Rotation:

Let us consider the transformation matrix

$$[T] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Consider a plane triangle ABC

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 4 & 1 \\ 2 & 1 \end{bmatrix}$$

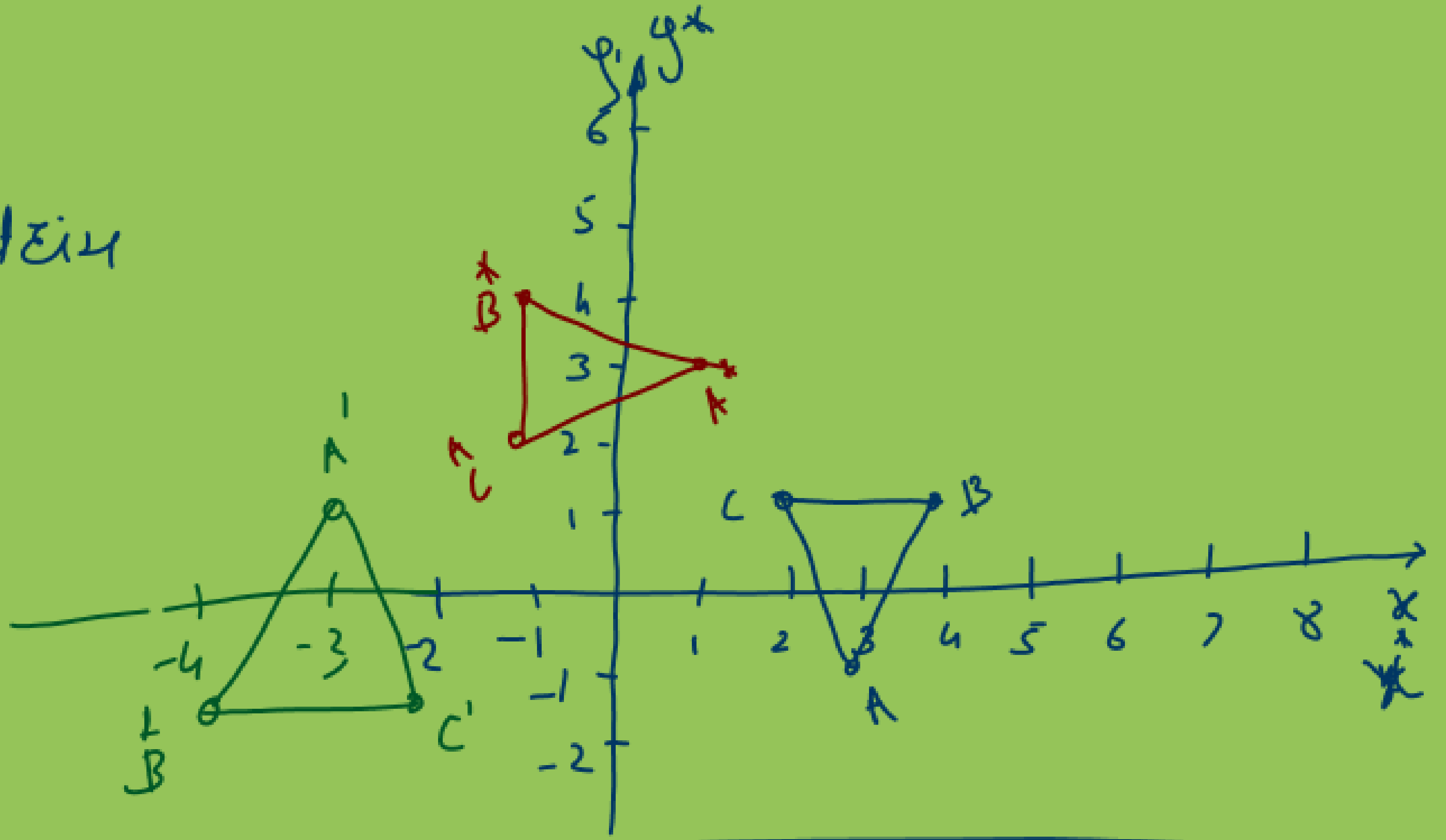
After transforming we will get

$$\begin{bmatrix} 3 & -1 \\ 4 & 1 \\ 2 & 1 \end{bmatrix} * \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & 4 \\ -1 & 2 \end{bmatrix}$$

– Rotate through 90° about the origin in counterclockwise direction.

$$\begin{bmatrix} 3 & -1 \\ 4 & 1 \\ 2 & 1 \end{bmatrix} * \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -4 & -1 \\ -2 & -1 \end{bmatrix}$$

– 180° about origin



For 180° about origin

$$[T] = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

For 270° about origin

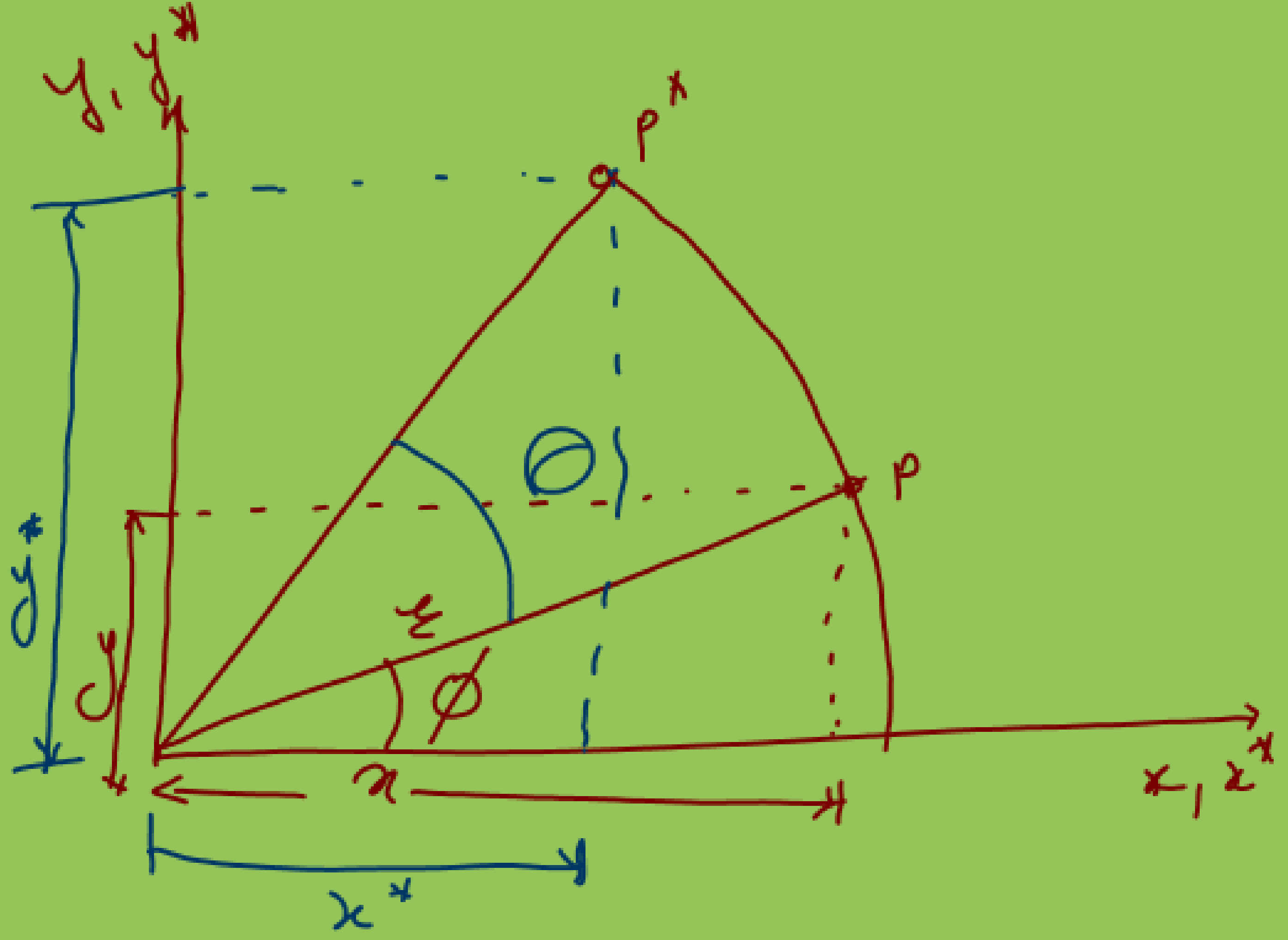
$$[T] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

For 0° or 360° about origin

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Rotation about in arbitrary angle θ:

The length of P to the x-axis - R at angle ϕ.
P rotate about the origin by angle θ to P^*
Writing position vector for P & P^*



$$P = [x \quad y] = [r\cos\phi \quad r\sin\phi]$$

$$P^* = [x^* \quad y^*] = [r\cos(\phi+\theta) \quad r\sin(\phi+\theta)]$$

$$\cos(\phi \pm \theta) = \cos\phi\cos\theta \mp \sin\phi\sin\theta$$

$$\sin(\phi \pm \theta) = \cos\phi\sin\theta \pm \sin\phi\cos\theta$$

So,

$$P^* = [x^* \quad y^*] = [r(\cos\phi\cos\theta - \sin\phi\sin\theta) \quad r(\cos\phi\sin\theta + \sin\phi\cos\theta)]$$

$$= [r\cos\phi\cos\theta - r\sin\phi\sin\theta \quad r\cos\phi\sin\theta + r\sin\phi\cos\theta]$$

Using definitions of x & y

$$P^* = [x^* \quad y^*] = [x\cos\theta - y\sin\theta \quad x\sin\theta + y\cos\theta]$$

Hence,

$$x^* = x\cos\theta - y\sin\theta \quad -(1)$$

$$y^* = x\sin\theta + y\cos\theta \quad -(2)$$

In matrix form

$$[X^*] = [x^* \quad y^*] = [X][T] = [x \quad y]\begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix} \quad -(3)$$

Transformation matrix for arbitrary angle θ is

$$[T] = \begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}$$ – positive counterclockwise rotation

Calculate determinant of $[T]$

$\underline{\det \text{ of } [T] = ad - bc = \cos^2\theta + \sin^2\theta = 1}$ – pure rotation

We wish to rotate P^* back to P, required rotation angle $(-\theta)$

$$[T] = \begin{bmatrix}\cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta)\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$$ – clockwise rotation

Ex: Rotate a point (2 -4) by an angle 30° by counterclockwise

$$P^* = [x^* \quad y^*] = [x \quad y]\,[T]$$

$$= [2 \quad -4] \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= [2 \quad -4] \begin{bmatrix} \cos 30 & \sin 30 \\ -\sin 30 & \cos 30 \end{bmatrix}$$

$$= [2 \quad -4] \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$$

$$[x^* \quad y^*] = [\sqrt{3}+2 \qquad 1-2\sqrt{3}]$$

Reflection :

A reflection is 180^0 rotation about an axis

① Case 1 : Reflection about x-axis, $y=0$

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

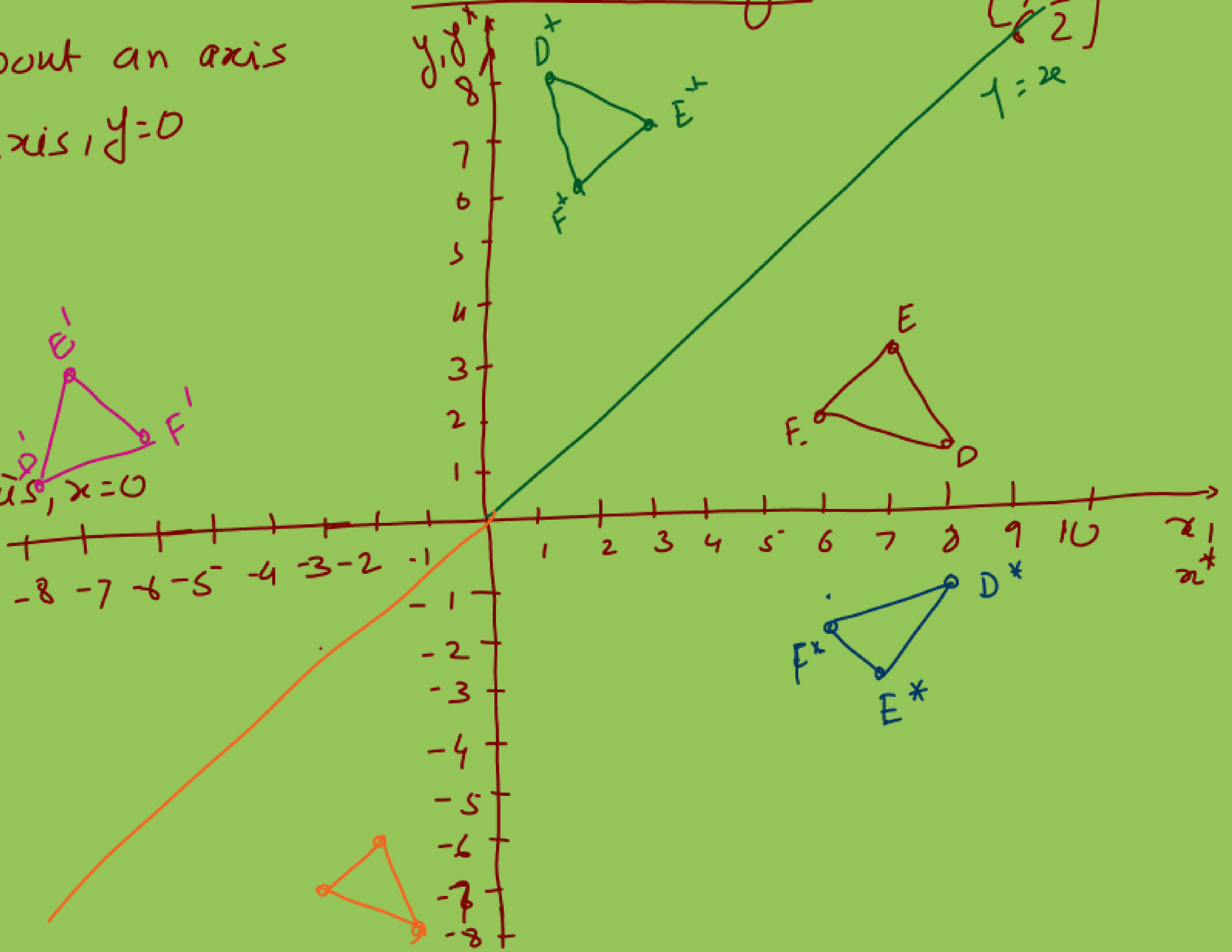
$$\begin{bmatrix} 8 & 1 \\ 7 & 3 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 8 & -1 \\ 7 & -3 \\ 6 & -2 \end{bmatrix}$$

② Case 2 : Reflection about y axis, $x=0$

$$[T] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 1 \\ 7 & 3 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -8 & 1 \\ -7 & 3 \\ -6 & 2 \end{bmatrix}$$

consider a triangle $DEF = \begin{bmatrix} 8 & 1 \\ 7 & 3 \\ 6 & 2 \end{bmatrix}$



case 3: Reflection about the line $y=x$

$$[T]=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

$$\begin{bmatrix}8 & 1\\ 7 & 3\\ 6 & 2\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}1 & 8\\ 3 & 7\\ 2 & 6\end{bmatrix}$$

Case 4: Reflection about the line $y=-x$

$$[T]=\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}$$

$$\begin{bmatrix}8 & 1\\ 7 & 3\\ 6 & 2\end{bmatrix}\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}=\begin{bmatrix}-1 & -8\\ -3 & -7\\ -2 & -6\end{bmatrix}$$

Each reflection matrices determinant is -1, then it produces pure reflection

Combined Reflections yields rotation:

Consider a triangle $ABC=\begin{bmatrix}4 & 1\\ 5 & 2\\ 4 & 3\end{bmatrix}$

(1) Reflect it about x-axis, $y=0$

$$[X^*] = [X][T] \begin{bmatrix} 4 & 1 \\ 5 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 5 & -2 \\ 4 & -3 \end{bmatrix}$$

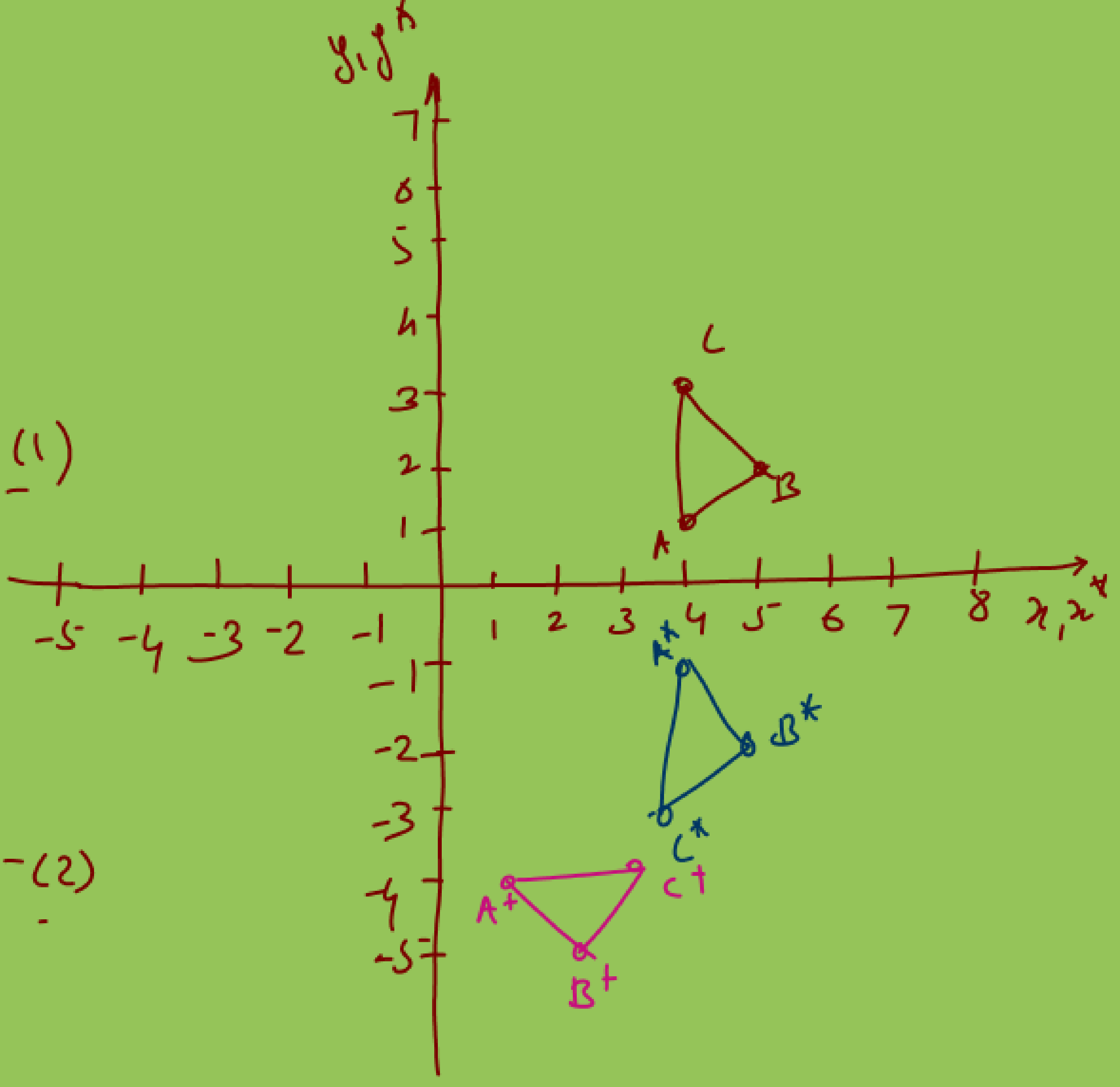
(2) Now reflect about line $y = -x$

$$[X^+] = [X^*][T] = \begin{bmatrix} 4 & -1 \\ 5 & -2 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 2 & -5 \\ 3 & -4 \end{bmatrix} \quad -(1)$$

If we rotate original triangle ABC by angle 270° then

$$[X^+] = [X][T] = \begin{bmatrix} 4 & 1 \\ 5 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 2 & -5 \\ 3 & -4 \end{bmatrix} \quad -(2)$$

(1) & (2) are identical



Scaling: To alter or change the size of object

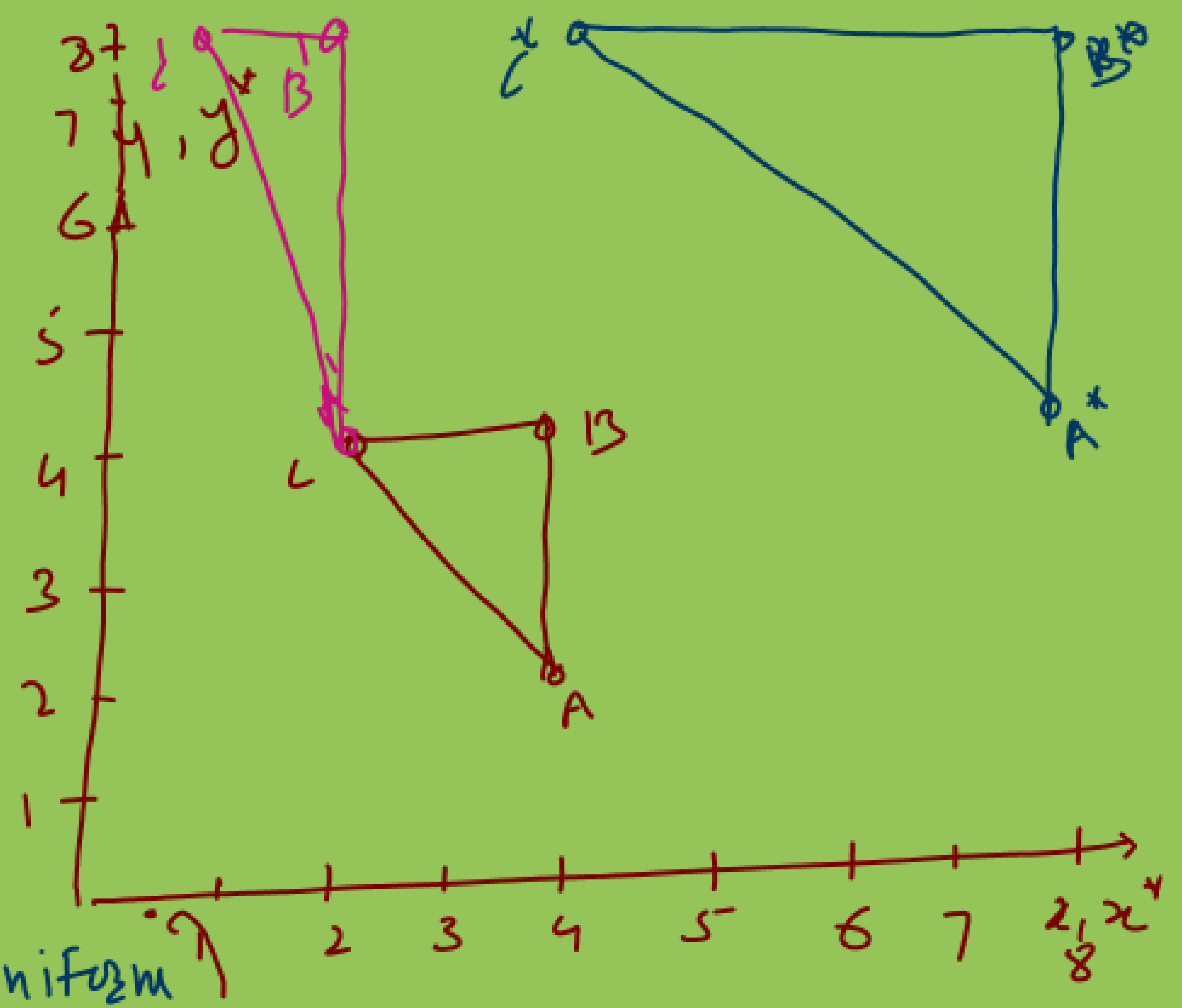
$[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ - primary diagonals, $b = c = 0$

eg: Consider a triangle ABC

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 4 & 4 \\ 2 & 4 \end{bmatrix} \quad \text{Let } [T] = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} [T] = \begin{bmatrix} 4 & 2 \\ 4 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 8 & 8 \\ 4 & 8 \end{bmatrix} = \begin{bmatrix} A^* \\ B^* \\ C^* \end{bmatrix}$$ · uniform

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} [T] = \begin{bmatrix} 4 & 2 \\ 4 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 2 & 8 \\ 1 & 8 \end{bmatrix} = \begin{bmatrix} A' \\ B' \\ C' \end{bmatrix}$$ - Distortion



$$[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

& $a = d,\ b = c = 0$ - uniform scaling

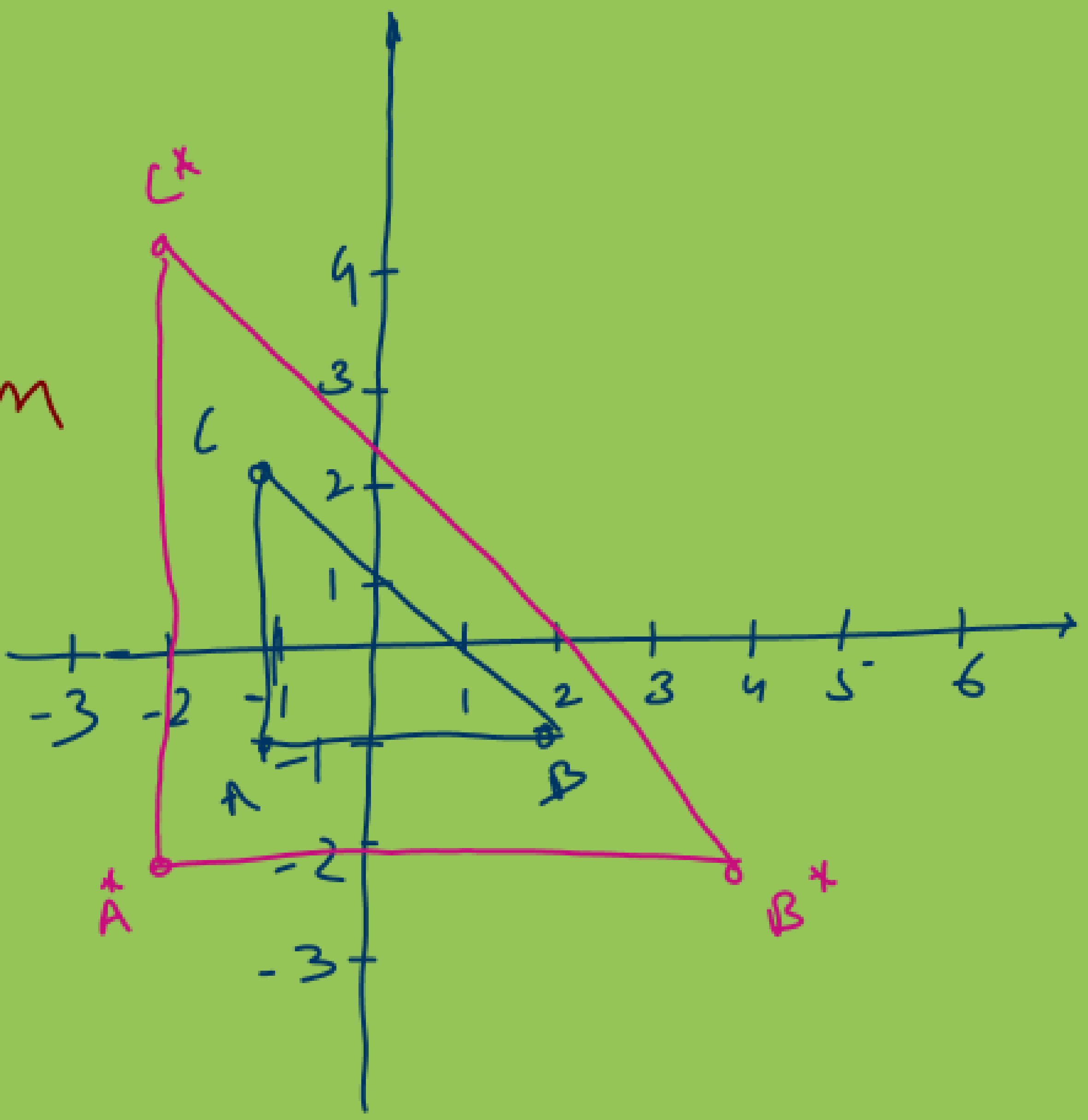
↳ 1) $a = d > 1$ - expansion

2) $a = d < 1$ - compression

$a \neq d,\ b = c = 0$ - non-uniform scaling or distortion

Pure scaling without apparent translation:

Consider the centroid of object at the origin



e.g. $\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 4 & -2 \\ -2 & 4 \end{bmatrix}$

Combined transformations:

$$[A][B] \neq [B][A]$$

① If a 90° rotation $[T_1]$

② Reflection through line $y = -x$: $[T_2]$

$$[X'] = [X][T_1] = [x \quad y]\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = [-y \quad x] \quad -(1)$$

$$[X^*] = [X'][T_2] = [-y \quad x]\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = [-x \quad y] \quad -(2)$$

If we perform in opposite

$$[X'] = [X][T_2] = [x \quad y]\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = [-y \quad -x] \quad -(3)$$

$$[X^*] = [X'][T_1] = [-y \quad -x]\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = [x \quad -y] \quad -(4)$$

eq^n (2) & (4) - order of application of matrix transformation is important

$$[T_1][T_2] = [T]$$

$$[X][T] = [X^*]$$

e.g. $\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & 2 \\ 4 & 4 \end{bmatrix}$

(1) 90° rotation about the origin

(2) Reflection through the line $y = -x$

Translation & Homogeneous coordinates:

2×2 [T] - rotation, scaling & reflection

But origin never chage.

For traslating origin in 2D plane, we need traslations factors m, n.

$$x^* = ax + cy + m$$

$$y^* = bx + dy + n$$

This difficulty overcome by homogeneous coordinates

$[x \quad y]$ are $[x' \quad y' \quad h]$

where $x' = hx$, $y' = yh$

eg. $[x \quad y] = [3 \quad 2]$

$\underset{h=1}{[3 \quad 2 \quad 1]}$ $\underset{h=2}{[6 \quad 4 \quad 2]}$ $\underset{h=3}{[9 \quad 6 \quad \overset{h}{3}]}$ - h is non-zero

Now the general transformation matrix is 3X3

$$[T] = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ m & n & 1 \end{bmatrix}$$

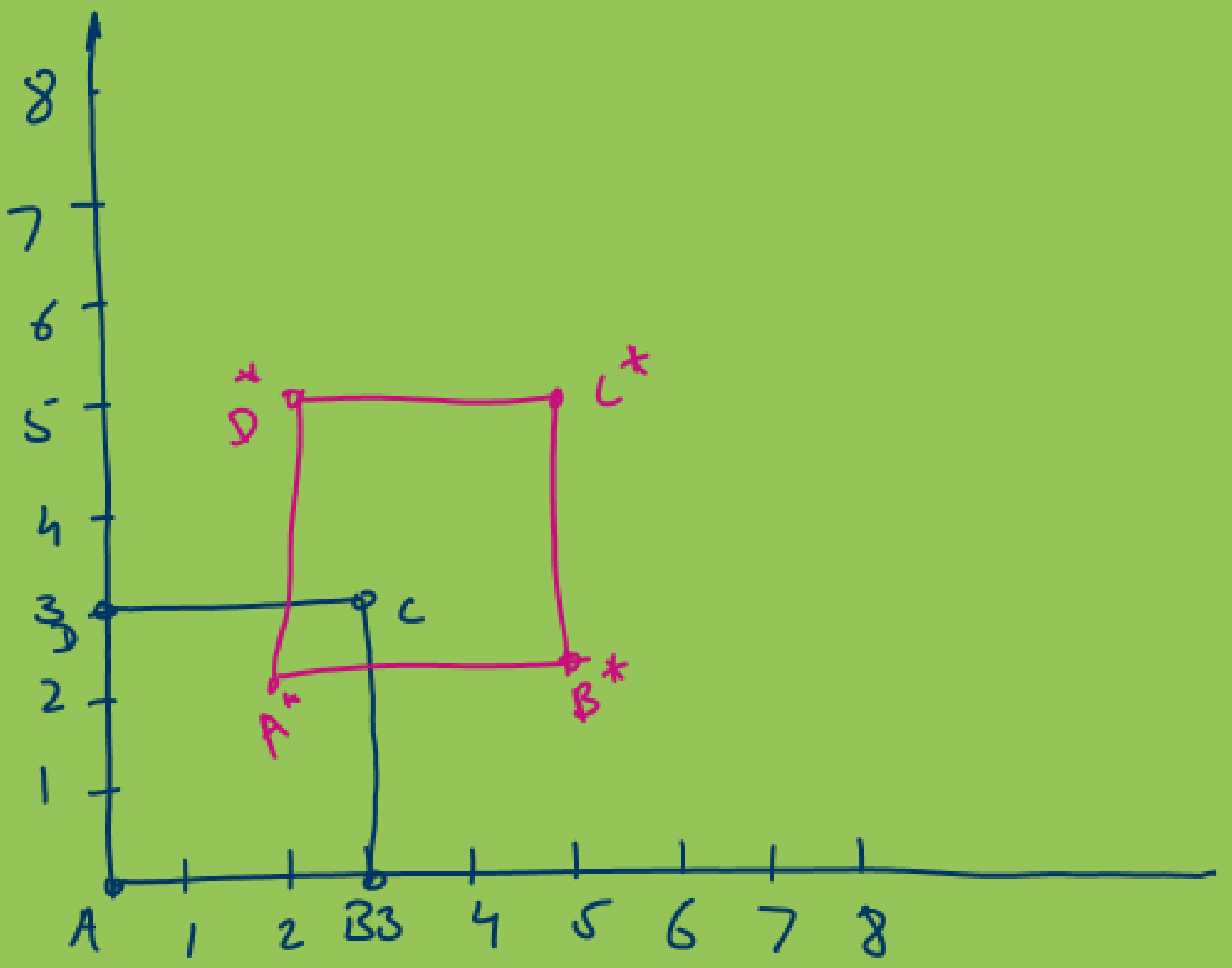
m & n are translation factors in x & y directions

eg. Traslate a square with coordinates A(0,0), B(3,0), C(3,3) & D(0,3), m=n=2

$$[X] = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{array}{c} \qquad\qquad h=1 \\ \begin{bmatrix} 0 & 0 & 1 \\ 3 & 0 & 1 \\ 3 & 3 & 1 \\ 0 & 3 & 1 \end{bmatrix} \end{array}, \quad [T] = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} \\ m= \quad n= \qquad \end{array}$$

3X3

$$[X^*] = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 0 & 1 \\ 3 & 3 & 1 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 5 & 2 & 1 \\ 5 & 5 & 1 \\ 2 & 5 & 1 \end{bmatrix} = \begin{bmatrix} A^* \\ B^* \\ C^* \\ D^* \end{bmatrix}$$



eg: Consider a triangle ABC with coordinates A(2,3), B(4,3) & C(3,2)

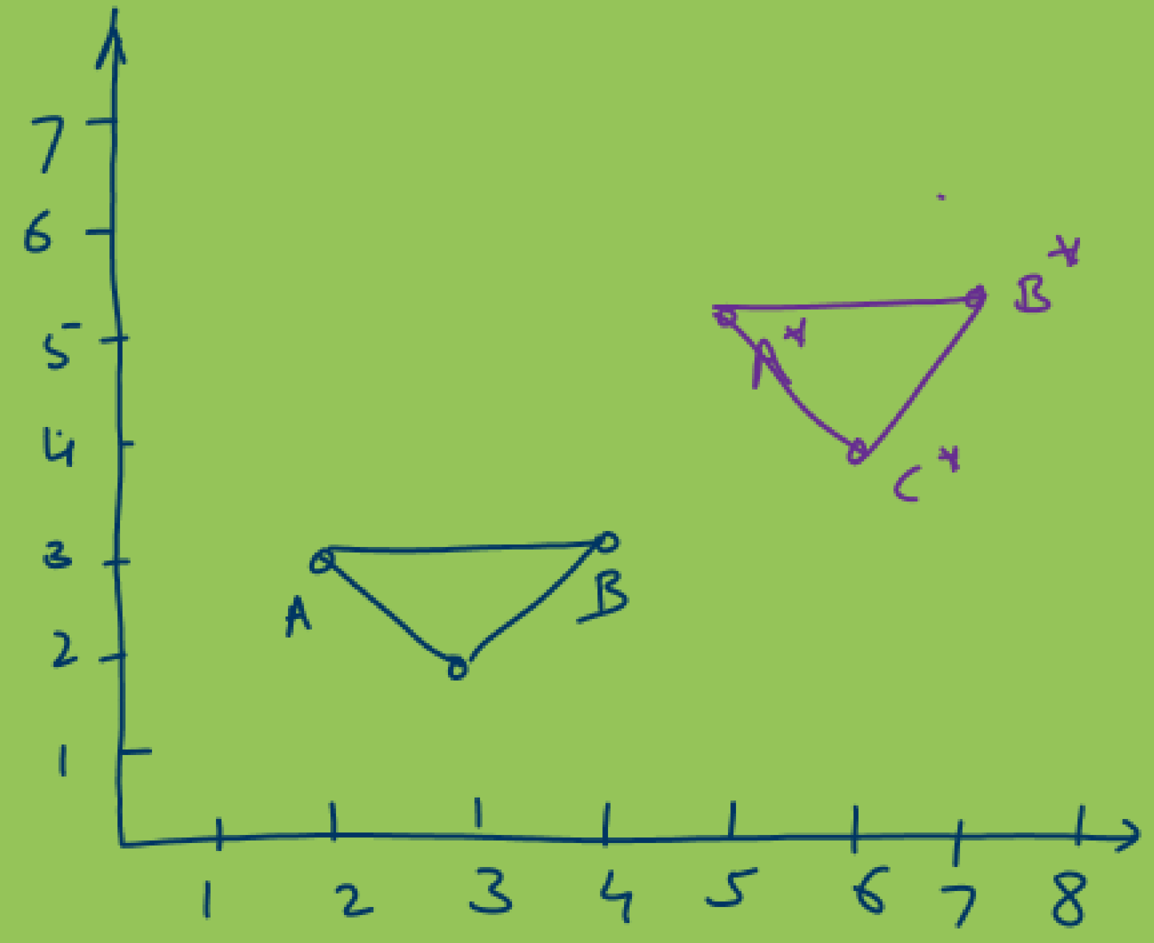
Translate it with $m=3$ & $n=2$

$$[X] = \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$[X][T] = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 5 & 1 \\ 7 & 5 & 1 \\ 6 & 4 & 1 \end{bmatrix} = \begin{bmatrix} A^* \\ B^* \\ C^* \end{bmatrix}$$



Following are matrices for two-dimensional transformation in homogeneous coordinate:

1. Translation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

2. Scaling

$$\begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Rotation (clockwise)

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Rotation (anti-clock)

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5. Reflection against X axis

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6. Reflection against Y axis

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Following are matrices for two-dimensional transformation in homogeneous coordinate:

7. Reflection against origin

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8. Reflection against line Y=X

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

9. Reflection against Y= –X

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

10. Shearing in X direction

$$\begin{bmatrix} 1 & 0 & 0 \\ Sh_x & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

11. Shearing in Y direction

$$\begin{bmatrix} 1 & Sh_y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

12. Shearing in both x and y direction

$$\begin{bmatrix} 1 & Sh_y & 0 \\ Sh_x & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation about an Arbitrary Point

- Homogenous coordinates provides a mechanism for rotation about an arbitrary point.
- To rotate an object about an arbitrary point, (Xp ,Yp) we have to carry out three steps:
  - Translate point (Xp, Yp) to the origin. [T1]
  - Rotate it about the origin and, [R]
  - Finally, translate the center of rotation back where it belongs (See figure 1.). [T2]

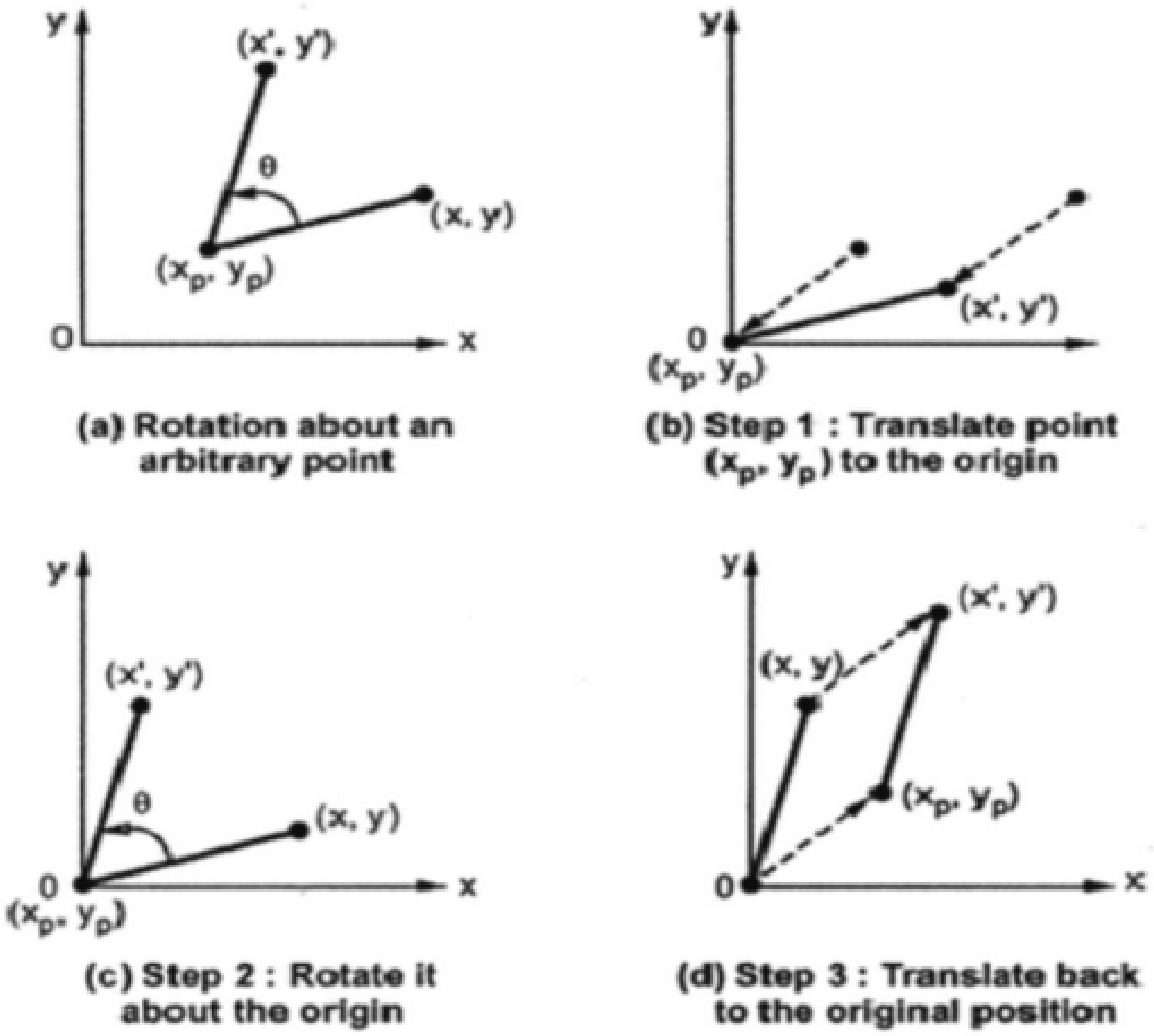


Fig. 1

The translation matrix to move point (x_p, y_p) to the origin is given as,

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_p & -y_p & 1 \end{bmatrix}$$

The rotation matrix for counterclockwise rotation of point about the origin is given as,

$$R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The translation matrix to move the center point back to its original position is given as,

$$T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_p & y_p & 1 \end{bmatrix}$$

eg: consider rotation of line AB, about point P(1,1) through 90°

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 5 & 5 \end{bmatrix} \qquad x_p = 1,\ y_p = 1$$

(1) $[T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

$$[X][T_1] = \begin{bmatrix} 2 & 2 & 1 \\ 5 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 4 & 1 \end{bmatrix} = [X']$$

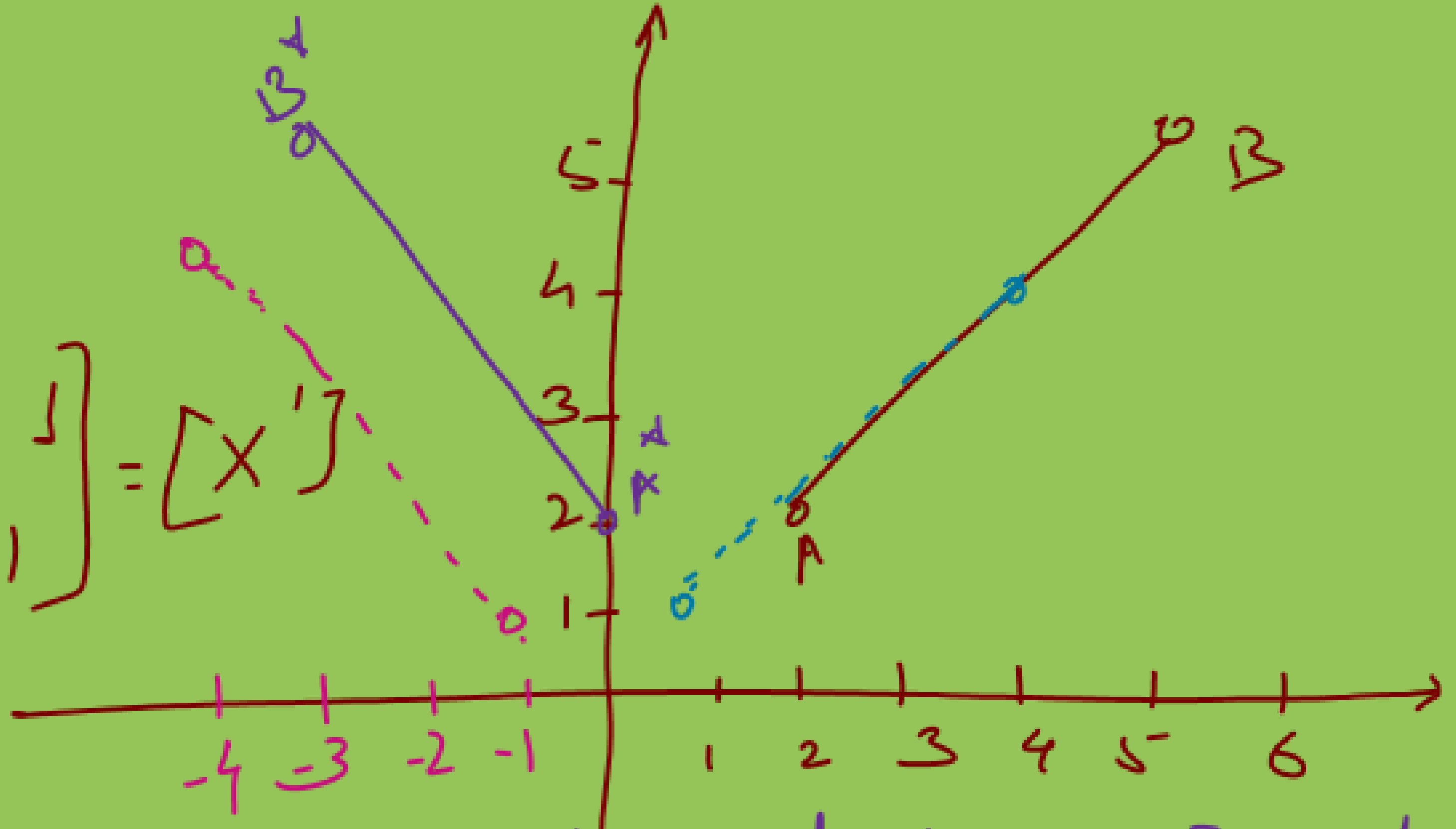
(2) Rotation about 90°

$[R] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$[X'][R] = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 4 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ -4 & 4 & 1 \end{bmatrix} = [X^+]$$

(3) Translate it back

$$[X^+][T_2] = \begin{bmatrix} -1 & 1 & 1 \\ -4 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ -3 & 5 & 1 \end{bmatrix} = [X^*]$$



Combined transformation

$$[T] = [T_1][R][T_2]$$

$$[X][T] = [X^*]$$

Reflection of An Object About An Arbitrary Line

1. Translate the line and the object so that the line passes through the origin
2. Rotate the line and the object about the origin until the line is coincident with one of the coordinate axis
3. Reflect object through coordinate axis
4. Rotate back about the origin (Inverse Rotation)
5. Translate back to the original location (Inverse translation)

$$[T] = [T'] [R] [R'] [R]^{-1} [T']^{-1}$$

[T’] – Translation Matrix

[R] – Rotation Matrix about the origin

[R’] – Reflection Matrix

Consider a triangle ABC with coordinates (2, 4), (4, 6) & (2, 6)

Reflect it about the line $y = \frac{1}{2}(x+4)$

$y = \frac{1}{2}x + 2$, y-intercept = 2 $\quad \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 6 & 1 \\ 2 & 6 & 1 \end{bmatrix}$

(1) Translate the line & object towards origin with translation factors

$m = 0 \qquad n = -2$

$$\begin{bmatrix} 2 & 4 & 1 \\ 4 & 6 & 1 \\ 2 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 4 & 4 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

(2) Rotate object & line to coincidant with x-axis

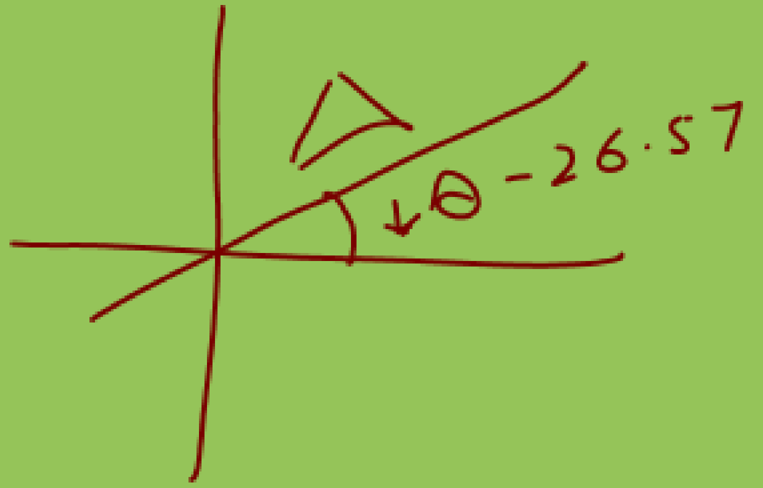
$m = \tan(-\theta)$

$m = -\tan(\theta)$

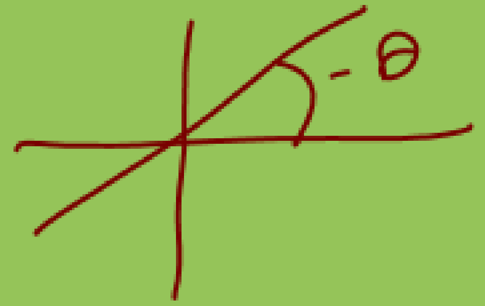
$\theta = -\tan^{-1}(m) \quad , m = \frac{1}{2}$

$\theta = -\tan^{-1}(1/2)$

$= -26.57^{\circ}$



$$\begin{bmatrix} 2 & 2 & 1 \\ 4 & 4 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} \cos(-26.57) & \sin(-26.57) & 0 \\ -\sin(-26.57) & \cos(-26.57) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 4 & 4 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} & 0 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6/\sqrt{5} & 2/\sqrt{5} & 1 \\ 12/\sqrt{5} & 4/\sqrt{5} & 1 \\ 8/\sqrt{5} & 6/\sqrt{5} & 1 \end{bmatrix}$$



$$= \begin{bmatrix} 2.68 & 0.89 & 1 \\ 5.36 & 1.78 & 1 \\ 3.57 & 2.68 & 1 \end{bmatrix}$$

(3) Reflect through x-axis

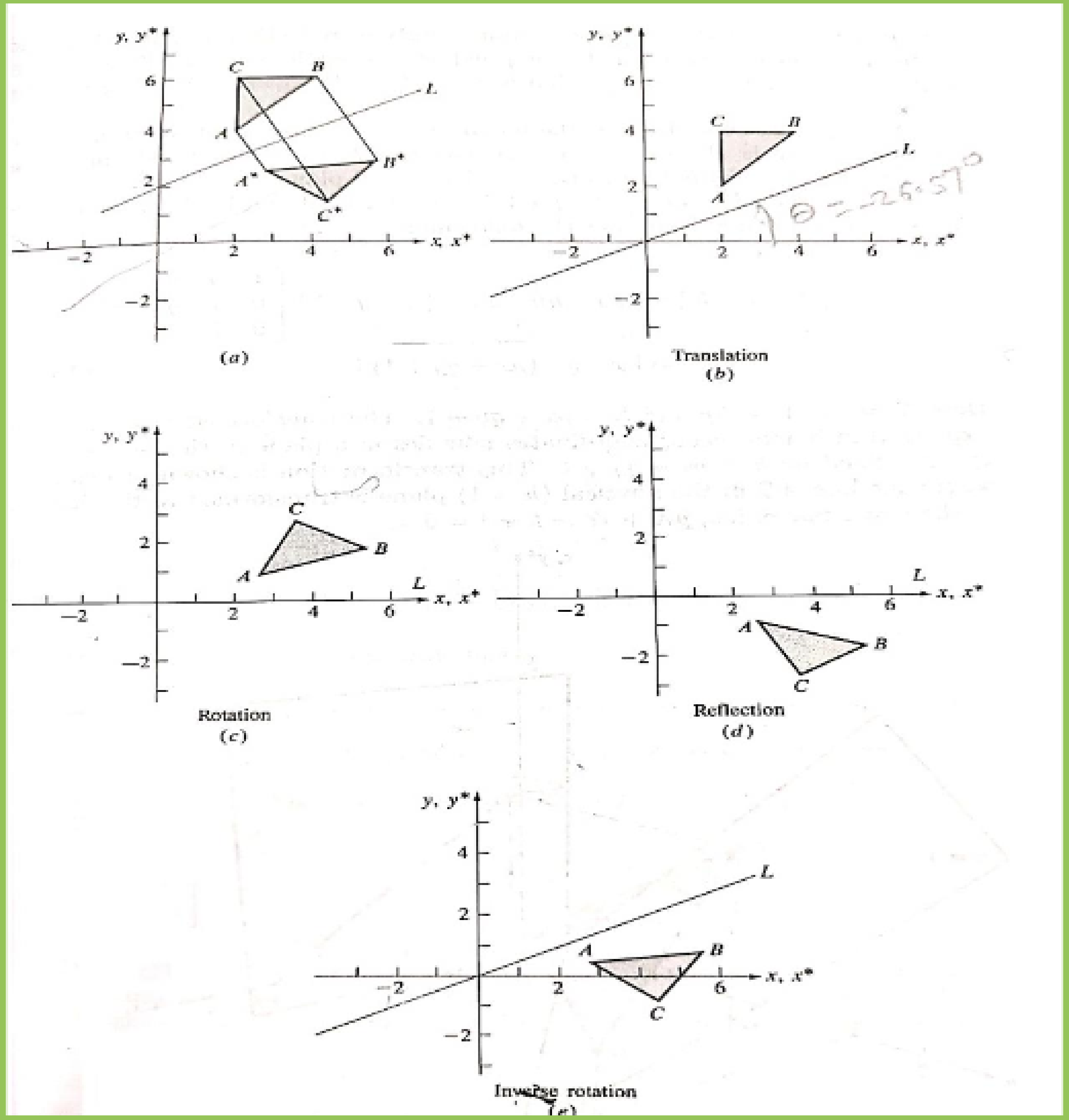
$$\begin{bmatrix} 6/\sqrt{5} & 2/\sqrt{5} & 1 \\ 12/\sqrt{5} & 4/\sqrt{5} & 1 \\ 8/\sqrt{5} & 6/\sqrt{5} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6/\sqrt{5} & -2/\sqrt{5} & 1 \\ 12/\sqrt{5} & -4/\sqrt{5} & 1 \\ 8/\sqrt{5} & -6/\sqrt{5} & 1 \end{bmatrix} = \begin{bmatrix} 2.68 & -0.89 & 1 \\ 5.36 & -1.78 & 1 \\ 3.57 & -2.68 & 1 \end{bmatrix}$$

(4) Inverse rotation ($\theta = 26.57$)

$$\begin{bmatrix} 6/\sqrt{5} & -2/\sqrt{5} & 1 \\ 12/\sqrt{5} & -4/\sqrt{5} & 1 \\ 8/\sqrt{5} & -6/\sqrt{5} & 1 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 14/5 & 2/5 & 1 \\ 28/5 & 4/5 & 1 \\ 22/5 & -4/5 & 1 \end{bmatrix} = \begin{bmatrix} 2.8 & 0.4 & 1 \\ 5.6 & 0.8 & 1 \\ 4.4 & -0.8 & 1 \end{bmatrix}$$

(5) Inverse translation $m = 0, n = 2$

$$\begin{bmatrix} 2.8 & 0.4 & 1 \\ 5.6 & 0.8 & 1 \\ 4.4 & -0.8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2.8 & 2.4 & 1 \\ 5.6 & 2.8 & 1 \\ 4.4 & 1.2 & 1 \end{bmatrix}$$



Projection : a geometric interpretation

$$3\times 3\,[T] = \left[\begin{array}{cc|c} a & b & p \\ c & d & q \\ \hline m & n & s \end{array}\right]$$
matrix

p & q - Projection $p=0\ q=0,\ h=1$ - physical plane

s - overall scaling $s=1$

To show the effect of $p\neq 0, q\neq 0$ in the 3rd column of general 3x3 transformation matrix, consider

$$[X\quad Y\quad h] = [hx\quad hy\quad h] = [x\quad y\quad 1]\begin{bmatrix} 1 & 0 & p \\ 0 & 1 & q \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{projection}$$

$$= [x\quad y\quad (px+qy+1)] \quad \text{— (1)}$$

where $h = px+qy+1$ - transformed position vector lies in a 3 dimensional plane $(h\neq 1)$

Results can be obtained by geometrically project $h\neq 1$ plane back to $h=1$ physical plane.

so $x^* = \dfrac{X}{h}$ & $y^* = \dfrac{Y}{h}$

or $$\begin{bmatrix} x^* & y^* & 1 \end{bmatrix} = \begin{bmatrix} \dfrac{x}{px+qy+1} & \dfrac{y}{px+qy+1} & 1 \end{bmatrix} \quad \text{— (2)}$$

Example: Consider line AB with $A = [1 \;\; 3]$, $B = [4 \;\; 1]$ & $p = q = 1$

$$\begin{bmatrix} A \\ B \end{bmatrix} [T] = \begin{bmatrix} 1 & 3 & 1 \\ 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} & & h= \end{array} \\ \begin{bmatrix} 1 & 3 & 5 \\ 4 & 1 & 6 \end{bmatrix} \\ \begin{array}{ccc} & & h \end{array} \end{array} = \begin{bmatrix} A' \\ B' \end{bmatrix}$$

$$h = px + qy + 1 = x + y + 1$$

$1 + 3 + 1 = 5$

$4 + 1 + 1 = 6$

$$\begin{bmatrix} A^* \\ B^* \end{bmatrix} = \begin{bmatrix} 1/5 & 3/5 & 1 \\ 4/6 & 1/6 & 1 \end{bmatrix}$$

Overall scaling:

s - produces overall scaling effect:

$$[X \quad Y \quad h] = [x \quad y \quad 1]\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & s\end{bmatrix} = [x \quad y \quad s] \quad - (h \neq 1)$$

Thus, the transformation is

$$[x \quad y \quad 1]\,[T] = \left[\frac{x}{s} \quad \frac{y}{s} \quad 1\right]$$

If $s<1$ - expansion & $s>1$ compression

eg consider line AB, $A = [2 \quad 2]$, $B = [6 \quad 6]$, $s = 2$

$$\begin{bmatrix}A\\ B\end{bmatrix}[T] = \begin{bmatrix}2 & 2 & 1\\ 6 & 6 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix} = \begin{bmatrix}2 & 2 & 2\\ 6 & 6 & 2\end{bmatrix} = \begin{bmatrix}A'\\ B'\end{bmatrix}$$

$$\begin{bmatrix}A^*\\ B^*\end{bmatrix} = \begin{bmatrix}2/2 & 2/2 & 2/2\\ 6/2 & 6/2 & 2/2\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\ 3 & 3 & 1\end{bmatrix}$$