§13.1 Vector functions and space curves

Vector function:

$$r(t) = \langle f(t), g(t), h(t) \rangle$$

Let

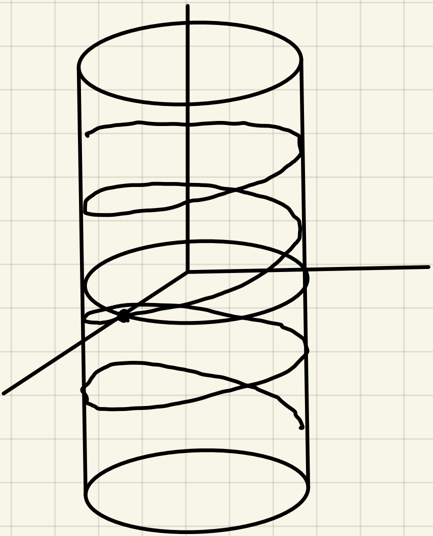
$$C = \{ (x, y, z) \in \mathbb{R}^3 : (x, y, z) = (f(t), g(t), h(t)) \text{ for some } t \}$$

Then C is called the space curve def. by the vector function $r(t) = (f(t), g(t), h(t))$.

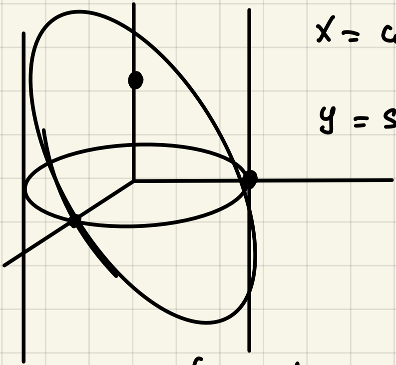
$\boxed{\text{Example}}$ $r(t) = \langle \cos t, \sin t, t \rangle$

$$\begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases} \quad \forall t \in \mathbb{R} \qquad \text{Helix}$$

parametric equation of Helix.



$\boxed{\text{Example}}$ Find the parametric equation of the curve of the intersection of cylinder $x^2+y^2=1$ and the plane $x+y+z=1$



$x = \cos t$

$y = \sin t$

$z = 1 - x - y = 1 - \cos t - \sin t$

So the parametric equation of the intersection is

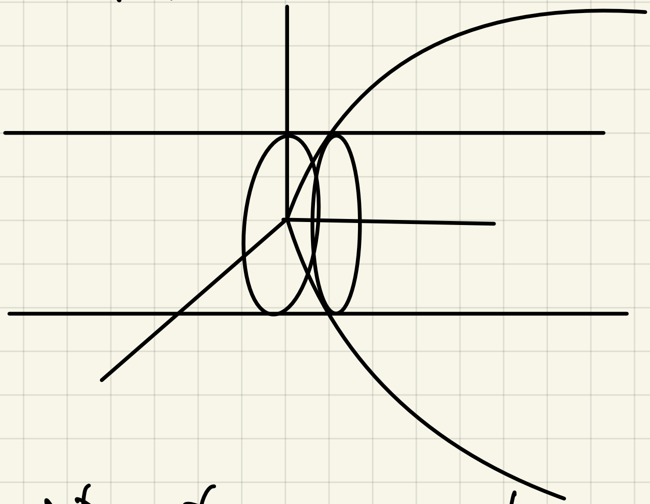
$$\begin{cases} x = \cos t \\ y = \sin t \\ z = 1 - \cos t - \sin t \end{cases} \quad \forall t \in \mathbb{R}.$$

Example

Paraboloid $y = x^2 + z^2$

Cylinder $x^2 + z^2 = 4$

Find the parametric equation of the intersection.



The intersection is a circle.

$x = 2\cos t$

$z = 2\sin t$

$y = x^2 + z^2 = 4\cos^2 t + 4\sin^2 t = 4$

So the equation for the intersection is

$$\begin{cases} x = 2\cos t \\ y = 4 \\ z = 2\sin t \end{cases}$$

§13.2. Derivatives and integrals

Def The derivative of a vector funct.

$$r(t) = \langle f(t), g(t), h(t) \rangle$$

with differentiable functions f, g, h is defined by

$$r'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

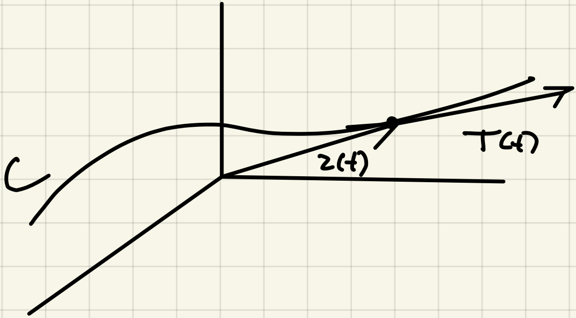
The vector $r'(t)$ is the tangent to

the curve C represented by $r(t)$.

the unit tangent is given by

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

provided that $r'(t) \neq 0$.



Example

$$\begin{cases} x = 2\cos t \\ y = 4 \\ z = 2\sin t \end{cases}$$

Find $T(0)$.

$$r'(t) = \langle -2\sin t, 0, 2\cos t \rangle$$

$$r'(0) = \langle 0, 0, 2 \rangle$$

$$|r'(0)| = \sqrt{0^2+0^2+2^2} = 2$$

$$T(0) = \frac{r'(0)}{|r'(0)|} = \frac{1}{2}\langle 0, 0, 2 \rangle$$

$$= \langle 0, 0, 1 \rangle$$

Differentiation laws:

Let $u(t)$ and $v(t)$ be two vector functions and c and $f(t)$ be, respectively, a scalar and a scalar function.

Then

1. $[u(t)+v(t)]' = u'(t)+v'(t)$

2. $[c\,u(t)]' = c\,u'(t)$

3. $[f(t)\,u(t)]' = f'(t)\,u(t)+f(t)\cdot u'(t)$

4. $[u(t)\cdot v(t)]' = u'(t)\cdot v(t)+u(t)\cdot v'(t)$

5. $[u(t)\times v(t)]' = u'(t)\times v(t)+u(t)\times v'(t)$

6. $[u(f(t))]' = f'(t)\,u'(f(t))$.

<u>Def</u> Let $r(t) = \langle f(t), g(t), h(t)\rangle$ $t\in[a,b]$, where f, g, h are continuous on $[a,b]$. Then the integral of $r(t)$ is defined by

$$\int_a^b r(t)\,dt = \langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt\rangle$$

Example Let

$r(t) = \langle 2\cos t, 4, 2\sin t \rangle, \quad t \in [0, \pi]$

$$\int_0^{\pi} r(t)\,dt = \left\langle \int_0^{\pi} 2\cos t\,dt, \int_0^{\pi} 4\,dt, \int_0^{\pi} 2\sin t\,dt \right\rangle$$

$$= \left\langle 2[\sin t]_0^{\pi}, 4\pi, [-2\cos t]_0^{\pi} \right\rangle$$

$$= \langle 0, 4\pi, 4 \rangle$$

§ 13.3 Arc length and curvature

Let C be a curve with param. equation $r(t) = \langle f(t), g(t), h(t) \rangle$

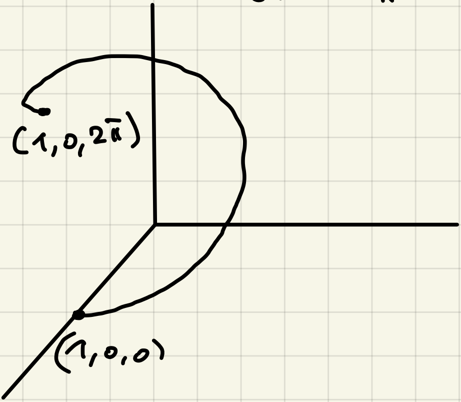
$t \in [a, b]$

It can be shown that the lenght of $C_{a,b}$ is given by

$$L = \int_a^b |z'(t)|\, dt = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'^2(t)}\, dt$$

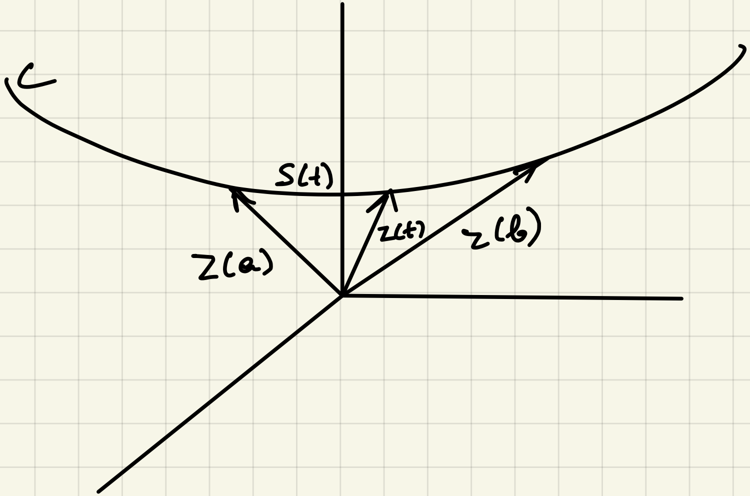
Example | Helix

$$z(t) = \langle \cos t, \sin t, t \rangle$$
$$0 \leq t \leq 2\pi$$



$$L = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}\, dt = \int_0^{2\pi} \sqrt{2}\, dt$$

$$= 2\sqrt{2}\pi.$$



$z(t),\ t \in [a,b]$

In this way we can define a lenght function $s(t)$.

$$s(t) = \int_a^t |z'(u)|\,du \qquad (\mathrm{I})$$

By the fundamental thm of calculus, for any continuous function $q(u)$, we have that

$$Q(t) = \int_a^t q(u)\,du$$ then

$$Q'(t) = q(t).$$

Thus (1) implies that

$$s'(t) = \frac{ds}{dt} = |\gamma'(t)| \quad (2)$$