midterm grades

CSC263 Wednesday Lecture Mar 7, 2018

- Review of BFS
  - Queue: out ← [ ][ ] to be explored [ ] ← in: newly discovered nodes
  - COLOUR[v] = WHITE : undiscovered
    = GREY : discovered, unexplored
    = BLACK : explored

* Today's topic: Proofs on graph algorithms
  - PROVE: BFS produces shortest paths

- Recall that during BFS, a "BFS discovery tree" is constructed
  - S : starting node is root of BFS d t
  - path from S to node is discovery path

- v's discovery path from S: $s \to v_1 \to v_2 \to \dots \to u \to v$
  distance of discovery path: $d(v) = d(u) + 1$
- v's shortest path from S: $s \to v_1 \to v_2 \to \dots \to v_k \to v$
  length of shortest path $s \to v$: $\delta(s,v)$

- Lemma 0: $\forall v \in V$, $d[v] \geq \delta(s,v)$ // trivial
* Theorem to show: after BFS(G, S), $\forall v \in V$ $d[v] = \delta(s,v)$
- Note: shortest path is not unique, there may be multiple paths with same shortest length. Thus, d[v] is a shortest path, not the shortest. However length of shortest path is unique
- Lemma 1: If u enters Q before v does, then $d[u] \leq d[v]$ //CLRS 22.4

Proof:
- suppose by contradiction that L1 is false. Let v be the first node that enters Q such that $d[u] > d[v]$ for some u that entered Q before v. (i.e. v is first node that violates the lemma).
- this node v cannot be S because something entered before v
- node u cannot be S because then $d[u] = 0 > d[v]$ requires negative d[v]
- let u and v be discovered by v' and u' ∴ $d[v] = d[v'] + 1$, $d[u] = d[u'] + 1$
- $(u' \neq v')$ because $(d[v] \neq d[u]) \to (d[v'] \neq d[u'])$

... continued

- u' was explored before v' because u in Q before v in Q
- then u' entered Q before v' entered Q
- because v is first node in Q that violates L1, we know L1 holds for u', v'; that is $d[u'] \leq d[v']$
- see that $d[u'] \leq d[v'] \rightarrow d[u']+1 \leq d[v']+1$
  $\rightarrow d[u] \leq d[v]$ ... but this violates our assumption that $d[v]$ violates L1 $(d[u] > d[v])$ $\therefore$ contradiction

$\therefore$ no node in BFS discovery tree that violates L1 ☐

* REFERENCE: Handout outlining following proof online:

Prove the theorem:

- suppose by contradiction that $\exists x \in V$ that violates theorem, that is, $\exists x \in V \mid d[x] \neq \delta(s,v)$. May be many such nodes.
- let v be closest node from s such that $d[v] \neq \delta(s,v)$

A • via L0, if $[d[v] \neq \delta(s,v)] \longrightarrow [d[v] > \delta(s,v)]$

? • suppose there is some shortest path from s to v, (u is the last node before v). Then length of this path $= \delta(s,v)$; $\therefore$ length of path from s to $u = \delta(s,u)$ because otherwise would be shorter from s to v.

C • $\therefore$ by definition of v: all ancestors of v satisfy $\boxed{d[u] = \delta(s,u)}$

B • via definition of our shortest path, $\boxed{\delta(s,v) = \delta(s,u)+1}$

- by A: $d[v] > \delta(s,v) \longrightarrow$ by B $d[v] > \delta(s,u)+1$

(*) $\longrightarrow$ by C: $\boxed{d[v] > d[u]+1}$

➤ consider the colour of v just before u is explored:

- case (i) v is WHITE $\longmapsto$ u will discover v $\therefore d[v] = d[u]+1$
  this contradicts (*) $\therefore$ it is impossible v is white
- case (ii) v is BLACK $\longmapsto$ then v in Q before u, by L1 $d[v] \leq d[u]$
  this contradicts (*) $\therefore$ it is impossible v is BLACK
- case (iii) v is GREY $\longmapsto$ $\exists w \neq u$ $P[v] = w$ $\therefore$ w in Q before u
  $\therefore$ by L1 $d[w] \leq d[u]$ $\therefore d[w]+1 \leq d[u]+1 \equiv d[v] \leq d[u]+1$
  this contradicts (*) $\therefore$ it is impossible v is grey
- then v does not exist $\therefore \forall v \in V, d[v] = \delta(s,v)$

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