

2. Master Equation at Work:

Examples of open system ensemble dynamics

We saw that, on the basis of three main assumptions, we were able to derive the most general form of a Markovian equation of motion for the reduced density matrix ρ of an open quantum system. This Lindblad master equation describes the ensemble dynamics of the system, which is governed by the system Hamiltonian H (which generally may differ from the bare system Hamiltonian and include Lamb-type shifts) and the Lindblad damping terms specified by a set of Lindblad operators $\{L_\mu\}$. How do we determine the appropriate Lindblad operators?

To answer this question, recall that there are two strategies for deriving the master equation :

- (1) the strategy we used in chapter 1 showed how the Lindblad master equation emerges generally based on a few assumptions
- (2) microscopic derivations specific for individual systems and couplings to an environment.

The second strategy is the appropriate one if we are given (or can determine) the Hamiltonian of the environment, H_E , and the system-environment coupling H_{SE} . One can show in this case that the relevant Lindblad operators are intimately related to the system operators appearing in H_{SE} , and can be constructed explicitly. Due to time limitations, we will skip over the microscopic derivation. (You are encouraged to read sections 3.3.1 and 3.4.1 in Breuer/Petruccione to see how the microscopic derivation works.)

Proceeding in the spirit of the first strategy, and hence in the absence of a microscopic model to guide us, we begin by studying the simplest

open quantum systems — the two-level system and the harmonic oscillator — for which the appropriate Lindblad operators can easily be inferred and studied.

2.1 The open two-level system

Consider, again, a quantum system composed of two levels with orthonormal states $|0\rangle$ and $|1\rangle$. This model is used in various contexts and may describe, e.g., the spin degree of freedom of an electron, a cartoon of an atom with only two atomic levels, the lowest eigenstates of a superconducting circuit or two discrete charge states of a quantum dot.

Before setting up the master equation and solving it, let us briefly review the concept of the Bloch sphere — the most convenient way to visualize the quantum state ρ of a two-level system.

The density matrix of the two-level system is a 2×2 hermitian matrix, and is a “vector” $|\rho\rangle$ in the $N^2 = 4$ dimensional \mathbb{R} -vector space $\mathcal{R}_{\mathbb{R}}$ (see PS#1). This vector space is spanned by the orthogonal basis

$$\mathcal{B} = \{|11\rangle, |\sigma_x\rangle, |\sigma_y\rangle, |\sigma_z\rangle\}$$

Recall from PS#1 that orthogonality in $\mathcal{R}_{\mathbb{R}}$ is based on the Hilbert-Schmidt inner product $(A|B) \equiv \text{tr}(AB)$. Since

$$(|11|11) = (\sigma_j | \sigma_j) = 2 \quad (j=x,y,z),$$

our basis \mathcal{B} is not normalized. Instead of normalization of the basis vectors, it is common practice to include a factor $\frac{1}{2}$ in the decomposition of $|\rho\rangle$:

$$|\rho\rangle = \frac{1}{2} [r_1 |11\rangle + r_x |\sigma_x\rangle + r_y |\sigma_y\rangle + r_z |\sigma_z\rangle]$$

where $r_1 = (|11|\rho)$, $r_x = (\sigma_x|\rho)$, $r_y = (\sigma_y|\rho)$, and $r_z = (\sigma_z|\rho)$ are all real-valued.

Normalization of the density matrix fixes r_i :

$$1 = \text{tr } \rho = (\mathbb{1}|\rho) = \frac{1}{2} r_i \underbrace{(\mathbb{1}|\mathbb{1})}_{=2} = r_i.$$

As a result, the density matrix of a two level system can always be expressed by a real-valued vector with three components $\vec{r} = (r_x, r_y, r_z)$. In general, such vector representations of the density matrix are called coherence vectors.

To interpret the coherence vector \vec{r} , recall the concept of purity from PS#1. We defined the purity as $\text{tr } \rho^2 = (\rho|\rho)$, which here gives

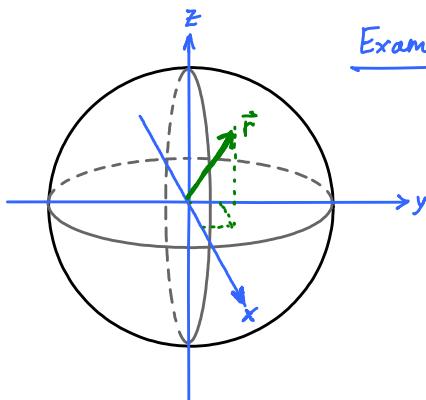
$$\begin{aligned} \text{tr } \rho^2 &= (\rho|\rho) \\ &= \frac{1}{4} [(\mathbb{1}| + r_x(x|x) + r_y(y| + r_z(z|))[(\mathbb{1}| + r_x(x|x) + r_y(y| + r_z(z|))] \\ &= \frac{1}{2} [1 + r_x^2 + r_y^2 + r_z^2]. \end{aligned}$$

Recall also that the purity satisfies $\frac{1}{N} \leq \text{tr } \rho^2 \leq 1$ where $\text{tr } \rho^2 = \frac{1}{N}$ for the completely mixed state and $\text{tr } \rho^2 = 1$ for any pure state.

From

$$\frac{1}{2} \leq \frac{1}{2}[1 + r_x^2 + r_y^2 + r_z^2] \leq 1$$

we infer $0 \leq |\vec{r}|^2 \leq 1$, i.e. the vector \vec{r} must lie within or on the surface of a sphere with unit radius. This sphere is called the Bloch sphere.



Examples: $|\rho\rangle = |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\vec{r} = \begin{pmatrix} (x|\rho) \\ (y|\rho) \\ (z|\rho) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$|\rho\rangle = |0\rangle\langle 0| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \vec{r} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

Both of these are pure states and hence, \vec{r} lies on the surface of the sphere. Note also that orthogonal states lie on opposing sides of the sphere ($\vec{r} \rightarrow -\vec{r}$).

Purity decreases as $|\vec{r}|$ decreases; the completely mixed state is reached when $\vec{r} = \vec{0}$.

With the Bloch representation at our disposal, we now turn to the system dynamics. Let us suppose the energy splitting between ground and excited state is E , and that the system Hamiltonian is simply given by

$$H_s = E\sigma^+\sigma^- \quad (\text{If you prefer, this can also be written as } H_s = \frac{1}{2}E\sigma_z + \frac{1}{2}E)$$

Relaxation at zero temperature

Consider the process of spontaneous relaxation, in which the two-level system undergoes a transition $|1\rangle \rightarrow |0\rangle$. Energy conservation dictates the transfer of the excitation energy into the environment — e.g. by emission of a photon or phonon with $\hbar\omega = E$.



Following our discussion of how to interpret the master equation and the Lindblad terms involved, it is clear that the appropriate Lindblad operator must effect the transition $|1\rangle \rightarrow |0\rangle$; no relaxation occurs when the system is in the ground state (think: bath in vacuum state, at $T=0$). This suggests:

$$L = \sqrt{\gamma} |0\rangle\langle 1| = \sqrt{\gamma} \sigma^-$$

where we have already made explicit the factor $\sqrt{\gamma}$ with $\gamma > 0$ having dimensions of a rate ($1/\text{time}$).

The resulting master equation is

$$\frac{d}{dt}\rho = -\frac{i}{\hbar} [\varepsilon\sigma^+\sigma^-, \rho] + \underbrace{\gamma(\sigma^-\rho\sigma^+ - \frac{1}{2}\sigma^+\sigma^-\rho - \frac{1}{2}\rho\sigma^+\sigma^-)}_{=\gamma \text{ID}[\sigma^-]\rho}$$

Understand if driving to $|0\rangle$ is making pure state

which we wish to rewrite as an equation for the coherence vector, i.e., an equation of the form $\frac{d}{dt}\vec{r}(t) = \underline{G}\vec{r}(t) + \vec{b}$. Here, \underline{G} is a matrix and \vec{b} a vector determined by the r.h.s. of our master equation. We will now extract \underline{G} and \vec{b} for the specific master equation at hand

— a more systematic approach applicable to any kind of master equation is subject of PS#3.

Start with the master equation, multiply from the left with σ_x , σ_y or σ_z and take the trace:

$$\frac{d}{dt} \text{tr}(\sigma_x \rho) = \frac{d}{dt} r_x(t) = -\frac{i}{\hbar} \text{tr}(\sigma_x [\epsilon \sigma^+ \sigma^- \rho]) + \gamma \text{tr}(\sigma_x \sigma^- \rho \sigma^+ - \frac{1}{2} \sigma_x \sigma^+ \sigma^- \rho - \frac{1}{2} \sigma_x \rho \sigma^+ \sigma^-)$$

Two ways to calculate this:

- (1) brute force, i.e. take $\rho = \frac{1}{2} \begin{pmatrix} 1+r_z & r_x-i r_y \\ r_x+i r_y & 1-r_z \end{pmatrix}$, plug in Pauli matrices and calculate.

For example:

$$\cdot \text{tr}(\sigma_x \sigma^- \rho \sigma^+) = \text{tr} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1+r_z & r_x-i r_y \\ r_x+i r_y & 1-r_z \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{1}{2} \text{tr} \begin{pmatrix} 0 & 1+r_z \\ 0 & 0 \end{pmatrix} = 0$$

- (2) short cuts

$$\cdot \text{tr}(\sigma_x \sigma^- \rho \sigma^+) = \langle 0 | \sigma_x \sigma^- \rho \sigma^+ | 0 \rangle \stackrel{\sigma^+ + \sigma^-}{\cancel{\downarrow}} \stackrel{\sigma - \sigma = 0}{\cancel{\downarrow}} = \langle 0 | \sigma^+ \sigma^- \rho \sigma^+ | 0 \rangle = 0$$

$$\cdot \text{tr}(\sigma_x [\sigma^+ \sigma^-, \rho]) = \langle 0 | \sigma^- [\sigma^+ \sigma^-, \rho] | 0 \rangle + \langle 1 | \sigma^+ [\sigma^+ \sigma^-, \rho] | 1 \rangle$$

$$\stackrel{\sigma^+ + \sigma^-}{\cancel{\downarrow}} = \langle 1 | \rho | 0 \rangle - \langle 0 | \rho | 1 \rangle = \frac{1}{2} (r_x - i r_y - (r_x + i r_y)) = -i r_y.$$

Result:

$$\begin{aligned} \frac{d}{dt} r_x(t) &= -\frac{\epsilon}{\hbar} r_y(t) - \frac{\gamma}{2} r_x(t) \\ \frac{d}{dt} r_y(t) &= \frac{\epsilon}{\hbar} r_x(t) - \frac{\gamma}{2} r_y(t) \\ \frac{d}{dt} r_z(t) &= -(1+r_z) \gamma \end{aligned}$$

Coherence vector equation:

$$\vec{r}(t) = \underline{G} \vec{r}(t) + \vec{b}$$

$$\text{with } \underline{G} = \begin{pmatrix} -\gamma/2 & -\epsilon/\hbar & 0 \\ \epsilon/\hbar & -\gamma/2 & 0 \\ 0 & 0 & -\gamma \end{pmatrix}$$

$$\text{and } \vec{b} = (0, 0, -\gamma)^t$$

Solution for initial state $\vec{r}(t=0) = \vec{r}_0$:

general homogeneous solution: $\dot{\vec{r}} = \underline{G} \vec{r}$

$$\vec{r}(t) = e^{\underline{G}t} \vec{c} = \left(\begin{bmatrix} g & 0 \\ 0 & 0 \\ 0 & e^{-\gamma t} \end{bmatrix} \right) \vec{c}$$

$$g = e^{-\frac{\gamma}{2} \frac{1}{2} \hbar t - \frac{i\epsilon}{\hbar} \sigma_y t} = 1 e^{-\gamma/2 t} (\cos(\frac{\epsilon}{\hbar} t) - i \sin(\frac{\epsilon}{\hbar} t) \sigma_y)$$

$$\therefore \vec{r}(t) = \begin{pmatrix} \cos(\frac{\epsilon}{\hbar} t) e^{-\gamma t/2} & -\sin(\frac{\epsilon}{\hbar} t) e^{-\gamma t/2} & 0 \\ \sin(\frac{\epsilon}{\hbar} t) e^{-\gamma t/2} & \cos(\frac{\epsilon}{\hbar} t) e^{-\gamma t/2} & 0 \\ 0 & 0 & e^{-\gamma t} \end{pmatrix} \vec{c}$$

particular inhomogeneous solution : (variation of constant)
 ansatz $\vec{r}(t) = e^{\underline{G}t} \vec{c}(t)$: $\dot{\vec{r}} = \underline{G}\vec{r} + e^{\underline{G}t} \dot{\vec{c}} = \underline{G}\vec{c} + \vec{b}$

$$\begin{aligned}\dot{\vec{c}} &= e^{-\underline{G}t} \vec{b} \quad \text{and} \quad \vec{c}(0) = \vec{r}_0 \\ \dot{\vec{c}} &= -e^{-\underline{G}t} \underline{G}^{-1} \vec{b} + \underline{G}^{-1} \vec{b} + \vec{r}_0 \\ \vec{r}(t) &= e^{\underline{G}t} \underline{G}^{-1} \vec{b} - \underline{G}^{-1} \vec{b} + e^{\underline{G}t} \vec{r}_0\end{aligned}$$

$$\begin{aligned}\text{full solution: } \vec{r}(t) &= e^{\underline{G}t} \vec{r}_0 + (e^{\underline{G}t} - 1) \underline{G}^{-1} \vec{b} \\ &= e^{\underline{G}t} (\vec{r}_0 + \underline{G}^{-1} \vec{b}) - \underline{G}^{-1} \vec{b} \\ &= \begin{pmatrix} \cos(\epsilon t/\hbar) e^{-\gamma t/2} & -\sin(\epsilon t/\hbar) e^{-\gamma t/2} & 0 \\ \sin(\epsilon t/\hbar) e^{-\gamma t/2} & \cos(\epsilon t/\hbar) e^{-\gamma t/2} & 0 \\ 0 & 0 & e^{-\gamma t} \end{pmatrix} \begin{pmatrix} |r_{ox}\rangle \\ |r_{oy}\rangle \\ |r_{oz}\rangle \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\end{aligned}$$

- Note:**
- For $\gamma=0$, the coherence vector precesses around the z -axis, as $\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$ and $\frac{1}{\sqrt{2}}(|0\rangle+i|1\rangle)$ are not eigenstates. (In the x and y basis, the Hamiltonian $\epsilon \sigma^+ \sigma^-$ acts as a drive term inducing Rabi oscillations.)
 - For $\gamma>0$, relaxation always drives the system into the $\vec{r}=(0,0,-1)$ state, i.e., the ground state $|0\rangle$. The decay towards the ground state follows an exponential law

$$\begin{aligned}P_{\text{exc}}(t) &= \text{tr}(\rho|1\rangle\langle 1|) = \text{tr}\left(\rho \frac{1}{2}(|1\rangle\langle 1| + \sigma_z)\right) = \frac{1}{2}(1 + \langle \sigma_z \rangle) = \frac{1}{2}(1 + r_z) \\ \text{initial } r_z &= 1 \rightarrow = \frac{1}{2}(1 + 2e^{-\gamma t} - 1) = e^{-\gamma t}\end{aligned}$$

with relaxation rate γ . The mean relaxation time is called $T_1 \equiv 1/\gamma$ in the contexts of NMR and quantum bits.

- The off-diagonal elements of the density matrix represented by $r_x = \langle \sigma_x \rangle$ and $r_y = \langle \sigma_y \rangle$ undergo damped oscillations. The rate for their decay towards zero is $\gamma/2$. Generally, this decay of off-diagonal elements is called dephasing. The corresponding

mean dephasing time is T_2 . Relaxation always induces dephasing with a rate $\gamma/2$. If other dephasing mechanisms cause additional dephasing with rate γ_φ ("pure dephasing"), then the total dephasing rate is $\gamma_2 = \gamma/2 + \gamma_\varphi$. Accordingly,

$$T_2 = (\gamma/2 + \gamma_\varphi)^{-1} \leq \frac{2}{\gamma} = 2T_1.$$

If equality holds, dephasing is dictated by relaxation alone and we say that T_2 is limited by $2T_1$.

■ Behavior of purity under relaxation

Considering the relaxation of $\vec{r} = (0, 0, 1)$ to $\vec{r}' = (0, 0, -1)$ it is clear that the purity may behave non-monotonically. In general, we obtain:

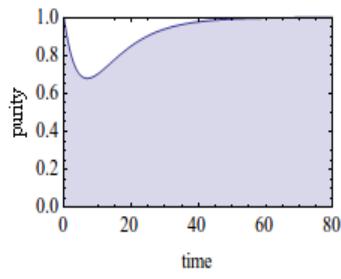
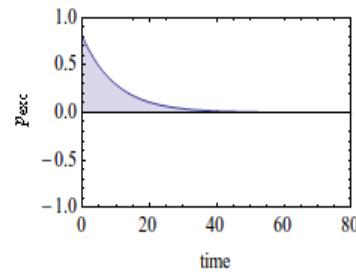
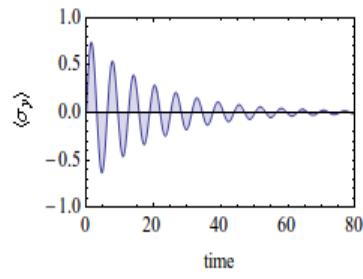
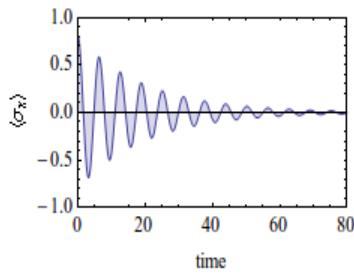
$$\begin{aligned} \text{tr } p^2 &= \frac{1}{4} \text{tr} [(\mathbb{1} + \vec{r} \cdot \vec{\sigma})(\mathbb{1} + \vec{r} \cdot \vec{\sigma})^\dagger] = \frac{1}{2} (1 + |\vec{r}|^2) \\ &= \frac{1}{2} \left\{ 1 + (r_{ox}c - r_{oy}s)^2 e^{-\gamma t} + (r_{ox}s + r_{oy}c)^2 e^{-\gamma t} + ([r_{oz} + 1]e^{-\gamma t} - 1)^2 \right\} \\ &= \frac{1}{2} \left\{ 1 + r_{ox}^2 e^{-\gamma t} + r_{oy}^2 e^{-\gamma t} + ([r_{oz} + 1]e^{-\gamma t} - 1)^2 \right\} \end{aligned}$$

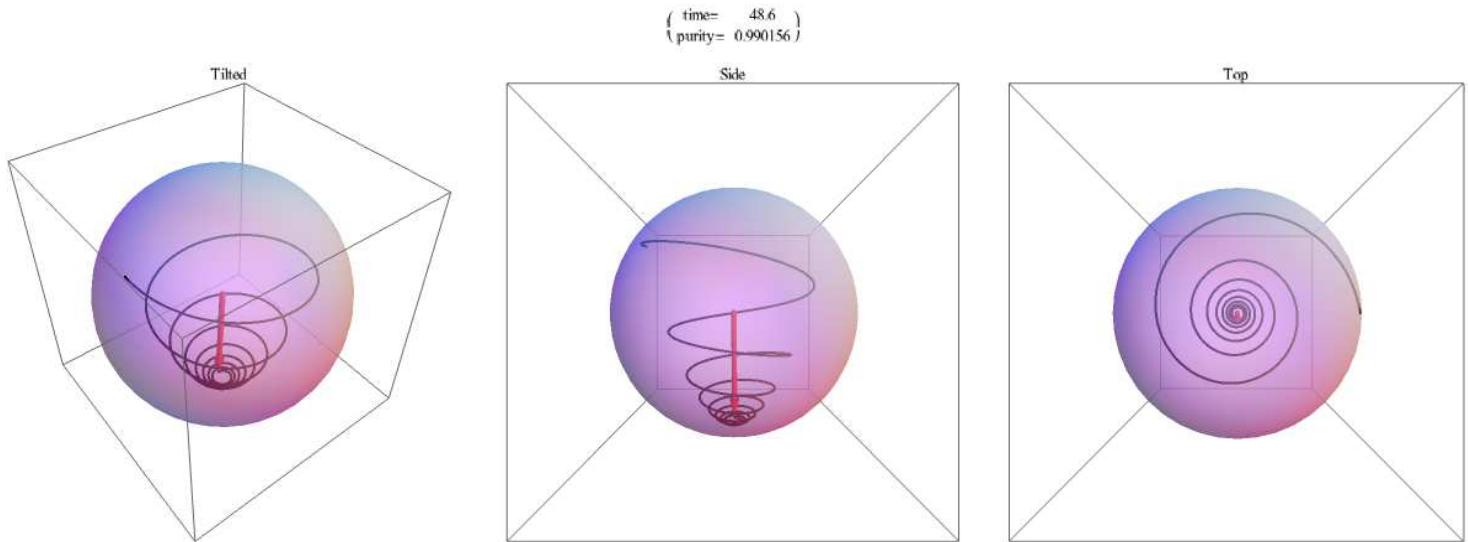
For example, for $r_{ox} = r_{oy} = 0$ and $r_{oz} = 1$:

$$\text{tr } p^2 = \frac{1}{2} + \frac{1}{2} (2e^{-\gamma t} - 1)^2 = \begin{cases} 1 & t=0 \\ 1 & t \rightarrow \infty \end{cases}$$

$$\text{Minimum: } 0 = \frac{d}{dt} \text{tr } p^2 = (2e^{-\gamma t} - 1)(-2\gamma)e^{-\gamma t} \Rightarrow e^{-\gamma t} = \frac{1}{2} \quad \therefore t = \frac{\ln 2}{\gamma}$$

Visualization: Mathematica Notebook





Relaxation at finite temperature

If the two-level system is coupled to a thermal bath with temperature $T > 0$, we expect that photon absorption is possible as well. Hence, we should consider the additional jump operator

$$L' = \sqrt{\gamma'} |1\rangle\langle 0| = \sqrt{\gamma'} \sigma^+$$

Here, γ' should be adjusted in such a way that the two-level system is driven towards a thermal equilibrium state

$$\rho_{eq} = e^{-\beta\epsilon\sigma_z} / \text{tr } e^{-\beta\epsilon\sigma_z} = \frac{1}{1 + e^{-\beta\epsilon}} \begin{pmatrix} e^{-\beta\epsilon} & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 + r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{pmatrix}$$

$$\therefore r_z = \frac{e^{-\beta\epsilon} - 1}{e^{-\beta\epsilon} + 1} = -\tanh(\beta\epsilon/2)$$

The master equation now reads:

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[H, \rho] + \gamma D[\sigma^-]\rho + \gamma' D[\sigma^+]\rho.$$

To rewrite this in the coherence vector language, consider

$$\text{tr } \sigma^\alpha [\sigma^+ \rho \sigma^- - \frac{1}{2} \sigma^- \sigma^+ \rho - \frac{1}{2} \rho \sigma^- \sigma^+].$$

Example: Calculate terms for $\frac{d}{dt} r_x$. Choose $\sigma^\alpha = \sigma_x = \sigma^+ + \sigma^-$.

$$\text{Term 1: } \text{tr}(\sigma_x \sigma^+ + \rho \sigma^-) = \text{tr}(\sigma^- \sigma^+ + \rho \sigma^-) = \langle | | \sigma^- \sigma^+ + \rho \sigma^- | | \rangle = 0$$

$$\text{Term 2: } \text{tr}(\sigma_x \sigma^- - \sigma^+ \rho) = \text{tr}(\sigma^+ \sigma^- - \sigma^+ \rho) = \langle | | \sigma^+ \sigma^- - \sigma^+ \rho | | \rangle = \rho_{01} = \frac{1}{2}(r_x - ir_y)$$

$$\text{Term 3: } \text{tr}(\rho \sigma^- \sigma^+ + \sigma_x) = \text{tr}(\rho \sigma^- \sigma^+ - \sigma^-) = \langle | | \rho \sigma^- \sigma^+ - \sigma^- | | \rangle = \rho_{10} = \frac{1}{2}(r_x + ir_y)$$

etc.

New coherence vector equation:

$$\dot{r}_x(t) = -\frac{\epsilon}{\hbar} r_y(t) - \frac{\gamma + \gamma'}{2} r_x(t)$$

$$\dot{r}_y(t) = \frac{\epsilon}{\hbar} r_x(t) - \frac{\gamma + \gamma'}{2} r_y(t)$$

$$\dot{r}_z(t) = -(\gamma + \gamma') r_z - \gamma + \gamma'$$

$$\dot{\vec{r}}(t) = \underline{G} \vec{r}(t) + \vec{b}$$

$$\underline{G} = \begin{pmatrix} -\gamma_\Sigma/2 & -\epsilon/\hbar & 0 \\ \epsilon/\hbar & -\gamma_\Sigma/2 & 0 \\ 0 & 0 & -\gamma_\Sigma \end{pmatrix}$$

$$\vec{b} = (0, 0, -\gamma + \gamma')$$

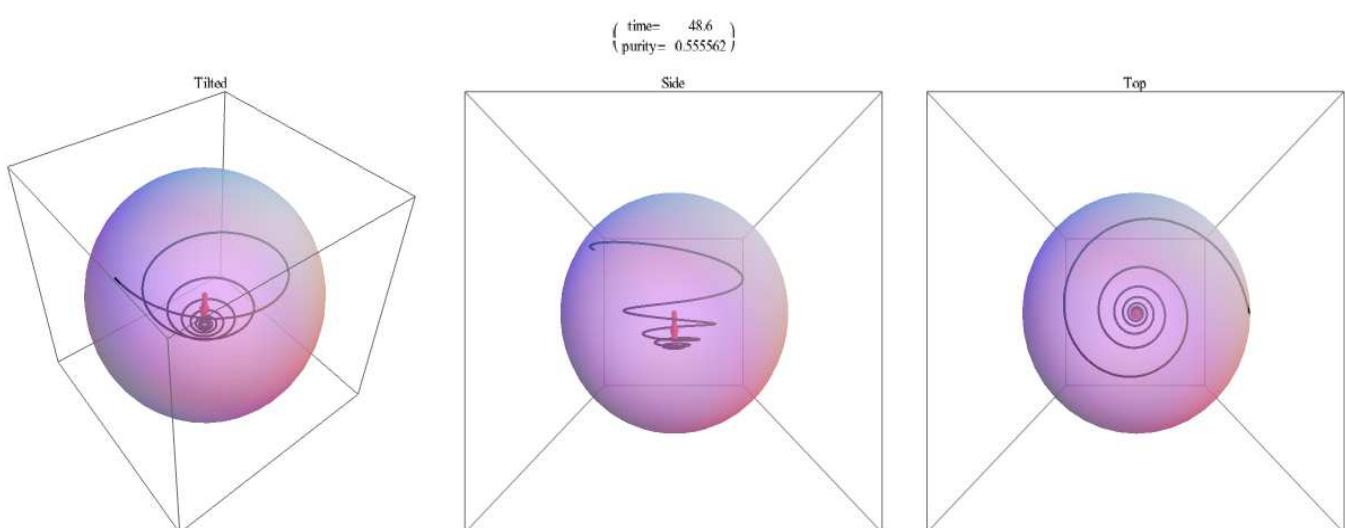
$$\text{For } t \rightarrow \infty, \quad \dot{r}_z = 0 = -(\gamma + \gamma') r_z - \gamma + \gamma' \stackrel{!}{=} (\gamma + \gamma') \tanh(\beta \epsilon/2) - \gamma + \gamma'$$

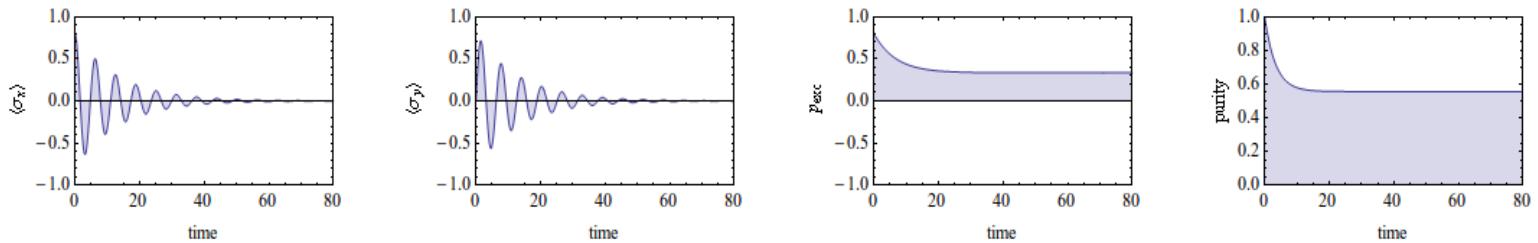
$$\Rightarrow \gamma'(1 + \tanh(\beta \epsilon/2)) = \gamma(1 - \tanh(\beta \epsilon/2))$$

$$\Rightarrow \gamma' = e^{-\beta \epsilon} \gamma$$

Note:

- For very large temperatures $k_B T \gg \epsilon$, $e^{-\beta \epsilon} \approx 1$. In this case, excitation and de-excitation occur with the same rate $\gamma = \gamma'$. The resulting steady-state reached for times $t \gg 1/\gamma$ is the completely mixed state ($\vec{r} = \vec{0}$).





Pure dephasing

Relaxation affects both the off-diagonal and the diagonal elements of the density matrix. Next, we will consider pure dephasing, which only induces decay of the off-diagonal elements. The appropriate Lindblad operator in this case is

$$L = \sqrt{\gamma_p/2} \sigma_z$$

The damping terms in the master equation now read:

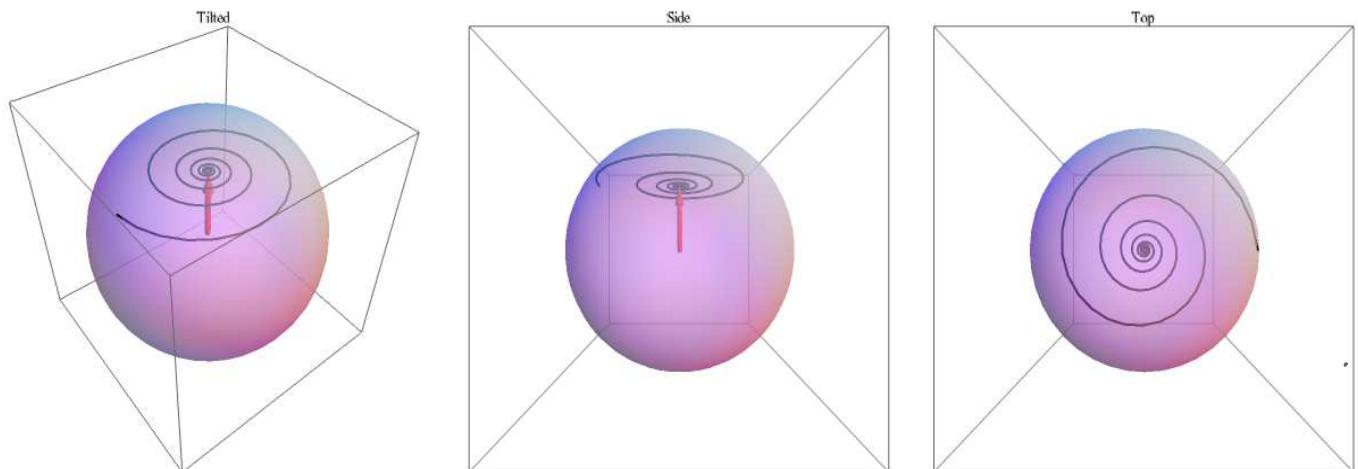
$$\mathbb{D}[L]\rho = \frac{\gamma_p}{2} \left(\sigma_z \rho \sigma_z - \frac{1}{2} \sigma_z^2 \rho - \frac{1}{2} \rho \sigma_z^2 \right) = \frac{\gamma_p}{2} (\sigma_z \rho \sigma_z - \rho)$$

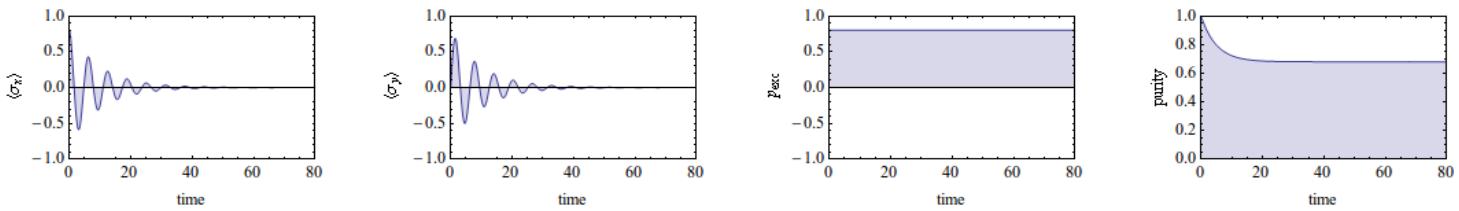
As previously, we determine the new terms in the coherence vector equation by calculating $\frac{d}{dt} r_\alpha = \text{tr}(\sigma_\alpha \cdot [\text{r.h.s. of master equation}])$

$$\begin{aligned} \dot{r}_x &= -\frac{\epsilon}{\hbar} r_y - \gamma_p r_x \\ \dot{r}_y &= \frac{\epsilon}{\hbar} r_x - \gamma_p r_y \\ \dot{r}_z &= 0 \end{aligned}$$

$$\Rightarrow \vec{r}(t) = \begin{pmatrix} [r_{ox} \cos(\epsilon t/\hbar) - r_{oy} \sin(\epsilon t/\hbar)] e^{-\gamma_p t} \\ [r_{ox} \sin(\epsilon t/\hbar) + r_{oy} \cos(\epsilon t/\hbar)] e^{-\gamma_p t} \\ r_{oz} \end{pmatrix}$$

{ time= 39.9
purity= 0.680131 }





NOTE:

- ☒ The Lindblad term $\frac{\gamma_\varphi}{2} \mathbb{D}[\sigma_z]$ leads to exponential decay of the off-diagonal elements of the density matrix on the time scale $T_\varphi = 1/\gamma_\varphi$. ("Off-diagonal" here refers to ρ expressed in the eigenbasis of the two-level system.)
- ☒ As a result of pure dephasing, coherence is lost — a coherent superposition of the $|0\rangle$ and $|1\rangle$ states, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, turns into a statistical mixture $\rho = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix}$. The final Bloch vector is on the z axis.
- ☒ $\langle \sigma_z \rangle$ (or, alternatively, the excited-state population $p_e = \langle |1\rangle \langle 1| \rangle$) remain unchanged under pure dephasing. Hence, the Bloch vector always remains in the plane defined by the initial value $z = \langle \sigma_z \rangle$.
- ☒ In the current setting, dephasing occurs under any QND-type measurement of the system energy.

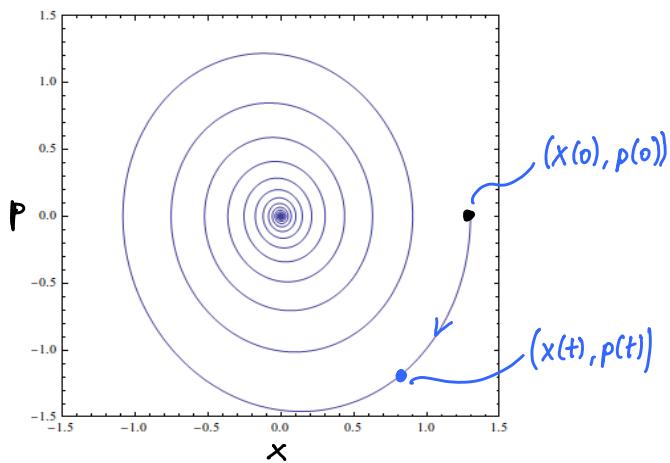
2.2 The open harmonic oscillator

The second example of an open quantum system is the harmonic oscillator. Together with the two-level systems, it represents one of the very few systems for which the master equation can be solved analytically and exactly. Similar to the development we followed in the previous section, we will discuss a useful way to visualize the density matrix of a harmonic oscillator.

Whatever the nature of the harmonic oscillator (mechanical vibrations in a molecule or solid, oscillations of the electromagnetic field inside a cavity etc.), its Hamiltonian takes the simple form $H = \hbar\omega_0 a^\dagger a$, where a and a^\dagger are lowering and raising operators for the excitation state of the oscillator; $a^\dagger a$ counts the number of quanta. For concreteness, we will discuss the example of photons inside a cavity. The operators a and a^\dagger annihilate a photon inside the cavity.

The Hilbert space \mathcal{H} in this case is infinite-dimensional, and the photon number states $\{|n\rangle\}_{n \in \mathbb{N}}$ represent an ONB for \mathcal{H} . Due to the dimensionality of \mathcal{H} , the attempt to visualize ρ in a way analogous to a Bloch sphere picture is doomed to fail. The idea behind achieving a useful visualization nonetheless is borrowed from an important method for visualizing system dynamics in classical mechanics: the phase space representation.

Consider a classical harmonic oscillator with damping: $m\ddot{x} = -kx - \gamma\dot{x}$ with solution $x(t) = x_0 \cos(\omega t + \varphi) e^{-rt/2m}$ w/ $\omega = \sqrt{\frac{k}{m} - \left(\frac{\gamma}{2m}\right)^2}$. The phase space picture is:



The idea is to try and imitate this representation for a quantum system. We will have to overcome two hurdles, however:

- (1) Heisenberg's uncertainty principle demands that $\Delta x \Delta p \gtrsim \hbar$. A single point (x, p) in phase space is not meaningful in quantum mechanics. (We will settle for a phase space distribution of some sort.)

- (2) All eigenstates of the harmonic oscillator have $\langle \hat{x}(t) \rangle_n = 0$ and $\langle \hat{p}(t) \rangle_n = 0$. eigenstates of H do not mimick the classical oscillatory behavior. Here, $\hat{x}(t)$ and $\hat{p}(t)$ are "position" and "momentum" operators in the Heisenberg picture. (For photons, $\hat{x} \sim a + a^\dagger$ and $\hat{p} \sim i(a - a^\dagger)$ represent the strength of electric field and magnetic vector potential.) As we will confirm in more detail later, photon number states are not "classical states" of light.

The second hurdle can be easily overcome, as you know, by considering a different class of states: coherent states. As this (hopefully!) is not your

first encounter with coherent states, we will merely review the most important properties and relations surrounding coherent states.

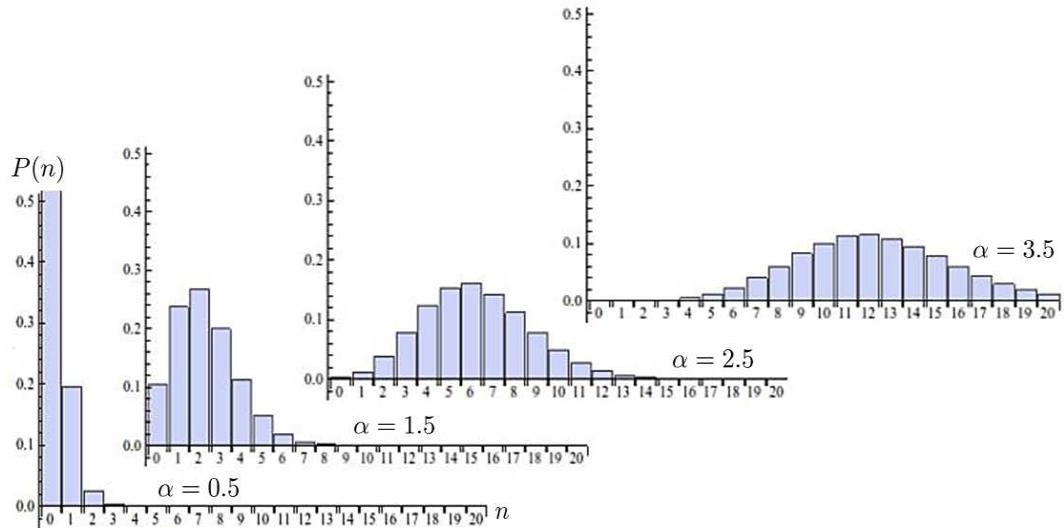
READER'S DIGEST: Coherent states

$$(1) |\alpha\rangle \text{ is a coherent state} \Leftrightarrow \underbrace{a|\alpha\rangle}_{\text{annihilation operator}} = \alpha|\alpha\rangle \quad (\alpha \in \mathbb{C})$$

$$(2) \text{ From (1) follows } |\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \text{ so that}$$

$$p_{\alpha}(n) = |\langle n|\alpha\rangle|^2 = \frac{1}{n!} e^{-|\alpha|^2} |\alpha|^{2n}.$$

This is a Poisson distribution with mean value $\langle n \rangle_{\alpha} = \langle \alpha | \hat{n} | \alpha \rangle = \sum_n n p_{\alpha}(n) = |\alpha|^2$.



(3) Coherent states are not orthogonal. Specifically,

$$\langle \alpha | \beta \rangle = e^{-|\alpha|^2/2} e^{-|\beta|^2/2} e^{\alpha^* \beta} \quad \text{and hence,} \quad |\langle \alpha | \beta \rangle|^2 = e^{-|\alpha - \beta|^2}.$$

(4) Every state $|\psi\rangle$ in the harmonic oscillator Hilbert space \mathcal{H} can be expressed in terms of coherent states by using $|\psi\rangle = \frac{1}{\pi} \int d\alpha |\alpha\rangle \langle \alpha | \psi \rangle$. For example:

$$|n\rangle = \frac{1}{\pi} \int d\alpha |\alpha\rangle \langle \alpha | n \rangle = \frac{1}{\sqrt{n!} \pi} \int d\alpha e^{-|\alpha|^2/2} (\alpha^*)^n |\alpha\rangle$$

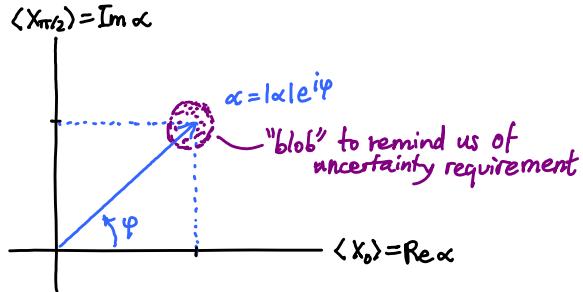
Note: even though each $|\alpha\rangle$ is normalized, a reduction factor of $\frac{1}{\pi}$ is necessary — signalling that $\{|\alpha\rangle\}_{\alpha \in \mathbb{C}}$ is overcomplete, i.e., the set spans \mathcal{H} but is not linearly independent. For instance: $|0\rangle$ is a coherent state as is, but is also given by $|0\rangle = \frac{1}{\pi} \int d\alpha e^{-|\alpha|^2/2} |\alpha\rangle$.

(5) The coherent-state expectation value for the two quadrature operators $X_0 = \frac{1}{2}(a + a^\dagger)$ and $X_{\pi/2} = \frac{1}{2}(ia - ia^\dagger)$ is given by

$$\langle X_0 \rangle_\alpha = \text{Re } \alpha \quad \text{and} \quad \langle X_{\pi/2} \rangle_\alpha = \text{Im } \alpha.$$

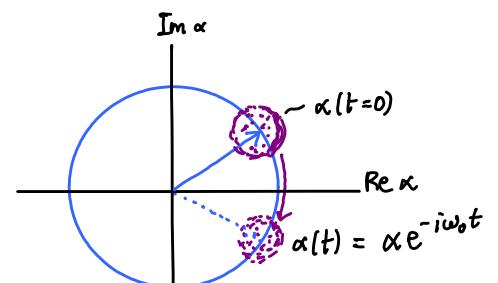
Up to prefactors, the quadrature operators correspond to \hat{x} and \hat{p} . Their commutator is $[X_0, X_{\pi/2}] = \frac{1}{2}i$. Heisenberg's uncertainty relation now reads: $\Delta X_0 \cdot \Delta X_{\pi/2} \geq \frac{1}{4}$.

Each coherent state can be represented in a phase space based on quadratures:



(6) The time evolution of coherent states nicely mimicks the classical oscillatory behavior:

$$\begin{aligned} |\psi(t)\rangle &= e^{-iHt/\hbar} |\alpha\rangle \quad \text{initial state} \\ &= e^{-i\omega_0 a^\dagger a t} e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \\ &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-i\omega_0 t \cdot n} |n\rangle \\ &= \frac{1}{\sqrt{n!}} (\alpha e^{-i\omega_0 t})^n \\ &= |\alpha e^{-i\omega_0 t}\rangle \end{aligned}$$



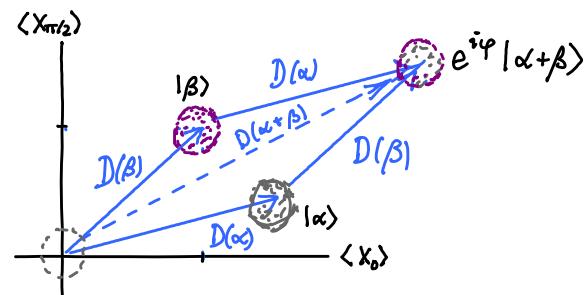
Time evolution of an initial coherent state under the harmonic oscillator Hamiltonian consists of a circular orbit in quadrature space. This matches the classical behavior in phase space.

- (7) The displacement operator $D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$ "shifts" the vacuum state to the coherent state $|\alpha\rangle$: $D(\alpha)|0\rangle = |\alpha\rangle$. Except for an additional phase factor, this shift also works when applied to coherent states $|\beta\rangle$ with $\beta \neq 0$:

$$D(\alpha)|\beta\rangle = e^{(\alpha\beta^* - \alpha^*\beta)/2} |\alpha + \beta\rangle,$$

or, in other words:

$$D(\alpha)D(\beta) = e^{(\alpha\beta^* - \alpha^*\beta)/2} D(\alpha + \beta)$$



- (8) In the case of photons in a certain mode of the EM field, a coherent state $|\alpha\rangle$ is a good description of a classical state generated by a microwave generator or laser, where the drive amplitude and phase (as well as its frequency and the photon loss rate) determine the coherent state's phase and amplitude. (PS#4 will walk us through the details.)

- (9) Quadrature representations $\langle X_o | \alpha \rangle$ and $\langle X_{\pi/2} | \alpha \rangle$

The quadrature representations of a coherent state $|\alpha\rangle$ correspond to position and momentum representations of $|\alpha\rangle$, up to certain factors of 2 and $\sqrt{2}$ coming from the convention $X_o = \frac{1}{2}(a + a^\dagger) = \frac{1}{\sqrt{2}}x$. The probability distributions $|\langle X_o | \alpha \rangle|^2$ and $|\langle X_{\pi/2} | \alpha \rangle|^2$ are Gaussians centered at $\text{Re } \alpha$ and $\text{Im } \alpha$ with equal width $1/\sqrt{2}$.

The two quadrature representations look promising but do not give a phase distribution yet — we can either choose $\langle X_o | \alpha \rangle$ or $\langle X_{\pi/2} | \alpha \rangle$, but not both at the same time! In the following, we will explore two ways to define phase space distributions: the Husimi Q function and the Wigner function.

The Husimi Q Function

Perhaps, the most obvious way to try and define a phase space distribution is the Husimi Q function. To motivate its definition, recall that a coherent state $|\alpha\rangle$ with $\alpha = \alpha_1 + i\alpha_2$ ($\alpha_1, \alpha_2 \in \mathbb{R}$) is centered at the point $X_0 = \alpha_1$, $X_{\pi/2} = \alpha_2$. Moreover, X_0 and $X_{\pi/2}$ correspond to generalized position and momentum of the oscillator (up to prefactors, which are not relevant to our present discussion). For a pure state $|\psi\rangle$, we define

$$Q^{|\psi\rangle\langle\psi|}(\alpha) = \frac{1}{\pi} |\langle\alpha|\psi\rangle|^2,$$

i.e. the Q function at the point (α_1, α_2) in phase space corresponds to the probability to find the system in the state $|\alpha\rangle$. Obviously, Q is non-negative. It is also normalized to 1 since

$$\int d^2\alpha Q^{|\psi\rangle\langle\psi|}(\alpha) = \frac{1}{\pi} \int d^2\alpha \langle\psi|\alpha\rangle \langle\alpha|\psi\rangle = \langle\psi| \underbrace{\frac{1}{\pi} \int d^2\alpha |\alpha\rangle\langle\alpha|}_{=1} |\psi\rangle = \langle\psi|\psi\rangle = 1.$$

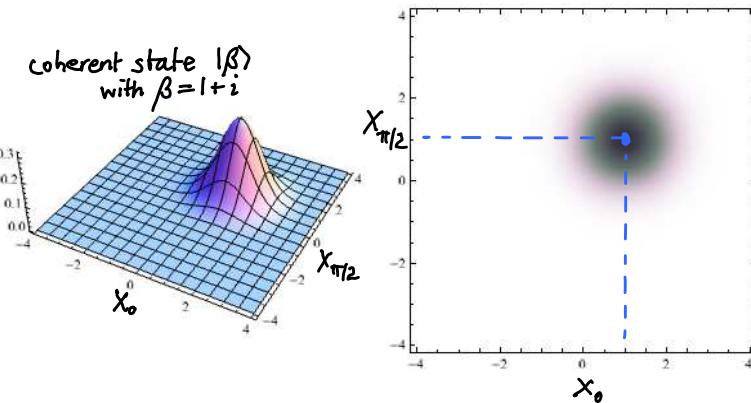
Have we succeeded in finding a proper probability distribution in phase space? The answer is "no" (and, as one can prove, it will remain "no" no matter what we do or define!). What is missing?

Mathematically, probability theory can be developed axiomatically, and is based on the concept of σ -algebras. Loosely speaking, the σ -algebra contains all possible sets E ("events") to which we can assign a probability $P(E)$. An important requirement of σ -algebras is that the probabilities of mutually exclusive events add up:

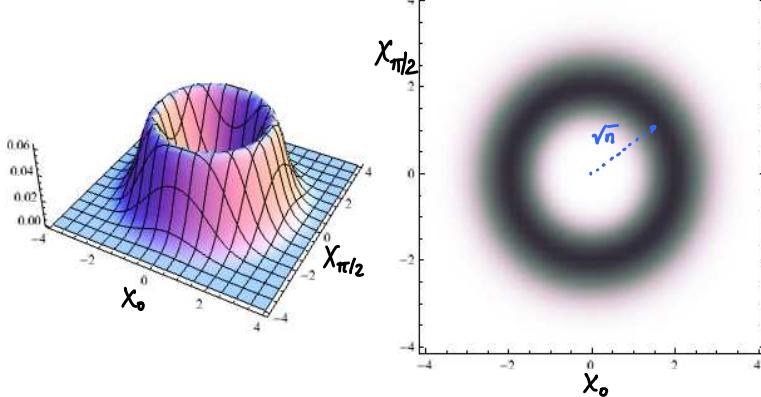
$$E_1 \cap E_2 = \emptyset : P(E_1 \cup E_2) = P(E_1) + P(E_2).$$

For the Husimi Q function it is this property that is violated because of the non-orthogonality of coherent states. Despite this unavoidable failure as a probability distribution, the Q function is very useful in visualizing harmonic oscillator states.

Examples: \blacksquare coherent state $|\beta\rangle \rightarrow Q^{|\beta\rangle\langle\beta|}(\alpha) = \frac{1}{\pi} e^{-|\alpha-\beta|^2}$
 (Gaussian centered at β)



\blacksquare Fock state $|n\rangle \rightarrow Q^{|n\rangle\langle n|}(\alpha) = \frac{1}{\pi n!} |\alpha|^{2n} e^{-|\alpha|^2}$
 (Poisson distribution in n , in α , approximately a Gaussian with center $|\alpha| = \sqrt{n}$)



Generalization of the Q function for mixed states

Our definition of Q above was tailored towards pure states. Here is how we can extend it to include mixed states as well:

$$Q^{|\psi\rangle\langle\psi|}(\alpha) = \frac{1}{\pi} |\langle\psi|\alpha\rangle|^2 = \frac{1}{\pi} \langle\alpha|\psi\rangle\langle\psi|\alpha\rangle \stackrel{\rho=|\psi\rangle\langle\psi|}{=} \frac{1}{\pi} \langle\alpha|\rho|\alpha\rangle$$

Here, the last expression expressed in terms of ρ points the way to the definition of Q for an arbitrary density matrix ρ :

$$\begin{aligned} Q^\rho(\alpha) &= \frac{1}{\pi} \langle\alpha|\rho|\alpha\rangle = \frac{1}{\pi} \text{tr}(\rho|\alpha\rangle\langle\alpha|) = \frac{1}{\pi} \langle 0|D(-\alpha)\rho D(\alpha)|0\rangle \\ &= \frac{1}{\pi} \text{tr}(|0\rangle\langle 0|D(-\alpha)\rho D(\alpha)) \end{aligned}$$

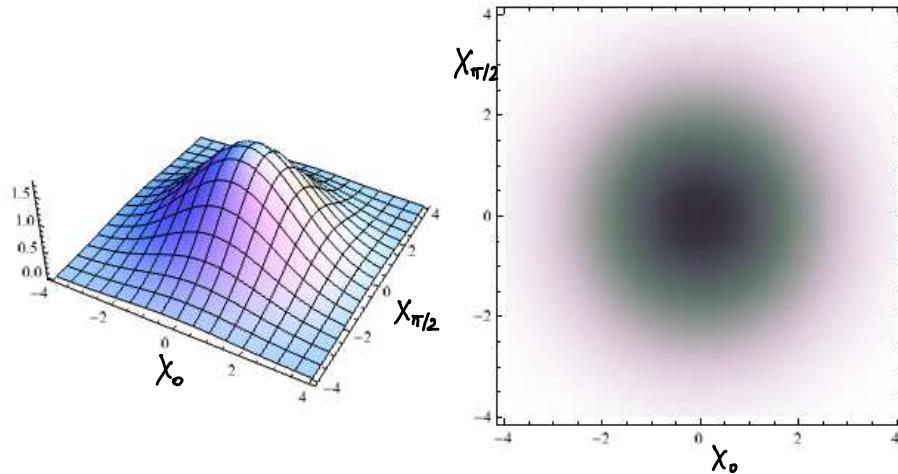
As a result, $Q^\rho(\alpha)$ can be understood as the expectation value of the projector onto the vacuum state with respect to the density matrix

when shifted by α in quadrature space.

Example: Thermal state $\rho_{\text{th}} = \frac{1}{Z} e^{-\beta \hbar \omega_0 a^\dagger a}$ where $Z = \sum_{n=0}^{\infty} (e^{-\beta \hbar \omega_0})^n = (1 - e^{-\beta \hbar \omega_0})^{-1}$ is the canonical partition function.

$$\begin{aligned} Q^{\text{Th}}(\alpha) &= \frac{1}{\pi} \langle 0 | D(-\alpha) \rho_{\text{th}} D(\alpha) | 0 \rangle = Z^{-1} \frac{1}{\pi} \sum_{n,m=0}^{\infty} \langle \alpha | n \rangle \langle n | e^{-\beta \hbar \omega_0 a^\dagger a} | m \rangle \langle m | \alpha \rangle \\ &= (Z\pi)^{-1} \sum_{n=0}^{\infty} |\langle \alpha | n \rangle|^2 e^{-\beta \hbar \omega_0 n} \\ &= (Z\pi)^{-1} \sum_{n=0}^{\infty} \frac{e^{-|\alpha|^2}}{n!} |\alpha|^2 e^{-\beta \hbar \omega_0 n} = (Z\pi)^{-1} e^{-|\alpha|^2} e^{|\alpha|^2 e^{-\beta \hbar \omega_0}} \\ &= \frac{1}{\pi} (1 - e^{-\beta \hbar \omega_0}) e^{-|\alpha|^2 (1 - e^{-\beta \hbar \omega_0})^{-1}} \end{aligned}$$

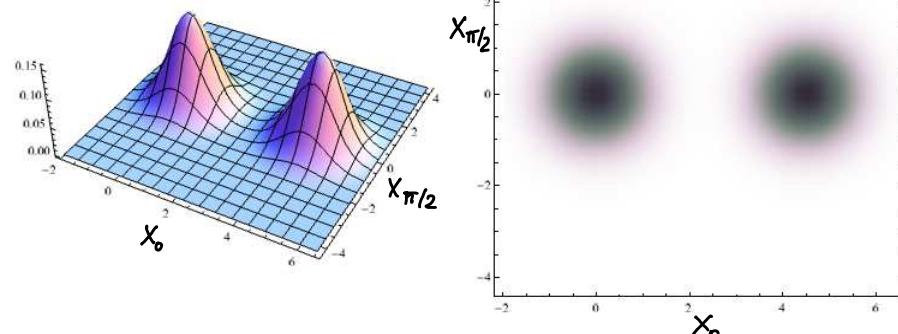
This is a Gaussian centered at the origin with a width that increases with temperature.



Mixture of two coherent states: $\rho = \frac{1}{2} (|0\rangle\langle 0| + |\beta\rangle\langle \beta|)$

$$\begin{aligned} Q^{\text{M}}(\alpha) &= \frac{1}{\pi} \text{tr} \langle 0 | D(-\alpha) \rho D(\alpha) | 0 \rangle = \frac{1}{2\pi} (|\langle \alpha | 0 \rangle|^2 + |\langle \alpha | \beta \rangle|^2) \\ &= \frac{1}{2\pi} (e^{-|\alpha|^2} + e^{-|\alpha-\beta|^2}) \end{aligned}$$

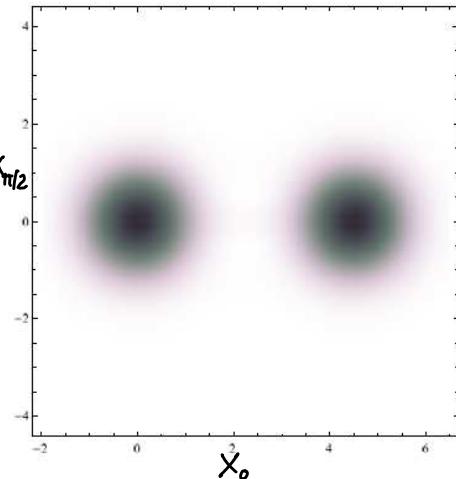
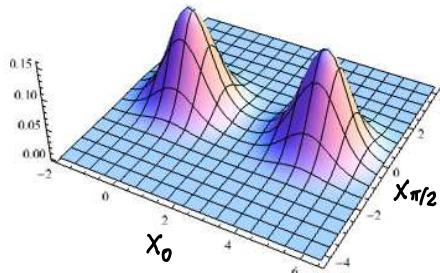
$$\rho = \frac{1}{2} (|0\rangle\langle 0| + |\beta\rangle\langle \beta|) \quad \omega/\beta = 4.5$$



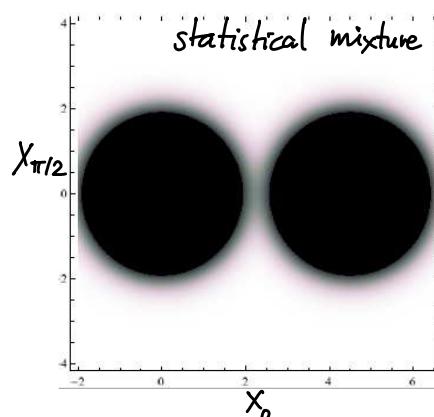
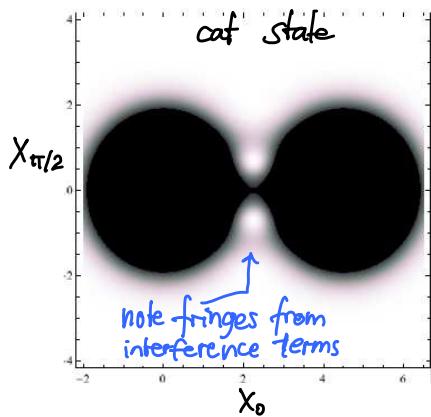
• "Cat state" $|\psi\rangle = \frac{1}{\sqrt{N_\beta}}(|0\rangle + |\beta\rangle)$ where $1 = (2 + 2\text{Re}(\beta|0\rangle))/N_\beta$
 $\therefore N_\beta = 2(1 + e^{-|\beta|^2/2})$

$$\begin{aligned} Q^{|\psi\rangle\langle\psi|}(\alpha) &= \frac{1}{\pi} |\langle\alpha|\psi\rangle|^2 = \frac{1}{\pi N_\beta} |\langle\alpha|0\rangle + \langle\alpha|\beta\rangle|^2 \\ &= \frac{1}{\pi N_\beta} \left(e^{-|\alpha|^2} + e^{-|\alpha-\beta|^2} + 2\text{Re}(\langle\alpha|0\rangle\langle\beta|\alpha\rangle) \right) \\ &= \frac{1}{\pi N_\beta} \left[e^{-|\alpha|^2} + e^{-|\alpha-\beta|^2} + 2\text{Re} \left(e^{-|\alpha|^2/2} e^{-|\alpha|^2/2 - |\beta|^2/2 + \beta^* \alpha} \right) \right] \\ &= \frac{1}{\pi N_\beta} \left[e^{-|\alpha|^2} + e^{-|\alpha-\beta|^2} + 2e^{-|\alpha|^2/2} e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2)} e^{\text{Re}(\beta^* \alpha)} \cos(\text{Im}(\beta^* \alpha)) \right] \end{aligned}$$

$$\rho = |\psi\rangle\langle\psi| \quad w/ \quad |\psi\rangle = \frac{1}{\sqrt{N_\beta}}(|0\rangle + |\beta\rangle)$$



This looks almost like the statistical mixture of two coherent states above — but only almost! The following two plots show the details of the tails between the two peaks:



The Husimi-Q function is certainly a useful representation to visualize the density matrix. As such, Q contains all information there is to know about ρ . (We acknowledge this fact here without going through the proof.) The last comparison between the statistical mixture and the cat state of two coherent states indicates a slight drawback of the Husimi-Q function: non-classicality of states

may only show up as subtle differences in the tails of the Q function. The Wigner function is advantageous in this specific aspect. It emphasizes features due to non-classicality of states — and also contains all information given by the density matrix ρ .

To see the connection between Q function and Wigner function, let us begin by rewriting Q :

$$\begin{aligned}
 Q^P(\alpha) &= \frac{1}{\pi} \text{tr}(\rho |\alpha\rangle\langle\alpha|) = \frac{1}{\pi} \text{tr} \left[\rho \int d^2\beta \delta(\beta - \alpha) |\beta\rangle\langle\beta| \right] \xrightarrow{\text{introduce integration over } \delta \text{ function}} \\
 &= \frac{1}{\pi^3} \text{tr} \left[\rho \int d^2\beta \int d^2\lambda e^{\lambda^*(\beta^* - \alpha^*) - \lambda^*(\beta - \alpha)} |\beta\rangle\langle\beta| \right] \xrightarrow{\text{use } \alpha|\beta\rangle = \beta|\alpha\rangle} \\
 &= \frac{1}{\pi^2} \text{tr} \left[\rho \int d^2\lambda e^{\lambda^*\alpha - \lambda\alpha^*} e^{-\lambda^*a} \underbrace{\frac{1}{\pi} \int d^2\beta |\beta\rangle\langle\beta| e^{\lambda a^\dagger}}_{=1} \right] = \frac{1}{\pi^2} \int d^2\lambda e^{\lambda^*\alpha - \lambda\alpha^*} \text{tr} \left[\rho e^{-\lambda^*a} e^{\lambda a^\dagger} \right] \\
 &\quad \text{anti-normal ordered characteristic function: } C_{an}^P(\lambda)
 \end{aligned}$$

Hence, the Husimi- Q function is the Fourier transform of the anti-normal ordered characteristic function $C_{an}^P(\lambda)$. Similarly, the Wigner function is defined as the Fourier transform of the symmetrically ordered characteristic function

$$C_{sym}^P(\lambda) = \text{tr}[\rho e^{\alpha a^\dagger - \alpha^* a}] = \text{tr} \left[\rho e^{-\alpha^* a} e^{-\alpha a^\dagger} e^{i\lambda a^\dagger/2} \right] = e^{i\lambda a^\dagger/2} C_{an}^P(\lambda),$$

namely:

$$e^A e^B = e^{A+B} e^{[A,B]/2} \text{ for } [A, [A, B]] = [B, [A, B]] = 0 \quad (\text{Glauber identity})$$

$$W^P(\alpha) = \frac{1}{\pi^2} \int d^2\lambda e^{\lambda^*\alpha - \lambda\alpha^*} C_{sym}^P(\lambda) = \frac{1}{\pi^2} \int d^2\lambda e^{\lambda^*\alpha - \lambda\alpha^* + i\lambda a^\dagger/2} C_{an}^P(\lambda)$$

Two common, fully equivalent expressions for the Wigner function are

$$W^P(\alpha = x + ip) = \frac{1}{\pi} \int dx' e^{-2ipx'} \langle x + x'/2 | \rho | x - x'/2 \rangle = \frac{2}{\pi} \text{tr} [\mathcal{P} D(-\alpha) \rho D(\alpha)]$$

where $\mathcal{P} = e^{i\pi a^\dagger a}$ is the photon number parity operator: $e^{i\pi a^\dagger a} |n\rangle = (-1)^n |n\rangle$.

For the derivation of these expressions see, e.g., Haroche/Raimond, appendix A.

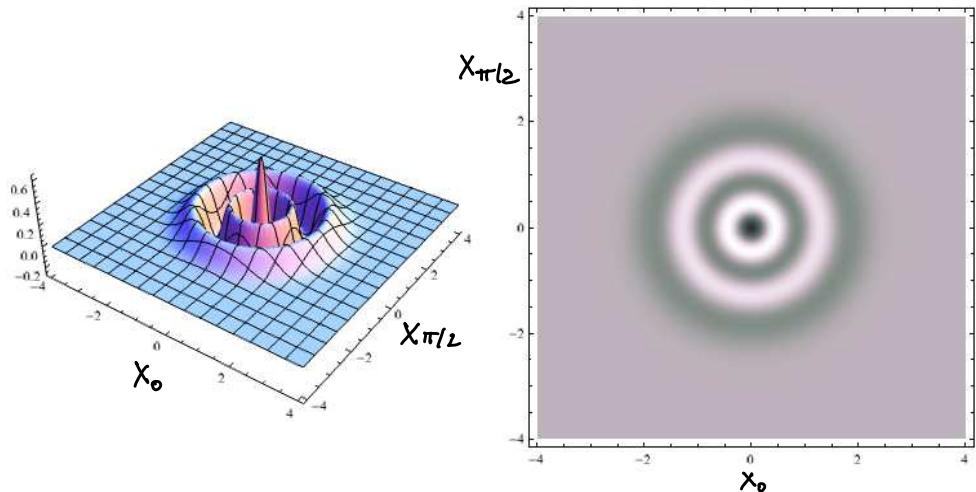
Examples: ■ Coherent state $\rho = |\beta\rangle\langle\beta|$

$$\begin{aligned}
 W^P(\alpha) &= \frac{2}{\pi} \text{tr} (\mathcal{P} D(-\alpha) \rho D(\alpha)) = \frac{2}{\pi} \sum_{n=0}^{\infty} \langle n | \mathcal{P} e^{(\alpha^*\beta - \alpha\beta^*)/2} |\beta - \alpha\rangle \langle \beta - \alpha | e^{(\alpha\beta^* - \alpha^*\beta)/2} |n\rangle \\
 &= \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n |\langle n | \beta - \alpha \rangle|^2 = \frac{2}{\pi} e^{-|\beta - \alpha|^2} \sum_{n=0}^{\infty} \frac{|(-1)|\beta - \alpha|^2|^n}{n!} \\
 &= \frac{2}{\pi} e^{-2|\beta - \alpha|^2} \quad \text{which is, again, a Gaussian.}
 \end{aligned}$$

■ Number state $\rho = |n\rangle\langle n|$

$$W^P(\alpha) = \frac{2}{\pi} \text{tr} [PD(-\alpha)|n\rangle\langle n|D(\alpha)] = \frac{2}{\pi} (-1)^n e^{-2|\alpha|^2} L_n(4|\alpha|^2)$$

Laguerre polynomial

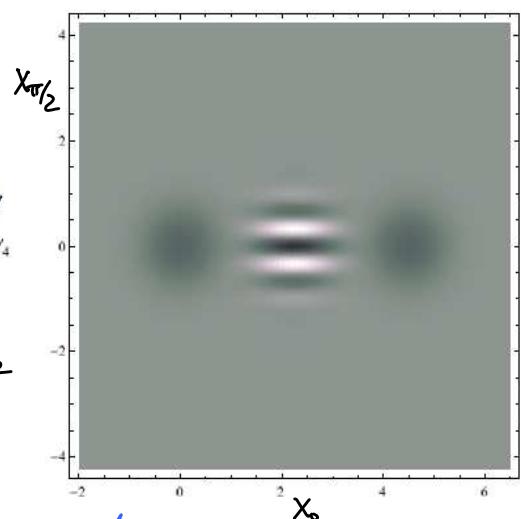
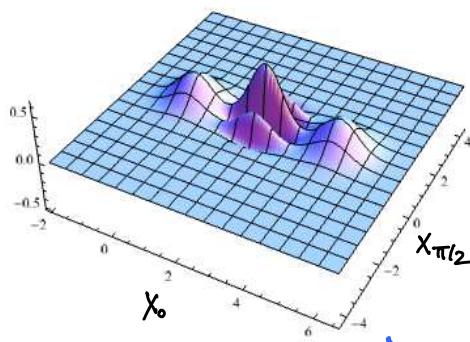


For Fock states, the Wigner function takes on negative values in certain regions of quadrature space — signalling that Fock states are not classical states. In fact, classicality of harmonic oscillator states is often defined by this criterion.

■ Cat state $\rho = |\psi\rangle\langle\psi|$ with $|\psi\rangle = \frac{1}{\sqrt{N_p}}(|0\rangle + |\beta\rangle)$ (as above)

$$W^P(\alpha) = \frac{2}{\pi(1+e^{-|\beta|^2/2})} \left[e^{-2|\alpha|^2} + e^{-2|\alpha-\beta|^2} + 2e^{-2|\alpha-\frac{1}{2}\beta|^2} \cos(2\text{Im } \alpha\beta^*) \right]$$

Cat state ($\beta=4.5$)



Note oscillations and negative values.

Photon decay at zero & finite temperature

Equipped with the Wigner function for visualizing ρ , we are ready to investigate the damped quantum harmonic oscillator. For finite temperature, the corresponding master equation is

$$\begin{aligned}\frac{d}{dt}\rho &= -\frac{i}{\hbar}[H, \rho] + \kappa \mathbb{D}[a]\rho + \kappa e^{-\beta\hbar\omega_0} \mathbb{D}[a^\dagger]\rho \\ &= -i[\omega_0 a^\dagger a, \rho] + \kappa(a a^\dagger - \frac{1}{2}a^\dagger a^\dagger - \frac{1}{2}a a) + \kappa e^{-\beta\hbar\omega_0}(a^\dagger \rho a - \frac{1}{2}a a^\dagger \rho - \frac{1}{2}\rho a a^\dagger).\end{aligned}$$

PHOTON OCCUPATION NUMBERS

The master equation above leads to a particularly simple solution for the time evolution of the probability distribution for finding n photons in the EM mode, $P(n, t) = \langle n | \rho(t) | n \rangle$:

$$\begin{aligned}\frac{d}{dt}P(n, t) &= \langle n | \frac{d}{dt}\rho | n \rangle \\ &= (-i\omega_0 n + i\omega_0 n)P(n, t) + \kappa(n+1)P(n+1, t) - \kappa n P(n, t) \\ &\quad + \kappa e^{-\beta\hbar\omega_0} n P(n-1, t) - \kappa e^{-\beta\hbar\omega_0} (n+1)P(n, t) \\ &= \underbrace{\kappa(n+1)P(n+1, t)}_{\substack{\text{decay from state} \\ \text{with } n+1 \text{ photons} \\ \text{increases } P(n, t)}} - \underbrace{\kappa n P(n, t)}_{\substack{\text{decay from state with } n \text{ photons decreases } P(n, t)}} + \kappa e^{-\beta\hbar\omega_0} n P(n-1, t) - \kappa e^{-\beta\hbar\omega_0} (n+1)P(n, t)\end{aligned}$$

This is a classical rate equation. Note: the effective decay rate from a state with n photons is $\kappa^{\text{eff}} = n \cdot \kappa$. The stationary distribution is obtained from

$$0 = \frac{d}{dt}P(n, t) = \kappa(n+1)P(n+1, t) - \kappa n P(n, t) + \underbrace{\kappa' n P(n-1, t)}_{\substack{=\kappa e^{-\beta\hbar\omega_0}}} - \kappa'(n+1)P(n, t)$$

$$\therefore n=0: \quad P(1) = \frac{\kappa'}{\kappa} P(0) = e^{-\beta\hbar\omega_0} P(0)$$

$$\begin{aligned}n=1: \quad P(2) &= \frac{1}{2\kappa} \left[+\kappa P(1) - \kappa' P(0) + 2\kappa' P(1) \right] \\ &= \frac{1}{2} (e^{-\beta\hbar\omega_0} P(0) - e^{-\beta\hbar\omega_0} P(0) + 2e^{-2\beta\hbar\omega_0} P(0)) \\ &= e^{-2\beta\hbar\omega_0} P(0) \\ &\vdots \\ P(n) &= e^{-n\beta\hbar\omega_0} P(0)\end{aligned}$$

Normalization: $\sum_{n=0}^{\infty} P(n) = P(0) \sum_{n=0}^{\infty} e^{-n\beta\hbar\omega_0} = \frac{P(0)}{1 - e^{-\beta\hbar\omega_0}} = 1$
 $\Rightarrow P(0) = (1 - e^{-\beta\hbar\omega_0}).$

We thus obtain the photon number distribution:

$$P(n) = (1 - e^{-\beta\hbar\omega_0}) e^{-n\beta\hbar\omega_0}$$

The average photon number then obeys the Planck distribution:

$$\langle n \rangle = \sum_n n P(n) = \frac{1}{e^{\beta\hbar\omega_0} - 1}$$

MORE GENERAL SOLUTION (including time dependence and off-diagonal elements)

Our treatment so far covers only the photon number distribution (i.e., the diagonal elements of the density matrix), and our explicit solution only described the steady state. Next, let us study the full dynamics of the density matrix ρ for two specific initial states: a simple coherent state $\rho_0 = |\beta_0\rangle\langle\beta_0|$ and a cat state $\rho_0 = |\psi\rangle\langle\psi|$ with $|\psi\rangle = (|\beta_1\rangle + |\beta_2\rangle)/\sqrt{N}$.

At zero temperature, damping only consists of photon loss and the master equation reads:

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[H, \rho] + \kappa D[a]\rho = -i[\omega_0 a^\dagger a, \rho] + \kappa(a a^\dagger - \frac{1}{2}a^\dagger a^\dagger - \frac{1}{2}\rho a^\dagger a)$$

■ Coherent initial state

Recalling the dynamics of the classical damped oscillator (spiraling towards the origin in phase space), it is reasonable to try the ansatz $\rho(t) = |\beta(t)\rangle\langle\beta(t)|$ with $\beta(t) \in \mathbb{C}$ and $\beta(0) = \beta_0$.

$$\begin{aligned} \frac{d}{dt}|\beta(t)\rangle &= \frac{d}{dt}e^{\beta(t)a^\dagger - \beta^*(t)a}|0\rangle \\ &= \frac{d}{dt}\left[e^{-(\beta(t))^{1/2}/2}e^{\beta(t)a^\dagger}e^{-\beta^*(t)a}|0\rangle\right] \end{aligned}$$

$$= \left[-\frac{1}{2}\frac{d}{dt}|\beta(t)|^2 + \dot{\beta}(t)a^\dagger\right]|\beta(t)\rangle = \left[-\frac{1}{2}\frac{d}{dt}|\beta(t)|^2 + \frac{\dot{\beta}(t)}{\beta(t)}a^\dagger a\right]|\beta(t)\rangle$$

$\Delta \frac{d}{dt}e^{f(t)A+g(t)B} \neq (fA+gB)e^{fA+gB}$
unless $[fA+gB, fA+gB] = (f_g - f_g)[A, B] = 0$.

In the last step, we have used $a^\dagger |\beta\rangle = \frac{1}{\beta} a^\dagger a |\beta\rangle$, which can easily be proven in the Fock basis representation. Plug this into the master equation:

$$\begin{aligned}\frac{d}{dt} \rho &= \frac{d}{dt} |\beta(t)\rangle \langle \beta(t)| = -|\beta\rangle \langle \beta| \frac{d}{dt} |\beta|^2 + \frac{\dot{\beta}}{\beta} a^\dagger a |\beta\rangle \langle \beta| + \frac{\dot{\beta}^*}{\beta^*} |\beta\rangle \langle \beta| a^\dagger a \\ &= -i\omega_0 a^\dagger a |\beta\rangle \langle \beta| + i\omega_0 |\beta\rangle \langle \beta| a^\dagger a + \kappa |\beta|^2 |\beta\rangle \langle \beta| - \frac{\kappa}{2} a^\dagger a |\beta\rangle \langle \beta| - \frac{\kappa}{2} |\beta\rangle \langle \beta| a^\dagger a\end{aligned}$$

This equation holds if

$$\frac{\dot{\beta}}{\beta} = -i\omega_0 - \kappa/2 \quad (\text{and} \quad -\frac{d}{dt} |\beta|^2 = \kappa |\beta|^2)$$

Here, the latter equation regards the magnitude of β . It follows directly from the first ODE:

$$\frac{d}{dt} |\beta|^2 = \dot{\beta} \beta^* + \beta \dot{\beta}^* = \frac{\dot{\beta}}{\beta} \beta \beta^* + \frac{\dot{\beta}^*}{\beta^*} \beta \beta^* = -\kappa |\beta|^2.$$

Hence, we find the solution $|\beta(t)\rangle$ with

$$\beta(t) = \beta_0 e^{-i\omega_0 t - \kappa t/2}$$

Accordingly, the Wigner function is given by a Gaussian of constant width and center point spiraling inwards towards the origin:

$$W^{p(t)}(\alpha) = \frac{2}{\pi} \exp \left(-2 \left| \beta_0 e^{-i\omega_0 t - \kappa t/2} - \alpha \right|^2 \right)$$

Initial state: cat state $\rho = |\psi\rangle \langle \psi|$ with $|\psi\rangle = (|\beta_1\rangle + |\beta_2\rangle)/\sqrt{N}$.

The proper normalization constant for the initial state is

$$N = 2(1 + \text{Re} \langle \beta_1 | \beta_2 \rangle) = 2(1 + e^{-|\beta_1 - \beta_2|^2/2})$$

Due to the coherent superposition of two coherent states, we cannot expect the density matrix to be of the form $\rho(t) = |\beta(t)\rangle \langle \beta(t)|$. Indeed, the initial state already has the structure

$$\rho(t=0) = \frac{1}{N} \left[|\beta_1\rangle \langle \beta_1| + |\beta_2\rangle \langle \beta_2| + |\beta_1\rangle \langle \beta_2| + |\beta_2\rangle \langle \beta_1| \right]$$

Since $\dot{\rho} = L\rho \Rightarrow \rho(t) = e^{Lt} \rho_0$ is linear, we can immediately make use of our previous solution — now with initial states $|\beta_i\rangle \langle \beta_i|$ ($i=1,2$). The new terms are $|\beta_1\rangle \langle \beta_2|$ and its hermitean conjugate. Let us attempt the ansatz

$$\sigma(t) = e^{Lt} |\beta_1\rangle \langle \beta_2| + \text{h.c.} \quad ? \quad f(t) |\beta_1(t)\rangle \langle \beta_2(t)| + \text{h.c.} \quad 74$$

In the following, temporarily drop the time dependence $\beta_i(t) \rightarrow \beta_i$ for simpler notation. Then, the ODE for the cross-terms only reads:

$$\begin{aligned}
\frac{d}{dt}\sigma(t) &= \dot{f}|\beta_1\rangle\langle\beta_2| + f\left(\frac{d}{dt}|\beta_1\rangle\right)\langle\beta_2| + f|\beta_1\rangle\left(\frac{d}{dt}\langle\beta_2|\right) + h.c. \\
&= \dot{f}|\beta_1\rangle\langle\beta_2| + f\left[-\frac{1}{2}\frac{d}{dt}|\beta_1|^2 + \frac{\beta_1^*}{\beta_1}a^\dagger a\right]|\beta_1\rangle\langle\beta_2| + \\
&\quad + |\beta_1\rangle\langle\beta_2|f\left[-\frac{1}{2}\frac{d}{dt}|\beta_2|^2 + \frac{\beta_2^*}{\beta_2}a^\dagger a\right] + h.c. \\
&= -i\omega_0 f a^\dagger a |\beta_1\rangle\langle\beta_2| + i\omega_0 f |\beta_1\rangle\langle\beta_2| a^\dagger a + \mathcal{K}\beta_1\beta_2^* f |\beta_1\rangle\langle\beta_2| \\
&\quad - \frac{1}{2}\mathcal{K}f a^\dagger a |\beta_1\rangle\langle\beta_2| - \frac{1}{2}\mathcal{K}f |\beta_1\rangle\langle\beta_2| a^\dagger a + h.c.
\end{aligned}$$

RHS of master equation

At this point, we restore our notation and make time dependence explicit again, i.e. $\beta_i \rightarrow \beta_i(t)$. We use β_i without time dependence for the initial superposition of the two coherent states $|\beta_1\rangle$ and $|\beta_2\rangle$. Comparison of coefficients in the above equation yields:

$$\therefore \dot{f}(t) = \frac{1}{2}f(t)\frac{d}{dt}(|\beta_1(t)|^2 + |\beta_2(t)|^2) + \mathcal{K}\beta_1(t)\beta_2^*(t)f(t) \quad \text{and } f(0)=1 \\ (\text{no other new equations})$$

$$\begin{aligned}
\Rightarrow \frac{\dot{f}(t)}{f(t)} &= \frac{d}{dt}(\ln f(t)) = e^{-\mathcal{K}t} \left[-\frac{1}{2}\mathcal{K}(|\beta_1|^2 + |\beta_2|^2) + \mathcal{K}\beta_1\beta_2^* \right] \\
\Rightarrow f(t) &= \exp \left[(1 - e^{-\mathcal{K}t}) \left(-\frac{1}{2}(|\beta_1|^2 + |\beta_2|^2) + \beta_1^*\beta_2 \right) \right] = \langle \beta_2 | \beta_1 \rangle^{(1 - e^{-\mathcal{K}t})}
\end{aligned}$$

In total, we thus obtain:

$$p(t) = \frac{1}{N} \left[|\beta_1(t)\rangle\langle\beta_1(t)| + |\beta_2(t)\rangle\langle\beta_2(t)| + f(t)|\beta_1(t)\rangle\langle\beta_2(t)| + f^*(t)|\beta_2(t)\rangle\langle\beta_1(t)| \right]$$

To compute the Wigner function $W(\alpha) = \frac{2}{\pi} \text{tr} [\mathcal{P} D(-\alpha) p D(\alpha)]$ of this, we make use of the previous expression for a coherent state. In addition, we need

$$\begin{aligned}
\text{tr} [\mathcal{P} D(-\alpha) |\beta_1\rangle\langle\beta_2| D(\alpha)] &= \text{tr} [\mathcal{P} |\beta_{-\alpha}\rangle\langle\beta_{-\alpha}|] e^{(-\alpha\beta_1^* + \alpha^*\beta_1)/2 + (-\alpha^*\beta_2 + \alpha\beta_2^*)/2} \\
D(\alpha)|\beta\rangle &= e^{(\alpha\beta^* - \alpha^*\beta)/2} |\alpha + \beta\rangle,
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} (-1)^n \langle n | \beta_1 - \alpha \rangle \langle \beta_2 - \alpha | n \rangle e^{\frac{1}{2}\alpha^*(\beta_1 - \beta_2) - \frac{1}{2}\alpha(\beta_1^* - \beta_2^*)} \\
&= e^{\frac{1}{2}(\dots)} \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} e^{-|\beta_1 - \alpha|^2/2 - |\beta_2 - \alpha|^2/2} (\beta_1 - \alpha)^n (\beta_2^* - \alpha^*)^n \\
&= \exp \left[\frac{1}{2}\alpha^*(\beta_1 - \beta_2) - \frac{1}{2}\alpha(\beta_1^* - \beta_2^*) - \frac{1}{2}|\beta_1 - \alpha|^2 - \frac{1}{2}|\beta_2 - \alpha|^2 - (\beta_1 - \alpha)(\beta_2^* - \alpha^*) \right] \\
&= \exp \left[-\frac{1}{2}|\beta_1|^2 - \frac{1}{2}|\beta_2|^2 - 2|\alpha|^2 \right] \exp \left[2\alpha^*\beta_1 + 2\alpha\beta_2^* - \beta_1\beta_2^* \right]
\end{aligned}$$

The resulting expression for the Wigner function is somewhat lengthy and I will refrain from giving it explicitly. It is straightforward to put everything together (compare Wigner function movie shown in lecture). The time evolution consists of the initial cat state with characteristic regions where $W^P(\alpha)$ is negative. As time progresses, the two gaussians spiral inward towards the origin. At the same time, coherences are lost as indicated by $W^F(\alpha)$ becoming positive everywhere. For small times $t \ll \kappa$, we find

$$f(t) = |\langle \beta_2 | \beta_1 \rangle^{(1-e^{-\kappa t})}| \simeq |\langle \beta_2 | \beta_1 \rangle^{\kappa t}| = e^{-|\beta_2 - \beta_1|^2 \kappa t}$$

$$\therefore \frac{T_2}{T_1} = \frac{(|\beta_2 - \beta_1|^2 \kappa)^{-1}}{1/\kappa} = \frac{1}{|\beta_2 - \beta_1|^2}$$

\Rightarrow For $|\beta_2 - \beta_1|^2 \gg 1$, dephasing is a lot faster than relaxation!

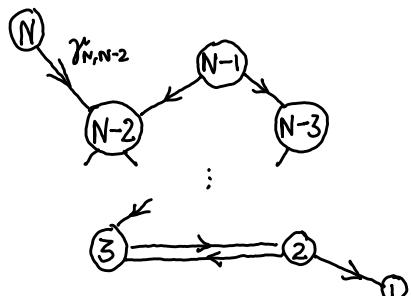
3. Unravelling the Master Equation: Quantum Trajectories

As it stands, the Lindblad master equation describes the dynamics (time dependent quantum state $\rho(t)$, expectation values of observables $\langle O(t) \rangle$) of an ensemble of quantum systems. What can we say about the dynamics of a single instance of a quantum system which is part of this ensemble?

This question will lead us to the concept of quantum trajectories, which obey a modified version of Schrödinger's equation including a stochastic term, i.e., a stochastic differential equation (SDE). Solutions of such a SDE are not deterministic but depend on the specific realization of the random terms entering the SDE. We will discuss how to generate such solutions, and how to use them for computing ensemble averages. Before delving into stochastic dynamics in Hilbert space, it is instructive to review the classical situation of a system described by rate equations and the unravelling in terms of a continuous-time Markov process.

3.1 Connection between Rate Equations and Continuous-Time Markov Processes

Consider a system which – at each instance of time –, may occupy one state n from a fixed set of available states $n \in \{1, 2, \dots, N\}$. Each state may connect to one or several other states $n \xrightarrow{\gamma_{nm}} m$, where γ_{nm} is the rate associated with this particular transition. A simple physics realization of such a system is the decay chain for a radioactive nucleus. The state space and associated transition rates can be visualized with a directed graph:



Note: bi-directional transitions do not occur for radioactive decay chains but do emerge for other types of processes.

The rate equations for this type of system take the form:

$$\frac{d}{dt} P_n = \underbrace{\sum_{m \neq n} \gamma_{nm} P_m}_{\text{"incoming" probability}} - \underbrace{\sum_{m \neq n} \gamma_{mn} P_n}_{\text{"outgoing" probability}} \quad (*)$$

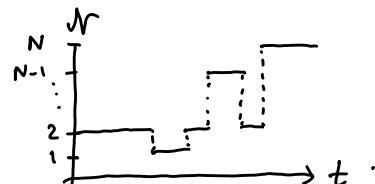


Under the assumption of the Markov property—loosely speaking, the behavior of the system at time t only depends on $\{P_m(t)\}_m$ at the same time t —one can show that $(*)$ is a special case of the Chapman-Kolmogorov equation. This equation governs the general behavior of classical continuous-time Markov processes.

The simulation of "sample paths" $N(t)$ of the Markov process is simple and is given as follows. The total decay rate for state $N=n$ is the sum of all individual decay rates: $\Gamma_n = \sum_{m \neq n} \gamma_{mn}$. If the system entered state $N=n$ at time $t=t_0$, then the next jump occurs at a random time $t_0+\tau$, where τ is exponentially distributed: $P_n(\tau) = \Gamma_n^{-1} \exp(-\Gamma_n \tau)$. The probability for the jump to a specific state m is given by its relative contribution to the decay rate: $P_{m \leftarrow n} = \gamma_{mn} / \Gamma_n$.

Accordingly, a sample path may look like :

By averaging over many sample paths, we recover the evolution of probabilities governed by the rate equation.



Remarks :

- Probabilities and ensemble averages can be computed by repeated generation of sample paths.
- The stochastic Markov process underlying the rate equations is a specific example of a piecewise deterministic process (PDP).

3.2 Piecewise Deterministic Processes

A stochastic process produces sample paths $X(t)$. For PDPs, such sample paths obey the stochastic differential equation

$$\underbrace{dX(t)}_{\substack{\text{increment} \\ \text{of } X}} = \underbrace{g(X(t))dt}_{\substack{\text{deterministic} \\ \text{contribution} \\ (\text{ordinary ODE})}} + \underbrace{dJ(X(t))}_{\substack{\text{jump term (not necessarily small!)}}} \quad (*)$$

In the case of rate equations, $g(x)=0$. In that case, there is a discrete set of possible states. Jumps switch the system from state to state. In between jumps "nothing happens". In the more general case described by (*), the dynamics in between jumps is governed by an ODE and $X(t)$ is not limited to discrete values but may take on any real value.

The crucial element that is new here (i.e., if stochastic differential equations are news to you) is the jump term $dJ(X(t))$. It is reasonable to expect that $dJ=0$ for most of the time (no jump). The resulting deterministic evolution is interrupted by instants of time where dJ is large, and thus produces a jump.

To give meaning to an equation of the type (*), there are two important questions we must answer:

1. What is the statistics of the jump times? How frequently do jumps occur, what is the distribution of waiting times?
2. How large are the jumps, what is the distribution of jump heights?

The answers to both questions lie in the conditional jump rates. Given an initial state $X=x_0$, let us assume that there is discrete set of states $\{z_\alpha = z_\alpha(x_0)\}$ that can be reached from x_0 . We will denote the rate for the jump to a specific state z_α by $W(z_\alpha|x_0)$. (This generalizes the rates γ_{mn} we encountered for rate equations.) The number of such jumps occurring

during the time interval $[t, t+dt]$ is given by

$$E[dN_\alpha(t)] = W(z_\alpha | x_0) dt$$

Here, $dN_\alpha(t)$ is an integer random variable. For sufficiently short dt , at most one jump will occur, so that $dN_\alpha(t) \in \{0, 1\}$ and

$$dN_\alpha(t) dN_\beta(t) = \delta_{\alpha\beta} dN_\alpha(t).$$

In these terms, the jump increments take the form

$$dJ(t) = \sum_\alpha [z_\alpha(X(t)) - X(t)] dN_\alpha(t).$$

With this, the stochastic differential equation for a PDP is fully determined and sample paths can easily be generated.

Our goal in the remaining sections will consist of demonstrating how open quantum systems can be described by a stochastic Schrödinger equation. The sample paths, in this case, are called quantum trajectories.

3.3 An Ensemble of Ensembles

— Probability Distributions in a Projective Hilbert Space

We already encountered the crucial idea behind quantum trajectories in Section 1.7. There, we saw that we may always explain the ensemble dynamics described by the Lindblad master equation in the following way. Time evolution under

$$\dot{\rho} = -\frac{i}{\hbar}[H, \rho] + \sum_{\mu>1} \gamma_\mu \mathbb{D}[L_\mu] \rho$$

corresponded to an operation of the form

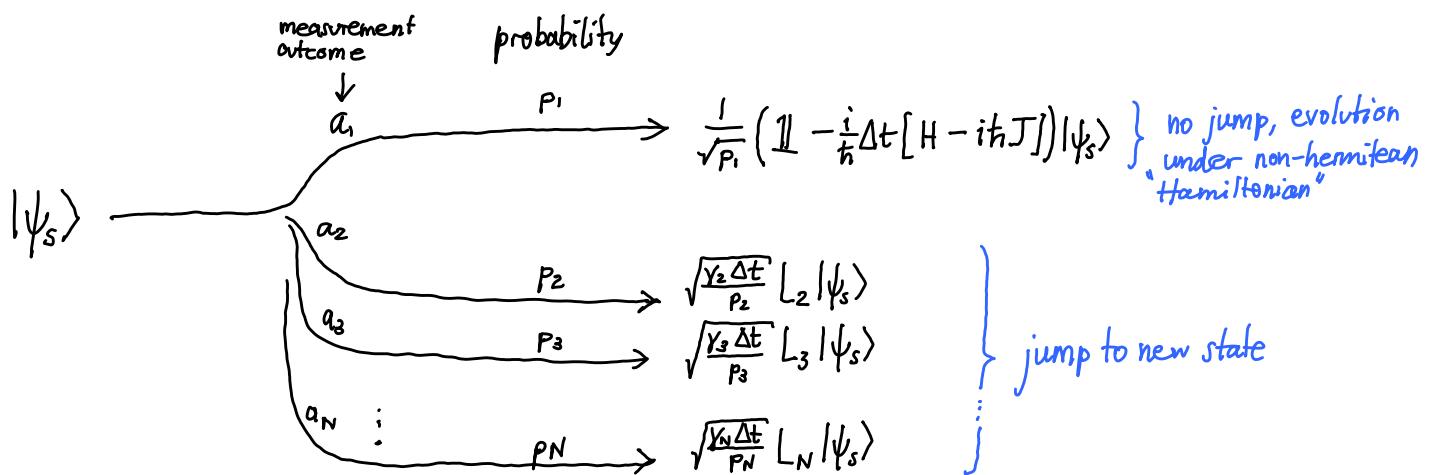
$$\rho(t+\Delta t) = \rho(t) + \Delta t \delta\rho + O(\Delta t^2) = \sum_\mu M_\mu \rho M_\mu^\dagger + O(\Delta t^2)$$

with $M_1 = \mathbb{I} - \frac{i}{\hbar} H \Delta t - \frac{1}{2} \sum_{\mu>1} \gamma_\mu L_\mu^\dagger L_\mu \Delta t$ and $M_\mu = \sqrt{\gamma_\mu \Delta t} L_\mu$ for $\mu>1$.

We then demonstrated that this exact operation could be achieved by coupling the system of interest to an auxiliary system (the "environment simulator") to which we applied an ideal projective measurement. Tracing over the environment then yielded the prescribed operation, i.e.

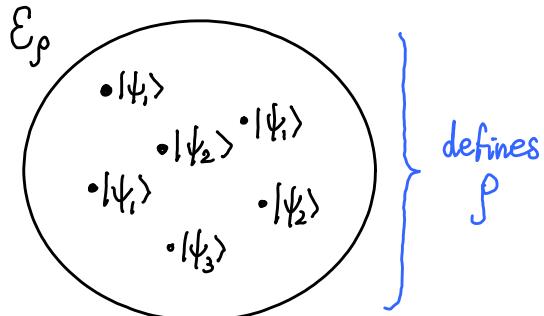
$$\rho(t+\Delta t) = \sum_{\mu} M_{\mu} \rho(t) M_{\mu}^+ = \text{tr}_E \rho_{\text{sys}+E}$$

In this sense, $\rho(t+\Delta t)$ corresponded to an unconditional post-measurement state. We noted that a different picture emerged when we worked with conditional post-measurement states instead. Namely, starting in a pure initial state $|\psi_s\rangle$ of the system, two distinct types of operations were possible:



Here, $J = -\frac{1}{2} \sum_{\mu>1} Y_{\mu} L_{\mu}^+ L_{\mu}$, $p_1 = 1 - \Delta t \sum_{\mu>1} Y_{\mu} \langle \psi_s | L_{\mu}^+ L_{\mu} | \psi_s \rangle$ and $p_{\mu} = Y_{\mu} \Delta t \langle \psi_s | L_{\mu}^+ L_{\mu} | \psi_s \rangle$. Keeping the information of individual measurement outcomes a_{μ} enables us to subdivide the full ensemble of post-measurement states into groups according to distinct a_{μ} . This leads to the "ensemble of ensembles" concept which is crucial for the development of quantum trajectories.

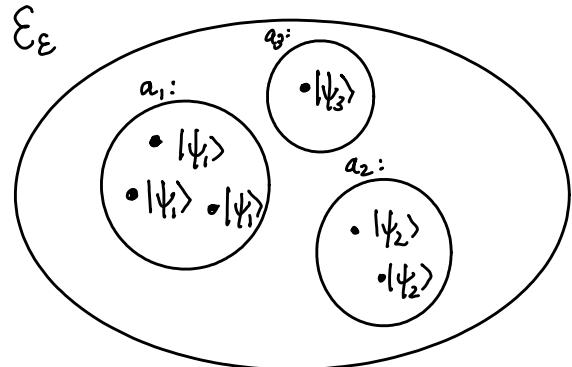
Ordinary ensemble of quantum states



Properties:

- unordered ensemble of states defines p
- specific ensemble for given p is not unique.

Ensemble of ensembles



Properties:

- for each measurement outcome there is an ensemble of conditional post-measurement states
- due to ordering into sub-ensembles, repeated measurements can determine each of the states $|\psi_1\rangle, |\psi_2\rangle$ etc.

The ensemble of ensembles concept allows us to associate probabilities with states in Hilbert space. Since $|\psi\rangle$ and $z|\psi\rangle$ ($z \in \mathbb{C}$) remain indistinguishable in this context, probabilities are associated with rays in Hilbert space (which make up the so-called projective Hilbert space). The probability for a certain subset $A \subset \mathcal{H}$ may be expressed in terms of a probability density functional $P[\psi]$:

$$\mu(A) = \int_A \mathcal{D}\psi \mathcal{D}\psi^* P[\psi].$$

Here, the right-hand side denotes a functional integral over states in the subset A . To avoid tedious notation "ket" character is suppressed here. The meaning of " $\mathcal{D}\psi \mathcal{D}\psi^*$ " will be explained next.

Example: Consider the Dirac measure $\mu_{\psi_0}(A) = \begin{cases} 1 & \text{if } \psi_0 \in A \\ 0 & \text{otherwise.} \end{cases}$

Its probability density functional is $P[\psi] = \delta[\psi - \psi_0]$ so that $\mu_{\psi_0}(A) = \int_A \mathcal{D}\psi \mathcal{D}\psi^* \delta[\psi - \psi_0]$.

Choose a fixed ONB $\{\phi_n\}$ of our Hilbert space \mathcal{H} . Every state $\psi \in \mathcal{H}$ can hence be decomposed as $\psi = \sum_n z_n \phi_n$ with $z_n = a_n + i b_n \in \mathbb{C}$.

The probability density can thus be interpreted as

$$P[\psi] = P[\{z_n\}, \{z_n^*\}] = P[\{a_n\}, \{b_n\}].$$

Functional integration thus implies integration over all a_n and b_n or z_n and z_n^* :

$$\mathcal{D}\psi \mathcal{D}\psi^* = \prod_n da_n db_n = \prod_n \frac{i}{2} dz_n d\bar{z}_n^*$$

In the last step, we have used $a_n = \frac{1}{2}(z_n + z_n^*)$, $b_n = \frac{1}{2i}(z_n - z_n^*)$ and

$$\prod_n da_n db_n = \det\left(\frac{\partial(a_n, b_n)}{\partial(z_n, \bar{z}_n^*)}\right) \prod_n dz_n d\bar{z}_n^*$$

Finally the proper definition of a probability distribution for rays $\bar{\psi}$ in projective Hilbert space may be achieved by bringing $P[\psi]$ into a form which uses normalized states $\|\psi\|=1$ and ignores phase factors:

$$P[\psi] = \delta(\|\psi\|-1) Q[\psi] \quad \text{with } Q[e^{i\chi}\psi] = Q[\psi]$$

Example: the proper Dirac distribution for projective Hilbert space is

$$P[\psi] = \int_0^{2\pi} d\chi \delta[\psi - e^{i\chi}\psi_0] \quad \text{where } \|\psi_0\|=1.$$

For any functional $F[\psi]$ — either complex-valued or operator valued —, we can use the distribution $P[\psi]$ to compute expectation values:

$$E(F[\psi]) = \int \mathcal{D}\psi \mathcal{D}\psi^* P[\psi] F[\psi]$$

- Examples:
- Expectation value of an observable A . Then $F[\psi] = \langle \psi | A | \psi \rangle$ and $E(F[\psi]) = \int \mathcal{D}\psi \mathcal{D}\psi^* P[\psi] \langle \psi | A | \psi \rangle$
 - The density matrix is obtained by choosing $F[\psi] = |\psi\rangle\langle\psi|$:
 $\rho = E(|\psi\rangle\langle\psi|) = \int \mathcal{D}\psi \mathcal{D}\psi^* P[\psi] |\psi\rangle\langle\psi|$.

3.4 Piecewise Deterministic Processes in Hilbert Space

We saw above: the system dynamics determined by the master equation and influenced by keeping a record of results from all indirect measurements fits neatly into the scheme of a PDP. To infer the explicit form of the SDE, recall:

- between jumps, the state evolves non-unitarily according to

$$\psi(t+dt) = \frac{1}{\sqrt{p_i}} \left(\mathbb{1} - i\hbar \frac{1}{2} \sum_{\mu>1} Y_\mu L_\mu^\dagger L_\mu \right) dt \psi(t) \quad (\text{no jump})$$

- jumps occur with probabilities $p_{\mu>1}$ and lead to

$$\psi(t+dt) = \sum_{\mu>1} \left(\frac{L_\mu \psi}{\|L_\mu \psi\|} - \psi \right) dN_\mu$$

To cast this into the standard form of a PDP, note that $p_i = 1 - \Delta t \underbrace{\sum_{\mu>1} Y_\mu \langle \psi_s | L_\mu^\dagger L_\mu | \psi_s \rangle}_x$
Hence, $\frac{1}{\sqrt{p_i}} \simeq \frac{1}{\sqrt{1-x}} \simeq 1 + \frac{1}{2}x = 1 + \frac{1}{2} \sum_{\mu>1} Y_\mu \|L_\mu \psi\|^2 \Delta t$

In total, we thus obtain

$$\begin{aligned} d\psi(t) &= \psi(t+dt) - \psi(t) = \\ &= -i\hbar \left(H + i\hbar \frac{1}{2} \sum_{\mu>1} Y_\mu L_\mu^\dagger L_\mu \right) \psi(t) dt + \frac{1}{2} \sum_{\mu>1} Y_\mu \|L_\mu \psi\|^2 \psi(t) dt + \sum_{\mu>1} \left(\frac{L_\mu \psi}{\|L_\mu \psi\|} - \psi \right) dN_\mu(t) \end{aligned}$$

Here, the Poisson increments satisfy

$$dN_\mu(t) dN_\nu(t) = \delta_{\mu\nu} dN_\mu(t)$$

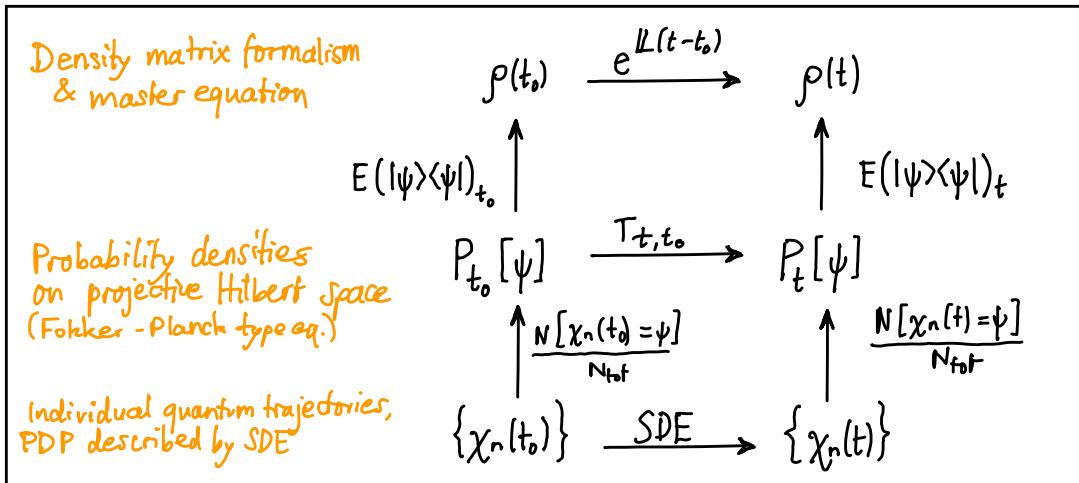
$$E[dN_\mu(t)] = \Gamma_\mu(t) dt = \frac{p_\mu}{dt} dt = Y_\mu \|L_\mu \psi(t)\|^2 dt$$

In the remainder of this chapter we will show

- how, precisely, this SDE is related to the master equation, and
- how to implement numerics for generating individual quantum trajectories.

Quantum trajectories "unravel" the master equation

The meaning of this statement can be succinctly summarized by the following diagram:



In other words: the density matrix at time t may always be obtained as the expectation value $E(|\psi\rangle\langle\psi|)_t$ with respect to the probability distribution $P_t[\psi]$ on projective Hilbert space. Unitary evolution of the density matrix is reflected by a transition probability density T_{t,t_0} which relates

$$P_t[\psi] = \int \mathcal{D}\bar{\psi} \mathcal{D}\bar{\psi}^* T_{t,t_0}[\psi|\bar{\psi}] P_{t_0}[\bar{\psi}].$$

Both $P_t[\psi]$ and $T_{t,t_0}[\psi|\bar{\psi}]$ can be extracted from averaging over many quantum trajectories (solutions of the SDE).

Numerical Implementation: Generation of Quantum Trajectories

Numerical solutions to the stochastic Schrödinger equation

$$d\psi(t) = g(\psi(t))dt + \sum_{\alpha} \left(\frac{L_{\alpha}\Psi}{\|L_{\alpha}\Psi\|} - \psi \right) dN_{\alpha}(t)$$

have to involve solving the deterministic equation $\frac{d\Psi}{dt} = g(\psi(t))$ which holds for time periods between jumps and the stochastic assignment of

jump times and jump targets.

(I) Deterministic Evolution

$$\frac{d\psi}{dt} = g(\psi(t)) = -\frac{i}{\hbar} \underbrace{\left(H + i\hbar \frac{1}{2} \sum_{\mu>1} \gamma_\mu L_\mu^\dagger L_\mu \right)}_{\text{non-hermitean "Hamiltonian"} \tilde{H}} \psi(t) + \underbrace{\frac{1}{2} \sum_{\mu>1} \gamma_\mu \langle \psi | L_\mu^\dagger L_\mu | \psi \rangle}_{\text{non-linear term}} \psi(t)$$

The first term on the r.h.s. resembles what we would expect from a Schrödinger equation. The operator \tilde{H} playing the role of the ordinary Hamiltonian lacks an important property: \tilde{H} is not hermitean. In a way this is to be anticipated. Namely, the absence of jumps reveals partial information about the system's quantum state. A weak measurement like this must cause evolution that is not fully reversible — and thus cannot be unitary. Indeed $e^{-i\tilde{H}t/\hbar}$ is not unitary if \tilde{H} is not hermitean.

The non-linear nature of the ODE brought on by the second term on the r.h.s. of our ODE might alarm us. Fortunately, it turns out that the non-linear term has a very specific form which merely ensures that $\psi(t)$ remains normalized to unity. The lack of hermiticity for H leads to a departure from $\langle \psi | \psi \rangle = 1$. The non-linearity reinstates normalization. To see this, let us make the ansatz

$$\psi(t) = \frac{\exp(-i\tilde{H}t) \psi_0}{\| \exp(-i\tilde{H}t) \psi_0 \|} = \exp(-i\tilde{H}t) \psi_0 \langle \psi_0 | e^{+i\tilde{H}t} e^{-i\tilde{H}t} | \psi_0 \rangle^{-1/2}$$

(here and in the following, $\hbar=1$)

Taking the time derivative, we obtain:

$$\begin{aligned}
 \frac{d\psi}{dt} &= -i\tilde{H}\psi(t) - \frac{1}{2} \exp(-i\tilde{H}t) \psi_0 \langle \psi_0 | e^{i\tilde{H}t} e^{-i\tilde{H}t} | \psi_0 \rangle^{-3/2} \frac{d}{dt} \langle \psi_0 | e^{i\tilde{H}t} e^{-i\tilde{H}t} | \psi_0 \rangle \\
 &= -i\tilde{H}\psi(t) - \frac{1}{2} \psi(t) \underbrace{\langle \psi_0 | e^{i\tilde{H}t} e^{-i\tilde{H}t} | \psi_0 \rangle^{-1}}_{\text{dashed line}} \underbrace{\langle \psi_0 | e^{i\tilde{H}t} (i\tilde{H}^+ - i\tilde{H}) e^{i\tilde{H}t} | \psi_0 \rangle}_{\text{dashed line}} \\
 &= -i\tilde{H}\psi(t) + \frac{1}{2} \sum_{\mu>1} \gamma_\mu \langle \psi | L_\mu^\dagger L_\mu | \psi \rangle \psi(t).
 \end{aligned}$$

Thus, the ansatz indeed solves the nonlinear equation $\frac{d\psi}{dt} = G(\psi(t))$.

From the steps above, we also gather:

$$\begin{aligned}
 \|e^{-i\tilde{H}t}\psi_0\|^{-2} \frac{d}{dt} \|e^{-i\tilde{H}t}\psi_0\|^2 &= - \sum_{\mu>1} \gamma_\mu \langle \psi | L_\mu^\dagger L_\mu | \psi \rangle \\
 \Leftrightarrow \frac{d}{dt} \|\tilde{\psi}\|^2 &= - \|\tilde{\psi}\|^2 \sum_{\mu>1} \gamma_\mu \|L_\mu \psi\|^2
 \end{aligned}$$

From this, we conclude that the norm of $\tilde{\psi}$ ($\tilde{\psi} = e^{-i\tilde{H}t}\psi_0$ is un-normalized!) is monotonically decreasing.

The numerical solution of the deterministic equation of motion, most straight-forwardly, proceeds by solving

$$\frac{d}{dt} \tilde{\psi} = -i\tilde{H}\tilde{\psi} \quad (\text{linear ODE})$$

and normalizing $\tilde{\psi}$ manually.

(2) Jumps

For a given (normalized) state $\psi(t)$, the instantaneous total jump rate is the sum of rates from all possible channels:

$$\begin{aligned}
 \Gamma(\psi(t)) &= \sum_{\mu>1} \Gamma_\mu(\psi(t)) = \sum_{\mu>1} \gamma_\mu \langle \psi | L_\mu^\dagger L_\mu | \psi \rangle = - \|e^{-i\tilde{H}t}\psi_0\|^{-2} \frac{d}{dt} \|e^{-i\tilde{H}t}\psi_0\|^2 \\
 &= - \frac{d}{dt} \ln \|e^{-i\tilde{H}t}\psi_0\|^2
 \end{aligned}$$

Let us assume a jump has occurred at $t=0$. Further let

$P(t)$ = probability that no second jump occurred in $[0, t]$.

$P(t)$ must obey the differential equation

$$\underbrace{\frac{dP(t)}{dt}}_{\text{change in } P \text{ from } t \rightarrow t+dt} = - \underbrace{\Gamma(t) P(t)}_{\substack{\text{jump probability} \\ \text{decrease}}} \quad \therefore \frac{dP}{dt} = - \Gamma(t) P(t)$$

probability that no jump occurred until t

$$\Rightarrow \frac{dP}{P} = - \Gamma(t) dt \quad \therefore \ln P = - \int_0^t \Gamma(\tau) d\tau \quad \therefore P(t) = \exp\left(- \int_0^t \Gamma(\tau) d\tau\right)$$

(a) $\Gamma(t) = \Gamma = \text{const.}$

Generate jump time by drawing t from the distribution

$$f_0(t) = \Gamma e^{-\Gamma t} \quad (f_0(t) dt = \text{probability for jump in } [t, t+dt])$$

Computers typically provide random numbers with uniform distribution between 0 and 1. To convert to a desired other distribution use:

$$f_0(t) dt = f_0(t) \frac{dt}{dx} dx = u(x) dx \quad u(x) = \begin{cases} 1 & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow f_0(t) \stackrel{!}{=} \frac{dx}{dt} = \Gamma e^{-\Gamma t} \quad \therefore x(t) = -e^{-\Gamma t} + 1$$

$$t = -\frac{1}{\Gamma} \ln(1-x). \quad (*)$$

I.e., draw x from $[0, 1]$ uniformly and obtain t via the last expression $(*)$.

(b) Time dependent $\Gamma(t)$

Now, the distribution of jump times is

$$f_0(t) = \Gamma(t) \exp\left[- \int_0^t \Gamma(\tau) d\tau\right].$$

With $\Gamma(t) = -\frac{d}{dt} \ln \|\exp(-i\tilde{H}t)\psi_0\|$, this reduces to

$$f_0(t) = \Gamma(t) \|\exp(-i\tilde{H}t)\psi_0\| = -\frac{d}{dt} \|\exp(-i\tilde{H}t)\psi_0\|$$

Employing the same strategy as in (a) to map from a uniform distribution, we obtain the relation

$$x = -\|\exp(-i\tilde{H}t)\psi_0\| + 1.$$

As a result, we can extract the next waiting time by

- ① Draw $x \in [0, 1]$ (uniform).
- ② While integrating the deterministic ODE, monitor the norm $\|\exp(-i\tilde{H}t)\psi_0\|$. Once it reaches $1-x$, induce a jump.
- ③ If multiple jump channels exist, determine target by another $x' \in [0, 1]$ and relative probabilities

$$P_x = \frac{\Gamma_\mu(t)}{\Gamma_{\text{tot}}(t)}.$$

Example: Driven Two-Level System with Relaxation ($T > 0$)

Hamiltonian: $H = \epsilon\sigma^+\sigma^- + \Omega\sigma_x$

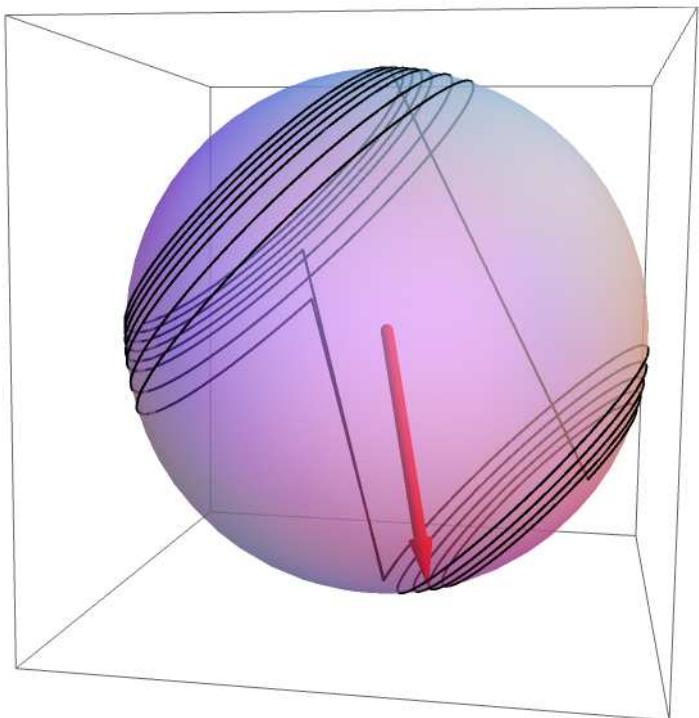
Master equation: $\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] + \gamma D[\sigma^-]\rho + \gamma e^{-\beta\epsilon} D[\sigma^+]\rho$

SDE:

$$\frac{d\psi(t)}{dt} = G(\psi(t)) + dN_+(t) \left(\frac{\sigma^+\psi}{\|\sigma^+\psi\|} - \psi \right) + dN_-(t) \left(\frac{\sigma^-\psi}{\|\sigma^-\psi\|} - \psi \right)$$

Deterministic evolution (linear part of G)

$$\frac{d\tilde{\psi}}{dt} = -i\tilde{H}\tilde{\psi} = -i \left(\epsilon\sigma^+\sigma^- + \Omega\sigma_x - \frac{i}{2}\gamma\sigma^+\sigma^- - \frac{i}{2}\gamma e^{\beta\epsilon}\sigma^-\sigma^+ \right) \tilde{\psi}$$



Bloch sphere representation of a quantum trajectory for the driven two-level system with relaxation (at $T>0$).