WINTER 2020 WRITEUP

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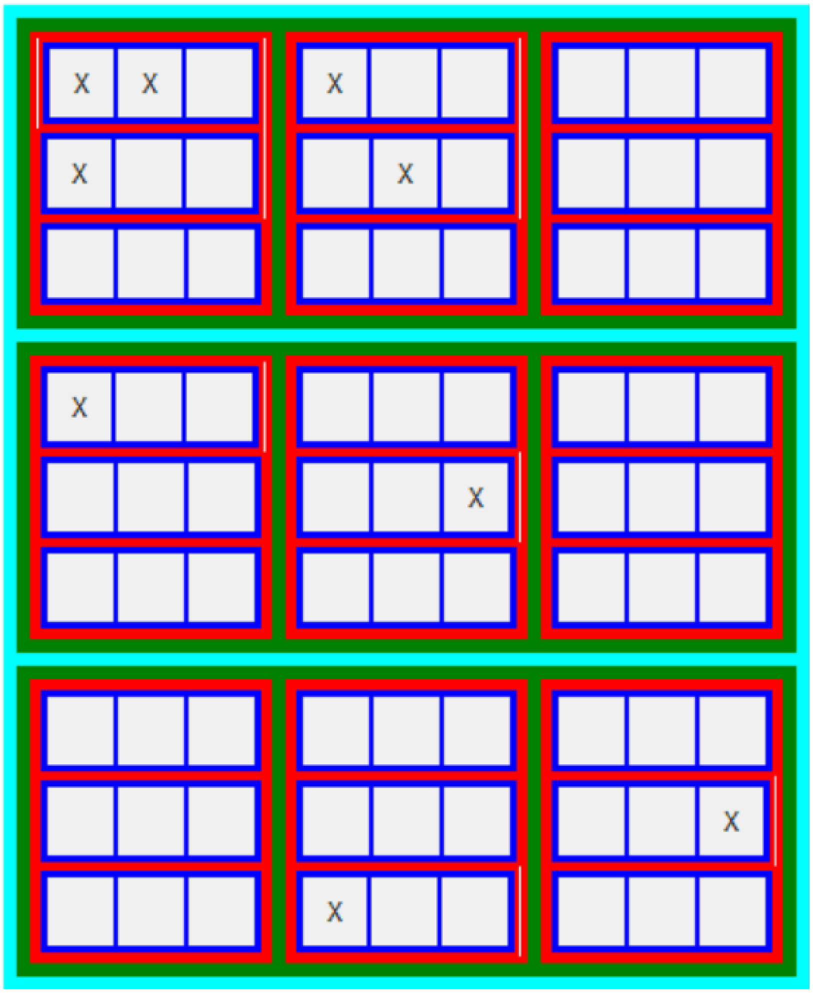
Introduction

The goal this quarter was to determine whether or not all maximal 2-caps in the affine space $\mathbb{F}_3^4$ are affinely equivalent. To do so we used code written by Jaron Wang, a past WXML researcher, as well as code we wrote this quarter. The conclusion we reached is that all 2-caps that we are interested in are indeed affinely equivalent.

2-caps

A 2-cap is any set of points in $\mathbb{F}_3^n$ such that no three points are collinear and no four points are coplaner. A maximal 2-cap is a 2-cap with as many points as possible. In $\mathbb{F}_3^4$ maximal 2-caps have size 9. An example of a maximal 2-cap in the space we are interested is given below.



Python Code

The first step in solving this problem was generating a list of maximal 2-caps. This was done using Jaron Wang's code, and the generated list includes all maximal 2-caps that include the vectors $(0,0,0,0),(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)$. This list does not include nearly all of the 2-caps, but it can be proven that all possible maximal 2-caps are affinely equivalent to at least one of the 13 2-caps on this list. The proof of this is given in a later section. Since all maximal 2-caps are equivalent to a 2-cap on this list, it's sufficient to prove that all the 2-caps in our list is equivalent.

We needed to prove that all 2-caps on this list are affinely equivalent to one another. To do this, we found 12 affine transformations, each of which is invertible and transforms the first 2-cap on the list to one of the other 2-caps on the list. It turns out that all of these affine transformations only needed to be linear transformations, so each one of them is just a 4×4 matrix. To find these matrices we first generated a list of all possible invertible matrices. We then looped over that list. On every iteration we applied the transformation encoded by the matrix to the first 2-cap on our list of 2-caps and checked if the resulting 2-cap was equal to any of the 2-caps on our list. If it was, we deleted the 2-cap that it was equal to from the list, and we continued this process until there were no more 2-caps on the list. From there it follows that every 2-cap on the list is affinely equivalent to every other 2-cap on the list. For example, if we wanted the affine transformation that sends the third 2-cap to the sixth 2-cap on the list, we can take the inverse of the transformation from the first 2-cap to third 2-cap and then multiply it by the transformation from the first to sixth. In this way we can come up with a transformation between any two 2-caps on the list.

Proofs

Denote $A = \{(0,0,0,0),(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$.

Lemma. *A maximal 2-cap in $\mathbb{F}_3^4$ must contain an affine basis of $\mathbb{F}_3^4$.*

Proof. Suppose for contradiction that there exists a maximal 2-cap B such that B does not contain an affine basis for $\mathbb{F}_3^4$. Since B does not contain a basis for $\mathbb{F}_3^4$, all of the vectors in B are contained an a lower dimensional subspace. Since the maximal size of a 2-cap in $\mathbb{F}_3^4$ is nine and nine is greater than the size of a maximal 2-cap in any lower dimension, B cannot be contained in a lower dimensional subspace, which is a contradiction, so B cannot exist. This means that every maximal 2-cap in $\mathbb{F}_3^4$ must contain a basis for $\mathbb{F}_3^4$.

□

Theorem. *All maximal 2-caps in $\mathbb{F}_3^4$ are affinely equivalent to at least one 2-cap on the list generated by flat-elim-search.*

Proof. We want to show that every possible maximal 2-cap is affinely equivalent to at least one of these 2-caps. Since any maximal 2-cap in $\mathbb{F}_3^4$ contains an affine basis for $\mathbb{F}_3^4$, and since any basis is affinely equivalent to any other basis, any maximal 2-cap is affinely equivalent to some maximal 2-cap with basis A. Since we are given all maximal 2-caps with basis A, any maximal 2-cap is affinely equivalent to one of the 2-caps generated by flat-elim-search. □