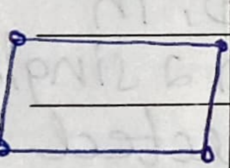
Graph

Defn :
A graph $G = (V, E)$ consists of V, a nonempty set of vertices (or nodes) & E, a set of edges. Each edge has either one or two vertices associated with it, called its endpoints. An edges is said to connect its endpoints.

Infinite Graph - A graph with an infinite vertex set or an infinite no. of edges is called an infinite graph.

Finite Graph : A graph with a finite vertex set and finite edge set is called a finite graph.



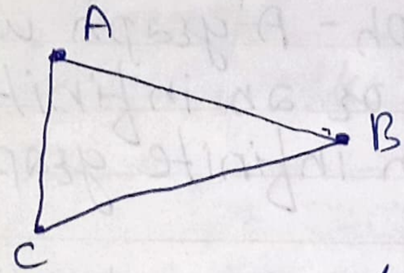
Simple Graph : A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called simple graph.
when
- For edges = $n(n-1)/2$ for vertices $n = 2^n$

Multigraphs : Graphs that may have multiple edges connecting the same vertices are called multigraphs.

loops: edges that connect a vertex to itself. Such edges are called loops.

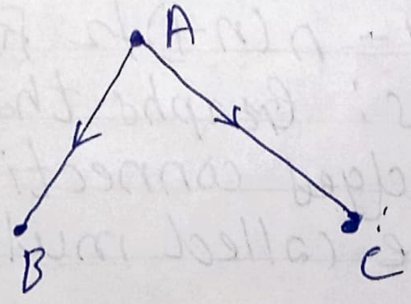
pseudographs :- Graphs that may include loops, and possibly multiple edges connecting the same pair of vertices or a vertex to itself. are called pseudographs.

Undirected Graphs : that have edges that do not have a direction. The edges indicate a two way relationship, in that each edge can be traversed in both directions.



This fig. shows a simple undirected graph with three nodes & three edges.

Directed graphs : have edges with directⁿ. The edges indicates a one-way relationship, in that each edge can only be traversed in a single directⁿ. This fig. shows a simple directed graph with three nodes and two edges.



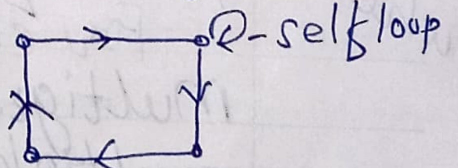
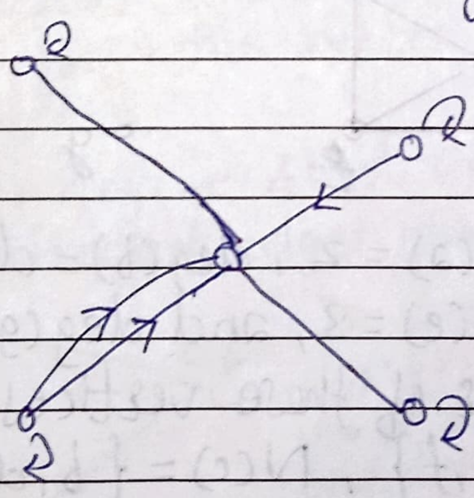


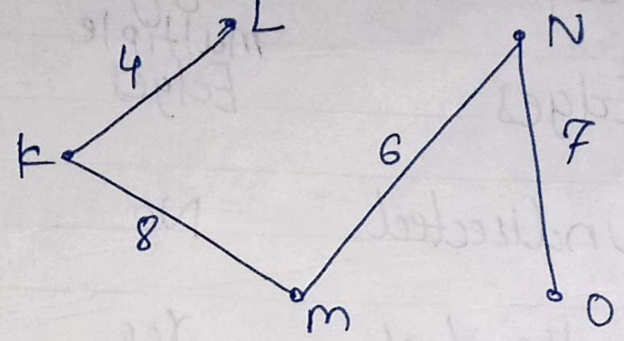
Table 1 : Graph Terminology

type	Edges	multiple Edges	Loops Allowed
simple graph	Undirected	No	No
multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
simple dire. graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
mixed graph	Directed & undirected	Yes	Yes

* Mixed graph: For some models we may need a graph were some edges are undirected, while others are directed. A graph with both directed & undirected edges is called mixed graph.



Weighted Graph : A graph $G=(V,E)$ is called a weighted graph if each of graph G is assigned a positive no. w called the weight of the edge e.

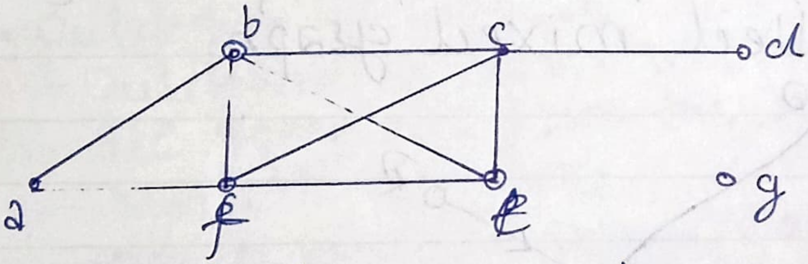


* Basic Terminology :

The degree of a vertex in an undirected graph is the no. of edges incident with it, except that a loop at vertex contributes twice to the degree of that vertex.

The degree of vertex v is denoted by $\deg(v)$.

Example 1: what are the degrees & what are the neighborhoods of the vertices in the graphs G

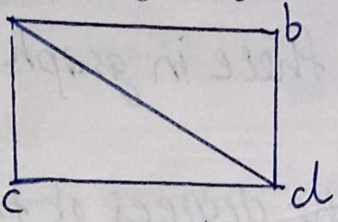


Soln: In G, $\deg(a)=2$, $\deg(b)=\deg(c)=\deg(f)=4$, $\deg(d)=1$, $\deg(e)=3$, and $\deg(g)=0$.

The neighborhoods of these vertices are $N(a)=\{b,f\}$, $N(b)=\{a,c,e,f\}$, $N(c)=\{b,d,e,f\}$, $N(d)=\{c\}$, $N(e)=\{b,c,f\}$, $N(f)=\{a,b,c,e\}$, $N(g)=\{\phi\}$

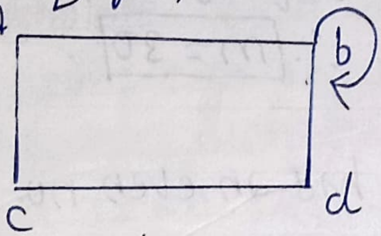
(3)

* A vetex of degree zero is called isolated.
* Pendant:- A vertex is pendant if & only if it had degree one.
* Degree of graph
  2 x no. of edges.



degree = 2 x 5
= 10

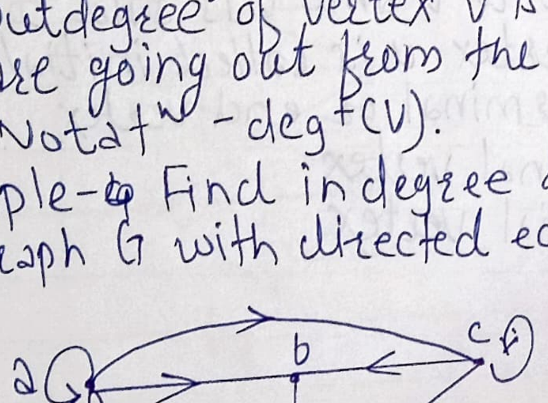
Total degree of graph is always even.



degree (b) = 4

* Degree of Directed graph.
  In a directed graph, each vertex has an indegree and an outdegree.
  - Indegree of a graph
    - Indegree of vertex V is the no. of edges which are coming into the vertex V.
    - Notation - $deg^-(V)$.
  - Outdegree of a graph
    - Outdegree of vertex V is the no. of edges which are going out from the Vertex V.
    - Notatⁿ - $deg^+(V)$.

Example-4 Find indegree & out-degree of each vertex in graph G with directed edges shown in fig.



In degrees - $deg^-(a) = 2$, $deg^-(b) = 2$, $deg^-(c) = 3$, $deg^-(d) = 2$, $deg^-(e) = 3$, $deg^-(f) = 0$.
out degrees, $deg^+(a) = 4$, $deg^+(b) = 1$
$deg^+(c) = 2$, $deg^+(d) = 2$, $deg^+(e) = 3$
$deg^+(f) = 0$.

* **The handshaking Theorem**:

Let $G=(V,E)$ be an undirected graph with m edges then,

$$2m=\sum_{v\in V} \deg(v)$$ (Note - This applies even if multiple edges & loops are present)

Example: How many edges are there in graph with 10 vertices each of degree six?

sol^n: Because the sum of the degrees of the vertices is $6 \cdot 10 = 60$, it follows that $2m = 60$, where m is the no. of edges. $\therefore \boxed{m=30}$

* **Theorem: 2**

An undirected graph has an even no. of vertices of odd degree.

Proof: V_1 & V_2 be the set of vertices of even degree & the set of vertices of odd degree, resp. in an undirected Graph $G=(V,E)$ with m edges. Then.

$$2m=\sum_{v\in V}\deg(v)=\sum_{v\in V_1}\deg(v)+\sum_{v\in V_2}\deg(v)$$

* **Vertex & terminal**

When (u,v) is an edge of the graph G with directed edges u is said to be adjacent to v and v is said to be adjacent from u. The vertex u is called initial vertex of (u,v) & v is called terminal or end vertex

$\deg^-(v)$ = terminal vertex

$\deg^+(v)$ = initial vertex

Theorem 4

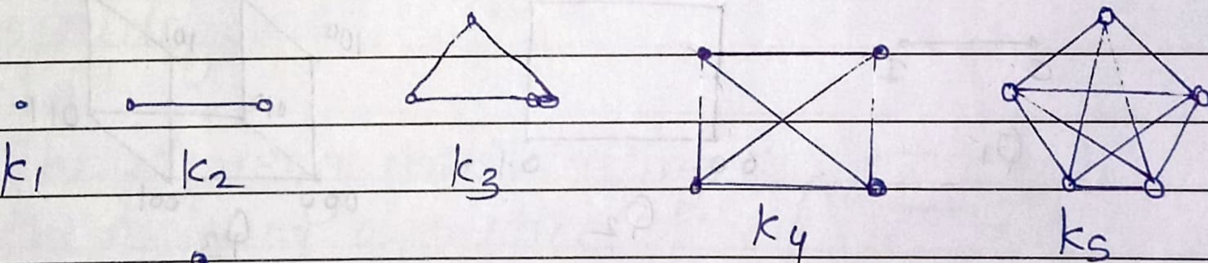
Let $G=(V,E)$ be a graph with directed edges.
Then,

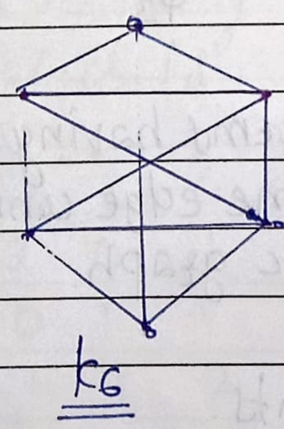
$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$$

* Complete Graph

A complete graph on n vertices, denoted by K_n, is a simple graph that contains exactly one edge between each pair of distinct vertices.
i.e. There is an edge betn any two vertices.

* A simple graph for which there is at least one pair of distinct vertex not connected by an edge is called noncomplete.

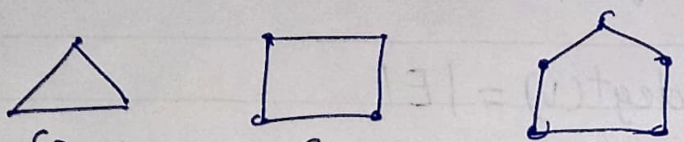




Graphs K_n for $1 \le n \le 6$

*** Cycle** - A cycle C_n, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\} \ldots \{v_{n-1}, v_n\}$ & $\{v_n, v_1\}$. The cycles C_3, C_4, C_5 & C_6 are displayed in fig.

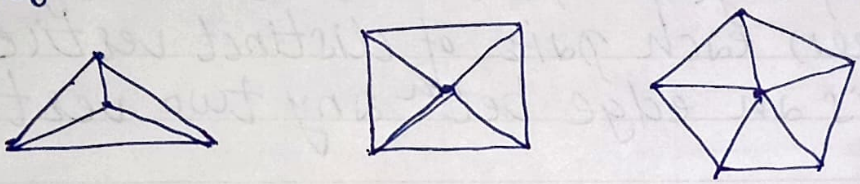
If graph consists of a single cycle, it is called cycle graph. The cycle graph with n vertices is denoted by C_n



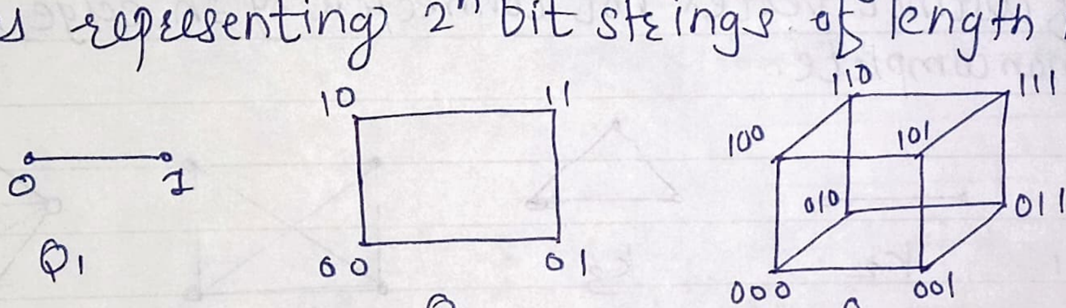
C_3 C_4 C_5

*** Wheels** : We obtain a wheel W_n when we add an additional vertex to a cycle C_n for $n \geq 3$, & connect this new vertex to each of n vertices in C_n by new edges.

no. of edges = $2(n-1)$



*** Cube** : n-cube, denoted by Q_n, is a graph that has vertices representing 2^n bit strings of length n.



Q_1 Q_2 Q_3

*** Isomorphic graphs**

A graph can exist in different forms having the same no. of vertices, edges and also the same edge connectivity. Such graphs are called isomorphic graph.

Isomorphic if

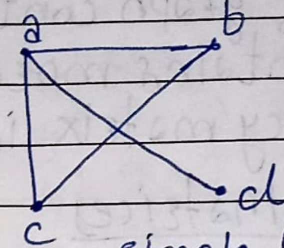
- Their no. of components.
- Their edge connectivity is retained.

Representing Graphs & Graph Isomorphism

· Adjacency Matrix

Example

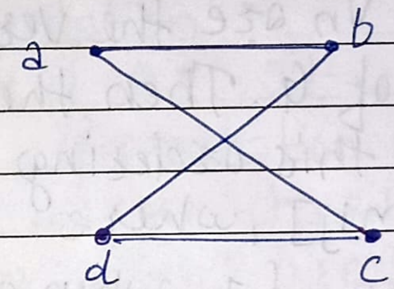
$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$



simple Graph

Example

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

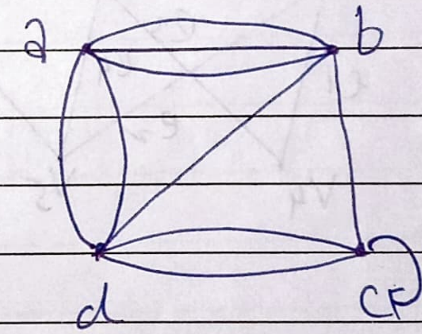


Example

Use an adjacency matrix to represent the pseudograph.

Sol^n:

The adjacency matrix using the ordering of vertices a, b, c, d is



$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

Trade-offs betn Adja. Lists & Adj. matrices.

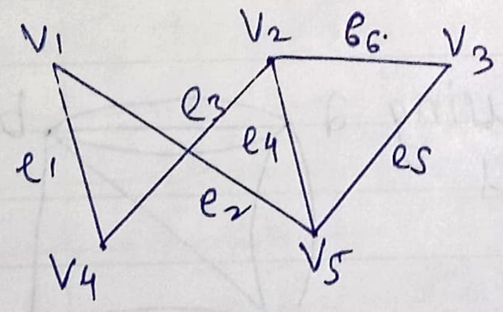
- Sparse - when a simple graph contains relatively few edges, that is.
  preferable to use adjacency lists rather than an adjacency matrix to represent the graph
- dense - If graph contains many edges, such that a graph contains more than half of all possible edges - adjacency matrix is preferrable.

* Incidence Matrices

Let $G = (V, E)$ be an undirected graph. Suppose that $v_1, v_2, \ldots v_n$ are the vertices and $e_1, e_2, \ldots e_m$ are the edges of G. Then the incidence matrix with respect to this ordering of V & E is the $m \times n$ matrix $M = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise} \end{cases}$$

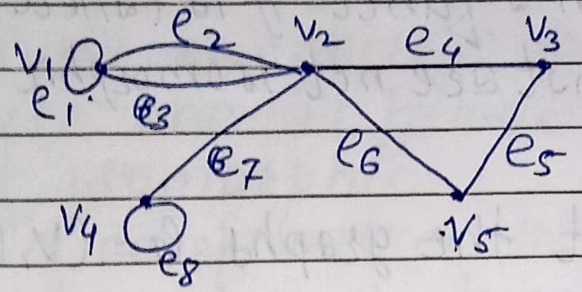
Example:



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ \cancel{v_6} \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ & & & & & \end{bmatrix} \end{array}$$

Undirected graph

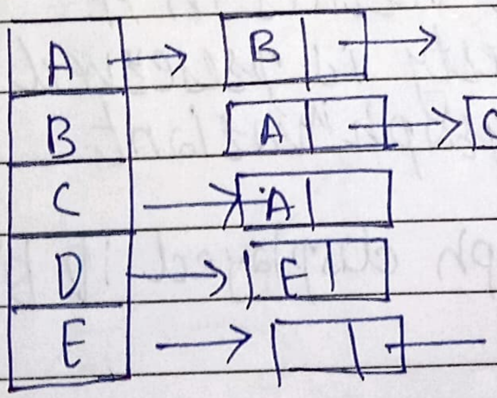
Example - Represent the pseudograph



	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
v_1	1	1	1	0	0	0	0	0
v_2	0	1	1	1	0	1	1	0
v_3	0	0	0	1	1	0	0	0
v_4	0	0	0	0	0	0	1	1
v_5	0	0	0	0	1	1	0	0

Adjacency List Representation

- This is another type of G.R. It is called the adjacency list. This representation is based Linked list. In this approach, each node is holding a list Nodes, which are Directly connected with that vertices, At the end of list, each node is connected with the null values to tell that it is the end node of that list.

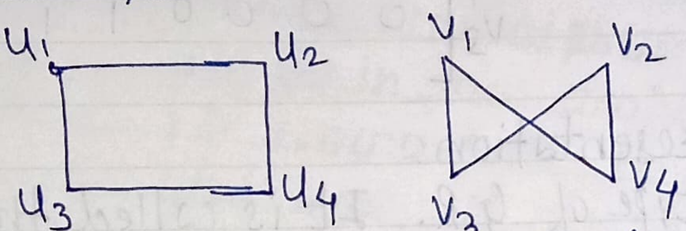


* **Isomorphism of Graphs.**

The simple graphs $G_1 = (V_1, E_1)$ & $G_2 = (V_2, E_2)$ are isomorphic if there exist one to one ~~fun~~ & onto function f from V_1 to V_2 with the property that a & b are adjacent in G_1 if & only if $f(a)$ & $f(b)$ are adjacent in G_2, for all a & b in V_1. Such a funct^n f is called an isomorphism.

* Two simple graphs that are not isomorphic are called non isomorphic.

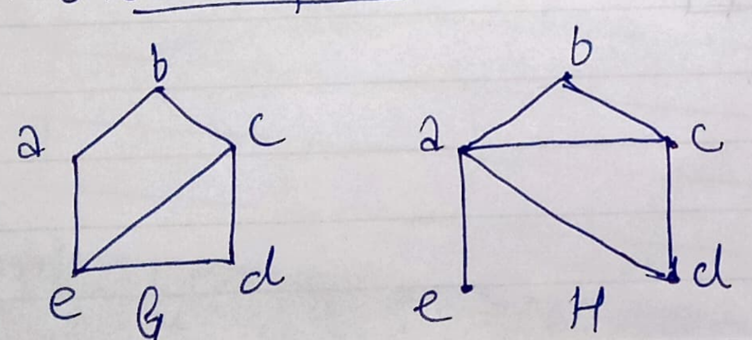
<u>Example</u>: Show that the graphs $G = (V, E)$ & $H = (W, F)$



sol^n: The funct^n f with $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, & $f(u_4) = v_2$ is a one-to-one correspondence bet^n V & W. To see that this corresponelence preserves adjacency, note that adjacent vertices in G are u_1, & u_2, ~~and~~ u_1 & u_3, u_2 & u_4 &, u_3 & u_4, and each of the pairs of $f(u_1) = v_1$ & $f(u_2) = v_4$, $f(u_1) = v_1$ & $f(u_3) = v_3$, $f(u_2) = v_4$ & $f(u_4) = v_2$ & $f(u_3) = v_3$ & $f(u_4) = v_2$ consists of two adjacent vertices in H.

* graph <u>invariant</u>: property is preserved by isomorphism of graphs is called a graph invariant.

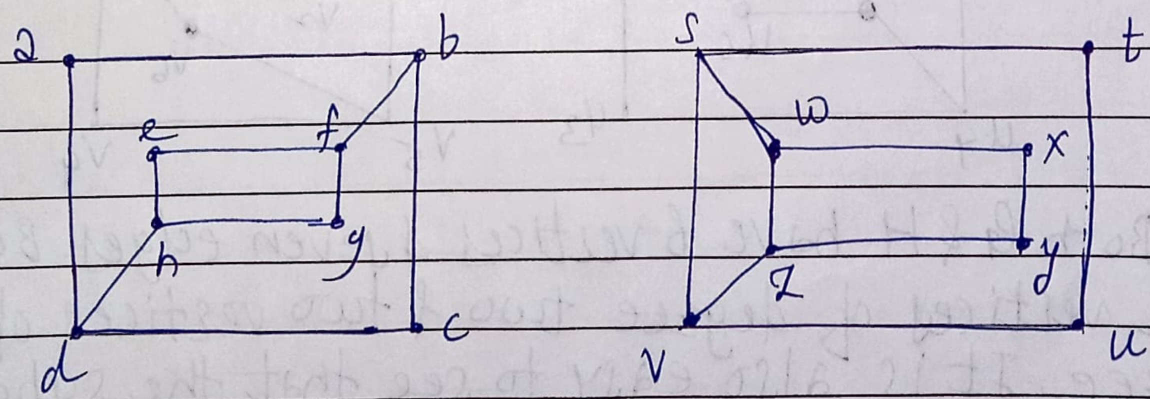
<u>Example</u>: Show that graph displayed in fig. 9. are not <u>isomorphic</u>.



Soln: Both G & H have five vertices & 6 edges. However, H has a vertex of a degree one, namely, e, whereas G has no vertices of degree one. It follows that G & H are not isomorphic.

The no. of vertices, the no. of edges, & the no. of vertices of each degree are all invariants under isomorphism. If any of these quantities differ in two simple graphs, these graphs can't be isomorphic. However, when invariants are the same, it does not necessarily mean that the two graphs are isomorphic. There are no useful sets of invariants currently known that can be used to determine whether simple graphs are isomorphic.

Example: Determine whether the graphs shown in fig. ~~10~~ are isomorphic.

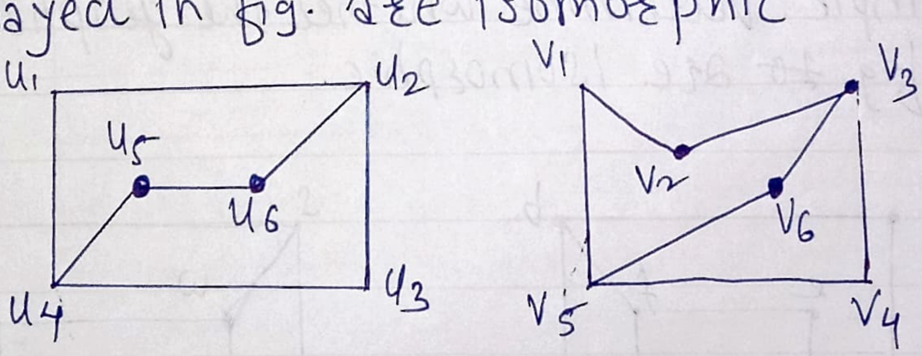


Sol^n: The graphs G and H both have eight vertices & 10 edges. They also both have four vertices of degree two & four of degree three. Because these invariants all agree it is still conceivable that these graphs are isomorphic.

However, G and H are not isomorphic. To see this: note that because deg(a) = 2 in G, a must correspond to either t, u, x or y in H, because these are the vertices of degree two in H. However, each of these four vertices in H is adjacent to another vertex of degree two in H, which is not true for a in G.

Another way to see that G & H are not isomorphic is to note that the subgraphs of G and H made up of vertices of degree three & the edges connecting them must be isomorphic if these two graphs are isomorphic.

<u>Example</u> - Determine whether the graphs G and H displayed in fig. are isomorphic



Both G & H have 6 vertices & seven edges. Both have four vertices of degree two & two vertices of degree three. It is also easy to see that the subgraphs of G & H consisting of all vertices of degree two & the edges connecting them are isomorphic. Because

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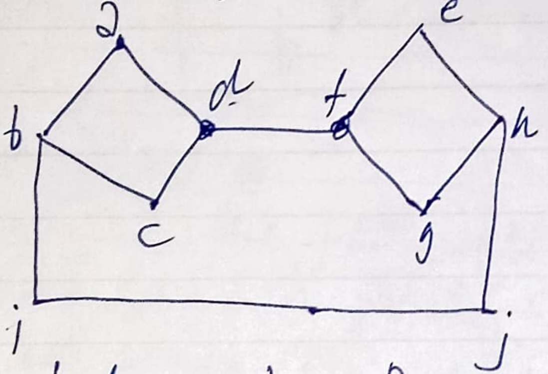
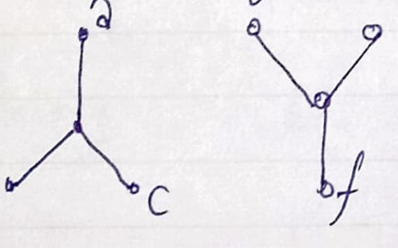
G & H agree with respect to these invariants, it is reasonable to try to find an isomorphism f.

Connected graph: A graph G is said to be connected if there exists a path between every pair of vertices. There should be at least one edge for every vertex in the graph. So that we can say that it is connected to some other vertex at the other side of the edge.

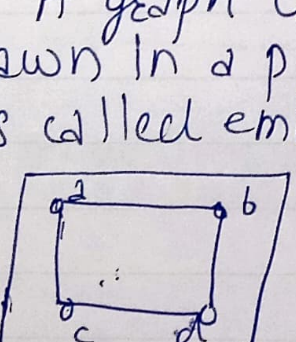


* Disconnected graph - A graph G is disconnected, if it does not contain at least two connected vertices.



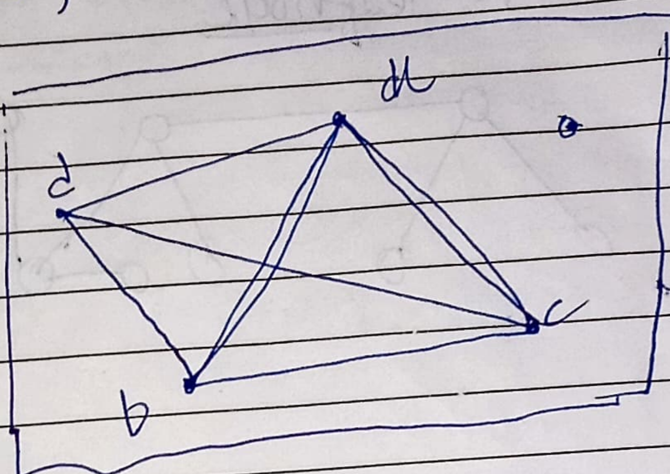
two component are independent & not connected to each other. Hence it is called disconnected graph.

* Planar Graph: A graph G is called a planar graph if it can be drawn in a plane without edge crossing, it is called embedding the graph in the plane



Non-planar Graph

- A graph is non-planar if it can't be drawn in a plane without graph edges crossing.



* Graph Coloring

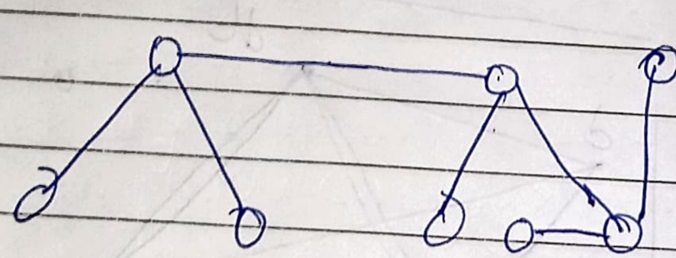
It is the procedure of assignment of colors to each vertex of a graph G such that no adjacent vertices get same color

Applicat^n:

Register Allocation
Map coloring
Making time table

No. :
Date:

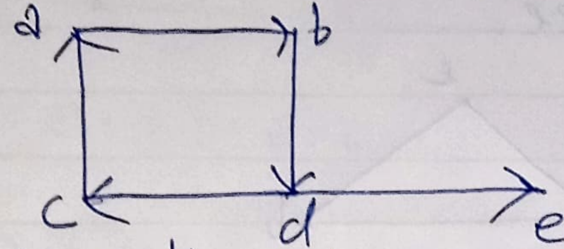
Tree:-
A Tree is a connected acyclic circuit undirected graph. There is a unique path betn every pair of vertices in G. A tree with N no. of vertices contains (N-1) no. of edges.
degree 0 - Root node
1 - leaf node



Euler path & Hamilton Paths

* Euler path & Circuit.

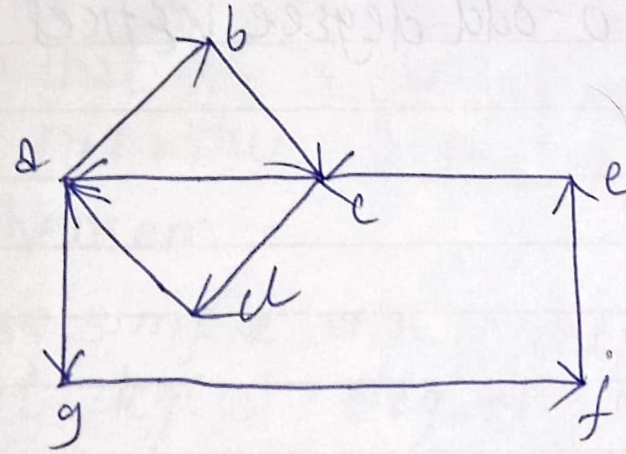
An Euler's path contains each edge of 'G' exactly once and each vertex of 'G' at least once. A connected graph G is said to be traversable. If it contains an Euler's path.



Euler's path = d - c - a - b - d - e

Euler's circuit

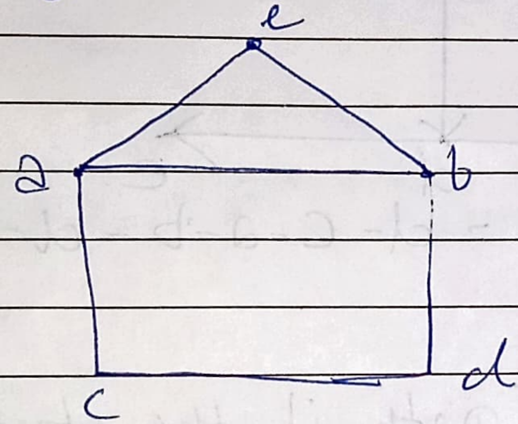
In an Euler's path, if the starting vertex is same as its ending vertex, then it is called an Euler's Circuit.



Euler's path = a - b - c - d - a - g - f - e - c - a.

Euler's circuit Theorem

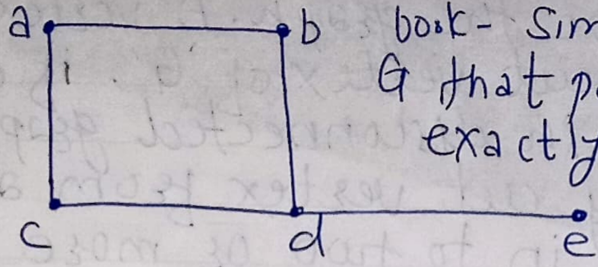
This Euler's path begins with a vertex with odd degree & ends with the other vertex of odd degree



b-e-a-b-d-c-a - not a circuit but path
2 odd degree vertices
0 - odd degree vertices - circuit.

Hamiltonian Path

A connected graph is said to be Hamiltonian if it contains each vertex of G exactly once.
Such path is called a Hamiltonian path.



book- Simple path in a graph G that passes through vertex exactly once

e-d-b-a-c

Hamilton circuits

A simple circuit in graph G that passes through every vertex exactly once is called hamilton circuit.

* Dirac's Theorem:

If G is a simple graph with n vertices. with $n \geq 3$ such that the degree of every vertex in G is at least $n/2$, then G has a hamilton circuit.

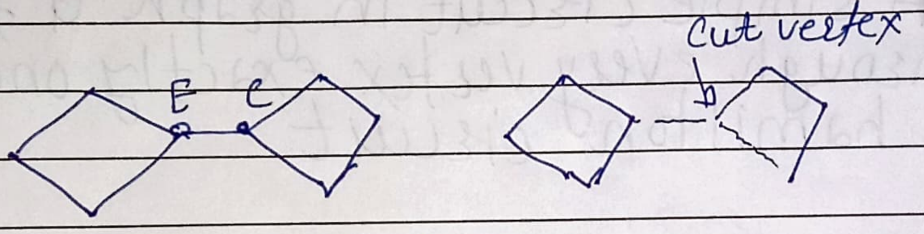
* ORE'S Theorem:

If G is a simple with n vertices with $n \geq 3$ such that $deg(u) + deg(v) \geq n$ for every pair of nonadjacent vertices u & v in G, then G has a hamilton circuit.

* Cut vertex

Let G be the graph. A vertex $v \in G$ is called a cut vertex of 'G'. if delete v from G results in disconnected graph.
Removing cut vertex from a graph breaks it in to two or more graphs.
- Note. Removing cut vertex may render a graph disconnected.

G - having $(n-2)$ cut vertices.



cut edge (Bridge)

Let 'G' be a connected graph. An edge 'e' $\in$ G is called a cut edge if 'G-e' results in a disconnected graph.