Q.2. Solve the system of non-linear equations.

i) $x^2+y=11$, $y^2+x=7$ with initial approximation $x_0=3.5$ and $y_0=-1.8$

→ let $u(x,y) = x^2+y-11$

$v(x,y) = y^2+x-7$

$$\frac{\partial u}{\partial x} = 2x$$

$$\frac{\partial u}{\partial y} = 1$$

$$\frac{\partial v}{\partial x} = 1$$

$$\frac{\partial v}{\partial y} = 2y$$

Here $x_0=3.5$, $y_0=-1.8$

$u_0 = (3.5)^2 - 1.8 - 11$

$= -0.55$

$v_0 = -0.26$

$\frac{\partial u_0}{\partial x} = 7.0$

$\frac{\partial u_0}{\partial y} = 1$

$\frac{\partial v_0}{\partial x} = 1$

$\frac{\partial v_0}{\partial y} = -3.6$

Now, $J = \begin{vmatrix} \frac{\partial u_0}{\partial x} & \frac{\partial u_0}{\partial y} \\ \frac{\partial v_0}{\partial x} & \frac{\partial v_0}{\partial y} \end{vmatrix}$

$= \begin{vmatrix} 7 & 1 \\ 1 & -3.6 \end{vmatrix}$

$= -26.2$

1st iteration,

$$x_1 = x_0 - \frac{\left[u_0 \frac{\partial v_0}{\partial y} - v_0 \frac{\partial u_0}{\partial y} \right]}{J}$$

$$x_1 = 3.5 - \frac{[(-0.55)(-3.6) - (-0.26)(1)]}{-26.2}$$

$= 3.5855$

$$y_1 = y_0 - \frac{\left[u_0 \frac{\partial v_0}{\partial x} + v_0 \frac{\partial u_0}{\partial x} \right]}{J}$$

$$= -1.8 - \frac{[(-0.55)(1) - (-0.26)(7)]}{-26.2}$$

$= -1.75153$

Now, $x_1 = 3.5855$, $y_1 = -1.75153$

$\therefore u_1 = (3.586)^2 - 1.752 - 11 = 0.1074$

$v_1 = 3.586 + (1.757)^2 - 7 = -0.3445$

$\frac{\partial u_1}{\partial x} = 7.172$

$\frac{\partial u_1}{\partial y} = 1$

$\frac{\partial v_1}{\partial x} = 1$

$\frac{\partial v_1}{\partial y} = -3.504$

$$J = \begin{vmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} \\ \frac{\partial v_1}{\partial x} & \frac{\partial v_1}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 7.172 & 1 \\ 1 & -3.504 \end{vmatrix}$$

$$= -25.131 - 1$$

$$= -26.131$$

2nd iteration :-

$$x_2 = 3.586 - \left[\frac{(0.1074)(-3.504) - (-0.3445)(1)}{-26.131}\right]$$

$$x_2 = 3.585$$

$$y_2 = -1.752 - \left[\frac{(0.1074)(1) - (-0.3445)(7.172)}{-26.131}\right]$$

$$= -1.653$$

∴ Solution is $x = 3.585$ & $y = -1.653$

ii) $x + y^2 = 5$, $x^2 + y = 11$ with initial approximation $x_0 = 1$ & $y_0 = 2$.

→ let, $u(x,y) = x + y^2 - 5$

$v(x,y) = x^2 + y - 11$

$\frac{\partial u}{\partial x} = 1$

$\frac{\partial u}{\partial y} = 2y$

$\frac{\partial v}{\partial x} = 2x$

$\frac{\partial v}{\partial y} = 1$

Given, $x_0 = 1$ & $y_0 = 2$

$\therefore u_0 = 1 + 4 - 5 = 0$

$v_0 = 1 + 2 - 11 = -8$

$\frac{\partial u_0}{\partial x} = 1$,

$\frac{\partial u_0}{\partial y} = 4$

$\frac{\partial v_0}{\partial x} = 2$

$\frac{\partial v_0}{\partial y} = 1$

$$J = \begin{vmatrix} \frac{\partial u_0}{\partial x} & \frac{\partial u_0}{\partial y} \\ \frac{\partial v_0}{\partial x} & \frac{\partial v_0}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix}$$

$$= 1 - 8$$

$$= -7$$

1st iteration,

$$x_1 = x_0 - \left[\frac{u_0 \partial v_0/\partial y - v_0 \partial u_0/\partial y}{J}\right]$$

$$= 1 - \left[\frac{0 - (-8)(4)}{-7}\right] = 1 + \frac{32}{7} = 5.5714$$

$$y_1 = y_0 - \left[\frac{u_0 \partial v_0/\partial x - v_0 \partial u_0/\partial x}{J}\right]$$

$$= 2 - \left[\frac{0 - (-8)}{-7}\right]$$

$$= 3.1429$$

$\therefore x_1 = 5.5714$ & $y_1 = 3.1429$

Now, $x_1 = 5.5714$ & $y_1 = 3.1429$

$u_1 = 5.571 + (3.143)^2 - 5$

$= 10.449$

$v_1 = (5.57)^2 + 3.143 - 11$

$= 23.179$

$\frac{\partial u_1}{\partial x} = 1$, $\frac{\partial u_1}{\partial y} = 6.286$, $\frac{\partial v_1}{\partial x} = 11.142$, $\frac{\partial v_1}{\partial y} = 1$

$$J = \begin{vmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} \\ \frac{\partial v_1}{\partial x} & \frac{\partial v_1}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 6.286 \\ 11.142 & 1 \end{vmatrix}$$

$$= 1 - 70.0386$$

$$= -69.039$$

2nd iteration :-

$$x_2 = x_1 - \left[\frac{u_1 \partial v_1/\partial y - v_1 \partial u_1/\partial y}{J}\right]$$

$$= 5.571 - \left[\frac{(10.449)(1) - (23.179)(6.286)}{-69.039}\right]$$

$= 3.6119$

$$y_2 = y_1 - \left[\frac{u_1 \partial v_1/\partial x - v_1 \partial u_1/\partial x}{J}\right]$$

$$= 3.143 - \left[\frac{(10.449)(11.142) - (27.179)}{-69.039}\right]$$

$= 4.4936$

$\therefore$ The solution is $x_1 = 3.6119$, $y_2 = 4.4936$

3) $x^2 - y^2 = 4$, $x^2 + y^2 = 16$ With initial approximation $x_0 = y_0 = 2.828$

→ let, $u(x,y) = x^2 - y^2 - 4$

$v(x,y) = x^2 + y^2 - 16$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 2y$$

Given, $x_0 = y_0 = 2.828$

$\therefore u_0 = (2.828)^2 - (2.828)^2 - 4$

$= -4$

$v_0 = (2.828)^2 + (2.828)^2 - 16$

$= -0.0048$

$$\frac{\partial u_0}{\partial x} = 5.656, \quad \frac{\partial u_0}{\partial y} = -5.656, \quad \frac{\partial v_0}{\partial x} = 5.656, \quad \frac{\partial v_0}{\partial y} = 5.656$$

$$J = \begin{vmatrix} \frac{\partial u_0}{\partial x} & \frac{\partial u_0}{\partial y} \\ \frac{\partial v_0}{\partial x} & \frac{\partial v_0}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 5.656 & -5.656 \\ 5.656 & 5.656 \end{vmatrix}$$

$= 63.980$

1st iteration :-

$$x_1 = x_0 - \left[\frac{u_0 \partial v_0/\partial y - v_0 \partial u_0/\partial y}{J}\right]$$

$$= 2.828 - \left[\frac{(-4)(5.656) - (-0.004)(-5.656)}{63.98}\right]$$

$x_1 = 3.182$

$$y_1 = y_0 - \left[\frac{u_0 \partial v_0/\partial x - v_0 \partial u_0/\partial x}{J}\right]$$

$$= 2.828 - \left[\frac{(-4)(5.656) + (-0.004)(5.656)}{63.98}\right]$$

$= 2.475$

2nd iteration :-

$x_1 = 3.182$, $y_1 = 2.475$

$u_1 = -0.0132$

$v_1 = 0.2507$

$\frac{\partial u_1}{\partial x} = 6.364$

$\frac{\partial u_1}{\partial y} = -4.95$

$\frac{\partial v_1}{\partial x} = 6.364$

$\frac{\partial v_1}{\partial y} = 4.95$

$$J = \begin{vmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} \\ \frac{\partial v_1}{\partial x} & \frac{\partial v_1}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 6.364 & -4.95 \\ 6.364 & 4.95 \end{vmatrix}$$

$= 62.876$

$$x_2 = x_1 - \left[\frac{u_1 \partial v_1/\partial y - v_1 \partial u_1/\partial y}{J}\right]$$

$$= 3.182 - \left[\frac{(-0.0132)(4.95) - (0.2507)(-4.95)}{62.876}\right]$$

$= 3.1633$

$$y_2 = y_1 - \left[\frac{u_1 \partial v_1/\partial x - v_1 \partial u_1/\partial x}{J}\right]$$

$$= 2.475 - \left[\frac{(-0.0132)(6.364) - (0.2507)(6.364)}{62.876}\right]$$

$= 2.4496$

∴ The solution is $x_2 = 3.1633$ & $y_2 = 2.4496$.

iv) $2x^2 + 3xy + y^2 = 3$, $4x^2 + 2xy + y^2 = 30$ with initial approximation $x_0 = -3$ and $y_0 = 2$.

→ let,

$u(x,y) = 2x^2 + 3xy + y^2 - 3$

$v(x,y) = 4x^2 + 2xy + y^2 - 30$

$\frac{\partial u}{\partial x} = 4x + 3y + 0$, $\frac{\partial u}{\partial y} = 3x + 2y$

$\frac{\partial v}{\partial x} = 8x + 2y$, $\frac{\partial v}{\partial y} = 2x + 2y$

$x_0 = -3$ & $y_0 = 2$.

$\therefore u_0 = 18 + 3(-3)(2) + 4 - 3 = 1$

$v_0 = 36 + 4 - 12 - 30 = -2$

$\frac{\partial u_0}{\partial x} = -12 + 6 = -6$, $\frac{\partial u_0}{\partial y} = -9 + 4 = -5$

$\frac{\partial v_0}{\partial x} = -24 + 4 = -20$, $\frac{\partial v_0}{\partial y} = -6 + 4 = -2$

$$J = \begin{vmatrix} \partial u_0/\partial x & \partial u_0/\partial y \\ \partial v_0/\partial x & \partial v_0/\partial y \end{vmatrix}$$

$$= \begin{vmatrix} -6 & -5 \\ -20 & -2 \end{vmatrix}$$

$= 88$

1st iteration,

$$x_1 = -3 - \left[\frac{1(-2) - (-2)(-5)}{88}\right]$$

$= -3 + 12/88$

$= -2.8636$

$$y_1 = 2 - \left[\frac{(-2)(-6) - (1)(-20)}{88}\right] = 2 - 32/88$$

$= 1.636$.

Now, $x_1 = -2.8636$ & $y_1 = 1.636$

$u_1 = 16.3935 + (-14.0516) + 2.6765 - 3 = 2.0184$

$v_1 = 32.787 - 9.3677 + 2.6765 - 30 = -3.9042$

$\frac{\partial u_1}{\partial y} = (3)(-2.863) + 2(1.636) = -5.317$

$\frac{\partial u_1}{\partial x} = -11.452 + 4.908 = -6.544$

$\frac{\partial v_1}{\partial x} = -19.632$

$\frac{\partial v_1}{\partial y} = 2(-2.8636) + 2(1.636) = -2.4552$

$$J = \begin{vmatrix} -6.544 & -5.317 \\ -19.632 & -2.4552 \end{vmatrix}$$

$J = 16.0668 - 104.383$

$= 88.316$.

2nd iteration,

$$X_2 = (-2.8636) - \left[\frac{(2.0184)(-2.4552) - (-3.9042)(-5.317)}{88.316}\right]$$

$$= -2.5724.$$

$$y_2 = 1.636 - \left[\frac{(-3.9042)(-6.544) - (2.0184)(-19.632)}{88.316}\right]$$

$$y_2 = 0.898$$

The solution is $x = -2.5724$ & $y = 0.898$.