Probability density functions

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1 Probability densty funcions

The pdf of a continuous random variable can be generated from a standard uniform random variable by using the formula below

$$F_X(x) = u \tag{1}$$

where $u \sim \mathcal{U}(0,1)$is the standard uniform random variable and $F_X(x)$ is the CDF of the random variable whose realizations we want to generate. For example for an exponential random variable, the CDF is given by

$$F_X(x) = 1 - e^{-\lambda x} \tag{2}$$

, so we just need to solve the equation

$$1 - e^{-\lambda x} = u \tag{3}$$

in order to generate realizations of exponential random variable.

1.1 Problem 1

In the same way, generate realizations of any uniform, laplace random variables.

1.2 Problem 2

X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 6(y-x), & 0 \le x \le y \le 1 \\ 0, & otherwise. \end{cases} \tag{4}$$

- (a) What is $f_{X|Y}(x|y)$?
- What is $\hat{X}_{MMSE}(Y)$, the minimum mean square error estimate of X given Y = y?
- What is $f_{Y|X}(y|x)$?

- What is $\hat{Y}_{MMSE}(X)$, the minimum mean square error estimate of Y given X = x?

We have already done this problem in the class. You just have to implement 2nd and 4th questions. In order to generate realizations of the X and Y random variables, you have to follow the steps in the first part of the lab manual (Generate realizations of X and Y from the standard uniform random variable). Following are the equations for the MMSE estimates of X and Y random variables.

$$\hat{X}_{MMSE}(Y) = \frac{Y}{3} \tag{5}$$

$$\hat{Y}_{MMSE}(X) = \frac{2 - 3X + X^3}{3(1 - 2X + X^2)} \tag{6}$$

After implementing these two estimators, you have to verify whether $E[\hat{X}_{MMSE}(Y)] = 1/4$ and $E[\hat{Y}_{MMSE}(X)] = 3/4$ for a large number of realizations of Y and X random variables, respectively.

1.3 Problem 3

A telemetry voltage V , transmitted from a position sensor on a ship's rudder, is a random variable with

$$p_V(v) = \begin{cases} \frac{1}{12}, & -6 \leq v \leq 6 \\ 0, & otherwise \end{cases} \tag{7}$$

A receiver in the ship's control room receives R = V + X, The random variable X is a Gaussian $(0, \sqrt{3})$ with mean zero and variance 3 noise voltage that is independent of V . The receiver uses R to calculate a linear estimate of the telemetry voltage

$$\hat{V} = aR + b \tag{8}$$

The LMMSE(Linear minimum mean squared error estimate) of V given R is given by the following equation

$$\hat{V}_{LMMSE}(R) = \frac{Cov[V, R]}{Var[R]}(R - E[R]) + E[V] \tag{9}$$

which is finally given by

$$\hat{V}_{LMMSE}(R) = 4R \tag{10}$$

Implement a program, which generate multiple realizations of R and V random variables, and finally estimate V from the observations R.

1.4 Problem 4

Let R be an exponential random variable with expected value $1/\mu$. If R = r , then over an interval of length T the number of phone calls N that arrive at a telephone switch has a Poisson PMF with expected value rT .

- Find the MMSE estimate of R given N.
- Find the MAP estimate of R given N.
- Find the ML estimate of R given N.

Write a program which implements these three estimators. MMSE estimator is given by

$$\hat{R}_{MMSE}(N) = \frac{N+1}{\mu + T} \tag{11}$$

MAP estimator is given by

$$\hat{R}_{MAP}(N) = \frac{N}{\mu + T} \tag{12}$$

ML estimator is given by

$$\hat{R}_{ML}(N) = \frac{N}{T} \tag{13}$$

1.5 Problem 5

For a certain coin, Q, is a uniform (0, 1) random variable. Given Q = q, each flip is heads with probability q, independent of any other flip. Suppose this coin is flipped n times. Let K denote the number of heads in n flips.

- What is the ML estimator of Q given K?
- What is the PMF of K ? What is E[K]?
- What is the conditional PDF $f_{Q|K}(q|k)$?
- Find the MMSE estimator of Q given K = k.

ML estimator of Q given K is

$$\hat{Q}_{ML}(K) = \frac{K}{n} \tag{14}$$

MMSE estimator of Q given K

$$\hat{Q}_{MMSE}(K) = \frac{K+1}{n+2} \tag{15}$$

Implement these two estimators.