Notation

A finite normal-form game with pure strategies is defined as a tuple

$$G = \big(N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}\big),$$

where:

- N is a finite set of players (e.g., $P_1, P_2, \ldots, P_n$);
- For each $i \in N$, S_i is the finite set of *pure strategies* available to player i;
- After the definition of pure strategies, note that player i's mixed-strategy space is
$$\Delta(S_i) = \big\{\, p_i : S_i \to [0,1] \;\big|\; \sum_{s_i \in S_i} p_i(s_i) = 1 \big\}.$$
- For each $i \in N$, the payoff function u_i is defined as
$$u_i : S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R}.$$

A pure strategy profile is denoted by

$$s = (s_1, s_2, \ldots, s_n), \quad s_i \in S_i,$$

and we write $s = (s_i, s_{-i})$ where s_{-i} denotes the strategies of all players except i.

Linking Preferences, Strategies, and Payoffs

1. **Preferences:** Each player i has a preference relation $\succeq_i$ over the set of outcomes, assumed to be complete and transitive.
2. **Utility Representation:** By the von Neumann–Morgenstern utility theorem [2], there exists a function
$$v_i : X \to \mathbb{R}$$
such that, for all $x, y \in X$,
$$x \succeq_i y \quad \Longleftrightarrow \quad v_i(x) \geq v_i(y).$$
3. Moreover, v_i is unique up to positive affine transformations.
4. **Mapping Strategies to Outcomes:** The game form
$$\gamma : S_1 \times \cdots \times S_n \to X$$
assigns an outcome to each strategy profile.
5. **Payoff Functions:** Player i's payoff is then
$$u_i(s) = v_i\big(\gamma(s)\big).$$

Nash Equilibrium in a 2×2 Matrix

Consider

$$S_1 = \{T, B\}, \quad S_2 = \{L, R\}.$$

A payoff matrix is

	L	R
T	(a, b)	(c, d)
B	(e, f)	(g, h)

A pure strategy profile $s^* = (s_1^*, s_2^*)$ is a *Nash equilibrium* [1] if, for each player i,

$$u_i(s_i^*, s_{-i}^*) \;\geq\; u_i(s_i, s_{-i}^*) \quad \text{for all } s_i \in S_i.$$

Remark on existence: While pure-strategy equilibria need not exist in all games, a mixed-strategy equilibrium always exists by Kakutani's fixed-point theorem.

Application 1: A 2×2 Game with a Unique Nash Equilibrium

Consider the payoff matrix

	L	R
T	(x_1, y_1)	(x_2, y_2)
B	(x_3, y_3)	(x_4, y_4)

Impose:

a. $x_1 > x_3$ and $x_2 > x_4$.

b. $y_1 > y_2$ and $y_3 > y_4$.

These imply Player 1 strictly prefers T and Player 2 strictly prefers L, so the unique Nash equilibrium is

$$s^* = (T, L).$$

References

[1] John F. Nash. Non-cooperative games. *Annals of Mathematics*, 54(2):286–295, 1951.

[2] John von Neumann and Oskar Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, 3rd edition, 1944.