3D Transformation:

position vector - P

$[x' \quad y' \quad z' \quad h]$ where $x' = hx$, $y' = hy$, $z' = hz$

3D Transformation with Homogeneous Coordinates

1. Upper Left 3 × 3 sub matrix – linear transformation such as scaling, shearing, reflection and rotation.
2. Lower Left 1 × 3 sub matrix – translation
3. Upper Right 3 × 1 sub matrix – perspective transformation
4. Lower Right 1 × 1 sub matrix – overall scaling

$$[T] = \begin{bmatrix} a & b & c & p \\ d & e & f & q \\ g & i & j & r \\ l & m & n & s \end{bmatrix}$$

scaling
rotation
reflection
shearing

$$\left[\begin{array}{c:c} 3 \times 3 & \begin{array}{c} 3 \\ \times \\ 1 \end{array} \\ \cdots \quad \cdots \quad \cdots & \cdots \\ 1 \times 3 & 1 \times 1 \end{array}\right]$$

perspective transformation

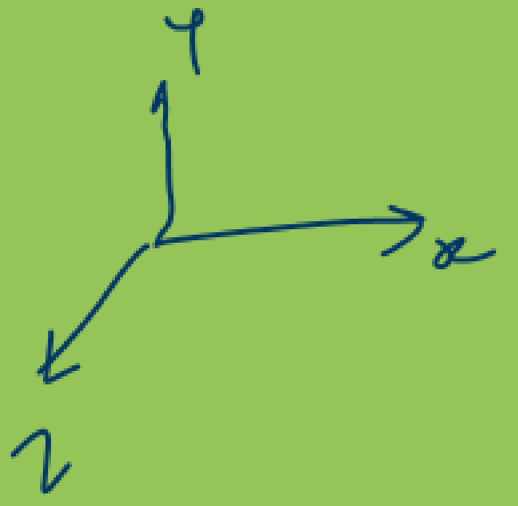
overall scaling

translation

Translation :

$$\begin{bmatrix} x^* & y^* & z^* & 1 \end{bmatrix} = \begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_x & t_y & t_z & 1 \end{bmatrix} = \begin{bmatrix} x+t_x & y+t_y & z+t_z & 1 \end{bmatrix}$$

eg: consider a point $(5, 6, 7)$. Translate it with $t_x = 3$, $t_y = 3$ & $t_z = 2$



$$\begin{bmatrix} x^* & y^* & z^* & 1 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 7 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 9 & 9 & 1 \end{bmatrix}$$

3D Rotation:

Specify the angle of rotation (θ) along with the axis of rotation

For rotation about x-axis

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

rotation about z-axis

$$[T] = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

rotation about y-axis

$$[T] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

eg. consider the rectangular parallelpiped (RPP) with following coordinates

Rotate it by $\theta = -90^\circ$ about x-axis

$$[X^*] = [X][T] = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 1 \\ 3 & 0 & 1 & 1 \\ 3 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 \\ 3 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 \\ 3 & 1 & 0 & 1 \\ 3 & 1 & -2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 \\ 3 & 0 & -2 & 1 \\ 0 & 0 & -2 & 1 \end{bmatrix}}_{A^*}$$

Combined Rotation:

$$[A][B] \neq [B][A]$$

Consider rotation about x-axis followed by equal rotation about y-axis

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ \sin^2\theta & \cos\theta & \sin\theta\cos\theta & 0 \\ \sin\theta\cos\theta & -\sin\theta & \cos^2\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{— (1)}$$

Now consider rotation about y-axis followed by equal rotation about x-axis

$$[T] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin^2\theta & -\sin\theta\cos\theta & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ \sin\theta & -\sin\theta\cos\theta & \cos^2\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{— (2)}$$

eq^n (1) & (2) are not identical

Reflection:

In 3D, Reflection - xy, yz or xz

Reflection through xy plane

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Reflection through yz plane

$$[T] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Reflection through xz plane

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

eg. Consider a RPP with following coordinate. Reflect it through xy plane.

$$[X^*] = [X][T] \begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 0 & -1 & 1 \\ 2 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & -2 & 1 \\ 2 & 1 & -2 & 1 \\ 1 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix}$$

3D scaling:

① **Local scaling**: primary diagonal terms

$$[X^*] = [X][T] = \begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x*s_x & y*s_y & z*s_z & 1 \end{bmatrix}$$

eg. Consider a RPP, scale it with factors $s_x = 1/2$, $s_y = 1/3$ & $s_z = 1$

$$[X^*] = [X][T] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 3 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 2 & 3 & 0 & 1 \\ 0 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

2) Overall Scaling: 4th diagonal element - S

$$[X'] = [X][T] = [x \;\; y \;\; z \;\; 1]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s \end{bmatrix} = [x \;\; y \;\; z \;\; s] = [x' \;\; y' \;\; z' \;\; \cancel{s}]$$

h = s means h ≠ 1 - not in physical plane

$$[X^*] = [x^* \;\; y^* \;\; z^* \;\; 1] = \left[\frac{x'}{s} \;\; \frac{y'}{s} \;\; \frac{z'}{s} \;\; 1\right]$$ - physical plane

eg. Consider Previous resulting RPP scale it by factor 2 (Double its size)

$s = 1/2$

$$[X][T] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1/2 \\ 1 & 0 & 1 & 1/2 \\ 1 & 1 & 1 & 1/2 \\ 0 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & 1/2 \\ 1 & 0 & 0 & 1/2 \\ 1 & 1 & 0 & 1/2 \\ 0 & 1 & 0 & 1/2 \end{bmatrix} \xrightarrow[\text{by } s = 1/2]{\text{divide}} \begin{bmatrix} 0 & 0 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 2 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

$s < 1$ expansion

$s > 1$ compression

Shearing: off-diagonal terms of upper left 3X3 matrix

$$[X][T] = [x \;\; y \;\; z \;\; 1]\begin{bmatrix} 1 & b & c & 0 \\ d & 1 & f & 0 \\ g & i & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [x+yd+gz \quad bx+y+iz \quad cx+yf+z \quad 1]$$

eg.

$$[X^*] = [X][T] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -0.85 & 0.25 & 0 \\ -0.75 & 1 & 0.7 & 0 \\ 0.5 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & 1 & 1 & 1 \\ 1.5 & 0.15 & 1.25 & 1 \\ 0.75 & 1.15 & 1.95 & 1 \\ -0.25 & 2 & 1.7 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & -0.85 & 0.25 & 1 \\ 0.25 & 0.15 & 0.95 & 1 \\ -0.75 & 1 & 0.7 & 1 \end{bmatrix}$$