DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING
Second Year B.Tech. (SEM - III)
COMPUTATIONAL MATHEMATICS (UCSE0301)

Unit No.1: Advanced Linear Algebra

Type1: Gauss-Jordan method:

Consider the linear equations with unknowns x, y, z *are*,

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a3x + b3y + c3z = d3$$

The system is called linear because each variable appears in the first power only. Write the system of equations in the matrix form and reduce the **coefficient matrix** to a **diagonal matrix** by elementary **row transformations** only.

Given equations can be expressed in matrix form as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d2 \\ d_3 \end{bmatrix}$$

i.e. $AX = B$

The augmented matrix is $(A;B) = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$

By elementary row transformations,

$$\xrightarrow{R_{ij}} \begin{bmatrix} e_1 & 0 & 0 & e_4 \\ 0 & e_2 & 0 & e_5 \\ 0 & 0 & e_3 & e_6 \end{bmatrix}$$

By back substitution,

$R_3 \Rightarrow \therefore z = z_1$ $\qquad$ $R_2 \Rightarrow \therefore y = y_1$ $\qquad$ $R_1 \Rightarrow \therefore x = x_1$

Examples:

1) Apply Gauss – Jordan elimination method to solve

$$x + y + z = 5; \quad 2x + 3y + z = 10; \quad 3x - 2y + 2z = 3.$$

$\Longrightarrow$ Given equations can be expressed in matrix form as $AX = B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 3 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 3 \end{bmatrix}$$

The augmented matrix is $(A;B) = \begin{bmatrix} 1 & 1 & 1 & 5 \\ 2 & 3 & 1 & 10 \\ 3 & -2 & 2 & 3 \end{bmatrix}$

$$\xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 1 & -1 & 0 \\ 0 & -5 & -1 & -12 \end{bmatrix}$$

$$\xrightarrow[R_3 + 5R2]{R1 - R_2} \begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -6 & -12 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{6}R_3} \begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow[R_2 + R3]{R1 - 2R_3} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Which is in diagonal form. By back substitution,

$R_3 \Rightarrow z = 2 \qquad R_2 \Rightarrow y = 2 \qquad R_1 \Rightarrow x = 1$

$\therefore$ The solution is $x = 1, y = 2 \;\&\; z = 2.$

2) Apply Gauss – Jordan elimination method to solve

$$x + y + z = 9; \quad 2x - 3y + 4z = 13; \quad 3x + 4y + 5z = 40.$$

$\Longrightarrow$ Given equations can be expressed in matrix form as $AX = B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$$

The augmented matrix is $(A;B) = \begin{bmatrix} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{bmatrix}$

$$\xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 1 & 2 & 13 \end{bmatrix}$$

$$\xrightarrow[5R_3 + R2]{5R1 + R_2} \begin{bmatrix} 5 & 0 & 7 & 40 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & 12 & 60 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{12}R_3} \begin{bmatrix} 5 & 0 & 7 & 40 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow[R_2 - 2R3]{R1 - 7R_3} \begin{bmatrix} 5 & 0 & 0 & 5 \\ 0 & -5 & 0 & -15 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Which is in diagonal form. By back substitution,

$R_3 \Rightarrow z = 5 \qquad R_2 \Rightarrow -5y = -15 \;\therefore y = 3 \qquad R_1 \Rightarrow 5x = 5 \quad \therefore x = 1$

$\therefore$ The solution is $x = 1, y = 3$ & $z = 5.$

3) Apply Gauss – Jordan elimination method to solve

$3x + 2y - 2z = 4; \;\; x - 2y + 3z = 6; \;\; 2x + 3y + 4z = 15.$

$\Longrightarrow$ Given equations can be expressed in matrix form as, AX = B

$$\begin{bmatrix} 3 & 2 & -2 \\ 1 & -2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 15 \end{bmatrix}$$

The augmented matrix is $(A; B) = \begin{bmatrix} 3 & 2 & -2 & 4 \\ 1 & -2 & 3 & 6 \\ 2 & 3 & 4 & 15 \end{bmatrix}$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -2 & 3 & 6 \\ 3 & 2 & -2 & 4 \\ 2 & 3 & 4 & 15 \end{bmatrix}$$

$$\xrightarrow[R_3 - 2R1]{R2 - 3R_1} \begin{bmatrix} 1 & -2 & 3 & 6 \\ 0 & 8 & -11 & -14 \\ 0 & 7 & -2 & 3 \end{bmatrix}$$

$$\xrightarrow{\mathrm{R}_2 - R_3} \begin{bmatrix} 1 & -2 & 3 & 6 \\ 0 & 1 & -9 & -17 \\ 0 & 7 & -2 & 3 \end{bmatrix}$$

$$\xrightarrow[R3 - 7R2]{R1 + 2R_2} \begin{bmatrix} 1 & 0 & -15 & -28 \\ 0 & 1 & -9 & -17 \\ 0 & 0 & 61 & 122 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{61}R_3}\begin{bmatrix}1 & 0 & -15 & -28\\0 & 1 & -9 & -17\\0 & 0 & 1 & 2\end{bmatrix}$$

$$\xrightarrow[R_2+9R3]{R1+15R_3}\begin{bmatrix}1 & 0 & 0 & 2\\0 & 1 & 0 & 1\\0 & 0 & 1 & 2\end{bmatrix}$$

Which is in diagonal form. By back substitution,

$R_3 \Rightarrow z = 2 \qquad R_2 \Rightarrow y = 1 \qquad R_1 \Rightarrow x = 2$

$\therefore$ The solution is $x = 2, y = 1$ & $z = 2.$

Exercise on Type - I:

Apply Gauss – Jordan elimination method to solve following examples.

1) $x + 3y + 3z = 16; \; x + 4y + 3z = 18; \; x + 3y + 4z = 19.$ ($Solution: x = 1, y = 2, z = 3$)

2) $10x + y + z = 12; \; 2x + 10y + z = 13; \; x + y + 5z = 7.$ ($Solution: x = 1, y = 1, z = 1$)

3) $2x - 6y + 8z = 24; \; 5x + 4y - 3z = 2; \; 3x + y + 2z = 16.$ ($Solution: x = 1, y = 3, z = 5$)

4) $x + 2y + 6z = 22; \; 3x + 4y + z = 26; \; 6x - y - z = 19.$ ($Solution: x = 4, y = 3, z = 2$)

5) $x + 2y + z = 8; \; 2x + 3y + 4z = 20; \; 4x + 3y + 2z = 16.$ ($Solution: x = 1, y = 2, z = 3$)

6) $10x + y + z = 12; \; x + 10y + z = 12; \; x + y + 10z = 12.$ ($Solution: x = 1, y = 1, z = 1$)

7) $3x + y - z = 4; \; -2x + 3y - 4z = -1; \; x - y + 2z = 2.$ ($Solution: x = 1, y = 3, z = 2$)

Type 2: Solution of Simultaneous linear equation using LU decomposition (Factorization) Method.

An LU-**factorization** of a given square matrix A is of the form

$$A = LU$$

Where L is ***lower triangular*** and U is ***upper triangular***.

For example, $$A = \begin{bmatrix} 2 & 3 \\ 8 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & -7 \end{bmatrix} = LU$$

It can be proved that for any **nonsingular matrix** the rows can be reordered so that the resulting matrix A has an LU-factorization in which L turns out to be the matrix of the ***multipliers*** m_{jk} of the Gauss elimination, with main diagonal 1,…,1 and U is the matrix of the triangular system at the end of the Gauss elimination.

As a specific instance of a simplification with LU-factorization, consider the linear equations are,

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a3x + b3y + c3z = d3$$

Given equations can be expressed in matrix form as AX=B

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d2 \\ d_3 \end{bmatrix}$$

$$whwre\ A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} d_1 \\ d2 \\ d_3 \end{bmatrix}$$

Suppose we want to solve a system AX=B….. (1)

If we write A=LU,

$$Where\ L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \qquad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

If A=LU then AX=B becomes,

(LU) X=L UX =B …… (2).

Let *UX=Y* and solve the system *LY=B*….. (3) for *Y*.

Once we know *Y*, then the solution of *AX=B* is the solution of *UX=Y*.

Both of these systems involve triangular coefficient matrices, hence may be easier to solve than the original system.

Examples:

1) Solve the following systems by *LU*-factorization method.

$2x_1 + 5x_2 + 7x_3 = 52;$ $\quad 2x_1 + x_2 - x_3 = 0;$ $\quad x_1 + x_2 + x_3 = 9;$

Solution: Given equations can be expressed in matrix form as,

$$\begin{bmatrix} 2 & 5 & 7 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 52 \\ 0 \\ 9 \end{bmatrix}$$

$$i.e. \quad AX = B$$

If we write $A=LU$, then the system is

$$AX=(LU)X=L(UX)=B.$$

The LU decomposition of matrix A is

$$\begin{bmatrix} 2 & 5 & 7 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 & 7 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

Equating component wise,

$$u_{11} = 2, \qquad u_{12} = 5, \qquad u_{13} = 7$$

$$l_{21}u_{11} = 2, \quad \Rightarrow \quad l_{21}(2) = 2 \quad \therefore l_{21} = 1$$

$$l_{21}u_{12} + u_{22} = 1 \quad \Rightarrow (1)(5) + u_{22} = 1 \quad \therefore u_{22} = 1 - 5 = -4,$$

$$l_{21}u_{13} + u_{23} = -1 \quad \Rightarrow (1)(7) + u_{23} = -1 \quad \therefore u_{23} = -1 - 7 = -8,$$

$$l_{31}u_{11} = 1 \quad \Rightarrow l_{31}(2) = 1 \quad \therefore l_{31} = \frac{1}{2},$$

$$l_{31}u_{12} + l_{32}u_{22} = 1 \quad \Rightarrow (\frac{1}{2})(5) + l_{32}(-4) = 1 \quad \therefore l_{32} = \frac{1 - 5/2}{-4} = \frac{3}{8},$$

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1 \quad \Rightarrow (\frac{1}{2})(7) + (\frac{3}{8})(-8) + u_{33} = 1 \quad \therefore u_{33} = \frac{1}{2}$$

$$\text{Thus, } \begin{bmatrix} 2 & 5 & 7 \\ 2 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 3/8 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & 7 \\ 0 & -4 & -8 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$i.\ e. \quad A = L\,U$$

Let $UX=Y$. Now, we solve the system $LY=B$ for Y.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 3/8 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 52 \\ 0 \\ 9 \end{bmatrix}$$

It gives $y_1 = 52,$ $\quad y_1 + y_2 = 0 \Rightarrow 52 + y_2 = 0, \Rightarrow y_2 = -52$

$$\frac{1}{2}y1 + \frac{3}{8}y_2 + y_3 = 9 \qquad \Rightarrow \frac{1}{2}(52) + \frac{3}{8}(-52) + y_3 = 9 \qquad \Rightarrow y_3 = \frac{5}{2}$$

As $UX=Y$ $\qquad \Rightarrow \begin{bmatrix} 2 & 5 & 7 \\ 0 & -4 & -8 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 52 \\ -52 \\ 5/2 \end{bmatrix}$

By back substitution, $R_3 \Rightarrow \quad \frac{1}{2}x_3 = \frac{5}{2} \quad \Rightarrow \quad x3 = 5,$

$$R_2 \Rightarrow \quad -4x_2 - 8x_3 = -52 \Rightarrow -4x_2 - 8(5) = -52 \quad x_2 = 3$$

$$R_1 \Rightarrow \quad 2x_1 + 5x_2 + 7x_3 = 52 \quad \Rightarrow 2x_1 + 5(3) + 7(5) = 52 \Rightarrow x_1 = 1$$

So, the solution of given system is $x_1 = 1, x_2 = 3, x_3 = 5$

2) Solve the following systems by *LU*-factorization method.

$$3x_1 + 5x_2 + 2x_3 = 8; \qquad 8x_2 + 2x_3 = -7; \qquad 6x_1 + 2x_2 + 8x_3 = 26;$$

Solution: Given equations can be expressed in matrix form as,

$$\begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 26 \end{bmatrix}$$

i.e. $\quad AX = B$

If we write $A=LU$, then the system is

$AX=(LU)X=L(UX)=B.$

The LU decomposition of matrix A is

$$\begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

Equating component wise,

$$u_{11} = 3, \qquad u_{12} = 5, \qquad u_{13} = 2$$

$$l_{21}u_{11} = 0, \quad \Rightarrow \quad l_{21}(3) = 0 \quad \therefore l_{21} = \frac{0}{3} = 0,$$

$$l_{21}u_{12} + u_{22} = 8 \quad \Rightarrow (0)(5) + u_{22} = 8 \quad \therefore u_{22} = 8 - 0 = 8,$$

$$l_{21}u_{13} + u_{23} = 2 \quad \Rightarrow (0)(2) + u_{23} = 2 \quad \therefore u_{23} = 2 - 0 = 2,$$

$$l_{31}u_{11} = 6 \quad \Rightarrow l_{31}(3) = 6 \quad \therefore l_{31} = \frac{6}{3} = 2,$$

$$l_{31}u_{12} + l_{32}u_{22} = 2 \quad \Rightarrow (2)(5) + l_{32}(8) = 2 \quad \therefore l_{32} = \frac{2-10}{8} = -1,$$

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 8 \quad \Rightarrow (2)(2) + (-1)(2) + u_{33} = 8 \quad \therefore u_{33} = 8 - 4 + 2 = 6$$

$$\text{Thus, } \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

i. e. $A = L\,U$

Let $UX=Y$. Now, we solve the system $LY=B$ for Y.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 26 \end{bmatrix}$$

It gives $y_1 = 8, \quad y_2 = -7, \quad 2y_1 - y_2 + y_3 = 26 \;\Rightarrow\; y_3 = 26 - 2(8) + (-7) = 3$

As $\quad UX=Y$

$$\therefore \begin{bmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 3 \end{bmatrix}$$

By back substitution, $R_3 \Rightarrow \quad 6x_3 = 3 \quad \Rightarrow \quad x3 = \frac{3}{6} = \frac{1}{2},$

$$R_2 \Rightarrow \; 8x_2 + 2x_3 = -7 \Rightarrow x_2 = \frac{1}{8}[-7 - 2(\frac{1}{2})] = -1,$$

$$R_1 \Rightarrow \; 3x_1 + 5x_2 + 2x_3 = 8 \;\Rightarrow\; x_1 = \frac{1}{3}[8 - 5(-1) - 2(\frac{1}{2})] = 4$$

So, the solution of given system is $x_1 = 4, x_2 = -1, x_3 = \frac{1}{2}$

Exercise on Type - II:

Solve the following systems by *LU*-factorization method.

1) $10x + y + z = 12; \qquad 2x + 10y + z = 13; \qquad x + y + 5z = 7;$

$(Solution: x = 1, y = 1, z = 1)$

2) $2x + y + z = 10; \qquad 3x + 2y + 3z = 18; \qquad x + 4y + 9z = 16;$

$(Solution: x = 7, y = -9, z = 5)$

3) $x + y + z = 9; \qquad 2x - 3y + 4z = 13; \qquad 3x + 4y + 5z = 40.$

$(Solution: x = 1, y = 3, z = 5)$

4) $5x_1 + 4x_2 + x_3 = 6.8; \quad 10x_1 + 9x_2 + 4x_3 = 17.6; \quad 10x_1 + 13x_2 + 15x_3 = 38.4;$

5) $2x_1 + x_2 + 2x_3 = 0; \qquad -2x_1 + 2x_2 + x_3 = 0; \qquad x_1 + 2x_2 - 2x_3 = 18;$

6) $4x_1 + 4x_2 + 2x_3 = 1; \qquad x_1 - x_2 + 3x_3 = 0; \qquad x_1 + 42x_2 + 2x_3 = 1;$

Type 3: Solution of non-linear simultaneous equations (Newton Raphson Method):

Consider the equations, $f(x,y)=0$, $g(x,y)=0$
If an initial approximation (x_0, y_0) to a solution (x_1, y_1) has been found by Newton Raphson Method,

$$f_0 + h\frac{\partial f}{\partial x_0} + k\frac{\partial f}{\partial y_0} = 0 \quad \text{and} \quad g_0 + h\frac{\partial g}{\partial x_0} + k\frac{\partial g}{\partial y_0} = 0$$

Where, $f_0 = f(x_0, y_0)$ $\quad \frac{\partial f}{\partial x_0} = \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}$ $\quad \frac{\partial f}{\partial y_0} = \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}$

$g_0 = g(x_0, y_0)$ $\quad \frac{\partial g}{\partial x_0} = \frac{\partial g}{\partial x}\Big|_{(x_0,y_0)}$ $\quad \frac{\partial g}{\partial y_0} = \frac{\partial g}{\partial y}\Big|_{(x_0,y_0)}$

Solving these equations for h and k, we get new approximation to the root as,

$$x_1 = x_0 + h, \quad y_1 = y_0 + k$$

Where, $h = \dfrac{\begin{vmatrix} -f_0 & f_{y_0} \\ -g_0 & g_{y_0} \end{vmatrix}}{\begin{vmatrix} f_{x_0} & f_{y_0} \\ g_{x_0} & g_{y_0} \end{vmatrix}}$ and $k = -\dfrac{\begin{vmatrix} -f_0 & f_{x_0} \\ -g_0 & g_{x_0} \end{vmatrix}}{\begin{vmatrix} f_{x_0} & f_{y_0} \\ g_{x_0} & g_{y_0} \end{vmatrix}}$

This process is repeating till we get the values to the desired accuracy.

Examples:

1) Solve the following non linear equation by using Newton Raphson Method with given initial condition in single step.

$y^2 + x = 5 \quad x^2 + y = 11 \quad \textit{with initial condition } x_0 = 1 \textit{ and } y_0 = 2$

Solution:

$\textit{Given initial conditions are } x_0 = 1 \textit{ and } y_0 = 2$

Let, $f(x,y) = y^2 + x - 5 = 0 \quad \textit{and} \quad g(x,y) = x^2 + y - 11 = 0 ------(1)$

Then Partial derivatives of equation (1) are,

$$\frac{\partial f}{\partial x} = f_x = 1 \qquad \frac{\partial f}{\partial y} = fy = 2y \qquad \frac{\partial g}{\partial x} = g_x = 2x \qquad \frac{\partial g}{\partial y} = g_y = 1 ----(2)$$

Step: Put the initial values in equation (1) and (2) then we get,

$f_0 = 4 + 1 - 5 = 0 \qquad f_{x_0} = 1 \qquad f_{y_0} = 2(2) = 4$

$g_0 = 1 + 2 - 11 = -8 \qquad g_{x_0} = 2 \qquad g_{y_0} = 1$

To find h and k:

$$h = \frac{\begin{vmatrix} -f_0 & f_{y_0} \\ -g_0 & g_{y_0} \end{vmatrix}}{\begin{vmatrix} f_{x_0} & f_{y_0} \\ g_{x_0} & g_{y_0} \end{vmatrix}} = \frac{\begin{vmatrix} 0 & 4 \\ 8 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix}} = \frac{-32}{-7} = \frac{32}{7} = 4.5714$$

$$k = -\frac{\begin{vmatrix} -f_0 & f_{x_0} \\ -g_0 & g_{x_0} \end{vmatrix}}{\begin{vmatrix} f_{x_0} & f_{y_0} \\ g_{x_0} & g_{y_0} \end{vmatrix}} = -\frac{\begin{vmatrix} 0 & 1 \\ 8 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix}} = -\left(\frac{-8}{-7}\right) = -\left(\frac{8}{7}\right) = -1.1429$$

Hence, $x_1 = x_{0+h} = 1 + 4.5714 = 5.5714 \qquad y_1 = y_{0+k} = 2 - 1.1429 = 0.8571$

Solution of given non linear equations are, x = 5.5714 and y = 0.8571.

2) solve the following non linear equation by using Newton Raphson Method with given initial condition in two steps.

$x^2 + y^2 = 16 \quad x^2 - y^2 = 9 \quad \textit{with initial condition } x_0 = 1 \textit{ and } y_0 = 1$

Solution:

$\textit{Given initial conditions are } x_0 = 1 \textit{ and } y_0 = 1$

Let, $f(x,y) = x^2 + y^2 - 16 = 0 \quad and \qquad g(x,y) = x^2 - y^2 - 9 = 0 ------(1)$

Then Partial derivatives of equation (1) are,

$$\frac{\partial f}{\partial x} = f_x = 2x \qquad \frac{\partial f}{\partial y} = fy = 2y \quad \frac{\partial g}{\partial x} = g_x = 2x \qquad \frac{\partial g}{\partial y} = g_y = -2y ----(2)$$

Step 1: Put the initial values in equation (1) and (2) then we get,

$f_0 = 1 + 1 - 16 = -14 \qquad f_{x_0} = 2 \qquad f_{y_0} = 2$

$g_0 = 1 - 1 - 9 = -9 \qquad g_{x_0} = 2 \qquad g_{y_0} = -2$

To find h and k:

$$h = \frac{\begin{vmatrix} -f_0 & f_{y_0} \\ -g_0 & g_{y_0} \end{vmatrix}}{\begin{vmatrix} f_{x_0} & f_{y_0} \\ g_{x_0} & g_{y_0} \end{vmatrix}} = \frac{\begin{vmatrix} 14 & 2 \\ 9 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 2 \\ 2 & -2 \end{vmatrix}} = \frac{-46}{-8} = \frac{46}{8} = 5.75$$

$$k = -\frac{\begin{vmatrix} -f_0 & f_{x_0} \\ -g_0 & g_{x_0} \end{vmatrix}}{\begin{vmatrix} f_{x_0} & f_{y_0} \\ g_{x_0} & g_{y_0} \end{vmatrix}} = -\frac{\begin{vmatrix} 14 & 2 \\ 9 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 2 \\ 2 & -2 \end{vmatrix}} = -\left(\frac{10}{-8}\right) = 1.25$$

Hence, $x_1 = x_{0+h} = 1 + 5.75 = 6.75 \qquad y_1 = y_{0+k} = 1 + 1.25 = 2.25$

Step 2: $\textit{Given initial conditions are } x_0 = 6.75 \textit{ and } y_0 = 2.25$

Put the initial values in equation (1) and (2) then we get,

$f_0 = (6.75)^2 + (2.25)^2 - 16 = 34.625$

$g_0 = (6.75)^2 - (2.25)^2 - 9 = 31.5$

$f_{x_0} = 2(6.75) = 13.5$ $\qquad$ $f_{y_0} = 2(2.25) = 4.5$

$g_{x_0} = 2(6.75) = 13.5$ $\qquad$ $g_{y_0} = -2(2.25) = -4.5$

To find h and k:

$$h = \frac{\begin{vmatrix} -f_0 & f_{y_0} \\ -g_0 & g_{y_0} \end{vmatrix}}{\begin{vmatrix} f_{x_0} & f_{y_0} \\ g_{x_0} & g_{y_0} \end{vmatrix}} = \frac{\begin{vmatrix} -34.625 & 4.5 \\ -31.5 & -4.5 \end{vmatrix}}{\begin{vmatrix} 13.5 & 4.5 \\ 13.5 & -4.5 \end{vmatrix}} = \frac{297.5625}{-121.5} = -2.4491$$

$$k = -\frac{\begin{vmatrix} -f_0 & f_{x_0} \\ -g_0 & g_{x_0} \end{vmatrix}}{\begin{vmatrix} f_{x_0} & f_{y_0} \\ g_{x_0} & g_{y_0} \end{vmatrix}} = -\frac{\begin{vmatrix} -34.625 & 13.5 \\ -31.5 & 13.5 \end{vmatrix}}{\begin{vmatrix} 13.5 & 4.5 \\ 13.5 & -4.5 \end{vmatrix}} = -\left(\frac{892.6875}{-121.5}\right) = 7.3472$$

Hence, $x_2 = x_{1+h} = 6.75 - 2.4491 = 4.3009$ $\qquad$ $y_1 = y_{0+k} = 2.25 + 7.3472 = 9.5972$

Solution of given non linear equations are, x = 4.3009 and y = 9.5972.

Exercise on Type – III:

Solve the following non linear equation by using Newton Raphson Method with given initial condition in two steps.

1) $x^2 + xy = -9.2 \quad y^2 - xy = -7.12 \quad$ *with initial condition* $x_0 = 2$ *and* $y_0 = 2$

2) $2x^2 + 3xy + y^2 = 3 \quad 4x^2 + 2xy + y^2 = 30 \quad$ *with initial condition* $x_0 = 3$ *and* $y_0 = 2$

3) $x^2 + y = 11 \quad y^2 + x = 7 \quad$ *with initial condition* $x_0 = 3.5$ *and* $y_0 = -1.8$

4) $x^2 + y^2 = x \quad x^2 - y^2 = y \quad$ *with initial condition* $x_0 = 0.8$ *and* $y_0 = 0.4$

Type 4: Determination of Eigen Value by Iteration method(Power Method):

A simple standard procedure for computing approximate values of the Eigen values of a $n \times n$ matrix $A = [a_{jk}]$ is the **power method**. In this method we start from any vector $X_0 (\neq 0)$ with n components and compute successively,

$$X_1 = AX_0, \quad X_2 = AX_1, \dots\dots\dots, X_m = AX_{m-1}$$

For simplifying notation, we denote X_{m-1} by X and X_m by Y, so that $Y = AX$.

The method applies to any $n \times n$ matrix A that has a **dominant eigenvalue** (a λ such that $|\lambda|$ is greater than the absolute values of the other Eigen values).

Examples:

1. Apply the power method (5 steps) with scaling, using $X_0 = [1 \quad 1]^T$ where matrix is.

$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

Solution: Let $A = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$ & $X_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$AX_0 = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 13 \\ 7 \end{bmatrix} = 13\begin{bmatrix} 1.0000 \\ 0.5385 \end{bmatrix} \quad \text{so let } X_1 = \begin{bmatrix} 1.0000 \\ 0.5385 \end{bmatrix}$$

$$AX_1 = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1.0000 \\ 0.5385 \end{bmatrix} = \begin{bmatrix} 11.1538 \\ 5.6154 \end{bmatrix} = 11.1538\begin{bmatrix} 1.0000 \\ 0.5035 \end{bmatrix} \quad \text{now let } X_2 = \begin{bmatrix} 1.0000 \\ 0.5035 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1.0000 \\ 0.5035 \end{bmatrix} = \begin{bmatrix} 11.0138 \\ 5.5104 \end{bmatrix} = 11.0138\begin{bmatrix} 1.0000 \\ 0.5003 \end{bmatrix} \quad \text{now let } X_3 = \begin{bmatrix} 1.0000 \\ 0.5003 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1.0000 \\ 0.5003 \end{bmatrix} = \begin{bmatrix} 11.0012 \\ 5.5009 \end{bmatrix} = 11.0012\begin{bmatrix} 1.0000 \\ 0.5000 \end{bmatrix} \quad \text{now let } X4 = \begin{bmatrix} 1.0000 \\ 0.5000 \end{bmatrix}$$

$$AX4 = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1.0000 \\ 0.5000 \end{bmatrix} = \begin{bmatrix} 11.0000 \\ 5.5000 \end{bmatrix} = 11.0000\begin{bmatrix} 1.0000 \\ 0.5000 \end{bmatrix}$$

Eigen value of given matrix is <u>11</u> and corresponding Eigen vector is $\begin{bmatrix} 1.0000 \\ 0.5000 \end{bmatrix}$

2. Find numerically largest Eigen value of the given matrix by using power method (Iteration method) (3 steps) with scaling, using $X_0 = [1 \quad 1 \quad 1]^T$ where matrix is.

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Solution: Let $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ & $X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$AX_0 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = 6\begin{bmatrix} 0.3333 \\ 0.6667 \\ 1.0000 \end{bmatrix} \quad \text{so let } X_1 = \begin{bmatrix} 0.3333 \\ 0.6667 \\ 1.0000 \end{bmatrix}$$

$$AX_1 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 0.3333 \\ 0.6667 \\ 1.0000 \end{bmatrix} = \begin{bmatrix} 1.0000 \\ 3.6667 \\ 4.6667 \end{bmatrix} = 4.6667\begin{bmatrix} 0.2143 \\ 0.7857 \\ 1.0000 \end{bmatrix} \text{ now let } X_2 = \begin{bmatrix} 0.2143 \\ 0.7857 \\ 1.0000 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 0.2143 \\ 0.7857 \\ 1.0000 \end{bmatrix} = \begin{bmatrix} 0.6429 \\ 4.1428 \\ 4.7857 \end{bmatrix} = 4.7857\begin{bmatrix} 0.1343 \\ 0.8657 \\ 1.0000 \end{bmatrix}$$

Eigen value of given matrix is <u>4.7857</u> and corresponding Eigen vector is $\begin{bmatrix} 0.1343 \\ 0.8657 \\ 1.0000 \end{bmatrix}$

3. Find numerically largest Eigen value of the given matrix by using power method (Iteration method) (5 steps) where matrix is.

$$\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution: Let $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ & $X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$AX_0 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{so let } X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$AX_1 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 0.4286 \\ 0 \end{bmatrix} \quad \text{so let } X2 = \begin{bmatrix} 1 \\ 0.4286 \\ 0 \end{bmatrix}$$

$$AX2 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.4286 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.5716 \\ 1.8572 \\ 0 \end{bmatrix} = 3.5716\begin{bmatrix} 1 \\ 0.5199 \\ 0 \end{bmatrix} \quad \text{so let } X3 = \begin{bmatrix} 1 \\ 0.5199 \\ 0 \end{bmatrix}$$

$$AX3 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5199 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.1194 \\ 2.0398 \\ 0 \end{bmatrix} = 4.1194\begin{bmatrix} 1 \\ 0.4952 \\ 0 \end{bmatrix} \quad \text{so let } X4 = \begin{bmatrix} 1 \\ 0.4952 \\ 0 \end{bmatrix}$$

$$AX4 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.4952 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.9712 \\ 1.9904 \\ 0 \end{bmatrix} = 3.9712\begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix}$$

Eigen value of given matrix is <u>3.9712</u> and corresponding Eigen vector is $\begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix}$

Exercise on Type - IV:

Find numerically largest (Dominant) Eigen value of the given matrix by using power method (Iteration method) (5 steps) where matrix is.

1) $\begin{bmatrix} 7 & -3 \\ -3 & -1 \end{bmatrix}$ 2) $\begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$ 3) $\begin{bmatrix} 4 & 2 & 3 \\ 2 & 7 & 6 \\ 3 & 6 & 4 \end{bmatrix}$ 4) $\begin{bmatrix} 2 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & 2 & 9 \end{bmatrix}$

5) $\begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$ 6) $\begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$ 7) $\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$ 8) $\begin{bmatrix} 10 & 2 & 1 \\ 2 & 10 & 1 \\ 2 & 1 & 10 \end{bmatrix}$

Summary

1. Solutions of simultaneous linear equations using Gauss-Jordan method.

2. Solutions of simultaneous linear equations using LU decomposition method.

3. Solution of non-linear simultaneous equations.

4. Determination of Eigen Value by Iteration method.