DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING
Second Year B.Tech. (SEM - III)
COMPUTATIONAL MATHEMATICS (UCSE0301)

Unit No. 3: Probability and Probability Distributions

Introduction:

Probability theory, a branch of statistics concerned with the analysis of random phenomena. The outcome of a random event cannot be determined before it occurs, but it may be any one of several possible outcomes. The actual outcome is considered to be determined by chance.

A numerical measurement of uncertainty is practiced by the important branch of statistics called the theory of probability. This is the basic probability theory which is also used in the probability distribution, where you will learn the possibility of outcomes for a random experiment.

Random Experiment:

A random experiment is a process characterized by the following properties:

(i) It is performed according to some set of rules,

(ii) It can be repeated arbitrarily often,

(iii) The result of each performance depends on chance and cannot be predicted uniquely.

E.g. tossing of a coin. Rolling of a die.

Trial:

Repetition of experiment is called trial.

E.g. Tossing of coin is an experiment and repeated number of times is called trial.

Sample Space:

A set of all possible outcomes from an experiment is called a sample space.

E.g.: Consider a random experiment E of throwing 2 coins at a time. The possible outcomes are HH, TT, HT, and TH.

These 4 outcomes constitute a sample space denoted by, S = {HH, TT, HT, TH}.

Event:

One or more outcome of an experiment is called event.

E.g.: Consider a random experiment E of throwing 2 coins at a time then sample space is

$$S = \{HH, TT, HT, TH\}.$$

Let A is the event that getting 2 heads at same time then its outcome is HH.

In other words, "Every non-empty subset of A of the sample space S is called an event"

Favourable Events:

The number of cases favourable to an event in a trial is the number of outcomes which entail the happening of the event.

E.g.: Consider a random experiment E of throwing 2 coins at a time then sample space is

$$S = \{HH, TT, HT, TH\}.$$

The numbers of cases favourable to the event of getting a head are 3, i.e. HH, HT, and TH.

Equally likely Events:

Two events are said to be equally likely if each one of them has an equal chance of happening.

E.g.: In throwing a coin, the events head & tail have equal chances of occurrence.

Mutually Exclusive Events:

Two events are said to be mutually exclusive when both cannot happen simultaneously in a single trial. In other words, if A & B are mutually exclusive events if A happens then B will not happen and vice versa.

E.g.: In tossing a coin the events head or tail are mutually exclusive, since both tail & head cannot appear in the same time.

Exhaustive Events:

The total number of possible outcomes in any trail is known as exhaustive events.

E.g.: In throwing a die the possible outcomes are getting 1 or 2 or 3 or 4 or 5 or 6. Hence we have 6 exhaustive events in throwing a die.

Independent and Dependent Events:

Two events are said to be independent when the actual happening of one does not influence in any way the happening of the other.

Events which are not independent are called dependent events.

E.g.: If we draw a card in a pack of well shuffled cards and again draw a card from the rest of pack of cards (containing 51 cards), then the second draw is dependent on the first. But if on the other hand, we draw a second card from the pack by replacing the first card drawn, the second draw is known as independent of the first.

Complementary Events:

Event A is complementary event of B, if A and B are mutually exclusive and exhaustive.

E.g.: In throwing a die occurrence of even number (2, 4, 6) and odd number (1, 3, 5) are complementary event.

Definition of Probability:

If there are n equally likely, mutually exclusive and exhaustive outcomes and m of them are favourable to an event A, the probability of the happening of A is defined as the ratio m/n.

$$P(A) = \frac{m}{n} = \frac{number\ of\ \ favourable\ \ \ cases}{Total\ number\ of\ \ exhaustive\ \ \ cases}$$

Also $P(\bar{A}) = 1 - \frac{m}{n} = 1 - P(A)$

Where $(\bar{A})$ is that A does not happening.

Note:

1) If P (A) = 0, the event A is called an **impossible event**.

2) If P (A) = 1, the event A is called a **certain event.**

3) If 0 < P (A) < 1, the event A is called an **uncertain event.**

4) With each event A associated a real number between **0 and 1**. i.e. $0 \le P(A) \le 1$

5) P (**sample space**) = P(S) = 1.

6) The number of favourable events can be counted with the help of **permutations and Combinations.**

7) If n cases are favourable to A and m are favourable to $(\bar{A})$ then $P(A) = \frac{n}{n+m}$ i.e. we say that **"odds in favor of A are n: m."**

Probability Rules

- Addition Rule (for simultaneous trials)
  - Events are Mutually exclusive
  - Events are not mutually exclusive
- Multiplication Rule (for consecutive trials)
  - Events are independent
  - Events are dependant (conditional)
- Bayes Rule (probability from joint & Conditional)

Laws of Probability

1) Addition Law of Probability:

Case 1: When events are mutually exclusive: If A and B are mutually exclusive events, then

$$P(A\,or\,B) = P(A \cup B) = P(A) + P(B)$$

Example 1: The probability that a company director will travel by train is 1/5 and by plane is 2/3. What is the probability of his travelling by train or plane?

Answer: Let A be the event for travelling by train i.e. P (A) = 1 / 5 and B be the event for travelling by plane i.e. P (B) = 2 / 3.

The probability of his travelling by train or plane is,

$$P(A\,or\,B) = P(A \cup B) = P(A) + P(B) \qquad \text{Events are mutually exclusive}$$

$$P(A \cup B) = \frac{1}{5} + \frac{2}{3} = \frac{13}{15} = 0.8666$$

Case 2: When events are not mutually exclusive: If A and B are not mutually exclusive events, then $P(A\,or\,B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Note: And means intersection and or means union.

Example 2: If the probability is 0.3 that a teaching job applicant has P.G. degree, 0.7 for his work experience and 0.2 for both, out of 300 applicants, how many will have either a P.G. degree or work experience?

Answer: Let A be the event that the applicant has a P.G. degree i.e. P (A) = 0.3 and B be the event that the applicant has a work experience i.e. P (B) = 0.7. Then $P(A \cap B) = 0.2$

The probability that the applicant is P.G degree or work experience is,

$$P(A\,or\,B) = P(A \cup B) = P(A) + P(B) - P(A \cap B) \qquad \textit{Events are not mutually exclusive}$$

$$P(A \cup B) \; = 0.3 + 0.7 - 0.2 = 0.8$$

The required number of applications having P.G. degree or work experience = 300*0.8=240.

2) Multiplication Law of Probability:

Case 1: When events are independent: If A and B are independent events, then the probability that both will happen is the product of the probabilities of their happening.

$$P(A\,and\,B) \; = P(A \cap B) \; = P(A) \times P(B)$$

Example 3: A bag contains 7 golden and 4 violet balls .Two successive drawings of 3 balls are made such that balls are replaced before the second trial. Find the probability that the first drawing will give 3 golden and the second drawing will give 3 violet balls.

Answer: In a bag contains 7 golden and 4 violet balls i.e. 11 balls and 3 balls can be taken out of 11 balls in ${}^{11}C_3\ ways$

Let A be the event that the 3 golden balls are at the first draw, then number of favourable outcomes to the event A in ${}^{7}C_3\ ways$

$$P(A) = \frac{{}^{7}C_3}{{}^{11}C_3} = 0.2121$$

Let B be the event that the 3 violet balls are at the second draw, then number of favourable outcomes to the event B in ${}^{4}C_3\ ways$

$$P(B) = \frac{{}^{4}C_3}{{}^{11}C_3} = 0.0242$$

Since, the balls at the first draw are replaced before the second draw, total number of outcomes for both the draw is the same and the **events are independent.**

The probability that the first drawing will give 3 golden and the second drawing will give 3 violet balls is,

$$P(A\,and\,B) = P(A \cap B) = P(A) \times P(B) \qquad \textit{Events are independent}$$

$$P(A \cap B) \; = 0.2121 \times 0.0242 = 0.0051$$

Case 2: When events are dependent: The probability of event B depending on the occurrence of events A is called conditional probability, then the probability that both the dependant events A and B will occur is given by,

$$P(A\,and\,B)\ = P(A \cap B)\ = P(A) \times P(B/A)$$

Example 4: A bag contains 7 golden and 4 violet balls .Two successive drawings of 3 balls are made such that balls are not replaced before the second trial. Find the probability that the first drawing will give 3 golden and the second drawing will give 3 violet balls.

Answer: In a bag contains 7 golden and 4 violet balls i.e. 11 balls and 3 balls can be taken out of 11 balls in ${}^{11}C_3\ ways$

Let A be the event that the 3 golden balls are at the first draw, then number of favourable outcomes to the event A in ${}^{7}C_3\ ways$

$$P(A) = \frac{{}^{7}C_3}{{}^{11}C_3} = 0.2121$$

Let B be the event that the 3 violet balls are at the second draw, but here the balls drawn at the first draw are not replaced before the second draw. Hence the outcome of event B is depends on outcome of event A i.e. the **events are dependent.** The number of favourable outcomes to the event B is still ${}^{4}C_3\ ways$ but total number of outcomes at the second draw is ${}^{8}C_3\ ways$

$$P(B/A) = \frac{{}^{4}C_3}{{}^{8}C_3} = 0.0714$$

The probability that the first drawing will give 3 golden and the second drawing will give 3 violet balls is,

$$P(A\,and\,B) = P(A \cap B) = P(A) \times P(B/A) \qquad Events\ are\ dependent$$

$$P(A \cap B)\ = 0.2121 \times 0.0714 = 0.0151$$

<u>Examples</u>

Example 5: A can hit a target 2 times in 5 shots whereas B can 3 times in 4 shots. Find the probability of the target being hit when they both try.

Answer: Given that P (A) = 2/5 = 0.4 and P (B) = 3/4 = 0.75. Here A and B are not mutually exclusive but they are independent events. Hence $P(A \cap B)\ = 0.4 \times 0.75\ = 0.3$

The probability of the target being hit when they both try is,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) = 0.4 + 0.75 - 0.3 = 0.85$$

OR

The probability that A and B hit the target is 0.4 and 0.75 respectively. Probability that A and B will not hit the target is 0.6 and 0.25. As the **events are independent**, the probability that the target will not hit =0.6 x 0.25 = 0.15.

Therefore, the probability that the target will be hit = 1- 0.15 = 0.85

Example 6: A problem in statistics is given to five students A, B, C, D and E. Their chance of solving it are1/2, 1/3, 1/4, 1/4 and 1/6.What is the probability that the problem will be solved?

Answer:

The probability that A, B, C, D and E will solve the problem are 1/2, 1/3, 1/4, 1/4 and 1/6 respectively.

The probability that A, B, C, D and E will not solve the problem are 1/2, 2/3, 3/4, 3/4 and 5/6 respectively.

The probability that problem will not solve by all 5 students

= ½ x 2/3 x ¾ x ¾ x 5/6 = 0.1562

Hence, the probability that problem will solve = 1- 0.1562 = 0.8438.

Example 7: Find the probability of selecting a black card or a 6 from a deck of 52 cards.

Answer:

Let A be the event for selecting a black card i.e. P (A) = 26 / 52 and B be the event for selecting a 6 card i.e. P (B) = 4 / 52, Then $P(A \cap B) = 2/52$

The probability of selecting a black card or a 6 from a deck of 52 cards is,

$$P(A\, or\, B) = P(A \cup B) = P(A) + P(B) - P(A \cap B) \qquad \textit{Events are not mutually exclusive}$$

$$P(A \cup B) = \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = 0.5384$$

Example 8: When two dice are rolled, find the probability of getting a greater number on the first die than the one on the second, given that the sum should equal 8.

Answer: Two dice are rolled then total number of outcomes are = 6^2 = 36.

Let A be the event for getting a greater number on first die than the second one and B be the event for getting sum on two die is 8.

There are 5 ways to get a sum of 8 when two dice are rolled = {(2,6),(3,5),(4,4), (5,3),(6,2)}.

i.e. P (B) = 5 / 36,

And there are two ways where the number on the first die is greater than the one on the second given that the sum should equal 8, G = {(5, 3), (6, 2)}.

$$P(A \cap B) = 2/36$$

The probability of getting a greater number on the first dies than the one on the second, given that the sum should equal 8 is,

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{2/36}{5/36} = \frac{2}{5}$$

Example 9: What is the probability of drawing a king and a queen consecutively from a deck of 52 cards, without replacement?

Answer: Let A be the event for getting a king from a deck of 52 cards i.e. P (A) = 4 / 52

After drawing one card and not replaced in pack then the number of cards are 51.

Let B be the event for getting a queen from a deck of 51 cards i.e. P (B /A) = 4 / 51.

Now, the probability of drawing a king and queen consecutively is,

$$P(A\,and\,B) = P(A \cap B) = P(A) \times P(B/A) \qquad Events\ are\ dependent$$

$$P(A \cap B) = 4/52 \times 4/51 = 0.006$$

Example 10: What is the probability of the occurrence of a number that is odd or less than 5 when a fair die is rolled?

Answer: Let A be the event of the occurrence of a number odd i.e. P (A) = 3 / 6

Let B be the event of the occurrence of a number less than 5 i.e. P (B) = 4 / 6.

Then P (A and B) = $P(A \cap B) = 2/6$

Now, P (A or B) = P (A) + P (B) – P (A and B) = 3/6 + 4/6 – 2/6 = 5/6.

Example 11: Consider another example where a pack contains 4 blue, 2 red and 3 black pens. If a pen is drawn at random from the pack, replaced and the process repeated 2 more times, what is the probability of drawing 2 blue pens and 1 black pen?

Answer: Here, total number of pens = 4+2+3= 9

Let A be the event of the occurrence of a first blue pen i.e. P (A) = 4 / 9

Let B be the event of the occurrence of a second blue pen i.e. P (B) = 4 / 9

Let C be the event of the occurrence of a one black pen i.e. P (C) = 3 / 9

Three events are independent

Probability of drawing 2 blue pens and 1 black pen = P (A) x P (B) x P (C)

= 4/9 * 4/9 * 3/9 = 48/729

= 0.0658

Example 12: A bag contains 4 white and 2 black balls and second bag contains 3 of each color. A bag is selected at random and a ball is then drawn at random from the bag selected. What the probability that the ball drawn is white?

Answer: Here, There are two bags in first bag 4 white and 2 black balls and in second bag 3 white and 3black balls.

Let A be the event of selecting first bag i.e. P (A) = 1 / 2

Let B be the event of the white ball drawn from first bag i.e. P (B) = 4 / 6

Let C be the event of selecting first bag i.e. P (C) = 1 / 2

Let D be the event of the white ball drawn from second bag i.e. P (D) = 3 / 6

The probability that the ball drawn is white = {P (A) x P (B)} + {P (C) x P (D)}

= {1/2 * 4/6} + {1/2 * 3/6}

= 1/3 + 1/4 = 0.5833

Example 13: A can hit a target 4 times in 5 shots; B can 3 times in 4 shots, C twice in 3 shots. They fire a volley. What is the probability that at least two shots hit?

Answer: Given $P(A) = 4/5, P(B) = 3/4, P(C) = 2/3$

Then $P(\bar{A}) = 1/5, P(\bar{B}) = 1/4, P(\bar{C}) = 1/3$

The probability that A and B hit the target but C not hit the target $= P(A) \times P(B) \times P(\bar{C})$

$= 4/5 \times 3/4 \times 1/3$

$= 0.2$

The probability that A and C hit the target but B not hit the target $= P(A) \times P(\bar{B}) \times P(C)$

$= 4/5 \times 1/4 \times 2/3$

$= 0.1333$

The probability that B and C hit the target but A not hit the target $= P(\bar{A}) \times P(B) \times P(C)$

$= 1/5 \times 3/4 \times 2/3$

$= 0.1$

The probability that A and B and C hit the target $= P(A) \times P(B) \times P(C)$

$$= \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3}$$
$$= 0.4$$

The probability that at least two shots hit a target = 0.2 + 0.1333 + 0.1 + 0.4 = 0.8333

Example 14: Given P(A)=1 /4, P(B)=1/3, and P(AUB)=1/2, evaluate P(A/B), P(B/A), P(A∩B') and P(A/B')

Answer: Given P (A) =1/4, P (B) =1/3, P (AUB) =1/2

P (A∩B) = P (A) +P (B) - P (AUB)

=1/4 + 1/3 -1/2 =1/12

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/12}{1/3} = \frac{1}{4}$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/12}{1/4} = \frac{1}{3}$$

$$P(A \cap \overline{B}) = P(A) - P(A \cap B) = \frac{1}{4} - \frac{1}{12} = \frac{1}{6}$$

$$P(A/\overline{B}) = \frac{P(A \cap \overline{B})}{P(\overline{B})} = \frac{1/6}{2/3} = \frac{1}{4} \qquad P(\overline{B}) = 1 - P(B) = \frac{2}{3}$$

Examples for Practice

1) Define probability and state its laws.

2) The odds that book will be favorably reviewed by three independent critics are 5 to 2, 4 to 3, and 3 to 4 respectively. What is the probability that of the three reviews, a majority will be favorable?

3) A die is thrown twice and the sum of the numbers appearing is observed to be 6. What is the conditional probability that number 4 has appeared at least once?

4) If P (A) = 1/3, P (B) =1/2, P (A/B) =1/6 then find P (B/A) And P (B/Ā).

5) Four cards are drawn from a pack of cards. Find the probability that

1. All are diamonds 2. There is one card of each suit, and 3. There are two spades and two hearts.

6) The probability that a student passes a chemistry test is 2/3 and the probability that he passes both chemistry and microbiology test is 14/45. The probability that he passes at least one test is 4/5, what is the probability that he passes the microbiology test?

7) A husband and wife appear in an interview for two vacancies in the same post. The probability of husband’s selection is 1/7 and that of wife selection is 1/5 .what is probability that 1) both of them will be selected. 2) Only one of them will be selected. 3) None of them will be selected.

8) The students in a class are selected at random, one after the other, for an examination. Find the probability p that the boys and girls in the class alternate if 1. The class consists of 4 boys and 3 girls 2. The class consists of 3 boys and 3 girls

9) In a given race, the odds in favor of four horses A, B, C, D are 1:3, 1:4, 1:5, and 1:6 respectively. Assuming that a dead heat is impossible; find the chance that one of them wins the race.

10) In a box, there are 6 balls of which 3 are black and 3 are white. They are drawn successively 1) with replacement 2) without replacement. What is the chance that colors are alternate?

11) Three groups of children contain respectively 3 girls and 1 boy; 2 girls and 2 boys; 1 girl and 3 boys. One child is selected at random from each group. Find the chance of selecting 1 girl and 2 boys.

12) An urn contains 10 white and 3 black balls, while another urn contains 3 white and 5 black balls. Two balls are drawn from the first urn & put into the second urn and then a ball is drawn from the latter. What is the probability that it is a white ball?

13) A university has to select an examiner from list of 50 persons, 20 of them women and 30 men, 10 of them knowing Hindi and 40 not, 15 of them being teachers and the remaining 35 not. What is the probability of the University selecting a Hindi knowing women teacher?

14)Find the chance of drawing 2 white balls in succession from a bag containing 5 red & 7 white balls, the balls drawn not being replaced.

15) A toy is rejected if the design is faulty or not. The probability that the design is faulty is 0.1and that the toy is rejected if the design is faulty is 0.95and the otherwise 0.45. If a toy is rejected, what is the probability that it is due to faulty design?

16)70% of the males subjects injected with certain drug show improvement where as only 25% of the female show the improvement .If any one subject selected at random is injected with drug what is the probability that it will show improvement ?

17) A bag contains 4 red and 3 blue balls. Two drawings of 2 balls are made. Find the probability of drawing first 2 red balls and second 2 blue balls, (i) If the balls are returned to the bags after the first draw, (ii) If the balls are not returned after the first draw.

18) If two dice are thrown, find the probability that the sum is neither 6 nor 10?

19) A bag contains 6 white, 4 red and 10 black balls. Two balls are drawn at random. Find the probability that they will both be black.

20) An urn contains 10 white and 3 black balls, while another urn contains 3 white and 5 black balls; two balls are drawn from the latter. What is the probability that it is a white ball?

21) A box contains 3 red and 7 white balls. One ball is drawn at random and in its place a ball of other color is put in the box. Now one ball is drawn at random from the box. Find the probability that it is red.

22) In a class, 40% of the students study math and science. 60% of the students study math. What is the probability of a student studying science given he/she is already studying math? (0.67)

23) A box contains 4 chocobars and 4 ice creams. Tom eats 3 of them one after another. What is the probability of sequentially choosing 2 chocobars and 1 ice-cream? (1/7)

24) A bag contains 2 yellow, 3green and 2 blue balls. Two balls are drawn at random. What is the probability that none of the balls drawn is blue?

25) Two cards are drawn at random from a pack of 52 cards. What is the probability that either both are black or both are queen?

26) Two dice are thrown simultaneously. What is the probability of getting two numbers whose product is even?

<u>Baye's Theorem:</u>

Let $A_1, A_2, \ldots, A_n$ are mutually exclusive events with $P(A_i) \neq 0, i = 1,2,3\ldots..n$. Let B be any event from the same sample space, such that P (B) > 0. Then,

$$P(A_i / B) = \frac{P(A_i)\,P(B/A_i)}{\sum_{i=1}^{n} P(A_i)\,P(B/A_i)} \qquad \text{OR} \qquad P(A_i / B) = \frac{P(A_i)\,P(B/A_i)}{P(A)}$$

Example 15: An Example to develop Baye's Theorem:

This example concerns diagnose of disease by doctor and patient had died.

1) It is known that roughly the chances that a doctor will diagnose a disease correctly 60%.

Consider two events,

A_1 = Doctor diagnoses a correct disease.

A_2 = Doctor does not diagnoses a correct disease.

[A_1 and A_2 events are mutually exclusive]

$P(A_1) = 0.6$ and $P(A_2) = 0.4$

2) Suppose the chance that a patient will die after correct diagnosis is 40% and the chance of death by wrong diagnosis is 70 %.

Let B is event that patient will die.

P $(B/A_1) = 0.4$ and P $(B/A_2) = 0.7$

With these definitions consider the following probability questions.

1) What is the probability of a patient will die and correct diagnoses by doctor.

$P(A_1 \cap B) = P(A_1).\ P(B/A_1) = 0.6 \times 0.4 = 0.24$

2) What is the chance of a patient will die and doctor diagnoses wrongly.

$P(A_2 \cap B) = P(A_2).\ P(B/A_2) = 0.4 \times 0.7 = 0.28$

3) What is the chance of patient will die.

$P(B) = P(A_1 \cap B) + P(A_2 \cap B)$

$= 0.24 + 0.28 = 0.52$

Finally with the knowledge of patient had die find the probability that his disease diagnoses correctly by doctor?

$$P(A_1 / B) = \frac{P(A_1)\, P(B/A_1)}{\sum_{i=1}^{n} P(A_i)\, P(B/A_i)} = \frac{0.6 \times 0.4}{(0.6 \times 0.4) + (0.4 \times 0.7)} = 0.4615$$

Example 16: Arun speaks truth 4 out of 5 times. A die is tossed and Arun report that it is 6. What is the probability that actually there were 6?

Answer: Consider two events,

Let A_1 is the event Arun speaks truth.

Let A_2 is event Arun tells lie.

[A_1 and A_2 events are mutually exclusive]

$P(A_1) = 4/5 = 0.8$ and $P(A_2) = 1/5 = 0.2$

Let B is the event that getting 6 for rolling of die.

$P(B/A_1) = 1/6 = 0.1666$ and $P(B/A_2) = 5/6 = 0.8333$

The probability that actually there were 6 is,

$$P(A_1 / B) = \frac{P(A_1)\, P(B/A_1)}{\sum_{i=1}^{n} P(A_i)\, P(B/A_i)} = \frac{0.1666 \times 0.8}{(0.1666 \times 0.8) + (0.8333 \times 0.2)} = 0.4444$$

Example 17: The contents of Urn I, II and III are as follows,

Urn I: 1 white, 2 black, 3 red balls.

Urn II: 2 white, 1 black, 1 red ball.

Urn III: 4 white, 5 black, 3 red balls.

One urn is selected at random and two balls are drawn these are found to be one white and one red. What is the probability that the balls are drawn from urn second?

Answer: Consider three events,

Let A_1 is the event that the urn I is selected.

Let A_2 is the event that the urn II is selected.

Let A_3 is the event that the urn III is selected.

[A_1, A_2 and A_3 events are mutually exclusive

P (A_1) = 1/3 = 0.3333, P (A_2) = 1/3 = 0.3333, P (A_3) = 1/3 = 0.3333,

Let B is the event that two balls taken from the selected urn are white and red.

$$P(B/A_1) = \frac{{}^{1}C_1 \times {}^{3}C_1}{{}^{6}C_2} = \frac{3}{15} = 0.2 \qquad P(B/A_2) = \frac{{}^{2}C_1 \times {}^{1}C_1}{{}^{4}C_2} = \frac{2}{6} = 0.3333$$

$$P(B/A_3) = \frac{{}^{4}C_1 \times {}^{3}C_1}{{}^{12}C_2} = \frac{2}{11} = 0.1818$$

The probability that two balls are drawn from urn second is,

$$P(A_2/B) = \frac{P(A_2)\, P(B/A_2)}{\sum_{i=1}^{n} P(A_i)\, P(B/A_i)}$$

$$= \frac{0.3333 \times 0.3333}{(0.3333 \times 0.2) + (0.3333 \times 0.3333) + (0.3333 \times 0.1818)}$$

$$= 0.4660$$

Example 18: Marie is getting married tomorrow, at an outdoor ceremony in the desert. In recent years, it has rained only 5 days each year. Unfortunately, the weatherman has predicted rain for tomorrow. When it actually rains, the weatherman correctly forecasts rain 90% of the time. When it doesn't rain, he incorrectly forecasts rain 10% of the time. What is the probability that it will rain on the day of Marie's wedding?

Answer: Consider two events,

Let A_1 is the event that it rains on Marie’s wedding.

Let A_2 is the event that it does not rain on Marie’s wedding.

[A_1 and A_2 events are mutually exclusive]

P (A_1) = 5/365 = 0.0136 (It rains 5 days out of the year)

P (A_2) = 360/365 = 0.9863 (It does not rains 360 days out of the year)

Let B is the event that weatherman predicts rain.

P (B/A_1) = 0.9 (When it rains, the weatherman predicts rain 90 % of the time)

P (B/A_2) = 0.1 (When it does not rain, the weatherman predicts rain 10 % of the time)

We want to know P (A_1 / B), the probability it will rain on the day of Marie's wedding, given a forecast for rain by the weatherman. The answer can be determined from Bayes' theorem, as shown below.

$$P(A_1 / B) = \frac{P(A_1)\, P(B/A_1)}{\sum_{i=1}^{n} P(A_i)\, P(B/A_i)} = \frac{0.0136 \times 0.9}{(0.0136 \times 0.9) + (0.9863 \times 0.1)} = 0.1103$$

Note the somewhat unintuitive result. Even when the weatherman predicts rain, it rains only about 11% of the time. Despite the weatherman's gloomy prediction, there is a good chance that Marie will not get rained on at her wedding.

Examples for Practice

Example 1: There are three bags: first containing 1 white, 2 red, 3 green balls; second 2 white, 3 red, 1 green balls and third 3 white, 1 red, 2 green balls. Two balls are drawn from a bag chosen at random. These are found to be one white and one red. Find the probability that the balls are drawn from the second bag.

Example 2: In a bolt factory machines A, B, and C manufactures respectively 25, 35 and 40 percent of the total, of their output 5, 4 and 2 percent are defective bolts respectively. A bolt is selected at random from the product and is found to be defective. What is the probability that it was manufactured my machines C?

Example 3: In a bolt factory, there are four machines A, B, C, D manufacturing 20%, 15%, 25% and 40% of the total output respectively. Of their outputs 5%, 4%, 3% and 2% in the same order are defective bolts. A bolt is chosen at random from the factory's production and is found defective. What is the probability that the bolt was manufactured by machine B?

Example 4: In a factory manufacturing pens, machines X, Y and Z manufacture 20, 30 and 40 per cent of the total production of pens, respectively. Of their output 4, 5 and 10 per cent of the pens are defective. If one pen is selected at random, and is found to be defective, what is the probability that it is manufactured by machine Z?

Example 5: Two urns contain 4 white, 6 blue and 4 white, 5 blue balls respectively. One of urns is selected at random and a ball is drawn from it. If the ball drawn is white, find the probability that it is drawn from the i) first urn ii) second urn.

Example 6: A bag X contains 2 white and 3 red balls and a bag Y contains 4 white and 5 red balls. One ball is drawn at random from one bag and is found to be red. Find the probability that it was from bag Y.

Example 7: In a post office, three clerks are assigned to process incoming mail. The first clerk A, processes 40 per cent; the second Clerk B, processes 35 per cent; and the third clerk C, process 25 per cent of the mail. The first clerk has an error rate of 0.04, the second has an error rate of 0.06, and the third has an error rate of 0.03. A mail selected at random from a day's output is found to have an error. The postmaster wishes to know the probability that it was processed by clerk A?

Probability Distributions

Variable: A variable is a quantity that may change within the context of a mathematical problem or experiment. Typically, we use a single letter to represent a variable. The letters x, y, and z are common generic symbols used for variables. For Example, Height, Weight, Temperature etc.

Random Variables:

A ***variable***, used to denote the numerical value of the outcomes of an experiment is called **random variable.** There are two types of random variables, **discrete** and **continuous**.

Discrete Random Variables:

If a random variable can take only a finite number of distinct values, then it must be **discrete random variable**. Examples of discrete random variables include the number of children in a family, the Friday night attendance at a cinema, the number of patients in a doctor's surgery, the number of defective light bulbs in a box of ten.

Continuous Random Variables:

A **continuous random variable** is one which takes an infinite number of possible values. Continuous random variables are usually measurements. Examples include Age, height, weight, the amount of sugar in an orange, the time required to run a mile.

For simplicity, we usually referred to years, kilograms (or pounds) and centimeters (or feet and inches) for age, weight and height respectively. However, a 28-year-old man could actually be 28 years, 7 months, 16 days, 3 hours, 4 minutes, 5 seconds, 31 milliseconds, 9 nanoseconds old.

Probability Distributions: It is Relationship between values of a random Variable and the probability of their occurrence is called probability distributions.

For example tossing of a die then probability distribution is,

X	1	2	3	4	5	6
P	1/6	1/6	1/6	1/6	1/6	1/6

Discrete Probability Distributions: The **probability distribution** of a discrete random variable is a list of probabilities associated with each of its possible values. It is also sometimes called the **probability function** or the **probability mass function (p. m. f)**. Suppose a random variable X may take n different values, with the probability that $X = x_i$ defined to be $P(X = x_i) = p_i$. The probabilities p_i must satisfy the following:

1: $0 \le p_i \le 1$ for each i

2: $p_1 + p_2 + ... + p_n = 1$. i. e. $\Sigma p_i = 1$.

Mean and Variance of a discrete random variable.

Mean = $E(X) = \Sigma x_i p_i$

$E(X^2) = \Sigma x_i^2 p_i$

Variance = $E(X^2) – [E(X)]^2$

Example 1: A discrete random variable has the following probability distribution,

X	0	1	2	3	4	5	6
P(x)	k	3k	5k	7k	9k	11k	13k

i) Find the value of k, ii) Find $P(X > 3)$,

iii) Find $P(1 \le X < 4)$, iv) mean and variance of the distribution.

Answer: i) Given probability function is discrete probability function and it satisfies two properties,

1: $0 \le p \le 1$

2: $\Sigma p = 1$.

Hence, $k + 3k + 5k + 7k + 9k + 11k + 13k = 1$

$\therefore 49k = 1$

$\therefore k = 1/49$

$\therefore k = 0.0204$

Hence we can write the probability distribution,

X	0	1	2	3	4	5	6
P(x)	0.0204	0.0612	0.1020	0.1428	0.1836	0.2244	0.2652

ii) $P(X > 3) = P(X = 4) + P(X = 5) + P(X = 6)$

$= 0.1836+0.2244 + 0.2652 \quad = 0.6732$

iii) $P(1 \le X < 4) = P(X = 1) + P(X = 2) + P(X = 3)$

$= 0.0612 + 0.1020 + 0.1428$

$= 0.306$

iv) Mean $= E(X) = \Sigma x_i p_i$

$= (0 \times 0.0204) + (1 \times 0.0612) + (2 \times 0.1020) + (3 \times 0.1428) + (4 \times 0.1836) +$

$(5 \times 0.2244) + (6 \times 0.2652) = 4.1412$

$E(X^2) = \Sigma x_i^2 p_i$

$= (0 \times 0.0204) + (1^2 \times 0.0612) + (2^2 \times 0.1020) + (3^2 \times 0.1428) + (4^2 \times 0.1836) +$

$(5^2 \times 0.2244) + (6^2 \times 0.2652) = 19.8492$

Variance $= E(X^2) - [E(X)]^2$

$= 19.8492 - (4.1412)^2 = 2.6956.$

Example 2: A discrete random variable has the following probability distribution

X	1	2	3	4	5	6	7
P(x)	k	2k	2k	3k	k^2	$2k^2$	$7k^2+k$

i) Find the value of k, ii) $P(X > 2)$ iii) $P(1.5 < X < 4.5 / X > 2)$,

iv) The smallest value of α for which $P(X \le \alpha) > 0.5$

Answer: i) Given probability function is discrete probability function and it satisfies two properties,

1: $0 \le p \le 1$

2**:** $\Sigma p = 1$.

Hence, $k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$\therefore 10k^2 + 9k = 1$ $\therefore 10k^2 + 9k - 1 = 0$

$\therefore 10k^2 + 10k - k - 1 = 0$ $\therefore 10k(k + 1) - 1(k + 1) = 0$

$\therefore (10k - 1)(k + 1) = 0$

$\therefore k = 1/10 = 0.1$ and $k = -1$ (cannot possible because $0 \le p \le 1$)

Hence we can write the probability distribution,

X	1	2	3	4	5	6	7
P(x)	0.1	0.2	0.2	0.3	0.01	0.02	0.17

ii) $P(X > 2) = P(X = 3, 4, 5, 6, 7) = 1 - P(X = 1, 2)$

$= 1 - P(X = 1) - P(X = 2) = 1 - 0.1 - 0.2 = 0.7$

iii) $P(A / B) = P(A \cap B) / P(B)$

$\therefore P(1.5 < X < 4.5 / x > 2) = P(1.5 < X < 4.5 \cap x > 2) / P(x > 2)$

$= P(X = 3, 4) / P(X = 3, 4, 5, 6, 7)$

$= P(X = 3) + P(X = 4) / 0.7 = 0.5 / 0.7 = 0.7142$

iv) Now from the table we find that,

$P(X \leq 3) = P(X = 1) + P(X = 2) + P(X = 3) = 0.1 + 0.2 + 0.2 = 0.5$

Hence $P(X \leq 4) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) = 0.5 + 0.3 = 0.8 > 0.5$

Hence, $\alpha = 4$.

Example 3: A discrete random variable has the following probability distribution

X	1	2	3	4	5	6	7
P(x)	k	2k	3k	k^2	k^2+k	$2k^2$	$4k^2$

i) Find the value of k, ii) Find $P(X < 5)$ iii) $P(X < 5 / 2 < X \leq 6)$

Answer: i) Given probability function is discrete probability function and it satisfies two properties,

1: $0 \leq p \leq 1$

2**:** $\Sigma p = 1$.

Hence, $k + 2k + 3k + k^2 + k^2 + k + 2k^2 + 4k^2 = 1$

$\therefore 8k^2 + 7k = 1$ $\quad\therefore 8k^2 + 7k - 1 = 0$

$\therefore 8k^2 + 8k - k - 1 = 0$ $\quad\therefore 8k(k + 1) - 1(k + 1) = 0$

$\therefore (8k - 1)(k + 1) = 0$

$\therefore k = 1/8 = 0.125$ and $k = -1$ (cannot possible because $0 \leq p \leq 1$)

Hence we can write the probability distribution,

X	1	2	3	4	5	6	7
P(x)	1/8	2/8	3/8	1/64	9/64	2/64	4/64

ii) $P(X < 5) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) = 1/8 + 2/8 + 3/8 + 1/64$

$= 49/64 = 0.7656$

iii) $P(X < 5 / 2 < X \leq 6) = P(X < 5 \cap 2 < X \leq 6) / P(2 < X \leq 6)$

$= P(X = 3, 4) / P(X = 3, 4, 5, 6,)$

$= (3/8 + 1/64) / (3/8 + 1/64 + 9/64 + 2/64) = 0.6944$

Examples for Practice

Example 1: A discrete random variable has the following probability distribution

X	0	1	2	3	4	5	6
P(x)	k	2k	3k	4k	5k	6k	7k

i) Find the value of k, ii) Find $P(X > 2)$,

ii) Find $P(2 \leq X < 4)$, iv) mean and variance of the distribution.

Example 2: A random variable x has the following probability distributions

X	0	1	2	3	4	5	6	7	8
P(x)	a	3a	5a	7a	9a	11a	13a	15a	17a

Determine (i) the value of a, ii) Find $P(X < 3)$ iii) $P(X < 5 / 4 < X \leq 8)$

iv)The largest value of x for which $P(X \geq x) < (1/2)$.

Example 3: A random variable x has the following probability distributions

X	-2	-1	0	1	2	3
P(x)	0.1	k	0.2	2k	0.3	k

Find (i) the value of k (ii) mean (iii) variance

(iv) $P(X \geq 2)$ (v) $P(X<2)$ (vi) $P(-1<X<3)$.

Continuous Probability Distributions: The **probability distribution** of a continuous random variable is defined over an interval of values, and is represented by the **area under a curve** .The curve, which represents a function f(x), must satisfy the following:

1: The curve has no negative values ($f(x) \geq 0$ for all x)

2: The total area under the curve is equal to 1. i. e. $\int_{-\infty}^{\infty} f(x)\, dx = 1$

A curve meeting these requirements is known as a **density curve** or the **probability density function (p. d. f)**.

Mean and Variance of a continuous random variable.

$$Mean = E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2\, f(x)\, dx$$

$$Variance = E(x^2) - [E(x)]^2$$

Example 1: The diameter of an electric cable is assumed to be continuous random variate with probability density function:

$f(x) = k(1 - x^2)$, $0 < x < 1$

$= 0$, otherwise

i) Find the value of k. ii) Find $P(x > 0.5)$ iii) $P(0.1 \leq X < 0.2)$

Answer: i) Given probability function is continuous probability function and it satisfies two properties,

1: $f(x) \geq 0$ for all x

2: $\int_{-\infty}^{\infty} f(x)\, dx = 1$

Hence, $\int_{0}^{1} f(x)\, dx = k\int_{0}^{1}(1 - x^2)\, dx = k\left[x - \frac{x^3}{3}\right]_0^1 = k\left[1 - \frac{1}{3}\right] = k\,\frac{2}{3}$

But this must be equal to 1 $\therefore 2k/3 = 1 \therefore k = 3/2$

Hence, $f(x) = 3/2(1 - x^2)$, $0 < x < 1$

ii) $P(x > 0.5) = \int_{0.5}^{1} f(x)\, dx = \frac{3}{2}\int_{0.5}^{1}(1 - x^2)\, dx = \frac{3}{2}\left[x - \frac{x^3}{3}\right]_{0.5}^{1}$

$$= \frac{3}{2}\left[\left\{1-\frac{1}{3}\right\}-\left\{0.5-\frac{(0.5)^3}{3}\right\}\right]_{0.5}^{1} \quad = \frac{3}{2}(0.6666-0.4533) \quad = 0.3199$$

iii) $P(0.1 \le x < 0.2) = \int_{0.1}^{0.2} f(x)\, dx \quad = \frac{3}{2}\int_{0.1}^{0.2} (1-x^2)\, dx \quad = \frac{3}{2}\left[x-\frac{x^3}{3}\right]_{0.1}^{0.2}$

$$= \frac{3}{2}\left[\left\{0..2-\frac{(0.2)^3}{3}\right\}-\left\{0.1-\frac{(0.1)^3}{3}\right\}\right]_{0.1}^{0.2} = \frac{3}{2}(0.1973-0.0996) \quad = 0.1465$$

Example 2: Following is the probability density function of a random variable x.

$$f(x) = 6x(1- x)\ ,\ 0 < x < 1$$
$$= \quad 0 \qquad , \text{otherwise}$$

i) Verify that above is a probability density function.

ii) Find P(x < 0.9)

iii) Find the mean and variance of the function.

Answer: i) The function is probability density function it satisfies two properties,

1: f(x) $\geq$ 0 for all x

2: $\int_{-\infty}^{\infty} f(x)\, dx = 1$

Here for $0 < x < 1$, f (x) $\geq$ 0

Also, $\int_{0}^{1} f(x)\, dx = 1 \quad \therefore 6\int_{0}^{1}(x-x^2)\, dx \quad = 6\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1 \quad = 6\left[\frac{1}{2}-\frac{1}{3}\right] \quad = 1$

i.e. given function is a probability density function.

ii) $P(x < 0.9) = \int_{0}^{0.9} f(x)\, dx \quad \therefore 6\int_{0}^{0.9} (x-x^2)\, dx \quad = 6\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^{0.9} \quad = 6\left(\frac{0.9^2}{2}-\frac{0.9^3}{3}\right) \quad = 0.972$

iii) Mean and variance of the function,

$$Mean = E(x) = \int_{-\infty}^{\infty} x\ f(x)\ dx$$

$$= \int_{0}^{1} x\ f(x)\, dx \quad = 6\int_{0}^{1}(x^2 - x^3)\, dx \quad = 6\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1 \quad = 6\left[\frac{1}{3}-\frac{1}{4}\right] \quad = 0.5$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 \, f(x)\, dx$$

$$= \int_0^1 x^2 f(x)\, dx \quad = 6\int_0^1 (x^3 - x^4)\, dx \quad = 6\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 \quad = 6\left[\frac{1}{4} - \frac{1}{5}\right] \quad = 0.3$$

$$Variance = E(x^2) - [E(x)]^2 \quad = (0.5) - (0.3)^2 \quad = 0.41$$

Example 3: Following is the probability density function of a random variable x.

$$f(x) = \begin{cases} kx, & 0 \le x \le 2, \\ 2k, & 2 \le x \le 3, \\ 5k - kx, & 3 \le x \le 4, \end{cases}$$

Find the value of k.

Answer: Given probability function is continuous probability function and it satisfies two properties,

1: f(x) $\geq$ 0 for all x

2: $\int_{-\infty}^{\infty} f(x)\, dx = 1$

Hence, $\int_0^4 f(x)\, dx \quad = k\int_0^2 x\, dx + 2k\int_2^3 dx + k\int_3^4 (5-x)dx$

$$= k\left[\frac{x^2}{2}\right]_0^2 + 2k[x]_2^3 + k\left[5x - \frac{x^2}{2}\right]_3^4$$

$$= 2k + 2k(3-2) + k\left[5(4-3) - (8 - \frac{9}{2})\right]$$

$$= 2k + 2k + k\left[5 - \frac{7}{2}\right] \quad = 2k + 2k + k\left[\frac{3}{2}\right] \quad = k\left[\frac{11}{2}\right]$$

But this must be equal to 1 $\quad \therefore k\left[\frac{11}{2}\right] = 1 \quad \therefore k = \frac{2}{11}$

Example 4: Following is the probability density function of a random variable x.

f(x) = k x^4 . $e^{-x/2}$ x > 0 Find the value of k.

Answer: i) Given probability function is continuous probability function and it satisfies two properties, 1: f(x) $\geq$ 0 for all x   2: $\int_{-\infty}^{\infty} f(x)\, dx = 1$

Hence, $\int_0^{\infty} f(x)\, dx = k\int_0^{\infty} x^4 e^{-x/2}\, dx$ $\therefore$ put $\frac{x}{2} = t$ $\therefore x = 2t \Rightarrow dx = 2dt$ $\therefore x = 0$ then $t = 0$, $x = \infty$ then $t = \infty$

$$= k\int_0^{\infty} (2t)^4 e^{-t}\, 2dt = 32k\int_0^{\infty} e^{-t} t^4\, dt = 32k\, \Gamma 5 = 32k\,(4!) = 768k$$

But this must be equal to 1 $\therefore 768k = 1$ $\therefore k = \frac{1}{768}$

Examples for Practice

1) Following is the probability density function of a random variable x.

$$f(x) = \begin{cases} ax, & 0 \le x \le 1, \\ a, & 1 \le x \le 2, \\ -ax+3a, & 2 \le x \le 3, \end{cases}$$ find the value of k.   (Ans: 0.5)

2) Following is the probability density function of a random variable *x*.

$f(x) = k.\, x^{-1/2}$, $4 < x < 16$

= 0   , otherwise

i) Find the value of *k*. ii) Find P($x \leq 7$) iii) mean and variance.

3) Following is the probability density function of a random variable x.

f(x) = k $\frac{1}{1+x^2}$, $-\infty < x < \infty$ Find the value of k. (Ans: $= \frac{1}{\pi}$)

4) Following is the probability density function of a random variable x.

f(x) = k e^{-2x} , x > 0   Find the value of k.

5) Following is the probability density function of a random variable x.

f(x) = $kx^2(1-x^3)$ 0 < x <1

i) Find the value of k ii) find P (X < 0.9)

6) Verify the following function is probability density function or not?

$i)\, f(x) = \frac{1}{2}e^{-|x|} \quad -\infty < x < \infty \qquad ii)\, f(x) = \frac{2}{9}\left(2 - \frac{x}{2}\right) \quad 0 < x < 3$

Binomial Distribution

Binomial Experiment:

A **binomial** experiment is one that possesses the following properties:

1) The experiment consists of ***n*** repeated trials;

2) Each trial results in an outcome that may be classified as a **success** or a **failure**; the probability of a success, denoted by ***p*,** remains constant from trial to trial and probability of a failure, denoted by ***q***, **(i.e. *p + q = 1*)**

3) Repeated trials are **independent**.

Binomial Distribution:

The number of successes r in n trials of a binomial experiment is called a **binomial random variable** and the probability distribution of such random variable X is called a **binomial distribution**, and is given by the formula,

$$P(X = r) = {}^nC_r \; p^r \; q^{n-r} \qquad where, r = 0\,, 1\,, 2\,, ..., n$$

Note: There are two parameters in a binomial distribution, namely ***n*** and ***p***.

Mean and Variance of Binomial Distribution:

Mean = ***np*** Variance = ***npq***

Example 1: If the probability of defective bulbs is 0.2, find the mean and variance of defective bulbs in a lot of 1000 bulbs.

Answer: Let X is random variable i.e. number of defective bulbs.

Given, p = probability of defective bulbs = 0.2

i.e. q = 1– p = 1– 0.2 = 0.8 and

n = 1000

X ~ B (n =1000, p = 0.2)

In binomial distribution,

Mean = np = (1000) (0.2) = 200

Variance = npq = (1000) (0.2) (0.8) = 160

Example 2: The mean and variance of a binomial distribution are 16 and 8. Find its probability function.

Answer: Let X is random variable and it follows binomial distribution,

$$X \sim B\,(n, p)$$

Given, Mean = np = 16

Variance = npq = 8

By solving we get q = 0.5 and p = 0.5 and n=32

$$X \sim B\,(n=32,\ p = 0.5)$$

The probability mass function (p. m. f) is,

$$P(X=r) = {}^{32}C_r\,(0.5)^r\,(0.5)^{32-r} \qquad where,\ r = 0, 1, 2, \ldots\ldots, 32$$

Example 3: If the mean of the binomial distribution is 2 and variance is 4/3. Find the probability of i) exactly 2 success ii) less than 2 successes.

Answer: Let X is random variable and it follows binomial distribution,

$$X \sim B\,(n, p)$$

Given, Mean = np = 2 Variance = npq = 4/3

By solving we get q = 2/3 and p = 1/3 and n=6

$$X \sim B\,(n=6,\ p = 1/3)$$

The probability mass function (p. m. f) is,

$$P(X=r) = {}^{6}C_r\left(\frac{1}{3}\right)^r\left(\frac{2}{3}\right)^{6-r} \qquad where,\ r = 0, 1, 2, \ldots\ldots, 6$$

i) The probability of exactly 2 successes,

$$P(X=2) = {}^{6}C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^{4} = 0.3292$$

ii) The probability of less than 2 successes,

$$P(X<2) = P(X=0) + P(X=1)$$

Therefore, $P(X=0) = {}^{6}C_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^{6} = 0.0877$ $\qquad P(X=1) = {}^{6}C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^{5} = 0.2633$

$$\therefore P(X<2) = 0.0877 + 0.2633 = 0.3510$$

Example 4: 10% bolts produced by a machine are defective. Calculate the probability that out of a sample selected at random of 10 bolts, i) at most 3 bolts are defective. ii) at least 2 bolts are defective.

Answer: Let X is random variable i.e. number of defective bolts.

Given, p = probability of defective bolts = 0.1

i.e. q = 1– p = 1– 0.1 = 0.9 and

n = 10

$$X \sim B\ (n = 10,\ p = 0.1)$$

The probability mass function (p. m. f) is,

$$P(X = r) = {}^{10}C_r\ (0.1)^r\,(0.9)^{10-r} \qquad where,\ r = 0, 1, 2, ..., 10$$

i) The probability of at most 3 bolts is defective, (not more than 3)

$$P(X \le 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$

Therefore, $P(X = 0) = {}^{10}C_0\ (0.1)^0 (0.9)^{10} \ = 0.3487$

$$P(X = 1)\ = {}^{10}C_1\ (0.1)^1 (0.9)^9 \ = 0.3874$$

$$P(X = 2) = {}^{10}C_2\ (0.1)^2 (0.9)^8 \ = 0.1937$$

$$P(X = 3) = {}^{10}C_3\ (0.1)^3 (0.9)^7 \ = 0.0573$$

$$\therefore P(X \le 3) = 0.3487 + 0.3874 + 0.1937 + 0.0573 \ = 0.9871$$

ii) The probability of at least 2 bolts are defectives,

$$P(X \ge 2) = 1 - \left(P(X = 0) + P(X = 1)\right) \ = 1 - (0.3487 + 0.3874) \ = 0.2639$$

Example 5: The probability that a bomb dropped from a plane will strike the target is 1/5. If six bombs are dropped, find the probability that (i) exactly two will strike the target, (ii) at least two will strike the target.

Answer: Let X is random variable i.e. number of bombs hit the target.

Given, p = probability of bombs hit the target = 1/5 =0.2

i.e. q = 1– p = 1– 0.2 = 0.8 and n = 6

$$X \sim B\ (n = 6,\ p = 0.2)$$

The probability mass function (p. m. f) is,

$$P(X = r) = {}^{6}C_r\ (0.2)^r\,(0.8)^{6-r} \qquad where,\ r = 0, 1, 2, ..., 6$$

i) The probability of exactly 2 bombs hit the target,

$$P(X=2)={}^6C_2\,(0.2)^2(0.8)^4 \quad =0.2458$$

ii) The probability of at least 2 bombs hit the target,

$$P(X\ge 2)=1-\left(P(X=0)+P(X=1)\right)$$

Therefore, $P(X=0)={}^6C_0\,(0.2)^0(0.8)^6 \;=0.2621$

$$P(X=1)={}^6C_1\,(0.2)^1(0.8)^5 \;=0.3932$$

$$P(X\ge 2)=1-\left(P(X=0)+P(X=1)\right)$$

$$P(X\ge 2)=1-\left(0.2621+0.3932\right) \quad =0.3447$$

Example 6: Fit a binomial distribution to following data:

x	0	1	2	3	4	5	6
f	5	18	28	12	7	6	4

Answer: Let X is random variable,

To fit binomial distribution we required parameters of binomial distribution,

Here, n = 6 (No of trials)

To find p: N = ∑f = 80 and ∑fx = 192

In discrete distribution, $Mean=\dfrac{\sum fx}{\sum f}=\dfrac{192}{80}=2.4$

But in binomial distribution, $Mean=np$

$$6p=2.4 \qquad \Rightarrow p=\frac{2.4}{6}=0.4 \qquad q=1-p=0.6$$

X ~ B (n =6, p = 0.4)

The probability mass function (p. m. f) is,

$$P(X=r)={}^6C_r\,(0.4)^r(0.6)^{6-r} \qquad where,\, r=0,1,2,......,6$$

Expected frequencies = N X P(X = r)

$$E(X=r)=N\times P(X=r)=80\times {}^6C_r\,(0.4)^r(0.6)^{6-r}$$

$$E(X=0)=80\times {}^6C_0\,(0.4)^0(0.6)^6=80\times 0.0466=3.7324\approx 4$$

$$E(X=1)=80\times {}^6C_1\,(0.4)^1(0.6)^5=80\times 0.1866=14.9299\approx 15$$

$$E(X=2)=80\times {}^6C_2\,(0.4)^2(0.6)^4=80\times 0.3110=24.8832\approx 25$$

$$E(X=3)=80\times {}^6C_3\,(0.4)^3(0.6)^3=80\times 0.2764=22.1184\approx 22$$

$$E(X=4)=80\times {}^6C_4\,(0.4)^4(0.6)^2=80\times 0.1382=11.0592\approx 11$$

$$E(X=5)=80\times {}^6C_5\,(0.4)^5(0.6)^1=80\times 0.0368=2.9491\approx 3$$

$$E(X=6)=80\times {}^6C_6\,(0.4)^6(0.6)^0=80\times 0.0040=0.3276\approx 0$$

Hence, binomial distribution is,

x	0	1	2	3	4	5	6
f	4	15	25	22	11	3	0

Example 7: Assuming that 80% the population is the literate so that the chance of an individual being literate is 4/5 and assuming that 100 investigators can take a sample of 10 individuals to see whether they are illiterates, how many investigators would you expect to report that three or less were illiterate?

Answer: Let X is random variable i.e. number of illiterate individuals.

Given, q = probability of individuals being literate = 0.8

i.e. p = 1– q= 1– 0.8 = 0.2 and

n = 10 and N=100 , X ~ B (n =10, p = 0.2)

The probability mass function (p. m. f) is,

$$P(X=r)={}^{10}C_r\,(0.2)^r(0.8)^{10-r}\quad where,\ r=0,1,2,......,10$$

i) The probability that 3 or less were illiterate individuals,

$$P(X\le 3)=P(X=0)+P(X=1)+P(X=2)+P(X=3)$$

Therefore, $P(X=0)={}^{10}C_0\,(0.2)^0(0.8)^{10}=0.1073$

$P(X=1)={}^{10}C_1\,(0.2)^1(0.8)^9=0.2684$ $P(X=2)={}^{10}C_2\,(0.2)^2(0.8)^8=0.3019$

$P(X=3)={}^{10}C_3\,(0.2)^3(0.8)^7=0.2013$

$$\therefore P(X\le 3)=0.1073+0.2684+0.3019+0.2013=0.8789$$

Expected number of investigators having 3 or less illiterate individuals = 100*0.8789 = 87.89 ≈ 88

Examples for Practice

Example 1: In 100 sets of 10 tosses of a unbiased coin, in how many cases do you expect (i) 7 heads, (ii) at least 7 head.

Example 2: The probability that on, joining Engineering College, a student will successfully complete the course of studies is 3/5. Determine the probability that out of 5 students joining the college (i) none, (ii) all 5 and (iii) at least two will complete the course successfully.

Example 3: The incidence of occupational disease in an industry is such that the workmen have a 10% chance of suffering from it. Find the probability that in a group of 7, (i) exactly three and (ii) five or more will suffer from it.

Example 4: If the sum of mean and variance of a Binomial distribution for 5 trials is 4.8 find the distribution.

Example 5: During war, one ship out of 9 was sunk on an average in making certain voyage. What is the probability that exactly 3 out of a convoy of 6 ships would arrive safely?

Example 6: 1380 welded joints produced by a certain type of welding machine have 276 defective joints. If 8 of these joints are selected for inspection, what is the probability that there are (i) no defectives (ii) not more than 2 defective joints?

Example 7: The probability that a man hits the target is1/3. How many times must he fire so that the probability of hitting the target at least once is more than 90%?

Example 8: If mean and variance of a binomial distribution are 4 and 2 respectively. Find the probability of i) exactly 2 success ii) less than 2 success.

Example 9: Six dice are thrown 729 times. How many times do you expect at least 3 dice to show a 5 or 6?

Example 10: Let X be a binomial random variable with parameter 5 and 0.7 compute

1) $P(X = 2)$ 2) $P(X \leq 5)$ 3) $P(X > 2)$

Example 11: Define binomial distribution and give any two examples where such a distribution is ideally suited.

Example 12: Define binomial distribution. If $n=10$ and $p=1/5$ then find i) $p(x=0)$ ii) $p(x=2)$.

Example 13: It is known from the experience that 2% of the injection needles produced by a company are defective. What is the probability that a sample of 10 needles will be free from defective needles?

Example 14: Seeds of certain plant have 60% germination rate. Calculate the probability when 8 of these seeds are planted, 6 or more will germinate.

Example 15: If 10% of people suffer from certain disorder, what is the probability that in a sample of 15 people 1) Exactly 3 people will have disorder? 2) At least 2 people will have disorder? 3) At the most two people will have disorder?

Example 16: In a sampling the mean number of defective bolts manufactured by a machine in a sample of 20 is 2. Determine the expected number of samples out of such 500 samples to contain at least 2 defective bolts.

Example 17: Assume that on the average one telephone number out of fifteen called between 2p.m. and 3p.m. on week days is busy. What is the probability that if 6 randomly selected telephone numbers are called i) not more than three ii) at least three of them will be busy?

Example 18: Out of 800 families with 4 children each, how many families would be expected to have 1) at least one boy, 2) at most two girls?

Example 19: The probability that a man aged 60 will live up to 70 is 0.65. What is the probability that out of 10 men now 60, at least 7 would live to be 70?

Example 20: Find the probability that in 10 tosses of a fair coin; a head appears 1) At no time 2) At 5 times 3) at 10 times.

Example 21: The probability that a boy will get scholarship is 0.9 and that a girl will get 0.8. What is the probability that at least one of them will get scholarship?

Example 22: If 10% of patients show certain side effect to the drug, what is the probability that out of 25 patients injected with this drug at least 23 will not show side effect.

Example 23: Fit the binomial distribution to the following data of number of boys observed in 200 samples of 5 children each:

No. of Boys :	0	1	2	3	4	5
No. of samples:	90	52	33	11	8	6

Example 24: Fit binomial distribution to following data.

x	0	1	2	3	4	5
f	2	14	20	34	22	8

Example 25: Fit binomial distribution to following data

x	0	1	2	3	4	5	6
f	6	20	28	12	8	6	0

Example 26: A set of 8 coins were tossed 256 times to produce the following distribution,

Number of heads	0	1	2	3	4	5	6	7	8
Frequency observed	2	6	24	63	64	50	36	10	1

Fit a binomial distribution i) if the coin is unbiased. ii) If the coin is biased.

Example 27: Fit binomial distribution to following data.

x	0	1	2	3	4	5	6
f	13	25	52	58	32	16	4

Example 28: With the usual notations find the probability of success for binomial random variable if n = 6 and 9P(X = 4) = P(X = 2).

Poisson distribution

Poisson distribution is a limiting case of binomial distribution under the following;

1) The number of **independent trials** (***n***) is indefinitely large, $i.e.\ n \to \infty$
2) The probability of a **success (*p*)** is very small, $i.e.\ p \to 0$
3) The average success is finite say λ, $i.e.\ np = \lambda$.

Poisson distribution:

Poisson experiment gives numerical values of a random variable X which represent the number of outcomes occurring during a given time interval and the probability distribution of such random variable *X* is called a **Poisson distribution**, and is given by the formula,

$$P(X = r) = \frac{e^{-\lambda}\lambda^{r}}{r\,!} \quad where\ \lambda > 0,\ and\ r = 0, 1, 2, ...$$

$$In\ notation\ X \sim P(\lambda)$$

Note: There is only one parameter in a Poisson distribution namely λ.

Mean and Variance of Poisson distribution: $mean = \lambda \qquad variance = \lambda$

A few Examples of Poisson variate are given below:

1) The number of defective articles in a box of 200 units.
2) The number of breakdowns of a printing machine in a day.
3) The number of deaths due to accidents in a month on national highway 4.
4) The number of telephone calls received at a particular office.
5) The number of cars passing a particular point on a road during a period of time.
6) The number of printing mistakes on a page of a book.
7) The number of deaths due to a disease such as COVID -19 etc.

Example 1: If a random variable is Poisson such that 3P(*X*=1) = 2*P*(*X*=2). Find i) the probability distribution of variable X ii) mean and iii) *P*(*X*=1)?

Answer: Let X is Poisson random variable and it's the probability mass function (p. m. f) is,

$$P(X = r) = \frac{e^{-\lambda}\lambda^{r}}{r\,!} \quad where\ \lambda > 0,\ and\ r = 0, 1, 2, ...$$

Given that, $3P(X = 1) = 2P(X = 2)$.

$$\therefore 3\frac{e^{-\lambda}\lambda}{1!}=2\frac{e^{-\lambda}\lambda^2}{2!}$$

$$\Rightarrow \lambda = 3$$

$$i.e.\ \ X \sim P(\lambda = 3)$$

i) The probability mass function (p. m. f) is,

$$P(X=r)=\frac{e^{-3}(3)^r}{r\,!}\quad ,r=0\,,1\,,2\,,......$$

ii) The mean of Poisson distribution is λ i.e. $mean = 3$

iii) P(X = 1), $P(X=1)=\frac{e^{-3}(3)^1}{1!}\quad =0.1493$

Example 2: If the probability that an individual suffers a bad reaction from injection of a give serum is 0.001. Determine the probability that out of 2000 individuals i) Exactly 2 ii) at least 2 iii) at most 2 will suffer a bad reaction.

Answer: Let X is random variable i.e. number of individual suffers a bad reaction from injection,

Given, p = probability of individual suffers a bad reaction from injection = 0.001 and

n = 2000

Here, $n \to \infty$ and $p \to 0$ then, $\lambda = np$.

$\therefore \lambda = 2000 \times 0.001 = 2 \quad i.e.\ \ X \sim P(\lambda = 2)$

The probability mass function (p. m. f) is,

$$P(X=r)=\frac{e^{-2}(2)^r}{r\,!}\quad ,r=0\,,1\,,2\,,......$$

i) The probability of exactly 2 individual suffers a bad reaction from injection,

$$P(X=2)=\frac{e^{-2}(2)^2}{2\,!}\quad =0.2706$$

ii) The probability of at least 2 individual suffers a bad reaction from injection,

$$P(X\geq 2)=1-\big(P(X=0)+P(X=1)\big)$$

Therefore, $P(X=0)=\frac{e^{-2}(2)^0}{0\,!}\quad =0.1353 \qquad P(X=1)=\frac{e^{-2}(2)^1}{1!}\quad =0.2706$

$$P(X\geq 2)=1-\big(0.1353+0.2706\big)\quad =0.5941$$

iii) The probability of at most 2 individual suffers a bad reaction from injection,

$$P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$P(X \le 2) = 0.1353 + 0.2706 + 0.2706 = 0.6765$$

Example 3: A manufacturer who produces medicine bottles finds that 0.1 % of the bottles are defective. The bottles are packed in boxes containing 500 bottles. A drug manufacturer purchases such 100 boxes from the producer of bottles. Find how many boxes will contain

i) no defective ii) at least 2 defective bottles.

Answer: Let X is random variable i.e. number of defective bottles in a box.

Given, p = probability of defective bottles in a box. = 0.001 and

n = 500

Here, $n \to \infty$ and $p \to 0$ then, $\lambda = np$.

$\therefore \lambda = 500 \times 0.001 = 0.5$ *i.e.* $X \sim P(\lambda = 0.5)$

The probability mass function (p. m. f) is,

$$P(X = r) = \frac{e^{-0.5}(0.5)^r}{r!} \quad , r = 0, 1, 2, ...$$

i) The probability of no defective bottles in a box,

$$P(X = 0) = \frac{e^{-0.5}(0.5)^0}{0!} = 0.6065$$

Expected number of boxes containing no defective bottles=100*0.6065 =60.65≈ 61

ii) The probability of at least 2 defective bottles in a box,

$$P(X \ge 2) = 1 - \left(P(X = 0) + P(X = 1)\right)$$

Therefore, $P(X = 1) = \dfrac{e^{-0.5}(0.5)^1}{1!} = 0.3032$

$$P(X \ge 2) = 1 - \left(0.6065 + 0.3032\right) = 0.0903$$

Expected number of boxes containing at least 2 defective bottles=100*0.0903 =9.03≈ 9

Example 4: A book of 480 pages contains 480 misprints. The number of misprints in a page is having Poisson distribution. Find the probability that a page contains exactly 3misprints.

Answer: Let X is random variable i.e. number of misprints in a page.

Given, Number of misprints in a book = 480 and n = 480

Average number of misprints in a page =480/480 = 1.

$$\therefore \lambda = 1 \quad i.e. \ X \sim P(\lambda = 1)$$

The probability mass function (p. m. f) is,

$$P(X = r) = \frac{e^{-1}(1)^r}{r\,!} \quad , r = 0, 1, 2, ...$$

The probability of exactly 3 misprints in a page,

$$P(X = 3) = \frac{e^{-1}(1)^3}{3\,!} \quad = 0.0613$$

Example 5: Between 2 and 4 P. M. the average of phone calls per minute coming into the switch board of a company is 2.5. Use Poisson distribution to find the probability that during one particular minute there will be (i) exactly 3 calls. (ii) At most 3 call.

Answer: Let X is random variable i.e. number of phone calls per minute coming into the switch board of a company,

Given, Average number of phone calls per minute coming into the switch board of a company = 2.5.

$$\therefore \lambda = 2.5 \quad i.e. \ X \sim P(\lambda = 2.5)$$

The probability mass function (p. m. f) is,

$$P(X = r) = \frac{e^{-2.5}(2.5)^r}{r\,!} \quad , r = 0, 1, 2, ...$$

i) The probability of exactly 3 phone calls,

$$P(X = 3) = \frac{e^{-2.5}(2.5)^3}{3\,!} \quad = 0.2137$$

ii) The probability of at most 3 phone calls,

$$P(X \le 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$

$$P(X = 0) = \frac{e^{-2.5}(2.5)^0}{0\,!} \quad = 0.0820 \qquad P(X = 1) = \frac{e^{-2.5}(2.5)^1}{1\,!} \quad = 0.2052$$

$$P(X = 2) = \frac{e^{-2.5}(2.5)^2}{2\,!} \quad = 0.2565$$

$$P(X \le 3) = 0.0820 + 0.2052 + 0.2565 + 0.2137 \quad = 0.7574$$

Example 6: Data was collected over a period of 10 years, showing number of deaths from horse kicks in each of the 20 army corps. The distribution of deaths was as follows:

Number of deaths (x)	0	1	2	3	4	Total
Frequency(f)	109	65	22	3	1	200

Fit a Poisson distribution to the data and calculate the theoretical frequencies

Answer: Let X is random variable, i.e. Number of deaths from horse kicks.

To fit Poisson distribution we required parameter of Poisson distribution,

To find λ : N = ∑f = 200 and ∑fx = 122

In discrete distribution, $Mean = \frac{\sum fx}{\sum f} = \frac{122}{200} = 0.61$

But in Poisson distribution, $Mean = \lambda \quad \Rightarrow \lambda = 0.61$

Hence,

$$i.e.\;\; X \sim P(\lambda = 0.61)$$

The probability mass function (p. m. f) is,

$$P(X = r) = \frac{e^{-0.61}(0.61)^r}{r\,!} \quad , r = 0, 1, 2, ...$$

Expected frequencies = N X P(X = r)

$$E(X = r) = N \times P(X = r) = 200 \times \frac{e^{-0.61}(0.61)^r}{r\,!}$$

$$E(X = 0) = 200 \times \frac{e^{-0.61}(0.61)^0}{0\,!} = 200 \times 0.5433 = 108.6702 \approx 109$$

$$E(X = 1) = 200 \times \frac{e^{-0.61}(0.61)^1}{1!} = 200 \times 0.3314 = 66.2881 \approx 66$$

$$E(X = 2) = 200 \times \frac{e^{-0.61}(0.61)^2}{2\,!} = 200 \times 0.1010 = 20.2180 \approx 20$$

$$E(X = 3) = 200 \times \frac{e^{-0.61}(0.61)^3}{3\,!} = 200 \times 0.0205 = 4.1110 \approx 4$$

$$E(X = 4) = 200 \times \frac{e^{-0.61}(0.61)^4}{4\,!} = 200 \times 0.0031 = 0.6269 \approx 1$$

Number of deaths (x)	0	1	2	3	4	Total
Frequency(f)	109	66	20	4	1	200

Examples for Practice

Example 1: Define Poisson distribution and write a short note on Poisson distribution.

Example 2: Suppose that X has Poisson distribution. If P(x=2) = 2/3 P(x=1), Find 1) P(x=0) 2) P(x=3).

Example 3: If X is a Poisson variate such that $P(x=2) = 9P(x=4) + 90P(x=6)$. Find mean of distribution and the probability for at most 2 successes.

Example 4: Using Poisson distribution find the probability that ace of spade will be drawn from a pack of well shuffled cards at least once in 104 consecutive draws.

Example 5: Find the probability that at most 4 defective bulbs will be found in a box of 400 bulbs if it is known that 1 % of the bulbs are defective.

Example 6: If average four serious cases report in a hospital every day, what is the probability that there will be no serious case on some specific day?

Example 7: Assume that on the average on telephone number out of fifteen called between 2 p.m. to 3 p.m. on week days is busy. What is the probability that if 6 randomly selected telephone numbers are called 1) not more than three? 2) At least three of them will busy?

Example 8: A car hire firm has two cars which it hires out day by day. The number of demands for car on each day with means 1.5. Calculate the proportion of days 1) on which there is no demand 2) on which demand is refused.

Example 9: It is known from the experience that on an average 2 accident cases are registered in a hospital every day. What is the probability that on a specific day? No accident case is registered. Only one accident case is registered.

Example 10: Suppose that a book of 600 pages contains 40 printing mistakes. Assuming that these errors are randomly distributed throughout the book and X, the number of errors per page has a Poisson distribution. What is the probability that 10 pages selected at random will be free from errors?

Example 11: If the number of accidents in a industry during a month has Poisson probability distribution with mean of 4, the find probability that in coming month there will be at least 3 accidents

Example 12: A workshop has 2 lathes which are hired out every day. The number of demands for lathes on each day is distributed a Poisson with mean 1.5. In an year of 300 working days, find the number of days in which there was no demand for lathes.

Example 13: A controlled manufacturing process is 0.2% defective. What is the probability of taking 2 or more defective from a lot of 100 pieces? i) Using binomial distribution ii) using Poisson approximation.

Example 14: Assume that the chance of a traffic accident on a day in a street is 0.001. On how many days out of a total of 1000 days selected can we expect (i) no accident (ii) more than 3 accidents, if there are 500 such streets.

Example 15: If 3% of tablets of certain drug manufactured by a company are defective find the probability that in a sample of 100 tablets i) exactly 2 are defective ii) None or one defective.

Example 16: It is known that on an average 0.3% of the results obtained on a computer are not reliable. Assuming that Poisson distribution is followed find the probability that from total of 1000 results i) exactly two are not reliable, ii) at the most 3 results are not reliable.

Example 17: Assume that the probability of an individual coalminer being killed in a mine accident during a year is $1/2400$. Use Poisson's distribution to calculate the probability that in a mine employing 200 miners there will be at least one fatal accident in a year.

Example 18: The distribution of typing mistakes committed by a typist is given below. Assuming a Poisson model, find out the expected frequencies.

Mistakes per page (x)	0	1	2	3	4	5
No. of pages (f)	142	156	69	27	5	1

Example 19: Fit Poisson distribution to following data.

x	0	1	2	3	4
f	122	60	5	2	1

Example 20: Fit Poisson distribution to following data.

x	0	1	2	3	4
f	193	99	24	3	1

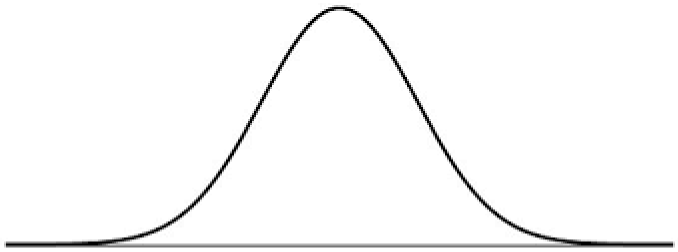
Normal Distribution:

Definition: A ***continuous random variable*** X is called to follow normal distribution with parameter **Mean** (μ) and **Variance** (σ^2), if its probability density function (p. d .f.) is given by,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty < X < \infty, \quad -\infty < \mu < \infty, \quad \sigma^2 > 0$$

In notation. $X \sim N(\mu, \sigma^2)$

A normal distribution is also called Gaussian distribution or Bell shaped curve.



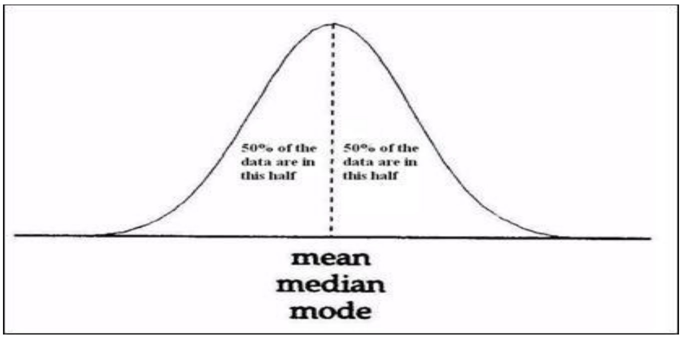
Properties of a normal distribution:

1) The **mean, mode and median** are all equal.

2) The curve is symmetric at the center (i.e. around the mean, μ).

3) Exactly half of the values are to the left of center and exactly half the values are to the right. P(X < mean) = P(X > mean) = 0.5

4) The total area under the curve is 1.



Areas of Application:

1) The variables such as height, weight, intelligence, age etc. follows normal distribution.

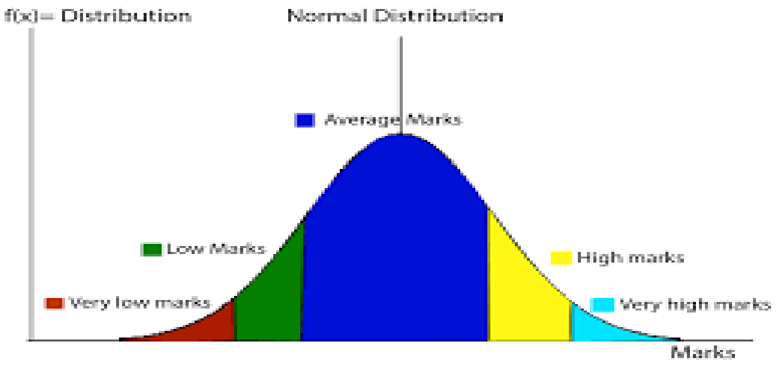
2) Normal distribution has wide applications in Statistical Quality Control.

3) It is also useful in psychological and educational research.

4) It describes many phenomena that occur in nature like rainfall and meteorological studies.

5) Under certain conditions normal distribution provides a good continuous approximation to the Binomial, Poisson etc. distributions.

For example, the distribution of percentage of marks of students in particular examination is shown in the following normal curve diagram,

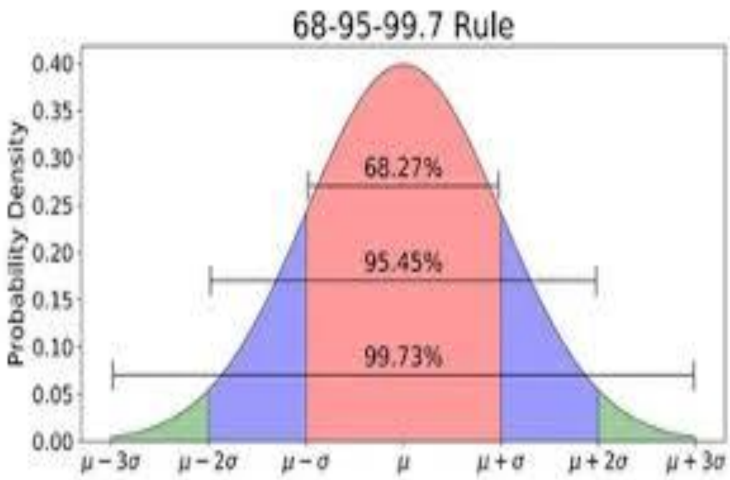


Note: The empirical rule tells you what percentage of your data falls within a certain number of standard deviations from the mean:

1) The area between $(\mu-\sigma)$ and $(\mu+\sigma)$ is 68.27 %.

2) The area between $(\mu-2\sigma)$ and $(\mu+2\sigma)$ is 95.45%.

3) The area between $(\mu-3\sigma)$ and $(\mu+3\sigma)$ is 99.73%.

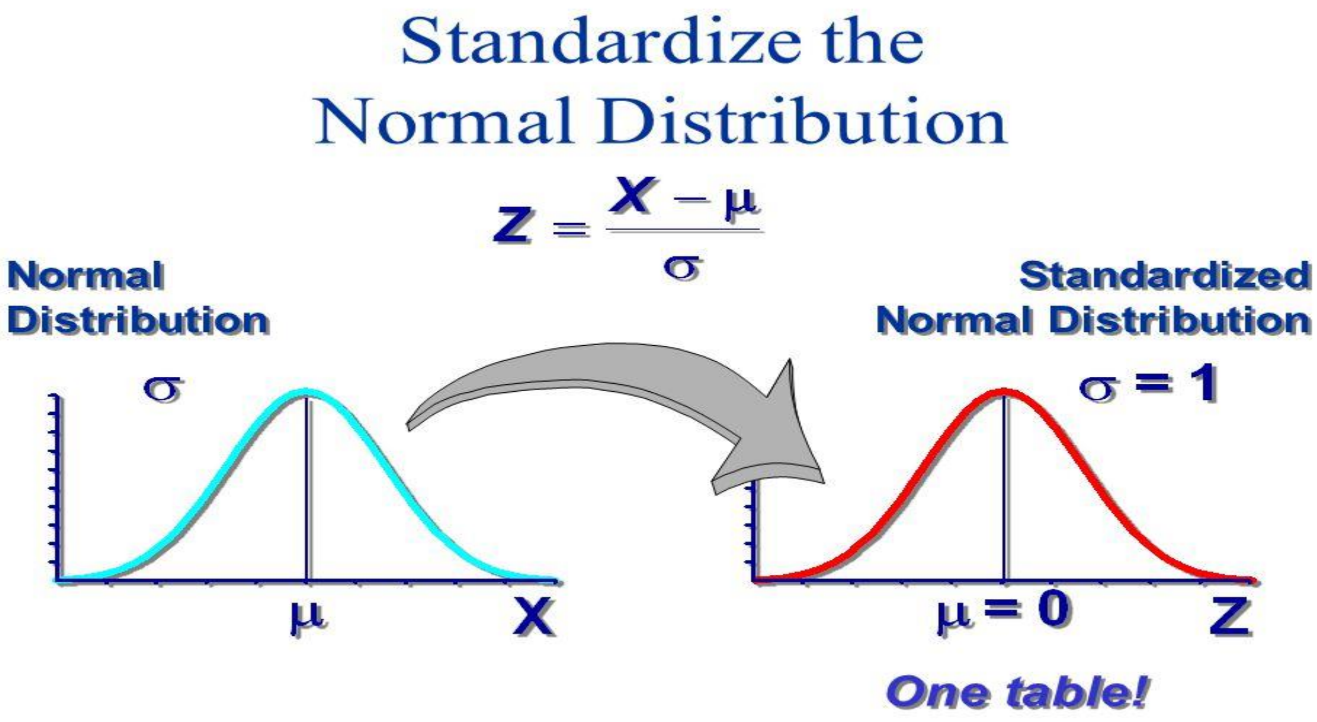


Standard Normal Distribution:

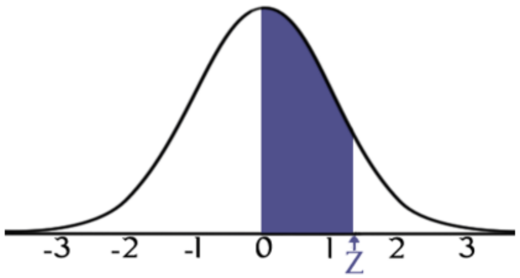
Definition: A ***continuous random variable*** X is normal distribution with parameter **Mean** (μ) and **Variance** (σ^2), then the distribution of $z = \dfrac{x-\mu}{\sigma}$ is also normal distribution with mean 0 and variance 1. Hence Z is called standard normal variable (S.N.V.) and its distribution is called standard normal distribution and its probability density function (p. d .f.) is given by,

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \qquad -\infty < Z < \infty,$$

$$\textit{In notation. } X \sim N(\mu, \sigma^2) \quad \textit{then} \quad Z \sim N(0,1)$$



Area of a Standard Normal Distribution:



STANDARD NORMAL TABLE (Z)

Entries in the table give the area under the curve between the mean and *z* standard deviations above the mean. For example, for *z* = 1.25 the area under the curve between the mean (0) and *z* is 0.3944.

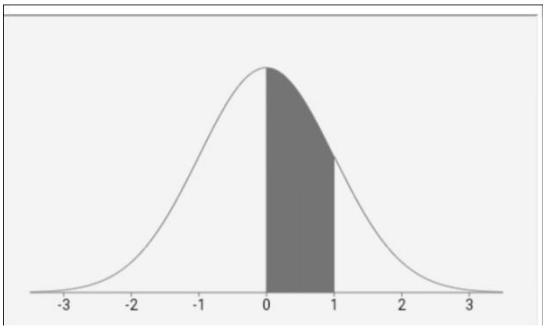
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0190	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2969	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3513	0.3554	0.3577	0.3529	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993
3.2	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995
3.3	0.4995	0.4995	0.4995	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998

Example 1: Find the probability that a random variable having the **standard normal distribution** will take a value between (1) between 0 and 1 (2) between -1 and 0 (3) greater than 1 (4) less than -1 (5) between -1 and 1 (6) between 1 and 2. (7) between -2 and -1.

Answer: Z is random variable having standard normal distribution i.e. $z \sim N(0,1)$

1) Probability that standard random variable between 0 and 1:

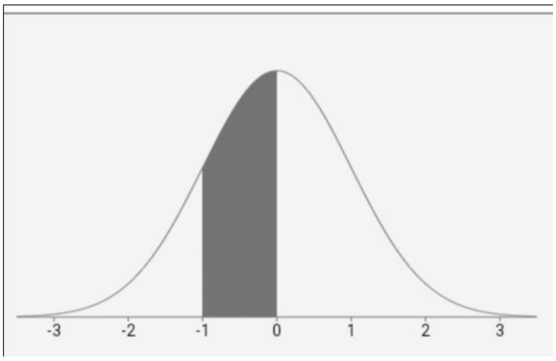
P (0 < Z < 1) = area between Z = 0 to Z = 1 = 0.3413.



2) Probability that standard random variable between -1 and 0:

P (-1 < Z < 0) = area between Z = -1 to Z = 0 (by symmetrical curve)

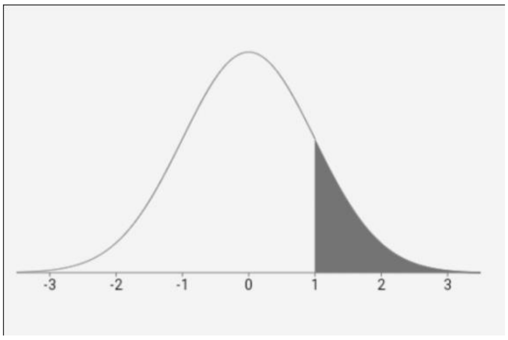
= area between Z = 0 to Z = 1 = 0.3413.



3) Probability that standard random variable greater than 1:

P (Z > 1) = area right side of 1. = 0.5 – (area between Z = 0 to Z = 1)

= 0.5 – 0.3413 = 0.1587.

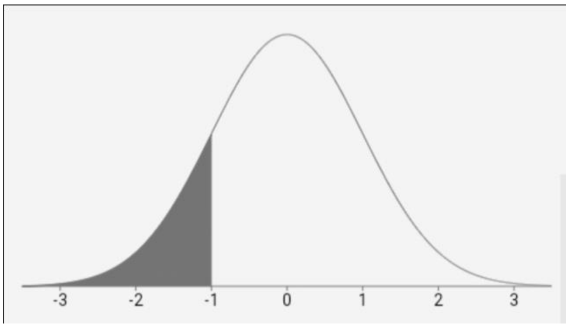


4) Probability that standard random variable less than -1:

P (Z < -1) = area left side of -1. = area right side of 1. (by symmetrical curve)

= 0.5 – (area between Z = 0 to Z = 1)

= 0.5 – 0.3413 = 0.1587.

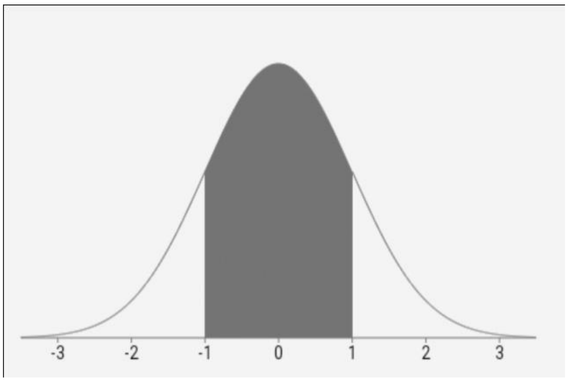


5) Probability that standard random variable between -1 and 1:

P (-1 < Z < 1) = area between Z = -1 to Z = 1. (by symmetrical curve)

= 2(area between Z = 0 to Z = 1)

= 2(0.3413) = 0.6826

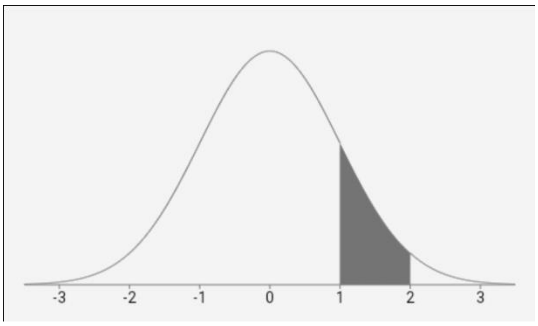


6) Probability that standard random variable between 1 and 2:

P (1 < Z < 2) = area between Z = 1 to Z = 2.

= (area between Z = 0 to Z = 2) - (area between Z = 0 to Z = 1)

= 0.4772 - 0.3413 = 0.1359



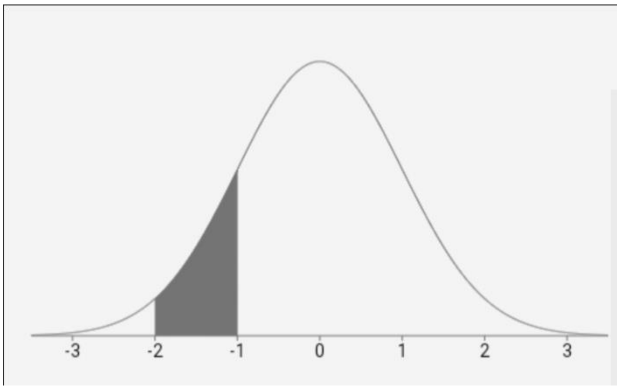
7) Probability that standard random variable between -2 and -1:

P (-2 < Z < -1) = area between Z = - 2 to Z = - 1

= area between Z = 1 to Z = 2. (by symmetrical curve)

= (area between Z = 0 to Z = 2) - (area between Z = 0 to Z = 1)

= 0.4772 - 0.3413 = 0.1359



Example 2: The mean height of 500 students is 151 cm. & the standard deviation is 15 cm. Assuming that the heights are normally distributed, find how many students heights lay between 120 & 155 cm.

Answer: Let X is random variable i.e. height of the students, follows normal distribution with mean 151 cm. and standard deviation 15 cm. and N = 500

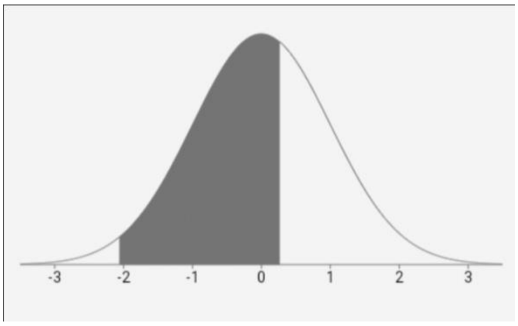
$$In \quad notation.\ X \sim N(\mu = 151, \sigma^2 = 15^2)$$

The Standard Normal Variable $z = \dfrac{x-\mu}{\sigma} = \dfrac{x-151}{15} \sim N(0,1)$

Probability that the height of students between 120 & 155 cm. i.e. P (120 < x < 155)

$$when \quad x = 120\ then\ z = \frac{120-151}{15} = -2.06$$

$$when \quad x = 155\ then\ z = \frac{155-151}{15} = 0.26$$



Therefore P (120 < x < 155) = P (-2.06 < Z < 0.26)

= area between Z = -2.06 to Z = 0.26.

= (area between Z = -2.06 to Z = 0) + (area between Z = 0 to Z = 0.26)

= (area between Z =0 to Z = 2.06) + (area between Z = 0 to Z = 0.26)

= 0.4803+0.1026 = 0.5829

Expected number of students having height between 120 & 155 cm = 500 x 0.5829

= 291.45 ≈ 291

Example 3: A manufacturer produces bolts that are specified to be between 1.19 and 1.21 inches in diameter. If his production process results in a bolt's diameter being normally distributed with mean 1.20 inches and S.D 0.005, what percentage of bolts will not meet Specifications?

Answer: Let X is random variable i.e. diameter of the bolts, follows normal distribution with mean 1.2 inches and standard deviation 0.005 inches.

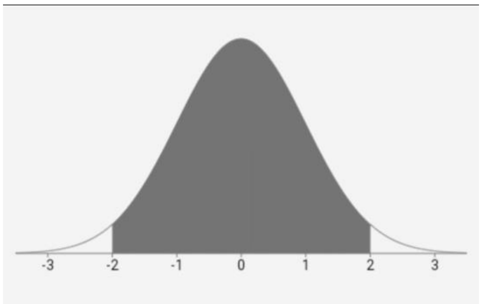
$$In \quad notation. \; X \sim N(\mu = 1.2\,, \sigma^2 = 0.005^2)$$

The Standard Normal Variable $z = \dfrac{x-\mu}{\sigma} = \dfrac{x-1.2}{0.005} \sim N(0,1)$

A manufacturer produces bolts that are specified to be between 1.19 and 1.21 inches in diameter.

Probability that the bolts diameter in specified range between 1.19 & 1.21inches.

i.e. P (1.19 < x < 1.21)



$$when \quad x = 1.19 \; then \;\; z = \frac{1.19-1.2}{0.005} = -2$$

$$when \quad x = 1.21 \; then \;\; z = \frac{1.21-1.2}{0.005} = 2$$

Therefore P (1.19 < x < 1.21) = P (-2 < Z < 2)

= area between Z = - 2 to Z = 2

= 2(area between Z = 0 to Z = 2) (by symmetrical curve)

= 2 (0.4772) = 0.9544

Probability that the bolts not meet specifications = 1 – 0.9544 =0.0456

Therefore percentage of bolts will not meet specifications = 100 x 0.0456 = 4.56%

Example 4: In an examination given by 500 candidates the average and standard deviation of marks obtained are 40 and 10 respectively. Assuming the distribution of marks to be normal find approximately, i) how many will pass if 50 is fixed as a minimum ii) what should be the minimum if 350 candidates are to pass.

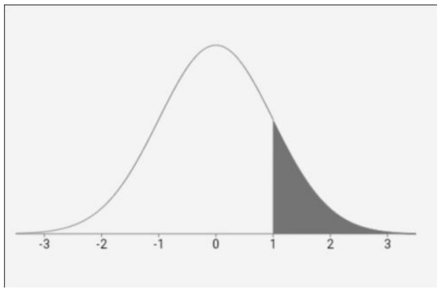
Answer: Let X is random variable i.e. marks of the candidates, follows normal distribution with mean 40 marks and standard deviation 10 marks.

$$In \quad notation. \; X \sim N(\mu = 40\,, \sigma^2 = 10^2)$$

The Standard Normal Variable $z = \dfrac{x-\mu}{\sigma} = \dfrac{x-40}{10} \sim N(0,1)$

i) Probability that the marks are 50 or greater than 50 i.e. P (X ≥ 50)

$$when \quad x = 50 \; then \;\; z = \frac{50-40}{10} = 1$$



Therefore P (X ≥ 50) = P (Z ≥ 1) = area right side of 1.

= 0.5 – (area between Z = 0 to Z = 1)

= 0.5 – 0.3413 = 0.1587.

Expected number of candidates marks are 50 or greater than 50 = 500 x 0.1587 = 79.35≈ 79

ii) There are 350 candidate are pass then probability of passing =350/500 = 0.7

Here we required a value of z that right side area is 0.7

By using statistical table the value of Z = - 0.525 (negative value because it's left side part of mean Z = 0)

$$-0.525 = \frac{X-40}{10} \quad \Rightarrow -5.25 = X - 40 \qquad \therefore X = 34.75$$

Here minimum marks should be 34.75 then 350 candidates pass.

Example 5: Find mean and S.D of normal distribution of marks in an examination where 58 % of the candidates obtained marks below 75 & 4% got above 80 and rest between 75 and 80.

Answer: Let μ and σ be the mean and the standard deviation of the normal variable.

Since 58% students are marks below 75, i.e. 58 – 50 = 8% students between mean and 75.

Since 4% students are marks above 80, i.e. 50 – 4 = 46% students between mean and 80.

Hence, the area between Z = 0 and Z = 0.2 is 0.08 and the area between Z = 0 and Z = 1.75 is 0.46

$$0.2 = \frac{75-\mu}{\sigma} \quad \Rightarrow \mu + 0.2\,\sigma = 75 \ldots\ldots\ldots(1)$$

$$1.75 = \frac{80-\mu}{\sigma} \quad \Rightarrow \mu + 1.75\,\sigma = 80 \ldots\ldots\ldots(2)$$

Solving equation (1) and (2) then we get,

$$\mu = 74.3548 \quad and \quad \sigma = 3.2258$$

Examples for Practice

Example 1: Define normal probability distribution and state its properties.

Example 2: If the heights of 300 students are normally distributed with mean 172 cm. and S. D. 8 cm. how many students have heights i) greater than 184 cm. ii) less than or equal to 160 cm. iii) equal to 172 cm means between 171.5 and 172.5. (Area under S. N. V. Z = 0 to Z = 1.5 is 0.4332, and Z = 0 to Z = 0.06 is 0.0239).

Example 3: An aptitude test for selecting engineers in an industry is conducted on 100 candidates. The average score is 42 and S. D. is 24. Assuming normal distribution for the scores find the number of candidates whose score is i) more than 60, ii) lies between 30 and 60. (Area under S. N. V. Z = 0 to Z = 0.75 is 0.2735 and Z = 0 to Z = 0.5 is 0.1915)

Example 4: A manufacturer of envelopes knows that the weight of the envelope is normally distributed with mean 1.9 gm and variance 0.1 gm. Find how many envelopes weighing (i) 2 gm or more, (ii)2.1 gm or more, can be expected in a given packet of 1000 envelopes.

Example 5: In a newly constructed township 2,000 lamps were installed. If the lamps have an average life of 1000 burning hours with a S.D of 200 hours. (a) What is the number of lamps expected to fail in the first 700 burning hours (b) after, what period 10% of the lamps would have been failed (Assume normal distribution for the life of lamps)

Example 6: Assuming that the diameter of 1000 plugs taken at random follows a normal distribution N $(0.7515, 0.002^2)$, how many of the plugs are likely to be rejected if the approved diameter is 0.75 ± 0.004cm?

Example 7: The lifetime of interactive computer chips produced by a certain semi-conductor manufacturer is normally distributed having mean 4.4×10^6 hours with a S.D of 3×10^5 hours. If a mainframe manufacturer requires that at least 90% of the chips from a large batch will have lifetime of at least 3.8×10^6 hours should he contract with the semiconductor firm?

Example 8: The mean and S.D of the marks obtained by 1000 students in an examination are 34.4 &16.5 respectively. Assuming the normality of the distribution, find the approximate number of students expected to obtain marks between 30 & 60.

Example 9: Pipes for tobacco are being packed in fancy plastic boxes. The length of pipes is normally distributed with mean 5’’ and S.D. 0.1”.The internal length of the boxes is 5.2’’.What is the probability that the box would be small for pipe.

Example 10: In sample of 1000 student the mean and standard deviation of marks obtained by student in certain test are 14 and 2.5.Assuming the distribution to be normal find the number of student getting marks i)Between 12 and 15 ii)Above 18 iii)Below 8.

Example 11: Assuming that the mean height of a group of men is 64.25 inches with S.D. of 2.81 inches. How many in a group of 400 men would you expect to be over 6 ft. Tall?

Example 12: In a regiment of 1000, the mean height of the soldiers is 68.12 units and the S.D. is 3.374 units. How many soldiers could be expected to be more than 72 units? Assume normal distribution.

Example 13: The sprout time of certain seed is normally distributed with the mean 48 hr. and standard deviation 2.5 hrs. If 100 seeds are observed for spout time, how many seeds will have 1) Spout time more than 50 hrs? 2) Spout time between 47 to 50 hrs?

Example 14: The height of certain plant is normally distributed with mean 32 inches and the standard deviation of 2.5 inches. If 100 such plants are inspected, how many plants will have 1) Height more than 38 inches? 2) Height in between 30 to 32 inches?

Example 15: In a sample of 1000 cases, the mean of a certain test is 14 and standard deviation is 2.5.Assuming the distribution to be normal find 1) How many score between 12 and 15? 2) How many score above 18.

Example 16: A sample of 100 dry battery cells tested to find the length of life produced the following results. X = 12 hours, σ= 3 hours Assuming the data to be normally distributed, what percentage of battery cells are expected to have life, i)more than 15hours ii)less than 6 hours iii)between 10 and 14 hours?

Example 17: The blood sugar content of human blood is normally distributed with the mean 175 units and the standard distributed with the mean 175 units and the standard deviation 10 units. What percentage of human being has blood sugar content in between 160 to 190 units?

Example 18: Height of the college students follows normal distribution with mean 130 cm. and standard deviation 5cms ,if there are 500 students in a college how many students will have height more than 140cms and how many will have height less than 120cms?

Example 19: In a sample of 1000 students the mean and S.D. of marks obtained by the students in a certain test are 14 and 2.5. Assuming the distribution to be normal find the number of students getting marks 1) Between 12 and 15 2) Above 18 3) Below 8

(Given for S. N. V. z, area between z = 0 &z = 0.4 is 0.1554, area between z = 0 & z = 0.8 is 0.2881, area between z = 0 & z = 1.6 is 0.4452   z = 0 & z = 2.4 is 0.4918)

Example 20: Weights of 4000 students are found to be normally distributed with mean 50 kgs and standard deviation 5kgs. Find the number of students with weights. i) Less than 45 kgs. ii) Between 45 and 60 kgs.

Example 21: From a box containing 100 transistors 20 of which are defective, 10 are selected at random. Find the probability that 1)all will be defective 2)all are non defective 3)at least one is defective.

Example 22: It is known from the experience that weights of babies are normally distributed with mean 2.5Lb and standard deviation 0.15Lb. What is the probability that the newly born baby will have weight, i) More than 3.00Lb and ii) Weight between 2.75 and 2.90Lb.

Example 23: Pipes for tobacco are being packed in fancy plastic boxes. The length of pipes is normally distributed with mean 5” and standard deviation 0.1”. The internal length of boxes is 5.2”. What is the probability that the box would be small for pipes?

Example 24: 2000 students appeared in a examination, distribution of marks is assumed to Be normal with mean 30 and S.D.625.How many students are expected to get Marks

i) Between 20 and 40 ii) Less than 35

Example 25: A random sample of size 100 is selected from large group of wage caners. The wages are normally distributed with mean annual income of Rs. 10,500 with S.D. of Rs.800. find the probability that sample mean of income falls between Rs.10,100 to Rs.10,600

Example 26: Of a large group of men, 5% are less than 60 inches in height & 40 % are between 60 & 65 inches. Assuming a normal distribution, find the mean & standard deviation.

Example 27: In a normal distribution, 31% of the items are under 45 and 8% are over 64.Find the mean and standard deviation of the distribution.

Example 28: The income distribution of workers in a certain factory was found to be normal with mean of Rs.500 & S.D Rs.50.There was 228 persons above Rs.600. How many persons were there in all?

Example 29: The marks obtained by students in an examination are known to follow a normal distribution of which 30% got under 35 and 10% got above 60. Find the mean and S.D.

Example 30: A random variable has N $(62.4, \sigma^2)$. Find the S.D if the probability is 0.20 that it will take on a value greater than 79.2.

Example 31: Of a large group of men, 10 % are less than 60 inches in height & 45 % are between 60 & 65 inches. Assuming a normal distribution, find the mean & standard deviation.

Example 32: Find mean and S.D of normal distribution of marks in an examination where 42% of the candidates obtained marks above 75 & 4% got above 80.