DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING
Second Year B.Tech. (SEM - III)
COMPUTATIONAL MATHEMATICS (UCSE0301)

Unit No. 5: Introduction to Fuzzy sets

Crisp (Classic) Sets:

A set is a collection of objects together with some rule to determine whether a given object belong to this collection. Any object of this collection is called an element of the set. Usually, sets are denoted in uppercase (e.g., a set A, B, ……), whereas objects are in lowercase (e.g., an object x, y, ……).

In the case of classic sets, a given object x may belong to a set A (be a member of a set A), or not belong to this set (not be a member of this set), and these two options are denoted by $x \in A$ and $x \notin A$.

A classic set may be described by means of the characteristic function (χ_A) that takes two values: 1 (for the object belonging to a set A), and 0 (for the object not belonging to a set A).

$$\chi_A(x) = 1 \quad , \ if \quad x \in A$$
$$= 0 \quad , \ if \quad x \notin A$$

Note: Crisp means sharp and clear.

Types of set:

1. Subset: A set A is said to be a subset of B if every element of A is an element of B, we use the expression $A \subseteq B$.

2. Equal set: Two sets, A and B are said to be equal if and only if A is a subset of B and B is a subset of A. We use the symbol $A = B$. Also $A \neq B$ means that A and B are not equal sets.

3. Empty set: A set containing no element is called the empty set or null set and is denoted by the symbol $A = \phi$.

4. Proper set: A is said to be proper subset of B if and only if:

(a) $A \subseteq B$ (b) $A \neq B$ (c) $A \neq \phi$. Also it is denoted by $A \subset B$.

5. Universal Set: A set that contains all the possible elements we interested in.

6. Power set: The set of all subsets of A is called the power set of A and denoted by P (A).

Operation on Sets:

1. Union: The set of elements which belong to A or B or both is called the union of A and B and it is denoted by $A \cup B$. It is defined as,

$$A \cup B = \{x \in X / x \in A \text{ or } x \in B\}$$

2. Intersection: The set of elements which belong to A both B is called the intersection of A and B and it is denoted by $A \cap B$. It is defined as

$$A \cap B = \{x \in X / x \in A \text{ and } x \in B\}$$

3. Complement of a set: Let A be a subset of a universal set X, then the set of all those elements of X which do not belong to A is called the complement of A, it is denoted by A^C or $\overline{A}$. It is defined as,

$$\overline{A} = \{x / x \notin A \text{ and } x \in X\}$$

Some Properties of Crisp set:

Sr. No.	Properties	Let A, B and C is finite sets. Then
1	Involution	$\overline{\overline{A}} = A$
2	Commutative	$A \cup B = B \cup A$ and $A \cap B = B \cap A$
3	Associative	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$
4	Distributive	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
5	Idempotent	$A \cup A = A$ and $A \cap A = A$
6	Absorption	$A \cup (A \cap B) = A$ and $A \cap (A \cup B) = A$
7	Identity	$A \cup \phi = A$ and $A \cap X = A$
8	Law of contradiction	$A \cap \overline{A} = \phi$
9	Law of Excluded Middle	$A \cup \overline{A} = X$
10	De' Morgan's Law	$\overline{A \cup B} = \overline{A} \cap \overline{B}$ and $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Fuzzy Set:

A function that maps elements of a given universal set X in to **real number in [0, 1]** that function is called **membership function** and it represented by following notations,

$$\mu_A : X \to [0, 1] \text{ or } A : X \to [0, 1]$$

A set defined by membership functions is called **Fuzzy set**.

Thus a fuzzy set is a pair (X, μ_A) where X is a reference set and is called **universe of discourse**, and for each $x \in X$ the value $\mu_A(x)$ is called the **grade** of membership of x in (X, μ_A).

A fuzzy set is simply denoted by A instead of μ_A.

Let $x \in X$ Then x is called

a) Not included in the fuzzy set (X, μ_A) if A (x) = 0 (no member),

b) Fully included if A (x) = 1 (full member),

c) Partially included if $0 < A(x) < 1$ (fuzzy member).

For example:

1. We are represented a fuzzy set that person is very tall.

Let A is height of person in feet, then universal set X ={ 0,1,2,3,4,5,6,7}

$$A: X \to [0, 1]$$

$$A(x) = \begin{cases} 1 & if\ x \geq 6 \\ \dfrac{x-3}{3} & if\ 3 < x < 6 \\ 0 & if\ x \leq 3 \end{cases}$$

2. We are represented a fuzzy set that student is highly irregular.

Let B is set of irregular student, then universal set X = {0, 1, 2……., 50}

$$B: X \to [0, 1]$$

$$B(x) = \begin{cases} 1 & if\ x < 10 \\ \dfrac{40-x}{30} & if\ 10 \leq x \leq 40 \\ 0 & if\ x > 40 \end{cases}$$

Note: 1.A notation for fuzzy sets for discrete universe X: $A = \sum_{x \in X} \frac{\mu_A(x)}{x}$

2. A notation for fuzzy sets for continuous universe X: $A = \int_{x \in X} \frac{\mu_A(x)}{x}$

General Definitions:

1. α-cut: Let a fuzzy set A defined on universal set X and any number $\alpha \in [0, 1]$ we define α - cut of A as, $^{\alpha}A = \{x \in X / A(x) \geq \alpha\}$

2. Strong α-cut: Let a fuzzy set A defined on universal set X and any number $\alpha \in [0, 1]$ we define strong α - cut of A as, $^{\alpha+}A = \{x \in X / A(x) > \alpha\}$

3. Level set of fuzzy set: The set of all levels $\alpha \in [0, 1]$ that represent distinct α - cuts of a given fuzzy A is called a level set of A, we define level set of A as,

$$\Lambda A = \{\alpha / A(x) = \alpha \ \textit{for some}\ x \in X\}$$

4. Support of fuzzy set: Let a fuzzy set A defined on universal set X. The set of all elements whose membership value are non negative is called a Support of fuzzy set A and defined as,

$$Supp\ A = {}^{0+}A = \{x \in X / A(x) > 0\}$$

5. Core of fuzzy set: Let a fuzzy set A defined on universal set X. The set of all elements whose membership value is one is Core of A and defined as,

$$Core\ A = {}^{1}A = \{x \in X / A(x) = 1\}$$

6. Height of fuzzy set: Let a fuzzy set A defined on universal set X. The height of fuzzy set A is the largest membership grade obtained by any element in that set A and defined as,

$$h(A) = \sup_{x \in X} A(x)$$

A fuzzy set A is called **normal** when h (A) =1 otherwise it is called **subnormal**.

7. Crossover point: A crossover point of a fuzzy set A is a point $x \in X$ at which A(x) = 0.5. This is also referred as **equilibrium point**.

8. Cardinality: The number of elements in a set is the cardinality of that set and it is noted as |A| or n (A).

9. Scalar Cardinality: The scalar cardinality of a fuzzy set is defined on a finite universal set X is the summation of the membership grades of all the elements of A and it is defined as

$$|A| = \sum_{x \in X} A(x)$$

10. Relative Cardinality: The scalar cardinality of a fuzzy set A is defined on a finite universal set X is the ratio of Scalar Cardinality and Cardinality and it is defined as,

$$\| A \| = \frac{|A|}{|x|}$$

Examples

Example 1: If fuzzy set $A(x) = 1 - \left(\frac{x}{10}\right)$, X = [0, 1, 2…10] then find,

1) α-cut of A for α = 0.6
2) Strong α-cut of A for α = 0.7
3) Level set of fuzzy set A.
4) Support of fuzzy set A.
5) Core of fuzzy set A.
6) Height of fuzzy set A.
7) Crossover point of fuzzy set A.
8) Cardinality of fuzzy set A.
9) Scalar Cardinality of fuzzy set A.
10) Relative Cardinality of fuzzy set A.

Solution: Given fuzzy set, $A(x) = 1 - \left(\frac{x}{10}\right)$ for universal set X = [0, 1, 2…10]

Fuzzy set A(x) can be represented as,

$$A(x) = \left\{ \frac{1}{0} + \frac{0.9}{1} + \frac{0.8}{2} + \frac{0.7}{3} + \frac{0.6}{4} + \frac{0.5}{5} + \frac{0.4}{6} + \frac{0.3}{7} + \frac{0.2}{8} + \frac{0.1}{9} + \frac{0}{10} \right\}$$

1) α-cut of A for α = 0.6:

We define α-cut of fuzzy set A as, ${}^{\alpha}A = \{ x \in X / A(x) \geq \alpha \}$

$${}^{0.6}A = \{ x \in X / A(x) \geq 0.6 \} \qquad {}^{0.6}A = \{ 0, 1, 2, 3, 4 \}$$

2) Strong α-cut of A for α = 0.7:

We define strong α-cut of fuzzy set A as, ${}^{\alpha+}A = \{ x \in X / A(x) > \alpha \}$

$${}^{0.7+}A = \{ x \in X / A(x) > 0.7 \} \qquad {}^{0.7+}A = \{ 0, 1, 2 \}$$

3) Level set of fuzzy set A:

We define Level set of fuzzy set A as, $\Lambda A = \{ \alpha / A(x) = \alpha \ \textit{for some } x \in X \}$

$$\Lambda A = \{ 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1 \}$$

4) Support of fuzzy set A:

We define support of fuzzy set A as, $Supp\ A = {}^{0+}A = \{ x \in X / A(x) > 0 \}$

$$Supp\ A = {}^{0+}A = \{ 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 \}$$

5) Core of fuzzy set A:

We define core of fuzzy set A as, $Core\ A = {}^{1}A = \{ x \in X / A(x) = 1 \}$

$$Core\ A = {}^{1}A = \{ 0 \}$$

6) Height of fuzzy set A:

We define height of fuzzy set A as, $h(A) = \sup_{x \in X} A(x)$

$$h(A) = 1$$

Here, fuzzy set A is **normal fuzzy set.**

7) Crossover point of fuzzy set A:

A crossover point of a fuzzy set A is a point $x \in X$ at which A(x) = 0.5.

Here, x = 5 is crossover point (equilibrium point) of fuzzy set A.

8) Cardinality of fuzzy set A:

The number of elements in a set is the cardinality of that set.

$|x| = 11.$

9) Scalar Cardinality of fuzzy set A:

We define scalar cardinality of fuzzy set A as, $|A| = \sum_{x \in X} A(x)$

$$|A| = 0 + 0.1 + 0.2 + 0.3 + 0.4 + 0.5 + 0.6 + 0.7 + 0.8 + 0.9 + 1$$

$$\therefore |A| = 5.5$$

10) Relative Cardinality of fuzzy set A:

We define Relative Cardinality of fuzzy set A as, $\| A \| = \frac{|A|}{|x|}$

$$\therefore \| A \| = \frac{5.5}{11} = 0.5$$

Example 2: Determine α-cut and Strong α-cut of fuzzy set A for α = 0.2.

$$Where, \quad A(x) = \begin{cases} \frac{x+1}{2} & if\ -1 < x \le 1 \\ \frac{3-x}{2} & if\ 1 < x \le 3 \\ 0 & otherwise \end{cases}$$

Solution: Given fuzzy set, $A(x) = \begin{cases} \frac{x+1}{2} & if\ -1 < x \le 1 \\ \frac{3-x}{2} & if\ 1 < x \le 3 \\ 0 & otherwise \end{cases}$

We know that α-cut of fuzzy set A as, ${}^{\alpha}A = \{ x \in X / A(x) \geq \alpha \}$

$$A(x) \geq \alpha \qquad\qquad A(x) \geq \alpha$$

$$\frac{x+1}{2} \geq \alpha \qquad\qquad \frac{3-x}{2} \geq \alpha$$

$$x+1 \geq 2\alpha \qquad\qquad 3-x \geq 2\alpha$$

$$x \geq 2\alpha - 1 \text{..........(1)} \qquad\qquad 3-2\alpha \geq x \text{..........(2)}$$

From equation (1) and (2), we get

$${}^{\alpha}A = [2\alpha - 1, \quad 3 - 2\alpha]$$

By putting α = 0.2 we get,

$${}^{0.2}A = [-0.6, \quad 2.6]$$

We know that strong α-cut of fuzzy set A as, ${}^{\alpha+}A = \{ x \in X / A(x) > \alpha \}$

$$A(x) > \alpha \qquad\qquad A(x) > \alpha$$

$$\frac{x+1}{2} > \alpha \qquad\qquad \frac{3-x}{2} > \alpha$$

$$x+1 > 2\alpha \qquad\qquad 3-x > 2\alpha$$

$$x > 2\alpha - 1 \text{..........(3)} \qquad\qquad 3-2\alpha > x \text{..........(4)}$$

From equation (3) and (4), we get

$${}^{\alpha+}A = (2\alpha - 1, \quad 3 - 2\alpha)$$

By putting α = 0.2 we get,

$${}^{0.2+}A = (-0.6, \quad 2.6)$$

Example 3: Determine α-cut and Strong α-cut of fuzzy set A for α = 0.3.

$$Where, \quad A(x) = \frac{x}{x+2} \qquad x \in X\,[0,10]$$

Solution: We know that α-cut of fuzzy set A as, ${}^{\alpha}A = \{ x \in X / A(x) \geq \alpha \}$

$$A(x) \geq \alpha$$

$$\frac{x}{x+2} \geq \alpha$$

$$x \geq x\alpha + 2\alpha \quad \Rightarrow x - x\alpha \geq 2\alpha$$

$$x(1-\alpha) \geq 2\alpha$$

$$x \geq \frac{2\alpha}{1-\alpha}$$

$${}^{\alpha}A = \left[\frac{2\alpha}{1-\alpha}, 10\right]$$

By putting α = 0.3 we get,

$$^{0.3}A = [0.8571, 10]$$

We know that strong α-cut of fuzzy set A as, $^{\alpha+}A = \{x \in X / A(x) > \alpha\}$

$$A(x) > \alpha$$

$$\frac{x}{x+2} > \alpha$$

$$x > x\alpha + 2\alpha \quad \Rightarrow x - x\alpha > 2\alpha$$

$$x(1-\alpha) > 2\alpha$$

$$x > \frac{2\alpha}{1-\alpha}$$

$$^{\alpha+}A = (\frac{2\alpha}{1-\alpha}, 10]$$

By putting α = 0.3 we get,

$$^{0.3+}A = (0.8571, 10]$$

Example 4: Find the Scalar Cardinality of fuzzy sets A and B which are defined as follows,

$$A(x) = 2^{-x}, \quad B(x) = \frac{3x+5}{4x+7} \quad \text{for } x \in \{0, 1, 2, 3, \ldots\ldots, 10\}.$$

Solution: Fuzzy set A(x) can be represented as,

$$A(x) = \left\{\frac{1}{0} + \frac{0.5}{1} + \frac{0.25}{2} + \frac{0.125}{3} + \frac{0.0625}{4} + \frac{0.0313}{5} + \frac{0.0156}{6} + \frac{0.0078}{7} + \frac{0.0039}{8} + \frac{0.0020}{9} + \frac{0.0010}{10}\right\}$$

Scalar cardinality of fuzzy set A as, $|A| = \sum_{x \in X} A(x) = 1.9991$

Fuzzy set B(x) can be represented as,

$$B(x) = \left\{\frac{0.7142}{0} + \frac{0.7272}{1} + \frac{0.7333}{2} + \frac{0.7368}{3} + \frac{0.7391}{4} + \frac{0.7407}{5} + \frac{0.7419}{6} + \frac{0.7429}{7} + \frac{0.7436}{8} + \frac{0.7441}{9} + \frac{0.7447}{10}\right\}$$

Scalar cardinality of fuzzy set B as, $|B| = \sum_{x \in X} B(x) = 8.1089$

Standard Operations on Fuzzy Sets:

1. Complement of fuzzy set A: Let A be a fuzzy set defined on universal set X then its complement is denoted as $\overline{A}$ and defined as,

$$\overline{A}(x) = 1 - A(x)$$

2. Union of fuzzy set : Let A and B be a fuzzy set defined on universal set X then union of fuzzy set A and B are denoted by A∪B and defined as,

$$A \cup B(x) = Max\{\, A(x), B(x)\,\}$$

3. Intersection of fuzzy set : Let A and B be a fuzzy set defined on universal set X then intersection of fuzzy set A and B are denoted by A∩B and defined as,

$$A \cap B(x) = Min\{\, A(x), B(x)\,\}$$

4. Degree of Subset hood: Let A and B be a fuzzy set defined on finite universal set X. The degree of subset hood of A in B is denoted by S (A, B) and defined as,

$$S(A,B) = \frac{1}{|A|}\left\{ |A| - \sum_{x \in X} \max\{0, A(x) - B(x)\} \right\}$$

More conveniently, $S(A,B) = \dfrac{|A \cap B|}{|A|}$

The degree of subset hood of B in A is denoted by S (B, A) and defined as, $S(B,A) = \dfrac{|A \cap B|}{|B|}$

Examples

Example 5: Determine intersection, union and complement of fuzzy set A and B

A= {(2, 0.4), (3, 0.6), (4, 0.8), (5, 1), (6, 0.8), (7, 0.6), (8, 0.4)}

B= {(2, 0.4), (4, 0.8), (6, 0.8), (8, 0.4)} where, X = [1, 2…10] Also find $\overline{A} \cup \overline{B}$, $\overline{A} \cap \overline{B}$

Solution: Consider the fuzzy set A and B as,

$$A(x) = \left\{ \frac{0}{1} + \frac{0.4}{2} + \frac{0.6}{3} + \frac{0.8}{4} + \frac{1}{5} + \frac{0.8}{6} + \frac{0.6}{7} + \frac{0.4}{8} + \frac{0}{9} + \frac{0}{10} \right\}$$

$$B(x) = \left\{ \frac{0}{1} + \frac{0.4}{2} + \frac{0}{3} + \frac{0.8}{4} + \frac{0}{5} + \frac{0.8}{6} + \frac{0}{7} + \frac{0.4}{8} + \frac{0}{9} + \frac{0}{10} \right\}$$

1. Union of fuzzy set A and B defined as,

$$A \cup B(x) = Max\{\, A(x), B(x)\,\}$$

$$A \cup B(x) = \left\{ \frac{0}{1} + \frac{0.4}{2} + \frac{0.6}{3} + \frac{0.8}{4} + \frac{1}{5} + \frac{0.8}{6} + \frac{0.6}{7} + \frac{0.4}{8} + \frac{0}{9} + \frac{0}{10} \right\}$$

2. Intersection of fuzzy set A and B defined as,

$$A \cap B(x) = Min\{A(x), B(x)\}$$

$$A \cap B(x) = \left\{\frac{0}{1} + \frac{0.4}{2} + \frac{0}{3} + \frac{0.8}{4} + \frac{0}{5} + \frac{0.8}{6} + \frac{0}{7} + \frac{0.4}{8} + \frac{0}{9} + \frac{0}{10}\right\}$$

3. Complement of fuzzy set A and B defined as,

$$\overline{A}(x) = 1 - A(x)$$

$$\overline{A}(x) = \left\{\frac{1}{1} + \frac{0.6}{2} + \frac{0.4}{3} + \frac{0.2}{4} + \frac{0}{5} + \frac{0.2}{6} + \frac{0.4}{7} + \frac{0.6}{8} + \frac{1}{9} + \frac{1}{10}\right\}$$

$$\overline{B}(x) = 1 - B(x)$$

$$\overline{B}(x) = \left\{\frac{1}{1} + \frac{0.6}{2} + \frac{1}{3} + \frac{0.2}{4} + \frac{1}{5} + \frac{0.2}{6} + \frac{1}{7} + \frac{0.6}{8} + \frac{1}{9} + \frac{1}{10}\right\}$$

4. $\overline{A} \cup \overline{B}(x) = Max\{\overline{A}(x), \overline{B}(x)\}$

$$\overline{A} \cup \overline{B}(x) = \left\{\frac{1}{1} + \frac{0.6}{2} + \frac{1}{3} + \frac{0.2}{4} + \frac{1}{5} + \frac{0.2}{6} + \frac{1}{7} + \frac{0.6}{8} + \frac{1}{9} + \frac{1}{10}\right\}$$

5. $\overline{A} \cap \overline{B}(x) = Min\{\overline{A}(x), \overline{B}(x)\}$

$$\overline{A} \cap \overline{B}(x) = \left\{\frac{1}{1} + \frac{0.6}{2} + \frac{0.4}{3} + \frac{0.2}{4} + \frac{0}{5} + \frac{0.2}{6} + \frac{0.4}{7} + \frac{0.6}{8} + \frac{1}{9} + \frac{1}{10}\right\}$$

Example 6: Two fuzzy sets A and B defined on universal set X are, $A(x) = \left\{\frac{0.1}{x_1} + \frac{0.6}{x_2} + \frac{0.8}{x_3} + \frac{0.9}{x_4} + \frac{0.7}{x_5} + \frac{0.1}{x_6}\right\}$, $B(x) = \left\{\frac{0.9}{x_1} + \frac{0.7}{x_2} + \frac{0.5}{x_3} + \frac{0.2}{x_4} + \frac{0.1}{x_5} + \frac{0}{x_6}\right\}$

Find the following,

(1) ${}^{0.2+}A \cap B$, (2) ${}^{0.5}\overline{A \cap B}$, (3) Degree of subset hood A in B.

Solution: Consider the fuzzy set A and B as,

$$A(x) = \left\{\frac{0.1}{x_1} + \frac{0.6}{x_2} + \frac{0.8}{x_3} + \frac{0.9}{x_4} + \frac{0.7}{x_5} + \frac{0.1}{x_6}\right\}$$

$$B(x) = \left\{\frac{0.9}{x_1} + \frac{0.7}{x_2} + \frac{0.5}{x_3} + \frac{0.2}{x_4} + \frac{0.1}{x_5} + \frac{0}{x_6}\right\}$$

Intersection of fuzzy set A and B defined as,

$$A \cap B(x) = Min\{A(x), B(x)\}$$

$$A \cap B(x) = \left\{ \frac{0.1}{x_1} + \frac{0.6}{x_2} + \frac{0.5}{x_3} + \frac{0.2}{x_4} + \frac{0.1}{x_5} + \frac{0}{x_6} \right\}$$

1. $^{0.2+}A \cap B\,(x) = \left\{ x \in X \,/\, A \cap B(x) > 0.2 \right\}$

$$^{0.2+}A \cap B(x) = \left\{ x_2\,, x_3 \right\}$$

2. $\overline{A \cap B}(x) = 1 - A \cap B(x)$

$$\overline{A \cap B}(x) = \left\{ \frac{0.9}{x_1} + \frac{0.4}{x_2} + \frac{0.5}{x_3} + \frac{0.8}{x_4} + \frac{0.9}{x_5} + \frac{1}{x_6} \right\}$$

$$^{0.5}\overline{A \cap B}\,(x) = \left\{ x \in X \,/\, \overline{A \cap B}(x) \geq 0.5 \right\}$$

$$^{0.5}\overline{A \cap B}\,(x) = \left\{ x_1\,, x_3, x_4\,, x_5, x_6 \right\}$$

3. Degree of subset hood A in B is defined as, $S(A,B) = \dfrac{|A \cap B|}{|A|}$

Scalar cardinality of fuzzy set A as, $|A| = \sum_{x \in X} A(x)$

$|A| = 0.1 + 0.6 + 0.8 + 0.9 + 0.7 + 0.1 \qquad \therefore |A| = 3.2$

Scalar cardinality of fuzzy set A∩B as, $|A \cap B| = \sum_{x \in X} A \cap B(x)$

$|A \cap B| = 0.1 + 0.6 + 0.5 + 0.2 + 0.1 + 0 \qquad \therefore |A \cap B| = 1.5$

$$S(A,B) = \frac{|A \cap B|}{|A|} \qquad \therefore S(A,B) = \frac{1.5}{3.2} = 0.4687$$

Example 7: Find Degree of subset hood S (A, B) and S (B, A) for x ∈ {0, 1, 2, 3, …10} for fuzzy sets $A(x) = \dfrac{2x}{3x+5}$, $B(x) = \dfrac{3x+7}{5x+9}$

Solution: Fuzzy set A(x) can be represented as,

$$A(x) = \left\{ \frac{0}{0} + \frac{0.25}{1} + \frac{0.3636}{2} + \frac{0.4286}{3} + \frac{0.4706}{4} + \frac{0.5}{5} + \frac{0.5217}{6} + \frac{0.5384}{7} + \frac{0.5517}{8} + \frac{0.5625}{9} + \frac{0.5714}{10} \right\}$$

Scalar cardinality of fuzzy set A as, $|A| = \sum_{x \in X} A(x) = 4.7586$

Fuzzy set B(x) can be represented as,

$$B(x) = \left\{ \frac{0.7778}{0} + \frac{0.7143}{1} + \frac{0.6842}{2} + \frac{0.6667}{3} + \frac{0.6552}{4} + \frac{0.6471}{5} + \frac{0.6410}{6} + \frac{0.6364}{7} + \frac{0.6327}{8} + \frac{0.6297}{9} + \frac{0.6271}{10} \right\}$$

Scalar cardinality of fuzzy set B as, $|B| = \sum_{x \in X} B(x) = 7.3120$

By definition, $A \cap B(x) = Min\,\{\, A(x), B(x)\,\}$

$$A \cap B(x) = \left\{ \frac{0}{0} + \frac{0.25}{1} + \frac{0.3636}{2} + \frac{0.4286}{3} + \frac{0.4706}{4} + \frac{0.5}{5} + \frac{0.5217}{6} + \frac{0.5384}{7} + \frac{0.5517}{8} + \frac{0.5625}{9} + \frac{0.5714}{10} \right\}$$

Scalar cardinality of fuzzy set A∩B as, $|A \cap B| = \sum_{x \in X} A \cap B(x) = 4.7586$

Degree of Subset hood, $S(A,B) = \dfrac{|A \cap B|}{|A|} = \dfrac{4.7586}{4.7586} = 1$

And $S(B,A) = \dfrac{|A \cap B|}{|B|} = \dfrac{4.7586}{7.3120} = 0.6508$

Example 8: Let the fuzzy sets A and B defined on X by the membership functions

$$A(x) = \left\{ \frac{0}{1} + \frac{0.2}{1.5} + \frac{0.35}{2} + \frac{0.15}{2.5} + \frac{0.5}{3} + \frac{0.25}{3.5} + \frac{0.4}{4} \right\}$$

$$B(x) = \left\{ \frac{1}{1} + \frac{0.15}{1.5} + \frac{0.2}{2} + \frac{0.35}{2.5} + \frac{0.4}{3} + \frac{0.15}{3.5} + \frac{0}{4} \right\}$$

Calculate i) $\overline{A}$ *and* $\overline{B}$ ii) $(A \cap \overline{B}) \cup A$ iii) $\overline{A \cup B}$

iv) ${}^{0.6}\overline{A \cup B}$ v) Height of a fuzzy set $\overline{A \cup B}$.

Solution:

i) Complement of fuzzy set A and B defined as,

$$\overline{A}(x) = 1 - A(x)$$

$$\overline{A}(x) = \left\{ \frac{1}{1} + \frac{0.8}{1.5} + \frac{0.65}{2} + \frac{0.85}{2.5} + \frac{0.5}{3} + \frac{0.75}{3.5} + \frac{0.6}{4} \right\}$$

$$\overline{B}(x) = 1 - B(x)$$

$$\overline{B}(x) = \left\{ \frac{0}{1} + \frac{0.85}{1.5} + \frac{0.8}{2} + \frac{0.65}{2.5} + \frac{0.6}{3} + \frac{0.85}{3.5} + \frac{1}{4} \right\}$$

ii) $(A \cap \overline{B})(x) = Min\,\{\, A(x), \overline{B}(x)\,\}$

$$(A \cap \overline{B})(x) = \left\{ \frac{0}{1} + \frac{0.2}{1.5} + \frac{0.35}{2} + \frac{0.15}{2.5} + \frac{0.5}{3} + \frac{0.25}{3.5} + \frac{0.4}{4} \right\}$$

$$(A \cap \overline{B}) \cup A(x) = Max\,\{\, A \cap \overline{B}(x), A(x)\,\}$$

$$(A \cap \overline{B}) \cup A(x) = \left\{ \frac{0}{1} + \frac{0.2}{1.5} + \frac{0.35}{2} + \frac{0.15}{2.5} + \frac{0.5}{3} + \frac{0.25}{3.5} + \frac{0.4}{4} \right\}$$

iii) By De Morgan's Law, $\overline{A \cup B} = \overline{A} \cap \overline{B}$,

$$(\overline{A} \cap \overline{B})(x) = Min\,\{\,\overline{A}(x),\ \overline{B}(x)\,\}$$

$$(\overline{A} \cap \overline{B})(x) = \overline{A \cup B}(x) = \left\{\frac{0}{1} + \frac{0.8}{1.5} + \frac{0.65}{2} + \frac{0.65}{2.5} + \frac{0.5}{3} + \frac{0.75}{3.5} + \frac{0.6}{4}\right\}$$

iv) By Definition, ${}^{0.6}\overline{A \cup B} = \left\{x \in X \,/\, \overline{A \cup B} \geq 0.6\right\}$,

$${}^{0.6}\overline{A \cup B} = \{1.5, 2, 2.5, 3.5, 4\}$$

v) Height of fuzzy set $\overline{A \cup B}$:

We define height of fuzzy set $\overline{A \cup B}$ as, $h(A) = \sup\limits_{x \in X} \overline{A \cup B}$

$$h(A) = 0.8$$

Here, fuzzy set $\overline{A \cup B}$ is **subnormal fuzzy set.**

Examples for Practice

Example 1: Define Fuzzy set and explains it with an example.

Example 2: Define: i) Degree of Subset hood ii) Scalar Cardinality of fuzzy set.

iii) Height of fuzzy set. iv) α - Cut and strong α - cut of a fuzzy set.

Example 3: If fuzzy set $A(x) = \frac{x}{x+3}$, X = [0, 1, 2…10] then find,

1) α-cut of A for α = 0.6
2) Strong α-cut of A for α = 0.7
3) Level set of fuzzy set A.
4) Support of fuzzy set A.
5) Core of fuzzy set A.
6) Height of fuzzy set A.
7) Crossover point of fuzzy set A.
8) Cardinality of fuzzy set A.
9) Scalar Cardinality of fuzzy set A.
10) Relative Cardinality of fuzzy set A.

Example 4: Let the fuzzy sets A and B defined on X by the membership functions,

$X:$	x_1	x_2	x_3	x_4	x_5	x_6
$A:$	0.1	0.6	0.8	0.9	0.7	1
$B:$	0.9	0.7	0.5	0.2	0.1	0

Express the following α cuts, 1) ${}^{0.7}(\overline{A})$ 2) ${}^{0.4}(B)$ 3) ${}^{0.7}(A \cup B)$ 4) ${}^{0.6}(A \cap B)$
5) ${}^{0.7}(A \cup \overline{A})$ 6) ${}^{0.5}(B \cap \overline{B})$ 7) ${}^{0.7}(\overline{A \cap B})$ 8) ${}^{0.5}(\overline{A \cup B})$

Example 5: Determine α-cut and Strong α-cut of fuzzy set A for α = 0.2, 0.5, 0.8

$$Where, \quad A(x) = \begin{cases} \frac{x+3}{3} & if\ -3 < x \le 0 \\ \frac{3-x}{3} & if\ 0 < x \le 3 \\ 0 & otherwise \end{cases}$$

Example 6: Define Degree of subset hood and hence find S (A, B) and S (B, A) for fuzzy sets, $A(x) = 3^{-x}$, $B(x) = \frac{x+5}{x+7}$ for x ∈ {0, 1, 2, 3, ……, 10}.

Example 7: Define Degree of subset hood and hence find S(B, A) of fuzzy sets,

$$A(x) = \frac{x}{x+2}, \quad B(x) = \frac{3x+5}{4x+7} \text{ for } x \in \{0, 1, 2, 3, \ldots\ldots, 10\}.$$

Example 8: Define Degree of subset hood and hence find S(B, A) of fuzzy sets,

$$A(x) = 2^{-x}, \quad B(x) = \frac{3x+5}{4x+7} \qquad \text{for } x \in \{0, 1, 2, 3\ldots 10\}.$$

Example 9: Find the degree of subset hood S (A, B) and S (B, A) for the fuzzy sets

$$A(x) = \frac{x}{x+3}, \quad B(x) = \frac{2x+5}{3x+7}, \quad x \in \{0,1,2,...,10\}$$

Example 10: Find the degree of subset hood $S(\overline{A}, \overline{B})$ for the fuzzy sets,

$$A(x) = \frac{x}{x+3}, \quad B(x) = 5^{-x}, \quad x \in \{0,1,2,...,5\}$$

Example 11: Consider fuzzy sets,

$$S_1(x) = \left\{\frac{0}{0} + \frac{0.5}{20} + \frac{0.65}{40} + \frac{0.08}{60} + \frac{1}{80} + \frac{1}{100}\right\}, \quad S_2(x) = \left\{\frac{0}{0} + \frac{0.45}{20} + \frac{0.66}{40} + \frac{0.04}{60} + \frac{0.95}{80} + \frac{1}{100}\right\}$$

Find i) $(S_1 \cup S_2)(x)$ ii) $(S_1 \cap S_2)(x)$ iii) $(\overline{S_1 \cup S_2})(x)$

iv) $(\overline{S_1 \cap S_2})(x)$ v) $(S_1 \cup \overline{S_2})(x)$ vi) $(\overline{S_1} \cap S_2)(x)$

Example 12: Find Scalar Cardinality of fuzzy sets A and B where, $A(x) = \dfrac{2x}{3x+5}$,

$$B(x) = \frac{3x+7}{5x+9} \qquad \text{for } x \in \{0, 1, 2, 3, \ldots\ldots, 10\}.$$

Example 13: Consider two fuzzy sets,

$$D_1(x) = e^{-x} \quad \text{and} \quad D_2(x) = \frac{x}{x+2}, \qquad \text{for } x \in \{0, 1, 2, 3, 4, 5\}.$$

Find 1) α-cut of D_1 and D_2 for $\alpha = 0.2, 0.5, 1$. 2) $\overline{D_1 \cap D_2}$ 3) $D_1 \cup \overline{D_2}$

Example 14: Verify De' Morgan's law for the fuzzy sets A and B defined on X by the membership functions,

$$A(x) = \left\{\frac{0}{1} + \frac{0.2}{1.5} + \frac{0.35}{2} + \frac{0.15}{2.5} + \frac{0.5}{3} + \frac{0.25}{3.5} + \frac{0.4}{4}\right\}$$

$$B(x) = \left\{\frac{1}{1} + \frac{0.15}{1.5} + \frac{0.2}{2} + \frac{0.35}{2.5} + \frac{0.4}{3} + \frac{0.15}{3.5} + \frac{0}{4}\right\}$$

Example 15: Verify Commutative law for the fuzzy sets,

$$D_1 = \left\{ \frac{1}{1} + \frac{0.75}{1.5} + \frac{0.3}{2} + \frac{0.15}{2.5} + \frac{0}{3} \right\}, \quad D_2 = \left\{ \frac{1}{1} + \frac{0.6}{1.5} + \frac{0.2}{2} + \frac{0.1}{2.5} + \frac{0}{3} \right\}$$

Example 16: Consider fuzzy sets, $A(x)=\dfrac{x+2}{x+5}$, $B(x)=\dfrac{1}{1+10(x-1)^2}$ for x ∈ {0, 1,…,10}.

Find i) $A\cup \overline{B}$ ii) $\overline{A\cap B}$

Example 17: Consider two fuzzy sets,

$$D_1(x)=1-\frac{x}{10} \text{ and } D_2(x)=\frac{x}{x+3} \text{ for } x \in \{0,1,2,...,10\}. \text{ Find } S(D_1, D_2).$$

Example 18: Find $^{\alpha}A$ for α = 0.2, 0.5, 0.7 for the fuzzy set $A(x)=3^{-x}$ for x ∈ [0, 10].

Example 19: Let the fuzzy sets A and B defined on X by the membership functions,

$$A(x)=\left\{\frac{0}{1}+\frac{0.2}{1.5}+\frac{0.35}{2}+\frac{0.15}{2.5}+\frac{0.5}{3}+\frac{0.25}{3.5}+\frac{0.4}{4}\right\},$$

$$B(x)=\left\{\frac{1}{1}+\frac{0.15}{1.5}+\frac{0.2}{2}+\frac{0.35}{2.5}+\frac{0.4}{3}+\frac{0.15}{3.5}+\frac{0}{4}\right\}$$

Calculate,

1) $(\overline{A})$ 2) $(A\cup B)$ 3) $(A\cap B)$ 4) $(A\cup \overline{A})$

5) $(A\cap \overline{B})$ 6) $(\overline{A\cap B})$ 7) $(\overline{A\cup B})$ 8) $(A\cap \overline{B})\cup A$

Example 20: Consider fuzzy sets,

$$A(x)=\frac{2x}{2x+5},\quad B(x)=\frac{x}{x+1} \text{ for } x \in \{6, 7, \dots, 10\}.$$

Find i) $A\cup(\overline{A\cap B})$ ii) α cut of $A\cup(\overline{A\cap B})$ for α =0.5, 0.7, 0.9

iii) Scalar cardinality of $A\cup(\overline{A\cap B})$.

Example 21: Let the fuzzy sets C and D defined on X by the membership functions,

$$C(x)=\left\{\frac{0.1}{x_1}+\frac{0.6}{x_2}+\frac{0.8}{x_3}+\frac{0.9}{x_4}+\frac{0.7}{x_5}+\frac{0.1}{x_6}\right\}$$

$$D(x)=\left\{\frac{0.9}{x_1}+\frac{0.7}{x_2}+\frac{0.5}{x_3}+\frac{0.2}{x_4}+\frac{0.1}{x_5}+\frac{0}{x_6}\right\}$$

Calculate,

1) $(\overline{C})$ 2) $(\overline{D})$ 3) Scalar cardinality of $(\overline{C}\cap D)$ 4) Height of $(C\cup \overline{C})$.

5) $(C\cap D)$ 6) $(\overline{C\cap D})$ 7) $(\overline{C\cup D})$

Representation of fuzzy set by Crisp sets:

Let fuzzy set $A(x)=\left\{\frac{0.2}{x_1}+\frac{0.4}{x_2}+\frac{0.6}{x_3}+\frac{0.8}{x_4}+\frac{1}{x_5}\right\}$ represents fuzzy sets by it's α cuts. Then given fuzzy set A associated with only five α cuts which are defined by the following characteristic function.

$$^{0.2}A=\{x\in X/A(x)\geq 0.2\}\quad=\left\{\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}+\frac{1}{x_5}\right\}\quad=\{x_1,x_2,x_3,x_4,x_5\}$$

$$^{0.4}A=\{x\in X/A(x)\geq 0.4\}\quad=\left\{\frac{0}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}+\frac{1}{x_5}\right\}\quad=\{x_2,x_3,x_4,x_5\}$$

$$^{0.6}A=\{x\in X/A(x)\geq 0.6\}\quad=\left\{\frac{0}{x_1}+\frac{0}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}+\frac{1}{x_5}\right\}\quad=\{x_3,x_4,x_5\}$$

$$^{0.8}A=\{x\in X/A(x)\geq 0.8\}\quad=\left\{\frac{0}{x_1}+\frac{0}{x_2}+\frac{0}{x_3}+\frac{1}{x_4}+\frac{1}{x_5}\right\}\quad=\{x_4,x_5\}$$

$$^{1}A=\{x\in X/A(x)\geq 1\}\quad=\left\{\frac{0}{x_1}+\frac{0}{x_2}+\frac{0}{x_3}+\frac{0}{x_4}+\frac{1}{x_5}\right\}\quad=\{x_5\}$$

Special fuzzy set:

The representation of an arbitrary fuzzy set A in terms of the special fuzzy set $_{\alpha}A$ which are defined in terms of α- cuts of A by $_{\alpha}A=\alpha\ {}^{\alpha}A$ ....(*)

Theorem: First Decomposition Theorem.

For every $A\in F(x)$,Where F(x) is the set of all ordinary fuzzy set. $A=\bigcup_{\alpha\in[0,1]}{}_{\alpha}A$

Where $_{\alpha}A$ is special fuzzy set defined by (*) and $\cup$ is defined by standard fuzzy union.

Example 9: Find α – cuts for distinct values of α of the fuzzy set A and hence find special fuzzy set where, $A(x)=\left\{\frac{0.2}{x_1}+\frac{0.4}{x_2}+\frac{0.6}{x_3}+\frac{0.8}{x_4}+\frac{1}{x_5}\right\}$ and hence show that the standard union of these special fuzzy set is exactly the original fuzzy set A.

Solution: Here α – cuts for distinct values of α of the fuzzy set A

$$^{0.2}A=\{x\in X/A(x)\geq 0.2\}\ =\{x_1,x_2,x_3,x_4,x_5\}$$

$$^{0.4}A=\{x\in X/A(x)\geq 0.4\}\ =\{x_2,x_3,x_4,x_5\}\qquad {}^{0.6}A=\{x\in X/A(x)\geq 0.6\}=\{x_3,x_4,x_5\}$$

$$^{0.8}A=\{x\in X/A(x)\geq 0.8\}\ =\{x_4,x_5\}\qquad {}^{1}A=\{x\in X/A(x)\geq 1\}\ =\{x_5\}$$

Special fuzzy set for A is,

$$_{0.2}A = 0.2\ ^{0.2}A \quad = 0.2\left\{\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}+\frac{1}{x_5}\right\} \qquad = \left\{\frac{0.2}{x_1}+\frac{0.2}{x_2}+\frac{0.2}{x_3}+\frac{0.2}{x_4}+\frac{0.2}{x_5}\right\}$$

$$_{0.4}A = 0.4\ ^{0.4}A \quad = 0.4\left\{\frac{0}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}+\frac{1}{x_5}\right\} \qquad = \left\{\frac{0}{x_1}+\frac{0.4}{x_2}+\frac{0.4}{x_3}+\frac{0.4}{x_4}+\frac{0.4}{x_5}\right\}$$

$$_{0.6}A = 0.6\ ^{0.6}A \quad = 0.6\left\{\frac{0}{x_1}+\frac{0}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}+\frac{1}{x_5}\right\} \qquad = \left\{\frac{0}{x_1}+\frac{0}{x_2}+\frac{0.6}{x_3}+\frac{0.6}{x_4}+\frac{0.6}{x_5}\right\}$$

$$_{0.8}A = 0.8\ ^{0.8}A \quad = 0.8\left\{\frac{0}{x_1}+\frac{0}{x_2}+\frac{0}{x_3}+\frac{1}{x_4}+\frac{1}{x_5}\right\} \qquad = \left\{\frac{0}{x_1}+\frac{0}{x_2}+\frac{0}{x_3}+\frac{0.8}{x_4}+\frac{0.8}{x_5}\right\}$$

$$_{1}A = 1\ ^{1}A \quad = 1\left\{\frac{0}{x_1}+\frac{0}{x_2}+\frac{0}{x_3}+\frac{0}{x_4}+\frac{1}{x_5}\right\} \qquad = \left\{\frac{0}{x_1}+\frac{0}{x_2}+\frac{0}{x_3}+\frac{0}{x_4}+\frac{1}{x_5}\right\}$$

By First Decomposition Theorem,

$$A = \bigcup_{\alpha \in [0,1]}\ {}_{\alpha}A$$

Where $_{\alpha}A$ is special fuzzy set and $\cup$ is defined by standard fuzzy union,

i.e. $_{0.2}A \cup\ _{0.4}A \cup\ _{0.6}A \cup\ _{0.8}A \cup _{1}A = Max\left\{\frac{0.2}{x_1}+\frac{0.4}{x_2}+\frac{0.6}{x_3}+\frac{0.8}{x_4}+\frac{1}{x_5}\right\} = A = \bigcup\ _{\alpha}A$

We observed that, the standard union of these special fuzzy set is exactly the original fuzzy set A.

Example 10: Find α – cuts for distinct values of α of the fuzzy set A and hence find special fuzzy set where, $A(x) = \left\{\frac{0.4}{-2}+\frac{0.5}{-1}+\frac{0.3}{0}+\frac{1}{1}+\frac{0.5}{2}\right\}$

Solution: Here α – cuts for distinct values of α of the fuzzy set A

$^{0.3}A = \{x \in X / A(x) \geq 0.3\} = \{-2, -1, 0, 1, 2\}$ $\quad ^{0.4}A = \{x \in X / A(x) \geq 0.4\} = \{-2, -1, 1, 2\}$

$^{0.5}A = \{x \in X / A(x) \geq 0.5\} = \{-1, 1, 2\}$ $\quad ^{1}A = \{x \in X / A(x) \geq 1\} = \{1\}$

Special fuzzy set for A is,

$$_{0.3}A = 0.3\ ^{0.3}A \quad = 0.3\left\{\frac{1}{-2}+\frac{1}{-1}+\frac{1}{0}+\frac{1}{1}+\frac{1}{2}\right\} \qquad = \left\{\frac{0.3}{-2}+\frac{0.3}{-1}+\frac{0.3}{0}+\frac{0.3}{1}+\frac{0.3}{2}\right\}$$

$$_{0.4}A = 0.4\ ^{0.4}A \quad = 0.4\left\{\frac{1}{-2}+\frac{1}{-1}+\frac{0}{0}+\frac{1}{1}+\frac{1}{2}\right\} \qquad = \left\{\frac{0.4}{-2}+\frac{0.4}{-1}+\frac{0}{0}+\frac{0.4}{1}+\frac{0.4}{2}\right\}$$

$$ {}_{0.5}A = 0.5\ {}^{0.5}A \qquad = 0.5\left\{\frac{0}{-2}+\frac{1}{-1}+\frac{0}{0}+\frac{1}{1}+\frac{1}{2}\right\} \qquad = \left\{\frac{0}{-2}+\frac{0.5}{-1}+\frac{0}{0}+\frac{0.5}{1}+\frac{0.5}{2}\right\} $$

$$ {}_{1}A = 1\ {}^{1}A \qquad = 1\left\{\frac{0}{-2}+\frac{0}{-1}+\frac{0}{0}+\frac{1}{1}+\frac{0}{2}\right\} \qquad = \left\{\frac{0}{-2}+\frac{0}{-1}+\frac{0}{0}+\frac{1}{1}+\frac{0}{2}\right\} $$

Examples for Practice

Example 1: Find α – cuts for distinct values of α of the fuzzy set A and hence find special fuzzy set where, $A(x)=\left\{\frac{0.3}{-1}+\frac{0.5}{0}+\frac{0.7}{1}+\frac{1}{2}+\frac{0.4}{3}\right\}$

Example 2: Find α – cuts for distinct values of α of the fuzzy set A and hence find special fuzzy set where, $A(x)=\left\{\frac{0.2}{1}+\frac{0.5}{2}+\frac{0.6}{3}+\frac{0.8}{4}+\frac{0.4}{5}\right\}$ and hence show that the standard union of these special fuzzy set is exactly the original fuzzy set A.

Example 3: Find α – cuts for distinct values of α of the fuzzy set A and hence find special fuzzy set where, $A(x)=\left\{\frac{0.1}{5}+\frac{0.3}{6}+\frac{0.5}{7}+\frac{1}{8}+\frac{0.8}{9}\right\}$

Example 4: Find α – cuts for distinct values of α of the fuzzy set A and hence find special fuzzy set where, $A(x)=\left\{\frac{0.1}{-3}+\frac{0.3}{-2}+\frac{0.4}{-1}+\frac{0.6}{0}+\frac{0.4}{1}\right\}$ and hence show that the standard union of these special fuzzy set is exactly the original fuzzy set A.

Example 5: Find α-cuts for distinct values of α for the following fuzzy set,

$D_1 = \{\ \frac{1}{1}+\frac{0.75}{1.5}+\frac{0.3}{2}+\frac{0.15}{2.5}+\frac{0}{3}\ \}$ and hence find special fuzzy sets.

Example 6: Find α-cuts for distinct values of α for the following fuzzy set,

$B(x) = \{\ \frac{0.1}{a}+\frac{0.55}{b}+\frac{0.8}{c}+\frac{0.35}{d}+\frac{0.2}{e}\ \}$ and hence find special fuzzy sets.

Example 7: Find α – cuts for distinct values of α of A = $\left\{\frac{0.4}{x_1}+\frac{0.5}{x_2}+\frac{0.4}{x_3}+\frac{1}{x_4}+\frac{0.8}{x_5}\right\}$ and hence find special fuzzy sets.

Extension principle for fuzzy set:

One of the most basic concepts of fuzzy set theory that can be used to generalize crisp mathematical concepts to fuzzy sets is the extension principle.

A principle for fuzzifield crisp function is called an extension principle. Mathematically $f: X \to Y$ is fuzzifield when it is extended to act on fuzzy set defined on X and Y has the form $f: F(X) \to F(Y)$ and $f^{-1}: F(Y) \to F(X)$ which are defined by

$$f(A)(x) = \underset{x/Y=f(x)}{Sup}\left[A(x)\right]$$

Example 11: Let the membership grade function at fuzzy set A define on X = [0, 1, 2…10] be given by $A(x) = \frac{x}{x+2}$, $f: X \to N$ such that $y = f(x) = x^2 \quad \forall x \in X$ Use the extension principle and find f (A).

Solution: Given fuzzy set $A(x) = \frac{x}{x+2}$ on X = [0, 1, 2…10]

$y = f(x) = x^2 \quad \forall x \in X$ **Y = {0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100}**

$$f(A)(0) = \underset{x=0/Y=0}{Sup}\left[A(0)\right] = \frac{0}{0+2} = 0 \qquad f(A)(1) = \underset{x=1/Y=1}{Sup}\left[A(1)\right] = \frac{1}{1+2} = 0.3333$$

$$f(A)(2) = \underset{x=2/Y=4}{Sup}\left[A(2)\right] = \frac{2}{2+2} = 0.5 \qquad f(A)(3) = \underset{x=3/Y=9}{Sup}\left[A(3)\right] = \frac{3}{3+2} = 0.6$$

$$f(A)(4) = \underset{x=4/Y=16}{Sup}\left[A(4)\right] = \frac{4}{4+2} = 0.6666 \qquad f(A)(5) = \underset{x=5/Y=25}{Sup}\left[A(4)\right] = \frac{5}{5+2} = 0.7142$$

$$f(A)(6) = \underset{x=6/Y=36}{Sup}\left[A(6)\right] = \frac{6}{6+2} = 0.75 \qquad f(A)(7) = \underset{x=7/Y=49}{Sup}\left[A(7)\right] = \frac{7}{7+2} = 0.7777$$

$$f(A)(8) = \underset{x=8/Y=64}{Sup}\left[A(8)\right] = \frac{8}{8+2} = 0.8 \qquad f(A)(9) = \underset{x=9/Y=81}{Sup}\left[A(9)\right] = \frac{9}{9+2} = 0.8181$$

$$f(A)(10) = \underset{x=10/Y=100}{Sup}\left[A(10)\right] = \frac{10}{10+2} = 0.8333$$

Hence,

$$f(A) = \left\{\frac{1}{0} + \frac{0.3333}{1} + \frac{0.5}{4} + \frac{0.6}{9} + \frac{0.6666}{16} + \frac{0.7142}{25} + \frac{0.75}{36} + \frac{0.7777}{49} + \frac{0.8}{64} + \frac{0.8181}{81} + \frac{0.8333}{100}\right\}$$

Example 12: Let the membership grade function at fuzzy set A define on X = [0, 1, 2…10] be given by $A(x)=\dfrac{1}{1+10(x-2)^2}$, $f: X \to N$ such that $y=f(x)=x^3 \quad \forall x \in X$ Use the extension principle and find f (A).

Solution: Given fuzzy set $A(x)=\dfrac{1}{1+10(x-2)^2}$ on X = [0, 1, 2…10]

$$y=f(x)=x^3 \quad \forall x \in X$$

Y = {0, 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000}

$$f(A)(0)=\underset{x=0/Y=0}{Sup}\left[A(0)\right]=0.0243 \qquad f(A)(1)=\underset{x=1/Y=1}{Sup}\left[A(1)\right]=0.0909$$

$$f(A)(2)=\underset{x=2/Y=8}{Sup}\left[A(2)\right]=1 \qquad f(A)(3)=\underset{x=3/Y=27}{Sup}\left[A(3)\right]=0.0909$$

$$f(A)(4)=\underset{x=4/Y=64}{Sup}\left[A(4)\right]=0.0243 \qquad f(A)(5)=\underset{x=5/Y=125}{Sup}\left[A(5)\right]=0.0109$$

$$f(A)(6)=\underset{x=6/Y=216}{Sup}\left[A(6)\right]=0.0062 \qquad f(A)(7)=\underset{x=7/Y=343}{Sup}\left[A(7)\right]=0.0039$$

$$f(A)(8)=\underset{x=8/Y=512}{Sup}\left[A(8)\right]=0.0027 \qquad f(A)(9)=\underset{x=9/Y=729}{Sup}\left[A(9)\right]=0.002$$

$$f(A)(10)=\underset{x=10/Y=1000}{Sup}\left[A(10)\right]=0.0015$$

Hence,

$$f(A)=\left\{\frac{0.0243}{0}+\frac{0.0909}{1}+\frac{1}{8}+\frac{0.0909}{27}+\frac{0.0243}{64}+\frac{0.0109}{125}+\frac{0.0062}{216}+\frac{0.0039}{343}+\frac{0.0027}{512}+\frac{0.002}{729}+\frac{0.0015}{1000}\right\}$$

Examples for Practice

Example 1: Let the membership grade function at fuzzy set A define on X = [0, 1, 2…5] be given by $A(x)=\dfrac{2x}{x+5}$, $f:X\to N$ such that $y=f(x)=\sqrt{x}\quad\forall x\in X$ Use the extension principle and find f (A).

Example 2: Let the membership grade function at fuzzy set A define on X = [0, 1, 2…5] be given by $A(x)=\dfrac{5}{x+7}$, $f:X\to N$ such that $y=f(x)=x\quad\forall x\in X$ Use the extension principle and find f (A).

Example 3: Let the membership grade function at fuzzy set A define on X = [0, 1, 2…10] be given by $A(x)=2^{-x}$, $f:X\to N$ such that $y=f(x)=x^2\quad\forall x\in X$ Use the extension principle and find f (A).

Example 4: Let the membership grade function at fuzzy set A define on X = [6, 7, 8…10] be given by $A(x)=\dfrac{5x}{x+2}$, $f:X\to N$ such that $y=f(x)=3x\quad\forall x\in X$ Use the extension principle and find f (A).

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