* Tutorial No : 3 *
* Title : Probability Distribution *

Q.1. The probability density function of a random variable x zero except at $x = 0, 1, 2$. At these points.

$p(0) = 3c^3$, $p(1) = 4c - 10c^2$, $p(2) = 5c - 1$

Find i) the value of c

ii) $P(0 < x \leq 2)$

→ i) $\Sigma P(x) = 1$

$$3c^3 + 4c - 10c^2 + 5c - 1 = 1$$
$$3c^3 - 10c^2 + 9c - 2 = 0$$
$$(c-1)(3c^2 - 7c + 2) = 0$$
$$(c-1)(3c^2 - 6c - c + 2) = 0$$
$$(c-1)(3c-1)(c-2) = 0$$
$$\therefore c = 1 \quad \text{or} \quad c = \frac{1}{3} \quad \text{or} \quad c = 2$$

Probability lies between 0 to 1

$$\therefore \boxed{c = \frac{1}{3}}$$

ii)
$$P(0 < x \leq 2) = P(1) + P(2)$$
$$= 4c - 10c^2 + 5c - 1$$
$$= 9c - 10c^2 - 1$$
$$= 9(1/3) - 10(1/3)^2 - 1$$

$$P(0 < x \leq 2) = 9\left(\frac{1}{3}\right) - 10\left(\frac{1}{3}\right)^2 - 1$$

$$= 3 - \frac{10}{9} - 1$$

$$P(0 < x \leq 2) = \frac{8}{9}$$

Q.2. A random variable x has the following probability distributions.

X	-2	-1	0	1	2	3
P(X)	0.1	k	0.2	2k	0.3	k

Find i) the value of k ii) mean iii) variance iv) $P(X \geq 1)$ v) $P(X < 1)$ vi) $P(-2 < x < 2)$.

→ i) $\Sigma P(x) = 1$

$$0.1 + k + 0.2 + 2k + 0.3 + k = 1$$

$$0.6 + 4k = 1$$

$$4k = 0.4$$

$$\boxed{k = 0.1}$$

ii) Mean $= \Sigma x_i P(x_i)$

$$= -0.2 \mp 0.1 + 0.2 + 0.6 + 0.3$$

$$= -0.3 + 1.1$$

Mean $= 0.80$

iii) Variance $= \Sigma E(x^2) - [E(X)]^2$

$\therefore E(X^2) = -2(0.1)^2 - 1(0.1)^2 + 4(0.1)^2 + 2(0.09) + 3(0.1)^2$

$\therefore E(X^2) = 4(0.1) + 0.1 + 0.2 + 1.2 + 0.9$

$E(X^2) = 2.80$

Variance $= 2.80 - 0.64$

Variance $= 2.16$

iv) $P(X \geq 1) = \cancel{1} \; P(X=1) + P(2) + P(3)$

$= 0.2 + 0.1 + 0.3$

$= \cancel{0.9} \; 0.6$

$\therefore P(X \geq 1) = 0.6$

v) $P(X < 1) = 1 - P(X \geq 1)$

$= 1 - 0.6$

$= 0.4$

vi) $P(-2 < x < 2) = P(-1) + P(0) + P(1)$

$= 0.1 + 0.2 + 0.2$

$P(-2 < x < 2) = 0.5$

Q.3. Following is the probability density function of a random variable x. $f(x) = k \cdot e^{-2x}$, $x > 0$. Find the value of k.

→ ∴ For probability density function,

$$\int_0^\infty f(x)\, dx = 1$$

$$\int_0^\infty (k \cdot e^{-2x})\, dx = 1$$

$$k \left[\frac{e^{-2x}}{-2}\right]_0^\infty = 1$$

$$\frac{k}{-2} \cdot (e^{-2\infty} - e^0) = 1$$

$$\boxed{k = 2}$$

Q.4. Following is the probability density function of a random variable x. $f(x) = kx^4 \cdot e^{-x/2}$. $0 \leq x < \infty$.

i) Find the value of k. ii) mean & variance.

→ ∴ For continuous probability density funct^n;

$$\int_0^\infty f(x)\, dx = 1$$

$$\int_0^\infty k \cdot x^4 \cdot e^{-x/2} dx = 1$$

put $\frac{x}{2} = t \Rightarrow x = 2t$

$dx = 2 \cdot dt$

x	0	∞
$t = x/2$	0	∞

$$\therefore k \int_0^{\infty} (2t)^4 e^{-t} \cdot 2\,dt = 1$$

$$(2)^5 k \int_0^{\infty} t^4 e^{-t}\,dt = 1$$

$$(2)^5 k \,\Gamma 5 = 1$$

$$4 \cancel{20} \times 32 \times 6k = 1$$

$$k = \frac{1}{192 \times 4}$$

$$k = \frac{1}{768}$$

$$\boxed{k = 0.001}$$

ii) Mean $= \int_{-\infty}^{\infty} x\, f(x)\,dx$

$$= \int_0^{\infty} x \cdot (0.001)\, x^4 e^{-x/2}\,dx$$

Mean $= (0.001) \int_0^{\infty} x^5 e^{-x/2}\,dx$

put $\frac{x}{2} = t \Rightarrow x = 2t$

$dx = 2 \cdot dt$

$$\text{Mean} = \int_0^\infty (0.001)\cdot(2t)^5\, e^{-2t/2}\, 2\,dt$$

$$= 0.004\,(2)^4 \int_0^\infty t^5 e^{-t}\,dt$$

$$= 0.004 \times 16\ \Gamma 6$$

$$= 0.004 \times 16 \times 5 \times 4 \times 3 \times 2 \times 1$$

$$= k \times 64 \times 20 \times 6$$

$$= \frac{7680}{768}$$

$$\boxed{\text{Mean} = 10}$$

$$\text{Variance} = \int_{-\infty}^{\infty} (x-\mu)^2 . f(x)\,dx = E(X^2) - [E(X)]^2$$

$$\text{Variance} = \int_0^\infty x^2 f(x)\,dx - (10)^2$$

$$= k \int_0^\infty x^6 e^{-x/2}\,dx - 100$$

$$= k. \int_0^\infty x^6\, e^{-x/2}\,dx - 100$$

put $\frac{x}{2} = t \Rightarrow x = 2t$

$dx = 2dt$

$$\therefore \text{Variance} = k. \int_0^\infty (2t)^6 e^{-t}\, 2\,dt - 100$$

$$\text{Variance} = \int_{-\infty}^{\infty} (x-\mu)^2 \, p(x)\, dx$$

$$= \int_{0}^{\infty} (x-10)^2 \, 0.001\, x^4 e^{-x/2} \, dx$$

$$= \frac{1}{768} \int_{0}^{\infty} (x-10)^2 x^4 e^{-x/2} \, dx$$

$$= \frac{1}{768} \int_{0}^{\infty} (x^2 - 10x + 100) x^4 e^{-x/2} \, dx$$

$$= \frac{1}{768} \int_{0}^{\infty} \left(x^6 e^{-x/2} - 10x^5 e^{-x/2} + 100 x^4 e^{-x/2}\right) dx$$

$$= \frac{1}{768} \left\{ \int_{0}^{\infty} x^6 e^{-x/2} \, dx - 10 \int_{0}^{\infty} x^5 e^{-x/2} \, dx + 100 \int_{0}^{\infty} x^4 e^{-x/2} \, dx \right\}$$

$$= \frac{1}{768} \left\{ 128 \int_{0}^{\infty} t^6 e^{-t} \, dt - 640 \int_{0}^{\infty} t^5 e^{-t} \, dt + 3200 \int_{0}^{\infty} t^4 e^{-t} \, dt \right\}$$

$$= \frac{1}{768} \left[128 \times 120 - 640 \times 24 + 3200 \times 6 \right]$$

$$= \frac{1}{768} \left[15360 - 15360 + 19200 \right]$$

$$= 25$$

If the sum of mean and variance of a Binomial distribution for 5 trials is 4.8. Find the distribution

Given $n = 5$. $np + npq = 4.8$

$$5p + 5pq = 4.8$$
$$5p(1+q) = 4.8$$
$$5p(1+1-p) = 4.8$$
$$5p(2-p) = 4.8$$
$$10p - 5p^2 = 4.8$$
$$5p^2 - 10p + 4.8 = 0$$
$$5p^2 - 4p - 6p + 4.8 = 0$$
$$p(5p-4) - 6(p-0.8) = 0$$

$$5(1-q)(1+q) = 4.8$$
$$5(1-q^2) = 4.8$$
$$1-q^2 = 0.96$$
$$1-0.96 = q^2$$
$$0.04 = q^2$$
$$q = 0.2$$
$$\therefore p = 1-q = 0.8$$

$\therefore$ $n = 5$. $p = 0.8$, $q = 0.2$

Let x be a binomial random variable with paramete

5 and 0.7 compute 1) $P(X=2)$ 2) $P(X \le 5)$
3) $P(X>2)$

Given, $n=5$, $p=0.7$ $\therefore q=0.3$

1) $P(X=2)$

$= {}^5C_2 (0.7)^2 (0.3)^3$

$= 10 \times 0.49 \times 0.027$

$= 4.9 \times 0.027$

$= 0.1323$

2) $P(X \le 5) = P(X=0) + P(1) + P(2) + P(3) + P(4) + P(5)$

$= {}^5C_0 (0.7)^0 (0.3)^5 + {}^5C_1 (0.7)(0.3)^4 +$
${}^5C_2 (0.7)^2 (0.3)^3 + {}^5C_3 (0.7)^3 (0.3)^2 +$
${}^5C_4 (0.7)^4 (0.3)^1 + {}^5C_5 (0.7)^5 (0.3)^0$

$= 0.0024 + 5 \times 0.7 \times 0.0081 + 10 \times 0.49 \times 0.027$
$+ 10 \times 0.343 \times 0.09 + 5 \times 0.2401 \times 0.3$
$+ 0.16807$

$= 0.0024 + 0.0283 + 0.1323 + 0.3087 +$
$0.36 + 0.16807$

$= 0.999 \sim 1$

$P(X \le 3) = P(0) + P(1) + P(2) + P(3) = 0.4717$

3) $p(X>2) = 1 - p(X \le 2)$

$= 1 - [p(0) + p(1) + p(2)]$

$= 1 - \{ {}^5C_0 (0.7)^0 (0.3)^5 + {}^5C_1 (0.7)^1 (0.3)^4 + {}^5C_2 (0.7)^2 (0.3)^3 \}$

$= 1 - \{ 0.00243 + 5 \times 0.7 \times 0.0081 + 10 \times 0.49 \times 0.027 \}$

$= 1 - (0.00243 + 0.02835 + 0.1323)$

$= 1 - (0.16308)$

$= 0.83692$

The probability that on joining Engineering College a student will successfully complete the course of studies is 3/5. Determine the probability that out of 5 students joining the college (i) none? (ii) all 5 and (iii) at least two will complete the course successfully

Given, $n = 5$, $p = \frac{3}{5}$, $q = \frac{2}{5}$

(i) none $= P(0) = {}^5C_0 \left(\frac{3}{5}\right)^0 \left(\frac{2}{5}\right)^5 = \frac{32}{3125}$

$= 0.01024$

ii) all 5 ($P(X=5)$)

$= {}^5C_5 \left(\frac{3}{5}\right)^5 \left(\frac{2}{5}\right)^0$

$= \frac{243}{3125} = 0.07776$

iii) at least two $= P(X \geq 2)$

$= 1 - P(X < 2)$

$= 1 - [P(0) + P(1) + \cancel{P(2)}]$

$= 1 - \left\{ {}^5C_0 \left(\frac{3}{5}\right)^0 \left(\frac{2}{5}\right)^5 + {}^5C_1 \left(\frac{3}{5}\right)^1 \left(\frac{2}{5}\right)^4 \right\}$

$= 1 - \left\{ \frac{32}{3125} + 5 \times \frac{3}{5} \times \frac{16}{625} \right\}$

$= 1 - \left\{ \frac{32}{3125} + \frac{240}{3125} \right\}$

$= 1 - \frac{272}{3125}$

$= \frac{2853}{3125}$

$= 0.91296$

out of 800 families with 5 children each how many would you except to have i) 3 boys ii) 5 girls

Given. $N = 800$, $n = 5$ $p = 1/2$, $q = 1/2$

i) 3 boys

$$p(x=3) = {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2$$

$$= 10 \times \frac{1}{8} \times \frac{1}{4}$$

$$= \frac{10}{32}$$

$$= 0.3125$$

Total No. of boys $= 800 \times 0.3125 = 250$

ii) 5 girls

$$P(x=0) = {}^5C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^5$$

$$= \frac{1}{32}$$

$$= 0.03125$$

Total No. of girls $= 800 \times 0.03125 = 25$

In a sampling the mean number of defective bulbs manufactured by a machine in a sample of 20 is 2. Determine the expected number of samples out of such 500 samples to contain at least 2 defective bolts

Given. $N : 500$, $n = 20$, $np = 2$

$p = \frac{1}{10} = 0.1$

$q = 0.9$

$P(X \geq 2) = 1 - P(X < 2)$

$= 1 - [P(0) + P(1)]$

$= 1 - \left\{ {}^{20}C_0 (0.1)^0 (0.9)^{20} + {}^{20}C_1 (0.1)^1 (0.9)^{19} \right\}$

$= 1 - \left\{ 0.1215 + 20 \times 0.1 \times 0.1350 \right\}$

$= 1 - \left\{ 0.1215 + 0.27 \right\}$

$= 1 - 0.3915$

$= 0.6085$

Number of samples $= 0.6085 \times 500 = 304$

A set of 4 coins were tossed 160 times to produce the following distribution.

Number of heads	0	1	2	3	4
Frequency observed	17	52	54	31	6

Fit a binomial distribution i) if the coin is unbiased ii) If the coin is baised

$n = 4$, $N = 160$, $p = q = \frac{1}{2}$

1) If the coin is unbiased

x	f	$P(X=r)$	$Np(x=r)$
0	–	0.0625	10
1	–	0.25	40
2	–	0.375	60
3	–	0.25	40
4	–	0.0625	10

11. Suppose that x has Poisson distribution If $P(x=2) = P(x=3)$, find 1) m 2) $P(x=4)$.

Given, $P(x=2) = P(x=3)$

$$\frac{e^{-m} m^2}{2!} = \frac{e^{-m} m^3}{3!}$$

$$\frac{m^2}{2} = \frac{m^3}{6}$$

$$6m^2 = 2m^3$$

$$3 = m \qquad \therefore \boxed{m=3}$$

$$P(x=4) = \frac{e^{-m} m^4}{4!}$$

$$= \frac{e^{-3}(3)^4}{24}$$

$$= \frac{0.049 \times 81}{24} = \frac{3.969}{24}$$

$$= 0.1653$$

12. Find the probability that at most 3 defective bulbs will be found in a box of 400 bulbs if it is known that 1 % of the bulbs are defective.

Given, $n = 400$, $p = \frac{1}{100} = 0.01$

$$m = np = 400 \times 0.01 = 4$$

P(at most 3 defective bulbs found) = $P(x \le 3)$

$P(x \le 3) = P(x=0) + P(x=1) + P(x=2) + P(x=3)$

$$= \frac{e^{-m}m^0}{0!} + \frac{e^{-m}m^1}{1!} + \frac{e^{-m}m^2}{2!} + \frac{e^{-m}m^3}{3!}$$

$$= e^{-m}\left(1 + 4 + 8 + \frac{64}{6}\right)$$

$$= e^{-4}\left(13 + \frac{64}{6}\right)$$

$$= 0.0183 \times 23.67$$

$$= 0.4331$$

13. A controlled manufacturing process is 0.2% defective. What is the probability of taking 2 or more defective from a lot of 100 pieces? i) Using binomial distribution ii) Using Poisson approximation

Given, $n = 100$, $p = \frac{0.2}{100} = 0.002$ $\therefore q = 0.998$

i) Using binomial distribution

$P(2 \text{ or more}) = P(x \geq 2) = 1 - P(x < 2)$

$= 1 - [P(x=0) + P(x=1)]$

$$\therefore P(x \geq 2) = 1 - \left\{ {}^{100}C_0 (0.002)^0 (0.998)^{100} + {}^{100}C_1 (0.002)(0.998)^{99} \right\}$$

$$= 1 - \{0.8185 + 100 \times 0.002 \times 0.82\}$$

$$= 1 - \{0.8185 + 0.164\}$$

$$= 1 - 0.9825$$

$$= 0.0175$$

ii) Using poisson distribution

$P(x \geq 2) = 1 - P(x < 2)$

$m = np$

$= 100 \times 0.002$

$= 0.2$

$P(x \geq 2) = 1 - [P(x=0) + P(x=1)]$

$= 1 - \left\{ \frac{e^{-0.2}(0.2)^0}{0!} + \frac{e^{-0.2}(0.2)^1}{1!} \right\}$

$= 1 - \{ 0.8187 + 0.8187 \times 0.2 \}$

$= 1 - \{ 0.8187 + 0.16374 \}$

$= 1 - 0.98244$

$= 0.01756$

14. If the number of accidents in a industry during a month has Poisson probability distribution with mean of 4, find probability that in coming month there will be at least 3 accidents.

Given, $m = np = 4$

Using Poisson distribution

$P(x \geq 3) = 1 - P(x < 3)$

$= 1 - [P(x=0) + P(x=1) + P(x=2)]$

$= 1 - \left[\frac{e^{-4}(4)^0}{0!} + \frac{e^{-4}(4)^1}{1!} + \frac{e^{-4}(4)^2}{2!} \right]$

$= 1 - [0.0183 + 0.0732 + 0.1465]$

$= 1 - 0.238$

$= 0.762$

Probability that coming month there will be at least 3 accidents is 0.762.