Unit-04 Algebraic Structure

Binary operation

Let A be a non-empty set, then a function $f: A \times A \to A$ is called binary operation on A.

A function $g: A \to A$ is called unary operation on A and a function $h: (A \times A \times A \to A)$ is called ternary operation on A, or in general, an n-ary operation is a function $\underbrace{A \times A \times A \times \cdots \times A}_{n\text{-times}}$ into A.

Thus binary operation on A is a function that assign each ordered pairs of element of A to an element of A. The symbol $+, \cdot, *, 0$ etc.. are used to dentote binary operation on a set.
$+$ will be a binary operation on A if and only if

$a+b \in A \ \forall \ a, b \in A$ and $a+b$ is unique.

similarly $*$ will be a binary operation on A if $a*b \in A \ \forall \ a, b \in A$ and $a*b$ is unique.

This is said to be the closure property of the binary operation on a set. is said to be closed with respect to binary operation.

Eg-1 Let $A = \{0, 1, -1\}$. then the addition is not a binary operation on A because

$$(-1) + (-1) = -2 \notin A$$

where the multiplication is a binary operation on A.

subtraction(−)

ep.2 The operation $\wedge$ on a set A is a binary operation on Z whereas it is not a binary operation on Z^+.

ep3 Define $a*b = a|b$ for $a,b \in R$. Then $*$ is not a binary operation on R since $a*b$ $(2*0 = 2|0)$ is not defined.

ep.3. Let $A = \{0, 1, 2\}$. Define $a*b = a+b$ if $a+b = \leq 2$ and $a*b = a+b-3$ if $a+b > 3$. Then $*$(binary operator) the table for $*$

b →

a ↓ \ *	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Row → a
Column → b

ep.4 If $A = \{a, b\}$, we shall now determine the number of binary operations that can be defined on A. Every binary operation $*$ on A can be described by a table

*	a	b
a	.	.
b	.	.

since every blank space can be filled with a and b because it should belong $\{a, b\}$ set.

So there are

$2 \times 2 \times 2 \times 2 = 2^4 = 16$ ways to complete the table. So there are 16 binary operations on A.

Properties of Binary operations

① Commutative Law

A binary operation $*$ on a set A is said to be commutative if

$$a * b = b * a \quad \forall\ a, b \in A$$

Ex.5 Addition and subtraction are commutative operations on the set Z of integers whereas subtraction is not commutative on the set Z of integers since $3-4 = -1 \neq 1 = 4-3$

Q.6 Consider the binary operation $*$ defined on the set $A = \{a, b, c\}$ by the following table.

$*$	a	b	c
a	b	c	b
b	a	b	c
c	c	a	b

then $*$ is not commutative since $a * b = c$ while $b * a = a$

② Associative Law :-

An operation $*$ on a set A is said to be associative or satisfy associative law, if $(a * b) * c = a * (b * c) \quad \forall\ a, b, c \in A$.

ep.7 Consider the set z^+ of non-negative integers. Check whether the operation $*$ defiend by

$a*b = a^2+b$ for all $a,b \in z^+$, is associative or not.

For $a,b,c \in z^+$

Soln $(a*b)*c = (a^2+b)*c$

$= (a^2+b)^2 + c$

$= a^4 + b^2 + 2a^2b + c$

$a*(b*c) = a^2 + (b*c)$

$= a^2 + b^2 + c$

$= a^2 + b^2 + c$

thus $(a*b)*c \neq a*(b*c)$ Ans

Q.8 Consider the operation $*$ defined on Z, the set of integers, as $a*b = a+b-3ab$. Determine whether $*$ is commutative or associative.

Soln: Since commutative and associative are both binary operations.

$\therefore \quad a*b = a+b-3ab$

$b*a = b+a-3ba$

$\therefore \quad a*b = b*a$ hence commutative

(ii) $(a*b)*c = (a+b-3ab)*c$

$= (a+b-3ab)+c-3\{a+b-3ab\}\times c$

$= a+b-3ab+c-3ac-3bc+9abc$

$= a+b+c-3ab-3bc-3ca+9abc$

2 $a*(b*c) = a*(b+c-3bc)$

$= a+(b+c-3bc)-3a(b+c-3bc)$

$= a+b+c-3ab-3bc-3ca+9abc$

$\therefore \quad (a*b)*c = a*(b*c)$ Ans

Q. Consider the operation $*$ defined on the set Z^+ of non-negative integers as

$$a*b = a+b+2 \text{ for } a,b \in Z^+$$

Show that it is commutative as well as associative

Soln $*$ is commutative

$\because$

$$a*b = a+b+2$$

& $$b*a = b+a+2$$

$\therefore$ $\boxed{a*b = b*a}$ commutative

(ii) Associative

$$(a*b)*c = (a+b+2)*c$$
$$= a+b+2+c+2$$
$$= a+b+c+4$$

Now $$a*(b*c) = a*(b+c+2)$$
$$= a+b+c+2+2$$
$$= a+b+c+4$$

$\therefore$ $\boxed{(a*b)*c = a*(b*c)}$

Q. Consider the binary operation $*$ defined on the set of integers Z as

$$a*b = a|b|$$

Determine whether operation $*$ is commutative or not. Check whether it is associative.

Soln. Since $3*-2 = 3|-2| = 3\times 2 = 6$

Let $a=3$, $b=-2$ $\quad -2*3 = -2|3| = (-2)(3) = -6$

$\therefore$ $*$ is not commutative.

(ii) $(a*b)*c = (a|b|)|c| = a|b||c|$

$a*(b*c) = a*(b|c|) = a|b|c||$ $= a|b||c|$

Thus $*$ is Associative.

Q. Consider the set N of Natural numbers, Define on operation * on N as

$a * b = a^b$ $\forall$ all $a, b \in N$.

Show that * is neither commutative nor associative.

Soln. Let $a=2$ & $b=3$

$a * b = 2^3 = 8$

$\therefore b * a = 3^2 = 9$

$\therefore \boxed{a * b \neq b * a}$ Hence not Commutative.

(ii) Let $a=2, b=2$ & $c=3$

$(a*b)*c = (2*2)*3 = (2^2)*3 = 4*3 = 4^3 = 64$

$a*(b*c) = 2*(2*3) = 2*(2^3) = 2*8 = 2^8 = 256$

$\therefore \boxed{(a*b)*c \neq a*(b*c)}$

Q. Show that the binary operation * defined on $(R, *)$, where $x*y = \max(x,y)$ is associative.

Soln. $x*y = \max(x,y)$

$\therefore (x*y)*z = \max(\max(x,y), z)$

$= \max(x,y) * z$

$= \max(\max(x,y), z)$

$= \max(\max(x,y), z)$

$= \max(x,y,z)$ ——— ①

$x*(y*z) = x * (\max(y,z))$

$= \max(x, \max(y,z))$

$= \max(x,y,z)$ ——— ②

$\therefore \boxed{(x*y)*z = x*(y*z)}$

is associative.

Property

Identity element

Let $*$ be a binary operation on a set A. An element $e \in A$ is called the identity element for $*$ if for any a in A,

$$\boxed{a * e = e * a = a}$$

if $a * e = a$, then e is called the right identity element for the operation $*$ and if $e * a = a$, then e is called the left identity element for the operation $*$.

eg: Consider the set Z of integers. For the operation of addition defined on Z, the integer 0 is an identity element since

$$0 + a = a + 0 = a \text{ for all } a \in Z.$$

Inverse element

Suppose a binary operation $*$ on a set A have an identity element e. An element $b \in A$ is said to be inverse of a if

$$\boxed{a * b = b * a = e}$$

eg: for the operation of multiplication defined on the set of integers Z, the integer 1 is an identity element since

$$1 \cdot a = a \cdot 1 = a \quad \forall a \in Z.$$

eg Consider the set Z of integers, for the operation of addition defined on Z, $-a \in Z$ is the inverse element of a since

$$a + (-a) = (-a) + a = 0 \quad \text{for all } a \in Z.$$

Q. Consider $A = \{1,2,3,4,5,6,7\}$. Define a binary operation $*$ on A as $a * b = \min\{a,b\}$ for all $a,b \in A$. Then prove that this binary operation $*$ is commutative and associative and also find the identity element.

Soⁿ.

Given $a * b = \min(a,b) \quad \forall\ a,b \in A$

Now $a * 7 = \min(a,7) = a$

$7 * a = \min(7,a) = a$

$\therefore \quad a * 7 = 7 * a = a$

$\therefore$ Binary operation is commutative and identity element is 7.

(ii)

$$(a*b)*c = \min(a,b) * c$$
$$= \min(\min(a,b), c)$$
$$= \min(a,b,c) \quad \text{——} (1)$$

$$a*(b*c) = a * (\min(b,c))$$
$$= \min(a, \min(b,c))$$
$$= \min(a,b,c) \quad \text{——} (2)$$

Now from (1) & (2)

$$\boxed{(a*b)*c = a*(b*c)}$$ Associative.

Group:- Let G be a non-empty set together with some operation $*$ then algebric structure $(G, *)$ is said to be a group if following 4 conditions are satisfied.

(i) G_1: **Closure Property**: If $a \in G, b \in G$ then $a * b \in G \quad \forall\ a, b \in G$.

(ii) G_2: **Associative Property**

If $a, b, c \in G$, then $a * (b * c) = (a * b) * c$ $\forall\ a, b, c \in G$.

(iii) G_3: **Existance of identity**:-
There exist $e \in G$, such that

$$a * e = e * a = a \quad \forall\ a \in G.$$

(iv) **Existance of Inverse element**:-
Every element of G has an inverse i.e for every $a \in G$ there exist $b \in G$ such that $a * b = b * a = e$, as b is called the inverse of a and b written as $(a^{-1}) = b$.

~~Ex~~:

Semigroup:

Let A be a non-empty set and * be a binary operation defined on A. The algebric system (A, *) is called a semigroup if the operation * is associative, that is (A, *) is a semigroup if for any $a, b, c \in A$

(i) $(a*b)*c = a*(b*c)$

(ii) closure property.

Note:- A semigroup may or may not have an identity element.

Monoid

An Algebric system (A, *) is called a monoid if for any $a, b, c \in A$

(i) $(a*b)*c = a*(b*c)$

(ii) There exist an identity element $e \in A$ such that for any $a \in A$

$$a*e = e*a = a$$

(iii) Closure - for every pair $(a,b) \in S$, $a*b$ must be present in S.

OR A semigroup (A, *) in which the binary operation * has an identity element is called a monoid.

eg. Consider a set of natural nos Then (N, +) and (N, *X) are semigroup since addition and and multiplication are associative on N

$a+(b+c) = (a+b)+c$ & $a \times (b \times c) = (a \times b) \times c$ } but (N, X) is monoid since ~~N has~~ with identity element 1. where as (N, +) is not monoid since N has no zero element.

eg: The algebric system $(Z, +)$ is a semigroup as well as monoid, whereas $(Z, -)$ is not a semigroup & not monoid.

Subsemigroup :-

Consider $(A, *)$ be a semigroup and B is a subset of A. Then $(B, *)$ is called a subsemigroup of $(A, *)$ if B is closed under the operation $*$, i.e for all $a, b \in B$, $a * b \in B$.

Exp: Let consider a semigroup $(N, +)$, where N is the set of natural numbers (set of natural numbers) and $+$ is an addition operation. The algebric system $(E, +)$ is a subsemigroup of $(N, +)$, where E is the set of even positive numbers

$N = \{1, 2, 3, 4, 5, \ldots\}$ $(N, +) \leftarrow$ semigroup

$E = \{2, 4, 6, 8, \ldots\}$ $(E, +)$ is subsemigroup

(because it is also satisfying associative and closure property.

$(2 * 4) * 8 = (6) + 8 = 14$
(+ , +)

$2 * (4 * 8) = 2 * (12) = 14$
(+ , +, +)

SubMonoid :-

Consider a monoid $(M, *)$ with identity element e and let T be the non-empty subset of M. Then $(T, *)$ is called submonoid of $(M, *)$ if $e \in T$ and T is closed under the operation $*$, i.e for all $a, b \in T$, $a * b \in T$.

→ Some first few non-negative even integers are
$\{0, 2, 4, 6, 8, \cdots\}$

→ Set of even integers are $= \{-4, -2, 0, 2, 4, \cdots\}$

Eg. If T be the set of all even integers $T = \{-4, -2, 0, 2, 4 \cdots\}$ then $(T, +)$ is subsemigroup of $(Z, +)$.
And it is even a submonoid because $(T, +)$ is the submonoid of monoid $(Z, +)$ with identity element $(e = 0)$.

→ Group &

→ Abelian Group - A Group G is said to be abelian group, if it satisfy the commutative property.
G5: Commutative Law: $a * b = b * a \; \forall a, b \in G$.

Groupoid	G_1
Semi Group	G_1, G_2
Monoid	$G_1\, G_2\, G_3$
Group	$G_1\, G_2\, G_3\, G_4$
Abelian Group	G_1, G_2, G_3, G_4, G_5

Q Prove that fourth roots of unity 1, −1, i and −i form an abelian (multiplicative) group.

Soln. Let $G = \{1, -1, i, -i\}$

$*$ → is multiplication

1: $\forall (1, -1, i, -i) \in G$
$a * b \in G$.
also

$*$	1	−1	i	−i
1	1	−1	i	−i
−1	−1	1	−i	i
i	i	−i	−1	1
−i	−i	i	1	−1

2: Associative Law

exp. $1[(-1)\, i] = -i$
$= [1\,(-1)] \cdot i = -i$

3: Commutative Law

From the table it is clear that elements in each row are the same as element in the corresponding column so that $ab = ba$.

4: Identity element $1 \in G$.

as $a \cdot 1 = 1 \cdot a = a \quad \forall a \in G$.

5: Inverse element

Inverse of 1 is 1, -1 is -1, i is $-i$
$-i$ is i.

$\therefore$ it is abelian group

Q Show that the set of all positive rational numbers forms an abelian group under the composition defined by $a * b = \left(\frac{ab}{2}\right)$.

Soln.

Let Q^+ be the set of all positive rational numbers. We have to show that $(Q^+, *)$ is a group with commutative property, under the composition $a*b = \left(\frac{ab}{2}\right)$

(i) Closure Property.

for all $a, b \in Q^+$, $\left(\frac{ab}{2}\right)$ is also in Q^+, therefore Q^+ is closed with respect to $*$ operation.

(ii) Associative Property

for every $a, b, c \in Q^+$

$$(a*b)*c = \left(\frac{ab}{2}\right)*c = \frac{abc}{4}$$

$$a*(b*c) = a*\left(\frac{bc}{2}\right) = \frac{a}{2}\cdot\frac{bc}{2} = \frac{abc}{4}$$

$\therefore$ therefore it is associative.

(iii) Commutative for any $a, b \in Q^+$

$$a*b = \frac{ab}{2} = \frac{ba}{2} = (b*a)$$

(iv) Identity element :- Let e be the identity element such that for all $a \in Q^+$

$$a*e = e*a = a.$$

$\therefore$ $a*e = a$

$$\left(\frac{ae}{2}\right) = a$$

$$ae = 2a \Rightarrow ae - 2a = 0$$

$$\Rightarrow a(e-2) = 0$$

$\because$ $a > 0$ (positive rational no. so it can't be zero)

$\therefore$ $\boxed{e = 2}$

(v) Inverse element :- Let $a \in Q^+$

Let if b is the inverse of a

$$a*b = b*a = e$$

$$a*b = e$$

$$\left(\frac{ab}{2}\right) = e$$

$$\frac{ab}{2} = 2$$

$b = \left(\frac{4}{a}\right)$ is the inverse element of $a \in Q^+$.

Properties of a Group

① The Identity element in a group is unique.

Let us suppose that $a \in G$ and e, e' be the two identities in G.

$\therefore$ $e \in G$ and $a \in G \Rightarrow ae = a$ — ①

$e' \in G$ and $a \in G \Rightarrow ae' = a$ — ②

From ① and ②

$ae = ae'$

$\therefore \boxed{e = e'}$

② The inverse of each element in a group is unique

Let $a \in G$ and $e \in G$

Let b and c be the two inverse of a in G.

$\because$ $a \in G$ and $b \in G \Rightarrow ab = e$ — ①

$\because$ $a \in G$ and $c \in G \Rightarrow ac = e$ — ②

From eqn ① and ②

$ab = ac$

$\Rightarrow \boxed{b = c}$

③ For every element 'a' in a Group G, prove that $(a^{-1})^{-1} = a$

Proof: ~~Let e be the identity element of the Group G.~~

For each element $a \in G$, there exist an element $b \in G$ such that $a * b = b * a = e$ (if b is the inverse of a)

उधार a का inverse b है तो

$b = a^{-1}$ — ①

$a = b^{-1}$ — ②

तो b का inverse b^{-1} होगा

put value of b from ① in ②

$\boxed{a = (a^{-1})^{-1}}$

4. The inverse of the product of two element of a group G is the product of the inverse taken in the reverse order i.e $(a*b)^{-1} = b^{-1} * a^{-1}$ for a Group $(G, *)$

Sol^n Prove that $(a*b)^{-1} = b^{-1} * a^{-1}$

Take $\Rightarrow (a*b)^{-1}$

Let $a, b \in G$ and suppose a^{-1} and b^{-1} are the inverse of a and b respectively.

$$\left.\begin{aligned} a \cdot a^{-1} &= e \\ b\ b^{-1} &= e \end{aligned}\right\} \text{ where } e \text{ is the identity.}$$

$$(a*b)\cdot(b^{-1}*a^{-1}) = (a(bb^{-1}))\,a^{-1} \qquad \text{Apply Associative Law.}$$
$$= a \cdot e\ a^{-1}$$
$$= a\,a^{-1} = e$$

$$(b^{-1}*a^{-1})\,(a*b) = b^{-1}(a^{-1}*a)\cdot b$$
$$= b^{-1} * e * b$$
$$= b^{-1} * b = e$$

$\because$ $b^{-1} * a^{-1}$ is the inverse of $a*b$.

$$\therefore \boxed{a*b = b^{-1}*a^{-1}}$$

⑤ Cancellation Law: If a, b, and c are any elements of $(G, *)$ then

(i) $a*c = b*c \Rightarrow \boxed{a=b}$ Right cancellation Law.

More detailed Solution

Let $x = b^{-1}*a^{-1}$. then we will prove that

(ii) x is the inverse of $a*b$.

$$\Rightarrow (a*b)*x = (a*b)*(b^{-1}*a^{-1})$$
$$= a*[(b*(b^{-1}*a^{-1}))] \quad \text{by associative}$$
$$= a*[(b*b^{-1})*a^{-1}] \quad " \quad "$$
$$= a*(e*a^{-1})$$
$$= (a*e)*a^{-1} \Rightarrow a*a^{-1} = e. \quad \text{---} \ (1)$$

$\because a*e = a$

Similarly we have to show that

$$x*(a*b)=e$$

$\Rightarrow (b^{-1}*a^{-1})*(a*b)$

$\Rightarrow [(b^{-1}*a^{-1})*a]*b$ by associative Law

$\Rightarrow [b^{-1}*(a^{-1}*a)]*b$ " "

$\Rightarrow [b^{-1}*e]*b$ $\because a^{-1}*a=e$

$\Rightarrow b^{-1}*(e*b)$ by associative

$\Rightarrow b^{-1}*b$ $e*b=b$

$\Rightarrow e$ —②

from eqⁿ ① and ②

$$\boxed{(a*b)^{-1} = b^{-1}*a^{-1}}$$

⑤ Cancellation Law

If a, b and c are any three element of G, then

(i) $ab=ac \Rightarrow b=c$ Left cancellation Law

(ii) $ba=ca \Rightarrow b=c$ Right cancellation Law

Proof (i) Let $a\in G \Rightarrow a^{-1}\in G$

Now $ab=ac$

$\because a^{-1}\in G$
$\therefore a^{-1}*(ab)\in G$

$a^{-1}(ab)=a^{-1}(ac)$

$(a^{-1}a)\cdot b=(a^{-1}a)c$ (associative Law)

$e\cdot b=e\cdot c$

$\boxed{b=c}$

(ii) $ba=ca$

$\Rightarrow (ba)a^{-1}=(ca)a^{-1}$

$b(aa^{-1})=c(aa^{-1})$ associatim

$\boxed{b=c}$

⑥ If G is a group and $a, b \in G$ be any element, then

(i) The equation $a * x = b$ has a unique solution $x = a^{-1} * b$ in G.

(ii) The equation $y * a = b$ has a unique solution $y = b * a^{-1}$ in G.

Proof: (i) $a * x = b$ ——①

Premultiply by a^{-1} $\Rightarrow$ on both side of eqn ——①

$$a^{-1} * (a * x) = a^{-1} * b \quad (\because a^{-1} \in G)$$

$$(a^{-1} * a) * x = a^{-1} * b \quad \text{associative.}$$

$$e * x = a^{-1} * b$$

$$x = a^{-1} * b \in G \ (\because a^{-1} \& b \in G).$$

therefore, the equation $a * x = b$ has a solution $x = a^{-1} * b$ in G.

Uniqueness:- Let if possible x_1 and x_2 be any two solution of the equation $a * x = b$, so that

$$a * x_1 = b \quad \text{and} \quad a * x_2 = b$$

$$\Rightarrow \quad a * x_1 = a * x_2$$

$$\Rightarrow \quad x_1 = x_2 \quad \text{(by left cancellation Law)}$$

therefore the equation $a * x = b$ has a unique solution.

Q. Let G be a group with identity element e, show that if $a^2 = e$ for all $a \in G$, then G is abelian.

Soln. Given $a^2 = e$ for all $a \in G$

$$a * a = e$$

Pre-multiply by a^{-1} on both size

$$a^{-1} * (a * a) = a^{-1} * e$$

$$(a^{-1} * a) * a = a^{-1} \quad \text{by associativity}$$

$$e * a = a^{-1}$$

$$a = a^{-1}$$

i.e every element is inverse itself.

Now for $a, b \in G$

$$a * b = a^{-1} * b^{-1}$$

$$= (b * a)^{-1} \quad \because (b*a)^{-1} = (b*a) \text{ every element is inverse itself.}$$

$$\boxed{a * b = b * a}$$

which is commutative

Q For any Group G, prove that G is abelian if and only if $(ab)^2 = a^2 b^2 \ \forall \ a, b \in G$

Soln. Suppose G is a abelian then $ab = b * a \ \forall \ a, b \in G$

$$(a*b)^2 = (b*a)(b*a)(a*b)(a*b)$$

$$= a * [b * (a * b)]$$

$$= a * [(b * a) * b]$$

$$= a * [(a * b) * b]$$

$$= a * [a * (b * b)]$$

$$= (a * a) * (b * a)$$

$$(a*b)^2 = a^2 * b^2$$

Q. For any group G, prove that G is abelian if and only if

$$(ab)^2 = a^2 b^2 \quad \forall\ a, b \in G$$

Sol^n Suppose G is abelian then

$$ab = ba \quad \forall\ a, b \in G.$$

Now $(ab)^2 = (ab)(ab)$

$= a[b(ab)]$

$= a[(ba)b]$ $\because$ G is abelian.

$= a[(ab)b]$

$= a[a(bb)]$

$= (a \cdot a)(b \cdot b)$

$= a^2 \cdot b^2$. prove.

Q Let $G = \{(a,b) \mid a,b \in R, a \neq 0\}$. Define a binary operation $*$ on G by $(a,b)*(c,d) = (ac, bc+d)$ for all $(a,b), (c,d) \in G$. Show that $(G,*)$ is a group

Soln

(G1)

(i) Closure Property :-

Let (a,b) and (c,d) be any two member (element) of G

Then $a \neq 0$ and $c \neq 0$

therefore $ac \neq 0$.

Now for all $(a,b)*(c,d) = (ac, bc+d)$ is also a member of G. Hence G is closed with respect to given composition.

(ii) G Associative Law

Let (a,b), (c,d) and (e,f) be any three member of S. Then

$$((a,b)*(c,d))*(e,f) = (ac, bc+d)*(e,f)$$
$$= ([ac]e, [bc+d]e + f)$$
$$= (ace, bce + de + f)$$

$$\therefore (a,b)*((c,d)*(e,f)) = (a,b)*[ce, de+f]$$
$$= (a[ce], bce + de + f)$$
$$= (ace, bce + de + f)$$

(iii) Identity Element. Suppose (x,y) is an element of G such that

$$(x,y)*(a,b) = (a,b)$$

then $(xa, ya+b) = (a,b) \quad \forall (a,b) \in G$.

Hence $xa = a$ and $ya + b = b$

$x = 1$ and $y = 0$

$\therefore (1,0) \in G$ is an identity element

(iv) Inverse Element:-

Let $(a,b) \in G$ be any member of G. Let (x,y) be a member of G such that

$$(x,y) * (a,b) = (1,0)$$

$$xa, ya+b = 1, 0$$

$$\Rightarrow xa = 1 \text{ and } ya+b = 0$$

$$\Rightarrow x = \frac{1}{a} \text{ and } y = \frac{-b}{a}$$

$\therefore \left(\frac{1}{a}, -\frac{b}{a}\right)$ is the inverse of (a,b). Hence G is a group.

Subgroup:-

Let G be a group then any non-empty subset H of G is called the complex of the group G.

A non-empty subset H of a group $(G,*)$ is said to be a subgroup if $(H,*)$ is also a group i.e if $(G,*)$ is a group then complex which satisfy all the axioms of the group is said to be subgroup i.e all subgroups are complexes but all complexes are not subgroups.

Any Group $(G,*)$ has at least two subgroups

(i) Trival subgroup $\{e,*\}$ as we know that

(ii) G itself. (because, every set is a subset of itself.)

Therefore, if G is a group, then G is also a subgroup of itself.

Examples

① $(Z,+)$ is a subgroup of $(Q,+)$

② $(Q,+)$ " " " " $(R,+)$

$\{1,-1\},\cdot\}$ is " " " $\{(1,-1,-i,i),\cdot\}$

Trival subgroup

Consider a subgroup S of $(\mathbb{Z}, +)$. Then 0 must be in S, as 0 is the identity of $(\mathbb{Z}, +)$. If S does not contain any element other than 0, then S is the trival (improper) subgroup. Otherwise, suppose S contains another elements are called proper or non-trival subgroups

therefore every group $(G, *)$ contain at least two subgroups $(G, *)$ and $\{e, *\}$. And these two subgroups are called trival subgroups.

Theorem ① The identity element of a subgroup is the same as that of the group.

Proof: Let H be the subgroup of the group G and suppose e and e' be the identity elements of G and H respectively.

Now, if $a \in H$ then $a \in G$ then $ae = a$ — ① since $e \in G$

and, $a \in H$ and $e' \in H \Rightarrow ae' = a$ — ② as $e' \in H$

① & ② $\Rightarrow ae = ae'$

$\Rightarrow \boxed{e = e'}$

After Theorem 2

② The necessary and sufficient condition for a non-empty subset H of a group $(G, *)$ to be a subgroup is

$$a \in H, b \in H \Rightarrow a * b^{-1} \in H$$

where b^{-1} is the inverse of b in G.

Theorem 2 A non-empty subset H of a group G is a subgroup of G if and only if

(i) $a \in H, b \in H \Rightarrow a * b \in H$

(ii) $a \in H \Rightarrow a^{-1} \in H$ where a^{-1} is the inverse of a in G.

Proof:- Suppose H is a subgroup of G, then H must be closed with respect to operation * i.e

$$a \in H, b \in H \Rightarrow a * b \in H$$

Let $a \in H$ and a^{-1} be the inverse of a in G. Then the inverse of a in H is also a^{-1}. As H itself is a group, each element of H will possess inverse in it, i.e $a \in H \Rightarrow a^{-1} \in H$.

Thus the condition is necessary, Now let us examine the sufficienty of this condition.

(i) Closure Property:- $a \in H, b \in H \Rightarrow a * b \in H$ (1) As we have been Given in condition (1) this show that H is closed under the operation *.

(ii) Associative Property:- As the element of H are also the element of G and the element of G satisfy the associative law for the binary operation, therefore, the elements of H will also satisfy the associative Law.

(iii) Existance of Inverse Identity

from (2) $a \in H$ and $a^{-1} \in H$

from (1) $a \in H, a^{-1} \in H \Rightarrow a a^{-1} \in H$

$\Rightarrow e \in H$

e is the identity of H

(iv) Inverse:- from (ii) $a \in H \Rightarrow a^{-1} \in H, \forall a \in H$.

therefore each element possess an inverse.

Theorem-3 A necessary and sufficient condition for a non-empty subset H of a group G to be a subgroup is that

$$a \in H, b \in H \Rightarrow a * b^{-1} \in H$$ where b^{-1} is the inverse of b in G.

Proof For necessary condition

Suppose H is a subgroup of G and let $a \in H, b \in H$. Since H is a group itself so each element of H must possess inverse in it.

$$b \in H \Rightarrow b^{-1} \in H.$$

Also, H is closed under the operation $*$ in G.

Therefore

$$a \in H, b^{-1} \in H \Rightarrow a * b^{-1} \in H$$

The condition is sufficient. If it is given that $a \in H, b^{-1} \in H \Rightarrow a * b^{-1} \in H$ then we have to prove that H is a subgroup.

(i) Closure Property:-

Let $a, b \in H$

then $b \in H \Rightarrow b^{-1} \in H$ (as above)

therefore by the given condition

$$a \in H, b^{-1} \in H \Rightarrow a * (b^{-1})^{-1} \in H$$

$$\Rightarrow a * b \in H.$$

Thus, H is closed with respect to the binary operation $*$.

(ii) Associative Property : Since the element of H are also the element of G, then the binary operation $*$ is associative in H.

(iii) Existance of identity:- Since

$$a \in H,\ a^{-1} \in H \Rightarrow a * a^{-1} \in H \Rightarrow e \in H$$

(iv) Existance of Inverse:-

Let $a \in H$, then

$e \in H,\ a \in H \Rightarrow e * a^{-1} \in H$ (by given condition)

$\Rightarrow a^{-1} \in H \quad [e*a = a*e = a]$

then each element of H possess inverse.

Hence H itself is a group for binary operation $*$ in G.

Q. The intersection of any two subgroup of a Group $(G,*)$ is again a subgroup of $(G,*)$.

Proof. Let H_1 and H_2 form any two subgroup of $(G,*)$. We have $H_1 \cap H_2 \neq \phi$. Since at least the identity element is common to both H_1 & H_2.

Now Let $a \in H_1 \cap H_2$ and $b \in H_1 \cap H_2$

$\Rightarrow$ $a \in H_1$ and $a \in H_2$ & $b \in H_1$ and $b \in H_2$

Since H_1 and H_2 form sub-groups under the group $(G,*)$, we have

$a \in H_1, b \in H_1 \Rightarrow a * b^{-1} \in H_1$

$a \in H_2, b \in H_2 \Rightarrow a * b^{-1} \in H_2$

Finally, $ab^{-1} \in H_1$, $ab^{-1} \in H_2 \Rightarrow a * b^{-1} \in H_1 \cap H_2$

thus

$a \in H_1 \cap H_2, b \in H_1 \cap H_2 \Rightarrow a * b^{-1} \in H_1 \cap H_2$

therefore, $H_1 \cap H_2$ forms a sub-group under $(G,*)$.

Cyclic Group

A group G is called cyclic group if for some $a \in G$, every element $x \in G$ is of the form a^n, where n is some integer. The element 'a' is then called a generator of G.

For example: The multiplicative group $G = \{1, -1, i, -i\}$ is cyclic.

We can write $G = \{i, i^2, i^3, i^4\}$
Thus G is cyclic group and i is a generator.

Similarly we can also write
$G = \{-i, (-i)^2, (-i)^3, (-i)^4\}$
Thus $-i$ also generator of G.

So there may be more than one generator of a cyclic group.

If G is a cyclic group generated by 'a' it is denoted by $G = (a)$. The elements of G are in the form of

$$\ldots a^{-3}, a^{-2}, a^{-1}, a^0, a, a^2, a^3 \ldots$$

Q. The Group $(G, +_6)$ is a cyclic group where $G = \{0, 1, 2, 3, 4, 5\}$
Sol^n $+_6$ is modulo 6 operator.

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Now:

$1^1 = 1$

$1^2 = 1 +_6 1 = 1 +_6 1 = 2$

$1^3 = 1 +_6 1^2 = 1 +_6 2 = 3$

$1^4 = 1 +_6 1^3 = 1 +_6 3 = 4$

$1^5 = 1 +_6 1^4 = 1 +_6 4 = 5$

$1^6 = 1 +_6 1^5 = 1 +_6 5 = 0$ (e)

Thus $G = \{1^6, 1^5, 1^4, 1^3, 1^2, 1^1\}$

Hence G is a cyclic group and 1 is generator. Similarly, it can be shown that 5 is another generator.

Q. Every cyclic group is an abelian group.

Proof:

Let G be a cyclic group and let a be a generator of G so that

$G = \{a^n : n \in Z\}$

If g_1 and g_2 be the two element of G, there exist integers r and s such that $g_1 = a^r$ and $g_2 = a^s$.

Then $g_1 g_2 = a^r \cdot a^s = a^{r+s} = a^s \cdot a^r = g_2 \cdot g_1$

$\therefore$ G is abelian.

Theorem

Q. If a is a generator of a cyclic group G, then a^{-1} is also generator of G.

Sol^{n}. Let $G = (a) =$ be a cyclic group generated by a.

Take r be any integer, then a^r be any element of G for some integer r.

So we can write $a^r = (a^{-1})^{-r}$

$= (a^{-1})^{-r}$

Since $-r$ is also some integer, therefore each element of G, is generated by a^{-1}. Thus a^{-1} is also a generator of G.

~~Theorem~~ - ~~If a cyclic group G is generated by a. Let a^r be any element of G, where r is some integer. We can write~~

Theorem :- If a cyclic group G is generated by an element a of order n, then a^m is a generator of G if and only if the greatest common divisor of m and n is 1 i.e if and only if m and n are relative primes.

Q How many generators are there of the cyclic group G of order 8.

Sol^{n} let a be the generator of G. Then $o(a) = 8$ and we can write $G = \{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8\}$

7 is prime to 8, therefor a^7 is also a generator of G

5 " " " 8, " a^5 " " " " " "

3 " " " " " a^3 " " " " "

1 " " " " a^1 " " " "

Thus there are only 4 generator of G

i.e a, a^3, a^5, a^7.

Cosets

Suppose G is a group and H is any subgroup of G. Let 'a' be any element of G, then the set

$$Ha = \{ha : h \in H\}$$

is called a right coset of H in G generated by 'a'.

Similarly, the set $aH = \{ah; a \in H\}$ is called left coset of H in G generated by a.

If e is the identity element of G then $\underline{He = eH = H}$. Therefore, H itself is right as well as left coset.

If the group G is abelian, then we have

$$ah = ha \quad \forall a \in H.$$

therfore right coset Ha will be equal to the corresponding left coset aH.

If the group operation be addition, then the right coset of H in G generated by 'a' is defined as

$$H + a = \{h + a : h \in H\}$$

and left coset is

$$(a + H) = \{a + h : h \in H\}$$

If H is a subgroup of group G the number of distinct left or right cosets of H in G is called the index of H in G and is denoted by $[G:H]$ or $i_G(H)$.

Properties of Cosets: Let H be a subgroup of G, and a & b belongs to G, then

1. $a \in aH$
2. $aH = H$ if and only if $a \in H$.
3. $aH = bH$ or $aH \cap bH = \phi$
4. $aH = bH$ if and only if $a^{-1}b \in H$.

Proof(i) $a = ae \in aH$, $\because e$ is identity element of G.

(ii) $aH = H$ if and only if $a \in H$.

If e be the identity in G and so is in H, then

$aH = H \Rightarrow ae \in H$

$\Rightarrow a \in H \quad \because e \in H.$ —— ①

Now $a \in H$ and $h \in H$ (let) then

$a \in H \Rightarrow ah \in H \ \forall \ h \in H$

$\therefore aH \subset H$ —— ② ✓

Also $a \in H \Rightarrow a^{-1} \in H$

$\Rightarrow a^{-1} h \in H \quad \forall h \in H$

~~$a^{-1} H \in H$~~

$a^{-1} h \in H \quad \forall h \in H$ by closure Law in H

$(a^{-1} \in H, h \in H \Rightarrow a^{-1} h \in H)$

$(a \in H, b \in H \Rightarrow a * b \in H)$

$a(a^{-1} h) \in aH \quad \forall h \in H$ by closure Law in H.

$h \in aH \ \forall \ h \in H$

$H \subseteq aH$ —— ③ ✓

So by ② and ③ $aH \subset H$ & $H \subset aH$.

$\boxed{aH = H}$

Now $a \in H \Rightarrow aH = H$

Hence $aH = H \Leftrightarrow a \in H$

$\boxed{aH = H}$ if and only if $a \in H$.

③ $aH = bH$ OR $aH \cap bH = \phi$

(Any two left cosets of a subgroup are either identical or disjoint)

Let H be a subgroup of a Group G and Let aH and bH be two its left cosets. Assume that $aH \cap bH \neq \phi$ and Let c be the common element of the two cosets.

Then we may write $c = ah$ and $c = bh'$ for $h, h' \in H$.

therefore $ah = bh'$, giving $a = bh'h^{-1}$

Since H is a subgroup, we have $h'h^{-1} \in H$

Now $aH = (bh'h^{-1})H = b(h'h^{-1}H) = bH$ → necessary & sufficient condition of a subgrp

$\because h'h^{-1} \in H$

$aH = bH$

Hence two cosets are identical if $aH \cap bH \neq \phi$

Thus two left coset aH & bH are identical if they are not disjoint $aH \cap bH \neq \phi$.

Thus $aH \cap bH = \phi$ or $aH = bH$.

Therefore, we have shown that any two left cosets which are not disjoint are identical.

if they will be disjoint then they will not be identical.

(4) If a and b are arbitrary distinct elements of a group G and H is any subgroup of G, then

$$Ha = Hb \Leftrightarrow ab^{-1} \in H$$
$$\& \; aH = bH \Leftrightarrow b^{-1}a \in H$$

Soln.

Let a and b be arbitrary elements of a group G such that $a \neq b$.

Let e be the identity element of G then $e \in H$.

Now

$$Ha = Hb$$
$$\Rightarrow (Ha)b^{-1} = (Hb)b^{-1}$$
$$\Rightarrow H(ab^{-1}) = H(bb^{-1})$$
$$\Rightarrow H(ab^{-1}) = He = H \quad (\because He = eH = H)$$
$$\Rightarrow H(ab^{-1}) = H$$
$$\Rightarrow ab^{-1} \in H \quad \because [Ha = H \text{ if and only if } a \in H]$$

Conversely:-

$$ab^{-1} \in H$$
$$\Rightarrow H(ab^{-1}) = H$$
$$\Rightarrow H(ab^{-1})b = Hb$$
$$\Rightarrow Ha(b^{-1}b) = Hb$$
$$\Rightarrow (Ha)e = Hb$$
$$\Rightarrow \boxed{Ha = Hb}$$

Therefore we have $Ha = Hb \Leftrightarrow ab^{-1} \in H$.

Lagrange's Theorem :-

The order of each subgroup of a finite group is a divisor (factor) of the group.

Proof. Let H be a subgroup of a finite Group G & let
$O(G)=n$ and $O(H)=m$.

To show that m is a divisor of n, we have to show the $n=mp$ for some $p \in N$.

Let Ha be any right coset of H in G.
Then $O(H)=m \Rightarrow \exists$ m distinct elements $h_1, h_2 \cdots h_m \in H$

$\Rightarrow$ there exist m distinct elements $h_1a, h_2a, \cdots h_ma \in Ha$

$\Rightarrow$ Every right coset of H in G has m distinct elements
Since, G is finite and therefore, number of distinct right coset of H in G will be finite say p.
Also, we know that any two right cosets of H in G will be disjoint. Hence p distinct right cosets of H in G will contain mp distinct elements

$\therefore G = Ha \cup Hb \cup Hc \cup \cdots$ where $a, b, c \cdots \in G$.

$O(G) = O(Ha) + O(Hb) + O(Hc) + \cdots$
$= m + m + m + \cdots$ p times
$= mp$

$O(G) = mp$
$\Rightarrow \boxed{n = mp}$

Order of an Element of a Group

Let G be a group under multiplication. Let e be the identity element in G. Suppose a is any element of G then the least positive integer n, if exist such that $a^n = e$ is said to be the order of an element $a \in G$, and can be written as

$$O(a) = n$$

In case, such a positive integer n does not exist, we say that the element a is of infinite or zero order.

If $a \in G$, & n be a positive integer such that $a^n = e$, then $O(a) \leq n$.

exp. Consider a multiplicative group $G = \{1, -1, i, -i\}$ of cube root of unity. find the order of each element of G.

Sol^n. Since 1 is the identity element, therefore

$$O(1) = 1$$

$(-1)^2 = 1 \Rightarrow O(-1) = 2$

$(i)^4 = 1 \Rightarrow O(i) = 4$

$(-i)^4 = 1 \Rightarrow O(-i) = 4.$

Theorem - 1

The order of every element of a finite group is finite and less than or equal to the order of the Group.

i.e $O(a) \leq O(G) \quad \forall \, a \in G.$

multiplicative.

Proof Let G be a finite group of order n and $a \in G$.

Now consider the elements $a^0 = e, a, a^2, a^3, \ldots a^r, \ldots a^n$.

By closure property, all these elements belongs to G.

These are n+1 numbers.

Since G is finite, therefore, all these element cannot be distinct.

Suppose

$a^r = a^s$, where $r > s$

then,

$a^r = a^s \Rightarrow a^r \cdot a^{-s} = a^s \cdot a^{-s}$ (or $a^{-s} \in G$)

multiply by a^{-s} on both sides

$a^{r-s} = a^0 = e$

$a^{r-s} = e$

$a^m = e$ where $m = r - s$.

Since $r > s$, therefore m is a positive integer less than or equal to n. Thus there exist a positive integer m such that

$$a^m = e$$

Hence the $O(a)$ is finite and less than or equal to $O(G)$.

Let $\exists\ a \in G$ such that $a^k \neq e$ $\forall\ k \in N$.

(ऐसा एक element लिया group में जिसका कोई power k करने पर e नहीं आता $k \in N$)

then we can write $k = k_1 - k_2$ where $k_1, k_2 \in Z$

$\Rightarrow a^{k_1 - k_2} \neq e \Rightarrow a^{k_1} \neq a^{k_2}$ $\forall$ $k_1 k_2 \in Z$.

So all power of a are distinct

$\{a, a^2, a^3, \ldots a^n \ldots\}$

Hence Group is infinite.

Theorem 2

The order of an element a of group G is the same as that of its inverse a^{-1}.

Soln:

Let $O(a)=n$ and $O(a^{-1})=m$

Now $O(a)=n$

$\therefore a^n = e$ (identity element)

$\Rightarrow (a^n)^{-1} = e^{-1}$ $[e^{-1} * e = e]$

$\Rightarrow (a^{-1})^n = e$

$\Rightarrow O(a^{-1}) \leq n$ [If $a \in G$ and n be a +ve integer such that $a^n = e$, then $O(a) \leq n$]

$\Rightarrow m \leq n$ —— (1)

Also $O(a^{-1}) = m$

$\Rightarrow (a^{-1})^m = e$

$\Rightarrow (a^m)^{-1} = e$

$\Rightarrow a^m = e^{-1} = e$

$\Rightarrow a^m = e$

$\Rightarrow O(a) \leq m$

$\Rightarrow n \leq m$ — (2)

From (1) & (2) $\boxed{n = m}$

Theorem-3 If the element a of group G is of order n, then $a^k = e$ iff n is a divisor of k.

Proof Let $a^k = e$ $O(a) = n$

Since k is an integer such that either $k = n$ or $k > n$.

(i) If $k = n$ then n is a divisor of k.

(ii) If $k > n$, then by division algorithm, there exist two integers p and q such that

$k = np + q$ where $0 \leq q < n$

Now, $a^k = a^{np+q}$

$= a^{np} \cdot a^q$

$= (a^n)^p \cdot a^q$

$= (e)^p \cdot a^q$

$= a^q$

Therefore $a^k = a^q = e$

Since $0 \leq q < n$ and $a^n = e$

then $a^q \neq e$ unless $q = 0$

$\therefore$ $k = np$ or n is a divisor of k when $q = 0$.

Conversely, Let n is a divisor of k, then there exist an integer p such that

$k = np$

Now $a^k = a^{np}$

$\Rightarrow = (a^n)^p$

$= (e)^p$

$= e$

Hence $a^k = e$ iff n is a divisor of k.

Normal Subgroup :-

Let G be a group and H is a subgroup of G. Subgroup H is called normal subgroup (or self conjugate subgroup or invariant subgroup) of G if and only if for every $x \in G$ and all $h \in H$, $xhx^{-1} \in H$.

clearly every subgroup of an abelian grap is a normal subgroup.

Theorem 1 Every subgroup of an abelian group is normal.

Proof

Let G be an abelian group & H be a subgroup of G.

let $h \in H$ and $x \in G$

then $xh = hx$ ($\because$ G is abelian)

$\Rightarrow xhx^{-1} = (hx)x^{-1}$

$= h \cdot e$

$= h$

Now $h \in H$

$\therefore \boxed{xhx^{-1} \in H}$

Hence H is a normal subgroup of G.

Theorem 2 A subgroup H of a group G is normal subgroup if and only if for every x in G, $xHx^{-1} = H$.

Proof Let H be a normal subgroup of G. Then we have to show that

$$xHx^{-1} = H \quad \forall\, x \in G.$$

Now $x \in G$ and $h \in H \Rightarrow xhx^{-1} \in H$ (from theorem)

$$\Rightarrow xHx^{-1} \subseteq H \quad \text{---} (1)$$

Again $xHx^{-1} \subseteq H \quad \forall\, x \in G.$

$$\Rightarrow x^{-1}H(x^{-1})^{-1} \subseteq H \quad [\because x^{-1} \in G]$$
$$\Rightarrow x^{-1}Hx \subseteq H$$
$$\Rightarrow x(x^{-1}Hx)x^{-1} \subseteq xHx^{-1}$$
$$\Rightarrow (xx^{-1})H(xx^{-1}) \subseteq xHx^{-1}$$
$$\Rightarrow eHe \subseteq xHx^{-1}$$
$$\Rightarrow H \subseteq xHx^{-1} \quad \text{---} (2)$$

From eq^n (1) & (2)

$$\boxed{xHx^{-1} = H}$$

Conversely:- Suppose that $xHx^{-1} = H \;\forall\, x \in G$. Then we have to show that H is a normal subgroup of G.

Now $xHx^{-1} = H$

$$\Rightarrow xHx^{-1} \subseteq H$$
$$\Rightarrow xhx^{-1} \in H \;\forall\, h \in H \;\&\; \forall\, x \in G.$$

It is the definition of normal subgroup, H is a normal subgroup of G. Thus a subgroup H is normal iff it is self conjugate.

Theorem The intersection of any two normal subgroup of a group is a normal subgroup

Proof Let G be a group & H_1 & H_2 be any two normal subgroup of a group G. Then $H_1 \cap H_2$ is a subgroup of G.

Now Let $x \in G$ and $h \in H_1 \cap H_2$. Then $h \in H_1 \cap H_2$

$\Rightarrow h \in H_1$ and $h \in H_2$

But H_1 is a normal subgroup of G, then by definition

$$xhx^{-1} \in H_1 \quad \forall \; x \in G \text{ and } h \in H_1 \quad \text{---} (1)$$

Also H_2 is a normal subgroup of G. then

$$xhx^{-1} \in H_2 \quad \forall \; x \in G \text{ and } h \in H_2 \quad \text{---} (2)$$

Now from (1) and (2) we get

$$\forall \; x \in G, \; h \in H_1 \cap H_2 \Rightarrow xhx^{-1} \in H_1 \cap H_2$$

thus $H_1 \cap H_2$ is a normal subgroup of G.

Homomorphism of a Group

$(G, \circ)$ & $(G', *)$

Suppose G and G' be any two groups and Let $\circ$ and $*$ be the respective binary operations. Then a mapping $f: G \to G'$ is called a homomorphism if $f(a \circ b) = f(a) * f(b) \quad \forall\, a, b \in G$.

Thus if f is a homomorphism from G to G' then f preserves the composition in G and G' i.e

image of the composition = compositions of images

A group G' is said to be a homomorphic of a group G, if there exists a homomorphism f of G onto G'.

Isomorphism

A homomorphism f of a group $(G, \circ)$ into $(G', *)$ is an isomorphism from G to G' if

(i) f is one-one i.e

$$f(a) = f(b) \Rightarrow a = b \quad \forall\, a, b \in G$$

(ii) f is onto means every $b \in G'$ there exist $a \in G$ such that $f(a) = b$

Q. Let $(I, +)$ be a group, Prove that the mapping $f: I \to I$ defined by $f(x) = 5x \quad \forall\, x \in I$ is homomorphism from I to itself.

Sol^n The mapping $f: I \to I$ will be homomorphism if it preserves the composition in I.

$$f(x_1 + x_2) = [\text{illegible}] = 5(x_1 + x_2)$$

$$= [\text{illegible}] = 5x_1 + 5x_2$$

$$= f(x_1) + f(x_2)$$

Thus f preserves compositions in I and hence homomorphism

Q. Show that the mapping $f: G \to G'$ given by $f(x) = 2x$ $\forall x \in G$ is an isomorphism of G onto G' where G is the additive group of integers and G' is the additive groups of even integers including zero.

Sol^n The mapping $f: G \to G'$ will be an isomorphism if it is

(i) one-one

(ii) Onto

(iii) preserves composition in G & G'.

Let $x_1, x_2 \in G$ then $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2$

$\Rightarrow x_1 = x_2$

(by cancelation Law)

Thus two elements in G have the same f-image in G' only if they are equal.

Therefore, distinct elements in G have distinct f-images in G'. Hence f is one-one.

f is onto :-

Let y is an element of G' i.e y is an even integer, then $\frac{y}{2} \in G$.

(For exp- $\frac{6}{2} = 3 \in G$, $\frac{8}{2} = 4 \in G$ ...)

$f(x) = 2x$

$f\left(\frac{y}{2}\right) = 2 \cdot \left(\frac{y}{2}\right) = y$

So, there exist $\frac{y}{2} \in G$ such that $f\left(\frac{y}{2}\right) = y$. Thus each element of G' is the f-image

Thus f is onto.

f preserves the composition in G & G'

Let $x_1, x_2 \in G$

$f(x_1 + x_2) = 2(x_1 + x_2) = 2x_1 + 2x_2$

$= f(x_1) + f(x_2)$

thus f preserves the composition in G & G'.

$\therefore f$ is homomorphism of G & G'.

Endomorphism:-

A homomorphism of a Group G into itself, is called endomorphism of G.

Monomorphism:-

A homomorphism f from a group G to a group G' is said to be monomorphism if f is one-one.

Epimorphism:-

A homomorphism f from a group G to a group G' is said to be epimorphism if f is onto.

Automorphism:-

An isomorphism of a group G onto itself is called an automorphism of G.

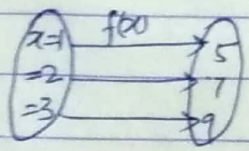
Exp Let a be an element of a group G. Then the mapping $f_a: G \to G$ given by $f_a(x) = axa^{-1}$, $\forall x \in G$ is an automorphism.

Sol^n. Let $a \in G$ and $x \in G$.

$\therefore a \cdot x \in G \Rightarrow a \cdot x \cdot a^{-1} \in G$.

hence f_a is a mapping from G to G.

Because if we take $y = 2x+3$



but in the above case $a \in G, x \in G$

$axa^{-1} \in G$ so it the mapping from G to G.

f_a is one-one

Let $x_1, x_2 \in G$, then $f_a(x_1) = f_a(x_2)$

$\Rightarrow ax_1a^{-1} = ax_2a^{-1}$

$\Rightarrow x_1a^{-1} = x_2a^{-1}$ (By left cancellation Law)

$\Rightarrow x_1 = x_2$ (By Right cancellation Law)

$\therefore f_a$ is one-one.

f_a is onto

For every $y \in G$, there exist $a^{-1}ya \in G$

$$f_a(x) = y = axa^{-1}$$

$$\Rightarrow \boxed{a^{-1}ya = x}$$

such that $f(x) = y$

$$f_a(x) = a(a^{-1}ya)a^{-1} = (aa^{-1})y(aa^{-1}) \quad \text{(Asso)}$$

$$\boxed{f_a(x) = y}$$

$\Rightarrow f_a$ is onto.

f_a preserve the composition in G.

$\Rightarrow$ Let $x_1, x_2 \in G$ then

$$f_a(x_1 x_2) = a(x_1 x_2)a^{-1}$$
$$= ax_1 x_2 a^{-1}$$
$$= ax_1 \cdot e\, x_2 a^{-1}$$
$$= ax_1 a^{-1}a\, x_2 a^{-1}$$
$$= (ax_1a^{-1})(ax_2a^{-1})$$
$$= f_a(x_1) \cdot f_a(x_2)$$

$\therefore f_a$ preserve the composition in G. Hence f_a is Isomorphism

Theorem 1

Let $(G,*)$ and $(G',\circ)$ be groups with respective identities e and e'. If $f: G \to G'$ is a homomorphism, then

(i) $f(e) = e'$ (ii) $f(a^{-1}) = [f(a)]^{-1}$ $\forall\, a \in G$.

(iii) $f(a^n) = [f(a)]^n$ $\forall\, a \in G$ and all $n \in Z$.

(iv) $f(S)$ be a subgroup of G' for each subgroup S of G.

Soln.

(i) We know that $e \in G$ then $f(e) \in G'$

$\because$ f is a mapping from $G \to G'$.

Now since $e' \in G'$ & $f(e) \in G'$

$e' \circ f(e) = f(e)$ $\because$ e' is identity element of G'.

$e' \circ f(e) = f(e * e)$ $\because$ e is the identity element of G.

$= f(e) \circ f(e)$ $\because$ f is homomorphic

$\Rightarrow$ $\boxed{e' = f(e)}$ By Right cancellation Law.

(If e is the identity of G then $f(e)$ is the identity of G')

(ii) We know that $a * a^{-1} = a^{-1} * a = e$ $\forall\, a \in G$

from (1)

$e' = f(e)$

$= f(a * a^{-1}) = f(a) \circ f(a^{-1})$ $\because$ f is homomorphism —(1)

and $e' = f(e) = f(a^{-1} * a) = f(a^{-1}) \circ f(a)$ —(2)

thus From (1) & (2)

$f(a) \circ f(a^{-1}) = f(a^{-1}) \circ f(a) = e'$

Hence, $f(a^{-1})$ is the inverse of $f(a)$

$\therefore$ $f(a^{-1}) = [f(a)]^{-1}$

Q. If R be the additive group of real numbers and R^+ the multiplicative group of positive real numbers show that the following mapping are isomorphism.

Sol^n. (i) $f: R \to R^+$ $f(x) = e^x$, $x \in R$ (ii) $f: R^+ \to R$ i.e $f(x) = \log x$, $x \in R^+$

For f is one-one. $|(R,+)$ & $(R^+,\times)$

Let $x_1, x_2 \in G$

then $f(x_1) = f(x_2)$

$e^{x_1} = e^{x_2}$

$\Rightarrow x_1 = x_2$

Hence two elements in G have same f image only if they are same.

for onto let $y = e^x \Rightarrow x = \log_e y$

$f(x) = e^{\log_e y} = y$

$\therefore$ f is onto.

f preserves the composition

$f(x_1 + x_2) = e^{x_1+x_2} = e^{x_1} \cdot e^{x_2} = f(x_1) \times f(x_2)$

Hence f preserves the composition in G & G'

(ii) $f(x) = \log x \quad \forall x \in R^+$

Since any two distinct real positive number have two different logarithms as real numbers

$\therefore$ f is one-one.

For $x_1 \neq x_2 \Rightarrow \log x_1 \neq \log x_2 \quad \forall x_1, x_2 \in R^+$

For f is onto let $y \in R^*$

let $y = \log_e x \Rightarrow x = e^y$.

For every $y \in R$, there exist a $x \in R^+$ such that

$f(x) = y$.

$f(x) = \log_e e^y = y$ $\therefore$ f is onto.

f preserve the composition $f: R^+ \to R$

$f(x_1 \cdot x_2) = \log(x_1 \cdot x_2) = \log x_1 + \log x_2 = f(x_1) + f(x_2)$

Hence f is homomorphism.

Ring - An algebric structure $(R, +, \cdot)$, where R is a non-empty set and + and · are two binary operations known as addition and multiplication respectively, is called a Ring if the following properties are satisfied.

(i) $(R, +)$ is an abelian group

(ii) $(R, \cdot)$ is semi-group

(iii) Multiplication is distributive with respect to Addition

OR

① Closed under addition

$a+b \in R \quad \forall\ a, b \in R$

② Associative under addition

$a+(b+c) = (a+b)+c \quad \forall\ a, b, c \in R$

③ Existence of identity

$0+a = a \quad \forall\ a \in R$

0 is the additive identity of R.

④ Existence of inverse

there exist an element $-a \in R$ such that

$-a+a = 0 = e \quad \forall\ a \in R$

⑤ Commutative under addition

$a+b = b+a \quad \forall\ a, b \in R$

⑥ Closed under multiplication

$a, b \in R \quad a \cdot b \in R$

⑦ Associative under multiplication

$a \cdot (b \cdot c) = (a \cdot b) \cdot c \quad \forall\ a, b, c \in R$

⑧ Multiplication is distributed over addition

(i) Left distributive law $a \cdot (b+c) = a \cdot b + a \cdot c$

(ii) Right " " $(b+c) \cdot a = b \cdot a + c \cdot a$

An algebraic structure $(R, +, \cdot)$ is said to be a ring provided
$a+b \in R$, $a \cdot b \in R$ $\forall\ a, b \in R$

Commutative Ring:-

A ring $(R, +, \cdot)$ is called a commutative Ring if $a \cdot b = b \cdot a \quad \forall \; a, b \in R$.

Ring with unity (if multiplicative semi-group $(R, \cdot)$ has identity element)

(such an identity element is called unity)

A ring $(R, +, \cdot)$ is said to be ring with unity if R has the identity element for its multiplicative composition. If there exists $e \in R$ such that

$$a \cdot e = e \cdot a = a \quad \forall \; a \in R$$ (here $e = 1$ for multiplicative composition.)

Some Examples:-

① Null Ring (or Zero Ring) - A set R having a single element 0 with two binary operations addition and multiplication defined by $0 + 0 = 0$ & $0 \cdot 0 = 0$ is called Null Ring.

② The set $R = \{a, b\}$ with addition and multiplication defined by the following table

+	a	b
a	a	b
b	b	a

·	a	b
a	a	a
b	a	b

is commutative ring because

$$a \cdot b = b \cdot a = a$$

It has an identity element $e = b$ such that

$$a \cdot e = e \cdot a = a$$

put $e = b$

$a \cdot b = b \cdot a = a$ hence $\boxed{e = b}$ is the identity element

hence it is commutative Ring with unity.

Ex. The set I of integers is a ring under addition and multiplication of integers. This ring is said to be the ring of Integers.

It is a commutative ring with unity because multiplication of integers is commutative and 1 is the unity element in I.

D. Show that the set $R = \{0,1,2,3,4,5\}$ form a commutative Ring with respect to $+_6$ and $\times_6$ as two binary composition.

Sol^n.

$R = \{0,1,2,3,4,5\}$ $\quad a +_6 b = (a+b) \bmod 6$

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$\times_6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

① Addition Modulo 6 is closed.
For $\forall a,b \in R$ $\quad a +_6 b \in R$.

② $+_6$ is associative. let $a,b,c \in R$ | $a +_6 b = (a+b) \bmod 6$

$a +_6 (b +_6 c) = (a +_6 b) +_6 c$
(non-negative)

= a reminder when $a+(b+c)$ is divided by 6

= a reminder when $(a+b)+c$ " " " "

$= (a +_6 b) +_6 c$

∴ Hence associative.

3. Existence of identity:- From the composition table 0 is the additive identity.

4. Existence of ~~ident~~ additive Inverse:- From the composition table

inverse of 0 is 0
" " 1 " 5
" " 2 " 4
" " 3 " 3
" " 4 " 2
" " 5 " 1

Hence the inverse axiom is true for $+_6$.

5) $+_6$ is commutative.

Let $a, b \in R$

$a +_6 b$ = a non negative reminder when $a+b$ is divided by 6.

= " non-negative reminder when $b+a$ is divided by 6.

$= b +_6 a$.

$\therefore$ Commutative

Thus the algebric structure $(R, +_6)$ is abelian group.

Now ⑥ X_6 is closed

(Multiplication Modulo 6 $+_6$) is closed.

$\because a, b \in R$, $a X_6 b \in R$.

⑦ X_6 is associative let $a, b, c \in R$

$a X_6 (b X_6 c)$ = a non-negative reminder when $a \times (b \times c)$ is divided by 6.

= a non-negative reminder when $(a \times b) \times c$ is divided by 6.

$= (a X_6 b) X_6 c$

8) $\times_6$ is distributive over $+_6$.

Let $a, b, c \in R$, then

$a \times_6 (b +_6 c) = (a \times_6 b) +_6 (a \times_6 c)$ [Left Dist Law]

$(b +_6 c) \times_6 a = (b \times_6 a) +_6 (c \times_6 a)$ [Right " "]

(i) $a \times_6 (b +_6 c)$ = a non-negative reminder when $a(b+c)$ is divided by 6.

= a non-negative reminder when $ab + ac$ is divided by 6.

$= (a \times_6 b) +_6 (a \times_6 c)$

Similarly $(b +_6 c) \times_6 a = (b \times_6 a) + (c \times_6 a)$

9) $\times_6$ is commutative

Let $a, b \in R$

$a \times_6 b$ = a non-negative reminder when $a \times b$ is divided by 6.

= a non-negative reminder when $b \times a$ is divided by 6

$= b \times_6 a$

Hence $\times_6$ is commutative.

Hence $(R, '+_6', '\times_6')$ is a commutative Ring.

8 If R is a Ring such that $a^2 = a \ \forall a \in R$ prove that

(i) $a + a = 0 \ \forall \ a \in R$ (ii) $a + b = 0 \Rightarrow a = b$

(iii) $a + b = 0 \ \forall \ a = b$

(iii) R is a commutative Ring.

$a \cdot b = a \cdot c \Rightarrow b = c$ Left cancell

$b \cdot a = c \cdot a \Rightarrow b = c$ (Right cancellation)

(i) Since $a \in R \Rightarrow a+a \in R$ (By closure Property)

$a^2 = a$ (given) — ①

Now $(a+a)^2 = a+a$ put $a = a+a$ in eq^n ①

$(a+a)\cdot(a+a) = (a+a)$

$(a+a)\cdot a + (a+a)\cdot a = a+a$ (By left Distributed Law)

$(a^2+a^2) + (a^2+a^2) = a+a$ (By Right Distributed Law)

$(a+a) + (a+a) = (a+a)$ ($\because a^2 = a$)

$(a+a) + (a+a) = (a+a) + 0$

$a+a = 0$ (By left cancellation Law)

Hence Prove.

(iii) If $a, b \in R$ then $a+b \in R$

Now $(a+b)^2 = a+b$ ($\because a^2 = a$)

$(a+b)(a+b) = (a+b)$

$(a+b)a + (a+b)\cdot b = (a+b)$ (By Left Distributive Law)

$a^2 + ba + ab + b^2 = a+b$ (By Right " ")

$a + ba + ab + b = a + b$

$a + ba + ab = a$ (By Right cancellation Law)

$ba + ab = 0$ (By Left " ")

$ab = ba$

[If $a+b=0$ then $a=b$]

(ii) $a+b=0$

$a+b = a+a$ from (i) $(a+a=0)$

$b = a$ (By left cancellation Law)

R is a commutative Ring.

(iii) $a+b=0 \Rightarrow a+b = a+a$ $\because a+a=0$ from eq ①

$b=a$ from Left cancellation Law) ① proved

Special Kind of Rings

① Integral Domain - A ring is called an integral domain if it is commutative has unit element and is without zero divisor.

Field:- A ring $(F, +, \cdot)$ is said to be a field if it
(i) it has unity
(ii) it has commutative
(iii) has every non-zero element has multiplicative inverse.

Note:- Let R be a ring with unity 1. If $a \in R$ and there exist $b \in R$ such that $a \cdot b = b \cdot a = 1$, then b is called the multiplicative inverse of a and a is called unit of R.

exp $(Z, +, \cdot)$ is a ring with unity, 1 being the unity of the ring

Zero Divisor

→ If in a Ring R, there exist element a & b such that $a \cdot b = 0$ when neither $a = 0$ nor $b = 0$. Then, the element a and b are called zero divisors.

→ A ring $(R, +, \cdot)$ is said to be a ring without zero divisor if for all $a, b \in R$, $ab = 0 \Rightarrow a = 0$ or $b = 0$, where 0 is the additive identity of the ring R.

Exp The Ring $(Z, +, \cdot)$ is a ring without zero divisor since the product of two non-zero integers cannot be equal to zero.

Q. Show that $(Z_6, +_6, \times_6)$ is a ring with zero divisor.

Ans:

$\times_6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Note that $3 \times_6 2 = 0$

$3 \times_6 4 = 0$

the product of two non-zero elements are zero.

hence zero divisors are = 2, 3, 4.

Field

* A ring $(F, +, \cdot)$ is said to be a field if
  (i) It is commutative
  (ii) It has unity
  (iii) It is such that every non-zero elements has multiplicative inverse.

OR

✓ A field is a set $F(+, \cdot)$ under two operation which is closed + and × such that
  (i) F is an abelian group under +
  (ii) $F - \{0\}$ (the set F without the additive identity) is an abelian group under ×.

Exp The ring $(\{0,1,2,3,4\}, +_5, \times_5)$ is an example of finite field.

Ans.

$+_6$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\times_6$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

(i) $\{0,1,2,3,4\}$ is an abelian group for $+_5$. 0 is the additive identity and each element is invertible.

(ii) $\{1,2,3,4\}$ is an abelian group for $\times_5$. 1 is the unity and each element has its multiplicative inverse.