Represention of Point:

$[P] = [x \quad y]$ 1 row X 2 columns

$[P] = [x \quad y \quad z]$ 1 row X 3 columns

row matrix method

$[P] = \begin{bmatrix} x \\ y \end{bmatrix}$ 2 rows X 1 column.

$[P] = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ 3 rows X 1 column

column matrix method

Transformation & matrices.

transformation matrix [T]
- addition
- substraction
- multication
- Inverse of a matrix
- Transpose of a matrix

For 2D [T] is of size 2 X 2

$$[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Transformation of a point:

let $[X] = [x \quad y]$ represents a point P

& matrix $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a general 2x2 transformation matrix

$$[X][T] = [x \quad y]\begin{bmatrix} a & b \\ c & d \end{bmatrix} = [(ax+cy) \quad (bx+dy)] = [x^* \quad y^*] \quad -(1)$$

$$x^* = (ax+cy) \quad \& \quad y^* = (bx+dy)$$

Examples

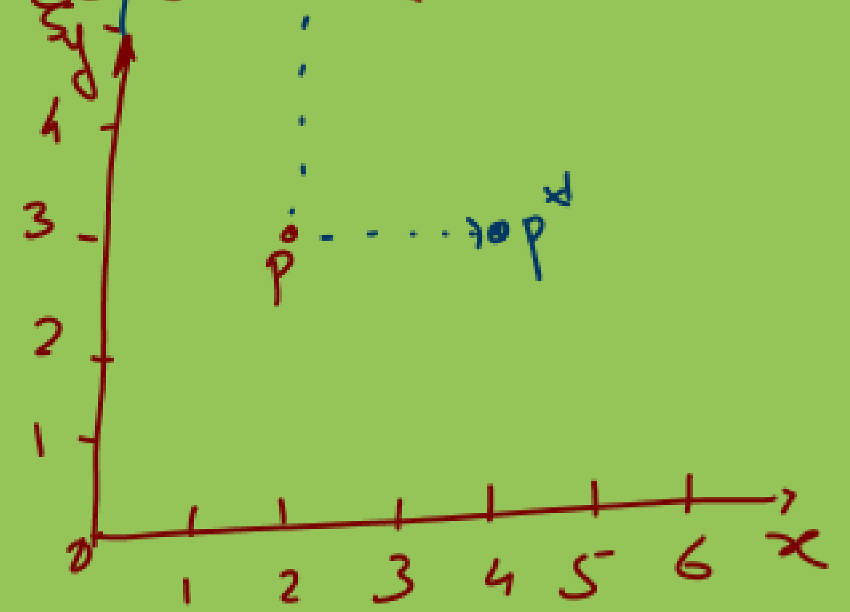
① case 1: $a = d = 1$ and $b = c = 0$

$$[X][T] = [x \quad y]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [x \quad y]$$, No change in position

Identity matrix

② **Case 2:** $d=1,\ b=c=0$

$$[X][T] = [x \quad y]\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} = [ax \quad y]$$ only change in x coordinate



$$[X][T] = [2 \quad 3]\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = [4 \quad 3] = [P^*]$$

$a=1,\ b=c=0$

$$[X][T] = [x \quad y]\begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix} = [x \quad dy]$$ only change in y coordinate

$$[X][T] = [2 \quad 3]\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = [2 \quad 6]$$

③ **Case 3:** $b=c=0$

$$[X][T] = [x \quad y] = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = [ax \quad dy] = [x^* \quad y^*]$$

Scaling of point in both direction.

$$[X][T] = [2 \quad 3]\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = [4 \quad 6] = [x^* \quad y^*]$$

If $a \neq d$ - scaling is not equal

if $a=d>1$ - pure enlargement or expansion

If $0 < a = d < 1$ - compression

Case 4: If a and/or d are negative, reflection through an axis

$b = c = 0, \ a = -1, \ d = 1$

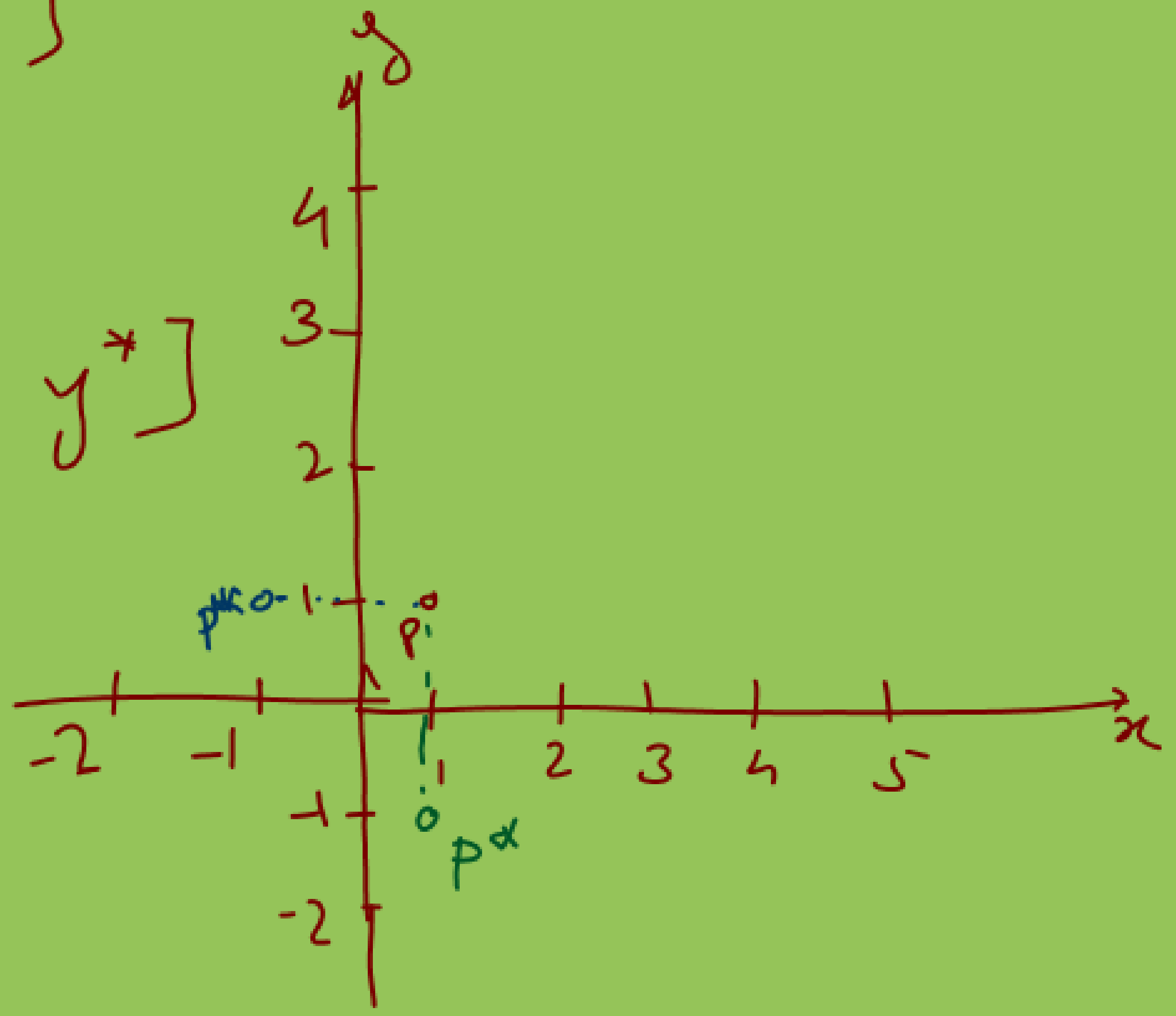
$$[X][T] = [x \quad y]\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = [-x \quad y] = [x^* \quad y^*]$$

Then a reflection through y-axis occurs

If $b = c = 0, \ a = 1, \ d = -1$

$$[X][T] = [x \quad y]\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = [x \quad -y] = [x^* \quad y^*]$$

Then a reflection through x-axis occurs.



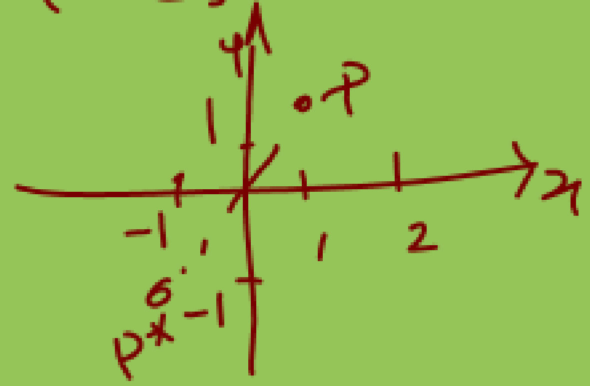
consider eg:

$$[X][T] = [1 \quad 1]\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = [-1 \quad 1]$$

$$[X][T] = [1 \quad 1]\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = [1 \quad -1]$$

(5) Case 5 : If $b = c = 0$ and $a = d < 0$ then a reflection through origin occurs

$$[X][T] = [1 \quad 1]\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = [-1 \quad -1] = [x^* \quad y^*]$$



(6) Case 6: Now consider the effect of off diagonal coordinates

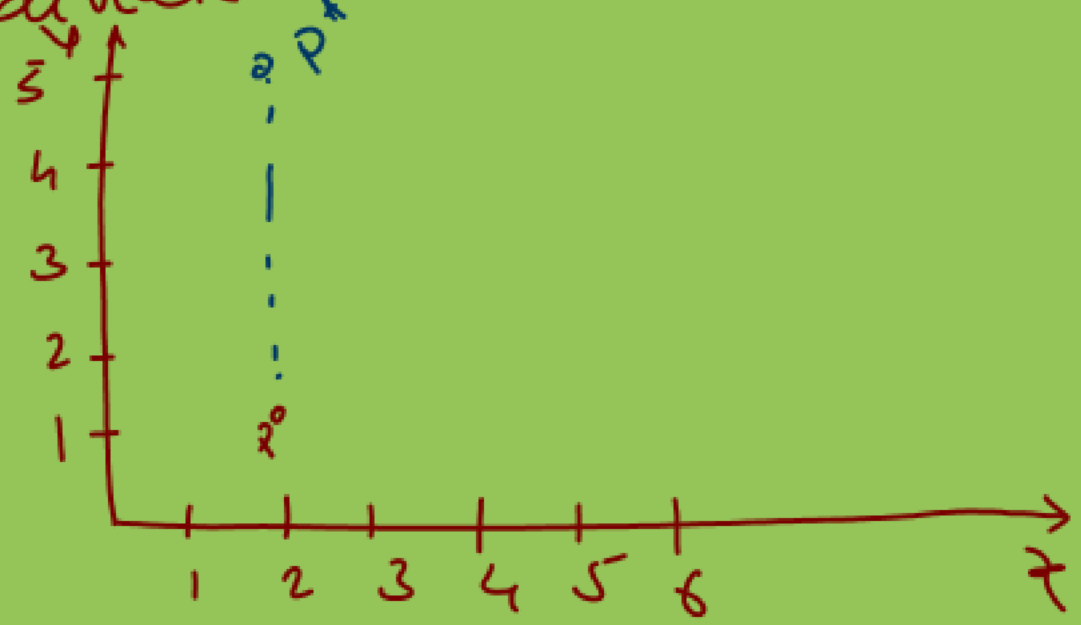
Let $a = d = 1$ and $c = 0$

$$[X][T] = [x \quad y]\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = [x \quad (bx + y)] = [x^* \quad y^*]$$

The x coordinate is unchanged while y^* depends on original coordinates x & y. - shearing effect proportional to x-coordinate.

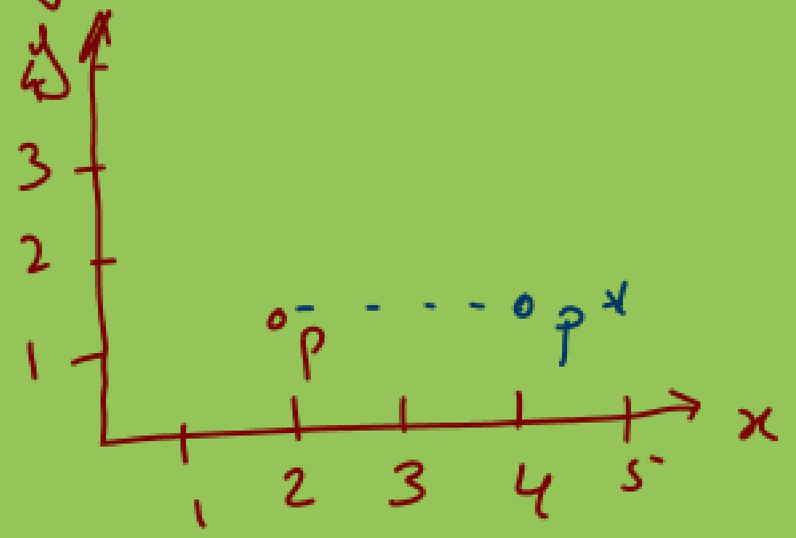
eg.

$$[X][T] = [2 \quad 1]\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = [2 \quad 5] = [x^* \quad y^*]$$



7 <u>Case 7</u>: $a=d=1, b=0$ produces shear effect proportional to y coordinates

$$[X][T] = [x \quad y]\begin{bmatrix}1 & 0\\ c & 1\end{bmatrix} = [(x+cy) \quad y] = [x^* \quad y^*]$$



e.g. $$[X][T] = [2 \quad 1]\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix} = [4 \quad 1] \; [x^* \quad y^*]$$

8 <u>Case 8</u>: Transformation of origin

$$[X][T] = [0 \quad 0]\begin{bmatrix}a & b\\ c & d\end{bmatrix} = [0 \quad 0]$$

Limitation - we cannot transfer the origin

↳ overcame - Homogeneous coordinates

Transformation of Line:

$$[L] = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}$$

eg $[A] = [0 \quad 1]$ & $[B] = [2 \quad 3]$

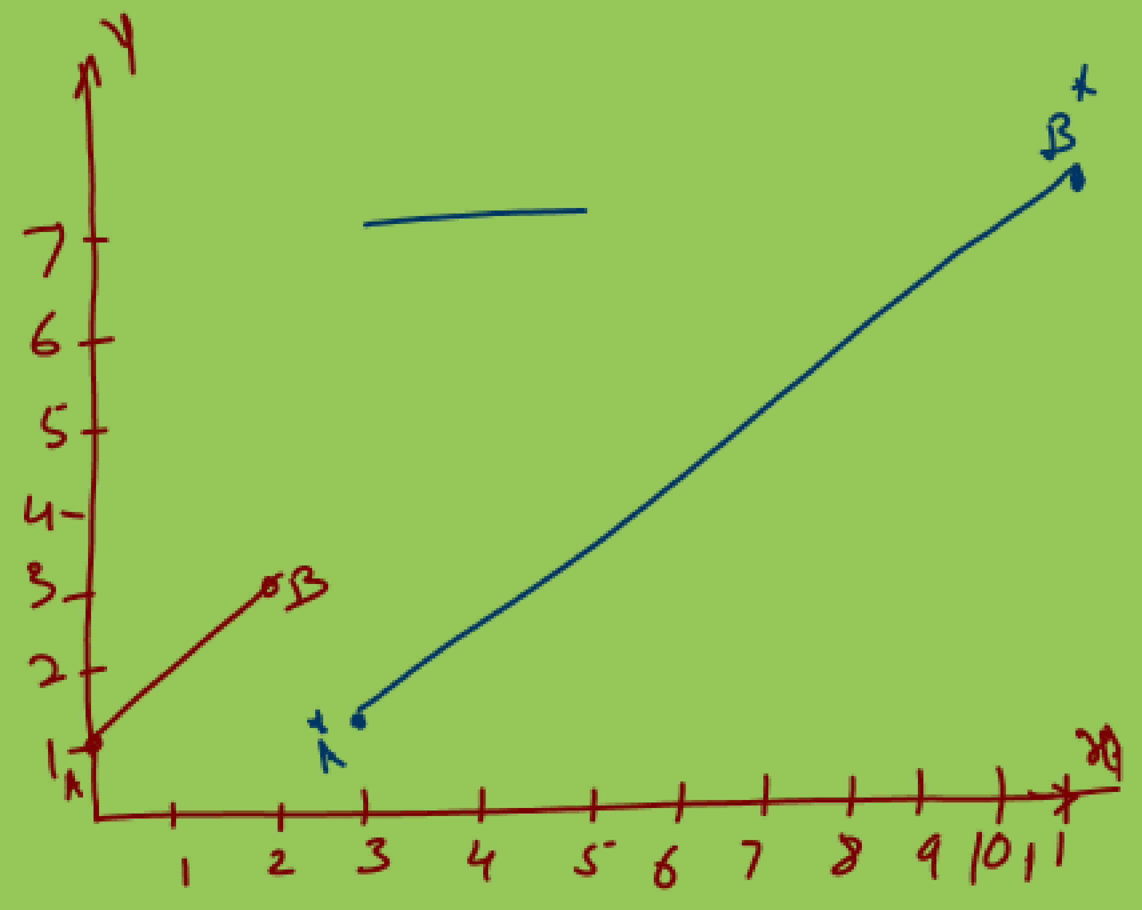
& $[T] = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$

$$[A][T] = [0 \quad 1] \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = [3 \quad 1] = [A^*]$$

$$[B][T] = [2 \quad 3] \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = [11 \quad 7] = [B^*]$$

OR

$$[L][T] = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 11 & 7 \end{bmatrix} = \begin{bmatrix} A^* \\ B^* \end{bmatrix} = [L^*]$$



Midpoint Transformation:

Points on 2nd line one-to-one correspondence with points on 1s line

Consider the transformation of midpoint

Let

$[A] = [x_1 \quad y_1]$, $[B] = [x_2 \quad y_2]$; & $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

After transforming both end points

$$\begin{bmatrix} A \\ B \end{bmatrix} [T] = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ax_1 + cy_1 & bx_1 + dy_1 \\ ax_2 + cy_2 & bx_2 + dy_2 \end{bmatrix} = \begin{bmatrix} A^* \\ B^* \end{bmatrix} \quad \text{--- (1)}$$

Hence, end points of transformed line are

$$\left.\begin{aligned} [A^*] &= [ax_1 + cy_1 \quad bx_1 + dy_1] \; [x_1^* \quad y_1^*] \\ [B^*] &= [ax_2 + cy_2 \quad bx_2 + dy_2] \; [x_2^* \quad y_2^*] \end{aligned}\right\} \text{ - (2)}$$

Now we calculate midpoint of transformed line $A^* B^*$

$$[x_m^* \quad y_m^*] = \left[\frac{x_1^* + x_2^*}{2} \quad \frac{y_1^* + y_2^*}{2}\right] = \left[\frac{(ax_1 + cy_1) + (ax_2 + cy_2)}{2} \quad \frac{(bx_1 + dy_1)(bx_2 + dy_2)}{2}\right]$$

$$[x_m^* \quad y_m^*] = \left[a\frac{(x_1 + x_2)}{2} + c\frac{(y_1 + y_2)}{2} \quad b\frac{(x_1 + x_2)}{2} + d\frac{(y_1 + y_2)}{2}\right] \text{ - (3)}$$

Now calculate midpoint of original line AB

$$\begin{bmatrix} x_m & y_m \end{bmatrix} = \begin{bmatrix} \frac{x_1+x_2}{2} & \frac{y_1+y_2}{2} \end{bmatrix} \quad \text{--- (4)}$$

Using [T] transform mid point

$$\begin{bmatrix} x_m & y_m \end{bmatrix}[T] = \begin{bmatrix} \frac{x_1+x_2}{2} & \frac{y_1+y_2}{2} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a\frac{(x_1+x_2)}{2} + c\frac{(y_1+y_2)}{2} & b\frac{(x_1+x_2)}{2} + d\frac{(y_1+y_2)}{2} \end{bmatrix} \quad \text{--- (5)}$$

Ex: Consider the line AB with

$[A] = [0 \quad 1]$, $B = [2 \quad 3]$

& the transformation matrix $[T] = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$

After transformation of original line

$$[L][T] = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 11 & 7 \end{bmatrix} = \begin{bmatrix} A^* \\ B^* \end{bmatrix}$$

Now calculate midpoint of A^*B^*

$$\begin{bmatrix} x_m^* & y_m^* \end{bmatrix} = \begin{bmatrix} \frac{3+11}{2} & \frac{1+7}{2} \end{bmatrix} = \begin{bmatrix} 7 & 4 \end{bmatrix} \quad \text{--- (1)}$$

Now calculate midpoint of original line AB

$$[x_m \quad y_m] = \left[\frac{0+2}{2} \quad \frac{1+3}{2}\right] = [1 \quad 2]$$

Transforming this mid point we will get

$$[x_m \quad y_m][T] = [1 \quad 2]\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = [7 \quad 4] \quad -(2)$$

* **How to Find eq^n of a line:**

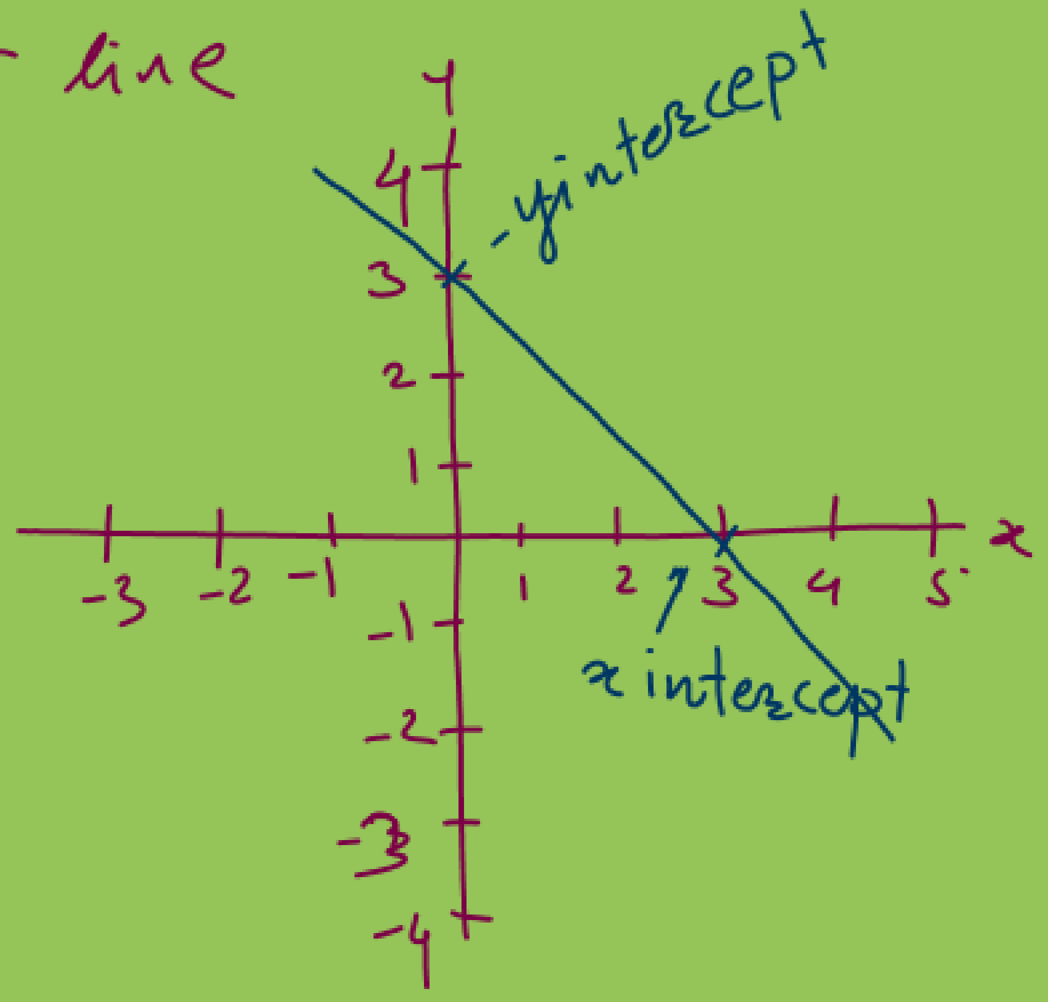
$(y - y_1) = m(x - x_1)$ where m is slope of line

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

eg: ① $\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$, $m = \frac{3-1}{2-0} = \frac{2}{2} = 1$

$(y - 1) = 1(x - 0)$

$y = x + \underline{1}$ 1 is y intercept



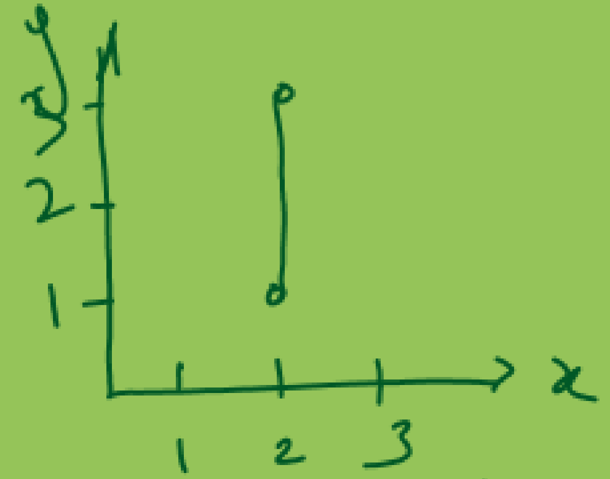
*Forms for eq^n of a line

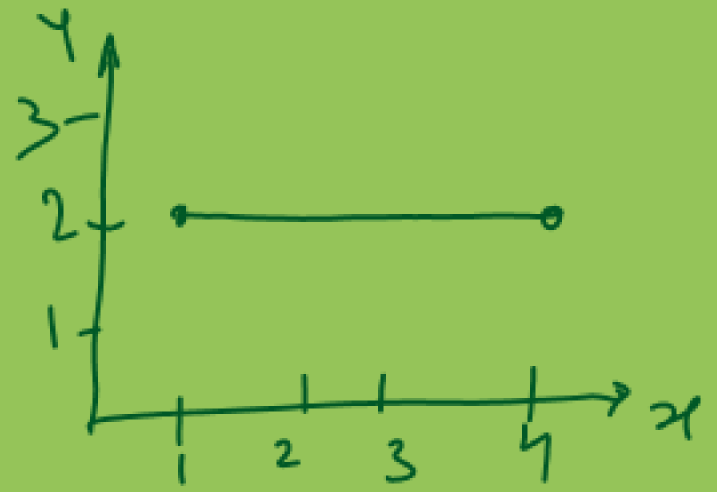
① $y = mx + b$ where m is slope & b is y-intercept - slope intercept

② Point slope - $(y - y_1) = m(x - x_1)$

③ Standard Form - $Ax + By = C$

④ Two intercept : $\frac{x}{a} + \frac{y}{b} = 1$





⑤ Vertical line : $x = a$ All points have x-coordinate value a

⑥ Horizontal line : $y = b$ All points have y-coordinate value b

*Transformation of parallel lines :

Consider $A = [x_1 \quad y_1]$ & $B[x_2 \quad y_2]$ & a line EF is parallel to AB

Examine slopes of AB, EF, A^*B^* & E^*F^*

Since AB & EF are parallel, the slope of both are equal

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{—— (1)}$$

Transforming original line AB,

$$\begin{bmatrix} A \\ B \end{bmatrix} [T] = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ax_1 + cy_1 & bx_1 + dy_1 \\ ax_2 + cy_2 & bx_2 + dy_2 \end{bmatrix} = \begin{bmatrix} A^* \\ B^* \end{bmatrix} = \begin{bmatrix} x_1^* & y_1^* \\ x_2^* & y_2^* \end{bmatrix} \text{ — (2)}$$

Using eq^n(2) calculate slope of transformed line A^*B^*

$$m^* = \frac{(bx_2 + dy_2) - (bx_1 + dy_1)}{(ax_2 + cy_2) - (ax_1 + cy_1)} = \frac{b(x_2 - x_1) + d(y_2 - y_1)}{a(x_2 - x_1) + c(y_2 - y_1)}$$

$$m^* = \frac{b + d\left(\frac{y_2 - y_1}{x_2 - x_1}\right)}{a + c\left(\frac{y_2 - y_1}{x_2 - x_1}\right)} = \frac{b + dm}{a + cm} \text{ — (3)}$$

Slope m^* is independent on x_1, y_1, x_2 & y_2 & dependent on a, b, c, d, m

Since a, b, c, d & m are same for AB & EF, it shows that m^* is same A^*B^* & E^*F^*

How to calculate inverse of matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Determinant of $A = |A| = ad - bc = 4 - 6 = -2$

Adjoint of $A = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} * \text{Adjoint of } A$$

$$= \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

To check

$$A * A^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[M] = \begin{bmatrix} -m_1 & -m_2 \\ 1 & 1 \end{bmatrix}$$

$$[M]^{-1} = \begin{bmatrix} \frac{1}{m_2 - m_1} & \frac{m_2}{m_2 - m_1} \\ \frac{-1}{m_2 - m_1} & \frac{-m_1}{m_2 - m_1} \end{bmatrix}$$

Transformation of Intersecting lines :

Let us consider a pair of straight lines, represented by

$$\left.\begin{aligned} y &= m_1 x + b_1 \\ y &= m_2 x + b_2 \end{aligned}\right\} \rightarrow \begin{aligned} b_1 &= -m_1 x + y \\ b_2 &= -m_2 x + y \end{aligned}$$

Reformulating the eq^n in matrix form

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} -m_1 & -m_2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \end{bmatrix}$$

$$[X]\,[M] = [B] \quad \text{— (1)}$$

If a sol^n to these pair of eq^{ns} exist, then the lines are intersect else parallel.

A sol^n can be obtained by matrix inverse

Let $[X_i]$ - be the intersection point of two lines

$$[X_i] = [x_i \quad y_i] = [B][M]^{-1} \quad \text{——} \ (2)$$

The inverse of $[M]$ is

$$[M]^{-1} = \begin{bmatrix} \frac{1}{m_2-m_1} & \frac{m_2}{m_2-m_1} \\ \frac{-1}{m_2-m_1} & \frac{-m_1}{m_2-m_1} \end{bmatrix} \quad \text{——} \ (3)$$

Hence intersection of 2 lines is

$$[X_i] = [x_i \quad y_i] = [b_1 \quad b_2] \begin{bmatrix} \frac{1}{m_2-m_1} & \frac{m_2}{m_2-m_1} \\ \frac{-1}{m_2-m_1} & \frac{-m_1}{m_2-m_1} \end{bmatrix}$$

$$[X_i] = [x_i \quad y_i] = \begin{bmatrix} \frac{b_1-b_2}{m_2-m_1} & \frac{b_1m_2-b_2m_1}{m_2-m_1} \end{bmatrix} \text{—} \ (4)$$ Intersection of original lines

Now we calculate intersection of transformed lines

$$[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The equation for transformed lines

$$y^* = m_1^* x^* + b_1^*$$

$$y^* = m_2^* x^* + b_2^*$$

Using eq^n of slope m^*

$$m_i^* = \frac{b + d m_i}{a + c m_i} \qquad i = 1 \text{ or } 2 \quad -(5)$$

So

$$m_1^* = \frac{b + d m_1}{a + c m_1} \quad \& \quad m_2^* = \frac{b + d m_2}{a + c m_2} \quad -(6)$$

Now the intersection of transformed line is

$$[X_i^*] = [x_i^* \quad y_i^*] = [B^*]\,[M^*]^{-1}$$

$$= [b_1^* \quad b_2^*] \begin{bmatrix} \dfrac{1}{m_2^* - m_1^*} & \dfrac{m_2^*}{m_2^* - m_1^*} \\ \dfrac{-1}{m_2^* - m_1^*} & \dfrac{-m_1}{m_2^* - m_1^*} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{b_1^* - b_2^*}{m_2^* - m_1^*} & \dfrac{b_1 m_2^* - b_2 m_1^*}{m_2^* - m_1^*} \end{bmatrix}$$

Rewriting the components of intersection point using eq^n (5) & (6)

$$[X_i^*] = [x_i^* \quad y_i^*] = \begin{bmatrix} \dfrac{a(b_1 - b_2) + b(m_2 - m_1)}{m_2 - m_1} & \dfrac{b(b_1 - b_2) + d(b_1 m_2 - b_2 m_1)}{m_2 - m_1} \end{bmatrix} \quad -(7)$$

This is intersection of transformed lines

Now apply transformation matrix on intersection of original lines

$$[X_i^{\#}] = [x_i^* \;\; y_i^*] = [x_i \;\; y_i] \; [T]$$

$$= \begin{bmatrix} \dfrac{b_1 - b_2}{m_2 - m_1} & \dfrac{b_1 m_2 - b_2 m_1}{m_2 - m_1} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{a(b_1 - b_2) + c(b_1 m_2 - b_2 m_1)}{m_2 - m_1} & \dfrac{b(b_1 - b_2) + d(b_1 m_2 - b_2 m_1)}{m_2 - m_1} \end{bmatrix} \quad - (8)$$

Comparing eq^n (7) & (8), show that they are identical. Means transformation of two intersecting lines generate another pair of intersecting lines.