Unit 2 :- Vector algebra

oct 22.

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Basic concepts :-

If n is positive integer then an ordered n-type is a sequence of n real numbers $(u_1, u_2, \ldots u_n)$

An n-vector is an $n \times 1$ matrix

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_n \end{bmatrix} \quad \text{or an } n\text{-tuple } u = (u_1, u_2, \ldots u_n)$$

where $u_1, u_2, \ldots u_n$ are real numbers, called components of u.

The set of all n-vectors is denoted by $\mathbb{R}^n$ and is called n-space. The space $\mathbb{R}^n$ consists of all column vectors with n components (we write $\mathbb{R}$ because the components are real numbers).

$\mathbb{R}^2$ is represented by the usual x-y plane, the two components of the vector becomes the x and y co-ordinate of corresponding point.

$\mathbb{R}^3$ is represented by the three dimensional space, the three components of the vector becomes x, y & z of corresponding point. The one-dimensional space $\mathbb{R}^1$ is a line.

** Vector operations

Let $u = (u_1, u_2, \ldots u_n)$, $v = (v_1, v_2, \ldots v_n)$ be two vectors in $\mathbb{R}^n$ then the sum of the vectors $\bar{u}$ and $\bar{v}$ is the vector $\bar{u} + \bar{v} = (u_1 + v_1, u_2 + v_2, \ldots u_n + v_n)$ and if c is scalar (real no) then the scalar multiple of u by c is the vector

$$c\bar{u} = (cu_1, cu_2, \ldots cu_n)$$

The operations of additions and scalar multiplication are called the standard operations in $\mathbb{R}^n$.

- The zero vector in $\mathbb{R}^n$ is a vector
  $\bar{0} = (0, 0, \cdots 0)$
- If $u = (u_1, u_2, \cdots u_n)$ is any vector in $\mathbb{R}^n$ then the negative (or additive inverse) of u is denoted by $-u$ and is defined by
  $-u = (-u_1, -u_2, \cdots -u_n)$
- Subtraction of vectors in $\mathbb{R}^n$ by
  $\bar{v} - \bar{u} = \bar{v} + (-\bar{u})$
  $= (v_1 - u_1, v_2 - u_2, \cdots v_n - u_n)$

** Vector space

Defn :- Let V be nonempty set of objects (e.g. - real matrices of any fixed order, vectors in a plane, vectors space, real valued functions etc.) on which two operations are defined, vector addition and scalar multiplication, then V is called a Vector space or linear space if the following axioms are satisfied

Closure axioms

C_1 : For u, v in V, $u+v$ is in V ie V is closed under addition

C_2 : For any scalar k and any object u in V
ku is in V
ie V is closed under multiplication

Addition axioms

A_1 : $u + v = v + u \quad \forall u, v \in V$ (commutativity)

A_2 : $(u+v)+w = u+(v+w) \quad \forall u, v, w \in V$ (Associativity)

A_3 : There is an object 0 in V, called zero vector for V, such that $u + 0 = u \quad \forall u \in V$

A_4: For each u in V there is an object $-u$ in V called negative of u (additive inverse) Such that

$u + (-u) = 0$

$-u$ is called negative of u.

Scalar multiplication axioms

M_1: $k(u+v) = ku + kv$

M_2: $(k+l)u = ku + lu$

M_3: $(kl)u = k(lu)$ where k, l are scalars.

M_4: $1u = u$

Note :- objects of V are called vectors.

Examples

① The set of real numbers $\mathbb{R}$ itself is a vector space with addition of any two members in $\mathbb{R}$ as usual addition of real numbers and multiplication by scalars as usual multiplication.

② Let $V = \mathbb{R}^n$ be set of all n-vectors of real numbers then $\mathbb{R}^n$ is a vector space under the standard vector operations in $\mathbb{R}^n$ ie vector addition & scalar multiplication of n-vectors.

③ $\mathbb{P}_n$ is the collection (set) of all polynomials of degree n or less [$\mathbb{P}$ is the collection of all polynomials of any degree]

$$\mathbb{P}_n = \{ a_0 + a_1x + a_2x^2 + \cdots + a_nx^n : a_i \in \mathbb{R} \}$$

Then $\mathbb{P}_n$ is a vector space under vector operation ie vector addition & scalar multiplication.

④ Let $V = F(a,b)$ be the set of all real valued functions defined on $[a,b]$ (set of real numbers).

** The zero polynomial is defined as

$$0t^n + 0t^{n-1} + \cdots + 0t + 0$$

Note that zero polynomial has no degree.

Examples of vector spaces

1) $\mathbb{R}$ – the set of real numbers
2) $\mathbb{R}^2$ – the set of all ordered pairs of real numbers
3) $\mathbb{R}^n$ – the set of all n-tuples of real numbers
4) $\mathbb{C}^n$ – the set of all n-tuples of complex numbers
5) P_n – the set of all polynomials of degree $\leq n$
6) $M_{m\times n}(\mathbb{R})$ = the set of all $m \times n$ matrices
7) $M_{n}(\mathbb{R})$ – the set of all $n \times n$ square matrices
8) $C^k[a,b]$ – set of continuous functions defined on $[a,b]$ that have atleast k continuous derivatives

Ex ① Let $V = R^2$, the set of all ordered pairs of real numbers; that is $R^2 = \{(x,y) / x, y \in \mathbb{R}\}$

Define for any $u = (x,y)$ & $v = (x', y')$ in $\mathbb{R}^2$

1) Equality : $u = v$ if and only if $x = x'$ & $y = y'$

2) Sum : $u + v = (x,y) + (x',y')$
$= (x+x', y+y')$

3) Scalar multiplication : For any scalar k
$ku = k(x,y) = (kx, ky)$

Is V a vector space.

$\Rightarrow$ The closure axioms clearly hold good as $x+x', y+y'$ and kx, ky are real numbers. hence $u+v$ and ku are again ordered pairs of real numbers & thus belongs to R^2

Let $u = (x,y)$, $v = (x',y')$ & $w = (x'',y'')$ be any vectors in $\mathbb{R}^2$

A_1: $u + v = (x,y) + (x',y')$
$= (x+x', y+y')$ by defn of sum in $\mathbb{R}^2$
$= (x'+x, y'+y)$ sum in $\mathbb{R}^2$ is commutative
$= (x',y') + (x,y)$ again by defn of sum in $\mathbb{R}^2$
$= v + u$

$\therefore u + v = v + u.$

A_2: $(u+v)+w = [(x+y)+(x',y')] + (x'',y'')$

$= (x+x', y+y') + (x'',y'')$ by sum in $\mathbb{R}^2$

$= ((x+x')+x'', (y+y')+y'')$ by sum in $\mathbb{R}^2$

$= (x+(x'+x''), y+(y'+y''))$

by associativity of sum in $\mathbb{R}^2$

$= (x,y) + (x'+x''; y'+y'')$ by sum in $\mathbb{R}^2$

$= (x,y) + [(x',y') + (x''+y'')]$ by sum in $\mathbb{R}^2$

$= u+(v+w)$

$\therefore (u+v)+w = u+(v+w)$

A_3: As $0 \in \mathbb{R}$ $(0,0)$ is in $\mathbb{R}^2$ and we have

$u+(0,0) = (x,y)+(0,0)$

$= (x+0, y+0)$ by sum in $\mathbb{R}^2$

$= (x,y)$

$= u$

Thus $(0,0)$ is a vector in $\mathbb{R}^2$

A4: For $u=(x,y)$ in $\mathbb{R}^2$ we have $-u=(-x,-y)$ in $\mathbb{R}^2$ such that

$u+(-u) = (x,y)+(-x,-y)$

$= (x+(-x); y+(-y))$ by sum in $\mathbb{R}^2$

$= (0,0)$ $\because x+(-x)=0=y+(-y)$ in $\mathbb{R}$

Thus for each u in $\mathbb{R}^2$ there is negative of u in $\mathbb{R}^2$.

M_1: Let k & l be any scalars.

Consider

$k(u+v) = k(x+x', y+y')$ by sum in $\mathbb{R}^2$

$= (k(x+x'), k(y+y'))$ by scalar multiplication in $\mathbb{R}^2$

$= (kx+kx', ky+ky')$

multiplication distributes over '+' in $\mathbb{R}$

$= (kx, ky) + (kx', ky')$ by sum in $\mathbb{R}^2$

$= (\cancel{kx}, \cancel{ky}) + (\cancel{kx}, \cancel{ky})$

$= k(x,y) + k(x',y')$ by scalar multiplication in $\mathbb{R}^2$

$= ku + kv$

Thus $k(u+v) = ku + kv$

M_2: Consider

$$(k+l)u = (k+l)(x,y)$$

$= ((k+l)x, (k+l)y)$ by scalar multiplication in R^2

$= (kx + lx, ky + ly)$

multiplication distributes over + in R

$= (kx, ky) + (lx, ly)$ by sum in R^2

$= k(x,y) + l(x,y)$ by scalar multiplication in R^2

$= ku + lu$

Thus $(k+l)u = ku + lu$.

M_3: consider

$$(kl)u = (kl)(x,y)$$

$= ((kl)x, (kl)y)$

$= (k(lx), k(ly))$ multiplication is associative in R^2

$= k[lx, ly]$

$= k[l(x,y)]$

$= k(lu)$

Thus $(kl)u = k(lu)$

M_4: $1u = 1(x,y)$

$= (1x, 1y)$

$= (x,y)$

$= u$

Thus R^2 is a vector space.

Ex ② Let $V = R$ the set of all real nos with the operations

$u + v = u - v$ (i.e. ordinary subtraction)

$c \cdot u = cu$ (i.e. ordinary multiplication)

Is V a vector space? If it is not which axioms fail to hold?

(4)

Soln: If $u, v \in V$ then

$u + v = u - v \in V$

$c \cdot \vec{u} = c\vec{u} \in V$

$\therefore V$ is closed under vector addition & scalar multiplication

However axiom A_1 fails to hold

For take $u = 2, v = 5 \in V (= R)$

$\therefore u + v = 2 + 5 = 2 - 5 = -3$

& $v + u = 5 + 2 = 5 - 2 = 3$

$u + v \neq v + u$

Similarly axioms A_2, A_3, A_4, M_2 does not hold but M_1, M_3, M_4 holds.

$\therefore$ The given vector V is not vectorspace.

Ex (3) Let V be the set of all real valued functions. defined on on a closed interval $[a, b]$ $a < b$

For any f and g in V & for any scalar k, define

(i) $f = g$ iff $f(x) = g(x)$ for all x in $[a, b]$

(ii) $(f+g)(x) = f(x) + g(x)$ $\quad x \in [a, b]$

(iii) $(kf)(x) = k f(x)$

Show that V is a vectorspace.

→ firstly C_1 & C_2 follows clearly as $f+g$ and kf are again real valued functions defined on $[a, b]$

A_1: For any x in $[a, b]$

$(f+g)(x) = f(x) + g(x)$ by sum in V

$= g(x) + f(x)$ $f(x)$ and $g(x)$ are real numbers.

$= (g+f)(x)$ → by sum in V

$\therefore (f+g)(x) = (g+f)(x)$ for all x in $[a, b]$ Hence by equality in V we have $f + g = g + f$.

A_2 : For any $x \in [a,b]$ we have

$[(f+g)+h](x) = (f+g)(x) + h(x)$ by defn of sum in V

$= [f(x)+g(x)] + h(x)$ by defn of sum in V

$= f(x) + [g(x)+h(x)]$

by associativity of + in R

$= f(x) + (g+h)(x)$ by defn of + in V

$= [f+(g+h)](x)$ by defn of + in V

$\therefore (f+g)+h = f+(g+h)$ by defn of equality.

A_3 : For any $x \in [a,b]$ we define a function

$\bar{0} : [a,b] \longrightarrow R$ by $\bar{0}(x) = 0$

then for any $f \in V$

$(f+\bar{0})(x) = f(x) + \bar{0}(x)$

$= f(x) + 0$

$= f(x)$

$\therefore f + \bar{0} = f$

$\bar{0}$ is zero element in V.

A_4 : For any $f \in V$ we define

$(-f)(x) = -f(x)$ then we have

$[f+(-f)](x) = f(x) + (-f(x))$

$= f(x) - f(x)$

$= 0$

Thus $[f+(-f)](x) = 0$

$= \bar{0}(x)$

$f+(-f) = \bar{0}$

M_1 : For any $x \in [a,b]$ and for any f, g in V & K any scalars consider

$[k(f+g)](x) = k(f+g)(x)$

$= k[f(x)+g(x)]$

$= kf(x) + kg(x)$

$= (kf)(x) + (kg)(x)$

$= (kf+kg)(x)$

$\therefore [k(f+g)](x) = (kf+kg)(x)$ for all $x \in [a,b]$ (8)

$\therefore k(f+g) = kf+kg$

M_3: For any scalars k and l and $f \in V$, consider

$[(kl)f](x) = (kl)f(x)$ by defn of scalar multiplication

$= k(l f(x))$ by associativity of multiplication in R

$= k[(lf)(x)]$

$= [k(lf)](x)$

Thus $[(kl)f](x) = [k(lf)](x)$ for all $x \in [a,b]$

$(kl)f = k(lf)$

M_2: For any scalars k and l and $f \in V$

consider

$[(k+l)f](x) = (k+l)f(x)$ by defn of scalar multiplication

$= kf(x) + lf(x)$

$= (kf)(x) + (lf)(x)$

$= (kf+lf)(x)$

Thus $[(k+l)f](x) = (kf+lf)(x)$ for all $x \in [a,b]$.

$\Rightarrow (k+l)f = kf+lf$

M_4: For any $f \in V$ and any $x \in [a,b]$

$(1f)(x) = 1 f(x)$ by defn of scalar multiplication

$= f(x)$

Thus $(1f)(x) = f(x)$ for all $x \in [a,b]$

$1f = f$

The above vector space is sometimes called a function space.

Ex (3) Let $V = M_{2\times2}(R)$, the set of all 2×2 matrices with real entries. ~~For any~~ Define for any

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ & $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ in V

1) $A = B$ iff $a=e$, $b=f$, $c=g$, $d=h$

2) $A+B = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$

iii) for any scalar k,

$kA = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$ Then show that V is a vector space.

$\Rightarrow A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$

C_1 : for A, B in V

$A+B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$

$\Rightarrow$ A+B is also in V

ie V is closed under vector addition

C_2 : For any scalar k & A in V

$kA = k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$

$\Rightarrow$ kA is also in V

ie V is closed under scalar multiplication.

A_1 : consider

$A+B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix}$

$= \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$

$= \begin{pmatrix} e+a & f+b \\ g+c & h+d \end{pmatrix}$

$= \begin{pmatrix} e & f \\ g & h \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$A+B = B+A$

A_2 : Let $C = \begin{pmatrix} i & j \\ k & l \end{pmatrix}$

consider $(A+B)+C = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} + \begin{bmatrix} i & j \\ k & l \end{bmatrix}$

$= \begin{bmatrix} a+e+i & b+f+j \\ c+g+k & d+h+l \end{bmatrix}$

$\& A+(B+C) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e+i & f+j \\ g+k & l+h \end{pmatrix}$

$$= \begin{pmatrix} a+e+i & b+f+j \\ c+g+k & d+l+h \end{pmatrix}$$

$\Rightarrow (A+B)+C = A+(B+C)$

A_3 :- There is zero matrix O in V such that

$$A+O = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\Rightarrow A+O = A$ for all A in V.

A_4 :- For each A in V there is $-A$ in V

such that $A+(-A) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$

$$= \begin{pmatrix} a-a & b-b \\ c-c & d-d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow A+(-A) = O$

M_1 : There is scalar k such that

$$k(A+B) = k\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} = \begin{pmatrix} k(a+e) & k(b+f) \\ k(c+g) & k(d+h) \end{pmatrix}$$

$$= \begin{pmatrix} ka+ke & kb+kf \\ kc+kg & kd+kh \end{pmatrix}$$

$$= \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix} + \begin{pmatrix} ke & kf \\ kg & kh \end{pmatrix}$$

$$= k\begin{pmatrix} a & b \\ c & d \end{pmatrix} + k\begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$= kA + kB$$

M_2: For any scalar k & l and A in V.

$$(k+l)A = (k+l)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (k+l)a & (k+l)b \\ (k+l)c & (k+l)d \end{pmatrix}$$

$$= \begin{pmatrix} ka+la & kb+lb \\ kc+lc & kd+ld \end{pmatrix}$$

$$= \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix} + \begin{pmatrix} la & lb \\ lc & ld \end{pmatrix}$$

$$= k\begin{pmatrix} a & b \\ c & d \end{pmatrix} + l\begin{pmatrix} a & b \\ c & d \end{pmatrix} = kA + lB.$$

M_3: consider

$$(kl)A = kl\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (kl)a & (kl)b \\ (kl)c & (kl)d \end{pmatrix}$$

$$= \begin{pmatrix} k(la) & k(lb) \\ k(lc) & k(ld) \end{pmatrix} = k\begin{pmatrix} la & lb \\ lc & ld \end{pmatrix}$$

$$= k(lA)$$

M_4: consider

$$1 \cdot A = 1 \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1\cdot a & 1\cdot b \\ 1\cdot c & 1\cdot d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

All axioms of vectorspace are satisfied

$\therefore$ Given set is a vectorspace.

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Subspace :-

Defn :- Let V be a vector space. A nonempty subset W of V is a subspace of V if W itself is a vector space under the addition & scalar multiplication defined on V.

Note (Thm) :- Let W be subset of a vector space V. Then W is subspace of V if and only if the following conditions hold.

i) $W \neq \phi$
ii) For any u and v in W, u+v is in W and
iii) For any u in W, for any scalar k, ku is also in W.

OR It can be stated as

A non empty subset W of a vector space V is a subspace if and only if for any $u, v \in W$ and scalars α, β

$$\alpha u + \beta v \in W.$$

Examples

① Let $W = \{(a,b,0) \mid a,b \in \mathbb{R}\}$ be the subset of vector space $\mathbb{R}^3$ with usual operations of vector addition and scalar multiplication. Is W a subspace ?

→ Take $u = (a_1, b_1, 0)$ $v = (a_2, b_2, 0) \in W$ & $k \in \mathbb{R}$

Then $u+v = (a_1, b_1, 0) + (a_2, b_2, 0)$
$= (a_1+a_2, b_1+b_2, 0) \in W$

and $ku = k(a_1, b_1, 0) = (ka_1, kb_1, 0) \in W$.

$\therefore W$ is a subspace of $\mathbb{R}^3$.

Ex ② $V = R^3$ is a subspace.

$W = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 \leq 1\}$

Is W is subspace of $V = R^3$.

$\Rightarrow$ Let $u = (1, 0, 0)$ $\quad v = (0, 1, 0)$

$u, v \in W$.

consider

$$u + v = (1, 0, 0) + (0, 1, 0)$$
$$= (1, 1, 0)$$

$\Rightarrow u + v \notin W$.

& $ku = k(1, 0, 0) = (k, 0, 0) \notin W$.

Hence W is not subspace of $V = R^3$.

Ex ③ Let $V = R^3$, the vector space of ordered triples of real numbers. If $W = \{(x, y, z) \in R^3 / 2x + 3y + 4z = 0\}$

$\Rightarrow$ Let $u = (x, y, z)$ & $v = (x', y', z')$ be any vectors in W

Then hypothesis for W, we have

$$2x + 3y + 4z = 0 \quad \text{— (i)}$$
&
$$2x' + 3y' + 4z' = 0 \quad \text{— (ii)}$$

Now $u + v = (x + x', y + y', z + z')$.

In order to show that $u + v$ is in W, we need to show that $2(x + x') + 3(y + y') + 4(z + z') = 0$

consider $2(x + x') + 3(y + y') + 4(z + z')$

$$= (2x + 3y + 4z) + (2x' + 3y' + 4z')$$
$$= 0 + 0 \qquad \text{by (i) \& (ii)}$$
$$= 0$$

$u + v \in W$

Also for any scalar k to show that ku is in W

i.e. (kx, ky, kz) is in W.

we have to show that

$\cancel{2(kx)+3(ky)+4(kz)=0}$

Consider $2(kx)+3(ky)+4(kz)$

$= k(2x+3y+4z)$

$= k \cdot 0$

$= 0$

Thus ku is in W. Hence W is a subspace of V.

Ex ④ Let V be a vector space of all real valued functions defined on closed interval $[0,1]$. Let

$W = \{ f \in V \mid f(1/2) = 0 \}$

Then show that W is a subspace of V.

$\Rightarrow$ Let $f, g \in W$

we have $f(1/2) = 0 = g(1/2)$

consider $(f+g)(\frac{1}{2}) = f(1/2) + g(1/2)$ by sum in V

$= 0 + 0$

Thus $(f+g)(1/2) = 0$

Hence $f+g$ is in W

Also for any scalar k, we have

$(kf)(1/2) = k f(1/2)$ by defn of scalar multiplication

$= k \cdot 0$

$= 0$

Thus $kf(1/2) = 0$

$\rightarrow kf \in W$.

Thus W is closed under addition and scalar multipli

hence W is a subspace of V.

Ex (5) Let $V = \mathbb{R}^3$ & W be the subset of V given by

$W = \{ (x, y, z) \in V \mid x = 0 \text{ or } y = 0 \}$

$\Rightarrow$ consider $W = \{ (x, y, z) \in V \mid x = 0 \text{ or } y = 0 \}$

Let $u = (0, 3, 5)$ & $v = (5, 0, -1) \in W$

$u + v = (0, 3, 5) + (5, 0, -1)$

$= (5, 3, 4)$

$\Rightarrow u + v \notin W$ since neither x is zero nor y is zero.

$\therefore$ W is not a subspace of V.

** Linear combination & spanning:

Defn (1) - Let $v_1, v_2, v_3 \ldots v_k$ be the vectors in a vector space V. A vector v in V is called a linear combination of $v_1, v_2, \ldots v_k$ if

$$\bar{v} = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$ for some real nos $c_1, c_2, \ldots \ldots c_k$.

Defn (2) - Let V be the vector space and $S = \{v_1, v_2, \ldots v_k\}$ be a set of vectors in V. Then the set of all linear combination of the vectors $v_1, v_2 \ldots v_k$ is called a linear span of the set S and is denoted by span S or L(S).

Thus

$$L(S) = \{ c_1 v_1 + c_2 v_2 + \cdots c_k v_k \mid c_1, c_2 \ldots c_k \text{ are scalar} \}$$

e.g consider the set S of 2x2 matrices given by

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Then span S is the set in M_{22} consisting of all vectors of the form $a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$

Thus span S is the subset of M_{22} consisting of all matrices of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ where $a, b, c \in \mathbb{R}$.

Note :- 1) Some we also say that $L(S)$ is spanned by the set S.

2) $\mathbb{R}^2$ is a linear span of just two vectors $(1,0)$ & $(0,1)$

lly $\mathbb{R}^3$ is a linear span of the vectors $(1,1,1)$, $(1,1,0)$ and $(1,0,0)$

3) Linear span of the empty set is defined to be zero vector that is $L(\phi) = \{0\}$.

Ex (1) Let $u = (1,1,1)$, $v = (1,1,0)$ and $w = (1,0,0)$ be three vectors in $\mathbb{R}^3$ show that $(3,2,1)$ is linear combination of u, v and w.

$\rightarrow$ consider

$$V = c_1 V_1 + c_2 V_2 + c_3 V_3$$

$$(3,2,1) = c_1(1,1,1) + c_2(1,1,0) + c_3(1,0,0)$$

$$\Rightarrow \quad c_1 + c_2 + c_3 = 3$$

$$c_1 + c_2 = 2$$

$$c_1 = 1$$

solving above eqns we get $c_1 = 1$, $c_2 = 1$ & $c_3 = 1$

$$\therefore (3,2,1) = 1(1,1,1) + 1(1,1,0) + 1(1,0,0)$$

This shows that vectors u, v, w are in linear combination.

Ex (2) Let $V_1 = (1,2,3)$ and $V_2 = (0,-1,2)$ be vectors in R^3 show that

i) $(-1,-4,1)$ is linear combination of V_1 & V_2

ii) $(4,7,15)$ is not a linear combination of V_1 & V_2

$\rightarrow$ Let c_1 & c_2 be the two scalar such that

$(-1,-4,1) = c_1 V_1 + c_2 V_2$

$\rightarrow (-1,-4,1) = c_1(1,2,3) + c_2(0,-1,2)$ — (1)

$\rightarrow c_1 = -1$

$2c_1 - c_2 = -4$

$3c_1 + 2c_2 = 1$

solving these equations we get $c_1 = -1$, $c_2 = 2$

$\therefore$ eqn (1) becomes

$(-1,-4,1) = -1(1,2,3) + 2(0,-1,2)$

Hence the vector $(-1,-4,1)$ is linear combination of V_1 & V_2.

ii) Let c_1 & c_2 be the two scalars such that

$(4,7,15) = c_1 V_1 + c_2 V_2$

$= c_1(1,2,3) + c_2(0,-1,2)$

$(4,7,15) = (c_1, 2c_1 - c_2, 3c_1 + 2c_2)$

$\rightarrow c_1 = 4$

$2c_1 - c_2 = 7$

$3c_1 + 2c_2 = 15$

solving these eqns, we get $c_1 = 4$, $c_2 = 1$

But $3c_1 + 2c_2 = 12 + 2 = 14 \neq 15$.

Hence u is not linear combination of V_1 & V_2.

** Linear Dependence & linear Independence –

Defn :- Linear Dependence :-

Let V be a vector space & $S = \{v_1, v_2, \dots v_k\}$ be nonempty set of vectors in V. If there exists scalars $c_1, c_2, \dots c_k$ not all zero such that

$c_1 v_1 + c_2 v_2 + \dots + c_k v_k = 0$

Then the set 's' is called linearly dependent set in V.

Defn :- linear independence :- Let V be a vector space & $S = \{v_1, v_2, \dots v_k\}$ be non empty scalars set of vectors in V. If there exists scalars $c_1, c_2, \dots c_k$ are all zero such that

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$c_1 V_1 + c_2 V_2 + \dots + c_r V_r = 0$

Then the set 'S' is called linearly independent.

Note :- If S is linearly independent then given set of eqns has only one solution ie $k_1 = k_2 = \dots = k_r = 0$

The empty set ϕ is defined to be independent.

Examples :-

① Let P_2 denotes vector space of polynomials of degree ≤ 2. Express $q = 2 + 6x^2$ as a linear combination of $P_1 = 2 + x + 4x^2$, $P_2 = 1 - x + 3x^2$, $P_3 = 3 + 2x + 5x^2$ in P_3

or Determine if the vector q belongs to span $\{P_1, P_2, P_3\}$.

$\Rightarrow$ Let C_1, C_2 & C_3 be the three scalars such that $q = C_1 P_1 + C_2 P_2 + C_3 P_3$ —— ①

ie $2 + 6x^2 = C_1(2 + x + 4x^2) + C_2(1 - x + 3x^2) + C_3(3 + 2x + 5x^2)$

$\Rightarrow 2 + 6x^2 = (2C_1 + C_2 + 3C_3) + (C_1 - C_2 + 2C_3)x + (4C_1 + 3C_2 + 5C_3)x^2$

comparing we get (equating the coefficients of like powers of x on both sides) we get system of linear equation as

$$2C_1 + C_2 + 3C_3 = 2$$
$$C_1 - C_2 + 2C_3 = 0$$
$$4C_1 + 3C_2 + 5C_3 = 6$$

Augmented matrix $[A : B]$

$$= \left[\begin{array}{ccc|c} 2 & 1 & 3 & 2 \\ 1 & -1 & 2 & 0 \\ 4 & 3 & 5 & 6 \end{array}\right]$$

Reducing to echelon form, we get

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Here $\rho[A] = \rho[A : B] = 3$

system is consistent & it has unique soln

By Back substitution we get

$c_1 = 4,\ c_2 = 0,\ c_3 = -2$

$\therefore$ vector q is expressed as a linear combination ~~vectors P_1, P_2 & P_3 as~~

$$q = 4P_1 + 0P_2 - 2P_3$$

$$\Rightarrow \boxed{q = 4P_1 - 2P_3}$$

Ex ② Let $S = \{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ be subset of vectors space B^3. Determine if the vector $u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ belongs to span S.

$\rightarrow$ If we can find scalars c_1, c_2, c_3 so that

$u = c_1 v_1 + c_2 v_2 + c_3 v_3$ then

$u \in$ span S.

Consider $u = c_1 v_1 + c_2 v_2 + c_3 v_3$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\rightarrow$

$$c_1 + 0c_2 + c_3 = 1$$
$$c_1 + c_2 + c_3 = 0$$
$$0c_1 + c_2 + c_3 = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The Augmented matrix $[A : B]$ is

$$\left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & & \end{array}\right]$$

Reducing to echelon form we get

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$\rho[A] = \rho[A:B] = 3$

$\Rightarrow c_1 = 0, c_2 = 1, c_3 = 1$

Thus $\bar{u} = 0V_1 + V_2 + V_3$

$\Rightarrow u \in \text{span } S.$

Ex (3) For which values of k the vector $V = (1, -2, k)$ in $\mathbb{R}^3$ is linear combination of the vectors $u = (3, 0, -2)$ & $w = (2, -1, -5)$?

$\Rightarrow$ Since vector V is linear combination of u & w

$\therefore$ there exist scalars c_1 & c_2 such that

$c_1 u + c_2 w = V$

ie $c_1(3, 0, -2) + c_2(2, -1, -5) = (1, -2, k)$

$\Rightarrow 3c_1 + 2c_2 = 1$ —— (1)

$-c_2 = -2$ —— (2)

$-2c_1 - 5c_2 = k$ —— (3)

Solving eqns (1) & (2) we get

$c_1 = -1$

putting the values of c_1 & c_2 in eqn (3) we get

$2 - 10 = k$

$\Rightarrow k = -8$

Thus if $k = -8$ then v is a linear combination of u & w.

Ex (4) Are the vectors $V_1 = (1, 0, 1, 2)$, $V_2 = (0, 1, 1, 2)$, $V_3 = (1, 1, 1, 3)$ in $\mathbb{R}^4$ linearly dependent or independent?

$\Rightarrow$ consider the equations

$c_1 V_1 + c_2 V_2 + c_3 V_3 = 0$

ie $c_1(1, 0, 1, 2) + c_2(0, 1, 1, 2) + c_3(1, 1, 1, 3) = (0, 0, 0, 0)$

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The resulting homogeneous system is

$c_1 + c_3 = 0$, $c_2 + c_3 = 0$, $c_1 + c_2 + c_3 = 0$, $2c_1 + 2c_2 + 3c_3 = 0$

This system has only solutions $c_1 = c_2 = c_3 = 0$

$\therefore$ Given vectors are linearly independent.

Ex (5) Is $S = \{V_1, V_2, V_3, V_4\} = \{(1,3,3), (0,1,4), (5,6,3), (7,2,-1)\} \subset R^3$ linearly dependent

→ consider the equation

$$c_1V_1 + c_2V_2 + c_3V_3 + c_4V_4 = 0$$

$$c_1(1,3,3) + c_2(0,1,4) + c_3(5,6,3) + c_4(7,2,-1) = (0,0,0)$$

This gives a homogeneous system as

$$c_1 + 5c_3 + 7c_4 = 0$$

$$3c_1 + c_2 + 6c_3 + 2c_4 = 0$$

$$3c_1 + 4c_2 + 3c_3 - c_4 = 0$$

This is homogeneous system of equations with more unknowns then the number of equations. Hence it has a nontrivial solution and so S is a linearly dependent set.

Ex (6) Let $S = \{P_1, P_2, P_3\}$ is set of vectors in P_2 where $P_1 = 6 - t^2$, $P_2 = 1 + t + t^2$, $P_3 = 3 - 3t - 4t^2$. Determine whether the set S is linearly independent.

→ consider $c_1P_1 + c_2P_2 + c_3P_3 = 0$ (zero polynomial)

ie $c_1(6 - t^2) + c_2(1 + t + t^2) + c_3(3 - 3t - 4t^2) = 0$

it gives homogeneous system as

$$6c_1 + c_2 + 3c_3 = 0$$

$$c_2 - 3c_3 = 0$$

$$-c_1 + c_2 - 4c_3 = 0$$

The augmented matrix is $\left[\begin{array}{ccc|c} 6 & 1 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ -1 & 1 & -4 & 0 \end{array}\right]$

which reduces to $\left[\begin{array}{ccc|c} 1 & -1 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Thus c_3 can be assigned arbitrary and we get infinitely many solutions. Hence S is linearly dependent

Basis and Dimension

Defn — Basis

If V is any vector space and $S = \{v_1, v_2, \ldots, v_n\}$ is a set of vectors in V then S is called a basis for V if ~~the foll.~~ i) S is linearly independent.

ii) S spans V.

Note:- If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space V, then every vector v in V can be expressed in the form $V = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$ in exactly one way.

examples:-

1) For V_2 the set $\{(1,0), (0,1)\}$ is a basis.
2) For V_2 the set $\{(1,1), (1,-1)\}$ is a basis.
3) For V_3 the set $\{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis
4) For V_4 the set $\{e_1 = (1,0,0,0), e_2 = (0,1,0,0), e_3 = (0,0,1,0), e_4 = (0,0,0,1)\}$ is a basis.

In general $S = \{e_1, e_2, \ldots, e_n\}$ form a basis for R^n.

If a vector space V has a basis consisting of finite number of elements then the vector space V is said to be finite dimensional.

The number of elements in the basis is called dimension of a vector space.

If $\dim V = n$ then V is said to be n dimensional

e.g. $\dim V_2 = 2$

$\dim V_3 = 3$

$\dim V_4 = 4$

If there is no such finite subset of V, then V is called infinite dimensional.

** Dimension

Defn :- The dimension of a non zero vector space V is the number of vectors in basis for V and it is denoted by $\dim(V)$

Note : The set $\{\bar{0}\}$ is linearly dependent. It is natural to say. The vector space $\{\bar{0}\}$ has dimension zero.

** $\dim \mathbb{R}^n = n$

** $\dim(M_{mn}) = mn$

** $\dim(P_n) = n+1$.

** If W is any non-zero subspace of finite dimensional vector space V then $\dim W \leq \dim V$.

Thm :- Let V be an n-dimensional vector space, and let $S = \{v_1, v_2, \ldots v_n\}$ be a set of n vectors in V

a) If S is linearly independent, then it is basis for V.

b) If S spans V, then it is basis for V.

Ex ① Show that the set of vectors $\{(1,0,0), (1,1,0), (1,1,1)\}$ forms a basis for $\mathbb{R}^3$.

Soln Let $S = \{(1,0,0), (1,1,0), (1,1,1)\}$

To show S is linearly independent

Consider $c_1v_1 + c_2v_2 + c_3v_3 = 0$

ie $c_1(1,0,0) + c_2(1,1,0) + (1,1,1)c_3 = 0$

which gives $c_1 + c_2 + c_3 = 0$

$c_2 + c_3 = 0$

$c_3 = 0$

This system has trivial solution $c_1 = c_2 = c_3 = 0$

$\therefore$ S is linearly independent

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To show S spans $\mathbb{R}^3$,

consider $u = (a, b, c)$ be any vector in $\mathbb{R}^3$.

Now we have to find scalars c_1, c_2, c_3 such that

$$u = c_1 v_1 + c_2 v_2 + c_3 v_3 \Rightarrow (a, b, c) = c_1(1,0,0) + c_2(1,1,0) + c_3(1,1,1)$$

which gives

$$c_1 + c_2 + c_3 = a$$

$$c_2 + c_3 = b$$

$$c_3 = c$$

This system has non-trivial solutions

$c_1 = a - b$, $c_2 = b - c$, $c_3 = c$

S spans $\mathbb{R}^3$.

Thus S forms basis for $\mathbb{R}^3$.

Ex ② Let $S = \{v_1, v_2, v_3, v_4\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Determine if S is a basis for $\mathbb{R}^4$.

we must determine if S spans B^4 & is linearly independent

If $w = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ where $a, b, c, d \in \{0, 1\}$ is any vector in B^4

Then we must find constants c_1, c_2, c_3 and c_4 such that

$$c_1 V_1 + c_2 V_2 + c_3 V_3 + c_4 V_4 = w$$

Substituting for V_1, V_2, V_3, V_4 & w we obtain linear system

$$c_1 + c_3 = a$$
$$c_2 + c_3 = b$$
$$c_2 + c_4 = c$$
$$c_1 + c_2 = d$$

consider Augmented matrix $[A : B]$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & a \\ 0 & 1 & 1 & 0 & b \\ 0 & 1 & 0 & 1 & c \\ 1 & 1 & 0 & 0 & d \end{array}\right]$$

$R_4 - R_1$, $R_3 - R_2$

$$\approx \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & a \\ 0 & 1 & 1 & 0 & b \\ 0 & 0 & -1 & 1 & c-b \\ 0 & 1 & -1 & 0 & d-a \end{array}\right]$$

$R_4 - R_2$

$$\approx \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & a \\ 0 & 1 & 1 & 0 & b \\ 0 & 0 & -1 & +1 & c-b \\ 0 & 0 & -2 & 0 & d-a-b \end{array}\right]$$

$R_4 - 2R_3$

$$\approx \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & \end{array}\right]$$

Ex ③ Determine whether the set
$B = \{(3,1,-4), (2,5,6), (1,4,8)\}$ is basis for $\mathbb{R}^3$

⇒ Since any basis of $\mathbb{R}^3$ contains three vectors, it is sufficient to show that either S is linearly independent or S spans $\mathbb{R}^3$.

To show that S is linearly independent, suppose we have scalars c_1, c_2 and c_3 such that

$$c_1(3,1,-4) + c_2(2,5,6) + c_3(1,4,8) = (0,0,0) \quad \text{--- ①}$$

ie $(3c_1 + 2c_2 + c_3,\ c_1 + 5c_2 + 4c_3,\ -4c_1 + 6c_2 + 8c_3) = (0,0,0)$

comparing both side we get,

$$3c_1 + 2c_2 + c_3 = 0$$
$$c_1 + 5c_2 + 4c_3 = 0$$
$$-4c_1 + 6c_2 + 8c_3 = 0$$

consider $|A| = \begin{vmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{vmatrix} = 3(40-24) - 2(8+16) + 1(6+20)$

$= 48 - 48 + 26$

$= 26 \neq 0$

∴ The system has a trivial soln (zero soln)
ie $c_1 = c_2 = c_3 = 0$

Therefore S is linearly independent & hence is basis of $\mathbb{R}^3$.

Ex ④ Show that the set $S = \{t^2+1, t-1, 2t+2\}$ is a basis for the vector space P_2.

⇒ To show S spans P_2, Take any vector (polynomial) $at^2 + bt + c$ in P_2 we must find constants c_1, c_2, c_3 such that

$$at^2 + bt + c = c_1(t^2+1) + c_2(t-1) + c_3(2t+2)$$
$$= c_1 t^2 + (c_2 + 2c_3)t + (c_1 - c_2 + 2c_3)$$

Comparing the coefficients we get

$$c_1 = a$$
$$c_2 + 2c_3 = b$$
$$c_1 - c_2 + 2c_3 = c$$

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EXAMINER'S REMARK							

Solving we have

$c_1 = a$, $a - c_2 + 2c_3 = c$

$\Rightarrow -c_2 + 2c_3 = c - a$

$$+\begin{array}{r} c_2 + 2c_3 = b \\ -c_2 + 2c_3 = c - a \\ \hline 4c_3 = b + c - a \end{array}$$

$c_3 = \frac{b+c-a}{4}$

& $c_2 + 2c_3 = b$

$\Rightarrow c_2 + \frac{b+c-a}{2} = b \Rightarrow 2c_2 + b + c - a = 2b$

$\Rightarrow c_2 = \frac{a+b-c}{2}$

Hence S spans P_2.

To show that S is linearly independent,

consider $c_1(t^2+1) + c_2(t-1) + c_3(2t+2) = 0$

we have $c_1 t^2 + (c_2 + 2c_3)t + (c_1 - c_2 + 2c_3) = 0$

$\Rightarrow c_1 = 0$, $c_2 + 2c_3 = 0$, $c_1 - c_2 + 2c_3 = 0$

which gives $c_1 = c_2 = c_3 = 0$ which implies that S

is linearly independent.

Thus S is a basis for P_2.

Ex ⑤ Is the set $S = \{(3,2,2), (-1,2,1), (0,1,0)\}$ forms a basis for $\mathbb{R}^3$?

Soln → To show S is linearly independent, consider

$$c_1V_1 + c_2V_2 + c_3V_3 = 0$$

$$c_1(3,2,2) + c_2(-1,2,1) + c_3(0,1,0) = (0,0,0)$$

$$\Rightarrow (3c_1 - c_2,\ 2c_1 + 2c_2 + c_3,\ 2c_1 + c_2) = (0,0,0)$$

we get linear system of equations

$$3c_1 - c_2 = 0$$
$$2c_1 + 2c_2 + c_3 = 0$$
$$2c_1 + c_2 = 0$$

Solving we get $c_1 = c_2 = c_3 = 0$

$\therefore$ Set S is linearly independent.

To show that S span $\mathbb{R}^3$ we let $V = (a,b,c)$ be any vector in $\mathbb{R}^3$. we now find scalars c_1, c_2, c_3 such that $c_1V_1 + c_2V_2 + c_3V_3 = V$

ie $c_1(3,2,2) + c_2(-1,2,1) + c_3(0,1,0) = (a,b,c)$

$$\Rightarrow \quad 3c_1 - c_2 = a$$
$$2c_1 + 2c_2 + c_3 = b$$
$$2c_1 + c_2 = c$$

consider Augmented matrix $[A : B]$

$$= \left[\begin{array}{ccc|c} 3 & -1 & 0 & a \\ 2 & 2 & 1 & b \\ 2 & 1 & 0 & c \end{array}\right]$$

$R_1 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -1 & a-b \\ 2 & 2 & 1 & b \\ 2 & 1 & 0 & c \end{array}\right]$$

$R_2 - 2R_1,\ R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -1 & \cancel{a-b}\,b-2a+3b \\ 0 & 8 & 3 & b-2a \\ 0 & 7 & 2 & c-2a+2b \end{array}\right]$$

$R_2 - R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -1 & b-2a+3b \\ 0 & 1 & 1 & -c-a \\ 0 & 7 & 2 & c-2a+2b \end{array}\right]$$

$R_3 - 7R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -1 & b-2a+3b \\ 0 & 1 & 1 & -c-a \\ 0 & 0 & -12 & 8c+5a+2b \end{array}\right]$$

$b-2a+2b \quad c-2a$

$b-2\cancel{a}-c+\cancel{2a}-2b$
$= -c-a$

$c-2a+2b+7c+7a$
$= 8c+5a+2b$

which gives (by directly solving)

$c_1 = \frac{a+c}{5}$ $\quad c_2 = \frac{3c-2a}{5}$ $\quad c_3 = \frac{2a+5b-8c}{5}$ 15

$\therefore$ set S spans $\mathbb{R}^3$

Hence S forms a basis for $\mathbb{R}^3$

** Four Fundamental Subspaces

1) Row space
2) column space
3) Null space
4) left null space.

* **Row Space** :- If A is $m \times n$ matrix then the subspace of $\mathbb{R}^n$ spanned by the row vectors of A is called the row space of A.

The row space of A is the column space of A^T. It is $C(A^T)$. It's dimension is the rank 'r.'

** **Column space** :- If A is $m \times n$ matrix then the subspace of $\mathbb{R}^n$ spanned by the column vectors of A is called the column space of A.

The column space of A is denoted by $C(A)$. It's dimension is also the rank 'r'.

** **Null space of A** :- The solution space of the homogeneous system of equation $AX=0$ which is a subspace of $\mathbb{R}^n$ is called the null space of A.

Null space of A is denoted by $N(A)$. It's dimension is $n-r$.

** **The left null space of A** :- The left null space of A is the null space of A^T. It is written as $N(A^T)$. It contain all vectors y such that $A^Ty = 0$ & ~~Note :- If A is any m×n~~. It's dimension is $m-r$.

Note:- If A is $m\times n$ matrix then

1) The null space $N(A)$ and row space $C(A^T)$ are subspaces of $\mathbb{R}^n$.

2) The left null space $N(A^T)$ and column space $C(A)$ are subspaces of $\mathbb{R}^m$.

Ex ① Find bases for the subspace of $\mathbb{R}^4$
$(1,1,-4,-3)$, $(2,0,2,-2)$, $(2,-1,3,2)$

⇒ The given vectors can be written in matrix form as

$$A = \begin{bmatrix} 1 & 1 & -4 & -3 \\ 2 & 0 & 2 & -2 \\ 2 & -1 & 3 & 2 \end{bmatrix}$$

$R_2 - 2R_1$, $R_3 - 2R_1$

$$\approx \begin{bmatrix} 1 & 1 & -4 & -3 \\ 0 & -2 & 10 & 4 \\ 0 & -3 & 11 & 8 \end{bmatrix}$$

$R_2 / -2$

$$\approx \begin{bmatrix} 1 & 1 & -4 & -3 \\ 0 & 1 & -5 & -2 \\ 0 & -3 & 11 & 8 \end{bmatrix}$$

$R_3 + 3R_2$

$$\approx \begin{bmatrix} 1 & 1 & -4 & -3 \\ 0 & 1 & -5 & -2 \\ 0 & 0 & -4 & 2 \end{bmatrix}$$

This is in row echelon form.

The non zero row vectors in this matrix are
$V_1 = (1,1,-4,-3)$, $V_2 = (0,1,-5,-2)$,
$V_3 = (0,0,-4,2)$.

This vectors form a basis for the row spaces of the matrix.

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Ex (2) Find the basis for the subspace of $\mathbb{R}^3$ spanned by the vectors $(1,2,-1), (4,1,3), (5,3,2), (2,0,2)$

$\Rightarrow$ Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \\ 5 & 3 & 2 \\ 2 & 0 & 2 \end{bmatrix}$$

$R_2-4R_1,\ R_3-5R_1,\ R_4-2R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 7 \\ 0 & -7 & 7 \\ 0 & -4 & 4 \end{bmatrix}$$

$R_2/-7,\ R_3/-7,\ R_4/4$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$R_3-R_2,\ R_4-R_2$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix is in row echelon form. The non-zero rows of this matrix are the vectors $(1,2,-1)$ & $(0,1,-1)$ these vectors form a basis for the row space of the matrix A.

& dimension

Ex ③ Find the basis for the null space of A

where $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

→ The null space of A is the solution space of the homogeneous system $AX=0$

i.e
$$x_1 + 4x_2 + 5x_3 + 6x_4 + 9x_5 = 0$$
$$3x_1 - 2x_2 + x_3 + 4x_4 - x_5 = 0$$
$$-x_1 - x_3 - 2x_4 - x_5 = 0$$
$$2x_1 + 3x_2 + 5x_3 + 7x_4 + 8x_5 = 0$$

To obtain the general solution of above system we reduce the coeff matrix A into row echelon form

consider $A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

$R_2 - 3R_1,\ R_3 + R_1,\ R_4 - 2R_1$

$\sim \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & -14 & -14 & -14 & -28 \\ 0 & 4 & 4 & 4 & 8 \\ 0 & -5 & -5 & -5 & -10 \end{bmatrix}$

$R_2/-14,\ R_3/4,\ R_4/-5$

$\sim \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \end{bmatrix}$

$R_3 - R_2,\ R_4 - R_2$

$\sim \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Here $\rho[A] = 2 <$ no of unknowns (5).

$\therefore$ no of parameters $= n - r = 5 - 2 = 3$.

Here x_3, x_4, x_5 are free variables.

put $x_3 = r,\ x_4 = s,\ x_5 = t$

By $R_2 \Rightarrow \quad x_2 + x_3 + x_4 + 2x_5 = 0$

$$x_2 = -r - s - 2t$$

By $R_1 \Rightarrow \quad x_1 + 4x_2 + 5x_3 + 6x_4 + 9x_5 = 0$

$$\Rightarrow x_1 + 4(-r - s - 2t) + 5r + 6s + 9t = 0$$

$$x_1 = -r - 2s - t$$

$\therefore$ The general solution of given system is given by

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -r-2s-t \\ -r-s-2t \\ r \\ s \\ t \end{bmatrix} = \begin{bmatrix} -r \\ -r \\ r \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2s \\ -s \\ 0 \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} -t \\ -2t \\ 0 \\ 0 \\ t \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

This shows that basis for this system is

$$B = \{(-1,-1,1,0,0),\ (-2,-1,0,1,0),\ (-1,-2,0,0,1)\}$$

Therefore the basis for the null space of A is given by B.

& dim of null space of $A = n - r = 5 - 2 = 3$.

Ex ④ Find a basis for and the dimension of the solution space of homogenous system $x_1 + x_2 + x_3 + x_4 = 0$

$$2x_1 + x_2 - x_3 + x_4 = 0$$

$\Rightarrow$ consider ~~the~~ matrix A as

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & 1 \end{bmatrix}$$

$R_2 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -3 & -1 \end{bmatrix}$$

$-R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix}$$

which is in echelon form.

$\rho[A] = 2 <$ no of unknowns (4).

Here x_3, x_4 are free variables

Let $x_3 = s$, $x_4 = t$

$R_2 \Rightarrow x_2 + 3x_3 + x_4 = 0$

$x_2 + 3s + t = 0$

$\Rightarrow x_2 = -3s - t$

$R_1 \Rightarrow x_1 + x_2 + x_3 + x_4 = 0$

$x_1 - 3s - \cancel{t} + s + \cancel{t} = 0$

$x_1 = 2s$

$\therefore$ The general solution of the given system is given by

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2s \\ -3s-t \\ s \\ t \end{bmatrix}$$

$$= \begin{bmatrix} 2s \\ -3s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -t \\ 0 \\ t \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ The basis for this system is

$B = \{(2, -3, 1, 0), (0, -1, 0, 1)\}$

dim of null space of $A = n - r = 4 - 2 = 2$.

Inner product space:-

Definition:- Let V be a real vector space suppose to each pair of vectors $u, v \in V$ there is assigned a real number denoted by $\langle u, v \rangle$ { $\langle \; \rangle$ - angle bracket } This function is called a (real) inner product on V. If it is satisfies the following axioms:-

1) Linear property: $\langle au_1 + bu_2, v \rangle = a\langle u_1, v\rangle + b\langle u_2, v\rangle$
2) Symmetric property: $\langle u, v \rangle = \langle v, u \rangle$
3) positive definite property: $\langle u, u \rangle \geq 0$; and $\langle u, u \rangle = 0$ iff $u = 0$

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The vector space V with an inner product is called a (real) inner product space.
other notations are for inner product (u, v), u / v.

** Norm of a vector :-
Let V be a inner product space & let $u \in V$
Then $\langle u, u \rangle$ is non-ve real no
Then norm of u is defined as $\|u\| = \sqrt{\langle u, u \rangle}$
$\Rightarrow \|u\|^2 = \langle u, u \rangle$
The norm of u is also called as length of u.

orthogonal vectors :- Let V be an inner product space. The vectors $u, v \in V$ are said to be orthogonal and u is said to be orthogonal to v if $\langle u, v \rangle = 0$
e.g consider the vectors $u = (1, 1, 1)$, $v = (1, 2, -3)$, $w = (1, -4, 3)$ in $\mathbb{R}^3$

Then $\langle u, v\rangle = 1+2-3=0$, $\langle u, w\rangle = 1-4+3=0$

$\langle v, w\rangle = 1-8-9 = -16$

Thus u is othogonal to v and w but v and w are not othogonal.

Remark : A vector $w=(x_1, x_2, \ldots x_n)$ is othogonal to $u=(a_1, a_2, \ldots a_n)$ in $\mathbb{R}^n$ if

$$\langle u, w\rangle = a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

Example :- Find a nonzero vector w that is othogonal to $u_1 = (1,2,1)$ & $u_2 = (2,5,4)$ in $\mathbb{R}^3$.

→ Let $w = (x, y, z)$. Since w is othogonal to u_1 & u_2

i.e $\langle u_1, w\rangle = 0$ & $\langle u_2, w\rangle = 0$

Thus ⇒ $\langle (1,2,1), (x,y,z)\rangle = 0$ &

$\langle (2,5,4), (x,y,z)\rangle = 0$

This yields the homogenous system

$$x+2y+z=0$$
$$2x+5y+4z=0$$

Reducing to echelon form we get

$$x+2y+z=0$$
$$y+2z=0$$

Here z is only free vasiables.

Let $z=1$. ∴ $y=-2$ & $x=3$.

Thus $w=(3,-2,1)$ is a desired nonzero vector othogonal to u_1 & u_2.

Remark : If $\|u\|=1$ or if $\langle u, u\rangle = 1$ then u is called a unit vector and is said to be normalised.

Every nonzero vector v in V can be multiplied by the reciprocal of its length to obtain the unit vector

$$\hat{v} = \frac{1}{\|v\|}v$$ which is positive multiple of v

This process is called normalizing v.

An inner product is defined by
$\langle u, v \rangle = a_1 b_1 + a_2 b_2 + \cdots$

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Definitions :-

① A set $S = \{u_1, u_2, \ldots u_k\}$ in $\mathbb{R}^n$ is called othogonal if every pair of distinct vectors in S are othogonal.
ie $u_i \cdot u_j = 0$ for $i \neq j$

② A set $S = \{u_1, u_2, \ldots u_k\}$ in $\mathbb{R}^n$ is called othonormal if
i) S is orthogonal ie $u_i \cdot u_j = 0 \ \forall\ i \neq j$ and
ii) every vector in S is unit vector.
ie $u_i \cdot u_j = 1 \ \forall\ i = 1, 2, \ldots k$

Gram-schmidt orthogonalization process

Suppose $\{v_1, v_2, \ldots v_n\}$ is a basis of an inner product space V. To construct an orthogonal basis $\{w_1, w_2, \ldots w_n\}$ of V we proceed as follows,

$$w_1 = v_1$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$$

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

$$\vdots$$

$$w_n = v_n - \frac{\langle v_n, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_n, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 - \cdots - \frac{\langle v_n - w_{n-1} \rangle}{\langle w_{n-1}, w_{n-1} \rangle} \cdot w_{n-1}$$

In other words for $k = 2, 3, \ldots n$ we define

$$w_k = v_k - c_{k1} w_1 - c_{k2} w_2 - \cdots - c_{k,k-1} w_{k-1}$$

where $c_{ki} = \langle v_k, w_i \rangle / \langle w_i, w_i \rangle$ is the component of v_k along w_i. each w_k is othogonal to the preceeding w's
Thus $w_1, w_2, \ldots w_n$ form an othogonal basis for V.
Normalizing each w_i will then yield an orthonormal basis for V. The above construction is known as Gram-schmidt othogonalization process.

Examples

स्मिथ

Ex ① Apply Gram schmid orthogonalization process to find orthogonal basis and then an orthonormal basis to the for the following vectors

$u_1 = (4,-3)$, $u_2 = (1,-1)$

$\Rightarrow$ Let $v_1 = u_1 = (4,-3)$

consider

$$v_2 = u_2 - \left(\frac{u_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1$$

$$= (1,-1) - \frac{(1,-1)\cdot(4,-3)}{(4,-3)\cdot(4,-3)} (4,-3)$$

$$= (1,-1) - \frac{7}{25}(4,-3)$$

$$= \left(\frac{-3}{25}, \frac{-4}{25}\right)$$

Thus v_1, v_2 form an orthogonal basis for given space. Normalize these vectors to obtain an orthonormal basis

$$w_1 = \frac{v_1}{\|v_1\|} = \frac{(4,-3)}{\sqrt{16+9}} = \left(\frac{4}{5}, \frac{-3}{5}\right)$$

$$w_2 = \frac{v_2}{\|v_2\|} = \frac{(-3/25, -4/25)}{\sqrt{\frac{9+16}{625}}} = 5\left(\frac{-3}{25}, \frac{-4}{25}\right)$$

$$= (-3/5, -4/5)$$

② $u_1 = (1,0,1)$, $u_2 = (1,0,-1)$, $u_3 = (0,3,4)$

$\Rightarrow$ Let $v_1 = u_1 = (1,0,1)$

$$v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$= (1,0,-1) - \frac{(1,0,-1)\cdot(1,0,1)}{(1,0,1)\cdot(1,0,1)}(1,0,1)$$

$$= (1,0,-1) - \frac{0}{2}(1,0,1)$$

$$= (1,0,-1) - (0,0,0)$$

$$= (1,0,-1)$$

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$$v_3 = u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

$$= (0,3,4) - \frac{(0,3,4)\cdot(1,0,1)}{(1,0,1)\cdot(1,0,1)}(1,0,1) - \frac{(0,3,4)\cdot(1,0,-1)}{(1,0,-1)\cdot(1,0,-1)}(1,0,-1)$$

$$= (0,3,4) - \frac{4}{2}(1,0,1) + \frac{4}{2}(1,0,-1)$$

$$= (0,3,4) - (2,0,2) + (2,0,-2)$$

$$= (0,3,0)$$

Thus v_1, v_2, v_3 form an orthogonal basis for given space. Normalize these vectors to obtain an orthonormal basis

$$w_1 = \frac{v_1}{\|v_1\|} = \frac{(1,0,1)}{\sqrt{1+1}} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$w_2 = \frac{v_2}{\|v_2\|} = \frac{(1,0,-1)}{\sqrt{1+1}} = \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$$

$$w_3 = \frac{v_3}{\|v_3\|} \quad \frac{(0,3,0)}{\sqrt{9}} = (0,1,0)$$

Ex ③ Let $S = \{u_1, u_2, u_3\} = \{(1,1,1), (-1,0,-1), (-1,2,3)\}$ be basis for $\mathbb{R}^3$. Transform S to an othonormal basis for $\mathbb{R}^3$, using Gram Schmidt process

→ Let $v_1 = u_1 = (1,1,1)$

$$v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$= (-1,0,-1) - \frac{(-1,0,-1)\cdot(1,1,1)}{(1,1,1)\cdot(1,1,1)}(1,1,1)$$

$$= (-1,0,-1) + \frac{2}{3}(1,1,1)$$

$$= \left(\frac{-1}{3}, \frac{2}{3}, \frac{-1}{3}\right)$$

$$v_3 = u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1} - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

$$= (-1,2,3) - \frac{(-1,2,3)\cdot(1,1,1)}{(1,1,1)\cdot(1,1,1)}(1,1,1) - \frac{(-1,2,3)\cdot(-1/3, 2/3, -1/3)}{(-1/3, 2/3, -1/3)\cdot(-1/3, 2/3, -1/3)}\left(\frac{-1}{3}, \frac{2}{3}, \frac{-1}{3}\right)$$

$$= (-1,2,3) - \frac{4}{3}(1,1,1) - \frac{1}{3}(-1,2,-1)$$

$$= (-2,0,2)$$

Thus v_1, v_2, v_3 forms an othogonal basis for $\mathbb{R}^3$
Now normalizing each vector we get

$$w_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}}(1,1,1)$$

$$w_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{6}}(-1,2,-1)$$

$$w_3 = \frac{1}{\|v_3\|} v_3 = \frac{1}{\sqrt{8}}(-2,0,2)$$

Thus w_1, w_2, w_3 forms an othonormal basis for $\mathbb{R}^3$.