Probability density functions (PDFs)

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October 17, 2022

1 Probability density functions

If X is a continuous random variable whose support is a subset of real numbers $X \subset R^1$, then there exists a CDF (Cumulative distribution function) defined by the following equation

$$F_X(x) = Pr[X \leq x] \tag{1}$$

then the PDF(Probability density function for that continuous random variable is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{2}$$

By using the above equation the CDF can also be defined as

$$F_X(x) = \int_{-\infty}^{x} p_X(t)dt \tag{3}$$

Probability that $a \leq X \leq b$ can be computed using the pdf of X random variable by using the equation

$$Pr[a \leq X \leq b] = \int_{a}^{b} p_X(x)dx \tag{4}$$

The integral of the pdf of X random variable over its support is always unity

$$\int_{S_X} p_X(x)dx = 1 \tag{5}$$

, where S_X is the support of range of X random variable. Probability of an event $A \subset S_X$ can be computed using the pdf of X as

$$Pr[A] = \int_{A} p_X(x)dx \tag{6}$$

1.1 Expectation and variance

Expectation of a continuous random variable X can be computed as below

$$E[X] = \int_{S_X} xp_X(x)dx \tag{7}$$

Variance of X can be computed as

$$Var[X] = E[(X - E[X])^2] = E[X^2] - (E[X])^2 \tag{8}$$

1.2 Some important pdfs

Following are the examples of a few important pdfs.

[1] R is a set of real numbers

1.2.1 Uniform distribution

The pdf of a uniform distribution of X random variable is defined as

$$p_X(x) = \frac{1}{b-a}, \qquad a \leq X \leq b \tag{9}$$

, whose mean and variance are defined by the following equations

$$E[X] = \frac{a+b}{2} \tag{10}$$

$$Var[X] = \frac{(b-a)^2}{12} \tag{11}$$

1.2.2 Exponential distribution

The pdf of the exponential distribution is defined as

$$p_X(x) = \lambda e^{-\lambda x}, \qquad x \geq 0 \tag{12}$$

, whose mean and variance are defined by the following equations

$$E[X] = \frac{1}{\lambda} \tag{13}$$

$$Var[X] = \frac{1}{\lambda^2} \tag{14}$$

1.2.3 Beta distribution

The pdf of the beta distribution is defined by the following equation

$$p_X(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}, \qquad 0 < x < 1 \tag{15}$$

, where the beta function $B(\alpha, \beta)$ is defined as

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{16}$$

, and the gamma function $\Gamma(x)$ is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \tag{17}$$

. Mean and variance of the Beta distribution are given by the following equations

$$E[X] = \frac{\alpha}{\alpha+\beta} \tag{18}$$

$$Var[X] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \tag{19}$$

1.2.4 Gaussian distribution

The pdf of the Gaussian distribution is defined by the following equation

$$p_X(x) = \frac{e^{-\frac{(x-m)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}, \qquad x \in \mathcal{R} \tag{20}$$

Mean and variance of the Gaussian distribution are as follow

$$E[X] = m \qquad Var[X] = \sigma^2 \tag{21}$$

1.2.5 Laplace distribution

The pdf of the Laplace distribution is defined by the following equation

$$p_X(x) = \frac{e^{-\frac{|x-m|}{\alpha}}}{2\alpha}, \qquad x \in \mathcal{R} \tag{22}$$

Mean and variance of the Gaussian distribution are as follow

$$E[X] = m \qquad Var[X] = 2\alpha^2 \tag{23}$$