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Abstract

There is a broad societal interest in the stability of financial markets. Individual investors, firms and regulators all deal with risk and need mathematical ways of quantifying the probability and the consequences of adverse development in financial markets. The aim of this project is to describe how risk is managed in the financial industry. To do that we are going to define and compare two of the most common risk measures used in the financial industry today, Value-at-Risk (VaR_α) and Expected shortfall (ES_α). We show how they are related. Using the concept of a coherent risk measure we also show why ES_α is theoretically preferable to VaR_α. In the empirical part of the project we measure the risk for 6 portfolios by estimating capital reserve requirements. The portfolios are constructed using the real world securities, the S&P500, Disney, Boeing and 1 year U.S. Treasury bonds. Using simulation with stochastic stock prices and stochastic interest rates we are going to test the reserve capital requirement estimated using the risk measures. Since the COVID-19 pandemic and the project coincided we also got to test the measures of risk on a real world financial downturn.

1 Introduction

Historic context and motivation

Financial markets are essential in a modern economy. Financial markets are markets where people who have capital can meet people who need capital. The exchange allows one party to borrow the capital from the other in exchange for a promise of future payment and a fee called interest. This allows the consumer to push consumption (spending of income) forward or backwards in time. Examples of this is saving for retirement. One might wish to not spend all income one earns now, but rather when one is retired and is not earning an income. To make the markets run more smoothly there are financial institutions. A bank for example takes a small fee to match the capital from the people who wish to save, with the people who want to borrow it. A different aspect of financial markets is that they allow us to share the risk of some accident. The probability of something bad happening might be low. However if an accident does occur it is going to be expensive. This allows a different sort of financial institutions, insurance companies, to share some of the risk over a larger group of people.

These financial institutions are usually firms. Therefore they can go bankrupt. This can have serious consequences. The consumer might loose their savings. A shareholder looses his equity in the financial firm. A regulator might be concerned with the activity in the economy slowing down when the access to credit or saving disappears. For all these reasons and more there is an interest in making sure the financial markets are stable [7]. Therefore it is beneficial to develop some mathematical tools to measure this risk.

One way to make sure the markets are stable is through regulations. Since Norway is part of the EEC EU regulation is of interest to us. To improve the stability of European financial institutions the EU has instituted Basel III for banking and Solvency II for insurance. The regulation framework has a goal to

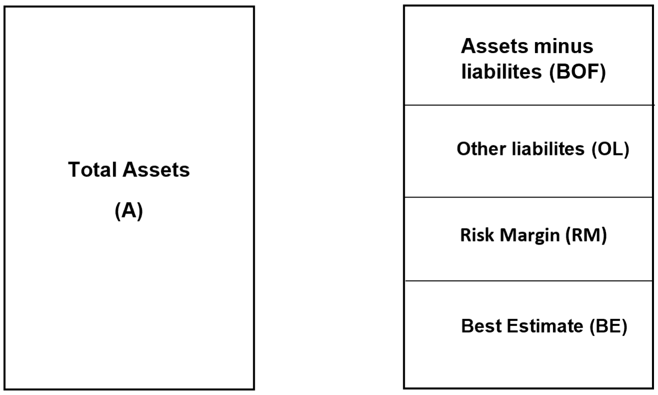


Figure 1: Balance sheet for an insurance company according to Solvency II, [11]

standardize a framework for regulating banking and insurance in Europe. To keep things simple we are concentrating on Solvency II.

Solvency II is a regulatory framework that is meant to regulate the capital reserves an insurance company has to have in order to operate. The goal is to make sure that the insurance company has enough liquid (easily sold) assets such as to be able to meet liabilities (payouts). Since if the liabilities are greater than the assets the company goes bankrupt.

To give an overview. Solvency II regulates the balance sheet of an insurance company. The insurer has liabilities and assets to cover them. To make sure the insurer can meet those liabilities the legislation mandates that the assets have to have value such that they can cover the liabilities in one year with a probability 99.5.

Figure 1 shows a general breakdown of the balance sheet of an insurance company. The assets on the left are generally the value of a portfolio of shares, bonds, real estate or may even be held in cash. All income from premiums and so on goes into the assets. The liabilities are even more complicated.

Best estimate is the best estimate of the liabilities the company has to pay out in the coming year. This is usually the largest post and represents the expected discounted value of all future cash flow from contracts. This is everything from claims being liquidated (payed out) to reinsurance premiums. This expectation should be interpreted as the expected value over some underlying stochastic model. Since contracts can have very long durations, up to seven or eight decades. There are very few risk free investments with such a long duration. Therefore one often needs to use stochastic interest rate models to find the estimate.

The risk margin in the price another company would have to be paid to take over the obligations. Since actors in a financial market are risk averse this is a positive number. This risk aversion means that the company taking over the liabilities would have to be payed some amount to assume the risk.

Other Liabilities are the liabilities not included in the best estimate. This can for example be tax obligations, loan arrangements or benefits for employees and shareholders. The sum of best estimate, risk margin and other liabilities are called *technical provisions*, that is $TP = BE + RM + OL$. These are obligations the company has to fulfill in order to operate.

Basic own fund is the difference between assets and technical provisions, that is $BOF = A - TP$. If BOF becomes negative the company is bankrupt.

Solvency II works by requiring capital buffers. Let $BOF_1 = A_1 - TP_1$ be the change in basic own funds over one year. Then the *solvency capital requirement, SCR* is the solution to the equation,

$$P(BOF_1 < 0 | A = TP + SCR) = 0.005$$

Even stricter requirement is the *Minimum capital requirement, MCR*. This is given by the equation:

$$P(BOF_1 < 0 | A = TP + MCR) = 0.15$$

An insurance company failing to comply with the MCR requirement is very dramatic. It's license to operate is immediately suspended [3].

Purpose and Result

The subject of this project is the methods of which actors in the financial markets can measure the risk in their portfolios. The practice of measuring risk is old as time. However during the last couple of decades the mathematical frameworks have developed fast. We are going to be introducing some of the most common methods of measuring financial risk, Value-at-Risk and expected shortfall. Then we are going to be comparing these two methods against some of the properties we would expect a measure of risk to have from an economic point of view. This is formalized in the definition of a coherent risk measure. We are then going to prove that Value-at-Risk is not a coherent risk measure, but that expected shortfall is. Lastly we are going to see what this difference means in the application of these methods. To do that we are going to be creating some portfolios, measure the risk with each method, and then simulate the movements over a year.

2 An introduction to mathematical finance

First of all we have to build up some of the theory. One of the basic assumptions in economics is that resources are finite. Resources can be renewable, but they are not in an infinite quantity at any one time. What implication this has for us is that prices have to be finite. In order to model risk we are going to need the concept of a probability space. From this point on when we talk about a probability space we talk about the finite probability space $(\Omega, \mathscr{F}, P)$, i.e. the state space Ω is finite. We are now ready to introduce some important concepts.

Definition 2.1. A *portfolio* is a collection of d financial assets. The value of a specific asset i at time t is modeled by a random variable $Z_i(t)$, and the vector $Z(t) = \{Z_1(t), Z_2(t), \dots, Z_d(t)\}$ stands for the prices of the different assets at time t. The value of a portfolio observed at time t is denoted V(t, Z), and is defined:

$$V(t, Z) = f(t, Z_t)$$

for some measurable function $f \colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$.

Definition 2.2. For a given time period Δ, the loss of the portfolio in a given time period $[s, s+\Delta]$ is called the *loss function*, and is given by:

$$L_{[s,s+\Delta]} \coloneqq -(V(s+\Delta) - V(s)).$$

Example 2.3. Suppose we own a number λ of shares in a company. A share is a "part" of the company that gives the owner the right to a part of the profits. The price of the shares at time t is denoted by the random variable $P(t)$. Thus the portfolio is just a collection of λ shares. Hence the value of the portfolio is $V(t) = f(t, P(t)) = \lambda P(t)$. The loss function for this portfolio is $L_{t+1} = -(V(t+1) - V(t)) = -\lambda(P(t+1) - P(t))$

3 Financial risk

The goal of this project is to discuss some of the common ways to measure risk and to compare them. Throughout history there has been need for measurement of financial risk.

A common way to measure risk is by using the variance of the profit/loss function. This method, although simple to calculate has some critical drawback. Firstly, it treats gains and losses the same. That is a portfolio that which gains a lot and a portfolio that which losses a lot can be deemed equally risky. Gains are usually something we want and losses is usually the thing we seek to limit. Therefore treating them the same is problematic. Secondly, the variance might not even exist. This can be a problem in life-insurance where the distributions can be very heavytailed.

These are only two potential problems we might face when dealing with methods of measuring risk. The following section formalizes what properties we want our risk measures to have.

Coherent risk measure

The criteria we will be concentrating on is *coherence*. To motivate the mathematical definition we need to discuss some of the economic properties we would expect from a measure of risk to have. What is the point of defining something that does not answer the questions we are interested in knowing?

Firstly, we want our measure of risk to take diversification into account. Diversification is the practice of buying different kinds of securities (the shares or bonds of different companies for example) to reduce the risk. Therefore we require the mathematical property of sub-additivity. Secondly, a sure future payment should reduce the risk. This makes sense if you have two portfolios one with a share and one with a share and a future guaranteed payout the one with the guaranteed payout should have lower risk. This is captured in the property translation invariance. Thirdly, we would like the risk to scale with the size of a portfolio. Suppose two otherwise equal portfolios with one having more of the same securities in the same proportion. Its reasonable that the risk of the larger portfolio to be greater. We can therefore set Positive homogeneity as a criterion. The last condition we are interested in is monotonicity. If we have two portfolios where one is always better or equal to the other. Then the "worse" portfolio should have higher risk.

These are the conditions introduced in [1, Artzner et. al. (1999)]. With the economic motivations down we have the following definition:

Definition 3.1 ([1])**.** Let $X, Y \in \mathscr{F}$, and $a \in \mathbb{R}$. A measurable function $\rho \colon \mathscr{F} \to \mathbb{R}$ is called a *coherent risk measure* if it has the following properties:

1. Sub-additivity: $\rho(X+Y) \leq \rho(X) + \rho(Y)$.
2. Translation invariance: $\rho(X+a) = \rho(X) - a$.
3. Positive homogeneity: $\rho(aX) = a\rho(X)$.
4. Monotonicity: if $X \leq Y$, then $\rho(X) \geq \rho(Y)$.

Value-at-Risk

With the definition of a coherent risk measure in place we can begin to look at the methods commonly used in the financial industry to measure risk. The first is the very common Value-at-Risk or VaR. This method has even found it's way into regulations such as Solvency II.

Definition 3.2 ([10])**.** Given some confidence level $\alpha \in (0,1)$. The *Value-at-Risk*, or VaR of a portfolio at the confidence level α is the smallest number $l \in \mathbb{R}$ such that the probability that the loss L in a given time period exceeds l is not greater than $(1-\alpha)$. That is:

$$\mathrm{VaR}_\alpha = \inf\{l \in \mathbb{R} \colon P(L > l) \leq (1-\alpha)\} = \inf\{l \in \mathbb{R} \colon F_L(l) \geq \alpha\}.$$

Remark. We can see that VaR_α is the α-quantile of the loss function or the $(1-\alpha)$-percentile of the loss distribution.

The first matter we now have to concern ourselves with is whether VaR is a coherent risk measure. We can formalize this in a proposition:

Proposition 3.3. *Conditions translation invariance 2, positive homogeneity 3 and monotonicity 4 hold for* VaR_α*.*

Proof. Let $a \in \mathbb{R}$ and X and Y be random variables.

Translation invariance 2:

$$\begin{aligned}
\mathrm{VaR}_\alpha(X+a) &= \inf\{l \in \mathbb{R} : P(X+a > l) \leq 1-\alpha\} \\
&= \inf\{l \in \mathbb{R} : P(X > l-a) \leq 1-\alpha\} \\
&= \inf\{l-a \in \mathbb{R} : P(X > l-a) \leq 1-\alpha\} - a \\
&= \inf\{b \in \mathbb{R} : P(X > b) \leq 1-\alpha\} - a = \mathrm{VaR}_\alpha(X) - a.
\end{aligned}$$

Postive homogeneity 3:

$$\begin{aligned}
\mathrm{VaR}_\alpha(aX) &= \inf\{l \in \mathbb{R} : P(aX > l) \leq 1-\alpha \\
&= \inf\{l \in \mathbb{R} : P(X > l/a) \leq 1-\alpha\} \\
&= a \inf\{l/a \in \mathbb{R} : P(X > l/a) \leq 1-\alpha\} \\
&= a \inf\{c \in \mathbb{R} : P(X > c) \leq 1-\alpha\} = a\,\mathrm{VaR}_\alpha(X).
\end{aligned}$$

Monotonicity4:

Suppose X, Y are two random variables such that $X < Y$.

$$1 - \alpha \geq P(Y > \mathrm{VaR}_\alpha(Y)) \geq P(X > VaR_\alpha(Y)) \implies \mathrm{VaR}_\alpha(Y) \geq \mathrm{VaR}_\alpha(X)$$

The last inequality follows from the definition of VaR_α.

■

The astute reader might see that we have left one of the conditions for being a coherent risk measure out. This is not an oversight but rather because VaR is not sub-additive 1.

To illustrate this we can look at an example:

Example 3.4. [4, p. 5].

Suppose we have two assets X and Y that are independent normally distributed but are subject to shocks with probability p. That is:

$$X = \epsilon_X + \eta_X, Y = \epsilon_Y + \eta_Y \quad \epsilon_X, \epsilon_Y \sim N(0,1), \quad \eta_X, \eta_Y = \begin{cases} 0, & p = 0.991 \\ -10, & p = 0.009 \end{cases}.$$

We can see that for both of these assets $\mathrm{VaR}_{0.01}(X) = \mathrm{VaR}_{0.01}(Y) = \Phi^{-1}(0.01) = 3.1$ Now suppose we have an equally weighted portfolio of X and Y. Since the probability of getting a negative shock is higher for $(X+Y)$. Thus, $\mathrm{VaR}_{0.01}(X+Y) = 9.8 > \mathrm{VaR}_{0.01}(X) + \mathrm{VaR}_{0.01}(Y) = 6.2$. This demonstrates that VaR is not sub-additive and therefor does not take diversification into account.

Lets note that VaR is coherent in some circumstances, for example if the loss function is elliptical or normal [10, Theorem 6.8, p. 242].

Expected shortfall

Since VaR did not prove to be satisfactory according to our criteria we would wish to introduce a different measure.

Definition 3.5 ([10]). For a loss L with $\mathrm{E}(|L|) < \infty$ and density function F_L the *expected shortfall*, or *ES*, at confidence level $\alpha \in (0,1)$ is defined as

$$\mathrm{ES}_\alpha = \frac{1}{1-\alpha} \int_\alpha^1 q_u(F_L) du$$

where $q_u(F_L) = F_L^{\leftarrow}(u)$ is the quantile function of F_L

Remark. We can observe that this is closely related to the VaR in the following way:

$$\mathrm{ES}_\alpha = \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_u(L) du,$$

for some rv L.

To make the matter a bit simpler later on we would also like to use the following lemma:

Lemma 3.6 ([10])**.** *For an integrable loss L with continuous df F_L, existing inverse and any $\alpha \in (0,1)$ we have*

$$ES_\alpha = \frac{E(L|L \geq q_\alpha(L))}{1-\alpha} = E(L|L \geq VaR_\alpha),$$

where $q_\alpha(L)$ is the α-quantile.

Proof. [10, p. 45] Denote U as uniformly distributed rv on the interval [0,1]. From probability theory we know that $F_l^{\leftarrow}(U) = F_L$ in distribution, where $F_L^{\leftarrow}$ is the generalized inverse of the cdf of L, the loss distribution. We have to show that $\mathrm{E}(L|L \geq q_\alpha(L)) = \int_\alpha^1 F_L^{\leftarrow}(u)du$. Hence:

$$\mathrm{E}(L|L \geq q_\alpha(L)) = E(F_L^{\leftarrow}(U); F_L^{\leftarrow}(U) \geq F_L^{\leftarrow}(\alpha)) = E(F_L^{\leftarrow}(U); U \geq \alpha).$$

The last followed from the fact that $F_L^{\leftarrow}$ is strictly increasing and F_L is continuous. Thus we get $E(F_L^{\leftarrow}(U); U \geq \alpha) = \int_\alpha^1 F_L^{\leftarrow}(udu$. The second representation follows since we for a continuous loss distribution F_L has that $P(L \geq q_\alpha(L)) = 1 - \alpha$.

■

The reasoning for introducing ES was that it is a coherent risk measure. Therefore let us formalize this in a proposition.

Proposition 3.7. *The expected shortfall is a coherent risk measure.*

Proof. We have to prove that all the conditions from the definition 3.1 hold for ES.

Conditions translation invariance 2, positive homogeneity3 and monotonicity 4 follow from the fact that VaR has all of those properties, and the linearity of integrals.

The proof of sub-additivity can be found [10][243] 1. ■

We conclude the theoretical part of this project with a result highlighting an important relation between these two risk measures.

Corollary 3.8. *For a loss distribution L and a confidence level $\alpha \in (0,1)$ we have:*

$$\mathrm{VaR}_\alpha(L) \leq \mathrm{ES}_\alpha(L)$$

Proof. This follows from Lemma 3.6. ■

We now have our two measures of risk. The difference between them are simple to see from a theoretical point of view but how do they stack up in practice ? This is the subject of the next chapter of this project.

4 Empirical investigations

To investigate the difference between these two measures of risk we are going to be using some simulations. We are going to combine 3 different stocks and a

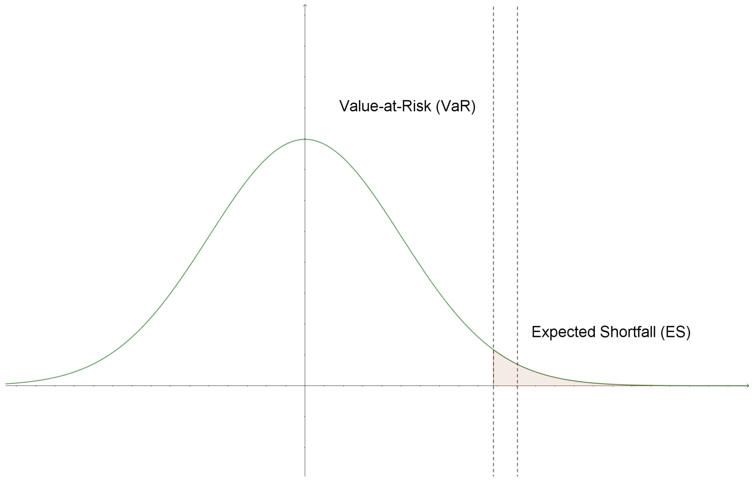


Figure 2: A loss function with distribution $\sim N(0,1)$ with $\mathrm{VaR}_{0.95}$ and ES 0.95[6, GeoGebra 5].

bond into 6 different portfolios. Using our two measures, VaR_α and ES_α, we are going to measure the risk in each. The measures of risk are going to be in the form of required reserve capital, i.e. the risk calculated by our two measures is the capital an investor would have to have in order to absorb a loss. Then we are going to model the change in price for 3 different securities and a stochastic interest rate 1000 times. Finally to investigate the difference between the two measures we are then going to see how many times the value of the portfolios falls below the measure of risk.

By a stroke of bad luck we can use a real downturn in the markets to see how well our estimates are. Financial markets sometimes take a tumble during disasters and the current COVID-19 pandemic is no different. The combination of uncertainty regarding the pandemic as well as policies instituted by national government have sent the financial markets into a nosedive. This is a prime candidate to test our measures of risk.

Construction of portfolios and securities

Before we start simulating we have to create some portfolios to measure the risk. The portfolios we are going to be using is in the table below. The numbers represent the weights of each security in each portfolio.

The securities we are going to be using are real and are as follows:

- S&P500 is a stock index. That is a collection of the 500 largest companies listed on American stock exchanges. The S&P500 is a popular proxy for the market portfolio. The market portfolio is efficient according to the Capital Asset Pricing Model (CAPM) [2, p. 464]. That means that it is the portfolio that give the highest expected return for a given risk.
- Boeing (BA) is a large American manufacturing company specializing in aircrafts.

- The Walt Disney company (DIS) is one of the largest media companies in the world. It is known for producing animated movies with characters such as Donald Duck and Mickey Mouse. The company has subsidiaries like Marvel Studios, Disney parks and the Disney cruise line.
- U.S. treasury bills (T-bills) are bonds bought from the U.S. Treasury. They have a maturity of up to a year and pay no coupon, but are sold at a discount. Since the U.S. Federal government is the issuer this form of securities are usually considered risk-free [13].

We construct the portfolios as follows:

Portfolio	SP500	BA	DIS	T-bills
1	1	0	0	0
2	0	1	0	0
3	0	0	1	0
4	0	0	0	1
5	0	0.5	0.5	0
6	0.5	0	0	0.5

Using data from the 5 year period 01.01.2015 to 31.12.2019. We can use the performance analytics package in R to calculate the $\mathrm{VaR}_{0.95}$ and $\mathrm{ES}_{0.95}$ for 251 trading days 6. The technical way to calculate the risk over a year is to take the log-returns, calculate the risk for one day and then scale this by multiplying with $\sqrt{(\Delta)}$ where Δ is the time period, i.e. number of days, here 251.

Since VaR and ES are statistics we have to estimate them. There are a couple of different way to do this, and they do give slightly different results. Firstly, we could assume that history repeats itself and use the empirical quantile and the empirical conditional mean from the dataset. This means that we assume the return is independent and identically distributed. The second is to assume that the returns are normal (or Gaussian) distributed, and then use the dataset to estimate the parameters of the distribution of the returns. The values are thus theoretical calculations based on the normal distribution. Lastly we can use a Monte Carlo method to simulate using the distribution of the dataset and give us empirical estimates based on this simulation.

The measurements of risk we get are the potential losses we would have to be able to absorb. One way to think if this is how large a portion of our original investment (of say 100 USD) would we loose if the 5% worst case happens (for VaR) or how much would we expect to loose if the loss is greater than the VaR. The table below describes these measurements as the loss on a 100 USD investment. A different way of looking at the risk measurements is to use them as a rule for the amount of reserve capital we should keep. That is how much cash would we need to keep at hand in order for the portfolio to not force us into insolvency.

The risk we have measured does appear to conform to what the theory would predict. Firstly, S&P500 is a broad index and should therefore have diversified away most of the individual risk. Hence portfolio 1 has a lower capital requirement then portfolio 2 and 3. Moreover sovereign bonds are usually very secure (assumed risk free). We are here assuming that we are buying a bond and holding until maturity. This means that the risk is associated with interest

	1	2	3	4	5	6
$\mathrm{VaR}_{0.95}$, historical	20.66	37.28	28.70	-0.22	30.93	10.22
$\mathrm{ES}_{0.95}$, historical	31.29	64.17	46.17	-0.01	53.21	15.64
$\mathrm{VaR}_{0.95}$, Gaussian	20.00	42.09	32.57	0.05	34.36	10.25
$\mathrm{ES}_{0.95}$, Gaussian	25.23	53.07	40.97	0.38	43.32	12.63
$\mathrm{VaR}_{0.95}$, Monte Carlo	20.39	37.12	28.87	0.02	31.12	10.21
$\mathrm{ES}_{0.95}$, Monte Carlo	26.99	58.39	45.25	0.14	47.24	13.57

we earn on this bond. This interest can in fact be negative. Lastly we see from portfolio 6 that we can reduce the risk in portfolio 1 by combining it with portfolio 4.

Moreover we can see that the Monte Carlo and Gaussian methods estimates lower risk than the historical method for ES. However they produce greater than or equal risk estimates for VaR. This is interesting as it could be taken as a sign that the true distribution of the returns have a heavier tail than the normal distribution.

Lastly we can note that ES_α always produce a higher capital requirement than VaR_α. This is as expected considering 3.8.

Components in the simulations

Stochastic interest rates We are going to be using a model for stochastic interest rates to simulate the interest rates in the market. The interest rates are very important in the valuing of bonds and options. We are going to be using a modified Cox-Ingersoll-Ross (CIR) model, which does not allow negative interest rates [9, p. 28]. It is given by the following SDE:

$$d\delta_t = \alpha(\delta_t - \delta)dt + \sigma\delta_t^{0.7}dW_t,$$

where δ_t is interest rate in continuous time t, δ is the long run interest rate, α is a constant describing the elastic pull towards δ, W_t is a Wiener process, σ is the standard deviation of that process. Note that in the classical CIR-model the diffusion coefficient in the SDE is given by $\sigma\delta_t^0.5$. However empirical evidence points to the modified version as preferable [12].

Stochastic stock price To model the random movement of shares we shall employ a mixed Poisson process. So the price of a share i at time t is given by the stochastic process in discrete time:

$$S_t = S_{t-1}\exp(X_1 + X_2 * N_t),$$

where $X_1 \sim N(\mu, \sigma^2)$ is a normal distributed variable from a multivariate distribution. Since the price of the securities are correlated to some degree not using a multivariate distribution would lead to errors. Jumps happen according to a Poisson process, N_t with some rate λ. So if a jump happens we simply add $X_2 \sim N(\mu_{outlier}, \sigma^2_{outlier})$.

Calibrating models

The models themselves are not the main focus of this paper. Therefore we are quickly going to explain the calibration without going into details. Finding the parameters of the CIR-model is simple enough. δ is estimated as the average interest over the period. σ is estimated very roughly as the empiric standard deviation in interest rates. For α we note that the discrete version of the CIR-model can be written as, [8]:

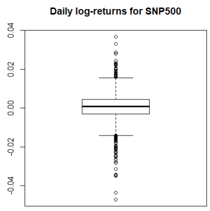
$$\delta_{t+1} - \delta_t = \alpha(\delta_t - \delta)dt + \sigma\delta_t^{0.7}dW_t = \alpha\delta_t - \alpha\delta_{t+1} + \sigma\delta_t^{0.7}dW_t \qquad (1)$$

$$\frac{\delta_{t+1} - \delta_t}{\delta_t^{0.7}} = \alpha\delta_t^{0.3} + \frac{\delta}{\delta_t^{0.7}} + \sigma W_t \qquad (2)$$

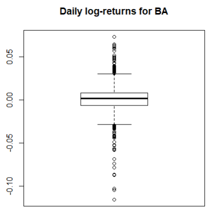
Since $W_t \sim N(0,1)$ we can use ordinary least squares (OLS) to estimate α. Hence, the parameters are as follows:

α	δ	σ
0.022753	0.01261472	0.03794099

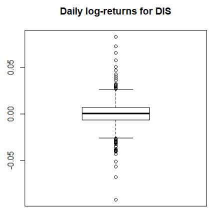
To model the stock prices we create a box plot over log gains. We then model the jumps as the outliers. So we estimate the rate of jumps (λ) from the mean number of outliers per year. X_2 is then normal with the mean and standard deviation of the outliers. As for the non-jump movement of the stocks we use the mean and covariate matrix between the log-gains.



(a) Box plot over log gains for the S&P 500.



(b) Box plot over the log gains for Boeign.



(c) Box plot over the log gains for Disney.

Simulation and results

To test the measurements of risk we are going to simulate a 100 USD position in each portfolio 1 year (251 trading days) 1000 times. We can then see how often the value of the portfolios fall below that of the estimates. An interpretation of this is how often would we go insolvent on our positions using the measurements above to determine capital reserves. I.e. how often would the capital reserves not be sufficient to offset the loss on the portfolio.

On running the simulation and tallying up the number of times a capital requirement is not sufficient we get the following table:

There are some numbers of interest. Firstly, we can note that using ES leads to fewer insolvencies than VaR. This is to expect. As we saw in section 3. for the same security (i.e. the same loss distribution) $\mathrm{VaR}_\alpha(L) \leq \mathrm{ES}_\alpha(L)$. Secondly, portfolio 5 has fewer insolvencies than either portfolio 2 or 3. This could point to some diversification benefit. This means that the event where both DIS and BA is having a bad simulation is lower than the probability of

	1	2	3	4	5	6
$\text{VaR}_{0.95}$, historical	34	74	54	11	28	28
$\text{ES}_{0.95}$, historical	1	0	1	0	0	1
$\text{VaR}_{0.95}$, Gaussian	38	35	28	0	11	28
$\text{ES}_{0.95}$, Gaussian	11	7	4	0	0	7
$\text{VaR}_{0.95}$, Monte Carlo	35	74	53	0	27	28
$\text{ES}_{0.95}$, Monte Carlo	7	1	1	0	0	5

either one having a bad simulation, even if they are covariant. Thirdly, we see some insolvencies for both measures, i.e. losses could be greater than even the most pessimistic measurement of risk. Lastly, the insolvencies in the bond portfolio is very low. The reason for this might be that the choice of model and the calibration is wrong. However bonds are very safe securities and interest rates does not tend to move a lot over such a short amount of time as a year.

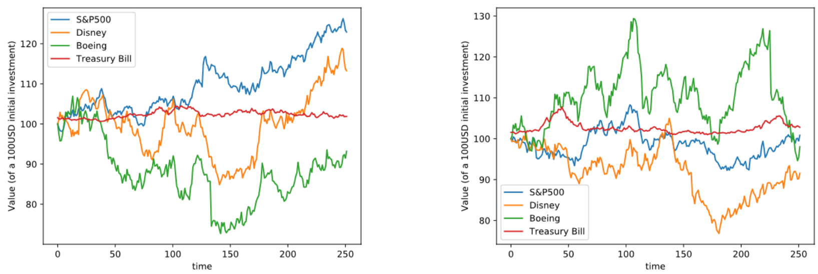


Figure 4: Two of the simulations

Real world data

At the time of writing in the middle of a pandemic and usually when the world is steeped in crisis there is a downturn in the financial markets. Therefore we can test our estimates on the real world and see if they are sufficient. As of time of writing (1st of may 2020) the following estimates have proven insufficient.

	1	2	3	4	5	6
$\text{VaR}_{0.95}$, historical	12.03	11.03	12.03	18.03	28.03	16.03
$\text{ES}_{0.95}$, historical	Solvent	18.03	Solvent	Solvent	12.03	Solvent
$\text{VaR}_{0.95}$, Gaussian	12.03	12.03	12.03	Solvent	06.02	16.03
$\text{ES}_{0.95}$, Gaussian	16.03	16.03	23.03	Solvent	09.03	24.03
$\text{VaR}_{0.95}$, Monte Carlo	12.03	11.03	12.03	Solvent	28.02	16.03
$\text{ES}_{0.95}$, Monte Carlo	20.03	16.03	Solvent	Solvent	11.03	23.03

5 Conclusions

Theoretical We stared this project by motivating why we need financial markets and an introduction to financial mathematics and the the study of financial risk. In building the theory we introduced portfolios, valuation of portfolios and lastly financial losses as a function of the change in value. Using the building blocks we tuned to the matter of financial risk. We motivated and introduced the concept of a coherent risk measure. Using this we found some theoretical differences between two of the most common measures of risk, Value-at-Risk and expected shortfall.

The most notable difference being that VaR_α is not a coherent risk measure, where as ES_α is. The reason for this is that VaR_α is not sub-additive. This means that diversification is always taken into account when calculating ES_α, but that for some distributions this is not the case for VaR_α.

The are some notable relation between these two risk measures. The most obvious being that $\mathrm{ES}_\alpha(L) = E(L|L \geq \mathrm{VaR}_\alpha(L)) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(L)du$. This means that $\mathrm{ES}_\alpha(L) \geq \mathrm{VaR}_\alpha(L)$. The practical implication being that when using these two methods to calculate required capital reserves ES_α is always going to suggest more reserve capital then VaR_α.

To give an practical intuition to the difference between these two measures we can think of them like this. VaR_α gives us a limit where the event of losing more than this has a probability of less than α. However we do not know just how big the loss is going to be. ES_α tells us that given a loss greater than VaR_α what is the expected value of this loss going to be.

Simulation and real data We decided to use 6 portfolios composed of a stock index, two stocks from large companies and a bond. This gave us some span to see the risk for different kinds of securities, from low risk treasury bills to stocks from single companies. To measure the risk we found VaR_α and ES_α for single day log-returns and scaled them up. We used three different methods to estimate VaR_α and ES_α, historical, normal/Gaussian, and Monte Carlo. The measurements we did confirmed what theory would suggest. Firstly, owning a broad index like the S&P 500 is less risky than owning stocks in single companies directly. Secondly, combining stocks from single companies gives a diversification benefit. We saw that the risk in the portfolio with both BA and DIS was less than the average risk of owning either individually. Thirdly, that the risk of owning a treasury bill was very low.

We decided to use a modified CIR-model to simulate the stochastic interest rates and a mixed-Poisson process to simulate the stochastic movement of stock prices. We then checked the results of the simulation to see if the loss of any of the portfolios exceeds the risk measurements and summed the number of times it did for each risk measure and each method. The results confirmed some of our expectations. The losses we would expect according to VaR_α were violated more often than the risk measurements we got using ES_α.

When we look at the real world data we can conclude that even the most pessimistic estimates of risk might not be enough in a crisis. Very few of the risk estimates are still solvent. This is a weakness in both the measures we have investigated here. It should be noted that the real world data describe an extreme and unlikely situation. However we can see from this that situations like this are very possible.

Criticisms The core theory in these results is pretty well established. The properties and relations between the above measures is straight forward. This does not mean that risk measures are not controversial. Both VaR_α and ES_α needs an underlying stochastic model. From a regulator's point of view this could be problematic as the different firms with different kinds of financial positions need to use different underlying distributions. This has been a major criticism of the Solvency II regulatory framework. Solvency II mandates the use of VaR_α but does not specify any underlying distribution.

Even if one does find a good underlying stochastic model there are still problems. None of these risk measures provide a size of the loss that is always correct. That means that even if one chooses the most pessimistic calculation there is still a chance that the loss in an extreme situation exceeds the estimate. Extreme or tail-area values are hard to predict, and require the use of extreme value theory. As we saw in the empirical part. The only measures of risk that did not cause insolvencies were the bonds. However that might be because of the risk measurements themselves or the CIR-model used to simulate interest rates.

There are some factors that could affect the result in the empirical part. Firstly, it is very difficult to estimate the risk over an entire year. Using returns over entire years is problematic because the sample size becomes to small to get good estimates. Therefore we choose to use daily returns and scale them up. However this is not unproblematic either. Empirical findings suggest that the true scaling of risk is different from $\sqrt{t}$ [10, p. 54].

Secondly, the simulation might be inaccurate. There could be multiple reasons for this. The models used to simulate the stocks or the interest rates might not be realistic. In other words the price of stocks might not act according to a mixed-Poisson process. A different problem is that the calibration of the models might not be good enough. This could be because the distribution estimated from the data changes over time. The data might not be good enough. Even that the methods used to estimate the parameters of the model are not precise enough could be problematic. A very good example of lack of data is in the estimates of parameters of jumps. No stock had more than 11 jumps (per our definition as clear outliers in a box plot). Hence the sample size was very small providing uncertain estimates of both rate, mean and variance during jumps. The modeling in itself is not a main focus in this project, but serves to provide a way to compare the methods.

There are multiple avenues of improvements and further research from this project. For one, economic theory would suggest that when the market for stocks goes down investors seek safer investments. This should lead to higher prices and lower interest rates for bonds. This is not modeled in the simulation. Furthermore the use of more advanced theory from extreme value statistics might have made the risk estimates more accurate and the simulation more realistic. A third possible improvement could have been to use some sort of GARCH model to simulate changing volatility in the simulations.

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6 Appendix

appendix I: R-script for calculating the risk of portfolios

```
#import
library(PerformanceAnalytics)
library(readxl)
library(plm)
library(lmtest)
library(sandwich)
library(haven)
library(car)

#Actual script
#dataset
SP500 <- read.csv("~/MAEC/MAT2000/data/^GSPC5Y.csv")
BA<-read.csv("~/MAEC/MAT2000/data/BA(1)5Y.csv")
DIS<-read.csv("~/MAEC/MAT2000/data/DIS5Y.csv")
View(SP500)

rates<- read_excel("MAEC/MAT2000/data/treasuryTbills5Y.xlsx",
                                        sheet = "Ark1")
```

```
#SNP500
#Finding gains
#normal, log gain daily adjusted up, very good, p1
SP500LogGain2 = log(SP500$Open[2:1258]/SP500$Open[1:1257])
VaR(SP500LogGain2, p = 0.95, method = "historical")*sqrt(251)
VaR(SP500LogGain2, p = 0.95, method = "gaussian")*sqrt(251)
VaR(SP500LogGain2, p = 0.95, method = "modified")*sqrt(251)
ES(SP500LogGain2, p=0.95, method="historical")*sqrt(251)
ES(SP500LogGain2, p=0.95, method="gaussian")*sqrt(251)
ES(SP500LogGain2, p=0.95, method="modified")*sqrt(251)
Monte_Carlo(SP500LogGain2, 0.05, 1000, 1000)


#BA, p2
#normal, log gain daily adjusted up, very good
length(BA$Open)
BALogGains = log(BA$Open[2:1258]/BA$Open[1:1257])

VaR(BALogGains, p = 0.95, method = "historical")*sqrt(251)
VaR(BALogGains, p = 0.95, method = "gaussian")*sqrt(251)
VaR(BALogGains, p = 0.95, method = "modified")*sqrt(251)
ES(BALogGains, p = 0.95, method = "historical")*sqrt(251)
ES(BALogGains, p = 0.95, method = "gaussian")*sqrt(251)
ES(BALogGains, p = 0.95, method = "modified")*sqrt(251)
Monte_Carlo(BALogGains, 0.05, 1000,1000)


#DIS, p3
#normal, log gain daily adjusted up, very good
length(DIS$Open)
DISLogGains = log(DIS$Open[2:1258]) - log(DIS$Open[1:1257])

VaR(DISLogGains, p = 0.95, method = "historical")*sqrt(251)
VaR(DISLogGains, p = 0.95, method = "gaussian")*sqrt(251)
VaR(DISLogGains, p = 0.95, method = "modified")*sqrt(251)

ES(DISLogGains, p = 0.95, method = "historical")*sqrt(251)
ES(DISLogGains, p = 0.95, method = "gaussian")*sqrt(251)
ES(DISLogGains, p = 0.95, method = "modified")*sqrt(251)
Monte_Carlo(DISLogGains, 0.05, 1000, 1000)



#T-bill,
#VaR
plot(density(r))
quantile(r, p = 0.05)
(1+0.22/100)*100
100*VaR(r, p=0.95, method="gaussian")
#ES
```

```
f = function(x){quantile(r, p=x)}
int = integrate(f, lower=0, upper=0.05)
int$value/0.05


100*ES(r, p = 0.95, method = "gaussian")
Monte_Carlo(r, 0.05, 1000, 1000)


#BA + DIS, p5
P5=DIS$Open + BA$Open
#P5Log = (DISLogGains[1:1258] + BALogGains[1:1258])
P5Log2 = log(P5[2:1258]/P5[1:1257])
VaR(P5Log2, p = 0.95, method="historical")*sqrt(251)
ES(P5Log2, p = 0.95, method="historical")*sqrt(251)
VaR(P5Log2, p = 0.95, method="gaussian")*sqrt(251)
ES(P5Log2, p = 0.95, method="gaussian")*sqrt(251)
Monte_Carlo(P5Log2, 0.05, 1000, 1000)



#SNP500+bonds, P6
#since both the log returns and the rate are symetrical we know that VaR and ES is less
VaR(SP500LogGain2, p = 0.95, method = "historical")
P6 = (100+rev(r))+SP500$Open[1:1250]
P6log = log(P6[2:1250]/P6[1:1249])
VaR(P6log, p = 0.95, method="historical")*sqrt(251)
ES(P6log, p=0.95, method = "historical")*sqrt(251)


#monte carlo method,
Monte_Carlo = function(data, alpha, n, R){
  v = c()
  es = c()
  for(i in 1:R){
    xstar = sample(data, n, TRUE)
    v = c(v, quantile(xstar, p = alpha))
    f = function(x){quantile(xstar, p=x)}
    int = integrate(f, lower=1-alpha, upper=1)
    es = c(es, 1/(alpha)*int$value)
    }
    #v = quantile(xstar, p = alpha)
    #f = function(x){quantile(r, p=x)}
    #es = integrate(f, lower=1-alpha, upper=1)
    print(mean(v)*sqrt(251))
    print(mean(es)*sqrt(251))
    #print(mean(es))
    #print(mean(v))
}
```

Appendix II: Code for the simulation:

In file called "interest.py"

```
#CIR-modell in seperate file
import numpy as np
import matplotlib.pyplot as plt


class CIR(object):
    def __init__(self, a, b, r0, s, n, T):
        self.a = a
        self.b = b
        self.r0 = r0
        self.s = s
        self.n =n
        self.T = T
        self.r = np.zeros(n)

    def __call__(self):
        #CIR stochstic interest modell
        a, b, r, r0, sigma = self.a, self.b, self.r, self.r0, self.s
        r[0] =r0
        dt = 1
        for i in range(n-1):
            dr = a*(b-r[i])*dt + sigma*((r[i])**0.7)*np.random.normal(0,1)
            r[i+1] = r[i]+dr
        return(r)
```

In file called "stock.py"

```
class Stock(object):
    def __init__(self, s0, clean_sd, clean_mu, outlier_sd, outlier_mu, num, n, T, cov =
        self.s0 = s0
        self.csd = clean_sd
        self.cmu = clean_mu
        self.osd = outlier_sd
        self.omu = outlier_mu
        self.l = num/(5*n)
        self.n = n
        self.T = T
        self.cov = None
        self.val = np.zeros(T)
        self.val[0] = s0

    def __call__(self):
        s0, csd, cmu, osd, omu, l, n = self.s0, self.csd, self.cmu, self.osd, self.omu,
        #rate = l/n
        s = np.zeros(n)
        s[0] = s0
        #poisson
```

```
        X_poi = np.random.poisson(l, size = n)

        direction = (-1)*np.ones(n)
        sign = np.random.randint(2, size = n)
        direction = np.power(direction, sign)
        #brownian
        ds = np.random.normal(cmu, csd, size = n)*(1-X_poi) + np.random.normal(omu, osd
        #sum
        for i in range(n-1):
            s[i+1] = s[i]*np.exp(ds[i])
        return(s)

    def jump(self):
        osd, omu, l, n = self.osd, self.omu, self.l, self.n
        X_poi = np.random.poisson(l, size = n)
        jumps = np.random.normal(omu, osd, size = n)*X_poi
        return(jumps)

    def calc_val(self, ds):
        T, val = self.T, self.val
        for i in range(T-1):
            val[i+1] = val[i]*np.exp(ds[i])

    def return_val(self):
        return(self.val)




class Portfolio(object):

    def __init__(self, portfolio, cov, T):
        """
        Portfolio is an array of stock objects.
        """
        self.portfolio = portfolio
        self.T = T
        self.cov = cov

    def __call__(self):
        #importing things
        p, cov, T = self.portfolio, self.cov, self.T

        #getting jumps
        val = np.zeros((len(p), T))
        jumps = np.zeros((len(p), T))
        for i in range(len(p)):
            jumps[i] = p[i].jump()
        #print(jumps)
        #simulating
```

```
        mus = [p[k].cmu for k in range(len(p))]
        ds = np.transpose(np.random.multivariate_normal(mus, cov, size = T))+jumps
        #print(ds)
        for i in range(len(p)):
            p[i].calc_val(ds[i])
```

From file called "model.py"

```
import numpy as np
import matplotlib.pyplot as plt
from interest import CIR
from stock import Stock, Portfolio

#parameters for the entire model
T = 251
n = 251

t = np.linspace(0, T, n)


#parameters for securities

#T bills
a = 0.022753
r0 = 0.0155
b = 0.01261472
s = 0.03794099
r = np.zeros(n)

rmod = CIR(a, b, r0, s, n, T)

#SNP500, clean
SPmu = 0.0004656113
SPsd = 0.007183155
#all
SPmu = 0.0003545835
SPsd = 0.007896655


#mu_outlier =  0.0003545835
SPmu_outlier = -0.01222183
SPsd_outlier = 0.03510873
#abs outlier
#SPmu_outlier = 0.03521084
#SPsd_outlier = 0.005757614

SPl = 11

SP = Stock(100, SPsd, SPmu, SPsd_outlier, SPmu_outlier, SPl, n, T)
```

```
#Dis, clean
#DISmu = 0.0003048244
#DISsd = 0.01125567
#all
DISmu =0.0003298249
DISsd = 0.01270292



#normla
DISmu_outlier = 0.003447386
DISsd_outlier = 0.07048988
#abs outliers
#DISmu_outlier = 0.06896989
#DISsd_outlier = 0.009409284
DISl = 11

DIS = Stock(100, DISsd, DISmu, DISsd_outlier, DISmu_outlier, DISl, n, T)

#BA, clean
BAmu = 0.001246122
BAsd = 0.01479049
#all
BAmu = 0.0007234323
BAsd = 0.01659743


BAmu_outlier = -0.07175627
BAsd_outlier  = 0.05655012
#abs outlier
#BAmu_outlier = 0.08804126
#BAsd_outlier = 0.01644162
BAl = 9

BA = Stock(100, BAsd, BAmu, BAsd_outlier, BAmu_outlier, BAl, n, T)


varhis = np.array([20.66, 37.28, 28.70, -0.22, 30.93, 10.22])
eshis = np.array([31.29, 64.17, 46.17, -0.01, 53.21, 15.64])
vargaus = np.array([20.00, 42.09, 32.57, 0.05, 34.36, 10.25])
esgaus = np.array([25.23, 53.07, 40.97, 0.38, 43.32, 12.63])
varmone = np.array([20.39, 37.12, 28.87, 0.02, 31.12, 10.21])
esmonte = np.array([26.99, 58.39, 45.25, 0.14, 47.24, 13.57])
risk = 100 + (-1)*np.array([varhis, eshis, vargaus, esgaus, varmone, esmonte])

insolv = np.zeros((6,6))

print(risk)
#print(insolv[0])
cov = [[6.235715e-05, 6.838463e-05, 4.692309e-05],[6.838463e-05, 2.754746e-04, 7.088641
```

```
p = Portfolio([SP, BA, DIS], cov, T)
for i in range(1000):
    val = p()
    #print(val)
    #val= np.transpose(val)
    sp = SP.return_val()
    dis = DIS.return_val()
    ba = BA.return_val()

    rate = rmod()
    TBill = 100*(1+rate[-1])
    #plt.plot(t, sp)
    #plt.plot(t, dis)
    #plt.plot(t, ba)
    #plt.show()
    #plt.plot(t, Tbill)
    #print(Tbill[-1])
    for k in range(6):
        if np.amin(sp) <= risk[k][0]:
            insolv[k][0] += 1

        if np.amin(ba) <= risk[k][1]:
            insolv[k][1] +=1

        if np.amin(dis)<=risk[k][2]:
            insolv[k][2] +=1

        if np.amin(TBill)<=risk[k][3]:
            insolv[k][3] += 1

        if np.amin(0.5*ba+0.5*dis) <= risk[k][4]:
            insolv[k][4] +=1

        if np.amin(0.5*sp + 0.5*TBill)<=risk[k][5]:
            insolv[k][5] += 1
```

Appendix III: R code to calibrate the models

```
#
#Calibration
#tbills, rates. calibrating model
View(rates)
r <- rates$`52 WEEKS BANK DISCOUNT`
r <-na.omit(r)
length(r)
r <- as.numeric(r[2:1251])
r = r/100
dr <- (r[1:1250] - r[2:1251])
```

```
mu_r = mean(r, na.rm=TRUE)
sigm_r = sd(r/sqrt(r))

sigmdt = 1:1250
y = dr/r[1:1250]^0.7
z1 = r[1:1250]^0.3
z2 = mu_r/r[1:1250]^0.7

length(y)

alpha_model = lm(y~z1+z2)
coeftest(alpha_model,  vcov = vcovHC(alpha_model, type = "HC1"))
summary(alpha_model)

#Calibration poisson
#DIS
boxplot(DISLogGains)
DISindex = c(1076, 22, 162, 1226, 1222, 148, 160, 655, 676, 161)
DISoutlier = DISLogGains[DISindex]
DISclean = DISLogGains[-DISindex]
mean(abs(DISoutlier))
sd(abs(DISoutlier))
mean(DISclean)
sd(DISclean)


#BA
boxplot(BALogGains)
BAindex = c(980, 1052, 979, 1208, 964, 161, 989, 280, 779)
BAoutliers = BALogGains[BAindex]
BAclean = BALogGains[-BAindex]
mean(abs(BAoutliers))
sd(abs(BAoutliers))
mean(BAclean)
sd(BAclean)

#SP500
boxplot(SP500LogGain2)
SPindex = c(164, 1003, 780, 819, 779, 989, 373, 162, 161, 951, 782)
SPoutliers = SP500LogGain2[SPindex]
SPclean = SP500LogGain2[-SPindex]
mean(abs(SPoutliers))
sd(abs(SPoutliers))
mean(SPclean)
sd(SPclean)
```