Tutorial No 5

Vector Algebra

1. Let V be the set of all real valued functions defined on a closed interval [a,b] a < b. For any f and g in V and for any scalar K, define
i) $f = g$ iff $f(x) = g(x)$ for all x in [a,b]
ii) $(f+g)x = f(x) + g(x)$
iii) $(kf)x = kf(x)$. Show that V is a vector space

→ Firstly C_1 & C_2 follow clearly as f+g and kf are again real valued functions defined on [a, b]

A1 : For any x in [a,b]

$(f+g)(x) = f(x) + g(x)$ → by sum in V

$= g(x) + f(x)$ → f(x) & g(x) are real no's

$= (g+f)x$ → by sum in V

$\therefore (f+g)(x) = (g+f)(x)$ for all x in [a,b]

Hence by equality in V we have

$f+g = g+f$

A2: For any $x \in [a,b]$ we have

$[(f+g) + h](x) = (f+g)(x) + h(x)$ → by def^n of sum in V

$= [f(x) + g(x)] + h(x)$

$= f(x) + [g(x) + h(x)]$ → By associativity

$= f(x) + (g+h)(x)$

$= [f + (g+h)](x)$

$\therefore (f+g)+h = f+(g+h) \rightarrow$ By equality

A3: For any $x \in [a,b]$ we define a function

$o : [a,b] \rightarrow R$ by $o(x) = 0$

then for any $f \in V$

$(f+o)(x) = f(x) + o(x)$

$= f(x) + 0$

$= f(x)$

$\therefore f + o = f$

$\therefore o$ is zero element in V.

A4: For any $f \in V$ we define

$(-f)(x) = -f(x)$

$\therefore [f+(-f)](x) = f(x) + (-f(x))$

$= f(x) - f(x)$

$= 0$

Thus $[f+(-f)](x) = 0$

$= \bar{0}(x)$

$f + (-f) = \bar{0}$

M1: For any $x \in [a,b]$ and for any f, g in V & K is any scalar consider

$[k(f+g)](x) = k(f+g)(x)$

$= k[f(x) + g(x)]$

$= kf(x) + kg(x)$

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$= (kf)(x) + (kg)(x)$
$= (kf + kg)(x)$
$\therefore k(f+g) = kf + kg$

M3 : For any scalars k & l and $f \in V$
Consider $[(kl)f](x) = (kl)f(x)$
$= k(l f(x))$
$= k((lf)(x))$
$= [k(lf)](x)$

$\therefore$ For all $x \in [a,b]$
$(kl)f = k(lf)$

M2 : For any scalars k and l & $f \in V$
consider
$[(k+l)f](x) = (k+l)f(x)$
$= k f(x) + l f(x)$
$= (kf)(x) + (lf)(x)$
$= (kf + lf)(x)$

Thus for all $x \in [a,b]$
$(k+l)f = kf + lf$

M4: For any $f \in V$ and any $x \in [a,b]$
$(1f)(x) = 1f(x)$
$= f(x)$

Thus $(1f)(x) = f(x)$ for all $x \in [a,b]$
$1f = f$

$\therefore$ V is a vector space
and is sometimes called a function space

2. Let $V = \mathbb{R}$ the set of all real no.s with the operations
$u + v = u - v$ (i.e ordinary subtraction)
$c \cdot u = cu$ (i.e ordinary multiplication)
Is V a vector space? If it is not which axioms fail to hold?

→ Let,
$u, v \in V$
If $(u,v) \in V$, $u+v = u-v$ is also in V
i.e $u+v \in V$
$c \cdot u = cu$ is also in V
i.e $cu \in V$
$\therefore$ V is closed under vector addition and scalar multiplication

A1: Consider $u = 2$, $v = 4$
$u+v = u-v$
$= 2-4$
$= -2$
$\therefore v+u = v-u$
$= 4-2 = 2$
$\therefore u+v \neq v+u$

A2: Let $u = 2$, $v = 4$, $w = 6$
Consider $(u+v)+w$
$\Rightarrow (u-v)+w$
$= u-v-w$

$= 2 - 4 - 6$
$= -8$

Now $u + (v + w) = u + (v - w)$
$= u - v + w$
$= 2 - 4 + 6$
$= 4$

$\therefore (u+v)+w \neq u+(v+w)$

A3 : Consider $0 \in R$

$\therefore u + 0 = u - 0$
$= u$

$\therefore u + 0 = u$

A4 : Consider $-u$ in V

$\therefore u + (-u) = u - (-u)$
$= u + u$
$= 2u$

$\therefore u + (-u) \neq 0$

M1: Consider any scalar k

Now, $k(u+v) = k(u-v)$
$= ku - kv$

$\therefore k(u+v) \neq ku + kv$

M2 : Consider any scalars k & l

$\Rightarrow (k+l)u = ku + lu$

$\therefore$ for any 2 scalars k and l

$(k+l)u = ku + lu$

M3: Consider scalars k & l

Now, $(kl).u = (kl)u$

$\therefore (kl)u \neq k(lu)$

M4: Consider $1 \in \mathbb{R}$

$\therefore 1.u = 1u$

$\Rightarrow 1.u \neq u$

$\therefore$ V is not a vector space as axioms A1, A2, A4, M1, M3, M4 fail to hold.

3. Which of the following subsets of $\mathbb{R}^2$ with the usual operations of vector addition and scalar multiplication are subspaces?

a) $W_1 = \{(x,y) \mid x \geq 0\}$

b) $W_2 = \{(x,y) \mid x, y \geq 0\}$

c) $W_3 = \{(x,y) \mid x = 0\}$

a) $W_1 = \{(x,y) \mid x \geq 0\}$

1) $W_1 \neq \phi$

2) Let $u = (2,1)$ & $v = (1,1)$

$\therefore$ Consider $u+v \Rightarrow (2,1)+(1,1)$

$= (2+1, 1+1)$

$= (3, 2)$

$\therefore$ For all $u, v \in W_1$

$u+v \in W_1$

iii) Consider any scalar k

let $u = (2,1)$

$\therefore ku = k(2,1)$

$= (2k, k)$

$\therefore$ For any $u \in W_1$, $ku \in W_1$

$\therefore W_1$ is a subspace of R^2.

b) $W_2 = \{(x,y) \mid x, y \geq 0\}$

can be equal to

1) $W_2 = \phi$ as $\rightarrow x, y \not\geq 0$

$\therefore W_2$ is not a subspace of R^2

c) $W_3 = \{(x,y) \mid x = 0\}$

1) $W_3 \neq \phi$

2) let $u = (0,1)$ $v = (0,2)$

Consider $u+v \Rightarrow (0,1) + (0,2)$

$= (0+0, 1+2)$

$= (0,3)$

$\therefore$ For all $u, v \in W_3$, $u+v \in W_3$

3) consider scalar k, let $u = (0,1)$

$\therefore ku = k(0,1) = (0,k)$

$\therefore$ For all $u \in W_3$, $ku \in W_3$

$\therefore W_3$ is a subspace of R^2

4. Let V be the set of all polynomials of degree exactly equal to 2, then show that V is not a subspace of P_2

→ V is a set of all polynomials of degree exactly equal to 2

$\therefore V = \{ax^2 + bx + c \mid a \neq 0\}$

$\therefore$ 1) $V \neq \phi$

2) Consider $u, v \in V$

let $u = 2x^2 + x + 3$, $v = -2x^2 + 4$

$\therefore$ Consider $u + v \Rightarrow 2x^2 + x + 3 + (-2x^2 + 4)$

$= 2x^2 + x + 3 - 2x^2 + 4$

$= x + 7$

$\therefore$ For any $u, v \in V$, $u + v \notin V$

$\therefore$ V is not a subspace of P_2.

5. Let $v_1 = 2t^2 + t + 2$, $v_2 = t^2 - 2t$, $v_3 = 5t^2 - 5t + 2$, $v_4 = -t^2 - 3t - 2$ be vectors in vector space P_2. Determine if the vector $u = t^2 + t + 2$ belongs to span $\{v_1, v_2, v_3, v_4\}$.

→ To show $u = t^2 + t + 2$ belongs to span $\{v_1, v_2, v_3, v_4\}$

we have to show that u can be expressed as a linear combination of v_1, v_2, v_3, v_4.

let c_1, c_2, c_3, c_4 be any 4 scalars such that

$$u = c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4$$

$$t^2 + t + 2 = c_1(2t^2 + t + 2) + c_2(t^2 - 2t) + c_3(5t^2 - 5t + 2) + c_4(-t^2 - 3t - 2)$$

Comparing coefficients

$$2c_1 + c_2 + 5c_3 - c_4 = 1$$
$$c_1 - 2c_2 - 5c_3 - 3c_4 = 1$$
$$2c_1 + 2c_3 - 2c_4 = 2$$

Augmented matrix is

$$[A : B] = \begin{bmatrix} 2 & 1 & 5 & -1 & 1 \\ 1 & -2 & -5 & -3 & 1 \\ 2 & 0 & 2 & -2 & 2 \end{bmatrix}$$

By $R_1 - R_2$,

$$\sim \begin{bmatrix} 1 & 3 & 10 & 2 & 0 \\ 1 & -2 & -5 & -3 & 1 \\ 2 & 0 & 2 & -2 & 2 \end{bmatrix}$$

By $R_2 - R_1$, $R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 3 & 10 & 2 & 0 \\ 0 & -5 & -15 & -5 & 1 \\ 0 & -6 & -18 & -6 & 2 \end{bmatrix}$$

By $R_2 - R_3$

$$\sim \begin{bmatrix} 1 & 3 & 10 & 2 & 0 \\ 0 & 1 & 3 & 1 & -1 \\ 0 & -6 & -18 & -6 & 2 \end{bmatrix}$$

$$\Rightarrow \sim \begin{bmatrix} 1 & 3 & 10 & 2 & 0 \\ 0 & 1 & 3 & 1 & -1 \\ 0 & -6 & -18 & -6 & 2 \end{bmatrix}$$

By $R_3 + 6R_2$,

$$\sim \begin{bmatrix} 1 & 3 & 10 & 2 & 0 \\ 0 & 1 & 3 & 1 & -1 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix}$$

$\therefore \rho(A) = 2 \neq \rho(A:B) \neq n$

$\therefore$ The system is inconsistent and has infinitely many solutions for C_1, C_2, C_3, C_4.

$\therefore$ u does not belong to span $\{v_1, v_2, v_3, v_4\}$

6. Are the vectors $v_1 = (1,0,1,2)$, $v_2 = (0,1,1,2)$, $v_3 = (1,1,1,3)$ in R^4 linearly dependent or independent

→ Let c_1, c_2, c_3 be 3 scalars such that

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

$$c_1(1,0,1,2) + c_2(0,1,1,2) + c_3(1,1,1,3) = (0,0,0,0)$$

$$c_1 + c_3 = 0$$

$$c_2 + c_3 = 0$$

$$c_1 + c_2 + c_3 = 0$$

$$2c_1 + 2c_2 + 3c_3 = 0$$

Consider augmented matrix

$$[A : B] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 3 & 0 \end{bmatrix}$$

By $R_3 - R_1$, $R_4 - 2R_1$,

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

By $R_3 - R_2$, $R_4 - 2R_2 \Rightarrow \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$

By $R_4 - R_3$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 3 = n$

$\therefore C_1 = C_2 = C_3 = 0$

$\therefore$ The vectors V_1, V_2, V_3, V_4 are linearly independent

7. Is $S = (V_1, V_2, V_3, V_4) = \{(1,3,3), (0,1,4), (5,6,3), (7,2,-1)\} \subset R^3$ linearly dependent or independent?

→ Assume 4 scalars C_1, C_2, C_3, C_4 such that

$$C_1V_1 + C_2V_2 + C_3V_3 + C_4V_4 = \bar{0}$$

$$C_1(1,3,3) + C_2(0,1,4) + C_3(5,6,3) + C_4(7,2,-1) = (0,0,0)$$

$$C_1 + 5C_3 + 7C_4 = 0$$

$$3C_1 + C_2 + 6C_3 + 2C_4 = 0$$

$$3C_1 + 4C_2 + 3C_3 - C_4 = 0$$

Consider augmented matrix

$$[A:B] = \begin{bmatrix} 1 & 0 & 5 & 7 & 0 \\ 3 & 1 & 6 & 2 & 0 \\ 3 & 4 & 3 & -1 & 0 \end{bmatrix}$$

By $R_2 - 3R_1$, $R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 0 & 5 & 7 & 0 \\ 0 & 1 & -9 & -19 & 0 \\ 0 & 4 & -12 & -22 & 0 \end{bmatrix}$$

By $R_3 - 4R_2$,

$$\sim \begin{bmatrix} 1 & 0 & 5 & 7 & 0 \\ 0 & 1 & -9 & -19 & 0 \\ 0 & 0 & 24 & 54 & 0 \end{bmatrix}$$

By $\frac{1}{6} R_3$,

$$\sim \begin{bmatrix} 1 & 0 & 5 & 7 & 0 \\ 0 & 1 & -9 & -19 & 0 \\ 0 & 0 & 4 & 9 & 0 \end{bmatrix}$$

$\rho(A) = 3 < n = 4$

$\therefore$ The system has non-trivial solution,

$\therefore$ $S = (V_1, V_2, V_3, V_4)$ is linearly dependent.