- **Spanning sets :-**

Let V be a vector space over K. Vectors $u_1, u_2, \cdots u_m$ in V are said to span V or to form a spanning set of V if every v in V is a linear combination of the vectors $u_1, u_2, \cdots, u_m$ i.e. if there exist scalars $a_1, a_2, \cdots, a_m$ in K such that

$$v = a_1u_1 + a_2u_2 + \cdots + a_mu_m.$$

Note: 1) Suppose $u_1, u_2, \cdots, u_m$ span V. Then for any vector w, the set $w, u_1, u_2, \cdots u_m$ also spans V.

2) Suppose $u_1, u_2, \cdots, u_m$ span V and suppose u_k is a linear combination of some of the other u's. Then the u's without u_k also span V.

3) Suppose $u_1, u_2, \cdots, u_m$ span V and suppose one of the u's is zero vector. Then the u's without the zero vector also span V.

Ex.1) Consider the vector space $V = \mathbb{R}^3$. Prove that the following vectors form a spanning set of $\mathbb{R}^3$.

i) $e_1 = (1,0,0)$ $e_2 = (0,1,0)$ $e_3 = (0,0,1)$

ii) $w_1 = (1,1,1)$, $w_2 = (1,1,0)$ $w_3 = (1,0,0)$

Soln: i) The vector space $V = \mathbb{R}^3$

$e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

For any $u = (a,b,c) \in \mathbb{R}^3$

let

$$u = k_1e_1 + k_2e_2 + k_3e_3$$

$$(a,b,c) = k_1(1,0,0) + k_2(0,1,0) + k_3(0,0,1)$$

$$= (k_1,0,0) + (0,k_2,0) + (0,0,k_3)$$

$$(a,b,c) = (k_1,k_2,k_3)$$

$$\therefore a = k_1, \ b = k_2, \ c = k_3.$$

$$\therefore u = ae_1 + be_2 + ce_3$$

$\therefore$ Given set of vectors form spanning set of $\mathbb{R}^3$.

ii) The vector space $V = \mathbb{R}^3$

$w_1 = (1,1,1)$, $w = (1,1,0)$, $w_3 = (1,0,0)$.

For any $u \in \mathbb{R}^3$ let $u = (a,b,c)$
let

$$u = k_1 w_1 + k_2 w_2 + k_3 w_3 \quad \text{— (I)}$$

$(a,b,c) = k_1(1,1,1) + k_2(1,1,0) + k_3(1,0,0)$

$(a,b,c) = (k_1, k_1, k_1) + (k_2, k_2, 0) + (k_3, 0, 0)$

$(a,b,c) = (k_1+k_2+k_3, k_1+k_2, k_1)$

$\therefore k_1+k_2+k_3 = a$

$k_1+k_2 = b$

$\boxed{k_1 = c}$

$\therefore \cancel{k_1+c=b}$ $\therefore c+k_2 = b$ $\therefore \boxed{k_2 = b-c}$

$c+b-c+k_3 = a$ $\therefore \boxed{k_3 = a-b}$

$\therefore k_1 = c,\ k_2 = (b-c),\ k_3 = (a-b)$

$\therefore u = c w_1 + (b-c) w_2 + (a-b) w_3$

$\therefore$ Given set of vectors form a spanning set for $V = \mathbb{R}^3$.

2) Show that $v = (2,7,8)$ cannot be written as a linear combination of the vectors

$u_1 = (1,2,3)$, $u_2 = (1,3,5)$ $u_3 = (1,5,9)$.

Also show that u_1, u_2, u_3 do not span $\mathbb{R}^3$.

Sol^n:- Given

$v = (2,7,8)$

$u_1 = (1,2,3)$ $u_2 = (1,3,5)$, $u_3 = (1,5,9)$

let

$$v = k_1 u_1 + k_2 u_2 + k_3 u_3 \quad \text{— (I)}$$

$(2,7,8) = k_1(1,2,3) + k_2(1,3,5) + k_3(1,5,9)$.

$(2,7,8) = (k_1, 2k_1, 3k_1) + (k_2, 3k_2, 5k_2) + (k_3, 5k_3, 9k_3)$

$(2,7,8) = (k_1+k_2+k_3,\ 2k_1+3k_2+5k_3,\ 3k_1+5k_2+9k_3)$.

$$\left.\begin{aligned} \therefore k_1+k_2+k_3 &= 2 \\ 2k_1+3k_2+5k_3 &= 7 \\ 3k_1+5k_2+9k_3 &= 8 \end{aligned}\right\} \text{— (II)}$$

Above eq^n can be written in matrix form as $AX = B$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 3 & 5 & 9 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 8 \end{bmatrix}$$

consider

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 5 & 7 \\ 3 & 5 & 9 & 8 \end{bmatrix}$$

$R_2 - 2R_1$, $R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & +1 & 3 & 3 \\ 0 & 2 & 6 & 2 \end{bmatrix}$$

$R_3 - 2R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$\therefore \rho([A:B]) = 3 \qquad \rho(A) = 2$

$\therefore \rho([A:B]) \neq \rho(A)$.

$\therefore$ ~~Given~~ system of eq^{ns} - (II) does not have a sol^n.

$\therefore$ V cannot be written as linear combination of u_1, u_2, u_3.

For $v = (2,7,8) \in \mathbb{R}^3$

$\therefore$ u_1, u_2, u_3 do not span $\mathbb{R}^3$.

- **Subspaces** :-

Def^n :- Let V be a vector space over a field K and let W be a subset of V. Then W is a subspace of V if W is itself a vector space over K with respect to the operations of vector addition and scalar multiplication on V.

Theorem :- Suppose W is a subset of a vector space V. Then W is a subspace of V if the following two conditions hold:

a) The zero vector 0 belongs to W

b) For every $u, v \in W$, $k \in K$

i) ~~b~~ Th sum $u+v \in W$. ii) The multiple $ku \in W$.

e.g. 1) Let $V = M_{n \times n}$ the vector space of $n \times n$ matrices. Let W_1 be the subset of all (upper) triangular matrices and let W_2 be the subset of all symmetric matrices. The W_1 is subspace of V, Since W_1 contains the zero matrix '0' and W_1 is closed under matrix addition and scalar multiplication. i.e. the sum and scalar mutiple of such triangular matrices are also triangular. Similarly W_2 is subspace of V.

2) Consider the vector space $V = \mathbb{R}^3$. Let U consist of all vectors in $\mathbb{R}^3$ whose entries are equal.

i.e. $U = \{(a,b,c) : a=b=c\}$.

Show that U is a subspace of V.

<u>Sol^n</u> For the vector $V = \mathbb{R}^3$

The subset of V is $U = \{(a,b,c) : a=b=c\}$

clearly,

$(0,0,0) \in U$.

For any $u, v \in U$

let $u = (a,a,a)$. $v = (b,b,b)$

i) $u+v = (a+b, a+b, a+b)$

$\therefore u+v \in U$

ii) For any $k \in \mathbb{R}$,

$ku = (ka, ka, ka)$

$\therefore ku \in U$.

$\therefore U$ is a subspace V.

3). Let $V = \mathbb{R}^3$. Show that W is not a subspace of V, where
a) $W = \{(a,b,c) : a \geq 0\}$ b) $W = \{(a,b,c) : a^2+b^2+c^2 \leq 1\}$

<u>Sol^n</u> Let $V = \mathbb{R}^3$

a) $W = \{(a,b,c) : a \geq 0\}$

Clearly $0 = (0,0,0) \in W$.

W consist of those vectors whose first entry is nonnegative

For $V=(1,2,3) \in W$ let $k=-2$

$-2V = (-2,-4,-6) \notin W$

$\therefore W$ is not a subspace of $\mathbb{R}^3$.

b)

$W = \{(a,b,c) : a^2+b^2+c^2 \leq 1\}$

clearly $(0,0,0) \in W$

let $u, v \in W$

$u=(1,1,0)$ $v=(0,1,1)$

$\therefore u+v = (1,2,1)$

$\because u+v \notin W$ $[\because 1^2+2^2+1 = 6 > 1]$

$\therefore W$ is not a subspace of $V=\mathbb{R}^3$.

Theorem: The intersection of any number of subspaces of a vector space V is a subspace of V.

Theorem: The solⁿ set W of a homogeneous system $AX=0$ in n unknowns is a subspace of K^n.

- **Linear spans**

Suppose $u_1, u_2, \dots, u_m$ are any vectors in a vector space V. Any vector of the form $a_1u_1+a_2u_2+\dots+a_mu_m$ where the a_i are scalars is called a linear combination of $u_1, u_2, \dots, u_m$. The collection of all such linear combinations, denoted by $\text{span}(u_1, u_2, \dots, u_m)$ or $\text{span}(u_i)$ is called the linear span of $u_1, u_2, \dots, u_m$.

Ex. Suppose $u_1, u_2, \dots, u_m$ are any vectors in a vector space V. Show that $\text{span}(u_i)$ is a subspace of V.

Solⁿ let $u_1, u_2, \dots, u_m \in V$ where V be a vector space.

clearly the zero vector 0 belongs to $\text{span}(u_i)$.

$0 = 0u_1+0u_2+\dots+0u_m$.

For any $u, v \in \text{span}(u_i)$.

$u = a_1u_1+a_2u_2+\dots+a_mu_m$ $v = b_1u_1+b_2u_2+\dots+b_mu_m$.

$$u+v = (a_1+b_1)u_1 + (a_2+b_2)u_2 + \cdots + (a_m+b_m)u_m$$

$\therefore u+v \in span(u_i)$

For any ~~any~~ scalar $k \in K$,

$$ku = ka_1u_1 + ka_2u_2 + \cdots + ka_mu_m.$$

$\therefore ku \in span(u_i)$

$\therefore span(u_i)$ is a subspace of vector space V.

Theorem : Let S be a subset of a vector space V.

i) Then $span(S)$ is a subspace of V that contains S.

ii) If W is a subspace of V containing S, then $span(S) \subseteq W$

- Row space of a matrix :-

Let $A = [a_{ij}]$ be an arbitrary $m \times n$ matrix over a field K. The rows of A,

$$R_1 = (a_{11}, a_{12}, \cdots, a_{1n}), \quad R_2 = (a_{21}, a_{22}, \cdots, a_{2n}) \cdots$$

$$R_m = (a_{m1}, a_{m2}, \cdots, a_{mn})$$

may be viewed as vectors in K^n, hence they span a subspace of K^n called the row space of A and denoted by $rowsp(A)$. That is,

$$rowsp(A) = span(R_1, R_2, \cdots, R_m).$$

Theorem :- Row equivalent matrices have the same row space.

Theorem :- Suppose A and B are row canonical matrices. Then A and B have the row space iff they have the same nonzero rows.

Corollary : Every matrix A is row equivalent to a unique matrix in row canonical form.

Ex. 1) Consider the following two sets of vectors in $\mathbb{R}^4$

$$u_1 = (1,2,-1,3), \quad u_2 = (2,4,1,-2), \quad u_3 = (3,6,3,-7)$$

$$w_1 = (1,2,-4,11), \quad w_2 = (2,4,-5,14).$$

$U = span(u_i)$ and $W = span(w_i)$

show that $U = W$.

$\underline{\underline{Sol^n}}$ Let the matrix A ~~whose~~ whose rows are $u_1; u_2, u_3$ and the matrix B whose rows are w_1, w_2.

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{bmatrix} \underset{R_3-3R_1}{\overset{R_2-2R_1}{\sim}} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 6 & -16 \end{bmatrix}$$

$$\overset{R_2/3}{\sim} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & -8/3 \\ 0 & 0 & 6 & -16 \end{bmatrix} \underset{R_1+R_2}{\overset{R_3-6R_2}{\sim}} \begin{bmatrix} 1 & 2 & 0 & 1/3 \\ 0 & 0 & 1 & -8/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & -4 & 11 \\ 2 & 4 & -5 & 14 \end{bmatrix} \overset{R_2-2R_1}{\sim} \begin{bmatrix} 1 & 2 & -4 & 11 \\ 0 & 0 & 3 & -8 \end{bmatrix}$$

$$\overset{R_2/3}{\sim} \begin{bmatrix} 1 & 2 & -4 & 11 \\ 0 & 0 & 1 & -8/3 \end{bmatrix} \overset{R_1+4R_2}{\sim} \begin{bmatrix} 1 & 2 & 0 & 1/3 \\ 0 & 0 & 1 & -8/3 \end{bmatrix}$$

The nonzero rows of the matrices in row canonical form are identical, the row spaces of A and B are equal.

$\therefore U = W$.

Ex Show that the vectors $u_1 = (1, 1, 1)$, $u_2 = (1, 2, 3)$, $u_3 = (1, 5, 8)$ span $\mathbb{R}^3$.

$\underline{\underline{Sol^n}}$ For any $v = (a, b, c) \in \mathbb{R}^3$

Given,

$u_1 = (1, 1, 1)$ $\quad u_2 = (1, 2, 3)$ $\quad u_3 = (1, 5, 8)$.

Let

$$v = xu_1 + yu_2 + zu_3 \quad - \text{ (I)}$$

$$(a, b, c) = x(1, 1, 1) + y(1, 2, 3) + z(1, 5, 8)$$

$\therefore$ $\cancel{(a, b, c) = (x+y+z, \; y+2z}$.

$$(a, b, c) = (x+y+z, \; x+2y+5z, \; x+3y+8z)$$

$$\therefore \left.\begin{array}{l} x+y+z = a \\ x+2y+5z = b \\ x+3y-8z = c \end{array}\right\} - \text{ (II)}$$

above eq^{ns} can be written in matrix form $AX=B$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 3 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

consider,

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & a \\ 1 & 2 & 5 & b \\ 1 & 3 & 8 & c \end{bmatrix}$$

$$R_2 - R_1,\ R_3 - R_1 \sim \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 4 & b-a \\ 0 & 2 & 7 & c-a \end{bmatrix}$$

$$R_3 - 2R_2$$

$$\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 4 & b-a \\ 0 & 0 & -1 & c-2b+a \end{bmatrix}$$

$\therefore$ $x+y+z=a$

$y+4z=b-a$

$-z = c-2b+a$ $\therefore z = 2b-a-c$.

$\therefore$ $y + 4(2b-a-c) = b-a$

$\therefore y = 3a-7b+4c$

$x + (3a-7b+4c) + 2b-a-c = a$

$x + 2a - 5b + 3c = a$

$\therefore x = 5b-a-3c$

$\therefore$ $v = (5b-a-3c)u_1 + (3a-7b+4c)u_2 + (2b-a-c)u_3$.

$\therefore$ u_1, u_2, u_3 span $\mathbb{R}^3$.

- <u>Linear dependance and independance</u> :-

Let V be a vector space over ~~k~~ a field K.

<u>Def^n</u> :- The vectors $v_1, v_2, \dots, v_m$ in V are linearly dependent if there exist scalars $a_1, a_2, \dots, a_m$ in K, not all of them 0, such that

$$a_1v_1 + a_2v_2 + \dots + a_mv_m = 0.$$

otherwise we say that the vectors are linearly independent.

Suppose,

$$x_1 v_1 + x_2 v_2 + \cdots + x_m v_m = 0 \quad \text{implies} \quad x_1 = x_2 = \cdots = x_m = 0,$$

then the vectors $v_1, v_2, \ldots, v_m$ are linearly independent.

Ex. Show that $u = (1,1,0)$, $v = (1,3,2)$, $w = (1,3,5)$ are linearly independent.

Sol^n ~~th~~ Given:

$$u = (1,1,0), \quad v = (1,3,2), \quad w = (1,3,5)$$

Let

$$xu + yv + zw = 0 \quad \text{— (I)}$$

$$x(1,1,0) + y(1,3,2) + z(1,3,5) = (0,0,0)$$

$$(x+y+z,\ x+3y+3z,\ 2y+5z) = (0,0,0)$$

$$\left.\begin{aligned} x+y+z &= 0 \\ x+3y+3z &= 0 \\ 2y+5z &= 0 \end{aligned}\right\} \quad \text{— (II)}$$

Above eq^n can be written in matrix form $AX = 0$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 0 & 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

consider,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 0 & 2 & 5 \end{bmatrix} \underset{\sim}{R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix} \underset{\sim}{R_3 - R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & +1 \end{bmatrix}$$

$\rho(A) = 3 = r \quad n = 3 \qquad \therefore r = n$

$\therefore$ ~~Eq^n~~ system of eq^{ns} (II) have trivial sol^n.

$$x = y = z = 0.$$

$\therefore$ Given vectors are linearly independent.

2) Determine whether or not each of the following lists of vectors in $\mathbb{R}^3$ is linearly dependent.

$u_1 = (1,3,5)$, $u_2 = (1,3,1)$, $u_3 = (2,5,7)$, $u_4 = (3,1,4)$.

Soln Given:

$u_1 = (1,3,5)$, $u_2 = (1,3,1)$, $u_3 = (2,5,7)$, $u_4 = (3,1,4)$.

Let

$$xu_1 + yu_2 + zu_3 + wu_4 = 0 \quad \text{— (I)}$$

$$x(1,3,5) + y(1,3,1) + z(2,5,7) + w(3,1,4) = (0,0,0).$$

$$(x+y+2z+3w,\ 3x+3y+5z+w,\ 5x+y+7z+4w) = (0,0,0).$$

$$\left.\begin{aligned} x+y+2z+3w &= 0 \\ 3x+3y+5z+w &= 0 \\ 5x+y+7z+4w &= 0 \end{aligned}\right\} \text{— (II)}$$

Above eqns can be written as $AX=0$.

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 3 & 5 & -1 \\ 5 & 1 & 7 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 3 & 5 & 1 \\ 5 & 1 & 7 & 4 \end{bmatrix} \underset{R_3 - 5R_1}{\overset{R_2 - 3R_1}{\sim}} \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & -1 & -8 \\ 0 & -4 & -3 & -11 \end{bmatrix}$$

$$\overset{R_{23}}{\sim} \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & -4 & -3 & -11 \\ 0 & 0 & -1 & -8 \end{bmatrix}$$

$\rho(A) = 3 = r$, $n = 4$

$r < n$.

$\therefore$ Given are linearly dependent.

- Basis and dimension :-

Defn :- A set $S = \{u_1, u_2, \dots, u_n\}$ of vectors is a basis of V if it has the following two properties

i) S is linearly independent ii) S spans V.

Another definition :-

A set $S = \{u_1, u_2, \dots, u_n\}$ of vectors is a basis of V if every $v \in V$ can be written uniquely as a linear combination of the basis vectors.

The number of elements (vectors) in basis is called the dimension of a vector space V and denoted by dim(V).

Theorem :- Let V be a vector space such that one basis has m elements and another basis has n elements. Then $m=n$.

- A vector space V is said to be of finite dimension n or n-dimensional if V has a basis with n elements.

$$\dim V = n.$$

- The vector space $\{0\}$ is defined to have dimension 0.
- Suppose a vector space V does not have a finite basis, then V is said to be of infinite dimension or to be infinite-dimensional.

Lemma :- Suppose $\{v_1, v_2, \ldots, v_n\}$ spans V, and suppose $\{w_1, w_2, \ldots, w_m\}$ is linearly independent. Then $m \le n$ and V is spanned by a set of the form

$$\{w_1, w_2, \ldots, w_m, v_{i_1}, v_{i_2}, \ldots, v_{i_{n-m}}\}$$

Thus, in particular, n+1 or more vectors in V are linearly dependent.

Ex. 1) The n vectors $e_1 = (1,0,0,\ldots 0)$, $e_2 = (0,1,0,\ldots 0)$, ... $e_n(0,0,\ldots,1)$ form a basis for K^n.

2) For vector space $V = M_{2\times 3}$ set of all matrices of order 2×3 basis is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3) The set $S = \{1, t, t^2, \cdots, t^n\}$ of $n+1$ polynomials is a basis for vector space $P_n(t)$. (set of ~~all~~ all polynomials whose degree is less than equal to n.)

• To finding a basis for $W = \text{span}(u_1, u_2, \cdots, u_r)$.

Given a list $S = \{u_1, u_2, \cdots, u_r\}$ of vectors in K^n and we want to find a basis for the subspace W of K^n spanned by the given vectors, i.e a basis of

$$W = \text{span}(S) = \text{span}(u_1, u_2, u_r).$$

Step 1 : Form the matrix M whose rows are the given vectors

Step 2 : Row reduce M to echelon form

Step 3 : Output the nonzero rows of the echelon matrix.

The non-zero rows in echelon form are form basis for $W = \text{span}(u_i)$.

Ex. Let W be the subspace of $\mathbb{R}^5$ spanned by the following vectors $u_1 = (1, 2, 1, 3, 2)$, $u_2 = (1, 3, 3, 5, 3)$
$u_3 = (3, 8, 7, 13, 8)$, $u_4 = (1, 4, 6, 9, 7)$, $u_5 = (5, 13, 13, 25, 19)$

Find a basis for W.

Solⁿ Given

$u_1 = (1, 2, 1, 3, 2)$ $u_2 = (1, 3, 3, 5, 3)$ $u_3 = (3, 8, 7, 13, 8)$
$u_4 = (1, 4, 6, 9, 7)$ $u_5 = (5, 13, 13, 25, 19)$.

Form a matrix M whose rows are given vectors,

$$M = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 1 & 3 & 3 & 5 & 3 \\ 3 & 8 & 7 & 13 & 8 \\ 1 & 4 & 6 & 9 & 7 \\ 5 & 13 & 13 & 25 & 19 \end{bmatrix}$$

$R_2 - R_1,\ R_3 - 3R_1,\ R_4 - R_1,\ R_5 - 5R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 2 & 4 & 4 & 2 \\ 0 & 2 & 5 & 6 & 5 \\ 0 & 3 & 8 & 10 & 9 \end{bmatrix}$$

$R_3 - 2R_2,\ R_4 - 2R_2,\ R_5 - 3R_2$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 4 & 6 \end{bmatrix} \underset{\sim}{R_5 - 2R_4} \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

R_{34}

$$\sim \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Basis for $W = \text{span}(u_i) = \{(1,2,1,3,2),\ (0,1,2,2,1),\ (0,1,2,3)\}$.

- Four fundamental subspaces

1) Row space : If A is $m \times n$ matrix then the subspace of $\mathbb{R}^n$ spanned by the rows of A is called the row space of A.

The row space of A is the column space of A^T.

It's dimension is the rank of matrix $\rho(A) = r$.

2) Column space :- If A is $m \times n$ matrix then the subspace of $\mathbb{R}^m$ spanned by the columns (vectors) of A is called the column space of A.

Column space of A is denoted by $C(A)$ or columnsp(A).

It's dimension is also the rank r.

3) Null space of A : The solution space of the homogeneous system of equation $AX = 0$ which is a subspace of $\mathbb{R}^n$ is called the null space of A.

Null space of A is denoted by $N(A)$. It's dimension is $n - r$.

4) Left null space of A : The left null space of A is the null space of A^T. It is written as $N(A^T)$, ~~It contains~~
It contains all vectors Y such that $A^T Y = 0$ and its dimension is $m - r$.

Ex ~~Find row space~~, ~~column space~~
Find the basis for row space, column space, null space left null space of A

where $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 5 & 6 & 4 \\ 2 & 6 & 8 & 5 \\ 1 & 3 & 1 & 2 \end{bmatrix}$

Soln, consider

$$AX = 0$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 5 & 6 & 4 \\ 2 & 6 & 8 & 5 \\ 1 & 3 & 1 & 2 \end{bmatrix} \quad \begin{matrix} R_2 - R_1, R_3 - 2R_1, \\ R_4 - 2R_1, R_5 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 4 & 2 \\ 0 & 4 & 6 & 3 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

$R_3 - 3R_2, R_4 - 4R_2, R_5 - 2R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & -4 & -2 \end{bmatrix} \quad R_4 - R_3, R_5 - 2R_3 \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Basis for row space of A is $\{(1,1,1,1), (0,1,2,1), (0,0,-2,-1)\}$.

$\rho(A) = 3$ dim row space of A is 3. i.e $\dim(\text{rowsp}(A)) = 3$.

Basis for column space of A is $\{(1,0,0,0,0), (1,1,0,0,0), (1,2,-2,0,0)\}$.

dimension of column space A is 3 ~~i.e dim columnsp~~

i.e $\dim(\text{columnsp}(A)) = 3$.

$\rho(A) = 3$ $n = 4$

No of parameter $= n - r = 4 - 3 = 1$

$x + y + z + w = 0$

$y + 2z + w = 0$

$-2z - w = 0$

Take $w = t$ $\therefore z = -t/2$

$\therefore y + 2(-t/2) + t = 0$ $\therefore y = 0$

$x + 0 - \frac{t}{2} + t = 0$ $\therefore x = -t/2$.

Take $t = 2$ $\therefore x = -1, y = 0, z = -1, w = 1$

$\therefore$ Basis for null space of A is $\{(-1,0,-1,1)\}$

$\dim(N(A)) = 1$.

Consider

$A^T Y = 0$

$$A^T = \begin{bmatrix} 1 & 1 & 2 & 2 & 1 \\ 1 & 2 & 5 & 6 & 3 \\ 1 & 3 & 6 & 8 & 1 \\ 1 & 2 & 4 & 5 & 2 \end{bmatrix} \underset{R_4 - R_1,\, R_5 - R_1}{\overset{R_2 - R_1,\, R_3 - R_1}{\sim}} \begin{bmatrix} 1 & 1 & 2 & 2 & 1 \\ 0 & 1 & 3 & 4 & 2 \\ 0 & 2 & 4 & 6 & 0 \\ 0 & 1 & 2 & 3 & 1 \end{bmatrix}$$

$R_3 - 2R_2,\ R_4 - R_2$

$$\begin{bmatrix} 1 & 1 & 2 & 2 & 1 \\ 0 & 1 & 3 & 4 & 2 \\ 0 & 0 & -2 & -2 & -4 \\ 0 & 0 & -1 & -1 & -1 \end{bmatrix} \overset{2R_4 - R_3}{\sim} \begin{bmatrix} 1 & 1 & 2 & 2 & 1 \\ 0 & 1 & 3 & 4 & 2 \\ 0 & 0 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Another basis for column space of A is

$\{(1,1,2,2,1), (0,1,3,4,2), (0,0,-2,-2,-2)\}$.

$\rho(A^T) = 3 \qquad n = 5 \qquad$ (for system $A^T Y = 0$)

no. of parameter $= 5 - 3 = 2$

$y_1 + y_2 + 2y_3 + 2y_4 + y_5 = 0$

$y_2 + 3y_3 + 4y_4 + 2y_5 = 0$

$-2y_3 - 2y_4 - 2y_5 = 0$

Take $y_4 = a,\ y_5 = b \qquad \therefore y_3 = -(a+b)$

$y_2 + 3.(a+b) + 4a - 2b = 0 \qquad \therefore y_2 = -a + 5b$

$y_1 - a + 5b - 2.(a+b) + 2a + b = 0$

$\therefore y_1 = a - 4b.$

Take $a = 1 \quad b = 0 \qquad \therefore y_1 = 1,\ y_2 = -1,\ y_3 = -1,\ y_4 = 1,\ y_5 = 0$

$a = 0,\ b = 1 \qquad y_1 = -4,\ y_2 = 5,\ y_3 = -1,\ y_4 = 0,\ y_5 = 1$

$\therefore$ Basis for left null space is $\{(1,-1,-1,1,0), (-4,5,-1,0,1)\}$