 \oplus_1 a_2, b_1 \oplus_2 b_2)$
- $(a_1, b_1) * (a_2, b_2) = (a_1 *_1 a_2, b_1 *_2 b_2)$

eg. consider a lattice $(L, \leq)$ where $L = \{1, 2\}$. Determine the lattices $(L^2, \leq)$, where $L^2 = L \times L$.




$(L, \leq)$

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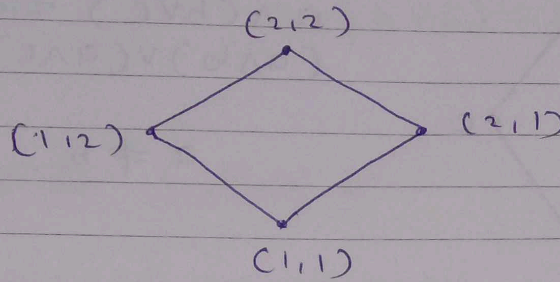
$\therefore L^2 = L \times L$

$\therefore \{1,2\} \oplus \{1,2\} = \{1 \oplus 1, 2 \oplus 2\} = \{2, 2\}$

$\{1,2\} * \{1,2\} = \{1 * 1, 2 * 2\} = \{1, 1\}$

$\therefore$ The lattice $(L^2, \leq)$

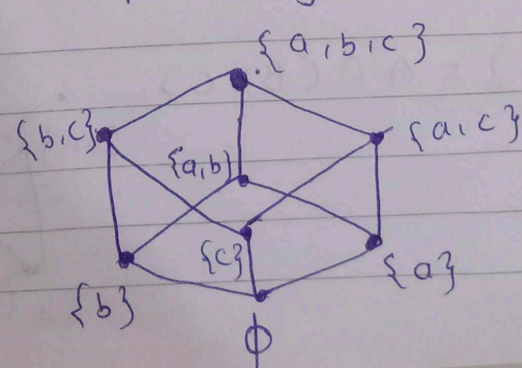



- **Bounded Lattices :**
  A lattice L is said to be bounded if it has a greatest element 1 and least element 0.

eg:
① The lattice P(S) of all subsets of a set S, with the relation containment is bounded. The greatest is S & the least element is empty set.

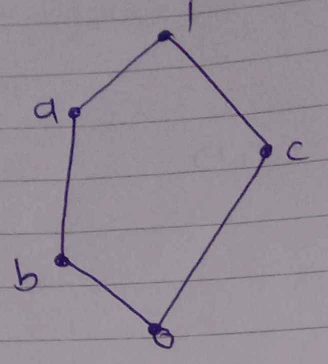
- **Distributive lattices :**
  A lattice $(L, \leq)$ is called distributive if for any elements a, b & c in L we have following distributive properties.
  1. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
  2. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

eg:
For a set S, the lattice P(S) is distributive, since join and meet each satisfy distributive property.



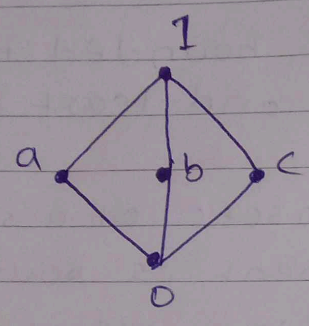
$\vee \rightarrow \oplus \rightarrow$ lowest upper bound
$\wedge \rightarrow * \rightarrow$ Greatest lower bound.

eg. show that the lattices as follows are non distributive.




$a \wedge (b \vee c) = a \wedge 1 = a$
$(a \wedge b) \vee (a \wedge c) = b \vee 0 = b$

$a \neq b$




$a \wedge (b \vee c) = a \wedge I = a$
$(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$

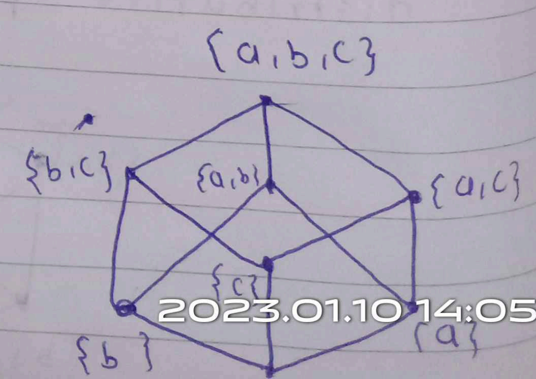
Note: The distributive property holds when,
a) any two of the elements a, b and c are equal (or)
b) when any one of the element is 0 or I

- **Modular Lattices :-**

A lattice $(L, \leq)$ is called modular if for any elements a, b and c in L if $b \leq a$ then
$b \vee (a \wedge c) = a \wedge (b \vee c)$

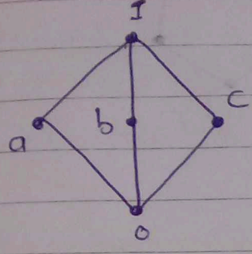
eg.
For a set S, the lattice P(S) is modular. (if $B \subseteq A$).

$B \cup (A \cap C) = A \cap (B \cup C)$



eg. ⑤ Every chain is a modular lattice

① Given Hasse diagram of a lattice which is modular.




$0 \le a$ i.e. taking $b = 0$

$b \vee (a \wedge c) = 0 \vee 0 = 0$

$a \wedge (b \vee c) = a \wedge c = 0$

- **Complemented Lattice :-**

- Complement of an element -

let L be bounded lattice with greatest element 1 and least element 0, and let a in L. An element b in L is called a complement of a if,

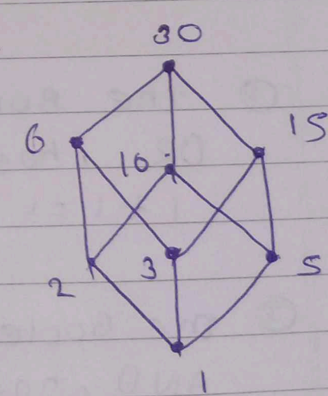
$a \vee b = 1$ and $a \wedge b = 0$ Note: $0' = 1$ & $1' = 0$

- Complemented Lattice :

A lattice L is said to be complemented if it is bounded and every element in it has a complement.

eg.

| Element | It's complement |
|---|---|
| 1 | 30 |
| 2 | 15 |
| ~~3~~ 5 | 10 |
| 5 | 6 |
| 6 | 5 |
| 10 | 3 |
| 15 | 2 |
| 30 | 1 |

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