1°. (1 point). Let $\mathbb{V} = \langle e^x, e^{2x}, \ldots, e^{100x}\rangle$, $\mathbb{W} = \langle 1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots, \sin 99x, \cos 99x\rangle$ be vector subspaces of $C^\infty(\mathbb{R})$. Using suitable operators $\varphi : \mathbb{V} \to \mathbb{V}$, $\psi : \mathbb{W} \to \mathbb{W}$ and their eigenvalues, find $\dim \mathbb{V}$ and $\dim \mathbb{W}$.

Hint: the dimensions are what you think they are but without eigenvalues you probably wouldn't be able to prove linear independence.

a) Consider $\varphi : \mathbb{V} \to \mathbb{V}$ as following:

$$\boxed{\varphi(e^{\lambda x}) = \lambda e^{\lambda x}}$$

$$\varphi(e^{x}) = e^{x}$$

$$\varphi(e^{2x}) = 2e^{x}$$

$$\vdots$$

$$\varphi(e^{100x}) = 100 e^{x}$$

Eigenvalue of each vector above is unique and is equal to $\frac{1}{\lambda}$, indeed $\varphi(e^{100x}) = \underbrace{\frac{100}{100}}_{} e^{100x}$.

$\frac{1}{100}$ is an eigenvalue.

Because eigenvalues are different, the vectors are L.I. $\Rightarrow \dim(\mathbb{V}) = 100$.

b) $\varphi : \mathbb{W} \to \mathbb{W}$:

$$\boxed{\begin{aligned} &\varphi(\sin \lambda x) = \lambda \sin \lambda x \\ &\varphi(\cos \lambda x) = \lambda \cos \lambda x \\ &\varphi(\lambda) = \lambda \end{aligned}}$$

$$\varphi(\sin x) = \sin x$$

$$\varphi(\sin 2x) = 2 \sin 2x$$

$$\varphi(\cos 2x) = 2 \cos 2x$$

$$\varphi(1) = 1$$

Here some vectors share the eigenvalue like „1", „sinx" and „cosx" and so on. But it's clear that 1, sinx and cosx are LI. Because every other vector has a unique eigenvalue, the $\dim W = 199$

2°. (0.5 points). Let $\varphi : \mathbb{V} \to \mathbb{V}$ be an operator and $\dim \operatorname{Ker} \varphi = k > 0$. Find geometric multiplicity of $0 \in Spec(\varphi)$.

Hint: it's trivial, just read the definition.

$Spec(\varphi) = \{\lambda \in \mathbb{F} \mid \varphi(v) = \lambda \cdot v,\ v \in \mathbb{V} \setminus \{0\}\}$

Clearly, it equals $k = \dim(\operatorname{Ker}\varphi)$

3°. (1 + 1 + 0.5 + 0.5 points). Let $\varphi : Mat_n(\mathbb{R}) \to Mat_n(\mathbb{R})$, $\varphi(A) = \frac{1}{2}(A + A^T)$.

(a) Check that $\varphi^2 = \varphi$ (b) Find $Spec(\varphi)$ (c) Find geometric multiplicities of all $\lambda \in Spec(\varphi)$ (d) Compute $\det \varphi$.

Hint: recall that $\det \varphi = \det T(\varphi, \mathcal{A})$ (doesn't depend on the choice of $\mathcal{A}$). Don't forget that for $n = 1$ the answer is different.

a)

$$\varphi^2(A) = \varphi\left(\frac{A+A^T}{2}\right) = \frac{1}{2}\varphi(A) + \frac{1}{2}\varphi(A^T) =$$

$$= \frac{1}{4}A + \frac{1}{4}A^T + \frac{1}{4}A^T + \frac{1}{4}A = \frac{2}{4}A + \frac{2}{4}A^T = \frac{1}{2}(A+A^T)$$

b) $spec(\varphi) = \{0, 1\}$

c) Geometric multiplicities of „1" from (b) is $\frac{(n^2+n)}{2}$ - basis for $Sym_n(\mathbb{R})$ and „0" is $\frac{(n^2-n)}{2}$ - $Skew_n^{\mathbb{R}}$

d) $n=1$: $\det \varphi = 1$ as nothing changes.

$n>1$: $\det \varphi = 0$ because $0 \cdot 1$ (eigenvalues) $= 0$.

4. (3 points). Let $\varphi : \mathbb{V} \to \mathbb{V}$ be an operator satisfying $\varphi^2 = \varphi$ and $\dim \mathbb{V} = n > 1$.

(a) Prove that $\mathbb{V} = \operatorname{Ker} \varphi \oplus \operatorname{Im} \varphi$

Hint: Recall that $\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_2$ if $\dim \mathbb{V} = \dim \mathbb{V}_1 + \dim \mathbb{V}_2$ and $\mathbb{V}_1 \cap \mathbb{V}_2 = \{0\}$. In our case the first condition is trivial and always true so you need to check the second one using $\varphi^2 = \varphi$.

(b) Find an example of $\psi : \mathbb{R}^4 \to \mathbb{R}^4$ such that $\dim(\ker \varphi \cap \operatorname{Im} \varphi) = 2$ to show that (a) is not true in general

(c) If φ is not equal to 0 or id, find $Spec(\varphi)$ and $\det \varphi$.

Hint: if $y \in \operatorname{Im} \varphi$ then $y = \varphi(x)$ for some $x \in \operatorname{Im} \varphi$ (we are using (a) here). You can show that actually $x = y$ using (a) and that $x - y \in \operatorname{Im} \varphi$ (because $\operatorname{Im} \varphi$ is a subspace).

a) $\varphi^2 = \varphi$ is a projection and its eigenvalues are 1 and 0. Their geometrical multiplicity gives us the $\dim \operatorname{Im} \varphi$ and $\dim \ker \varphi$. From (3) it's obvious that $\operatorname{Im}\varphi$ and $\ker \varphi$ has no intersection (skew and Sym matrices example)

b) $\psi(e_1) \to e_2$ $\qquad \operatorname{Im}(\psi) = \langle e_2, e_3 \rangle$

$\psi(e_2) \to 0$

$\psi(e_3) \to 0$ $\qquad \ker(\psi) = \langle e_2, e_3 \rangle$

$\psi(e_4) \to e_3$ $\qquad \operatorname{Im} \cap \ker = 2$

c) The projection's eigenvalues are: $\operatorname{spec} \varphi = \{0, 1\}$

$\det = 1 \cdot 0 = 0$

5°. (0.2 + 0.4 + 0.4 points). Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ $(n > 1)$ be the shift operator, $\varphi([x_1, \ldots, x_n]^T) = [0, x_1, \ldots, x_{n-1}]^T$.

(a) Check that $\varphi^n = 0$ (b) Find $Spec(\varphi)$ (c) Compute $Tr(\varphi)$.

a) Let $\varphi([x_1 \ldots x_n])$, then $\varphi^n([x_1 \ldots x_n]) = [\underbrace{0 \ldots 0}_{n \text{ zeros}}]$.

b) Consider $\ker \varphi$. Clearly $\varphi([0, \ldots, 0_{n-1}, x_n]) = [0 \ldots 0_n] \in \ker \varphi$. From (2) we know that $\operatorname{spec} \varphi = \{0\}$

with geometric multiplicity $=1=\dim\ker\varphi$

c) Trace equals to the sum of eigenvalues $=0$.

6. (0.5 + 1 points). Let $\varphi : \mathbb{V} \to \mathbb{V}$ be an operator satisfying $\varphi^n = 0$ for some n and $\varphi^{n-1} \neq 0$.

(a) Find $Spec(\varphi)$ (b) Let $v \in \mathbb{V}$ be a vector such that $\varphi^{n-1}(v) \neq 0$. Is the set $\{v, \varphi(v), \dots, \varphi^{n-1}(v)\}$ linearly independent?

a) given $\varphi^n = 0 \Rightarrow \ker\varphi$ is non-empty $\Rightarrow$

$\Rightarrow \operatorname{spec}\varphi = \{0\}$.

To prove there are no other values in spec consider $\lambda \neq 0, v \neq 0$:

$\varphi(v) = \lambda v \Rightarrow \varphi^n(v) = \lambda^n(v) \Rightarrow 0 = \lambda^n(v) \perp$.

b) Suppose the set is not LI.

$$c_1 v + c_2\varphi(v) + \dots + c_n\varphi^{n-1}(v) = 0 = \varphi^n(v)$$

iff $c_i \neq 0,\ i \in [n-1]$

Then $\varphi^{n-1}(0) = \varphi^{n-1}(c_1 v + c_2\varphi(v) + \dots + c_n\varphi^{n-1}(v)) =$

$= c_1\varphi^{n-1}(v) + c_2\varphi^n(v) + \dots + c_n\varphi^{2n-2}(v) = c_1\varphi^{n-1}$ because

$\varphi^n = 0$ so $m \geq n$ $\varphi^m = 0$. By assumption $\varphi^{n-1}(v) \neq 0$, so $c_1 = 0$. Continue for every c_n and we obtain that all c_i $i \in [n]$ are 0. So the set is LI.

7*. (1 bonus point). Let $L : \mathbb{R}^\infty \to \mathbb{R}^\infty,\ L([x_1, x_2, \ldots]) = [x_2, x_3, \ldots]$. Prove that $\mathbb{R} \subset Spec(L)$.

Consider a set of vectors $[\lambda, \lambda^2, \lambda^2 \ldots]$ for $\lambda \in \mathbb{R}$

$L([\lambda, \lambda^2, \lambda^3 \ldots]) = \lambda[1, \lambda, \lambda \ldots]$

Hence, $\forall \lambda \in \mathbb{R} \subset spec L$.