# Vector Space

**Introduction:**

Many familiar physics notions, such as forces, velocity and accelerations involve both a magnitude and a direction. Any such entity involving both magnitude and direction is called a vector. A vector is represented by an arrow whose length denotes the magnitude of the vector and whose direction represents the directions of the vector. In most physical situations involving vector, only the magnitude and direction of the vector are significant; consequently, we regard vectors with the same magnitude and directions as being equal irrespective of their positions. In this unit the geometry of vector is discussed.

**Euclidean Space:** Euclidean space is the fundamental space of classical geometry, it is a three dimensional space.

**Notations:**

| | |
|---|---|
| R- Real number | x |
| $R^2$ | $x=(x_1, x_2)$ or $P(x, y)$ |
| **$R^3$** | **$x=(x_1, x_2, x_3)$ or $P(x, y, z)$** |
| $R^n$ | $x=(x_1, x_2,x_3,\ldots,x_n)$ or $P(x, y, z,\ldots n)$ |

**Definition:** Let V be an arbitrary non-empty set of object (elements) on which two operations defined-addition and multiplication by scalars called sum and scalar multiplication respectively. If the following axioms are satisfied by all objects $u, v$ *and* $w$ in V and for all scalar $k$ *and* $l$, then V is called a **vector space or linear space.**

**Closure Axioms:**

C1: For all $u, v$ in $V$, $u+v\in V$ , addition is closed in $V$ .

C2: For all scalars k and any $u\in V$, $ku\in V$, scalar multiplication is closed in V.

**Addition Axioms:**

A1: $u+v=v+u$ Sum is commutative

A2: $(u+v)+w=u+(v+w)$ Sum is associative

A3: There is vector 0 in V, called zero vector, such that $u+0=u=0+u$, for all $u\in V$

A4: For each $u\in V$ there is $-u\in V$ called negative of u such that $u+(-u)=0$

**Scalar Multiplication axioms:**

M1: $k(u+v)=ku+kv$ Scalar multiplication distribution over vector addition

M2: $(k+l)u=ku+lu$ Vector multiplication distributes over scalar addition

M3: $(kl)u = k(lu)$ *M*ultiplication by scalars is associative

M4: $1u = u$

**Example 1:** Let $V = M_{2\times 2}(R)$, the set of all $2\times 2$ matrices with real entries.

For any $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

Define

1. $A = B$ if and only if $a = e, b = f, c = g, d = h$
2. $A + B = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$
3. $kA = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$ for any scalar k.

Show that V is vector space.

**Solution:** To show that V is a vector space, means to show that all the vector space axioms are satisfied by these two operations in $V = M_{2\times 2}(R)$.

**Closure Axioms:** The closure axioms clearly hold well as $A + B \, and \, kA$ are again $2\times 2$ matrices with real entries.

**Addition Axioms:** For any $A, B \in V = M_{2\times 2}(R)$

$$A1: \quad A + B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

$$= \begin{bmatrix} e+a & f+b \\ g+c & h+d \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} = B + A$$

$$A2: \quad \text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}, C = \begin{bmatrix} i & j \\ k & l \end{bmatrix}$$

$$(A+B)+C = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} \right\} + \begin{bmatrix} i & j \\ k & l \end{bmatrix} = \left\{ \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \right\} + \begin{bmatrix} i & j \\ k & l \end{bmatrix}$$

$$= \begin{bmatrix} a+e+i & b+f+j \\ c+g+k & d+h+l \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \left\{ \begin{bmatrix} e+i & f+j \\ g+k & h+l \end{bmatrix} \right\}$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \left\{ \begin{bmatrix} e & f \\ g & h \end{bmatrix} + \begin{bmatrix} i & j \\ k & l \end{bmatrix} \right\} = A + (B+C)$$

$$\therefore (A+B)+C = A+(B+C).$$

$$A3: \quad \text{For any } 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in V = M_{2\times 2}(R)$$

$$A + 0 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a+0 & b+0 \\ c+0 & d+0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$$

$\therefore 0$ is a zero matrix in $V = M_{2\times 2}(R)$

$A4$: For any $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V = M_{2\times 2}(R),\ -A=\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} \in V = M_{2\times 2}(R)$

$$\therefore A+(-A)=\begin{bmatrix} a & b \\ c & d \end{bmatrix}+\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}=\begin{bmatrix} a-a & b-b \\ c-c & d-d \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ For any $A \in V = M_{2\times 2}(R)$, there is $-A \in V = M_{2\times 2}(R)$.

**Scalar Multiplication Axioms:** Let $k$ *and* $l$ be any scalars

$M1$: For any $A,B \in V$

$$k(A+B)=k\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix}+\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right\}=k\begin{bmatrix} a & b \\ c & d \end{bmatrix}+k\begin{bmatrix} e & f \\ g & h \end{bmatrix}=kA+kB$$

$M2$: $(k+l)A=(k+l)\begin{bmatrix} a & b \\ c & d \end{bmatrix}=k\begin{bmatrix} a & b \\ c & d \end{bmatrix}+l\begin{bmatrix} a & b \\ c & d \end{bmatrix}=kA+lA$

$M3$: $(kl)A=(kl)\begin{bmatrix} a & b \\ c & d \end{bmatrix}=k\left\{\begin{bmatrix} la & lb \\ lc & ld \end{bmatrix}\right\}=k\left\{l\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right\}=k(lA)$

$M4$: $1A=1\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} 1.a & 1.b \\ 1.c & 1.d \end{bmatrix}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad\therefore 1A=A$

Hence V is a vector space.

**Example 2:** Let $V=M_{2\times 2}(R)$, the set of all $2\times 2$ matrices with real entries.

For any $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix},\ B=\begin{bmatrix} e & f \\ g & h \end{bmatrix}$. Define

1. $A=B$ if and only if $a=e, b=f, c=g, d=h$
2. $A+B=\begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$
3. $kA=\begin{bmatrix} ka & 0 \\ 0 & kd \end{bmatrix}$ for any scalar k.

Show that V is **not a vector space.**

**Solution: Solution:** To show that V is a vector space, means to show that all the vector space axioms are satisfied by these two operations in $V=M_{2\times 2}(R)$.

**Closure Axioms:** The closure axioms clearly hold well as $A+B$ *and* $kA$ are again $2\times 2$ matrices with real entries.

**Addition Axioms:** For any $A,B \in V = M_{2\times 2}(R)$

$A1$: $A+B=\begin{bmatrix} a & b \\ c & d \end{bmatrix}+\begin{bmatrix} e & f \\ g & h \end{bmatrix}=\begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$

$$=\begin{bmatrix} e+a & f+b \\ g+c & h+d \end{bmatrix}=\begin{bmatrix} e & f \\ g & h \end{bmatrix}+\begin{bmatrix} a & b \\ c & d \end{bmatrix}=B+A$$

$A2:$ Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}, B=\begin{bmatrix} e & f \\ g & h \end{bmatrix}, C=\begin{bmatrix} i & j \\ k & l \end{bmatrix}$

$$(A+B)+C=\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix}+\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right\}+\begin{bmatrix} i & j \\ k & l \end{bmatrix}=\left\{\begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}\right\}+\begin{bmatrix} i & j \\ k & l \end{bmatrix}$$

$$=\begin{bmatrix} a+e+i & b+f+j \\ c+g+k & d+h+l \end{bmatrix}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}+\left\{\begin{bmatrix} e+i & f+j \\ g+k & h+l \end{bmatrix}\right\}$$

$$=\begin{bmatrix} a & b \\ c & d \end{bmatrix}+\left\{\begin{bmatrix} e & f \\ g & h \end{bmatrix}+\begin{bmatrix} i & j \\ k & l \end{bmatrix}\right\}=A+(B+C)$$

$\therefore (A+B)+C=A+(B+C).$

$A3:$ For any $0=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in V=M_{2\times2}(R)$

$$A+0=\begin{bmatrix} a & b \\ c & d \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} a+0 & b+0 \\ c+0 & d+0 \end{bmatrix}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}=A$$

$\therefore 0$ is a zero matrix in $V=M_{2\times2}(R)$

$A4:$ For any $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V=M_{2\times2}(R),\ -A=\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix} \in V=M_{2\times2}(R)$

$$\therefore A+(-A)=\begin{bmatrix} a & b \\ c & d \end{bmatrix}+\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}=\begin{bmatrix} a-a & b-b \\ c-c & d-d \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ For any $A \in V=M_{2\times2}(R)$, there is $-A \in V=M_{2\times2}(R)$.

**Scalar Multiplication Axioms:** Let $k$ *and* $l$ be any scalars

$M1:$ For any $A,B \in V$

$$k(A+B)=k\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix}+\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right\}=k\begin{bmatrix} a & b \\ c & d \end{bmatrix}+k\begin{bmatrix} e & f \\ g & h \end{bmatrix}=kA+kB$$

$M2:$ $(k+l)A=(k+l)\begin{bmatrix} a & b \\ c & d \end{bmatrix}=k\begin{bmatrix} a & b \\ c & d \end{bmatrix}+l\begin{bmatrix} a & b \\ c & d \end{bmatrix}=kA+lA$

$M3:$ $(kl)A=(kl)\begin{bmatrix} a & b \\ c & d \end{bmatrix}=k\left\{\begin{bmatrix} la & lb \\ lc & ld \end{bmatrix}\right\}=k\left\{l\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right\}=k(lA)$

$M4:$ $1.A=1.\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} 1.a & 0 \\ 0 & 1.d \end{bmatrix}=\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\neq A$

As the M4 axiom is not satisfies, Therefore V is **not a vector space.**

**Example 3:** Consider $V=\left\{p(x)=a_0x^2+a_1x+a_2 \,/\, a_0,a_1,a_2 \text{ are any real numbers}\right\}$ is a set of polynomial of degree less then and equal to 2.

Define

$$(p+q)(x)=p(x)+q(x)=\left(a_0x^2+a_1x+a_2\right)+\left(b_0x^2+b_1x+b_2\right)$$
$$=\left((a_0+b_0)x^2+(a_1+b_1)x+(a_2+b_2)\right)$$
$$k\,p(x)=k\left(a_0x^2+a_1x+a_2\right)=\left(ka_0x^2+ka_1x+ka_2\right)$$

Addition of two polynomial and multiplication of a polynomial by a real number is taken as usual. Show that V is a vector space.

**Solution: Closure Axioms:** The closure axioms clearly hold well as $p+q$ and $kp$ are again polynomial of degree less then and equal to 2.

**Addition Axioms:** For any $u,v\in V$

$$A1:(p+q)(x)=p(x)+q(x)=\left(a_0x^2+a_1x+a_2\right)+\left(b_0x^2+b_1x+b_2\right)$$
$$=\left((a_0+b_0)x^2+(a_1+b_1)x+(a_2+b_2)\right)$$
$$=\left((b_0+a_0)x^2+(b_1+a_1)x+(b_2+a_2)\right)$$
$$=\left(\left(b_0x^2+b_1x+b_2\right)+\left(a_0x^2+a_1x+a_2\right)\right)=(q+p)(x)$$
$$\therefore (p+q)(x)=(q+p)(x)$$

$A2$: Let $r=c_0x^2+c_1x+c_2$

$$(p+q)(x)+r(x)=\left[\left(a_0x^2+a_1x+a_2\right)+\left(b_0x^2+b_1x+b_2\right)\right]+\left(c_0x^2+c_1x+c_2\right)$$
$$=\left[(a_0+b_0)x^2+(a_1+b_1)x+(a_2+b_2)\right]+\left(c_0x^2+c_1x+c_2\right)$$
$$=\left[(a_0+b_0+c_0)x^2+(a_1+b_1+c_1)x+(a_2+b_2+c_2)\right]$$
$$=\left[a_0+(b_0+c_0)x^2+a_1+(b_1+c_1)x+a_2+(b_2+c_2)\right]$$
$$=\left[a_0x^2+a_1x+a_2\right]+\left[(b_0+c_0)x^2+(b_1+c_1)x+(b_2+c_2)\right]$$
$$=\left[a_0x^2+a_1x+a_2\right]+\left[\left(b_0x^2+b_1x+b_2\right)+\left(c_0x^2+c_1x+c_2\right)\right]$$
$$\therefore (p+q)(x)+r(x)=p(x)+\left(q(x)+r(x)\right)=p(x)+(q+r)(x)$$

$A3$: For any $0(x)=0x^2+0x+0$

$$(p+0)(x)=\left[\left(a_0x^2+a_1x+a_2\right)+\left(0x^2+0x+0\right)\right]$$
$$=\left[(a_0+0)x^2+(a_1+0)x+(a_2+0)\right]$$
$$=\left(a_0x^2+a_1x+a_2\right)=p(x)$$

$\therefore 0$ is a zero polynomial of degree less then or equal to two in $V$

$A4$: For any $p(x)=\left(a_0x^2+a_1x+a_2\right)\in V$ there is $-p(x)=\left(-a_0x^2-a_1x-a_2\right)\in V$

$$p(x)+(-p(x))=\left(a_0x^2+a_1x+a_2\right)+\left(-a_0x^2-a_1x-a_2\right)$$
$$=\left[\left(a_0+(-a_0)\right)x^2+\left(a_1+(-a_1)\right)x+\left(a_2+(-a_2)\right)\right]$$
$$=\left(0x^2+0x+0\right)=0(x)=0$$

**Scalar Multiplication Axioms:** Let $k$ *and* $l$ be any scalars

$M1$: For any p,q $\in$ V

$$k\left[p(x)+q(x)\right]=k\left[\left(a_0x^2+a_1x+a_2\right)+\left(b_0x^2+b_1x+b_2\right)\right]$$
$$=k\left[(a_0+b_0)x^2+(a_1+b_1)x+(a_2+b_2)\right]$$
$$=\left[k(a_0+b_0)x^2+k(a_1+b_1)x+k(a_2+b_2)\right]$$
$$=\left[ka_0x^2+kb_0x^2+ka_1x+kb_1x+ka_2+kb_2\right]$$
$$=\left[\left(ka_0x^2+ka_1x+ka_2\right)+\left(kb_0x^2+kb_1x+kb_2\right)\right]$$
$$=\left[k\left(a_0x^2+a_1x+a_2\right)+k\left(b_0x^2+b_1x+b_2\right)\right]=kp(x)+kq(x)$$
$$\therefore k\left[p(x)+q(x)\right]=kp(x)+kq(x)$$

$M2$: For any p,q $\in$ V

$$(k+l)\,p(x)=(k+l)\left(a_0x^2+a_1x+a_2\right)=\left((k+l)a_0x^2+(k+l)a_1x+(k+l)a_2\right)$$
$$=\left[ka_0x^2+la_0x^2+ka_1x+la_1x+ka_2+la_2\right]$$
$$=\left[\left(ka_0x^2+ka_1x+ka_2\right)+\left(la_0x^2+la_1x+la_2\right)\right]$$
$$=\left[k\left(a_0x^2+a_1x+a_2\right)+l\left(a_0x^2+a_1x+a_2\right)\right]=kp(x)+lp(x)$$
$$\therefore (k+l)\,p(x)=kp(x)+lp(x)$$

$M3$: For any p,q $\in$ V

$$(kl)\,p(x)=(kl)\left(a_0x^2+a_1x+a_2\right)=\left((kl)a_0x^2+(kl)a_1x+(kl)a_2\right)$$
$$=\left[k(la_0)x^2+k(la_1)x+k(la_2)\right]$$
$$=\left[k\left((la_0)x^2+(la_1)x+(la_2)\right)\right]$$
$$=\left[k\left(l\left(a_0x^2+a_1x+a_2\right)\right)\right]=k\left(lp(x)\right)$$
$$\therefore (kl)\,p(x)=k\left(lp(x)\right)$$

$$M4: 1p(x)=1.\left(a_0x^2+a_1x+a_2\right)=\left(1a_0x^2+1a_1x+1a_2\right)=\left(a_0x^2+a_1x+a_2\right)$$
$$=p(x)$$
$$\therefore 1.p(x)=p(x)$$

Hence V is a vector space.

**Example 4:** Let $V=R^+$ be the set of all positive reals. Define addition of any two members x and y to be the usual multiplication of numbers that is $x+y=x.y$ .Define scalar multiplication by a scalar k to any is $x\in R^+$ to be $x^k$ that is $kx=x^k$. Then Show that V is a vector space.

**Solution:**

**Closure Axioms:**

$C1$: Since the product of two positive real number is again positive real number for any $x,y\in R^+$ , $x+y\in R^+$.

$C2$: For any real number k and $x\in R^+$, $x^k$ is also positive real number $\Rightarrow kx\in R^+$.

**Addition Axioms:** For any $u, v \in V$

$$
\begin{aligned}
A1: x+y &= x.y \\
&= y.x \\
&= y+x \\
x+y &= y+x
\end{aligned}
$$

$$
\begin{aligned}
A2: (x+y)+z &= (x.y)+z \\
&= (x.y).z \\
&= x.(y.z) \\
&= x+(y.z) \\
&= x+(y+z) \\
(x+y)+z &= x+(y+z)
\end{aligned}
$$

$A3$: For any $1 \in R^{+}$

$$x+1 = x.1 = x$$

$$\Rightarrow x+1 = x \quad \therefore \text{Thus 1 is zero element in } \mathrm{R}^{+}$$

**Note:** The zero element of a vector space may not be usual zero element, it is zero element w. r. t the operation defined in the set.

$$A4: \text{For any } x \in R^{+}, \frac{1}{x} \in R^{+} \because x > 0$$

$$\therefore x + \frac{1}{x} = x.\frac{1}{x} = 1 \text{ (Zero element)}$$

Thus, each element in $\mathrm{R}^{+}$ has negative element in $\mathrm{R}^{+}$.

**Scalar Multiplication Axioms:** Let $k$ *and* $l$ be any scalars

$M1$: For any $\mathrm{x,y} \in \mathrm{R}^{+}$

$$
\begin{aligned}
k(x+y) &= k(x.y) = (x.y)^{k} \\
&= x^{k}.y^{k} = x^{k} + y^{k} \\
&= kx + ky \\
k(x+y) &= kx + ky
\end{aligned}
$$

$M2$: For any $\mathrm{x,y} \in \mathrm{R}^{+}$

$$
\begin{aligned}
(k+l)x &= x^{(k+l)} \\
&= x^{k}.x^{l} = x^{k} + x^{l} \\
&= kx + lx \\
\therefore (k+l)x &= kx + lx
\end{aligned}
$$

$M3$: For any $\mathrm{x,y} \in \mathrm{R}^{+}$

$$
\begin{aligned}
(kl)x &= (kl)x \\
&= x^{kl} = x^{lk} = \left(x^{l}\right)^{k} \\
&= k\left(x^{l}\right) = k(lx) \\
(kl)x &= k\left(x^{l}\right) = k(lx)
\end{aligned}
$$

$$M4: 1x = 1.x$$
$$= x$$
$$\therefore 1.x = x$$

Hence V is a vector space.

**Example 5:** Let $V = R^2$ be the set of all ordered pairs of real numbers, that is $R^2 = \{(x,y)/x, y \in R\}$. Define for any $u = (x,y)$ *and* $v = (x', y')$ *in* $R^2$

$(i)$Equality: $u = v$ if and only if $x = x'$ and $y = y'$

$(ii)$Sum: $u + v = (x,y) + (x',y') = (x + x', y + y')$

$(iii)$ Scalar Multiplication: For any scalar k, $ku = k(x,y) = (kx, ky)$.

Show that V is a vector space.

**Solution:** To show that $V = R^2$ is a vector space, means to show that all the vector space axioms are satisfied by these two operations in $R^2$.

**Closure Axioms:** The closure axioms clearly hold well as $x + x', y + y'$ *and* $kx, ky$ are real numbers. Hence $u + v$ *and* $ku$ are again ordered pairs of real numbers and thus belong to $R^2$.

**Addition Axioms:** Let $u = (x,y), v = (x',y')$ *and* $w = (x'', y'')$ be any vector in $R^2$

$A1$: $u + v = (x,y) + (x',y') = (x + x', y + y')$ by definition of sum in $\mathrm{R}^2$

$= (x' + x, y' + y)$ Since sum in R is commutative

$= (x', y') + (x, y)$ Agian by definition of sum in $\mathrm{R}^2$

$= v + u$

$\therefore u + v = v + u$ for all u,v in $\mathrm{R}^2$.

$A2$: $(u+v) + w = [(x,y) + (x',y')] + (x'' + y'')$

$= (x + x', y + y') + (x'' + y'')$ by sum in $\mathrm{R}^2$

$= [(x + x') + x'', (y + y') + y'']$ by sum in $\mathrm{R}^2$

$= [x + (x' + x''), y + (y' + y'')]$ by associativity of sum in $\mathrm{R}^2$

$= (x,y) + [x' + x'', y' + y'']$ by sum in $\mathrm{R}^2$

$= (x,y) + [(x',y') + (x'',y'')]$ by sum in $\mathrm{R}^2$

$= u + (v + w)$

$\therefore (u+v) + w = u + (v + w)$ for all u,v,w in $\mathrm{R}^2$.

$A3$: As $0 \in R, (0,0)$ is in $\mathrm{R}^2$ and we have

$$u + (0,0) = (x,y) + (0,0) = (x + 0, y + 0) = (x,y) = u$$

$\therefore (0,0)$ is a zero vector in $\mathrm{R}^2$.

$A4:$ $\quad For\, u=(x,y)$ in $\mathrm{R}^2$ we have $-u=(-x,-y)$ in $\mathrm{R}^2$ *such that*

$$u+(-u)=(x,y)+(-x,-y)=\left[x+(-x),\,y+(-y)\right]=(0,0)$$

Thus for each u in $\mathrm{R}^2$ there is negative of u in $\mathrm{R}^2$.

**Scalar Multiplication Axioms:** Let $k$ *and* $l$ be any scalars

$M1:$

$$\begin{aligned}
k(u+v) &= k(x+x', y+y') \\
&= \left[k(x+x'), k(y+y')\right] \quad \text{by Scalar Multiplication in } R^2 \\
&= (kx+kx', ky+ky') \\
&= (kx,ky)+(kx',ky') \\
&= k(x,y)+k(x',y') \\
&= ku+kv
\end{aligned}$$

$$\therefore k(u+v)=ku+kv$$

$M2:$

$$\begin{aligned}
(k+l)u &= (k+l)(x,y) \\
&= \left[(k+l)x,(k+l)y\right] \quad \text{by Scalar Multiplication in } R^2 \\
&= (kx+lx, ky+ly) \\
&= (kx,ky)+(lx,ly) \\
&= k(x,y)+l(x,y) \\
&= ku+lu
\end{aligned}$$

$$\therefore (k+l)u=ku+lu$$

$M3:$

$$\begin{aligned}
(kl)u &= (kl)(x,y) \\
&= \left[(kl)x,(kl)y\right] \quad \text{by Scalar Multiplication in } R^2 \\
&= \left[k(lx),k(ly)\right] \\
&= k\left[l(x,y)\right] \\
&= k(lu)
\end{aligned}$$

$$\therefore (kl)u=k(lu)$$

$M4:$

$$\begin{aligned}
1u &= 1(x,y) \\
&= \left[1x,1y\right] \quad \text{by Scalar Multiplication in } R^2 \\
&= (x,y)=u
\end{aligned}$$

$$\therefore 1u=u$$

Hence $R^2$ is a vector space and $R^2$ is a real vector space.

**Theorem:** Let V be a vector space, $u$ a vector in V and k be any scalar, then

1. $0u=0$
2. $k0=0$
3. $(-1)u=-u$
4. If $ku=0,$ then $k=0$ *or* $u=0$

**Example 6: The Zero vector space:**

Let V consist of a single object, which we denoted by 0 and define

$0+0=0$ and $\alpha 0=0$ for all scalars $\alpha$. We call this the zero vector space.

Zero vector space is vector space, one can easily check that all the vector space axioms are satisfies.

**Note:** We will denote this vector space by the symbol $R^{\infty}$

**Example 7: The vector space of 2*2 matrices**

Let V be the set of 2*2 matrices with real entries and take the vector space operations on V to be usual operations of matrix addition and scalar multiplication that is

$$u = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \; and \;\; v = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$

$$u + v = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$

$$ku = k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$

**Example 8: A set that is not a vector space**

Let $V = R^2$ and define addition and scalar multiplication operations as follows:

If $u = (u_1, u_2), \quad v = (v_1, v_2)$ then define $u + v = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$

and if k is any real number, then define $ku = k(u_1, u_2) = (ku_1, 0)$

For example if $u = (2,4), \quad v = (-3,5) \therefore u + v = (2,4) + (-3,5) = (2 + (-3), 4 + 5) = (-1,9)$

Let $k = 7 \therefore ku = 7u = 7(2,4) = (14,0)$

The addition operation is the standard one from $R^2$ , but the scalar multiplication is not. The first nine axioms are satisfies. However axioms number 10 fails. For example , if $u = (u_1, u_2)$ is such that $u_2 \neq 0$ then $1.u = 1.(u_1, u_2) = (1.u_1, 0) \neq u$. Thus V is not a vector space.