Orthogonal Subspaces:

Two Subspaces W_1 and W_2 of a vector space V are said to be ***orthogonal*** if every vector $v_1 \in W_1$ is perpendicular to every vector $v_2 \in W_2$. i.e., their inner product is zero. $v_1^T v_2 = 0$.

E.g. $W_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right\}$, $W_2 = \left\{ \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\}$ are orthogonal to each other.

Example 1 : Find the vector perpendicular to the row space $\begin{bmatrix} 1 & 3 & 4 \\ 5 & 2 & 7 \end{bmatrix}$ Ans: $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

Example 2 : Find the vector perpendicular to the row space $\begin{bmatrix} 1 & -2 & 5 \\ 3 & -1 & 5 \end{bmatrix}$ Ans: $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

Orthogonal Complement :

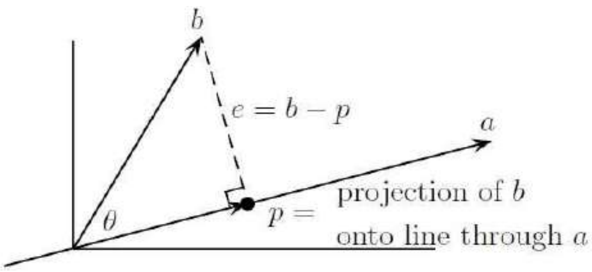
The orthogonal complement of a subspace V contains every vector that is perpendicular to V. This orthogonal subspace is denoted by $V^\perp$.

Example 1: Calculate $V^\perp$ if V = span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 3 \\ 12 \end{bmatrix} \right\}$. Ans: $V^\perp$ = span $\left\{ \begin{bmatrix} -29 \\ 12 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -9 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Example 2: Calculate $V^\perp$ if V = span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 6 \\ -9 \end{bmatrix} \right\}$. Ans: $V^\perp$ = span $\left\{ \begin{bmatrix} 13 \\ 17 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Cosines and Projections onto lines

Suppose we want to find the distance from a point b to the line in the direction of the vector a. We are looking along that line for the point p closest to b. The line connecting b to p is perpendicular to a. This fact will allow us to find the projection p.



So the projection matrix is P = $\dfrac{a^T a}{\left\| a a^T \right\|}$. To project b onto a at point p, $p^T = Pb^T$.

Example 1: Find the projection matrix that projects onto the line through a = (1, 1, 1). Hence find projection point p that projects b = (2, 3, 4) onto a. Ans: p = (3, 3, 3)

Example 2: Find the projection matrix that projects onto the line through a = (0, 1, 4). Hence find projection point p that projects b = (-1, 3, -2) onto a. Ans: p = (0, -5/17, -20/17)

Projection Operator

We define the ***projection operator*** by, $proj_u(v) = \frac{\langle u, v\rangle}{\langle u, u\rangle} u$

where $\langle u, v\rangle$ denotes the inner product of the vectors **u** and **v**. This operator projects the vector **v** orthogonally onto the line spanned by vector **u**. If **u** = **0**, we define $proj_0(v) = 0$, i.e., the projection map $proj_0$ is the zero map, sending every vector to the zero vector.

Gram–Schmidt process

A finite, linearly independent set of vectors $S = \{v_1, ..., v_k\}$ for $k \le n$ and generates an orthogonal set $S' = \{u_1, ..., u_k\}$ that spans the same k-dimensional subspace of R^n as S.

The Gram–Schmidt process then works as follows:

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1, & \mathbf{e}_1 &= \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \\
\mathbf{u}_2 &= \mathbf{v}_2 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_2), & \mathbf{e}_2 &= \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \\
\mathbf{u}_3 &= \mathbf{v}_3 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_3) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_3), & \mathbf{e}_3 &= \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} \\
\mathbf{u}_4 &= \mathbf{v}_4 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_4) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_4) - \mathrm{proj}_{\mathbf{u}_3}(\mathbf{v}_4), & \mathbf{e}_4 &= \frac{\mathbf{u}_4}{\|\mathbf{u}_4\|} \\
&\;\;\vdots & &\;\;\vdots \\
\mathbf{u}_k &= \mathbf{v}_k - \sum_{j=1}^{k-1} \mathrm{proj}_{\mathbf{u}_j}(\mathbf{v}_k), & \mathbf{e}_k &= \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}.
\end{aligned}$$

The sequence $\mathbf{u}_1, ..., \mathbf{u}_k$ is the required system of orthogonal vectors, and the normalized vectors $\mathbf{e}_1, ..., \mathbf{e}_k$ form an orthonormal set. The calculation of the sequence $\mathbf{u}_1, ..., \mathbf{u}_k$ is known as Gram–Schmidt orthogonalization, while the calculation of the sequence $\mathbf{e}_1, ..., \mathbf{e}_k$ is known as Gram–Schmidt orthonormalization as the vectors are normalized.

Example 1 : Consider the following set of vectors in $\mathbf{R}^2$

$$S = \left\{ \mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}.$$

Perform Gram–Schmidt, to obtain an orthogonal and orthonormal set of vectors.

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_2) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \mathrm{proj}_{\left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right]} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \frac{8}{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 6/5 \end{bmatrix}.$$

We check that the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are indeed orthogonal:

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \left\langle \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2/5 \\ 6/5 \end{bmatrix} \right\rangle = -\frac{6}{5} + \frac{6}{5} = 0,$$

For non-zero vectors, we can then normalize the vectors by dividing out their sizes as shown above:

$$\mathbf{e}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\mathbf{e}_2 = \frac{1}{\sqrt{\frac{40}{25}}} \begin{bmatrix} -2/5 \\ 6/5 \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

Orthogonal Set = { u_1, u_2 } and Orthonormal Set = { e_1, e_2 }.

Example 2 : Consider the following set of vectors in $\mathbf{R}^4$. x_1 = (1, 0, 1, 0), x_2 = (1 , 1 , 1 , 1) , x_3 = (−1, 2, 0, 1). Perform Gram–Schmidt, to obtain an orthogonal and orthonormal set of vectors.

Example 3 : Consider the following set of vectors in $\mathbf{R}^4$. {(1,3,−1,1),(−1,1,1,−1),(1,0,2,1)}

Perform Gram–Schmidt, to obtain an orthogonal and orthonormal set of vectors.

Example 4 : Consider the following set of vectors in $\mathbf{R}^3$. {(2,−1,1),(1,1,−1),(0,1,1)}.

Perform Gram–Schmidt, to obtain an orthogonal and orthonormal set of vectors.