KOLHAPUR INSTITUTE OF TECHNOLOGY'S,

COLLEGE OF ENGINEERING (AUTONOMOUS), KOLHAPUR

DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING

Second Year B.Tech. (SEM - III)

COMPUTATIONAL MATHEMATICS (UCSE0301)

Unit No. 6: Fuzzy Arithmetic

1. Fuzzy Number:

A fuzzy number is a fuzzy set 'A' define on set of real numbers (R) must satisfies the following three properties.

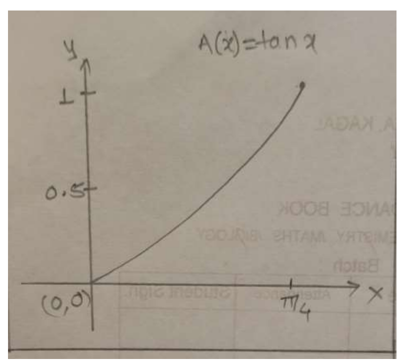
1. Fuzzy set (A) must be a normal fuzzy set. i. e. h (A) = 1.
2. α cut of fuzzy set A is a closed interval for all $\alpha \in [0,1]$.
3. The support of fuzzy set A must be bounded.

Examples

Example 1: Determine following fuzzy sets is a fuzzy numbers or not.

$$A(x) = \begin{cases} \tan x & , if\ 0 \le x \le \dfrac{\pi}{4} \\ 0 & , otherwise \end{cases}$$

Solution: We draw the diagram of given fuzzy set A(x) as,



1. We define height of fuzzy set A as, $h(A) = \sup\limits_{x \in X} A(x)$

   $$here,\ A(x) = \tan x \ \Rightarrow A\left(\frac{\pi}{4}\right) = 1 \text{ as, } \frac{\pi}{4} \in \left[0, \frac{\pi}{4}\right]$$

   Here h (A) = 1 so A(x) is normal fuzzy set.

2. We define α-cut of fuzzy set A as,

   $$^{\alpha}A = \{ x \in X / A(x) \ge \alpha \}$$

   $$\tan x \ge \alpha \quad \Rightarrow x \ge \tan^{-1}\alpha$$

   $$^{\alpha}A = \left[\tan^{-1}\alpha, \frac{\pi}{4}\right] \ for\ every\ \alpha \in [\,0, 1\,]$$

   α cut of fuzzy set A is a closed interval for all $\alpha \in [0,1]$.

3. We define support of fuzzy set A as, $Supp\ A = {}^{0+}A = \{ x \in X / A(x) > 0 \}$

$$Supp\ A = {}^{0+}A = \left(0\,,\frac{\pi}{4}\right]$$

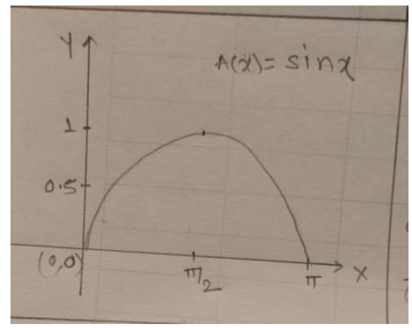
$${}^{0+}A \text{ is bounded.}$$

Here, A(x) has satisfied all three properties therefore A(x) is a fuzzy number.

Example 2: Determine following fuzzy sets is a fuzzy numbers or not.

$$A(x) = \begin{cases} \sin x & ,if\ 0 \le x \le \pi \\ 0 & ,otherwise \end{cases}$$

Solution: We draw the diagram of given fuzzy set A(x) as,



1. We define height of fuzzy set A as,

$$h(A) = \sup_{x \in X} A(x)$$

$$here,\ A(x) = \sin x \ \Rightarrow A\left(\frac{\pi}{2}\right) = 1 \text{ as, } \frac{\pi}{2} \in [0, \pi]$$

Here h (A) = 1 so A(x) is normal fuzzy set.

2. We define α-cut of fuzzy set A as, ${}^{\alpha}A = \{x \in X / A(x) \ge \alpha\}$

$$\sin x \ge \alpha \quad \Rightarrow x \ge \sin^{-1}\alpha$$

$${}^{\alpha}A = [\sin^{-1}\alpha, \pi - \sin^{-1}\alpha] \text{ for every } \alpha \in [0,1]$$

α cut of fuzzy set A is a closed interval for all α ∈[0,1].

3. We define support of fuzzy set A as, $Supp\ A = {}^{0+}A = \{x \in X / A(x) > 0\}$

$$Supp\ A = {}^{0+}A = (0, \pi)$$

$${}^{0+}A \text{ is bounded.}$$

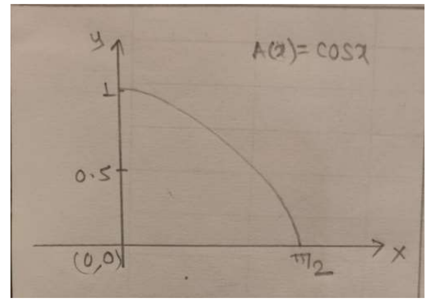
Here, A(x) has satisfied all three properties therefore A(x) is a fuzzy number.

Example 3: Determine following fuzzy sets is a fuzzy numbers or not.

$$A(x) = \begin{cases} \cos x & ,if\ 0 \le x \le \frac{\pi}{2} \\ 0 & ,otherwise \end{cases}$$

Solution: We draw the diagram of given fuzzy set A(x) as,

1. We define height of fuzzy set A as,



$$h(A) = \sup_{x \in X} A(x)$$

$$here, \; A(x) = \cos x \;\; \Rightarrow A(0) = 1 \text{ as, } 0 \in \left[0, \frac{\pi}{2}\right]$$

Here h (A) = 1 so A(x) is normal fuzzy set.

2. We define α-cut of fuzzy set A as, ${}^{\alpha}A = \{x \in X \,/\, A(x) \geq \alpha\}$

$$\cos x \;\geq\; \alpha \qquad \Rightarrow x \;\geq\; \cos^{-1}\alpha$$

$${}^{\alpha}A = [0\,, \cos^{-1}\alpha] \; for\ every\ \alpha \in [\,0,1\,]$$

α cut of fuzzy set A is a closed interval for all $\alpha \in [0,1]$.

3. We define support of fuzzy set A as, $Supp\ A = {}^{0+}A = \{x \in X \,/\, A(x) > 0\}$

$$Supp\ A = {}^{0+}A = \left[0, \frac{\pi}{2}\right) \qquad {}^{0+}A\ is\ bounded.$$

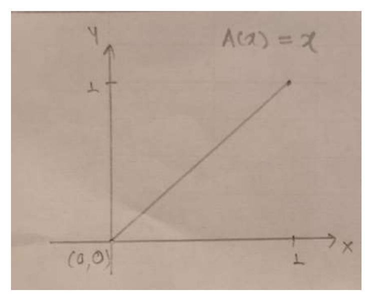
Here, A(x) has satisfied all three properties therefore A(x) is a fuzzy number.

Example 4: Determine following fuzzy sets is a fuzzy numbers or not.

$$A(x) = \begin{cases} x & ,if\ 0 \leq x \leq 1 \\ 0 & ,otherwise \end{cases}$$

Solution: We draw the diagram of given fuzzy set A(x) as,

1. We define height of fuzzy set A as,



$$h(A) = \sup_{x \in X} A(x)$$

$$here, \; A(x) = x \;\; \Rightarrow A(1) = 1 \text{ as, } 1 \in [0,1]$$

Here h (A) = 1 so A(x) is normal fuzzy set.

2. We define α-cut of fuzzy set A as,

$${}^{\alpha}A = \{x \in X \,/\, A(x) \geq \alpha\}$$

$$x \;\geq\; \alpha \qquad {}^{\alpha}A = [\alpha\,, 1] \; for\ every\ \alpha \in [\,0,1\,]$$

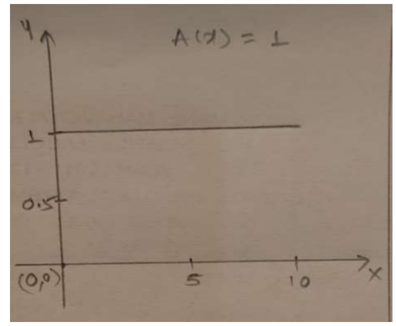
α cut of fuzzy set A is a closed interval for all $\alpha \in [0,1]$.

3. We define support of fuzzy set A as, $Supp\ A = {}^{0+}A = \{x \in X \,/\, A(x) > 0\}$

$$Supp\ A = {}^{0+}A = (0\,, 1] \qquad {}^{0+}A\ is\ bounded.$$

Here, A(x) has satisfied all three properties therefore A(x) is a fuzzy number.

Example 5: Determine following fuzzy sets is a fuzzy numbers or not.



$$A(x) = \begin{cases} 1 & ,if\ 0 \le x \le 10 \\ 0 & ,otherwise \end{cases}$$

Solution: We draw the diagram of given fuzzy set A(x) as,

1. We define height of fuzzy set A as,

$$h(A) = \sup_{x \in X} A(x) \qquad here,\ A(x) = 1$$

Here h (A) = 1 for $0 \le x \le 10$, so A(x) is normal fuzzy set.

2. We define α-cut of fuzzy set A as,

$${}^{\alpha}A = \{ x \in X \,/\, A(x) \ge \alpha \}$$

$${}^{\alpha}A = [0\,,10]\ for\ every\ \alpha \in [\,0,1\,]$$

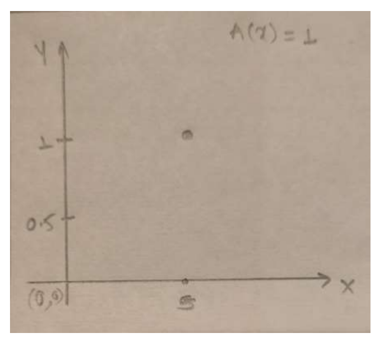
α cut of fuzzy set A is a closed interval for all α ∈[0,1].

3. We define support of fuzzy set A as, $Supp\ A = {}^{0+}A = \{ x \in X \,/\, A(x) > 0 \}$

$$Supp\,A = {}^{0+}A = [\,0,10\,] \qquad {}^{0+}A\ is\ bounded.$$

Here, A(x) has satisfied all three properties therefore A(x) is a fuzzy number.

Example 6: Determine following fuzzy sets is a fuzzy numbers or not.



$$A(x) = \begin{cases} 1 & ,if\ x = 5 \\ 0 & ,otherwise \end{cases}$$

Solution: We draw the diagram of given fuzzy set A(x) as,

1. We define height of fuzzy set A as,

$$h(A) = \sup_{x \in X} A(x)$$

$$here,\ A(x) = 1 \ \Rightarrow A(5) = 1 \text{ as, } 5 \in [0,1]$$

Here h (A) = 1, so A(x) is normal fuzzy set.

2. We define α-cut of fuzzy set A as,

$${}^{\alpha}A = \{ x \in X \,/\, A(x) \ge \alpha \}$$

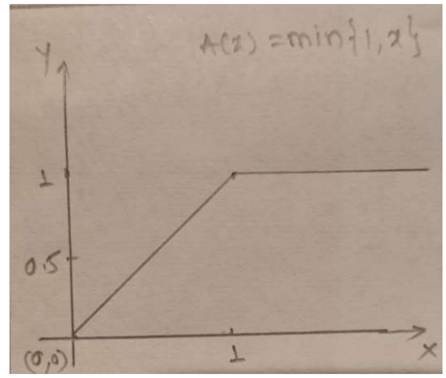
$${}^{\alpha}A = \{\,5\,\}\ for\ every\ \alpha \in [\,0,1\,]$$

α cut of fuzzy set A is a closed interval for all $\alpha \in [0,1]$.

3. We define support of fuzzy set A as, $Supp\ A = {}^{0+}A = \{x \in X / A(x) > 0\}$

$$Supp\ A = {}^{0+}A = \{5\} \qquad {}^{0+}A \text{ is bounded.}$$

Here, A(x) has satisfied all three properties therefore A(x) is a fuzzy number.



Example 7: Determine following fuzzy sets is a fuzzy numbers or not.

$$A(x) = \begin{cases} \min\{1, x\} & , if\ x \geq 0 \\ 0 & , otherwise \end{cases}$$

Solution: We draw the diagram of given fuzzy set A(x) as,

1. We define height of fuzzy set A as,

$$h(A) = \sup_{x \in X} A(x)$$

$$here,\ A(x) = \min\{1, x\}$$

Here h (A) = 1,for $x \geq 1$ so A(x) is normal fuzzy set.

2. We define α-cut of fuzzy set A as,

$$^{\alpha}A = \{x \in X / A(x) \geq \alpha\}$$

$$^{\alpha}A = [\alpha, \infty]\ for\ every\ \alpha \in [0,1]$$

α cut of fuzzy set A is not a closed interval for all $\alpha \in [0,1]$.

3. We define support of fuzzy set A as, $Supp\ A = {}^{0+}A = \{x \in X / A(x) > 0\}$

$$Supp\ A = {}^{0+}A = (0, \infty] \qquad {}^{0+}A \text{ is not bounded.}$$

Here, A(x) has not satisfied all three properties therefore A(x) is not a fuzzy number.

Examples for Practice

Example 1: What are the criteria's of a fuzzy set to be a fuzzy number. Determine whether the following fuzzy set is a fuzzy number.

$$A(x) = \begin{cases} \sin 2x & ,if \ 0 \le x \le \frac{\pi}{4} \\ 0 & ,otherwise \end{cases}$$

Example 2: Determine whether the following fuzzy set is a fuzzy number.

$$A(x) = \begin{cases} 1 + x & ,if \ -1 \le x \le 0 \\ 0 & ,otherwise \end{cases}$$

Example 3: Determine whether the following fuzzy set is a fuzzy number.

$$A(x) = \begin{cases} 1 - |x| & ,if \ -1 \le x \le 1 \\ 0 & ,otherwise \end{cases}$$

Example 4: Determine whether the following fuzzy set is a fuzzy number.

$$A(x) = \begin{cases} \frac{x}{10} & ,if \ 0 \le x \le 10 \\ 0 & ,otherwise \end{cases}$$

2.Fuzzy Cardinality:

Fuzzy cardinality is defined as a fuzzy number rather than as a real number, as is the case for a scalar cardinality. When a fuzzy set A has a finite support its fuzzy cardinality is denoted by $|\widetilde{A}|$ is a fuzzy set defined on N whose membership function is defined by,

$$|\widetilde{A}| = \frac{\alpha}{|{}^{\alpha}A|} \qquad \text{A: N} \rightarrow [0, 1]$$

Where, $|{}^{\alpha}A|$ is the cardinality of α - cut of A.

Example 1: Find Fuzzy Cardinality for the fuzzy set, $A(x) = 3^{-x}$ for x ∈ {0,1,..,5}

Solution: Given fuzzy set, $A(x) = 3^{-x}$ for universal set X = [0, 1, 2…5]

Fuzzy set A(x) can be represented as,

$$A(x) = \left\{ \frac{1}{0} + \frac{0.3333}{1} + \frac{0.1111}{2} + \frac{0.037}{3} + \frac{0.0123}{4} + \frac{0.0041}{5} \right\}$$

We define Level set of fuzzy set A as,

$$\Lambda A = \{ \alpha / A(x) = \alpha \ \ for\ some\ x \in X \}$$

$$\Lambda A = \{ \ 0.0041, 0.0123, 0.037, 0.1111, 0.3333, 1 \ \}$$

We define α-cut of fuzzy set A as, ${}^{\alpha}A = \{ x \in X / A(x) \geq \alpha \}$

$${}^{0.0041}A = \{ x \in X / A(x) \geq 0.0041 \} \qquad {}^{0.0041}A = \{ 0, 1, 2, 3, 4,5 \} \qquad \therefore |{}^{0.0041}A| = 6$$

$${}^{0.0123}A = \{ x \in X / A(x) \geq 0.0123 \} \qquad {}^{0.0123}A = \{ 0, 1, 2, 3, 4 \} \qquad \therefore |{}^{0.0123}A| = 5$$

$${}^{0.037}A = \{ x \in X / A(x) \geq 0.037 \} \qquad {}^{0.037}A = \{ 0, 1, 2, 3 \} \qquad \therefore |{}^{0.037}A| = 4$$

$${}^{0.1111}A = \{ x \in X / A(x) \geq 0.1111 \} \qquad {}^{0.1111}A = \{ 0, 1, 2 \} \qquad \therefore |{}^{0.1111}A| = 3$$

$${}^{0.3333}A = \{ x \in X / A(x) \geq 0.3333 \} \qquad {}^{0.3333}A = \{ 0, 1 \} \qquad \therefore |{}^{0.3333}A| = 2$$

$${}^{1}A = \{ x \in X / A(x) \geq 1 \} \qquad {}^{1}A = \{ 0 \} \qquad \therefore |{}^{1}A| = 1$$

Fuzzy cardinality of A (x) is, $|\widetilde{A}| = \frac{\alpha}{|{}^{\alpha}A|}$

$$|\widetilde{A}| = \left\{ \frac{1}{1} + \frac{0.3333}{2} + \frac{0.1111}{3} + \frac{0.037}{4} + \frac{0.0123}{5} + \frac{0.0041}{6} \right\}$$

Example 2: Let A and B be fuzzy sets defined on the universal set X=Z (set of integers) whose membership functions are given by,

$$A(x)=\left\{\frac{0}{1}+\frac{0.2}{1.5}+\frac{0.35}{2}+\frac{0.15}{2.5}+\frac{0.5}{3}+\frac{0.25}{3.5}+\frac{0.4}{4}\right\}$$

$$B(x)=\left\{\frac{1}{1}+\frac{0.15}{1.5}+\frac{0.2}{2}+\frac{0.35}{2.5}+\frac{0.4}{3}+\frac{0.125}{3.5}+\frac{0.4}{4}\right\}$$

Find Fuzzy Cardinality of $\overline{A\cup B}$.

Solution: Given the fuzzy set A and B as,

$$A(x)=\left\{\frac{0}{1}+\frac{0.2}{1.5}+\frac{0.35}{2}+\frac{0.15}{2.5}+\frac{0.5}{3}+\frac{0.25}{3.5}+\frac{0.4}{4}\right\}$$

$$B(x)=\left\{\frac{1}{1}+\frac{0.15}{1.5}+\frac{0.2}{2}+\frac{0.35}{2.5}+\frac{0.4}{3}+\frac{0.125}{3.5}+\frac{0.4}{4}\right\}$$

Union of fuzzy set A and B defined as,

$$A\cup B(x)=Max\ \{\ A(x), B(x)\ \}$$

$$A\cup B(x)=\left\{\frac{1}{1}+\frac{0.2}{1.5}+\frac{0.35}{2}+\frac{0.35}{2.5}+\frac{0.5}{3}+\frac{0.25}{3.5}+\frac{0.4}{4}\right\}$$

Complement of fuzzy set A $\cup$ B defined as,

$$\overline{A\cup B}(x)=1-A\cup B(x)$$

$$\overline{A\cup B}(x)=\left\{\frac{0}{1}+\frac{0.8}{1.5}+\frac{0.65}{2}+\frac{0.65}{2.5}+\frac{0.5}{3}+\frac{0.75}{3.5}+\frac{0.6}{4}\right\}$$

We define Level set of fuzzy set $\overline{A\cup B}$ as,

$$\Lambda\overline{A\cup B}\ (x)=\left\{\alpha\,/\,\overline{A\cup B}=\alpha\ \ for\ some\ x\in X\ \right\}$$

$$\Lambda\overline{A\cup B}\ (x)=\{\ 0, 0.5, 0.6, 0.65, 0.75, 0.8\,\}$$

We define α-cut of fuzzy set A as, ${}^{\alpha}A=\{\,x\in X\,/\,A(x)\geq\alpha\ \}$

$${}^{0}\overline{A\cup B}\ (x)=\left\{x\in X\,/\,\overline{A\cup B}(x)\geq 0\right\}$$

$${}^{0}\overline{A\cup B}\ (x)=\{1, 1.5, 2, 2.5, 3, 3.5, 4\,\} \qquad \therefore|\ {}^{0}\overline{A\cup B}\,|=7$$

$${}^{0.5}\overline{A\cup B}\ (x)=\left\{x\in X\,/\,\overline{A\cup B}(x)\geq 0.5\right\}$$

$${}^{0.5}\overline{A\cup B}\ (x)=\{1.5, 2, 2.5, 3, 3.5, 4\,\} \qquad \therefore|\ {}^{0.5}\overline{A\cup B}\,|=6$$

$${}^{0.6}\overline{A\cup B}\ (x)=\left\{x\in X\,/\,\overline{A\cup B}(x)\geq 0.6\right\}$$

$${}^{0.6}\overline{A\cup B}\ (x)=\{1.5, 2, 2.5, 3.5, 4\,\} \qquad \therefore|\ {}^{0.6}\overline{A\cup B}\,|=5$$

$${}^{0.65}\overline{A\cup B}\ (x)=\left\{x\in X\,/\,\overline{A\cup B}(x)\geq 0.65\right\}$$

$$^{0.65}\overline{A \cup B}\,(x) = \{1.5, 2, 2.5, 3.5\} \qquad \therefore |\,^{0.65}\overline{A \cup B}\,| = 4$$

$$^{0.75}\overline{A \cup B}\,(x) = \left\{x \in X \,/\, \overline{A \cup B}(x) \geq 0.75\right\}$$

$$^{0.75}\overline{A \cup B}\,(x) = \{1.5, 3.5\} \qquad \therefore |\,^{0.75}\overline{A \cup B}\,| = 2$$

$$^{0.8}\overline{A \cup B}\,(x) = \left\{x \in X \,/\, \overline{A \cup B}(x) \geq 0.8\right\}$$

$$^{0.8}\overline{A \cup B}\,(x) = \{1.5\} \qquad \therefore |\,^{0.8}\overline{A \cup B}\,| = 1$$

Fuzzy cardinality of A (x) is, $|\tilde{A}| = \dfrac{\alpha}{|\,^{\alpha}A|}$

$$|\tilde{A}| = \left\{\frac{0}{7} + \frac{0.5}{6} + \frac{0.6}{5} + \frac{0.65}{4} + \frac{0.75}{2} + \frac{0.8}{1}\right\}$$

Example 3: Find the degree of subset hood $S(|\tilde{A}|, |\tilde{B}|)$ for the fuzzy sets,

$$A(x) = \frac{x}{x+3}\ , \quad B(x) = \frac{x}{x+2}\ , \ x \in \{0,1,2\}$$

Solution: Given fuzzy set, $A(x) = \dfrac{x}{x+3}$ for universal set X = [0, 1, 2]

Fuzzy set A(x) can be represented as,

$$A(x) = \left\{\frac{0}{0} + \frac{0.25}{1} + \frac{0.4}{2}\right\}$$

We define Level set of fuzzy set A as,

$$\Lambda A = \{\alpha \,/\, A(x) = \alpha \ \text{for some}\ x \in X\}$$

$$\Lambda A = \{\ 0, 0.25, 0.4\ \}$$

We define α-cut of fuzzy set A as, $^{\alpha}A = \{x \in X \,/\, A(x) \geq \alpha\}$

$$^{0}A = \{x \in X \,/\, A(x) \geq 0\} \qquad ^{0}A = \{0, 1, 2\} \qquad \therefore |\,^{0}A| = 3$$

$$^{0.25}A = \{x \in X \,/\, A(x) \geq 0.25\} \qquad ^{0.25}A = \{1, 2\} \qquad \therefore |\,^{0.25}A| = 2$$

$$^{0.4}A = \{x \in X \,/\, A(x) \geq 0.4\} \qquad ^{0.4}A = \{2\} \qquad \therefore |\,^{0.4}A| = 1$$

Fuzzy cardinality of A (x) is, $|\tilde{A}| = \dfrac{\alpha}{|\,^{\alpha}A|}$

$$|\tilde{A}| = \left\{\frac{0}{3} + \frac{0.25}{2} + \frac{0.4}{1}\right\}$$

Scalar cardinality of fuzzy set A as, $\left\| \widetilde{A} \right\| = \sum_{x \in X} |\widetilde{A}|(x) = 0.65$

Given fuzzy set, $B(x) = \dfrac{x}{x+2}$ for universal set X = [0, 1, 2]

Fuzzy set A(x) can be represented as,

$$B(x) = \left\{ \frac{0}{0} + \frac{0.3333}{1} + \frac{0.5}{2} \right\}$$

We define Level set of fuzzy set B as,

$$\Lambda B = \{ \alpha / B(x) = \alpha \ \ for\ some\ x \in X \}$$

$$\Lambda B = \{ \ 0, 0.3333, 0.5 \ \}$$

We define α-cut of fuzzy set A as, ${}^{\alpha}B = \{ x \in X / B(x) \geq \alpha \}$

${}^{0}B = \{ x \in X / B(x) \geq 0 \}$ ${}^{0}B = \{ 0, 1, 2 \}$ $\therefore |{}^{0}B| = 3$

${}^{0.3333}B = \{ x \in X / B(x) \geq 0.3333 \}$ ${}^{0.3333}B = \{ 1, 2 \}$ $\therefore |{}^{0.3333}B| = 2$

${}^{0.5}B = \{ x \in X / B(x) \geq 0.5 \}$ ${}^{0.5}B = \{ 2 \}$ $\therefore |{}^{0.4}B| = 1$

Fuzzy cardinality of A (x) is, $|\widetilde{B}| = \dfrac{\alpha}{|{}^{\alpha}B|}$

$$|\widetilde{B}| = \left\{ \frac{0}{3} + \frac{0.3333}{2} + \frac{0.5}{1} \right\}$$

Scalar cardinality of fuzzy set B as, $\left\| \widetilde{B} \right\| = \sum_{x \in X} |\widetilde{B}|(x) = 0.8333$

By definition, $|\widetilde{A}| \cap |\widetilde{B}|(x) = Min\,\{ |\widetilde{A}|(x), |\widetilde{B}|(x) \}$

$$|\widetilde{A}| \cap |\widetilde{B}| = \left\{ \frac{0}{3} + \frac{0.25}{2} + \frac{0.4}{1} \right\}$$

Scalar cardinality of fuzzy set $|\widetilde{A}| \cap |\widetilde{B}|(x)$ as, $\left\| \widetilde{A} | \cap | \widetilde{B} \right\| = \sum_{x \in X} |\widetilde{A}| \cap |\widetilde{B}| = 0.65$

Degree of Subset hood, $S(|\widetilde{A}|, |\widetilde{B}|) = \dfrac{\left\| \widetilde{A} | \cap | \widetilde{B} \right\|}{|\widetilde{A}|} = \dfrac{0.65}{0.65} = 1$

Examples for Practice

Example 1: Find Fuzzy Cardinality for the fuzzy set, $A(x) = 3^{-x}$ for x $\in$ {0,1,..,5}

Example 2: Find Fuzzy Cardinality for the fuzzy set, $A(x) = \dfrac{x+2}{x+5}$ for x $\in$ {0,1,..,10}

Example 3: Find Fuzzy Cardinality for the fuzzy set, $A(x) = \dfrac{x}{x+2}$ for x $\in$ {6,7,..,10}

Example 4: Find Fuzzy Cardinality for the fuzzy set, $B(x) = \dfrac{1}{1+10(x-1)^2}$ for x $\in$ {0,1,..,5}

Example 5: Find Fuzzy Cardinality for the fuzzy set, $A(x) = \dfrac{2x+5}{3x+7}$ for x $\in$ {0,1,..,10}

Example 6: Find Fuzzy Cardinality for the fuzzy set, $A(x) = \dfrac{x+5}{4x+9}$ for x $\in$ {0,1,..,10}

Example 7: Find the degree of subset hood $S(|\widetilde{A}|,|\widetilde{B}|)$ for the fuzzy sets,

$$A(x) = \frac{x}{x+4} \;,\; B(x) = 3^{-x} \;,\; x \in \{0,1,2\ldots5\}$$

Example 8: Find the degree of subset hood $S(|\widetilde{A}|,|\widetilde{B}|)$ for the fuzzy sets,

$$A(x) = \frac{2x+1}{5x+2} \;,\; B(x) = \frac{x}{x+1} \;,\; x \in \{0,1,2\ldots10\}$$

Example 9: Let A and B be fuzzy sets defined on the universal set X=Z (set of integers) whose membership functions are given by,

$$A(x) = \left\{\frac{0}{1} + \frac{0.2}{1.5} + \frac{0.35}{2} + \frac{0.15}{2.5} + \frac{0.5}{3} + \frac{0.25}{3.5} + \frac{0.4}{4}\right\}$$

$$B(x) = \left\{\frac{1}{1} + \frac{0.15}{1.5} + \frac{0.2}{2} + \frac{0.35}{2.5} + \frac{0.4}{3} + \frac{0.125}{3.5} + \frac{0.4}{4}\right\}$$

Find Fuzzy Cardinality of $\overline{A \cap B}$

Example 10: Let fuzzy numbers A and B each defined on its own inverse be

$$A(x) = \left\{\frac{0.5}{1} + \frac{0.6}{2} + \frac{0.9}{3} + \frac{1}{4} + \frac{0.2}{5}\right\} \quad , \quad B(x) = \left\{\frac{0.3}{3} + \frac{0.4}{4} + \frac{0.7}{5} + \frac{1}{6} + \frac{0.3}{7}\right\}$$

Find Fuzzy Cardinality of $A \cup \overline{B}$.

3. Method of developing Fuzzy Arithmetic is based on extension Principle:

Let A is a function from $A: R \to [0,1]$ and $B: R \to [0,1]$ be two fuzzy numbers if * be binary operation defined on set of real numbers (R) then,

$$f(A,B)(z) = \underset{X*Y=Z}{Max}\{\min[A(x), B(x)]\}$$

Example 1: Let A and B be fuzzy sets defined on the universal set X=Z (set of integers) whose membership functions are given by,

$$A(x) = \left\{\frac{0.4}{-2} + \frac{0.5}{-1} + \frac{0.3}{0} + \frac{1}{1} + \frac{0.5}{2}\right\} \qquad B(x) = \left\{\frac{0.2}{1} + \frac{0.7}{2} + \frac{1}{3} + \frac{0.3}{4} + \frac{0.5}{5}\right\}$$

Find f (A, B) if f: X x X → X is defined by $f(x_1, x_2) = x_1 + x_2 \ \forall \ x_1, x_2 \in X$

Solution: Universal set values of A = {-2,-1, 0, 1, 2} and B = {1, 2, 3, 4, 5}

A+B	**1**	**2**	**3**	**4**	**5**
-2	**-1**	**0**	**1**	**2**	**3**
-1	**0**	**1**	**2**	**3**	**4**
0	**1**	**2**	**3**	**4**	**5**
1	**2**	**3**	**4**	**5**	**6**
2	**3**	**4**	**5**	**6**	**7**

Here, Z = {-1, 0, 1, 2, 3, 4, 5, 6, 7}

Z	**{ A(x_1), B(x_2) }**	**Min{ A(x_1), B(x_2) }**	**Max**
-1	(0.4,0.2)	0.2	0.2
0	(0.5,0.2), (0.4,0.7)	0.2, 0.4	0.4
1	(0.3,0.2), (0.5,0.7), (0.4,1)	0.2, 0.5, 0.4	0.5
2	(1,0.2), (0.3,0.7), (0.5,1), (0.4,0.3)	0.2, 0.3, 0.5, 0.3	0.5
3	(0.5,0.2), (1,0.7), (0.3,1), (0.5,0.3), (0.4,0.5)	0.2, 0.7, 0.3, 0.3, 0.4	0.7
4	(0.5,0.7), (1,1), (0.3,0.3), (0.5,0.5)	0.5, 1, 0.3, 0.5	1
5	(0.5,1), (1,0.3), (0.3,0.5)	0.5,0.3,0.3	0.5
6	(0.5,0.3), (1,0.5)	0.3,0.5	0.5
7	(0.5,0.5)	0.5	0.5

By, extension principle,

$$f(A,B)(z) = \underset{X*Y=Z}{Max}\{\min[A(x), B(x)]\}$$

$$f(A,B) = \left\{\frac{0.2}{-1} + \frac{0.4}{0} + \frac{0.5}{1} + \frac{0.5}{2} + \frac{0.7}{3} + \frac{1}{4} + \frac{0.5}{5} + \frac{0.5}{6} + \frac{0.5}{7}\right\}$$

Example 2: Let A and B be fuzzy sets defined on the universal set X=Z (set of integers) whose membership functions are given by,

$$A(x) = \left\{\frac{0.5}{-1} + \frac{1}{0} + \frac{0.5}{1} + \frac{0.3}{2}\right\} \qquad B(x) = \left\{\frac{0.5}{2} + \frac{1}{3} + \frac{0.5}{4} + \frac{0.3}{5}\right\}$$

Find f (A, B) if f: X x X → X is defined by $f(x_1, x_2) = x_1\, x_2 \ \forall\, x_1, x_2 \in X$

Solution: Universal set values of A = {-1, 0, 1, 2} and B = {2, 3, 4, 5}

A x B	**2**	**3**	**4**	**5**
-1	**-2**	**-3**	**-4**	**-5**
0	**0**	**0**	**0**	**0**
1	**2**	**3**	**4**	**5**
2	**4**	**6**	**8**	**10**

Here, Z = {-5, -4,-3,-2, 0, 2, 3, 4, 5, 6, 8, 10}

Z	**{ $A(x_1)$, $B(x_2)$ }**	**Min{ $A(x_1)$, $B(x_2)$ }**	**Max**
-5	(0.5, 0.3)	0.3	0.3
-4	(0.5, 0.5)	0.5	0.5
-3	(0.5, 1)	0.5	0.5
-2	(0.5, 0.5)	0.5	0.5
0	(1, 0.5), (1, 1), (1, 0.5), (1, 0.3)	0.5, 1, 0.5, 0.3	1
2	(0.5,0.5)	0.5	0.5
3	(0.5,1)	0.5	0.5
4	(0.5,0.5),(0.3,0.5)	0.5, 0.3	0.5
5	(0.5, 0.3)	0.3	0.3
6	(0.3,1)	0.3	0.3
8	(0.3,0.5)	0.3	0.3
10	(0.3,0.3)	0.3	0.3

By, extension principle, $f(A,B)(z) = \underset{X*Y=Z}{Max}\{\min[A(x), B(x)]\}$

$$f(A,B)=\left\{\frac{0.3}{-5}+\frac{0.5}{-4}+\frac{0.5}{-3}+\frac{0.5}{-2}+\frac{1}{0}+\frac{0.5}{2}+\frac{0.5}{3}+\frac{0.5}{4}+\frac{0.3}{5}+\frac{0.3}{6}+\frac{0.3}{8}+\frac{0.3}{10}\right\}$$

Example 3: Let A and B be two fuzzy sets defined on the universal set X = Z (the set of integers) whose membership function are given by,

$$A(x)=\left\{\frac{0.3}{-1}+\frac{0.5}{0}+\frac{0.7}{1}+\frac{1}{2}\right\} \text{ and } B(x)=\left\{\frac{0.2}{-3}+\frac{0.4}{-2}+\frac{0.8}{-1}+\frac{1}{0}\right\}.$$

Let a function $f : X \rightarrow X$ be defined by $f(x_1, x_2) = x_1 + x_2 \ \forall\, x_1, x_2 \in X$. Calculate f (A,B).

Solution: Universal set values of A = {-1, 0, 1, 2} and B = {-3, -2, -1, 0}

A+B	**-3**	**-2**	**-1**	**0**
-1	**-4**	**-3**	**-2**	**-1**
0	**-3**	**-2**	**-1**	**0**
1	**-2**	**-1**	**0**	**1**
2	**-1**	**0**	**1**	**2**

Here, Z = { - 4, - 3, - 2, - 1, 0, 1, 2}

Z	**{ A(x_1), B(x_2) }**	**Min{ A(x_1), B(x_2) }**	**Max**
-4	(0.3, 0.2)	0.2	0.2
-3	(0.5, 0.2), (0.3, 0.4)	0.2, 0.3	0.3
-2	(0.7, 0.2),(0.5,0.4),(0.3,0.8)	0.2, 0.4, 0.3	0.4
-1	(1, 0.2), (0.7, 0.4), (0.5, 0.8), (0.3, 1)	0.2, 0.4, 0.5, 0.3	0.5
0	(1, 0.4), (0.7, 0.8), (0.5, 1)	0.4, 0.7, 0.5	0.7
1	(1, 0.8), (0.7, 1)	0.8,0.7	0.8
2	(1,1)	1	1

By, extension principle,

$$f(A,B)(z)=\underset{X*Y=Z}{Max}\{\min[A(x),B(x)]\}$$

$$f(A,B)=\left\{\frac{0.2}{-4}+\frac{0.3}{-3}+\frac{0.4}{-2}+\frac{0.5}{-1}+\frac{0.7}{0}+\frac{0.8}{1}+\frac{1}{2}\right\}$$

Examples for Practice

Example 1: Let A and B be fuzzy sets defined on the universal set X whose membership functions are given by,

$$A(x)=\left\{\frac{0.5}{-1}+\frac{1}{0}+\frac{0.5}{1}+\frac{0.3}{2}\right\},\quad B(x)=\left\{\frac{0.5}{2}+\frac{1}{3}+\frac{0.5}{4}+\frac{0.3}{5}\right\}$$

Find f(A, B) if f: X x X → X is defined by $f(x_1, x_2) = x_1 + x_2 \ \forall\, x_1, x_2 \in X$

$$Ans: f(A,B)=\left\{\frac{0.5}{1}+\frac{0.5}{2}+\frac{1}{3}+\frac{0.5}{4}+\frac{0.5}{5}+\frac{0.3}{6}+\frac{0.3}{7}\right\}$$

Example 2: Let A and B be fuzzy sets defined on the universal set X whose membership functions are given by,

$$A(x)=\left\{\frac{0.4}{-2}+\frac{0.6}{-1}+\frac{1}{0}+\frac{0.5}{1}+\frac{0.3}{2}\right\}\quad,\quad B(x)=\left\{\frac{0.4}{-2}+\frac{0.6}{-1}+\frac{1}{0}+\frac{0.5}{1}+\frac{0.3}{2}\right\}$$

Find f(A, B) if f: X x X → X is defined by $f(x_1, x_2) = x_1 . x_2 \ \forall\, x_1, x_2 \in X$

$$Ans: f(A,B)=\left\{\frac{0.3}{-10}+\frac{0.4}{-8}+\frac{0.4}{-6}+\frac{0.3}{-5}+\frac{0.5}{-4}+\frac{0.6}{-3}+\frac{0.6}{-2}+\frac{0.4}{-1}+\frac{1}{0}+\frac{0.4}{1}+\frac{0.5}{2}+\frac{0.5}{3}+\frac{0.5}{4}+\frac{0.3}{5}+\frac{0.3}{6}+\frac{0.3}{8}+\frac{0.3}{10}\right\}$$

Example 3: Let fuzzy numbers A and B each defined on its own inverse be

$$A(x)=\left\{\frac{0.5}{1}+\frac{0.6}{2}+\frac{0.9}{3}+\frac{1}{4}+\frac{0.2}{5}\right\}\quad,\quad B(x)=\left\{\frac{0.3}{3}+\frac{0.4}{4}+\frac{0.7}{5}+\frac{1}{6}+\frac{0.3}{7}\right\}$$

Determine the membership grades for algebraic mapping f(A, B) =A .B (arithmetic product)

$$Ans: f(A,B)=\left\{\frac{0.3}{3}+\frac{0.4}{4}+\frac{0.5}{5}+\frac{0.5}{6}+\frac{0.3}{7}+\frac{0.4}{8}+\frac{0.3}{9}+\frac{0.6}{10}+\frac{0.6}{12}+\frac{0.3}{14}+\frac{0.7}{15}+\frac{0.4}{16}+\frac{0.9}{18}+\frac{0.7}{20}+\frac{0.3}{21}+\frac{1}{24}+\frac{0.2}{25}+\frac{0.3}{28}+\frac{0.2}{30}+\frac{0.2}{35}\right\}$$

Example 4: Let A and B be fuzzy sets defined on the universal set X=Z (set of integers) whose membership functions are given by,

$$A(x)=\left\{\frac{0.5}{1}+\frac{0.6}{2}+\frac{0.9}{3}+\frac{1}{4}+\frac{0.2}{5}\right\}\qquad B(x)=\left\{\frac{0.3}{3}+\frac{0.4}{4}+\frac{0.7}{5}+\frac{1}{6}+\frac{0.3}{7}\right\}$$

Find f (A, B) if f: X x X → X is defined by $f(x_1, x_2) = 2x_1 + x_2 \ \forall\, x_1, x_2 \in X$

$$Ans: f(A,B)=\left\{\frac{0.3}{5}+\frac{0.4}{6}+\frac{0.5}{7}+\frac{0.5}{8}+\frac{0.6}{9}+\frac{0.6}{10}+\frac{0.7}{11}+\frac{0.9}{12}+\frac{0.7}{13}+\frac{1}{14}+\frac{0.3}{15}+\frac{0.2}{16}+\frac{0.2}{17}\right\}$$

Example 5: Let fuzzy numbers A and B each defined on its own inverse be

$$A(x)=\left\{\frac{0.2}{1}+\frac{1}{2}+\frac{0.7}{4}+\frac{0.1}{8}\right\} \quad , B(x)=\left\{\frac{0.1}{1}+\frac{0.5}{2}+\frac{1}{4}+\frac{0.2}{8}\right\}$$

Determine the membership grades for algebraic mapping f (A, B) = A.B (arithmetic product)

Example 6: Let A and B be fuzzy sets defined on the universal set X whose membership functions are given by,

$$A(x)=\left\{\frac{0}{1}+\frac{0.2}{1.5}+\frac{0.35}{2}+\frac{0.15}{2.5}+\frac{0.5}{3}\right\}, \quad B(x)=\left\{\frac{0.2}{2}+\frac{0.35}{2.5}+\frac{0.4}{3}+\frac{0.15}{3.5}+\frac{0}{4}\right\}$$

Find f(A, B) if f: X x X → X is defined by $f(x_1, x_2) = x_1 + x_2 \;\; \forall \, x_1, x_2 \in X$

Example 7: Determine the membership grades for algebraic mapping f (A, B) = 2A - B where A and B are the fuzzy numbers.

$$A(x)=\left\{\frac{0.2}{1}+\frac{1}{2}+\frac{0.7}{4}+\frac{0.1}{8}\right\} \quad , \quad B(x)=\left\{\frac{0.1}{1}+\frac{0.5}{2}+\frac{1}{4}+\frac{0.2}{8}\right\}$$

Example 8: Let A and B be fuzzy sets defined on the universal set X whose membership functions are given by,

$$A(x)=\left\{\frac{0.35}{2}+\frac{0.15}{3}+\frac{0.5}{4}+\frac{0.25}{5}\right\} \quad , \quad B(x)=\left\{\frac{1}{0}+\frac{0.15}{1}+\frac{0.2}{2}+\frac{0.35}{3}\right\}$$

Find f(A, B) if f: X x X → X is defined by $f(x_1, x_2) = x_1 \cdot x_2 \;\; \forall \, x_1, x_2 \in X$

4. Method of developing Fuzzy Arithmetic is based on α – cut sets:

Let A and B are fuzzy numbers, if * be binary operation defined on set of real numbers (R) then, A* B by defining it's α – cut.

$$^{\alpha}A * B = {}^{\alpha}A * {}^{\alpha}B$$

The four arithmetic operations on closed intervals are defined as follows,

1) $[a, b] + [c, d] = [a + c, b + d]$

For example, $[2, 5] + [1, 3] = [2 + 1, 5 + 3] = [3, 8]$

2) $[a, b] - [c, d] = [a - d, b - c]$

For example, $[2, 5] - [1, 3] = [2 - 3, 5 - 1] = [-1, 4]$

3) $[a, b] \times [c, d] = \{\min [ac, ad, bc, bd], \max [ac, ad, bc, bd]\}$

For example,

$$[-1, 1] \times [-2, -0.5] = (\min[2, 0.5, -2, -0.5], \max[2, 0.5, -2, -0.5]) = (-2, 2)$$

4) $[a, b] / [c, d] = \left(\min\left[\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\right], \max\left[\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\right]\right)$

For example,

$$[4, 10] / [1, 2] = \left(\min\left[\frac{4}{1}, \frac{4}{2}, \frac{10}{1}, \frac{10}{2}\right], \max\left[\frac{4}{1}, \frac{4}{2}, \frac{10}{1}, \frac{10}{2}\right]\right) = (2, 10)$$

Example 1: Consider fuzzy number A and B defined by,

$$A(x)=\begin{cases}\dfrac{x+3}{3} & if\ -3<x\le 0\\ \dfrac{3-x}{3} & if\ 0<x\le 3\\ 0 & otherwise\end{cases}\qquad B(x)=\begin{cases}\dfrac{x-3}{3} & if\ 3<x\le 6\\ \dfrac{9-x}{3} & if\ 6<x\le 9\\ 0 & otherwise\end{cases}$$

Calculate fuzzy numbers i) A + B ii) A - B iii) A * B iv) A / B

Solution: We know that α-cut of fuzzy set A as, ${}^{\alpha}A=\{x\in X\,/\,A(x)\ge\alpha\}$

$A(x)\ge\alpha$	$A(x)\ge\alpha$
$\dfrac{x+3}{3}\ge\alpha$	$\dfrac{3-x}{3}\ge\alpha$
$x+3\ \ge 3\alpha$	$3-x\ge 3\alpha$
$x\ \ge 3\alpha-3$..........(1)	$3-3\alpha\ \ge x$..........(2)

From equation (1) and (2) we get, ${}^{\alpha}A=[3\alpha-3\,,\,3-3\alpha]$

We know that α-cut of fuzzy set B as, ${}^{\alpha}B=\{x\in X\,/\,B(x)\ge\alpha\}$

$B(x)\ge\alpha$	$B(x)\ge\alpha$
$\dfrac{x-3}{3}\ge\alpha$	$\dfrac{9-x}{3}\ge\alpha$
$x-3\ \ge 3\alpha$	$9-x\ge 3\alpha$
$x\ \ge 3\alpha+3$..........(3)	$9-3\alpha\ \ge x$..........(4)

From equation (3) and (4) we get, ${}^{\alpha}B=[3\alpha+3\,,\,9-3\alpha]$

1. for A+B:

We know that, ${}^{\alpha}A+B={}^{\alpha}A\ +{}^{\alpha}B$

$${}^{\alpha}A+B=[3\alpha-3\,,\,3-3\alpha]+[3\alpha+3\,,\,9-3\alpha]$$

$${}^{\alpha}A+B=[3\alpha-3+3\alpha+3\,,\,3-3\alpha+9-3\alpha]$$

$${}^{\alpha}A+B=[6\alpha\,,12-6\alpha]$$

$$i.e.\ \ 6\alpha\ \le\ \ x\ \ \le 12-6\alpha$$

$6\alpha\ =x$	$12-6\alpha\ =x$
$\alpha\ =\dfrac{x}{6}$	$\alpha=\dfrac{12-x}{6}$

$$\begin{array}{l|l} if\ \alpha = 0 \Rightarrow x = 0 & if\ \alpha = 1 \Rightarrow x = 6 \\ if\ \alpha = 1 \Rightarrow x = 6 & if\ \alpha = 0 \Rightarrow x = 12 \end{array}$$

$$Hence, \quad (A+B)(x) = \begin{cases} \dfrac{x}{6} & if\ 0 < x \le 6 \\ \dfrac{12-x}{6} & if\ 6 < x \le 12 \\ 0 & otherwise \end{cases}$$

2. for A - B:

We know that, ${}^{\alpha}A - B = {}^{\alpha}A \ - {}^{\alpha}B$

$${}^{\alpha}A - B = [3\alpha - 3\,, 3 - 3\alpha] - [3\alpha + 3\,, 9 - 3\alpha]$$

$${}^{\alpha}A - B = [3\alpha - 3 - 9 + 3\alpha\,, 3 - 3\alpha - 3\alpha - 3]$$

$${}^{\alpha}A - B = [6\alpha - 12\,, -6\alpha]$$

$$i.e.\ \ 6\alpha - 12 \le \quad x \quad \le -6\alpha$$

$$\begin{array}{l|l} 6\alpha - 12 = x & -6\alpha = x \\ \alpha = \dfrac{x+12}{6} & \alpha = \dfrac{-x}{6} \\ if\ \alpha = 0 \Rightarrow x = -12 & if\ \alpha = 1 \Rightarrow x = -6 \\ if\ \alpha = 1 \Rightarrow x = -6 & if\ \alpha = 0 \Rightarrow x = 0 \end{array}$$

$$Hence, \quad (A-B)(x) = \begin{cases} \dfrac{x+12}{6} & if\ -12 < x \le -6 \\ \dfrac{-x}{6} & if\ -6 < x \le 0 \\ 0 & otherwise \end{cases}$$

3. For A*B:

We know that, ${}^{\alpha}A\times B = {}^{\alpha}A \times {}^{\alpha}B$

$${}^{\alpha}A\times B = [3\alpha-3\,,3-3\alpha]\times[3\alpha+3\,,9-3\alpha]$$

$${}^{\alpha}A\times B = [\min\{(3\alpha-3)(3\alpha+3)\,,(3\alpha-3)(9-3\alpha),(3-3\alpha)(3\alpha+3),(3-3\alpha)(9-3\alpha)\},$$
$$\max\{(3\alpha-3)(3\alpha+3)\,,(3\alpha-3)(9-3\alpha),(3-3\alpha)(3\alpha+3),(3-3\alpha)(9-3\alpha)\}]$$

α	(3α-3) (3α+3)	(3α-3) (9-3α)	(3-3α) (3α+3)	(3-3α) (9-3α)
0.5	- 6.75	**- 11.25**	6.75	**11.25**

$${}^{\alpha}A\times B = [(3\alpha-3)(9-3\alpha),(3-3\alpha)(9-3\alpha)]$$

$${}^{\alpha}A\times B = [-9\alpha^2+36\alpha-27, 9\alpha^2-36\alpha+27]$$

$$i.e.\ -9\alpha^2+36\alpha-27\le x\le 9\alpha^2-36\alpha+27$$

$$-9\alpha^2+36\alpha-27=x$$

$$-9\alpha^2+36\alpha=x+27$$

$$\alpha^2-4\alpha=\frac{-x-27}{9}$$

$$\alpha^2-4\alpha+4=\frac{-x-27}{9}+4$$

$$(2-\alpha)^2=\frac{9-x}{9}$$

$$(2-\alpha)=\frac{\sqrt{9-x}}{3}$$

$$2-\frac{\sqrt{9-x}}{3}=\alpha$$

$$\frac{6-\sqrt{9-x}}{3}=\alpha$$

$$if\ \alpha=0\Rightarrow x=-27$$

$$if\ \alpha=1\Rightarrow x=0$$

$$9\alpha^2-36\alpha+27=x$$

$$9\alpha^2-36\alpha=x-27$$

$$\alpha^2-4\alpha=\frac{x-27}{9}$$

$$\alpha^2-4\alpha+4=\frac{x-27}{9}+4$$

$$(2-\alpha)^2=\frac{9+x}{9}$$

$$(2-\alpha)=\frac{\sqrt{9+x}}{3}$$

$$2-\frac{\sqrt{9+x}}{3}=\alpha$$

$$\frac{6-\sqrt{9+x}}{3}=\alpha$$

$$if\ \alpha=1\Rightarrow x=0$$

$$if\ \alpha=0\Rightarrow x=27$$

$$Hence,\quad (A\times B)(x)=\begin{cases}\dfrac{6-\sqrt{9-x}}{3} & if\ -27<x\le 0\\[2mm] \dfrac{6-\sqrt{9+x}}{3} & if\ 0<x\le 27\\[2mm] 0 & otherwise\end{cases}$$

4. for A/B:

We know that, $^{\alpha}A/B = {}^{\alpha}A \ / \ {}^{\alpha}B$

$$^{\alpha}A/B = [3\alpha-3\,,3-3\alpha]/[3\alpha+3\,,9-3\alpha]$$

$$^{\alpha}A/B = \left[\min\left(\frac{3\alpha-3}{3\alpha+3}\right),\left(\frac{3\alpha-3}{9-3\alpha}\right),\left(\frac{3-3\alpha}{3\alpha+3}\right),\left(\frac{3-3\alpha}{9-3\alpha}\right),\max\left(\frac{3\alpha-3}{3\alpha+3}\right),\left(\frac{3\alpha-3}{9-3\alpha}\right),\left(\frac{3-3\alpha}{3\alpha+3}\right),\left(\frac{3-3\alpha}{9-3\alpha}\right)\right]$$

α	(3α-3)/ (3α+3)	(3α-3)/ (9-3α)	(3-3α)/ (3α+3)	(3-3α) /(9-3α)
0.5	**- 0.3333**	-0.2	**0.3333**	0.2

$$^{\alpha}A/B = \left[\left(\frac{3\alpha-3}{3\alpha+3}\right),\left(\frac{3-3\alpha}{3\alpha+3}\right)\right]$$

$$i.e.\left(\frac{3\alpha-3}{3\alpha+3}\right) \le x \le \left(\frac{3-3\alpha}{3\alpha+3}\right)$$

$\frac{3\alpha-3}{3\alpha+3} = x$	$\frac{3-3\alpha}{3\alpha+3} = x$
$3\alpha-3 = 3\alpha x+3x$	$3-3\alpha = 3\alpha x+3x$
$3\alpha-3\alpha x = 3+3x$	$3-3x = 3\alpha x+3\alpha$
$\alpha(1-x) = 1+x$	$(1-x) = \alpha(1+x)$
$\alpha = \frac{1+x}{1-x}$	$\alpha = \frac{1-x}{1+x}$
$if\ \alpha = 0 \Rightarrow x = -1$	$if\ \alpha = 1 \Rightarrow x = 0$
$if\ \alpha = 1 \Rightarrow x = 0$	$if\ \alpha = 0 \Rightarrow x = 1$

$$Hence, \quad (A/B)(x) = \begin{cases} \frac{1+x}{1-x} & if\ -1 < x \le 0 \\ \frac{1-x}{1+x} & if\ 0 < x \le 1 \\ 0 & otherwise \end{cases}$$

Example 2: Consider fuzzy number A and B defined by,

$$A(x)=\begin{cases}\dfrac{x+4}{4} & if\ -4<x\le 0\\ \dfrac{4-x}{4} & if\ 0<x\le 4\\ 0 & otherwise\end{cases} \qquad B(x)=\begin{cases}\dfrac{x-4}{4} & if\ 4<x\le 8\\ \dfrac{12-x}{4} & if\ 8<x\le 12\\ 0 & otherwise\end{cases}$$

Calculate fuzzy numbers i) A + B ii) A - B iii) A * B iv) A / B

Solution: We know that α-cut of fuzzy set A as, ${}^{\alpha}A=\{x\in X / A(x)\ge \alpha\}$

$A(x)\ge\alpha$	$A(x)\ge\alpha$
$\dfrac{x+4}{4}\ge\alpha$	$\dfrac{4-x}{4}\ge\alpha$
$x+4\ge 4\alpha$	$4-x\ge 4\alpha$
$x\ge 4\alpha-4$..........(1)	$4-4\alpha\ge x$..........(2)

From equation (1) and (2) we get, ${}^{\alpha}A=[4\alpha-4\,,\,4-4\alpha]$

We know that α-cut of fuzzy set B as, ${}^{\alpha}B=\{x\in X / B(x)\ge \alpha\}$

$B(x)\ge\alpha$	$B(x)\ge\alpha$
$\dfrac{x-4}{4}\ge\alpha$	$\dfrac{12-x}{4}\ge\alpha$
$x-4\ge 4\alpha$	$12-x\ge 4\alpha$
$x\ge 4\alpha+4$..........(3)	$12-4\alpha\ge x$..........(4)

From equation (3) and (4) we get, ${}^{\alpha}B=[4\alpha+4\,,\,12-4\alpha]$

1. for A+B:

We know that, ${}^{\alpha}A+B={}^{\alpha}A+{}^{\alpha}B$

$${}^{\alpha}A+B=[4\alpha-4\,,\,4-4\alpha]+[4\alpha+4\,,\,12-4\alpha]$$

$${}^{\alpha}A+B=[4\alpha-4+4\alpha+4\,,\,4-4\alpha+12-4\alpha]$$

$${}^{\alpha}A+B=[8\alpha\,,\,16-8\alpha]$$

$$i.e.\ \ 8\alpha\le x\le 16-8\alpha$$

$8\alpha=x$	$16-8\alpha=x$
$\alpha=\dfrac{x}{8}$	$\alpha=\dfrac{16-x}{8}$

$$if\ \alpha = 0 \Rightarrow x = 0 \qquad\qquad if\ \alpha = 1 \Rightarrow x = 8$$

$$if\ \alpha = 1 \Rightarrow x = 8 \qquad\qquad if\ \alpha = 0 \Rightarrow x = 16$$

$$Hence, \quad (A+B)(x) = \begin{cases} \dfrac{x}{8} & if\ 0 < x \le 8 \\ \dfrac{16-x}{8} & if\ 8 < x \le 16 \\ 0 & otherwise \end{cases}$$

2. for A - B:

We know that, ${}^{\alpha}A - B = {}^{\alpha}A\ - {}^{\alpha}B$

$${}^{\alpha}A - B = [4\alpha - 4\,, 4 - 4\alpha] - [4\alpha + 4\,, 12 - 4\alpha]$$

$${}^{\alpha}A - B = [4\alpha - 4 - 12 + 4\alpha\,, 4 - 4\alpha - 4\alpha - 4]$$

$${}^{\alpha}A - B = [8\alpha - 16\,, -8\alpha]$$

$$i.e.\ \ 8\alpha - 16 \le \quad x \quad \le -8\alpha$$

$$8\alpha - 16 = x \qquad\qquad -8\alpha = x$$

$$\alpha = \frac{x+16}{8} \qquad\qquad \alpha = \frac{-x}{8}$$

$$if\ \alpha = 0 \Rightarrow x = -16 \qquad\qquad if\ \alpha = 1 \Rightarrow x = -8$$

$$if\ \alpha = 1 \Rightarrow x = -8 \qquad\qquad if\ \alpha = 0 \Rightarrow x = 0$$

$$Hence, \quad (A-B)(x) = \begin{cases} \dfrac{x+16}{8} & if\ -16 < x \le -8 \\ \dfrac{-x}{8} & if\ -8 < x \le 0 \\ 0 & otherwise \end{cases}$$

3. for A*B:

We know that, $^{\alpha}A\times B={}^{\alpha}A\ \times{}^{\alpha}B$

$$^{\alpha}A\times B=[4\alpha-4\,,4-4\alpha]\times[4\alpha+4\,,12-4\alpha]$$

$$^{\alpha}A\times B=[\min\{(4\alpha-4)(4\alpha+4)\,,(4\alpha-4)(12-4\alpha),(4-4\alpha)(4\alpha+4),(4-4\alpha)(12-4\alpha)\},$$
$$\max\{(4\alpha-4)(4\alpha+4)\,,(4\alpha-4)(12-4\alpha),(4-4\alpha)(4\alpha+4),(4-4\alpha)(12-4\alpha)\}]$$

α	(4α-4) (4α+4)	(4α-4) (12-4α)	(4-4α) (4α+4)	(4-4α) (12-4α)
0.5	- 12	**- 20**	12	**20**

$$^{\alpha}A\times B=[(4\alpha-4)(12-4\alpha),(4-4\alpha)(12-4\alpha)]$$

$$^{\alpha}A\times B=[-16\alpha^2+64\alpha-48,16\alpha^2-64\alpha+48]$$

$$i.e.\ -16\alpha^2+64\alpha-48\le x\ \le 16\alpha^2-64\alpha+48$$

$$-16\alpha^2+64\alpha-48=x$$
$$-16\alpha^2+64\alpha=x+48$$
$$\alpha^2-4\alpha=\frac{-x-48}{16}$$
$$\alpha^2-4\alpha+4=\frac{-x-48}{16}+4$$
$$(2-\alpha)^2=\frac{16-x}{16}$$
$$(2-\alpha)=\frac{\sqrt{16-x}}{4}$$
$$2-\frac{\sqrt{16-x}}{4}=\alpha$$
$$\frac{8-\sqrt{16-x}}{4}=\alpha$$
$$if\ \alpha=0\Rightarrow x=-48$$
$$if\ \alpha=1\Rightarrow x=0$$

$$16\alpha^2-64\alpha+48=x$$
$$16\alpha^2-64\alpha=x-48$$
$$\alpha^2-4\alpha=\frac{x-48}{16}$$
$$\alpha^2-4\alpha+4=\frac{x-48}{16}+4$$
$$(2-\alpha)^2=\frac{x+16}{16}$$
$$(2-\alpha)=\frac{\sqrt{16+x}}{4}$$
$$2-\frac{\sqrt{16+x}}{4}=\alpha$$
$$\frac{8-\sqrt{16+x}}{4}=\alpha$$
$$if\ \alpha=1\Rightarrow x=0$$
$$if\ \alpha=0\Rightarrow x=48$$

$$Hence,\quad (A\times B)(x)=\begin{cases}\dfrac{8-\sqrt{16-x}}{4} & if\ -48<x\le 0\\ \dfrac{8-\sqrt{16+x}}{4} & if\ 0<x\le 48\\ 0 & otherwise\end{cases}$$

4. for A/B:

We know that, $^{\alpha}A/B = {}^{\alpha}A\ /\ {}^{\alpha}B$

$$^{\alpha}A/B=[4\alpha-4\,,4-4\alpha]/[4\alpha+4\,,12-4\alpha]$$

$$^{\alpha}A/B=\left[\min\left(\frac{4\alpha-4}{4\alpha+4}\right),\left(\frac{4\alpha-4}{12-4\alpha}\right),\left(\frac{4-4\alpha}{4\alpha+4}\right),\left(\frac{4-4\alpha}{12-4\alpha}\right),\max\left(\frac{4\alpha-4}{4\alpha+4}\right),\left(\frac{4\alpha-4}{12-4\alpha}\right),\left(\frac{4-4\alpha}{4\alpha+4}\right),\left(\frac{4-4\alpha}{12-4\alpha}\right)\right]$$

α	(4α - 4)/ (4α + 4)	(4α - 4)/ (12 - 4α)	(4 - 4α)/ (4α + 4)	(4 - 4α) /(12 - 4α)
0.5	**- 0.3333**	- 0.2	**0.3333**	0.2

$$^{\alpha}A/B=\left[\left(\frac{4\alpha-4}{4\alpha+4}\right),\left(\frac{4-4\alpha}{4\alpha+4}\right)\right]$$

$$i.e.\left(\frac{4\alpha-4}{4\alpha+4}\right)\le x\ \le\left(\frac{4-4\alpha}{4\alpha+4}\right)$$

$$\frac{4\alpha-4}{4\alpha+4}=x$$
$$4\alpha-4=4\alpha x+4x$$
$$4\alpha-4\alpha x=4+4x$$
$$\alpha(1-x)=1+x$$
$$\alpha=\frac{1+x}{1-x}$$
$$if\ \alpha=0\Rightarrow x=-1$$
$$if\ \alpha=1\Rightarrow x=0$$

$$\frac{4-4\alpha}{4\alpha+4}=x$$
$$4-4\alpha=4\alpha x+4x$$
$$4-4x=4\alpha x+4\alpha$$
$$(1-x)=\alpha(1+x)$$
$$\alpha=\frac{1-x}{1+x}$$
$$if\ \alpha=1\Rightarrow x=0$$
$$if\ \alpha=0\Rightarrow x=1$$

$$Hence,\quad (A/B)(x)=\begin{cases}\dfrac{1+x}{1-x} & if\ -1<x\le 0\\ \dfrac{1-x}{1+x} & if\ 0<x\ \le 1\\ 0 & otherwise\end{cases}$$

Examples for Practice

Example 1: Consider fuzzy number A and B defined by,

$$A(x)=\begin{cases}\dfrac{x-1}{3} & if\ 1<x\le 4\\ \dfrac{7-x}{3} & if\ 4<x\le 7\\ 0 & otherwise\end{cases}\qquad B(x)=\begin{cases}\dfrac{x-7}{3} & if\ 7<x\le 10\\ \dfrac{13-x}{3} & if\ 10<x\le 13\\ 0 & otherwise\end{cases}$$

Calculate fuzzy numbers i) A + B ii) A - B iii) A * B iv) A / B

Solution:

$$(A+B)(x)=\begin{cases}\dfrac{x-8}{6} & if\ 8<x\le 14\\ \dfrac{20-x}{6} & if\ 14<x\le 20\\ 0 & otherwise\end{cases}\qquad (A-B)(x)=\begin{cases}\dfrac{x-12}{6} & if\ -12<x\le -6\\ \dfrac{-x}{6} & if\ -6<x\le 0\\ 0 & otherwise\end{cases}$$

$$(A\times B)(x)=\begin{cases}\dfrac{-4+\sqrt{9+x}}{3} & if\ 7<x\le 40\\ \dfrac{10-\sqrt{9+x}}{3} & if\ 40<x\le 91\\ 0 & otherwise\end{cases}\qquad (A/B)(x)=\begin{cases}\dfrac{13x-1}{3(x+1)} & if\ \dfrac{1}{13}<x\le \dfrac{2}{5}\\ \dfrac{7-7x}{3(1+x)} & if\ \dfrac{2}{5}<x\le 1\\ 0 & otherwise\end{cases}$$

Example 2: Consider fuzzy number A and B defined by,

$$A(x)=\begin{cases}\dfrac{x+2}{2} & if\ -2<x\le 0\\ \dfrac{2-x}{2} & if\ 0<x\le 2\\ 0 & otherwise\end{cases}\qquad B(x)=\begin{cases}\dfrac{x-2}{2} & if\ 2<x\le 4\\ \dfrac{6-x}{2} & if\ 4<x\le 6\\ 0 & otherwise\end{cases}$$

Calculate fuzzy numbers i) A + B ii) A - B iii) A * B iv) A / B

Solution:

$$(A+B)(x)=\begin{cases}\dfrac{x}{4} & if\ 0<x\le 4\\ \dfrac{8-x}{4} & if\ 4<x\le 8\\ 0 & otherwise\end{cases}\qquad (A-B)(x)=\begin{cases}\dfrac{x+8}{4} & if\ -8<x\le -4\\ \dfrac{-x}{4} & if\ -4<x\le 0\\ 0 & otherwise\end{cases}$$

$$(A \times B)(x) = \begin{cases} \dfrac{-4+\sqrt{4-x}}{-2} & if\ -12 < x \leq 0 \\ \dfrac{4-\sqrt{4+x}}{2} & if\, 0 < x \leq 12 \\ 0 & otherwise \end{cases}$$

$$(A/B)(x) = \begin{cases} \dfrac{2x+2}{2(1-x)} & if\ -1 < x \leq 0 \\ \dfrac{2-2x}{2(1+x)} & if\, 0 < x \leq 1 \\ 0 & otherwise \end{cases}$$

Example 3: Consider fuzzy number A and B defined by,

$$A(x) = \begin{cases} \dfrac{x-1}{2} & if\ 1 < x \leq 3 \\ \dfrac{5-x}{2} & if\ 3 < x \leq 5 \\ 0 & otherwise \end{cases} \qquad B(x) = \begin{cases} \dfrac{x-3}{2} & if\ 3 < x \leq 5 \\ \dfrac{7-x}{2} & if\ 5 < x \leq 7 \\ 0 & otherwise \end{cases}$$

Calculate fuzzy numbers i) A + B ii) A - B iii) A * B iv) A / B

Solution:

$$(A+B)(x) = \begin{cases} \dfrac{x-4}{4} & if\ 4 < x \leq 8 \\ \dfrac{12-x}{4} & if\ 8 < x \leq 12 \\ 0 & otherwise \end{cases}$$

$$(A-B)(x) = \begin{cases} \dfrac{x+6}{4} & if\ -6 < x \leq -2 \\ \dfrac{2-x}{4} & if\ -2 < x \leq 2 \\ 0 & otherwise \end{cases}$$

$$(A \times B)(x) = \begin{cases} \dfrac{-2+\sqrt{1+x}}{2} & if\ 3 < x \leq 15 \\ \dfrac{6-\sqrt{1+x}}{2} & if\, 15 < x \leq 35 \\ 0 & otherwise \end{cases}$$

$$(A/B)(x) = \begin{cases} \dfrac{7x-1}{2(1+x)} & if\ \dfrac{1}{7} < x \leq \dfrac{3}{5} \\ \dfrac{5-3x}{2(1+x)} & if\ \dfrac{3}{5} < x \leq \dfrac{5}{3} \\ 0 & otherwise \end{cases}$$

5. Fuzzy Equation:

Fuzzy equation is important area in fuzzy set theory and fuzzy numbers. In fuzzy equations coefficient and unknowns are fuzzy numbers and formulas are constructed by operation of fuzzy arithmetic.

1. Fuzzy equation of the type A + X = B:

To solve the fuzzy equation by representing it's α – cuts,

$$^{\alpha}A = \left[a_1 , a_2 \right] \quad \text{and} \quad ^{\alpha}B = \left[b_1 , b_2 \right]$$

$$\text{Then, } \; ^{\alpha}A + {}^{\alpha}X = {}^{\alpha}B \;\Rightarrow\; {}^{\alpha}X = {}^{\alpha}B - {}^{\alpha}A$$

$$\therefore {}^{\alpha}X = \left[b_1 , b_2 \right] - \left[a_1 , a_2 \right]$$

$$\therefore {}^{\alpha}X = \left[b_1 - a_1 , b_2 - a_2 \right]$$

Then solution of fuzzy equation by using Decomposition theorem,

$$X = \bigcup_{\alpha \in (0,1]} {}_{\alpha}X \;, \quad \text{Where, } {}_{\alpha}X = \alpha \, {}^{\alpha}X \quad \textit{i.e special fuzzy set}$$

2. Fuzzy equation of the type AX = B:

To solve the fuzzy equation by representing it's α – cuts,

$$^{\alpha}A = \left[a_1 , a_2 \right] \quad \text{and} \quad ^{\alpha}B = \left[b_1 , b_2 \right]$$

$$\text{Then, } \; ^{\alpha}A \times {}^{\alpha}X = {}^{\alpha}B \;\Rightarrow\; {}^{\alpha}X = \frac{^{\alpha}B}{^{\alpha}A}$$

$$\therefore {}^{\alpha}X = \frac{[b_1 , b_2]}{[a_1 , a_2]}$$

$$\therefore {}^{\alpha}X = \left[\frac{b_1}{a_1} , \frac{b_2}{a_2} \right]$$

Then solution of fuzzy equation by using Decomposition theorem,

$$X = \bigcup_{\alpha \in (0,1]} {}_{\alpha}X \;, \quad \text{Where, } {}_{\alpha}X = \alpha \, {}^{\alpha}X \quad \textit{i.e special fuzzy set}$$

Solved Examples

Example 1: A and B be two fuzzy numbers such that,

$$A(x) = \left\{ \frac{0.2}{[0,1)} + \frac{0.6}{[1,2)} + \frac{0.8}{[2,3)} + \frac{0.9}{[3,4)} + \frac{1}{4} + \frac{0.5}{(4,5]} + \frac{0.1}{(5,6]} \right\}$$

$$B(x) = \left\{ \frac{0.1}{[0,1)} + \frac{0.2}{[1,2)} + \frac{0.6}{[2,3)} + \frac{0.7}{[3,4)} + \frac{0.8}{[4,5)} + \frac{0.9}{[5,6)} + \frac{1}{6} + \frac{0.5}{(6,7]} + \frac{0.4}{(7,8]} + \frac{0.2}{(8,9]} + \frac{0.1}{(9,10]} \right\}$$

Then find the solution of the equation A + X = B.

Solution: To solve the fuzzy equation by representing it's α – cuts,

$$^{\alpha}A = \left[a_1 , a_2 \right] \quad \text{and} \quad ^{\alpha}B = \left[b_1 , b_2 \right]$$

Then, $^{\alpha}A + ^{\alpha}X = ^{\alpha}B \Rightarrow ^{\alpha}X = ^{\alpha}B - ^{\alpha}A$

$$\therefore ^{\alpha}X = \left[b_1 , b_2 \right] - \left[a_1 , a_2 \right]$$

$$\therefore ^{\alpha}X = \left[b_1 - a_1 , b_2 - a_2 \right]$$

The all relevant α – cuts of A, B and X are given in the following table,

α	$^{\alpha}A$	$^{\alpha}B$	$^{\alpha}X = {}^{\alpha}B - {}^{\alpha}A$
0.1	[0 , 6]	[0 , 10]	[0 , 4]
0.2	[0 , 5]	[1 , 9]	[1 , 4]
0.3	[1 , 5]	[2 , 8]	[1 , 3]
0.4	[1 , 5]	[2 , 8]	[1 , 3]
0.5	[1 , 5]	[2 , 7]	[1 , 2]
0.6	[1 , 4]	[2 , 6]	[1 , 2]
0.7	[2 , 4]	[3 , 6]	[1 , 2]
0.8	[2 , 4]	[4 , 6]	[2 , 2]
0.9	[3 , 4]	[5 , 6]	[2 , 2]
1	[4 , 4]	[6 , 6]	[2 , 2]

Therefore the range of X is [0, 4] is divided in to [0, 1), [1, 2), 2, (2, 3], (3, 4]

Then solution of fuzzy equation by using Decomposition theorem,

$$X = \bigcup_{\alpha \in (0,1]} {}_{\alpha}X \text{ ,} \quad \text{Where, } {}_{\alpha}X = \alpha \, ^{\alpha}X \;\; i.e \; special \;\; fuzzy \; set$$

$$X = \left\{ \frac{0.1}{[0,1)} + \frac{0.7}{[1,2)} + \frac{1}{2} + \frac{0.4}{(2,3]} + \frac{0.2}{(3,4]} \right\}$$

Example 2: A and B be two fuzzy numbers such that,

$$A(x)=\left\{\frac{0.2}{[0,1)}+\frac{0.5}{[1,2)}+\frac{0.6}{[2,3)}+\frac{0.8}{[3,4)}+\frac{0.9}{[4,5)}+\frac{1}{5}+\frac{0.4}{(5,6]}+\frac{0.3}{(6,7]}+\frac{0.1}{(7,8]}\right\}$$

$$B(x)=\left\{\frac{0.1}{[0,1)}+\frac{0.2}{[1,2)}+\frac{0.5}{[2,3)}+\frac{0.6}{[3,4)}+\frac{0.7}{[4,5)}+\frac{0.8}{[5,6)}+\frac{0.9}{[6,7)}+\frac{1}{7}+\frac{0.5}{(7,8]}+\frac{0.4}{(8,9]}+\frac{0.3}{(9,10]}+\frac{0.2}{(10,11]}+\frac{0.1}{(11,12]}\right\}$$

Then find the solution of the equation A + X = B.

Solution: To solve the fuzzy equation by representing it's α – cuts,

$$^{\alpha}A=\left[a_1\,,a_2\right] \quad \text{and} \quad {}^{\alpha}B=\left[b_1\,,b_2\right]$$

$$\text{Then,} \quad {}^{\alpha}A+{}^{\alpha}X={}^{\alpha}B \;\Rightarrow {}^{\alpha}X={}^{\alpha}B-{}^{\alpha}A$$

$$\therefore {}^{\alpha}X=\left[b_1\,,b_2\right]-\left[a_1\,,a_2\right]$$

$$\therefore {}^{\alpha}X=\left[b_1-a_1\,,b_2-a_2\right]$$

The all relevant α – cuts of A, B and X are given in the following table,

α	**α_A**	**α_B**	**$\alpha_X = \alpha_B - \alpha_A$**
0.1	[0 , 8]	[0 , 12]	[0 , 4]
0.2	[0 , 7]	[1 , 11]	[1 , 4]
0.3	[1 , 7]	[2 , 10]	[1 , 3]
0.4	[1 , 6]	[2 , 9]	[1 , 3]
0.5	[1 , 5]	[2 , 8]	[1 , 3]
0.6	[2 , 5]	[3 , 7]	[1 , 2]
0.7	[3 , 5]	[4 , 7]	[1 , 2]
0.8	[3 , 5]	[5 , 7]	[2 , 2]
0.9	[4 , 5]	[6 , 7]	[2 , 2]
1	[5 , 5]	[7 , 7]	[2 , 2]

Therefore the range of X is [0, 4] is divided in to [0, 1), [1, 2), 2, (2, 3], (3, 4]

Then solution of fuzzy equation by using Decomposition theorem,

$$X=\bigcup_{\alpha\in(0,1]}{}_{\alpha}X\,, \quad \text{Where, } {}_{\alpha}X=\alpha\,{}^{\alpha}X \;\; \textit{i.e special fuzzy set}$$

$$X=\left\{\frac{0.1}{[0,1)}+\frac{0.7}{[1,2)}+\frac{1}{2}+\frac{0.5}{(2,3]}+\frac{0.2}{(3,4]}\right\}$$

Example 3: Solve the fuzzy equation for A + X = B,

$$A(x) = \begin{cases} \dfrac{x+2}{2} & if\ -2 < x \le 0 \\ \dfrac{2-x}{2} & if\ 0 < x \le 2 \\ 0 & otherwise \end{cases} \qquad B(x) = \begin{cases} \dfrac{x-2}{2} & if\ 2 < x \le 4 \\ \dfrac{6-x}{2} & if\ 4 < x \le 6 \\ 0 & otherwise \end{cases}$$

Solution: We know that α-cut of fuzzy set A as, ${}^{\alpha}A = \{x \in X / A(x) \ge \alpha\}$

$A(x) \ge \alpha$	$A(x) \ge \alpha$
$\dfrac{x+2}{2} \ge \alpha$	$\dfrac{2-x}{2} \ge \alpha$
$x+2 \ge 2\alpha$	$2-x \ge 2\alpha$
$x \ge 2\alpha - 2$..........(1)	$2-2\alpha \ge x$..........(2)

From equation (1) and (2) we get, ${}^{\alpha}A = [2\alpha - 2 , 2 - 2\alpha]$

We know that α-cut of fuzzy set B as, ${}^{\alpha}B = \{x \in X / B(x) \ge \alpha\}$

$B(x) \ge \alpha$	$B(x) \ge \alpha$
$\dfrac{x-2}{2} \ge \alpha$	$\dfrac{6-x}{2} \ge \alpha$
$x-2 \ge 2\alpha$	$6-x \ge 2\alpha$
$x \ge 2\alpha + 2$..........(3)	$6-2\alpha \ge x$..........(4)

From equation (3) and (4) we get, ${}^{\alpha}B = [2\alpha + 2 , 6 - 2\alpha]$

Then, ${}^{\alpha}X = {}^{\alpha}B - {}^{\alpha}A \quad \therefore {}^{\alpha}X = [b_1 , b_2] - [a_1 , a_2]$

$\therefore {}^{\alpha}X = [2\alpha + 2 , 6 - 2\alpha] - [2\alpha - 2 , 2 - 2\alpha]$

$\therefore {}^{\alpha}X = [2\alpha + 2 - 2\alpha + 2 , 6 - 2\alpha - 2 + 2\alpha]$ *we know that* ${}^{\alpha}X = [b_1 - a_1 , b_2 - a_2]$

$\therefore {}^{\alpha}X = [4, 4]$

Then solution of fuzzy equation by using Decomposition theorem,

$$X = \bigcup_{\alpha \in (0,1]} {}_{\alpha}X \,, \quad \text{Where, } {}_{\alpha}X = \alpha\ {}^{\alpha}X \quad i.e\ special\ fuzzy\ set$$

$$X(x) = \begin{cases} 1 & if\ x = 4 \\ 0 & otherwise \end{cases}$$

Example 4: Solve the fuzzy equation for A+X = B,

$$A(x) = \begin{cases} x-4 & if\ 4 < x \le 5 \\ 6-x & if\ 5 < x \le 6 \\ 0 & otherwise \end{cases} \qquad B(x) = \begin{cases} \dfrac{x-10}{5} & if\ 10 < x \le 15 \\ \dfrac{25-x}{10} & if\ 15 < x \le 25 \\ 0 & otherwise \end{cases}$$

Solution: We know that α-cut of fuzzy set A as, ${}^{\alpha}A = \{x \in X / A(x) \ge \alpha\}$

$A(x) \ge \alpha$	$A(x) \ge \alpha$
$x - 4 \ge \alpha$	$6 - x \ge \alpha$
$x \ge \alpha + 4$...........(1)	$6 - \alpha \ge x$...........(2)

From equation (1) and (2) we get, ${}^{\alpha}A = [\alpha + 4, 6 - \alpha]$

We know that α-cut of fuzzy set B as, ${}^{\alpha}B = \{x \in X / B(x) \ge \alpha\}$

$B(x) \ge \alpha$	$B(x) \ge \alpha$
$\dfrac{x-10}{5} \ge \alpha$	$\dfrac{25-x}{10} \ge \alpha$
$x - 10 \ge 5\alpha$	$25 - x \ge 10\alpha$
$x \ge 5\alpha + 10$...........(3)	$25 - 10\alpha \ge x$...........(4)

From equation (3) and (4) we get, ${}^{\alpha}B = [5\alpha + 10, 25 - 10\alpha]$

Then, ${}^{\alpha}X = {}^{\alpha}B - {}^{\alpha}A \qquad \therefore {}^{\alpha}X = [b_1, b_2] - [a_1, a_2]$

$$\therefore {}^{\alpha}X = [5\alpha + 10, 25 - 10\alpha] - [\alpha + 4, 6 - \alpha]$$

we know that ${}^{\alpha}X = [b_1 - a_1, b_2 - a_2]$

$$\therefore {}^{\alpha}X = [5\alpha + 10 - \alpha - 4, 25 - 10\alpha - 6 + \alpha]$$

$$\therefore {}^{\alpha}X = [4\alpha + 6, 19 - 9\alpha]$$

$$i.e.\ \ 4\alpha + 6 \le x \le 19 - 9\alpha$$

$4\alpha + 6 = x$	$19 - 9\alpha = x$
$\alpha = \dfrac{x-6}{4}$	$\alpha = \dfrac{19-x}{9}$
$if\ \alpha = 0 \Rightarrow x = 6$	$if\ \alpha = 1 \Rightarrow x = 10$
$if\ \alpha = 1 \Rightarrow x = 10$	$if\ \alpha = 0 \Rightarrow x = 19$

Then solution of fuzzy equation by using Decomposition theorem,

$$X = \bigcup_{\alpha \in (0,1]} {}_{\alpha}X \,, \quad \text{Where, } {}_{\alpha}X = \alpha \, {}^{\alpha}X \quad \textit{i.e special fuzzy set}$$

$$\textit{Hence,} \quad X(x) = \begin{cases} \dfrac{x-6}{4} & \textit{if } 6 < x \le 10 \\ \dfrac{19-x}{9} & \textit{if } 10 < x \le 19 \\ 0 & \textit{otherwise} \end{cases}$$

Example 5: Solve the fuzzy equation for A . X = B,

$$A(x) = \begin{cases} x-3 & \textit{if } 3 < x \le 4 \\ 5-x & \textit{if } 4 < x \le 5 \\ 0 & \textit{otherwise} \end{cases} \qquad B(x) = \begin{cases} \dfrac{x-12}{8} & \textit{if } 12 < x \le 20 \\ \dfrac{32-x}{12} & \textit{if } 20 < x \le 32 \\ 0 & \textit{otherwise} \end{cases}$$

Solution: We know that α-cut of fuzzy set A as, ${}^{\alpha}A = \{ x \in X / A(x) \ge \alpha \}$

$A(x) \ge \alpha$	$A(x) \ge \alpha$
$x - 3 \ge \alpha$	$5 - x \ge \alpha$
$x \ge \alpha + 3$ ....(1)	$5 - \alpha \ge x$ ....(2)

From equation (1) and (2) we get, ${}^{\alpha}A = [\alpha + 3 \, , 5 - \alpha]$

We know that α-cut of fuzzy set B as, ${}^{\alpha}B = \{ x \in X / B(x) \ge \alpha \}$

$B(x) \ge \alpha$	$B(x) \ge \alpha$
$\dfrac{x-12}{8} \ge \alpha$	$\dfrac{32-x}{12} \ge \alpha$
$x - 12 \ge 8\alpha$	$32 - x \ge 12\alpha$
$x \ge 8\alpha + 12$ ....(3)	$32 - 12\alpha \ge x$ ....(4)

From equation (3) and (4) we get, ${}^{\alpha}B = [8\alpha + 12 \, , 32 - 12\alpha]$

$$\text{Then, } {}^{\alpha}A \times {}^{\alpha}X = {}^{\alpha}B \quad \Rightarrow {}^{\alpha}X = \frac{{}^{\alpha}B}{{}^{\alpha}A} \quad \therefore {}^{\alpha}X = \frac{[b_1 \, , b_2 \,]}{[a_1 \, , a_2 \,]}$$

$$\therefore {}^{\alpha}X = \frac{[8\alpha + 12 \, , 32 - 12\alpha]}{[\alpha + 3 \, , 5 - \alpha]} \qquad \textit{we know that } {}^{\alpha}X = \left[\frac{b_1}{a_1}, \frac{b_2}{a_2} \right]$$

$$\therefore {}^{\alpha}X = \left[\frac{8\alpha+12}{\alpha+3}, \frac{32-12\alpha}{5-\alpha}\right]$$

$$i.e.\left(\frac{8\alpha+12}{\alpha+3}\right) \le x \le \left(\frac{32-12\alpha}{5-\alpha}\right)$$

$$\frac{8\alpha+12}{\alpha+3} = x$$

$$8\alpha + 12 = \alpha x + 3x$$

$$8\alpha - \alpha x = 3x - 12$$

$$\alpha(8-x) = 3x - 12$$

$$\alpha = \frac{3x-12}{8-x}$$

$$if\ \alpha = 0 \Rightarrow x = 4$$

$$if\ \alpha = 1 \Rightarrow x = 5$$

$$\frac{32-12\alpha}{5-\alpha} = x$$

$$32 - 12\alpha = 5x - \alpha x$$

$$12\alpha - \alpha x = 32 - 5x$$

$$\alpha(12-x) = 32 - 5x$$

$$\alpha = \frac{32-5x}{12-x}$$

$$if\ \alpha = 1 \Rightarrow x = 5$$

$$if\ \alpha = 0 \Rightarrow x = \frac{32}{5}$$

Then solution of fuzzy equation by using Decomposition theorem,

$$X = \bigcup_{\alpha\in(0,1]} {}_{\alpha}X \quad , \quad \text{Where, } {}_{\alpha}X = \alpha\ {}^{\alpha}X \quad i.e\ special\ fuzzy\ set$$

$$Hence, \quad X(x) = \begin{cases} \dfrac{3x-12}{8-x} & if\ 4 < x \le 5 \\ \dfrac{32-5x}{12-x} & if\ 5 < x \le \dfrac{32}{5} \\ 0 & otherwise \end{cases}$$

Example 6: Solve the fuzzy equation for A . X = B,

$$A(x) = \begin{cases} \dfrac{x-2}{2} & if\ 2 < x \le 4 \\ \dfrac{6-x}{2} & if\ 4 < x \le 6 \\ 0 & otherwise \end{cases} \qquad B(x) = \begin{cases} \dfrac{x-6}{2} & if\ 6 < x \le 8 \\ \dfrac{10-x}{2} & if\ 8 < x \le 10 \\ 0 & otherwise \end{cases}$$

Solution: We know that α-cut of fuzzy set A as, ${}^{\alpha}A = \{x \in X / A(x) \ge \alpha\}$

$$A(x) \ge \alpha$$

$$\frac{x-2}{2} \ge \alpha$$

$$x - 2 \ge 2\alpha$$

$$x \ge 2\alpha + 2 \text{..........(1)}$$

$$A(x) \ge \alpha$$

$$\frac{6-x}{2} \ge \alpha$$

$$6 - x \ge 2\alpha$$

$$6 - 2\alpha \ge x \text{..........(2)}$$

From equation (1) and (2) we get, ${}^{\alpha}A = [2\alpha + 2\ , 6 - 2\alpha]$

We know that α-cut of fuzzy set B as, ${}^{\alpha}B = \{ x \in X / B(x) \ge \alpha \}$

$$B(x) \ge \alpha \qquad\qquad B(x) \ge \alpha$$

$$\frac{x-6}{2} \ge \alpha \qquad\qquad \frac{10-x}{2} \ge \alpha$$

$$x - 6 \ge 2\alpha \qquad\qquad 10 - x \ge 2\alpha$$

$$x \ge 2\alpha + 6 \text{..........(3)} \qquad\qquad 10 - 2\alpha \ge x \text{..........(4)}$$

From equation (3) and (4) we get, ${}^{\alpha}B = [2\alpha + 6\ , 10 - 2\alpha]$

Then, ${}^{\alpha}A \times {}^{\alpha}X = {}^{\alpha}B \qquad \Rightarrow {}^{\alpha}X = \dfrac{{}^{\alpha}B}{{}^{\alpha}A} \qquad \therefore {}^{\alpha}X = \dfrac{[b_1\ , b_2\ ]}{[a_1\ , a_2\ ]}$

$$\therefore {}^{\alpha}X = \frac{[2\alpha + 6\ ,10 - 2\alpha]}{[2\alpha + 2\ , 6 - 2\alpha]} \qquad \textit{we know that } {}^{\alpha}X = \left[\frac{b_1}{a_1}, \frac{b_2}{a_2} \right]$$

$$\therefore {}^{\alpha}X = \left[\frac{2\alpha + 6}{2\alpha + 2}, \frac{10 - 2\alpha}{6 - 2\alpha} \right]$$

$$i.e. \left(\frac{2\alpha + 6}{2\alpha + 2} \right) \le x \le \left(\frac{10 - 2\alpha}{6 - 2\alpha} \right)$$

$$\frac{2\alpha + 6}{2\alpha + 2} = x \qquad\qquad \frac{10 - 2\alpha}{6 - 2\alpha} = x$$

$$2\alpha + 6 = 2\alpha x + 2x \qquad\qquad 10 - 2\alpha = 6x - 2\alpha x$$

$$2\alpha - 2\alpha x = 2x - 6 \qquad\qquad 2\alpha x - 2\alpha = 6x - 10$$

$$2\alpha(1 - x) = 2(x - 3) \qquad\qquad 2\alpha(x - 1) = 6x - 10$$

$$\alpha = \frac{x - 3}{1 - x} \qquad\qquad \alpha = \frac{3x - 5}{x - 1}$$

$$\textit{if } \alpha = 0 \Rightarrow x = 3 \qquad\qquad \textit{if } \alpha = 1 \Rightarrow x = 2$$

$$\textit{if } \alpha = 1 \Rightarrow x = 2 \qquad\qquad \textit{if } \alpha = 0 \Rightarrow x = \frac{5}{3}$$

This condition is not valid hence given fuzzy number has no solution.

Examples for Practice

Example 1: A and B be two fuzzy numbers such that,

$$A(x)=\left\{\frac{0.2}{[0,1)}+\frac{0.6}{[1,2)}+\frac{0.9}{[2,3)}+\frac{1}{3}+\frac{0.4}{(3,4]}+\frac{0.1}{(4,5]}\right\}$$

$$B(x)=\left\{\frac{0.1}{[0,1)}+\frac{0.2}{[1,2)}+\frac{0.6}{[2,3)}+\frac{0.7}{[3,4)}+\frac{0.9}{[4,5)}+\frac{1}{5}+\frac{0.4}{(5,6]}+\frac{0.4}{(6,7]}+\frac{0.1}{(7,8]}\right\}$$

Then find the solution of the equation A + X = B.

Example 2: A and B be two fuzzy numbers such that,

$$A(x)=\left\{\frac{0.1}{[0,1)}+\frac{0.2}{[1,2)}+\frac{0.5}{[2,3)}+\frac{0.7}{[3,4)}+\frac{0.8}{[4,5)}+\frac{0.9}{[5,6)}+\frac{1}{6}+\frac{0.5}{(6,7]}+\frac{0.4}{(7,8]}+\frac{0.2}{(8,9]}+\frac{0.1}{(9,10]}\right\}$$

$$B(x)=\left\{\frac{0.1}{[0,1)}+\frac{0.2}{[1,2)}+\frac{0.4}{[2,3)}+\frac{0.6}{[3,4)}+\frac{0.7}{[4,5)}+\frac{0.8}{[5,6)}+\frac{0.9}{[6,7)}+\frac{1}{7}+\frac{0.5}{(7,8]}+\frac{0.5}{(8,9]}+\frac{0.3}{(9,10]}+\frac{0.2}{(10,11]}+\frac{0.1}{(11,12]}\right\}$$

Then find the solution of the equation A + X = B.

Example 3: Solve the fuzzy equation for A + X = B,

$$A(x)=\begin{cases}\dfrac{x-2}{2} & if\ 2<x\le 4\\ \dfrac{6-x}{2} & if\ 4<x\le 6\\ 0 & otherwise\end{cases}\qquad B(x)=\begin{cases}\dfrac{x-6}{2} & if\ 6<x\le 8\\ \dfrac{10-x}{2} & if\ 8<x\le 10\\ 0 & otherwise\end{cases}$$

Example 4: Solve the fuzzy equation for A +X = B,

$$A(x)=\begin{cases}x-3 & if\ 3<x\le 4\\ 5-x & if\ 4<x\le 5\\ 0 & otherwise\end{cases}\qquad B(x)=\begin{cases}\dfrac{x-12}{8} & if\ 12<x\le 20\\ \dfrac{32-x}{12} & if\ 20<x\le 32\\ 0 & otherwise\end{cases}$$

Example 5: Solve the fuzzy equation for A X = B,

$$A(x)=\begin{cases}\dfrac{x-2}{2} & if\ 2<x\le 4\\ \dfrac{6-x}{2} & if\ 4<x\le 6\\ 0 & otherwise\end{cases}\qquad B(x)=\begin{cases}\dfrac{x-6}{2} & if\ 6<x\le 8\\ \dfrac{10-x}{2} & if\ 8<x\le 10\\ 0 & otherwise\end{cases}$$

Example 6: Solve the fuzzy equation for AX = B,

$$A(x) = \begin{cases} x-4 & if\ 4 < x \le 5 \\ 6-x & if\ 5 < x \le 6 \\ 0 & otherwise \end{cases} \qquad B(x) = \begin{cases} \dfrac{x-10}{10} & if\ 10 < x \le 20 \\ \dfrac{35-x}{10} & if\ 20 < x \le 35 \\ 0 & otherwise \end{cases}$$

Example 7: Solve the fuzzy equation for A X = B,

$$A(x) = \begin{cases} \dfrac{x+2}{2} & if\ -2 < x \le 0 \\ \dfrac{2-x}{2} & if\ 0 < x \le 2 \\ 0 & otherwise \end{cases} \qquad B(x) = \begin{cases} \dfrac{x-2}{2} & if\ 2 < x \le 4 \\ \dfrac{6-x}{2} & if\ 4 < x \le 6 \\ 0 & otherwise \end{cases}$$

Example 8: Solve the fuzzy equation for A + X = B,

$$A(x) = \begin{cases} x-4 & if\ 4 < x \le 5 \\ 6-x & if\ 5 < x \le 6 \\ 0 & otherwise \end{cases} \qquad B(x) = \begin{cases} \dfrac{x-10}{5} & if\ 10 < x \le 15 \\ \dfrac{25-x}{10} & if\ 15 < x \le 25 \\ 0 & otherwise \end{cases}$$

Example 9: Solve the fuzzy equation for A + X = B,

$$A(x) = \begin{cases} \dfrac{x-2}{2} & if\ 2 < x \le 4 \\ \dfrac{6-x}{2} & if\ 4 < x \le 6 \\ 0 & otherwise \end{cases} \qquad B(x) = \begin{cases} \dfrac{x-6}{2} & if\ 6 < x \le 8 \\ \dfrac{10-x}{2} & if\ 8 < x \le 10 \\ 0 & otherwise \end{cases}$$

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