Unit 2: Transformation

Transformation

Basic 2D & 3D transformations

- Representation of Points
- Transformations and Matrices
- Transformation of Points
- Transformation of Straight Lines
- Midpoint Transformations
- Transformation of Parallel Lines
- Transformation of Intersecting Lines
- Rotation
- Reflection
- Scaling

Transformations and Homogeneous Coordinates

Rotation about an Arbitrary Point

Reflection through an Arbitrary Line

Rotation about an axis parallel to a coordinate axis

Rotation about an arbitrary axis in space

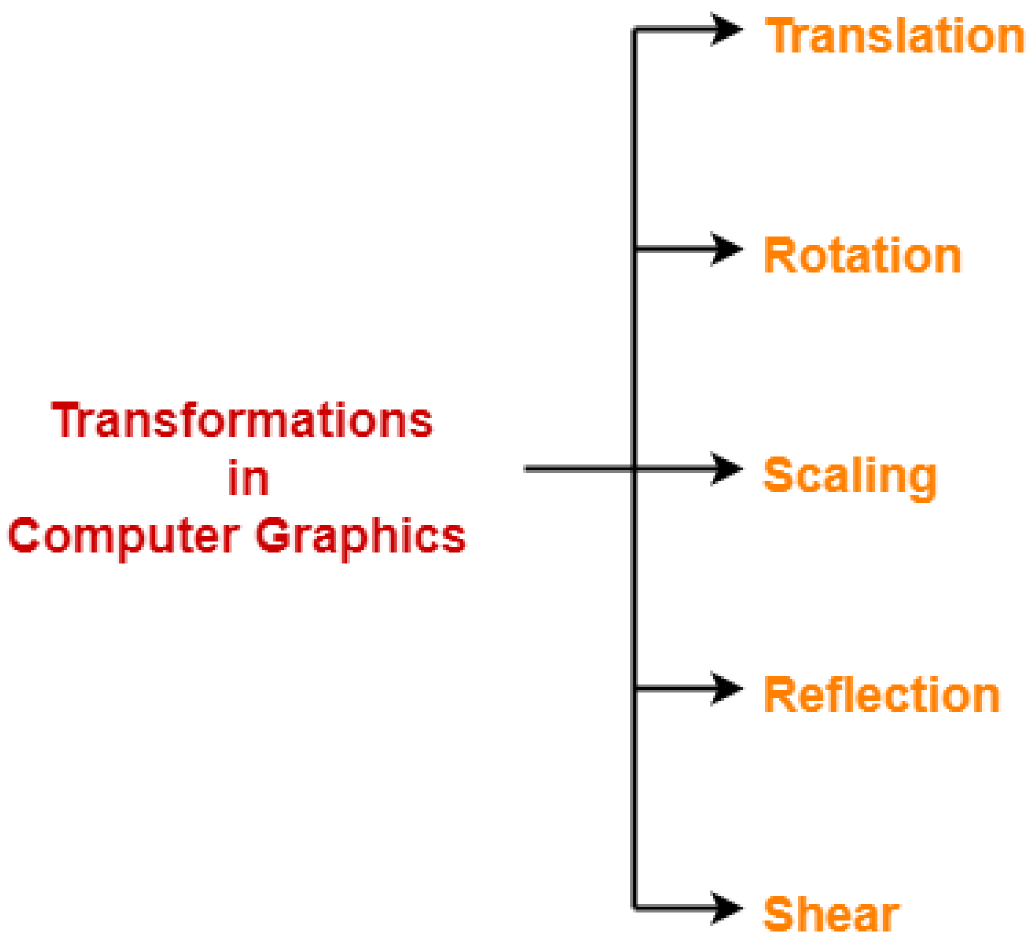
Reflection through an arbitrary plane

Windowing and View-porting

Sutherland - Cohen line clipping algorithm

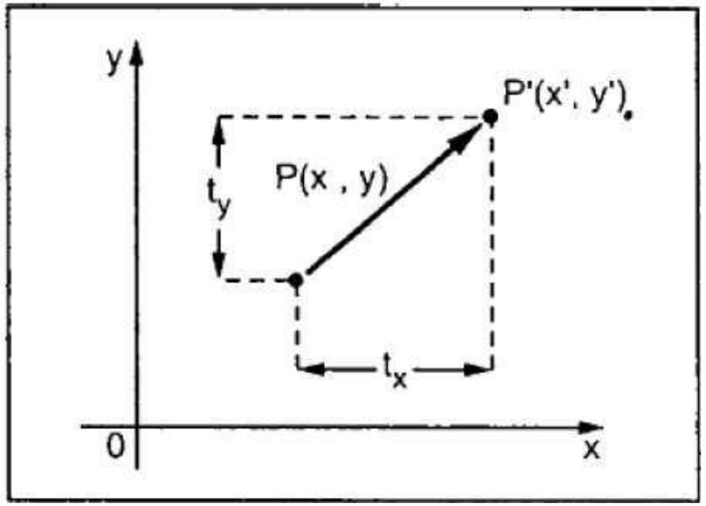
Transformation

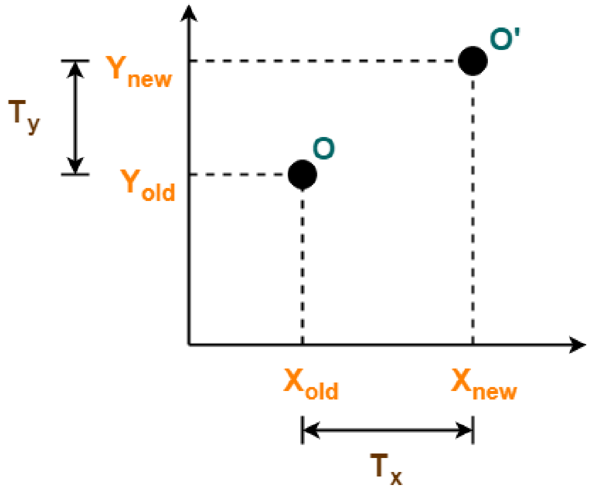
- Transformation changes the way object appears.
- Transformation means changing some graphics into something else by applying rules.
- Implementing changes in size of object, its position on screen or its orientation called **Transformation.**
- When a transformation takes place on a 2D plane, it is called 2D transformation.



Transformation

- **Translation**: A translation moves an object to a different position on the screen.

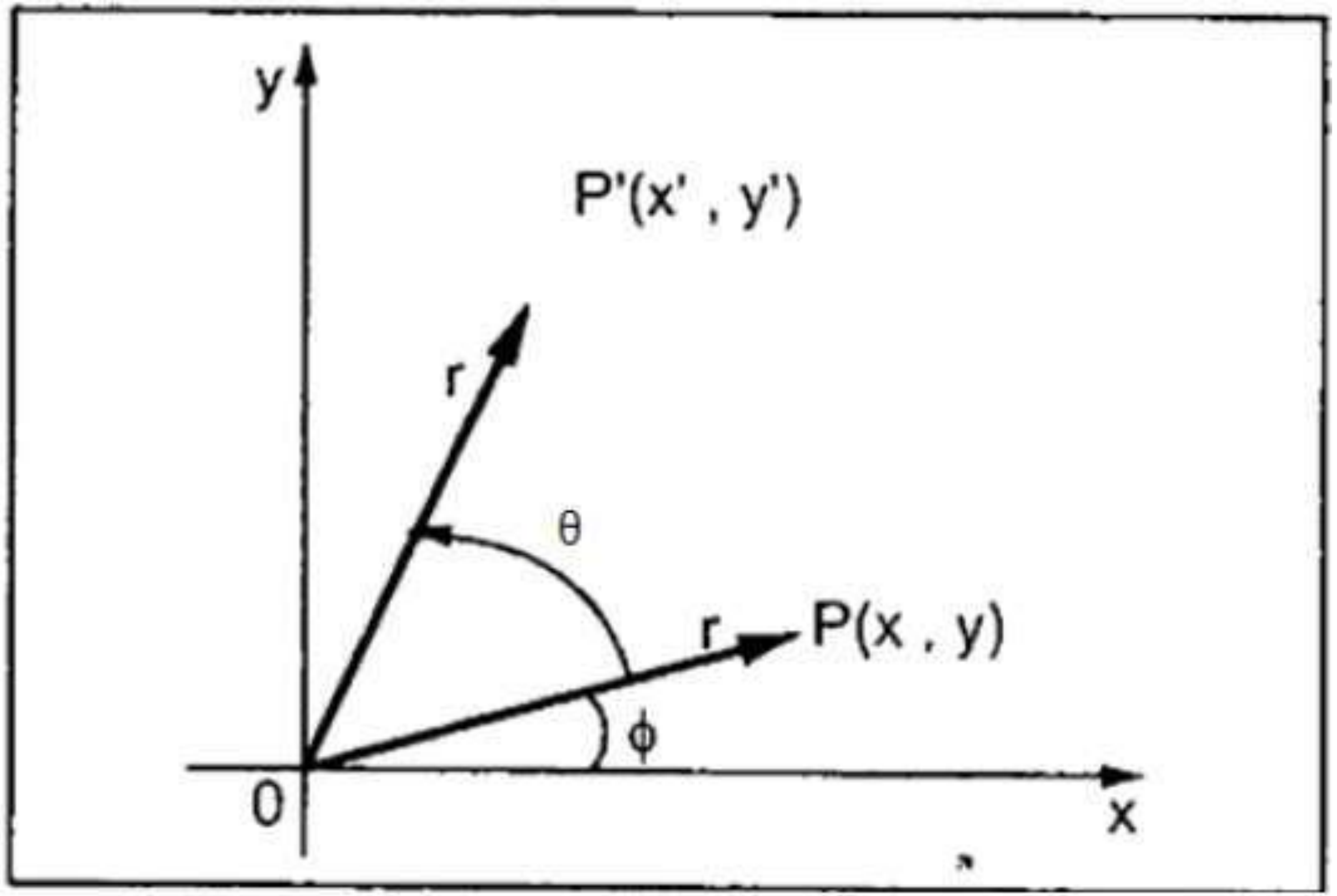




2D Translation in Computer Graphics

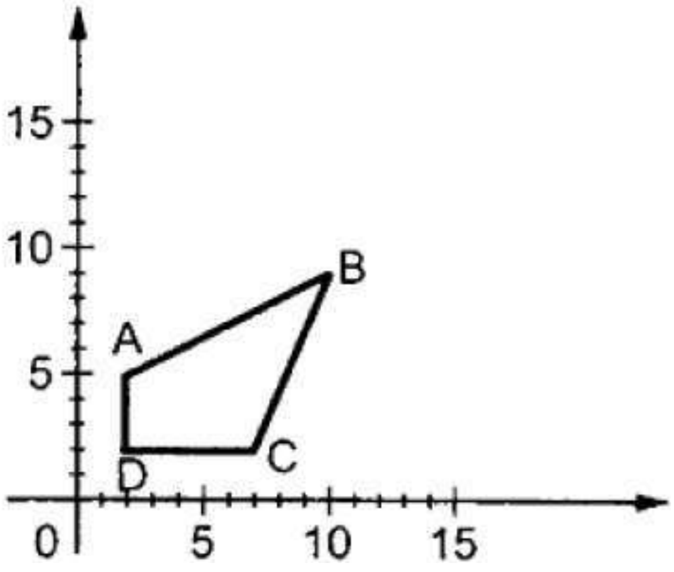
Transformation

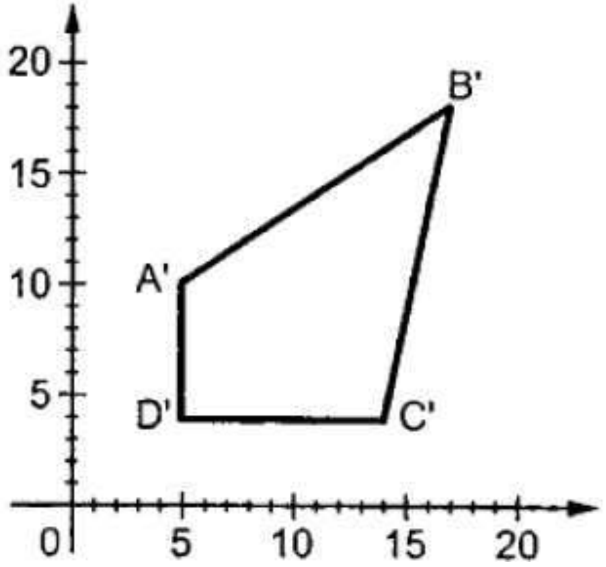
- Rotation: In rotation, we rotate the object at particular angle θ theta from its origin.



Transformation

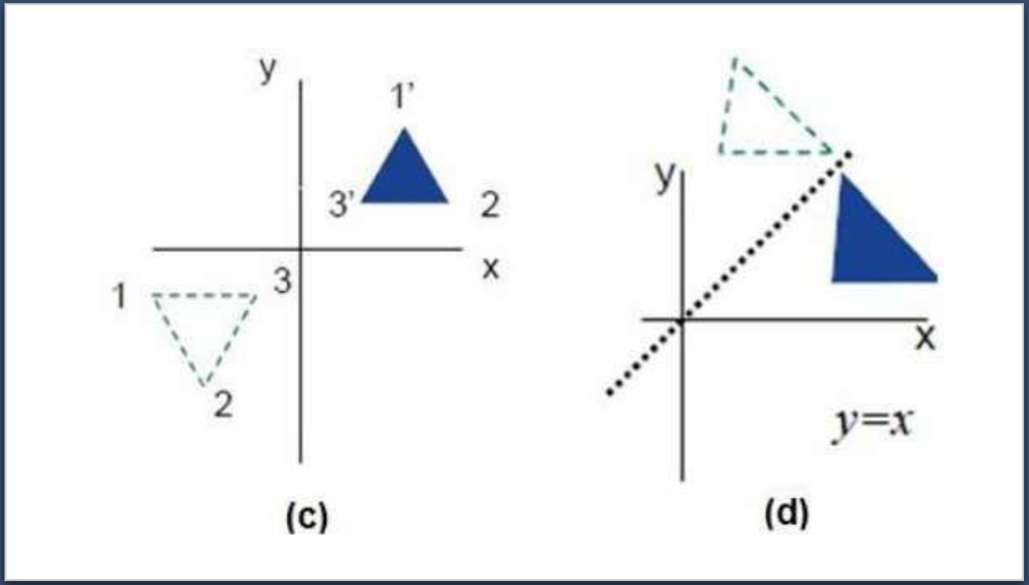
- **Scaling :** To change the size of an object, scaling transformation is used. In the scaling process, you either expand or compress the dimensions of the object using scaling factor.

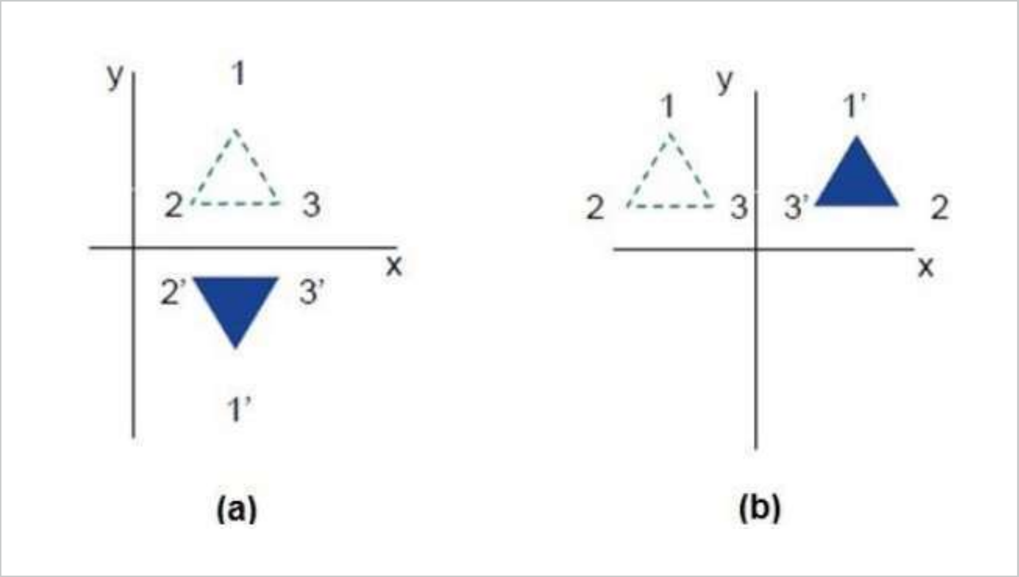




Transformation

- **Reflection:** Reflection is the mirror image of original object. In other words, we can say that it is a rotation operation with 180°. In reflection transformation, the size of the object does not change.



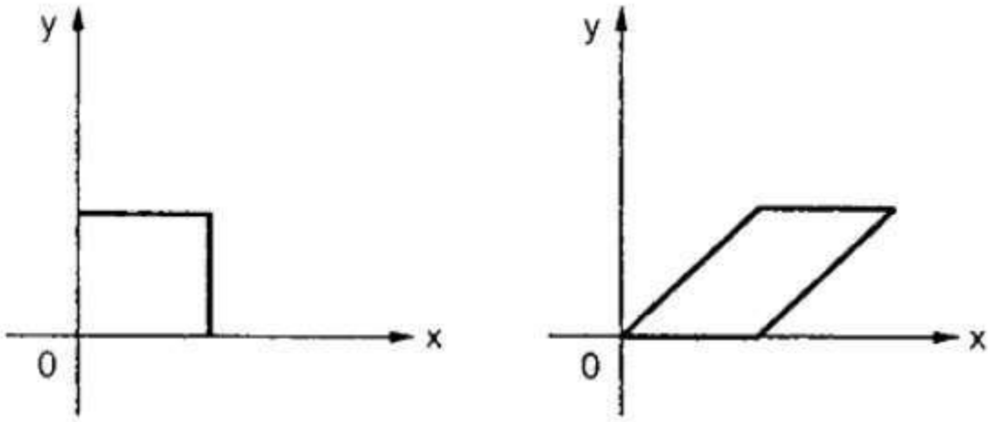


Transformation

- **Shear:** A transformation that slants the shape of an object is called the shear transformation.
- There are two shear transformations **X-Shear** and **Y-Shear**.
- One shifts X coordinates values and other shifts Y coordinate values.
- However; in both the cases only one coordinate changes its coordinates and other preserves its values.
- Shearing is also termed as **Skewing**.

Transformation

- **X-Shear:** The X-Shear preserves the Y coordinate and changes are made to X coordinates, which causes the vertical lines to tilt right or left as shown in below figure.

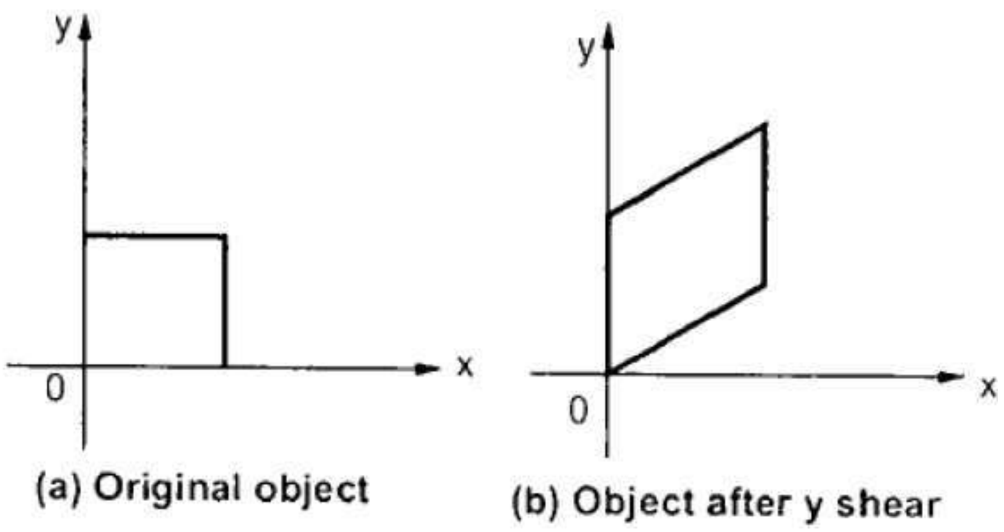


(a) Original object

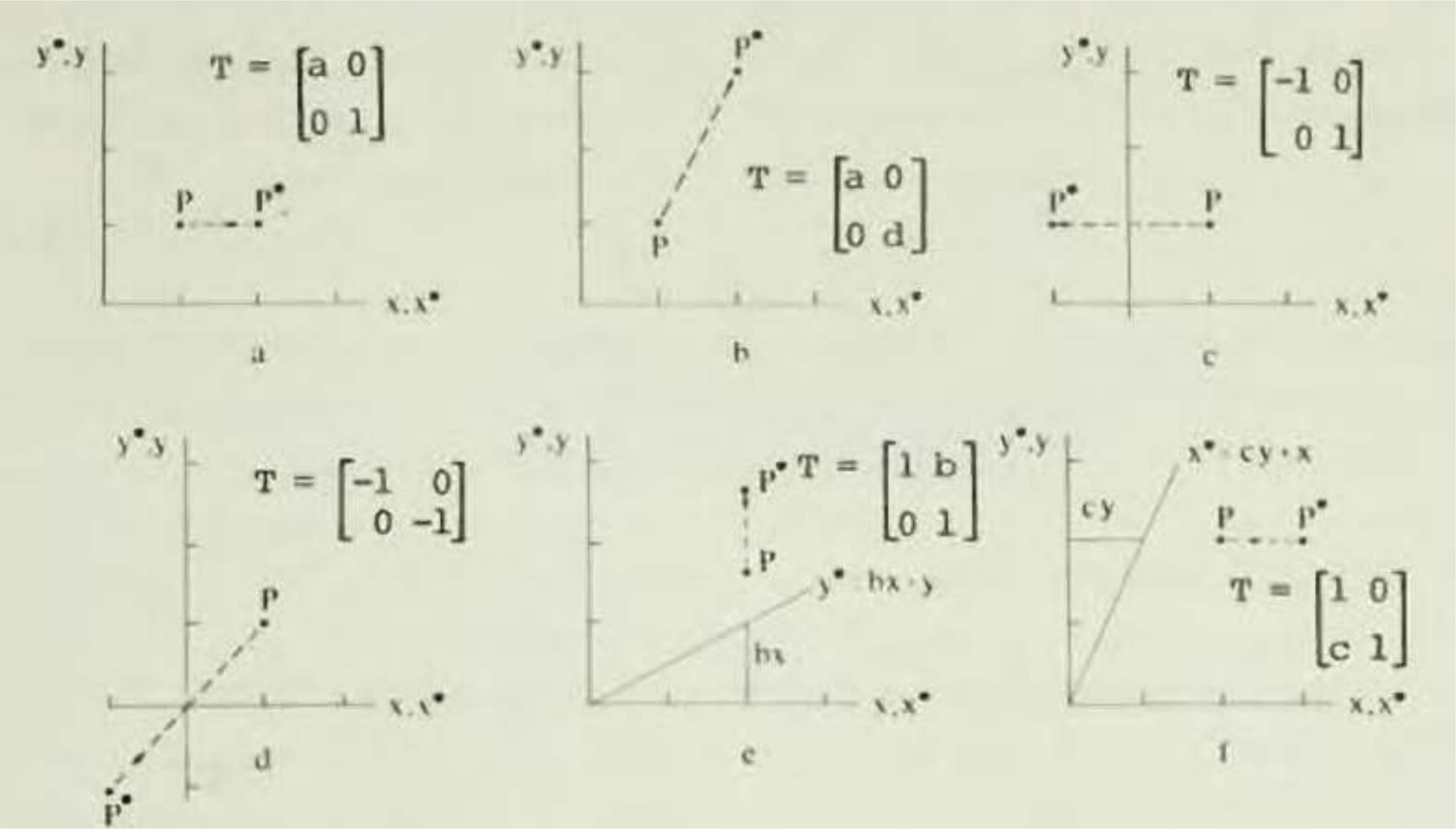
(b) Object after x shear

Transformation

- **Y-Shear:** The Y-Shear preserves the X coordinates and changes the Y coordinates which causes the horizontal lines to transform into lines which slopes up or down as shown in the following figure.



2D translation of point



Transformations of Intersecting Lines

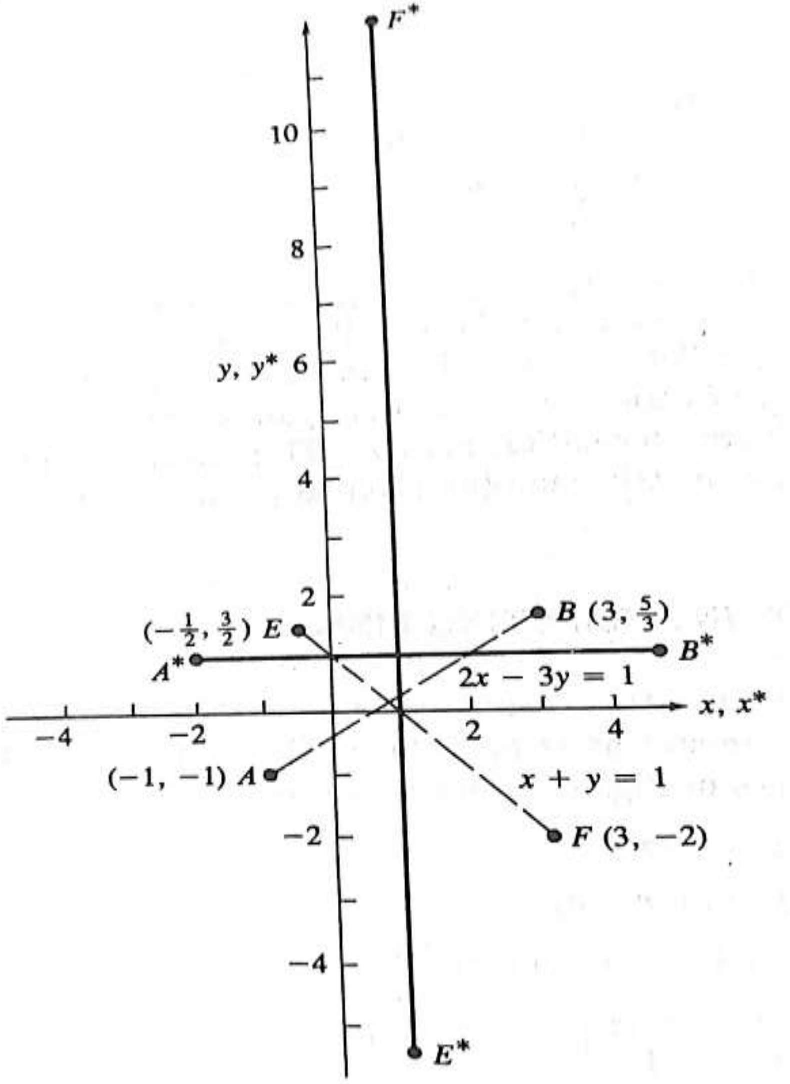


Figure 2–3 Transformation of intersecting lines.

Rotation

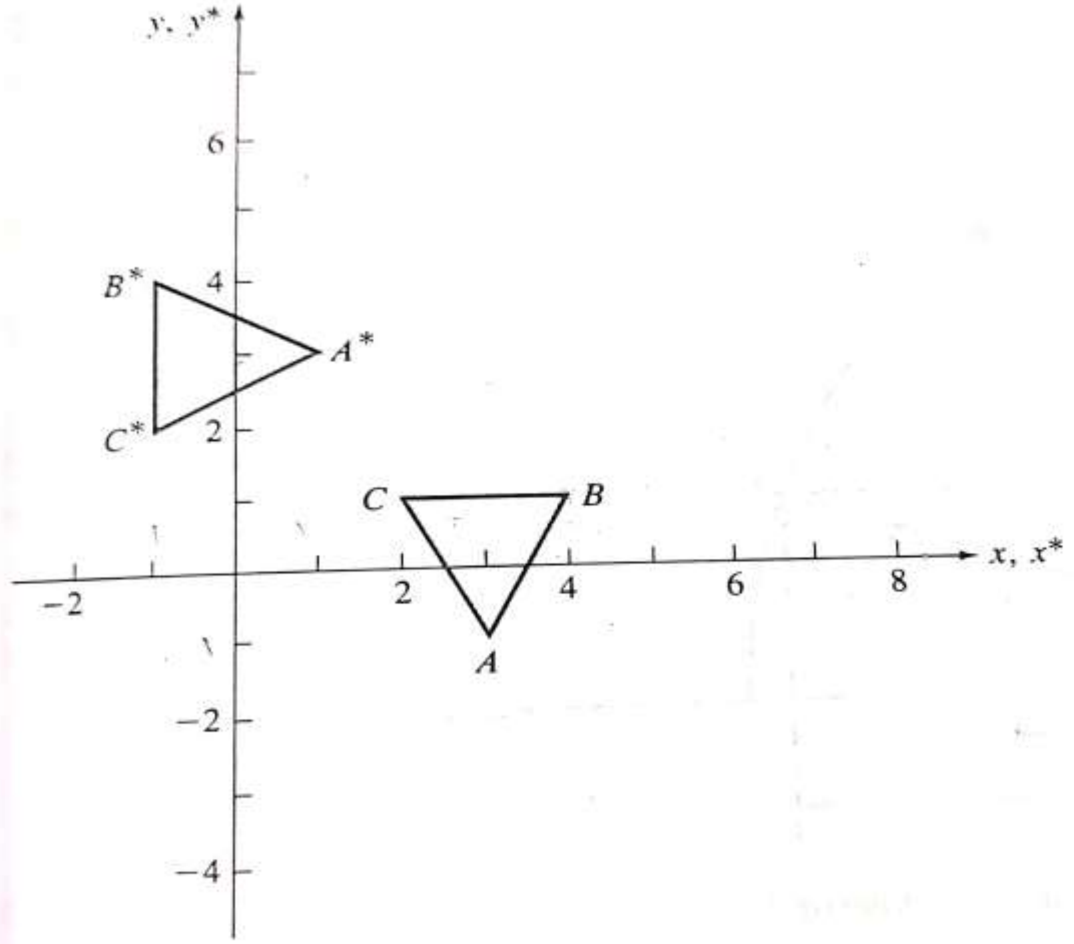


Figure 2–4 Rotation.

Rotation of a Position Vector

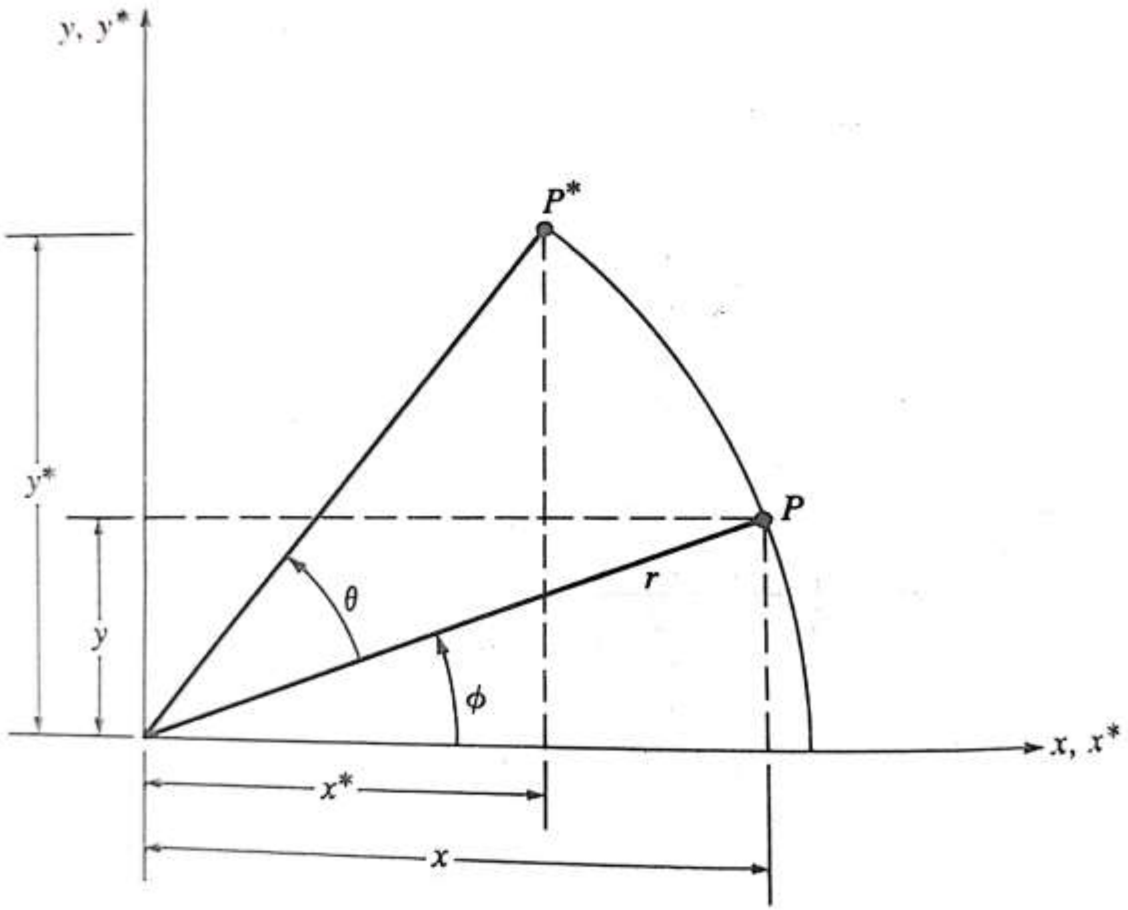


Figure 2–5 Rotation of a position vector.

Reflection

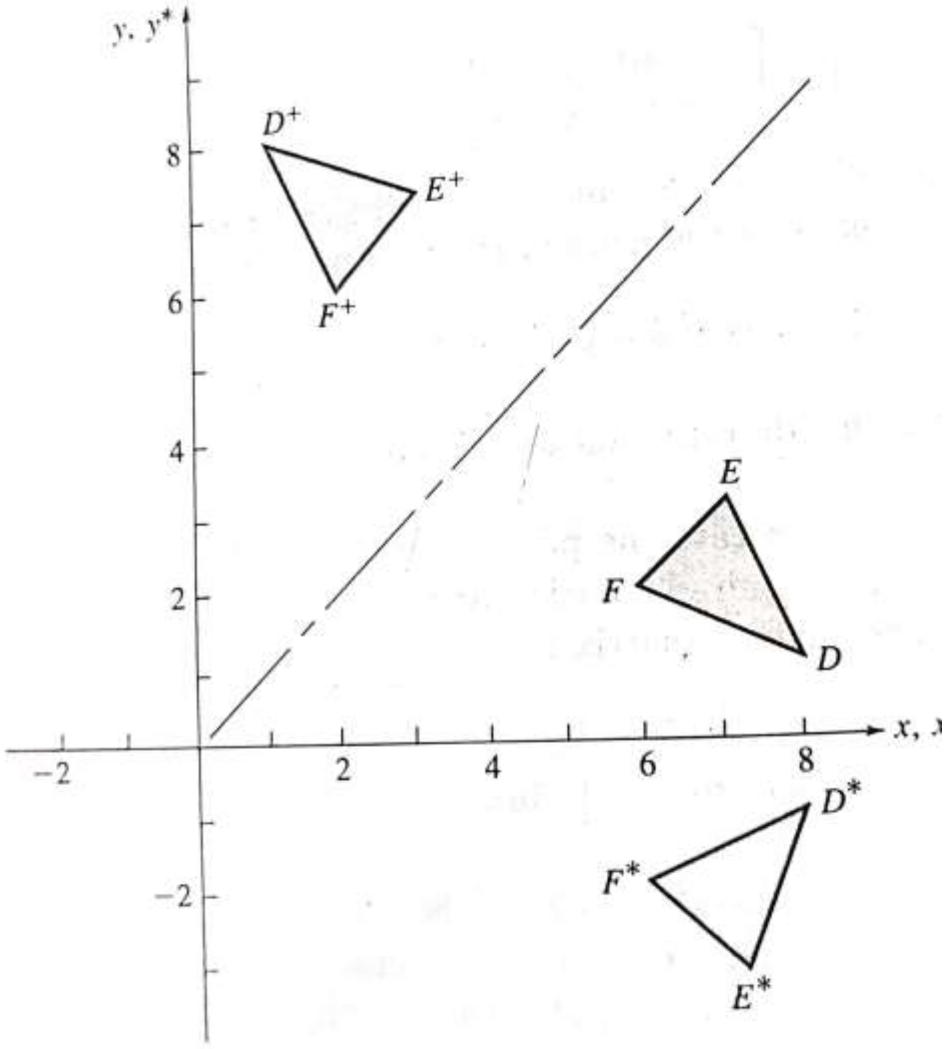


Figure 2–6 Reflection.

Combined Reflection yields Rotation

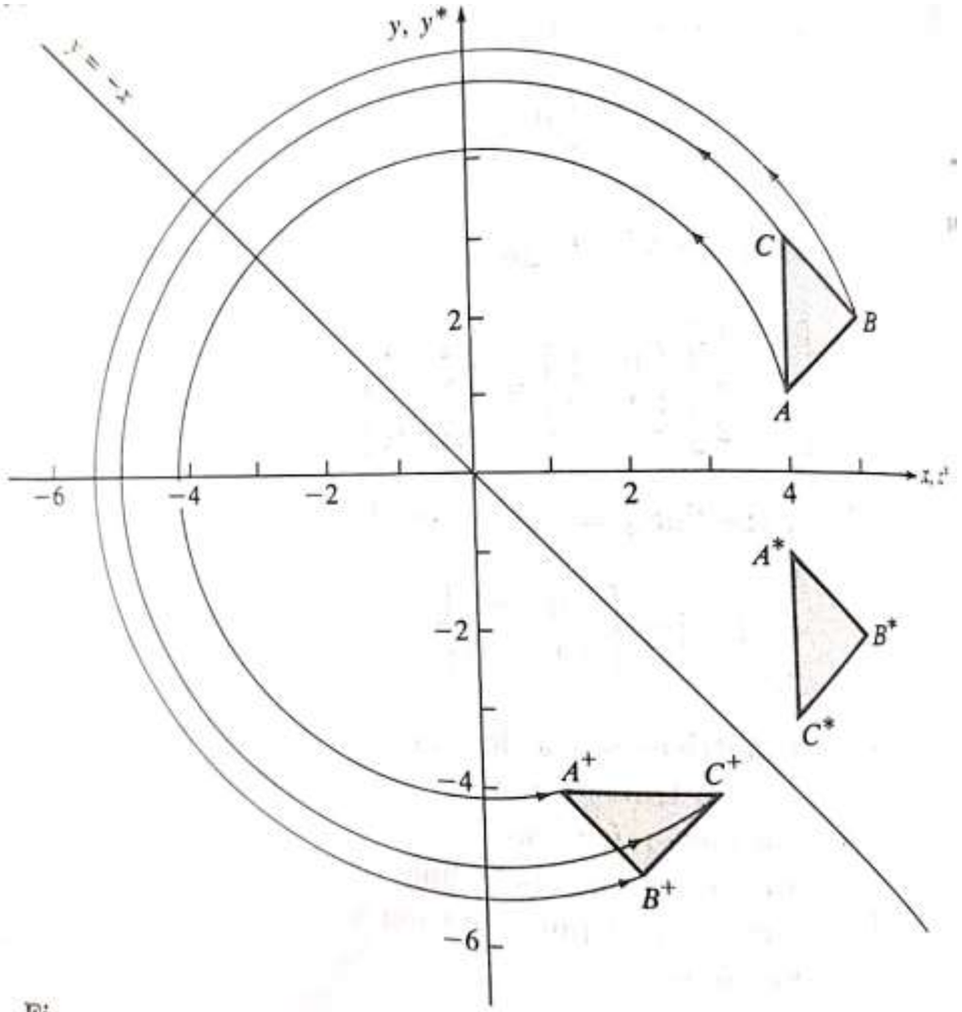


Figure 2–7 Combined reflections yield rotations.

Uniform and Nonuniform Scaling or Distortion

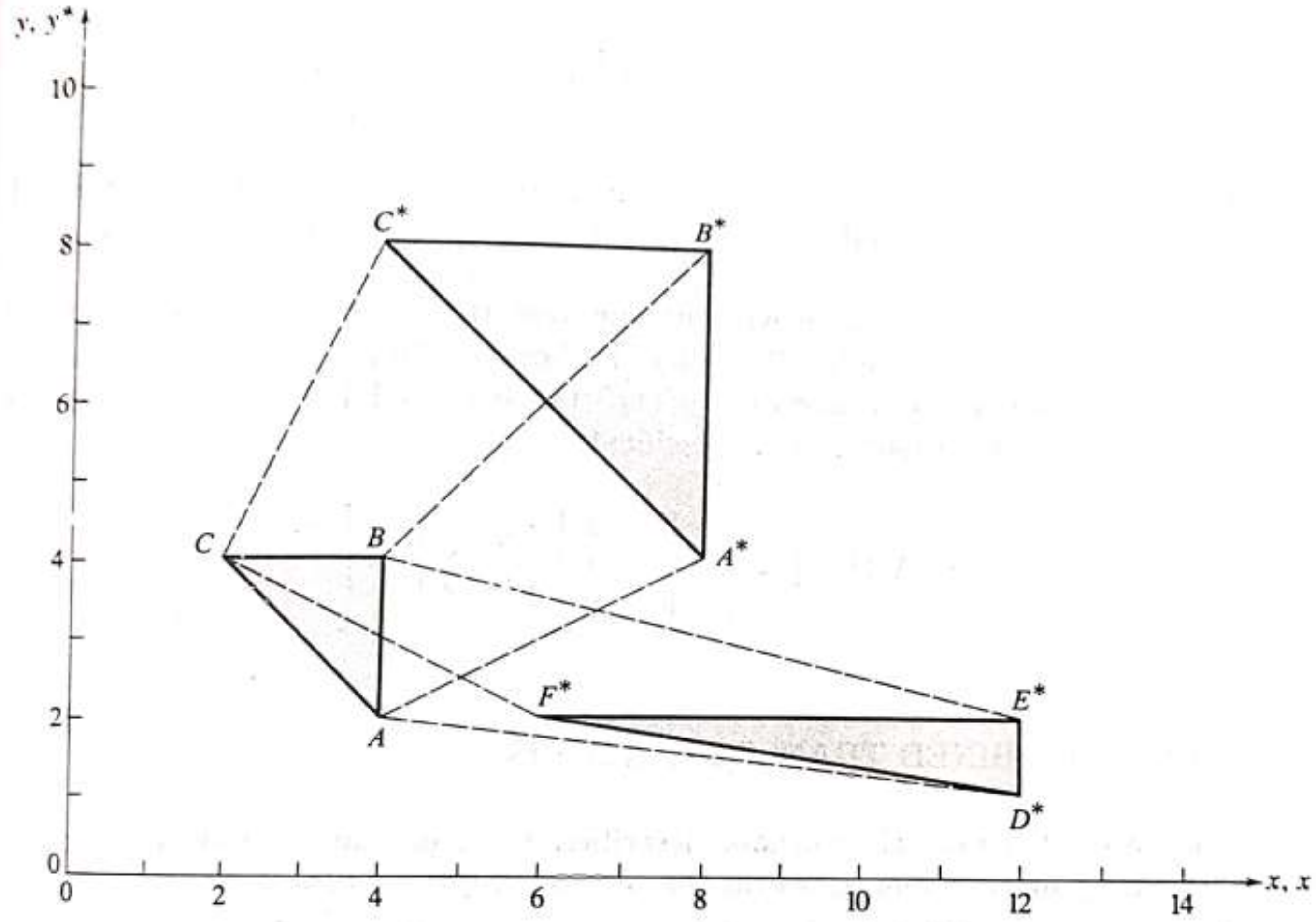


Figure 2–8 Uniform and nonuniform scaling or distortion.

Combined Transformation

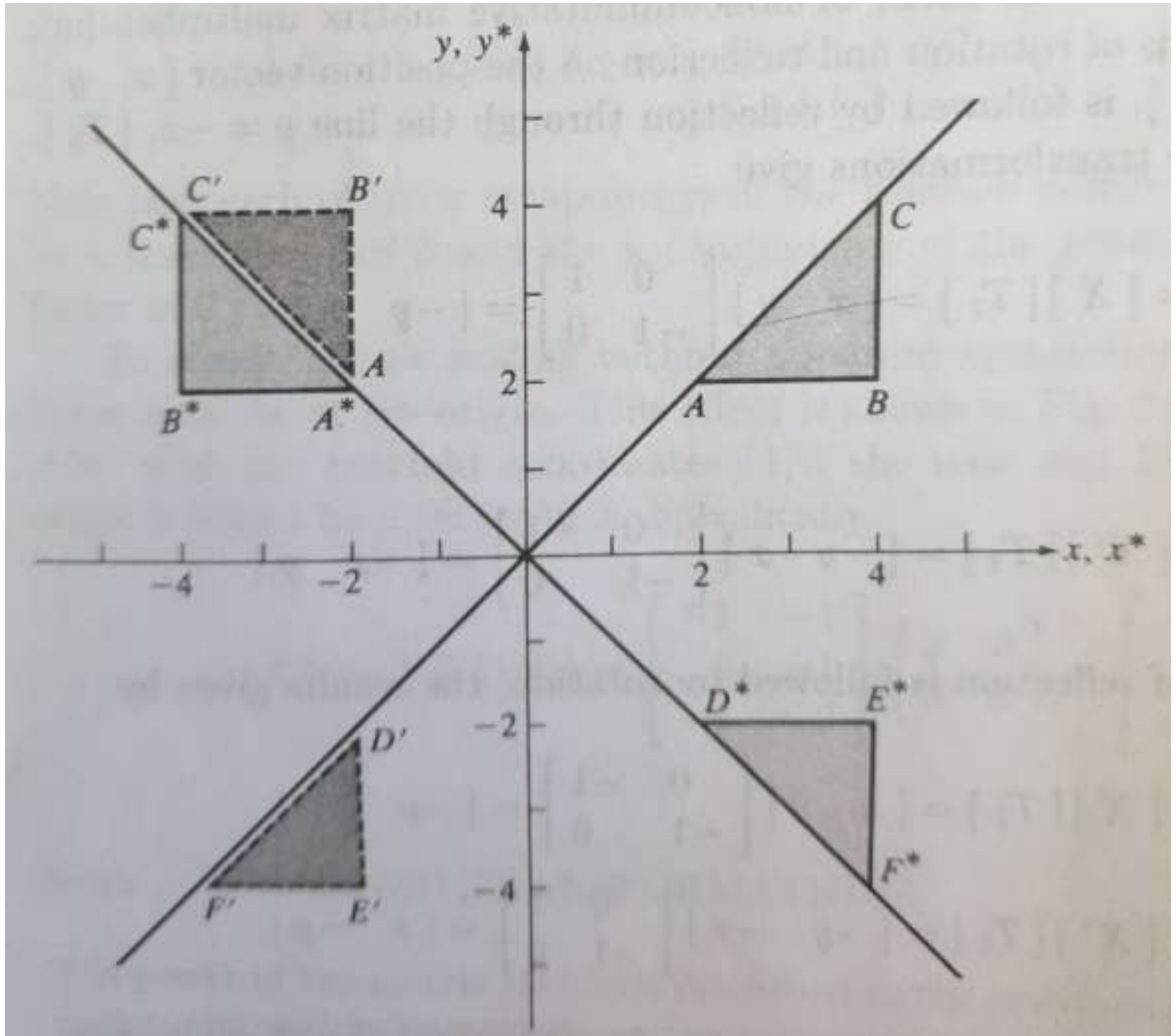


Figure 2–10 Combined two-dimensional transformations.

Following are matrices for two-dimensional transformation in homogeneous coordinate:

1. Translation

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ t_x & t_y & 1\end{bmatrix} \text{ or } \begin{bmatrix}1 & 0 & t_x\\ 0 & 1 & t_y\\ 0 & 0 & 1\end{bmatrix}$$

2. Scaling

$$\begin{bmatrix}S_x & 0 & 0\\ 0 & S_y & 0\\ 0 & 0 & 1\end{bmatrix}$$

3. Rotation (clockwise)

$$\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

4. Rotation (anti-clock)

$$\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

5. Reflection against X axis

$$\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

6. Reflection against Y axis

$$\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Following are matrices for two-dimensional transformation in homogeneous coordinate:

7. Reflection against origin

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8. Reflection against line Y=X

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

9. Reflection against Y= –X

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

10. Shearing in X direction

$$\begin{bmatrix} 1 & 0 & 0 \\ Sh_x & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

11. Shearing in Y direction

$$\begin{bmatrix} 1 & Sh_y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

12. Shearing in both x and y direction

$$\begin{bmatrix} 1 & Sh_y & 0 \\ Sh_x & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation about an Arbitrary Point

- Homogenous coordinates provides a mechanism for rotation about an arbitrary point.
- To rotate an object about an arbitrary point, (Xp ,Yp) we have to carry out three steps:
  - Translate point (Xp, Yp) to the origin.
  - Rotate it about the origin and,
  - Finally, translate the center of rotation back where it belongs (See figure 1.).

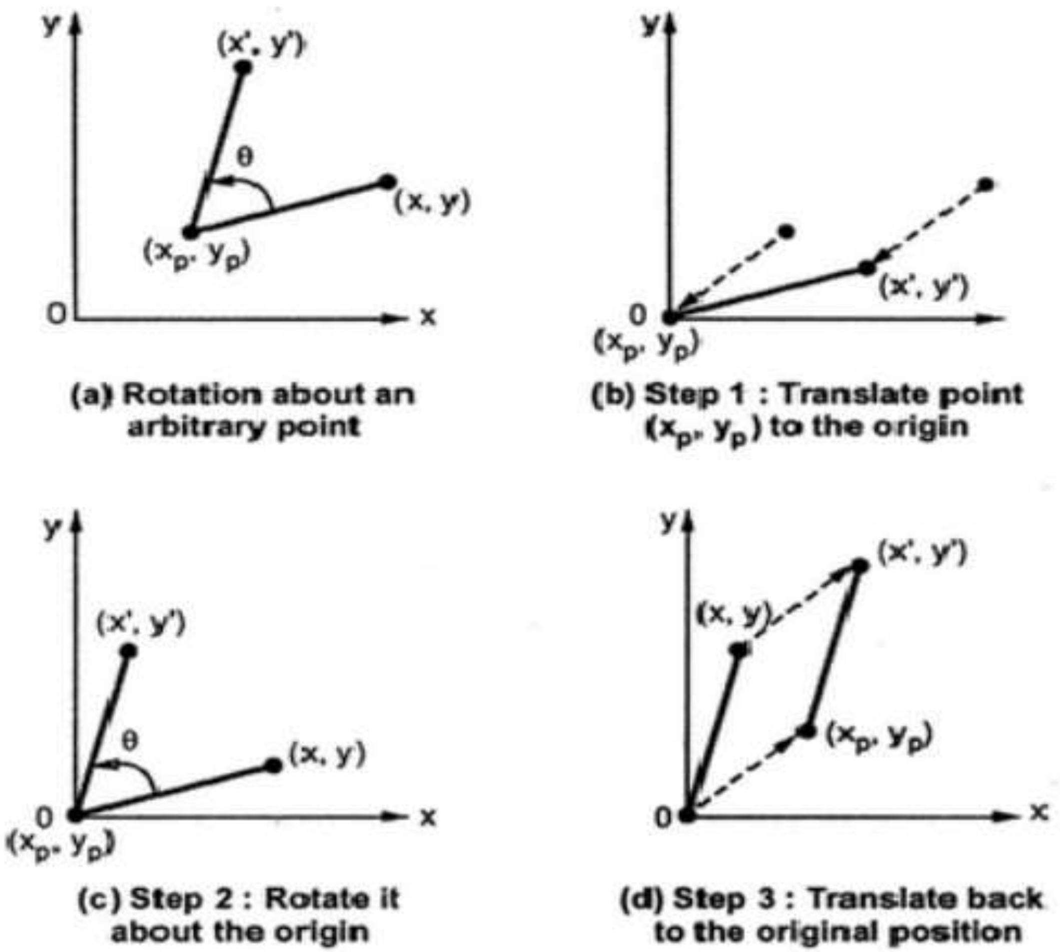


Fig. 1

Rotation about an Arbitrary Point

The translation matrix to move point (x_p, y_p) to the origin is given as,

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_p & -y_p & 1 \end{bmatrix}$$

The rotation matrix for counterclockwise rotation of point about the origin is given as,

$$R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The translation matrix to move the center point back to its original position is given as,

$$T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_p & y_p & 1 \end{bmatrix}$$

Rotation about an Arbitrary Point

Therefore, the overall transformation matrix for a counterclockwise rotation by an angle θ about the point (x_p, y_p) is given as,

$$T_1 \cdot R \cdot T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_p & -y_p & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_p & y_p & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ -x_p\cos\theta + y_p\sin\theta & -x_p\sin\theta - y_p\cos\theta & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_p & y_p & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ -x_p\cos\theta + y_p\sin\theta + x_p & -x_p\sin\theta - y_p\cos\theta + y_p & 1 \end{bmatrix}$$

Reflection of An Object About An Arbitrary Line

1. Translate the line and the object so that the line passes through the origin
2. Rotate the line and the object about the origin until the line is coincident with one of the coordinate axis
3. Reflect object through coordinate axis
4. Rotate back about the origin (Inverse Rotation)
5. Translate back to the original location (Inverse translation)

$$[T] = [T'] [R] [R'] [R]^{-1} [T']^{-1}$$

[T'] – Translation Matrix

[R] – Rotation Matrix about the origin

[R'] – Reflection Matrix

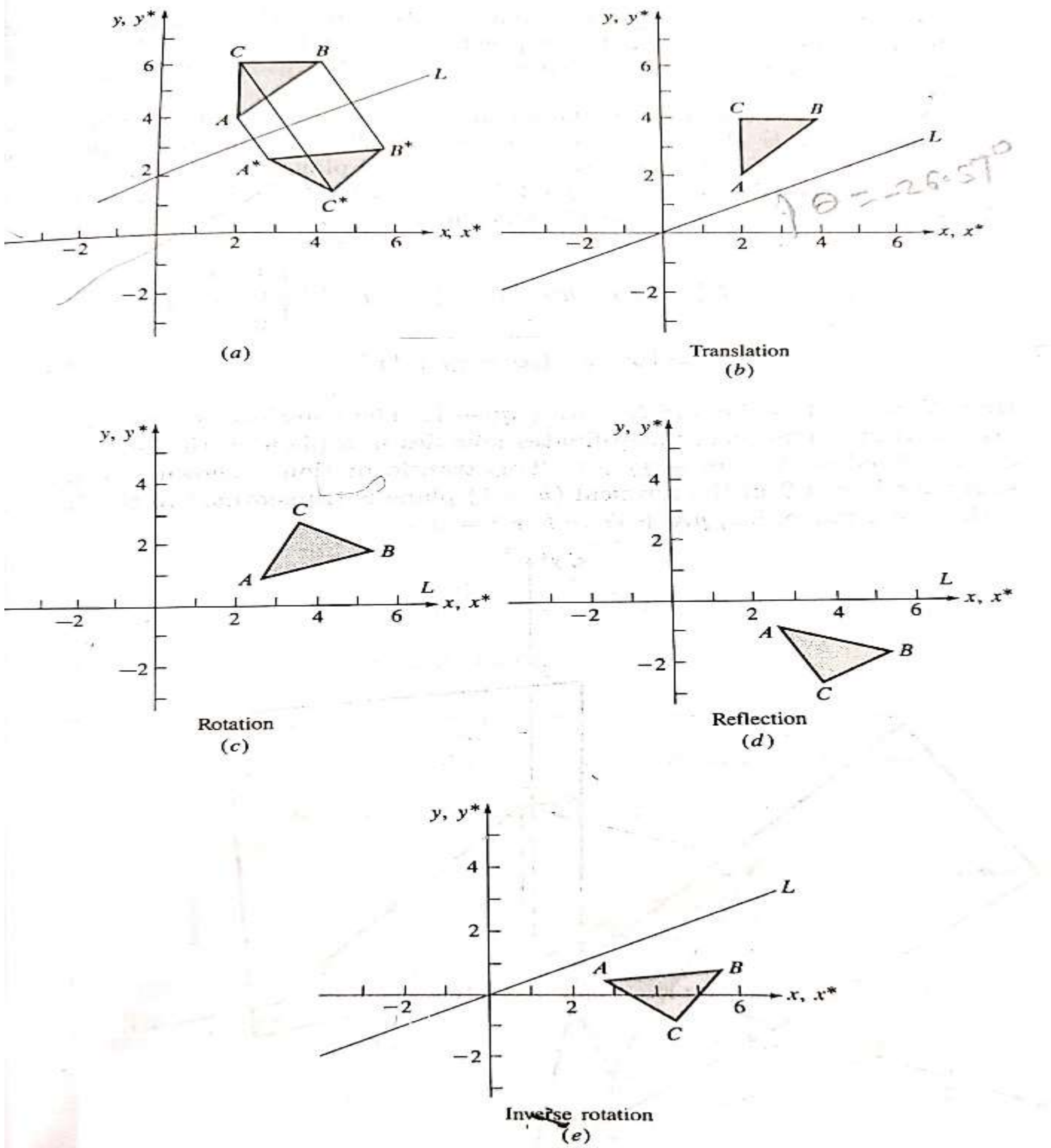


Figure 2–12 Reflection through an arbitrary line. (a) Original and final position; (b)

Projection – A Geometric Interpretation of Homogeneous Coordinates

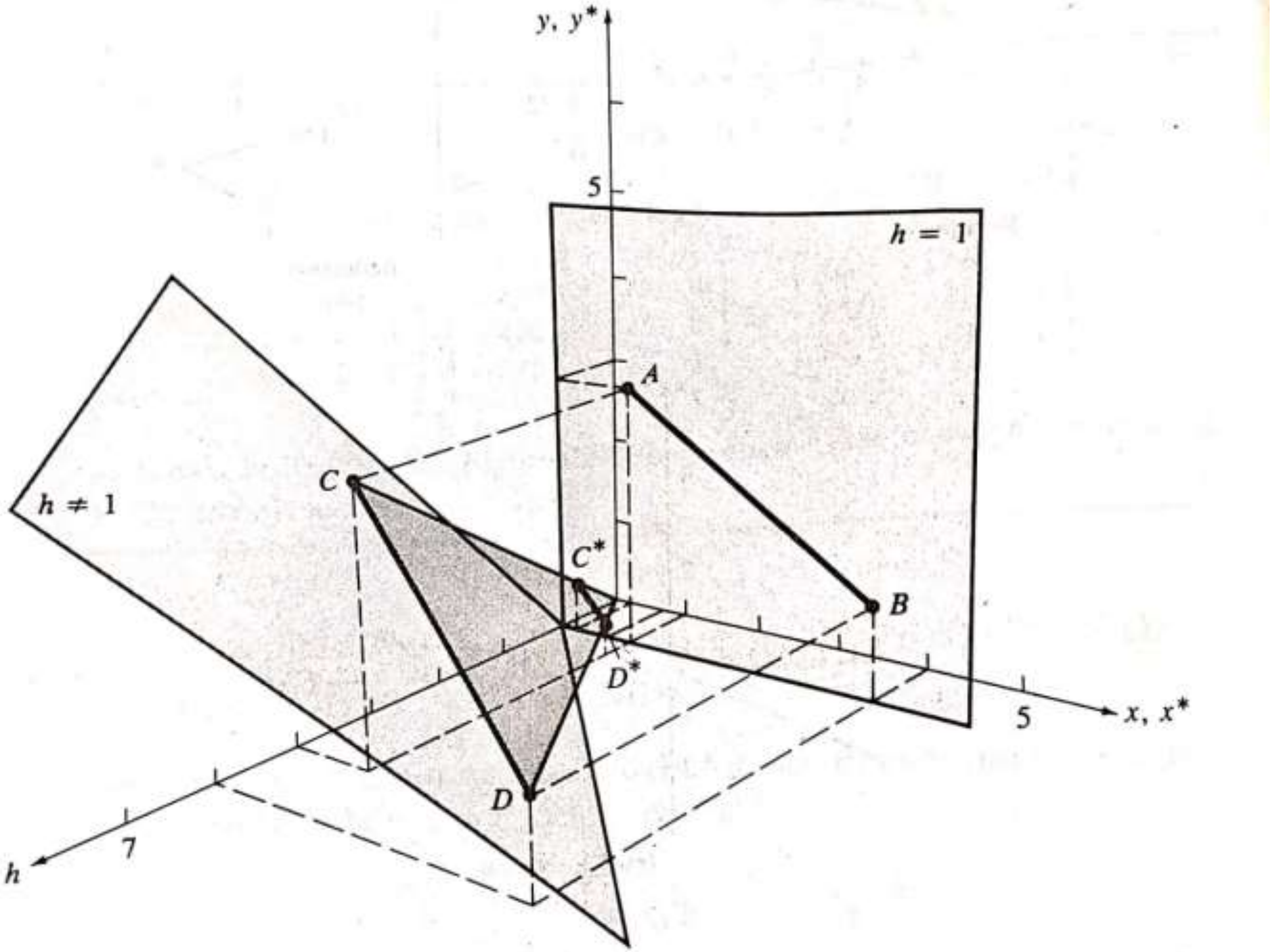


Figure 2–14 Transformation from the physical ($h = 1$) plane into the $h \neq 1$ pla
and projection from the $h \neq 1$ plane back into the physical plane.

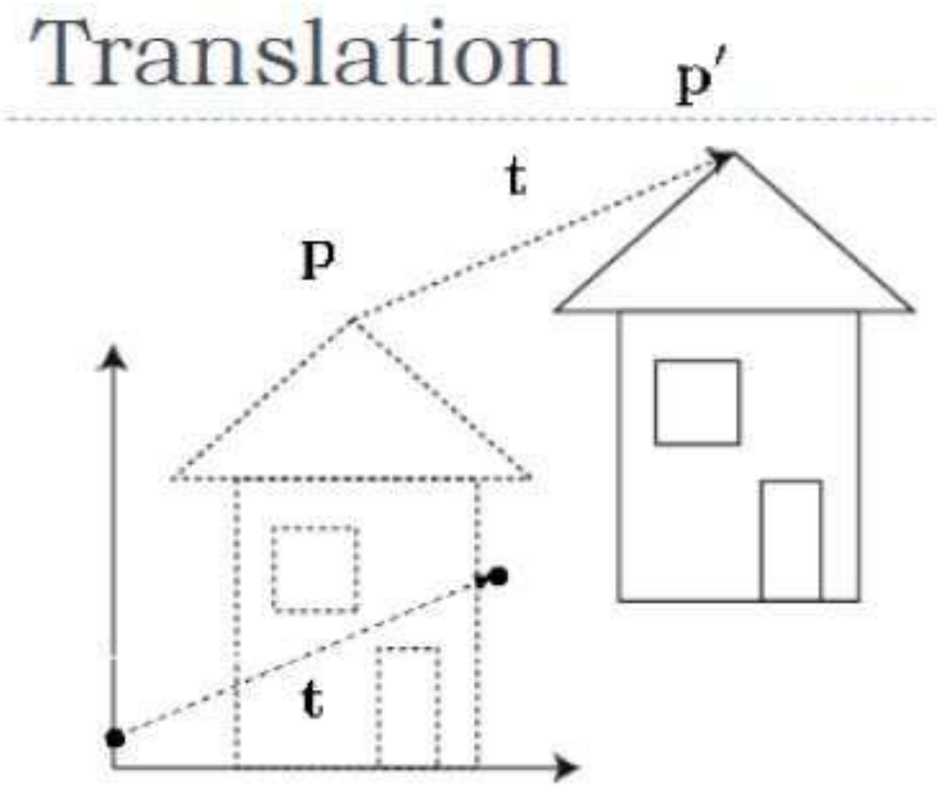
3D Transformation with Homogeneous Coordinates

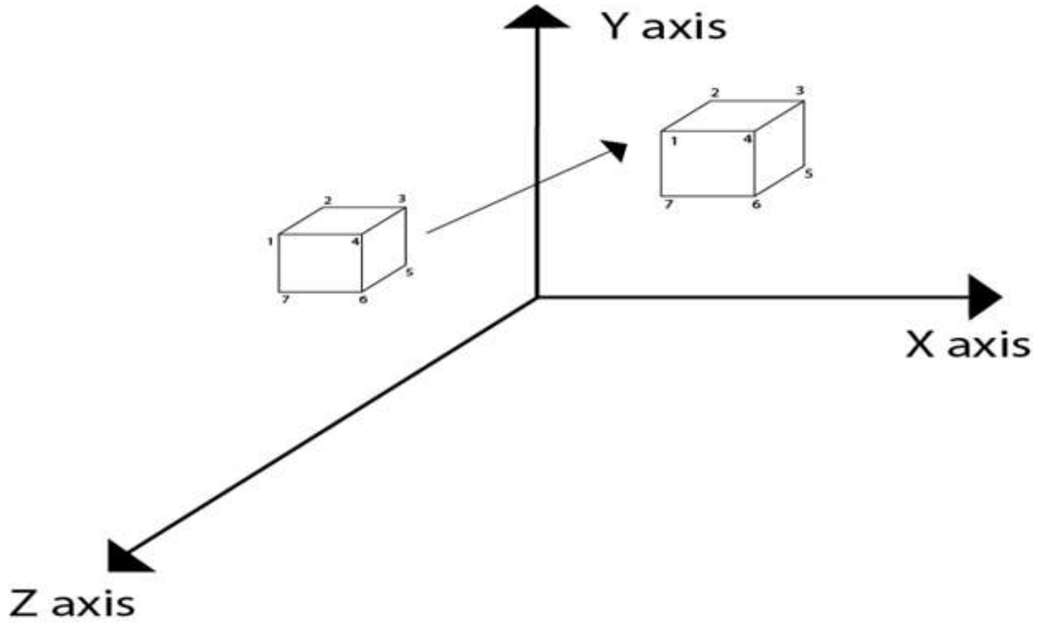
1. Upper Left 3×3 sub matrix – linear transformation such as scaling, shearing, reflection and rotation.
2. Lower Left 1×3 sub matrix – translation
3. Upper Right 3×1 sub matrix – perspective transformation
4. Lower Right 1×1 sub matrix – overall scaling

$$[T] = \begin{bmatrix} a & b & c & p \\ d & e & f & q \\ g & i & j & r \\ l & m & n & s \end{bmatrix}$$

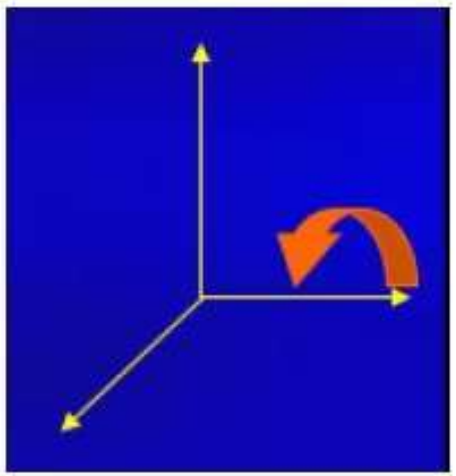
$$\left[\begin{array}{ccc:c} & & & 3 \\ & 3 \times 3 & & \times \\ & & & 1 \\ \cdots & \cdots & \cdots & \cdots \\ & 1 \times 3 & & 1 \times 1 \end{array}\right]$$

3D Translation

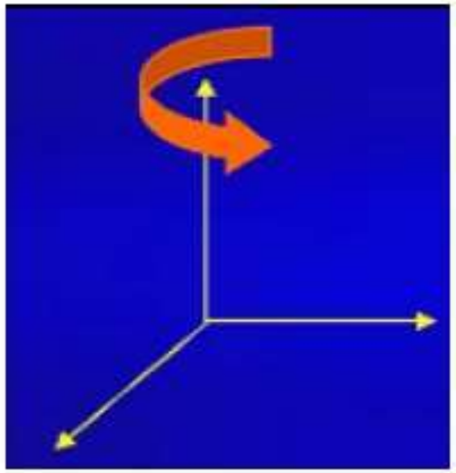




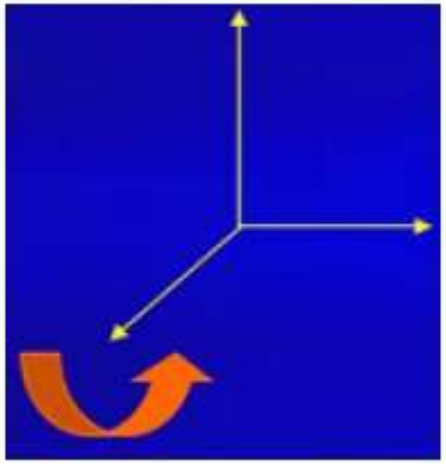
3D Rotation



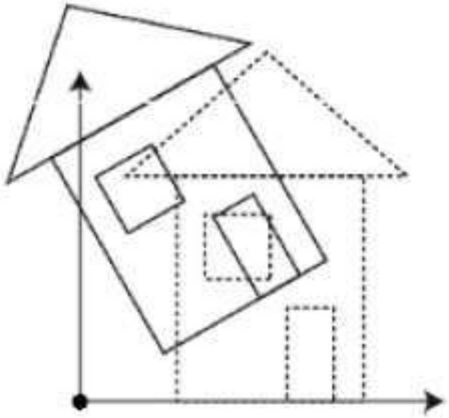
Rotation about x-axis



Rotation about y-axis



Rotation about z-axis



3 D Rotation

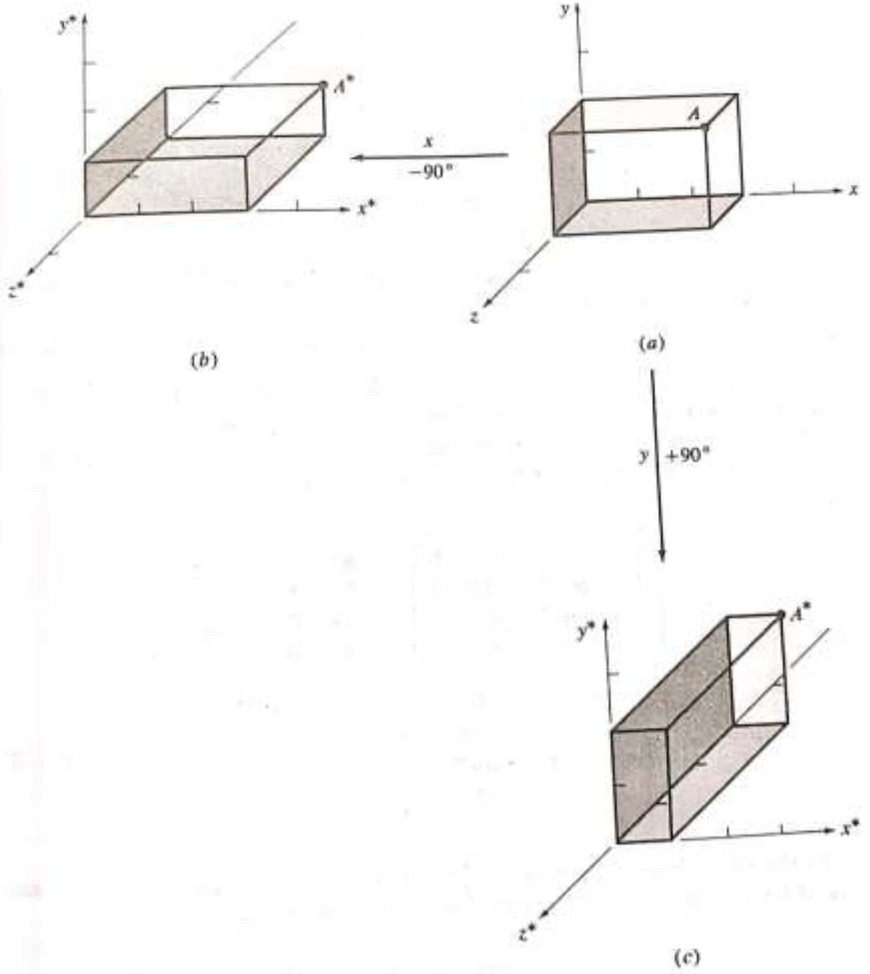


Figure 3–2 Three-dimensional rotations.

. . . . yields the new position

3 D Combined Rotation

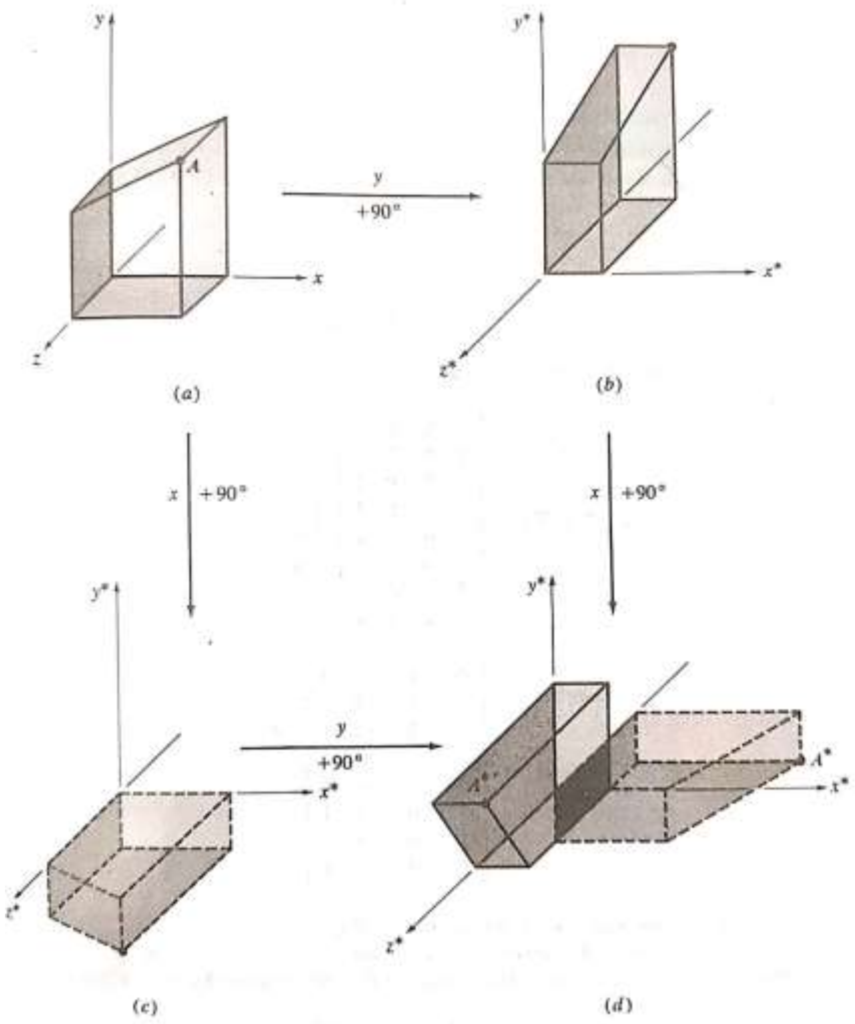


Figure 3–3 Three-dimensional rotations are noncommutative.

3D Reflection

- **Reflection relative to XY plane**

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- **Reflection relative to YZ plane**

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- **Reflection relative to XZ plane**

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3D Reflection

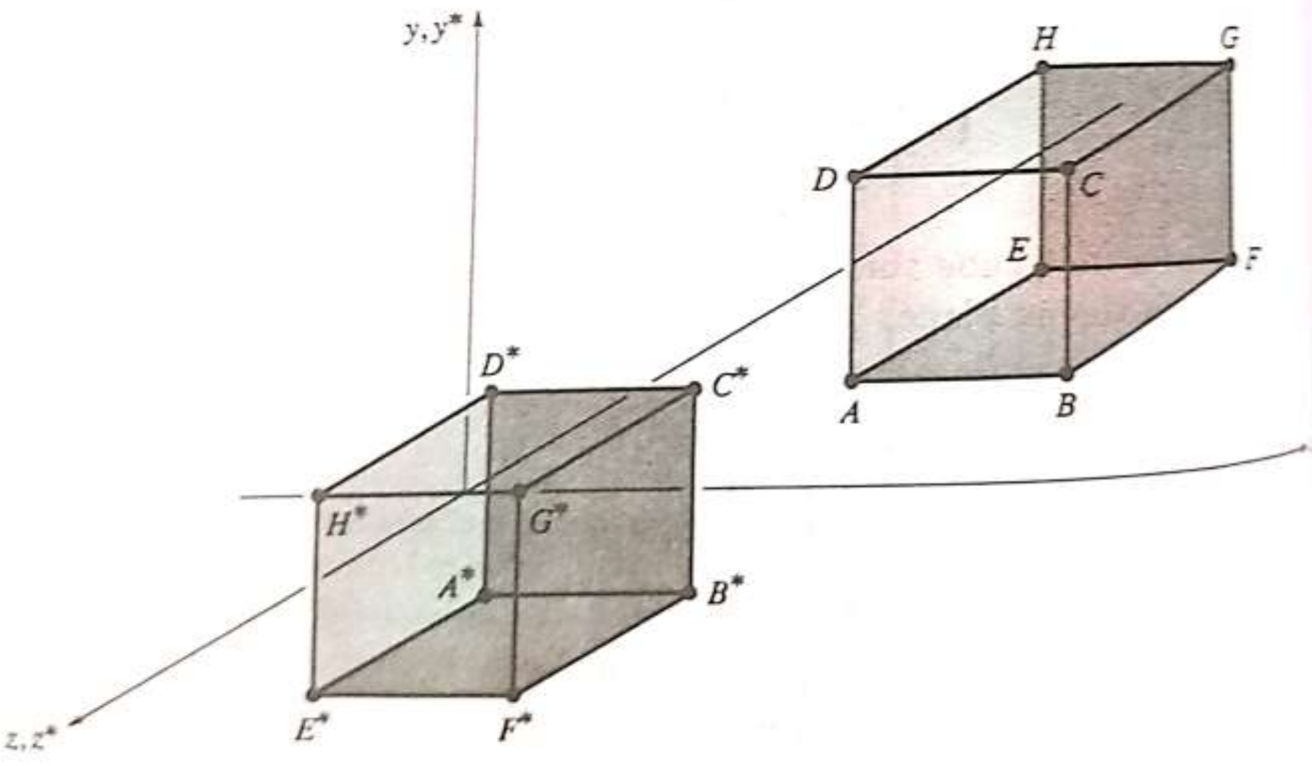


Figure 3–4 Three-dimensional reflection through the xy plane.

3D Scaling

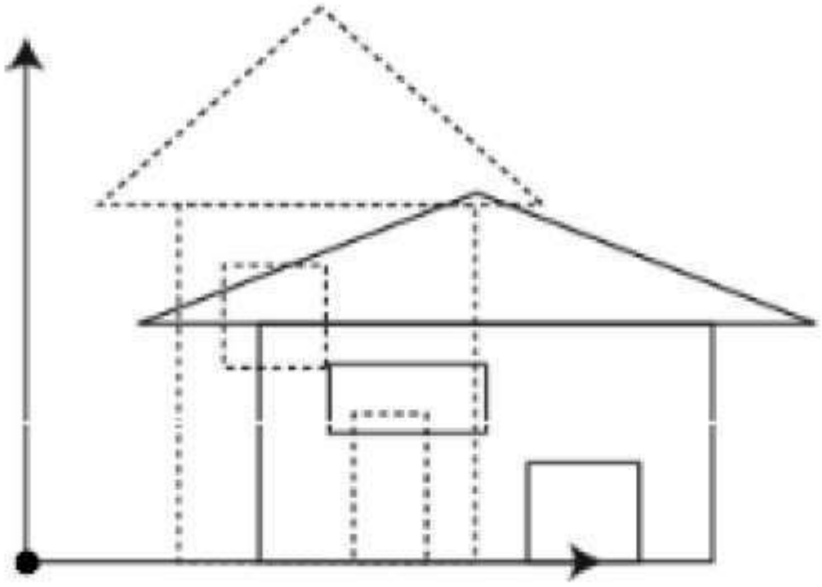
Scaling is used to change the size of an object. The size can be increased or decreased. The scaling three factors are required S_x S_y and S_z.

S_x=Scaling factor in x- direction

S_y=Scaling factor in y-direction

S_z=Scaling factor in z-direction

$$S = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

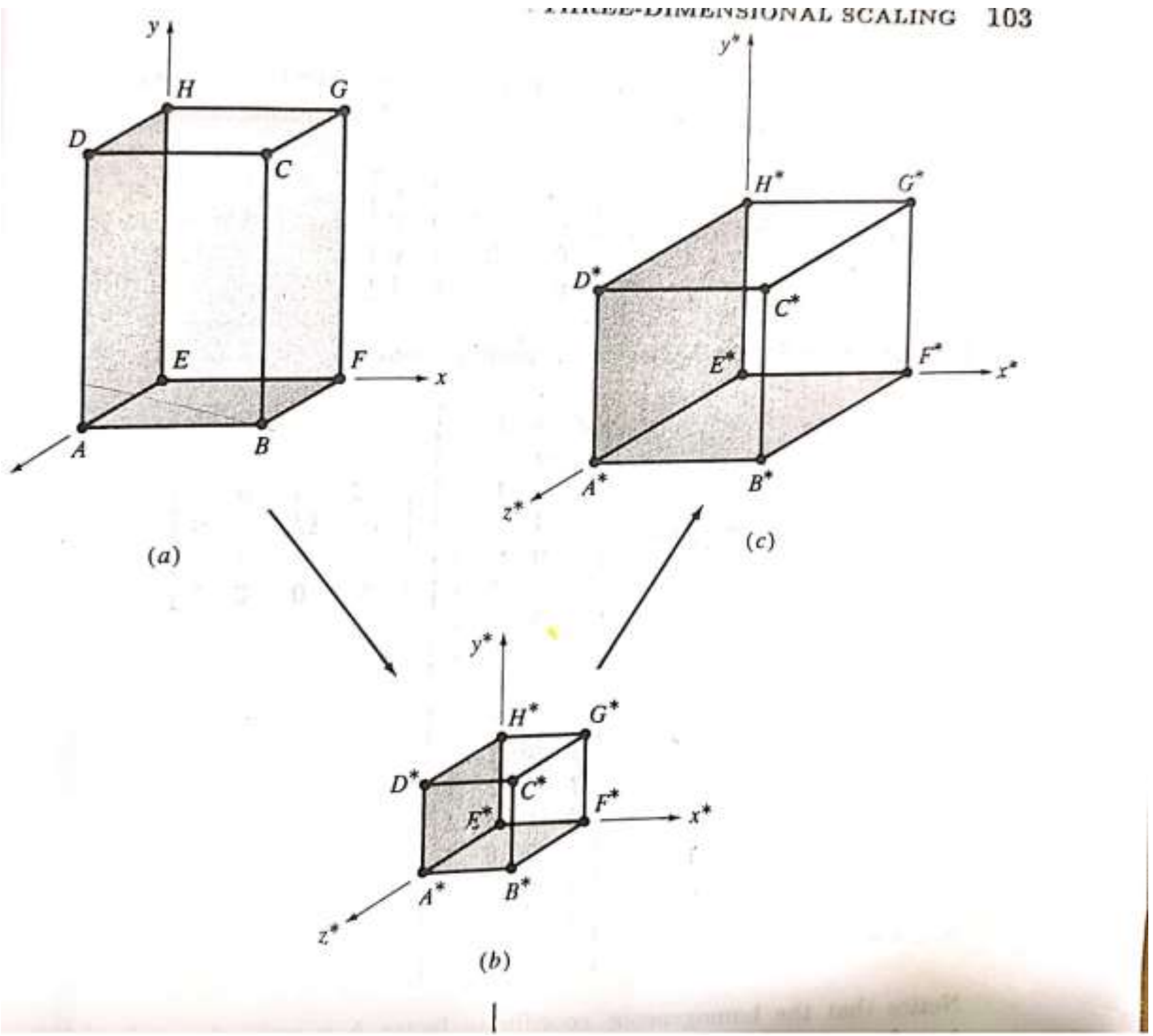


P' = P·S

$$[X' \;\; Y' \;\; Z' \;\; 1] = [X \;\; Y \;\; Z \;\; 1] \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= [X.S_x \;\; Y.S_y \;\; Z.S_z \;\; 1]$$

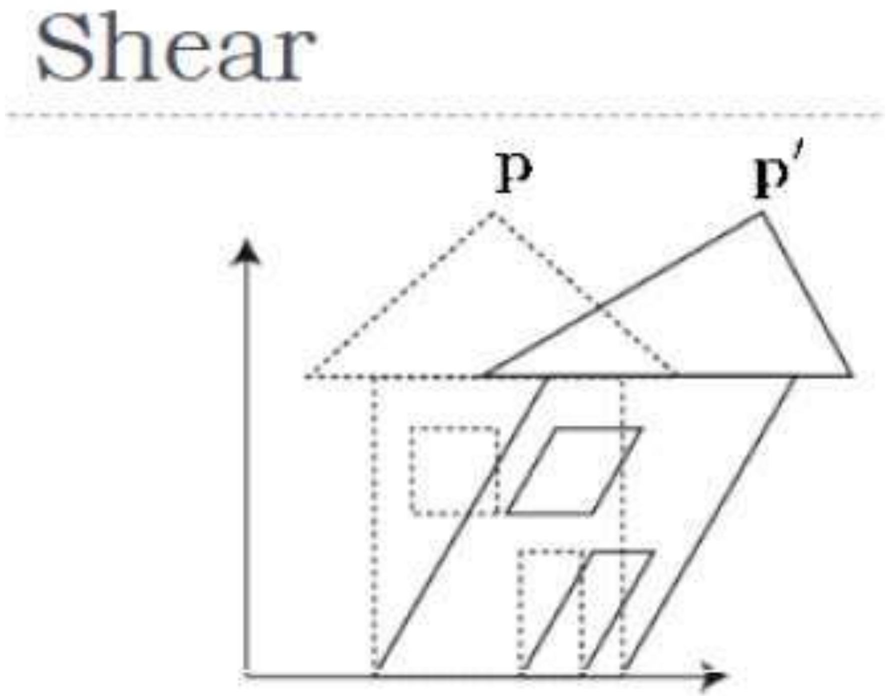
3D Scaling



THREE-DIMENSIONAL SCALING 103
y
H
G
D
C
E
F
x
A
B
(a)
y*
H*
G*
D*
C*
E*
F*
x*
z*
A*
B*
(c)
y*
H*
G*
D*
C*
E*
F*
x*
z*
A*
B*
(b)

3D Shearing

- A transformation that slants the shape of an object is called the **shear transformation**. Like in 2D shear, we can shear an object along the X-axis, Y-axis, or Z-axis in 3D.



$$Sh = \begin{bmatrix} 1 & sh_x^y & sh_x^z & 0 \\ sh_y^x & 1 & sh_y^z & 0 \\ sh_z^x & sh_z^y & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

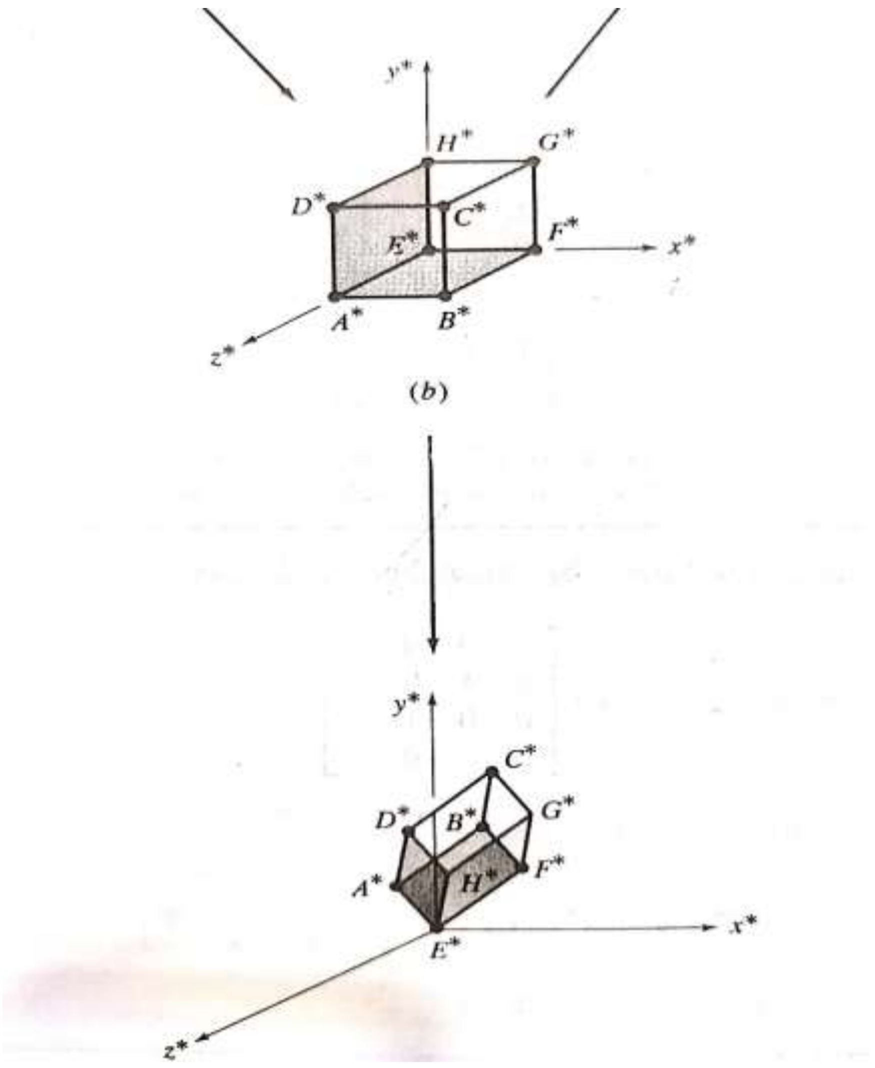
$$P' = P \cdot Sh$$

$$X' = X + Sh_x^y Y + Sh_x^z Z$$

$$Y' = Sh_y^x X + Y + sh_y^z Z$$

$$Z' = Sh_z^x X + Sh_z^y Y + Z$$

3D Shearing



Multiple Transformations

- Successive transformations can be combined or concatenated into a single 4×4 transformation that yields the same result.
- [X] [T] = [X] [T1] [T2] [T3] [T4] …
- [T] = [T1] [T2] [T3] [T4] …
- Where [Ti] are any combination of scaling, shearing, rotation, reflection, etc.
- Since matrix multiplication is non commutative, the order of application is important

Example 3–7 Multiple Transformations

Consider the effect of a translation in the x, y, z directions by -1, -1, -1, respectively, followed successively by a $+30°$ rotation about the x-axis, and a $+45°$ rotation about the y-axis on the homogeneous coordinate position vector $[\,3 \quad 2 \quad 1 \quad 1\,]$.

Multiple Transformations

$$[T] = [Tr][R_x][R_y]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ l & m & n & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ l & m & n & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ \sin\phi\sin\theta & \cos\theta & \cos\phi\sin\theta & 0 \\ \sin\phi\cos\theta & -\sin\theta & \cos\phi\cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ \sin\phi\sin\theta & \cos\theta & \cos\phi\sin\theta & 0 \\ \sin\phi\cos\theta & -\sin\theta & \cos\phi\cos\theta & 0 \\ \begin{array}{l} l\cos\phi \\ +m\sin\phi\sin\theta \\ +n\sin\phi\cos\theta \end{array} & \begin{array}{l} m\cos\theta \\ -n\sin\theta \end{array} & \begin{array}{l} -l\sin\phi \\ +m\cos\phi\sin\theta \\ +n\cos\phi\cos\theta \end{array} & 1 \end{bmatrix} \qquad (3-16)$$

where θ, ϕ are the rotation angles about the x- and y-axes, respectively; and l, m, n are the translation factors in the x, y, z directions, respectively.

For a general position vector we have

Multiple Transformations

$$[X][T] = [x \quad y \quad z \quad 1]\begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ \sin\phi\sin\theta & \cos\theta & \cos\phi\sin\theta & 0 \\ \sin\phi\cos\theta & -\sin\theta & \cos\phi\cos\theta & 0 \\ \begin{array}{l} l\cos\phi \\ +m\sin\phi\sin\theta \\ +n\sin\phi\cos\theta \end{array} & \begin{array}{l} m\cos\theta \\ -n\sin\theta \\ \end{array} & \begin{array}{l} -l\sin\phi \\ +m\cos\phi\sin\theta \\ +n\cos\phi\cos\theta \end{array} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{array}{l} (x+l)\cos\phi \\ +(y+m)\sin\phi\sin\theta \\ +(z+n)\sin\phi\cos\theta \end{array} & \begin{array}{l} (y+m)\cos\theta \\ -(z+n)\sin\theta \\ \end{array} & \begin{array}{l} -(x+l)\sin\phi \\ +(y+m)\cos\phi\sin\theta \\ +(z+n)\cos\phi\cos\theta \end{array} & 1 \end{bmatrix}$$

For specific values of $\theta = +30°$ ...

Multiple Transformations

$$[X][T] = [3 \quad 2 \quad 1 \quad 1]\begin{bmatrix} 0.707 & 0 & -0.707 & 0 \\ 0.354 & 0.866 & 0.354 & 0 \\ 0.612 & -0.5 & 0.612 & 0 \\ -1.673 & -0.366 & -0.259 & 1 \end{bmatrix}$$

$$[X][T] = [1.768 \quad 0.866 \quad -1.061 \quad 1]$$

To confirm that the concatenated matrix yields the same result as individually applied matrices consider

$$[X'] = [X][Tr]$$

$$= [3 \quad 2 \quad 1 \quad 1]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

$$= [2 \quad 1 \quad 0 \quad 1]$$

$$[X''] = [X'][R_x] = [2 \quad 1 \quad 0 \quad 1]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & 0.5 & 0 \\ 0 & -0.5 & 0.866 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= [2 \quad 0.866 \quad 0.5 \quad 1]$$

$$[X'''] = [X''][R_y] = [2 \quad 0.866 \quad .5 \quad 1]\begin{bmatrix} 0.707 & 0 & -0.707 & 0 \\ 0 & 1 & 0 & 0 \\ 0.707 & 0 & 0.707 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[X'''] = [1.768 \quad 0.866 \quad -1.061 \quad 1]$$

which confirms our previous result.

Rotation about an Axis Parallel to a Coordinate Axis

- **Single Relative Rotation**: Rotation of the object about any of the individual x', y' or z' local axes is accomplished using following procedure:
  - Translate the object until the local axis is coincident with the coordinate axis in the same direction.
  - Rotate about the specified axis.
  - Translate the transformed object back to its original position.

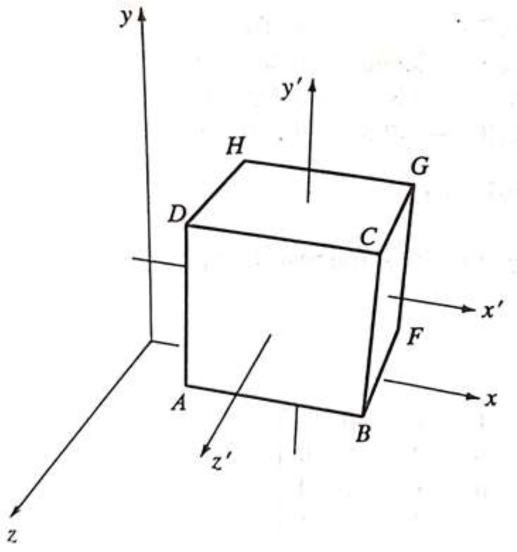
Mathematically,

$$[X^*] = [X][Tr][R_x][Tr]^{-1}$$

where

$[X^*]$	represents the transformed body
$[X]$	is the untransformed body
$[Tr]$	is the translation matrix
$[R_x]$	is the appropriate rotation matrix
$[Tr]^{-1}$	is the inverse of the translation matrix

An illustrative example is given below.



Rotation about an Axis Parallel to a Coordinate Axis

Example 3–8 Single Relative Rotation

Consider the block in Fig. 3–5a defined by the position vectors

$$[X] = \begin{array}{c} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array}$$

Rotation about an Axis Parallel to a Coordinate Axis

relative to the global xyz-axis system. Let's rotate the block $\theta = +30^\circ$ about the local x'-axis passing through the centroid of the block. The origin of the local axis system is assumed to be the centroid of the block.

The centroid of the block is $[\, x_c \quad y_c \quad z_c \quad 1 \,] = [\, 3/2 \quad 3/2 \quad 3/2 \quad 1 \,]$.

The rotation is accomplished by

$$[\, X^* \,] = [\, X \,][\, Tr \,][\, R \,][\, Tr \,]^{-1}$$

where

$$[\, Tr \,] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -y_c & -z_c & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3/2 & -3/2 & 1 \end{bmatrix}$$

$$[\, R_x \,] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & 0.5 & 0 \\ 0 & -0.5 & 0.866 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$[\, Tr \,]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & y_c & z_c & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3/2 & 3/2 & 1 \end{bmatrix}$$

Rotation about an Axis Parallel to a Coordinate Axis

The first matrix $[\,Tr\,]$ translates the block parallel to the $x = 0$ plane until the x'-axis is coincident with the x-axis. The second matrix $[\,R_x\,]$ performs the required rotation about the x-axis, and the third matrix $[\,Tr\,]^{-1}$ translates the x'-axis and hence the rotated block back to its original position. Concatenating these three matrices yields

$$[\,T\,] = [\,Tr\,][\,R_x\,][\,Tr\,]^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & y_c(1-\cos\theta) + z_c\sin\theta & z_c(1-\cos\theta) - y_c\sin\theta & 1 \end{bmatrix}$$

After substituting numerical values the transformed coordinates are

$$[\,X'\,] = [\,X\,][\,T\,] = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & 0.5 & 0 \\ 0 & -0.5 & 0.866 & 0 \\ 0 & 0.951 & -0.549 & 1 \end{bmatrix}$$

Rotation about an Axis Parallel to a Coordinate Axis

$$[X'] = \begin{array}{c} \begin{bmatrix} 1 & 0.817 & 1.683 & 1 \\ 2 & 0.817 & 1.683 & 1 \\ 2 & 1.683 & 2.183 & 1 \\ 1 & 1.683 & 2.183 & 1 \\ 1 & 1.317 & 0.817 & 1 \\ 2 & 1.317 & 0.817 & 1 \\ 2 & 2.183 & 1.317 & 1 \\ 1 & 2.183 & 1.317 & 1 \end{bmatrix} \end{array} \begin{array}{l} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array}$$

The result is shown in Fig. 3–5b.

Rotation about an Axis Parallel to a Coordinate Axis

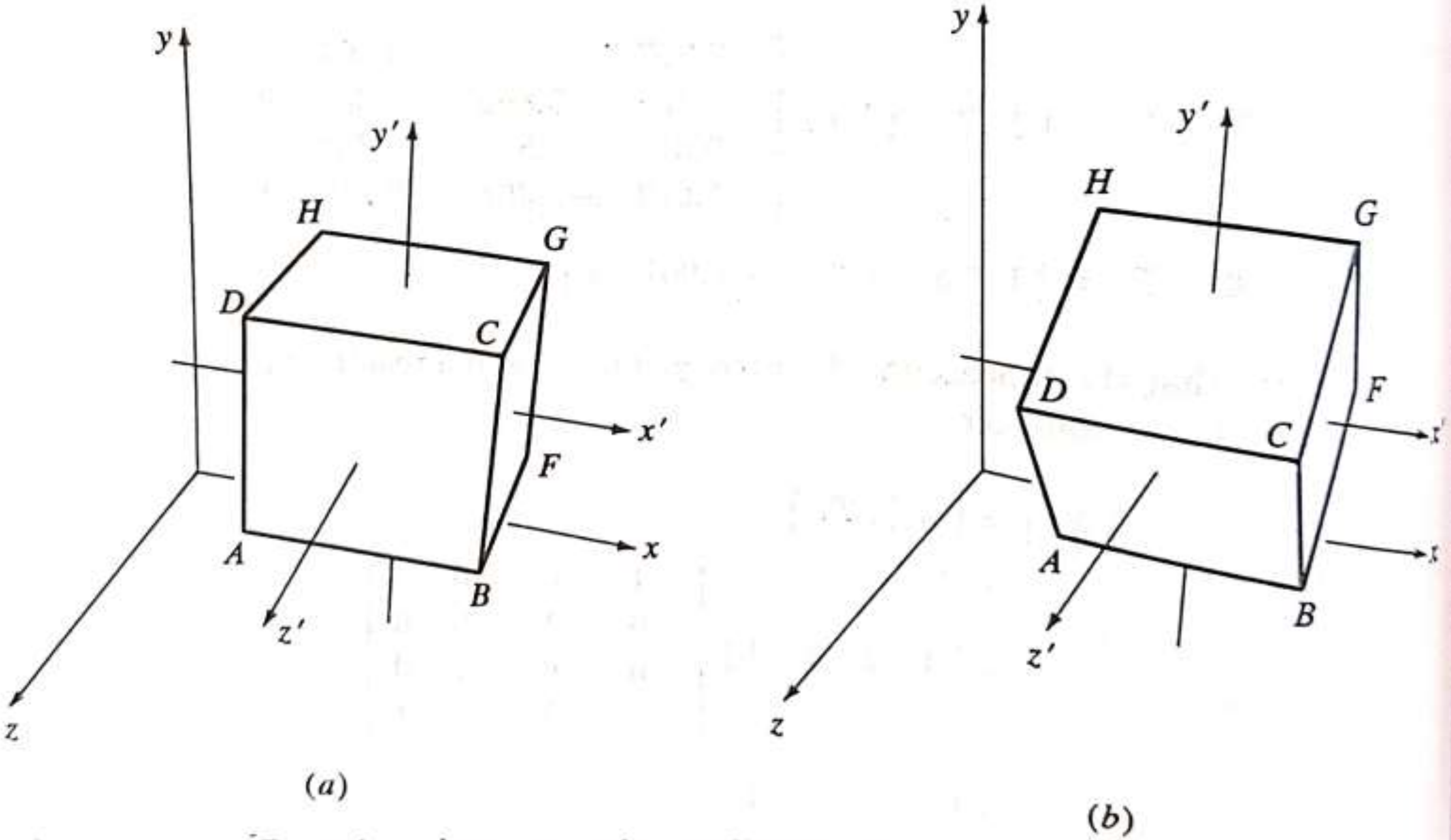
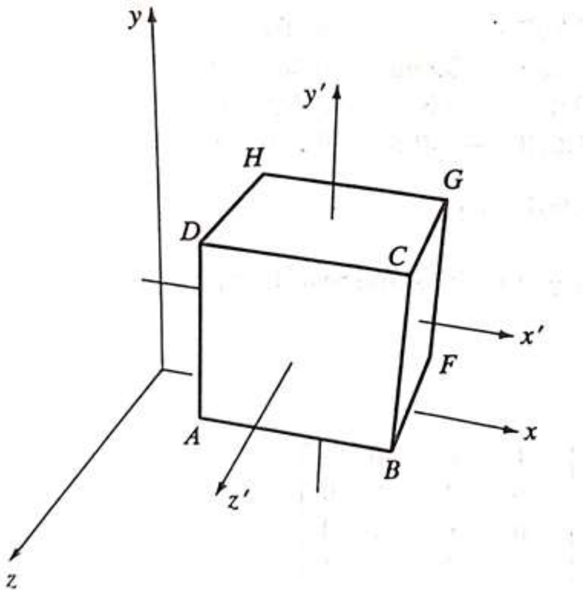


Figure 3–5 Rotation about an axis parallel to one of the coordinate axes.

Rotation about an Axis Parallel to a Coordinate Axis

- **Multiple Relative Rotation** Procedure:
  - Translate the origin of the local axis system to make it coincident with the global coordinate system.
  - Perform the required rotation.
  - Translate local axis back to its original position.



Rotation about an Axis Parallel to a Coordinate Axis

Example 3–9 Multiple Relative Rotations

Again consider the block shown on Fig. 3–5a. To rotate the block $\phi = -45°$ about the y'-axis, followed by a rotation of $\theta = +30°$ about the x'-axis, requires that the origin of the $x'y'z'$-axis system be made coincident with the origin of the xyz-axis system, the rotations performed and then the result translated back to the original position.

The combined transformation is

$$[X'] = [X][T] = [X][Tr][R_y][R_x][Tr]^{-1}$$

Specifically,

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -x_c & -y_c & -z_c & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x_c & y_c & z_c & 1 \end{bmatrix}$$

Rotation about an Axis Parallel to a Coordinate Axis

where ϕ and θ represent the rotation angle about the y'- and x'-axes respectively. Concatenating these matrices yields

$$[\,T\,] = \begin{bmatrix} \cos\phi & \sin\phi\sin\theta & -\sin\phi\cos\theta & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ \sin\phi & -\cos\phi\sin\theta & \cos\phi\cos\theta & 0 \\ \begin{matrix} x_c(1-cos\phi) \\ -z_c\sin\phi \\ \end{matrix} & \begin{matrix} -x_c\sin\phi\sin\theta \\ +y_c(1-\cos\theta) \\ +z_c\cos\phi\sin\theta \end{matrix} & \begin{matrix} x_c\sin\phi\cos\theta \\ -y_c\sin\theta \\ +z_c(1-\cos\phi\cos\theta) \end{matrix} & 1 \end{bmatrix} \qquad (3-17)$$

The transformed position vectors are then

$$[\,X'\,] = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.707 & -0.354 & 0.612 & 0 \\ 0 & 0.866 & 0.5 & 0 \\ -0.707 & -0.354 & 0.612 & 0 \\ 1.5 & 1.262 & -1.087 & 1 \end{bmatrix}$$

$$[\,X'\,] = \begin{bmatrix} 0.793 & 1.067 & 1.25 & 1 \\ 1.5 & 0.713 & 1.862 & 1 \\ 1.5 & 1.579 & 2.362 & 1 \\ 0.793 & 1.933 & 1.75 & 1 \\ 1.5 & 1.421 & 0.638 & 1 \\ 2.207 & 1.067 & 1.25 & 1 \\ 2.207 & 1.933 & 1.75 & 1 \\ 1.5 & 2.287 & 1.138 & 1 \end{bmatrix}$$

The result is shown in Fig. 3–6.

Rotation about an Axis Parallel to a Coordinate Axis

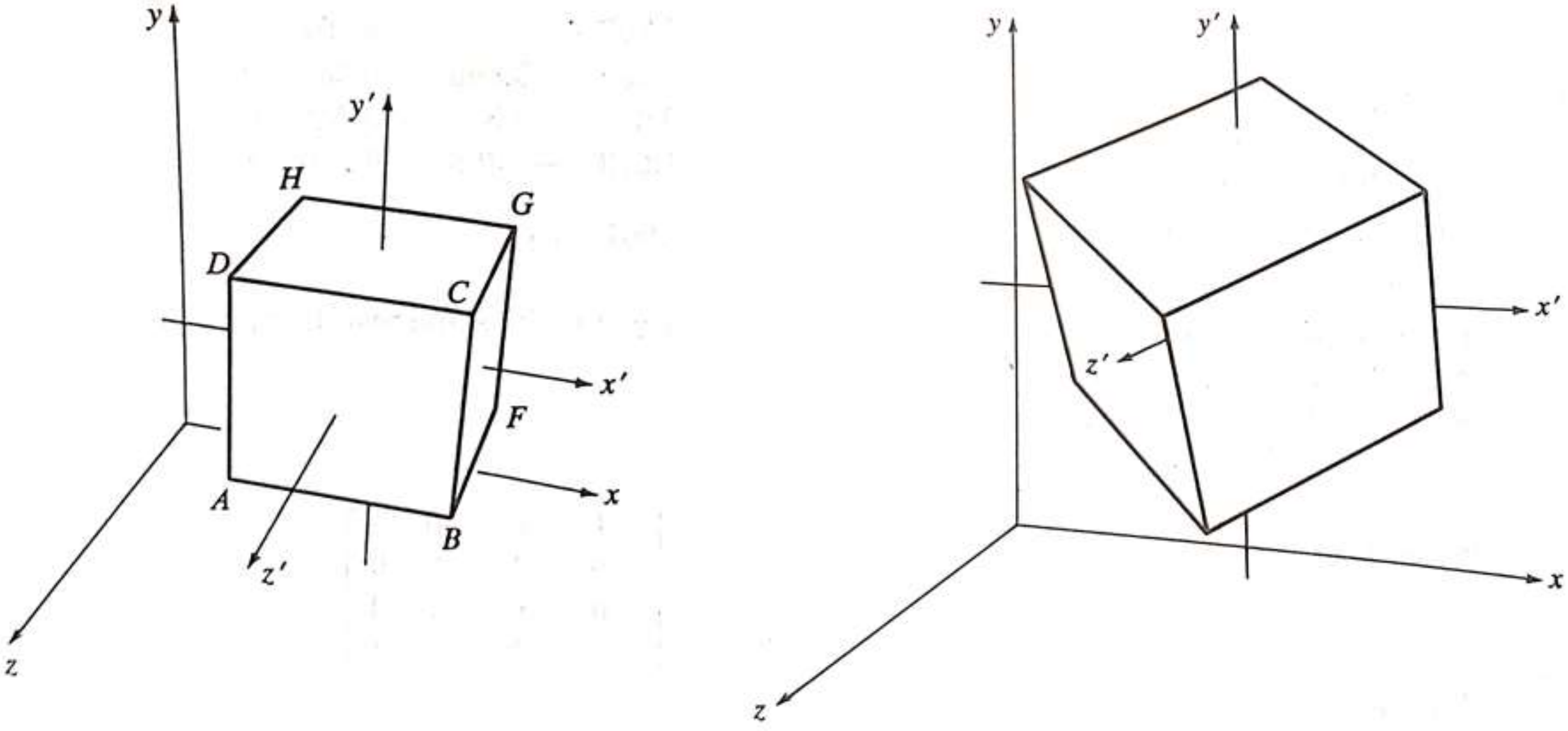


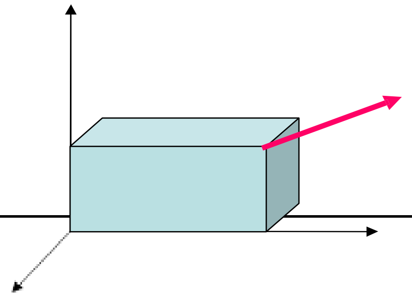
Figure 3–6 Multiple rotations about a local axis system.

ROTATION ABOUT AN ARBITRARY AXIS IN SPACE

- Used in robotics, animation and simulation
- Make the arbitrary axis coincide with one of the coordinate axes.

➢ Consider an arbitrary axis passing through a point (x_0, y_0, z_0) with direction cosines (c_x, c_y, c_z)

- Rotation about this axis by some angle б is accomplished with following Procedure:
    1. Translate (x_0, y_0, z_0) so that the point is at origin
    2. Make appropriate rotations to make the line coincide with one of the axes, say z-axis
    3. Rotate the object about z-axis by required angle б
    4. Apply the inverse of step 2 (Reverse Rotation)
    5. Apply the inverse of step 1 (Reverse Translation)



ROTATION ABOUT AN ARBITRARY AXIS IN SPACE

- **ROTATION ABOUT AN ARBITRARY AXIS IN SPACE**
  - In general, making an arbitrary axis passing through the origin coincident with one of the coordinate axes requires two successive rotations about the other two coordinate axes.

➢ To make arbitrary rotation axis coincident with z-axis, first rotate about x-axis and then y-axis

➢ To calculate the angles of rotations about the x and y axes consider direction cosines (c_x, c_y ,c_z)

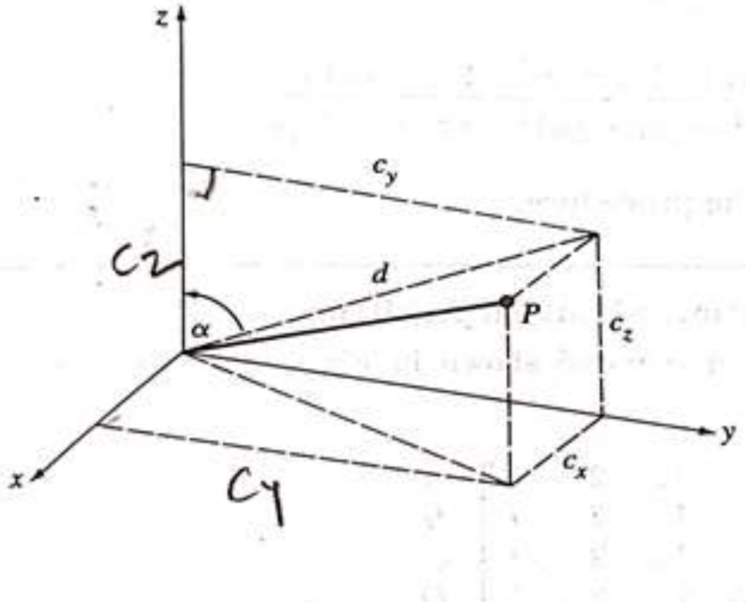
ROTATION ABOUT AN ARBITRARY AXIS IN SPACE

- **First rotate about x-axis by an angle α:**
  - To determine the rotation angle α, about x – axis, project the unit vector OP along yz plane.
  - The y and z components, c_y and c_z, are the direction cosines of the unit vector along the arbitrary axis.
  - From Diagram $d = \sqrt{c_y^2 + c_z^2}$
  - **Cos α= Base/ Hypotenuse**

$$\cos\alpha = \frac{c_z}{d}$$

  - **Sin α = Perpendicular/Hypotenuse**

$$\sin\alpha = \frac{c_y}{d}$$



(a)

ROTATION ABOUT AN ARBITRARY AXIS IN SPACE

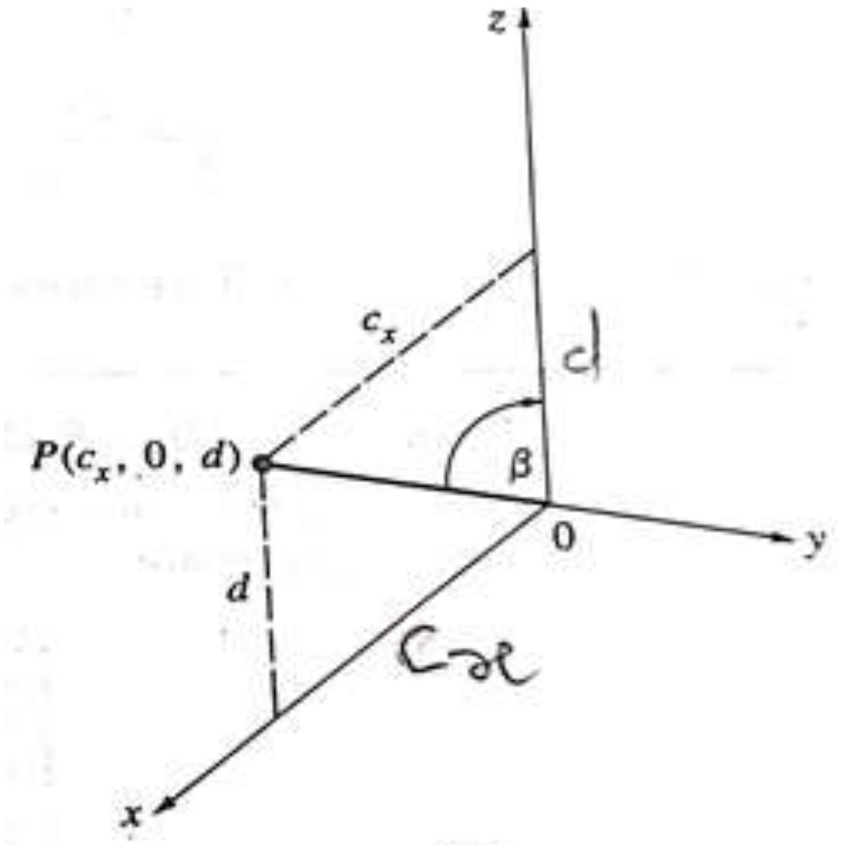
- **Now rotate about y-axis by an angle β:**
  - To determine the rotation angle **β**, about y – axis, project the unit vector OP along xz plane.
  - x component is c_x
  - y component is d
  - **Cos β** = **Base/ Hypotenuse**

    Cos β = d/ length of unit vector

    Cos β = d
  - **Sin β = Perpendicular/Hypotenuse**

    Sin β = c_x / length of unit vector

    Sin β = c_x
    - Its clockwise so considered –β during Calculations.



(b)

ROTATION ABOUT AN ARBITRARY AXIS IN SPACE

- Substituting for the respective angles we obtain the following matrices

$$[T]=[Tr][R_\alpha][R_\beta][R][R_\beta]^{-1}[R_\alpha]^{-1}[Tr]^{-1}$$

$$[Tr]=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\-x_0&-y_0&-z_0&1\end{bmatrix}[R_\alpha]=\begin{bmatrix}1&0&0&0\\0&\cos\alpha&\sin\alpha&0\\0&-\sin\alpha&\cos\alpha&0\\0&0&0&1\end{bmatrix}[R_\beta]=\begin{bmatrix}\cos(-\beta)&0&-\sin(-\beta)&0\\0&1&0&0\\\sin(-\beta)&0&\cos(-\beta)&0\\0&0&0&1\end{bmatrix}$$

$$\boxed{[Tr]^{-1}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\x_0&y_0&z_0&1\end{bmatrix}[R_\alpha]^{-1}=\begin{bmatrix}1&0&0&0\\0&\cos(-\alpha)&\sin(-\alpha)&0\\0&-\sin(-\alpha)&\cos(-\alpha)&0\\0&0&0&1\end{bmatrix}[R_\beta]^{-1}=\begin{bmatrix}\cos\beta&0&-\sin\beta&0\\0&1&0&0\\\sin\beta&0&\cos\beta&0\\0&0&0&1\end{bmatrix}}$$

ROTATION ABOUT AN ARBITRARY AXIS IN SPACE

- Substituting for the respective angles we obtain the following matrices:

$$[T]=[Tr][R_\alpha][R_\beta][R][R_\beta]^{-1}[R_\alpha]^{-1}[Tr]^{-1}$$

$$[R_\alpha]=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\alpha & \sin\alpha & 0\\ 0 & -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & c_z/d & c_y/d & 0\\ 0 & -c_y/d & c_z/d & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$[R_\beta]=\begin{bmatrix}\cos(-\beta) & 0 & -\sin(-\beta) & 0\\ 0 & 1 & 0 & 0\\ \sin(-\beta) & 0 & \cos(-\beta) & 0\\ 0 & 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}d & 0 & c_x & 0\\ 0 & 1 & 0 & 0\\ -c_x & 0 & d & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$[R]=\begin{bmatrix}\cos\delta & \sin\delta & 0 & 0\\ -\sin\delta & \cos\delta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

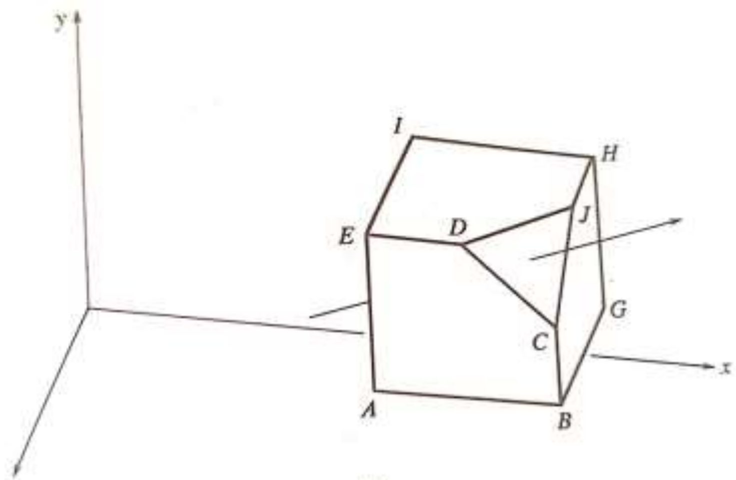
[Video](https://www.youtube.com/watch?v=75o5pmeXUMo)

https://www.youtube.com/watch?v=75o5pmeXUMo

ROTATION ABOUT AN ARBITRARY AXIS IN SPACE

Example 3–10 Rotation About an Arbitrary Axis

Consider the cube with one corner removed shown in Fig. 3–8a. Position vectors for the vertices are



(a)

$$[X] = \begin{array}{c} \begin{bmatrix} 2 & 1 & 2 & 1 \\ 3 & 1 & 2 & 1 \\ 3 & 1.5 & 2 & 1 \\ 2.5 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 3 & 2 & 1.5 & 1 \end{bmatrix} \end{array} \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \\ I \\ J \end{array}$$

ROTATION ABOUT AN ARBITRARY AXIS IN SPACE

The cube is to be rotated by $-45°$ about a local axis passing through the point F and the diagonally opposite corner. The axis is directed from F to the opposite corner and passes through the center of the corner face.

First, determine the direction cosines of the rotation axis. Observing that the corner cut off by the triangle CDJ also lies on the axis, Eq. (3–26) yields

$$[\,c_x \quad c_y \quad c_z\,] = \frac{[\,(3-2) \quad (2-1) \quad (2-1)\,]}{((3-2)^2+(2-1)^2+(2-1)^2)^{\frac{1}{2}}}$$

$$= [\,1/\sqrt{3} \quad 1/\sqrt{3} \quad 1/\sqrt{3}\,]$$

Using Eqs. (3–18) to (3–20) yields

$$d = \sqrt{(1/\sqrt{3})^2 + (1/\sqrt{3})^2} = \sqrt{2/3}$$

and

$$\alpha = \cos^{-1}(1/\sqrt{3} \big/ \sqrt{2/3}) = \cos^{-1}(1/\sqrt{2}) = 45°$$

$$\beta = \cos^{-1}(\sqrt{2/3}) = 35.26°$$

ROTATION ABOUT AN ARBITRARY AXIS IN SPACE

Since the point F lies on the rotation axis, the translation matrix is

$$[\,T\,] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & -1 & -1 & 1 \end{bmatrix}$$

The rotation matrices to make the arbitrary axis coincident with the z-axis are then

$$[\,R_x\,] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$[\,R_y\,] = \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} & 0 \\ 0 & 1 & 0 & 0 \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ROTATION ABOUT AN ARBITRARY AXIS IN SPACE

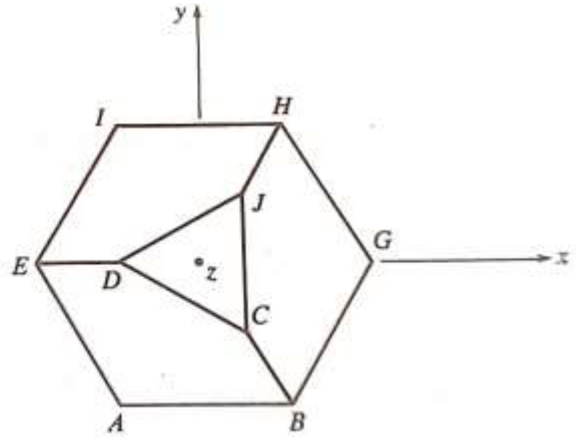
$[R_x]^{-1}$, $[R_y]^{-1}$, and $[T]^{-1}$ are obtained by substituting $-\alpha$, $-\beta$ and (x_0, y_0, z_0) for α, β and $(-x_0, -y_0, -z_0)$, respectively, in Eqs. (3–22) to (3–24).

Concatenating $[T][R_x][R_y]$ yields

$$[M] = [T][R_x][R_y] = \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} & 0 \\ -2/\sqrt{6} & 0 & -4/\sqrt{3} & 1 \end{bmatrix}$$

ROTATION ABOUT AN ARBITRARY AXIS IN SPACE

The transformed intermediate position vectors are



(b)

$$[X][M] = \begin{bmatrix} -0.408 & -0.707 & 0.577 & 1 \\ 0.408 & -0.707 & 1.155 & 1 \\ 0.204 & -0.354 & 1.443 & 1 \\ -0.408 & 0 & 1.443 & 1 \\ -0.816 & 0 & 1.155 & 1 \\ 0 & 0 & 0 & 1 \\ 0.816 & 0 & 0.577 & 1 \\ 0.408 & 0.707 & 1.155 & 1 \\ -0.408 & 0.707 & 0.577 & 1 \\ 0.204 & 0.354 & 1.443 & 1 \end{bmatrix}$$

This intermediate result is shown in Fig. 3–8b. Notice that point F is at $(0, 0, 0)$.

ROTATION ABOUT AN ARBITRARY AXIS IN SPACE

The rotation about the arbitrary axis is now given by the equivalent rotation about the z-axis. Hence (see Eq. 3–7)

$$[\,R_\delta\,] = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformed object is returned to its 'original' position in space, using

$$[\,M\,]^{-1} = [\,R_y\,]^{-1}\,[\,R_x\,]^{-1}\,[\,T\,]^{-1} = \begin{bmatrix} 2/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{6} & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} & 0 \\ 2 & 1 & 1 & 1 \end{bmatrix}$$

This result can be obtained either by concatenating the inverses of the individual component matrices of $[\,M\,]$ or by formally taking the inverse of $[\,M\,]$. Incidentally, notice that $[\,R_x\,]\,[\,R_y\,]$ is a pure rotation. The upper left 3×3 submatrix of $[\,M\,]^{-1}$ is just the transpose of the upper left 3×3 submatrix of $[\,M\,]$.

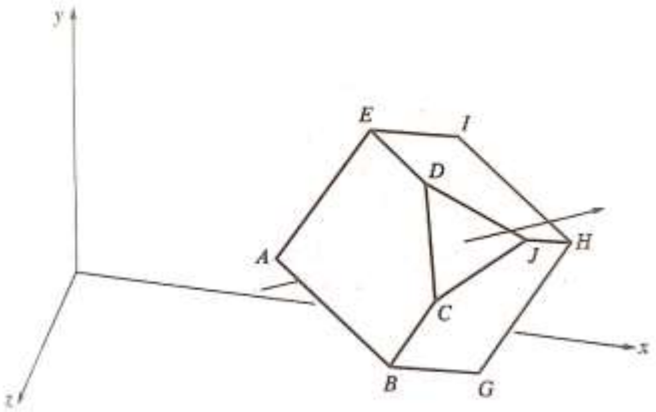
ROTATION ABOUT AN ARBITRARY AXIS IN SPACE

where

$$[\,M\,][\,R_\delta\,][\,M\,]^{-1} = \begin{bmatrix} 0.805 & -0.311 & 0.506 & 0 \\ 0.506 & 0.805 & -0.311 & 0 \\ -0.311 & 0.506 & 0.805 & 0 \\ 0.195 & 0.311 & -0.506 & 1 \end{bmatrix}$$

ROTATION ABOUT AN ARBITRARY AXIS IN SPACE

The resulting position vectors are



(c)

Figure 3–8 Rotation about an arbitrary axis.

$$[X][M][R_\delta][M]^{-1} = \begin{bmatrix} 1.689 & 1.506 & 1.805 & 1 \\ 2.494 & 1.195 & 2.311 & 1 \\ 2.747 & 1.598 & 2.155 & 1 \\ 2.598 & 2.155 & 1.747 & 1 \\ 2.195 & 2.311 & 1.494 & 1 \\ 2 & 1 & 1 & 1 \\ 2.805 & 0.689 & 1.506 & 1 \\ 3.311 & 1.494 & 1.195 & 1 \\ 2.506 & 1.805 & 0.689 & 1 \\ 3.155 & 1.747 & 1.598 & 1 \end{bmatrix}$$