1. (3 points) (a) Solve $x^2-(1+i)x+6+3i=0$ (b) Find $\sqrt[4]{-16i}$ (c) find $arg(z)$, $z=(10+10i)\cdot(\sqrt{3}+i)$.

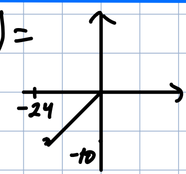
Hint: in (b) recall finding $\sqrt[5]{1}$ on the seminar and use similar approach. Do not forget that the set contains four elements, not one.

a) $x^2-(1+i)x+6+3i=0$

$D = 1+2i-1-4(6+3i) = 2i-24-12i = -24-10i$

$\sqrt{D} = \sqrt{26}\left(\sin\left(\frac{arctg(\frac{5}{12})}{2}\right)+i\cos\left(\frac{arctg(\frac{5}{12})}{2}\right)\right)$

$arctg\left(\frac{-10}{-24}\right) = arctg\left(\frac{5}{12}\right)$

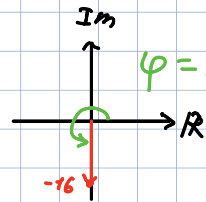


$=\sqrt{26}\cdot\left(-\frac{\sqrt{26}}{26}+i\,\frac{5\sqrt{26}}{26}\right)=-1+5i$

as sin is in the 3rd quarter

$x_1=\frac{1+i-1+5i}{2}=3i$; $x_2=\frac{1+i+1-5i}{2}=1-2i$

b) $\sqrt[4]{-16i} \Rightarrow z^4 = 0-16i$



$\varphi = 270^\circ = \frac{3\pi}{2}$

$z^4 = 16\cdot e^{i(\frac{3\pi}{2}+2\pi k)}$

$z = 2\cdot e^{i(\frac{3\pi}{8}+\frac{\pi k}{2})}$

$k=0 \Rightarrow z_1 = 2\cdot e^{i(\frac{3\pi}{8})}$

$k=1 \Rightarrow z_2 = 2\cdot e^{i(\frac{7\pi}{8})}$

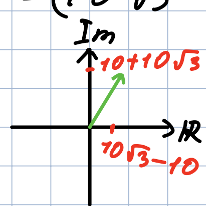
$k=2 \Rightarrow z_3 = 2\cdot e^{i(\frac{11\pi}{8})}$

$k=3 \Rightarrow z_4 = 2\cdot e^{i(\frac{15\pi}{8})}$

c) $arg(z)$, $z=(10+10i)(\sqrt{3}+i)$

$z = 10\sqrt{3}+10i+10\sqrt{3}i-10 =$

$=(10\sqrt{3}-10)+(10+10\sqrt{3})i$



$arctg\left(\frac{10+10\sqrt{3}}{10\sqrt{3}-10}\right) = arctg\left(\frac{1+\sqrt{3}}{\sqrt{3}-1}\right) =$

$= arctg\left(\frac{1+2\sqrt{3}+3}{2}\right) = arctg(2+\sqrt{3}) = 75^\circ$

2. (1 point) Find $(\lambda_1+\lambda_2+\lambda_3)^{50}$, where $\lambda_1,\lambda_2,\lambda_3$ are the roots of $ix^3+(1+i)x^2+(1-3i)x+3i=0$.

Hint: use Vieta's theorem but do not forget to divide the equation by i since the polynomial must be monic.

$$x^3+\frac{(1+i)}{i}x^2+\frac{(1-3i)}{i}x+3=0$$

$$\begin{cases} \frac{-1-i}{i}=\lambda_1+\lambda_2+\lambda_3 \quad \Rightarrow \\ \frac{1-3i}{i}=\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3 \\ 3=-\lambda_1\lambda_2\lambda_3 \end{cases}$$

$$\Rightarrow \left(\left(\frac{-1-i}{i}\right)^2\right)^{25}=(-2i)^{25}=(-2)^{25}i$$

3. (1 points) Let $\mathbb{M}=\left\{\begin{bmatrix} a & -b \\ b & a\end{bmatrix} \mid a,b\in\mathbb{R}\right\}$ be a subset of $Mat_2(\mathbb{R})$. Prove that $\mathbb{M}$ is a field, it means proving the following statements for all $A,B\in\mathbb{M}$:

- $A\pm B$ and AB are in $\mathbb{M}$
- If $A\neq 0$ then A^{-1} exists and belongs to $\mathbb{M}$
- $AB=BA$

Remark: actually we need 1, 0, distributivity and so on but all this follow from the matrix multiplication properties.

1) Consider $\begin{bmatrix} a & -b \\ b & a\end{bmatrix}+\begin{bmatrix} d & -c \\ c & d\end{bmatrix}=\begin{bmatrix} a+d & -c-b \\ b+c & a+d\end{bmatrix}$

$$\begin{bmatrix} a & -b \\ b & a\end{bmatrix}\cdot\begin{bmatrix} d & -c \\ c & d\end{bmatrix}=\begin{bmatrix} ad-bc & -ac-bd \\ ac+bd & ad-bc\end{bmatrix}$$

2) $\begin{bmatrix} a & -b \\ b & a\end{bmatrix}^{-1}=\frac{1}{a^2+b^2}\begin{bmatrix} a & b \\ -b & a\end{bmatrix}$; $\frac{1}{a^2+b^2}a:=z$, $\frac{1}{a^2+b^2}(-b):=p \Rightarrow \begin{bmatrix} z & -p \\ p & z\end{bmatrix}$

3) $\begin{bmatrix} a & -b \\ b & a\end{bmatrix}\cdot\begin{bmatrix} d & -c \\ c & d\end{bmatrix}=\begin{bmatrix} ad-bc & -ac-bd \\ ac+bd & ad-bc\end{bmatrix}=\begin{bmatrix} d & -c \\ c & d\end{bmatrix}\cdot\begin{bmatrix} a & -b \\ b & a\end{bmatrix}$

4. (2 points) Consider a function $f : \mathbb{C} \to \mathbb{M}$, $f(a+bi) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Check that

(a) f "respects"addition and multiplication, i.e. $f(z_1 \pm z_2) = f(z_1) \pm f(z_2)$, $f(z_1 \cdot z_2) = f(z_1) \cdot f(z_2)$ for all $z_1, z_2 \in \mathbb{C}$;

(b) $f(\frac{1}{z}) = f(z)^{-1}$ for all nonzero $z \in \mathbb{C}$; *Hint:* show that $f(1) = I$ and use (a). $I = f(z \cdot \frac{1}{z}) = f(z) \cdot f(\frac{1}{z}) = \dots$

(c) $f(\bar{z}) = f(z)^T$;

(d) $|z|^2 = \det f(z)$;

(e) $f(z) = \begin{bmatrix} |z| & 0 \\ 0 & |z| \end{bmatrix} \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$, where $\varphi = arg(z)$;

(f) It's a bijection.

a) $$f(a_1+b_1 i \pm a_2+b_2 i) = \begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix} \pm \begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix} = \begin{bmatrix} a_1 \pm a_2 & -b_1 \pm b_2 \\ b_1 \pm b_2 & a_1 \pm a_2 \end{bmatrix}$$

$$f((a_1+b_1 i)(a_2+b_2 i)) = f(a_1 a_2 + a_1 b_2 i + a_2 b_1 i - b_1 b_2) =$$

$$= f(\underbrace{a_1 a_2 - b_1 b_2}_{a} + \underbrace{a_2 b_1 i + a_1 b_2 i}_{b}) \quad \begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix} \cdot \begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix} = \begin{bmatrix} a_1 a_2 - b_1 b_2 & -a_1 b_2 - a_2 b_1 \\ a_2 b_1 + a_1 b_2 & -b_1 b_2 + a_1 a_2 \end{bmatrix}$$

b) $$f(1+0 \cdot i) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underbrace{f\left((a+bi) \cdot \frac{1}{(a+bi)}\right) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{a} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$\cancel{f(z)} \cdot f(\tfrac{1}{z}) \cdot \cancel{f(z)^{-1}} = \cancel{I} \cdot f(z)^{-1} \Rightarrow f(\tfrac{1}{z}) = f(z)^{-1}$$

c) $$f(\bar{z}) = f(a-bi) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = f(z)^T$$

d) $$|z|^2 = a^2+b^2 = \det f(z) = a^2 - (-b^2) = a^2+b^2$$

e) $$f(z) = \begin{bmatrix} \sqrt{a^2+b^2} & 0 \\ 0 & \sqrt{a^2+b^2} \end{bmatrix} \begin{bmatrix} \cos(\operatorname{arctg}(\frac{b}{a})) & -\sin(\operatorname{arctg}(\frac{b}{a})) \\ \sin(\operatorname{arctg}(\frac{b}{a})) & \cos(\operatorname{arctg}(\frac{b}{a})) \end{bmatrix} =$$

$$= \begin{bmatrix} \sqrt{a^2+b^2}\cos(\operatorname{arctg}(\frac{b}{a})) & -\sqrt{a^2+b^2}\sin(\operatorname{arctg}(\frac{b}{a})) \\ \sqrt{a^2+b^2}\sin(\operatorname{arctg}(\frac{b}{a})) & \sqrt{a^2+b^2}\cos(\operatorname{arctg}(\frac{b}{a})) \end{bmatrix} =$$

$$= \sqrt{a^2+b^2}\cos(\operatorname{arctg}(\tfrac{b}{a})) + \sqrt{a^2+b^2}\sin(\operatorname{arctg}(\tfrac{b}{a}))i =$$

$$= \sqrt{a^2+b^2} \cdot \frac{a}{\sqrt{a^2+b^2}} + \sqrt{a^2+b^2} \cdot \frac{b}{\sqrt{a^2+b^2}} i = a+bi$$

f) Consider $\underbrace{\begin{matrix} z_1 = a_1+b_1 i \\ \neq \\ z_2 = a_2+b_2 i \end{matrix}}_{\text{bijective}}$ $\left.\begin{matrix} f(z_1) = \begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix} \\ f(z_2) = \begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix} \end{matrix}\right\} \Rightarrow$ surjective

5. (1 points) Find a matrix $X \in Mat_2(\mathbb{R})$ (with real coefficients !!!) such that

(a) $X^2 = -I$ (b) $X^2 + X + I = 0$ *Hint:* use the previous problem.

Remark: finding all possible values of X *is not* required, it is suficcient to find just one.

a) $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \cdot \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $\begin{cases} x_1^2 + x_2 \cdot x_3 = -1 \\ x_1 x_2 + x_2 x_4 = 0 \\ x_3 x_1 + x_4 x_3 = 0 \\ x_3 x_2 + x_4^2 = -1 \end{cases}$ $\begin{matrix} x_1 = 0 \\ x_2 = -1 \\ x_3 = 1 \\ x_4 = 0 \end{matrix}$

$$X = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

b) $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}^2 + \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$

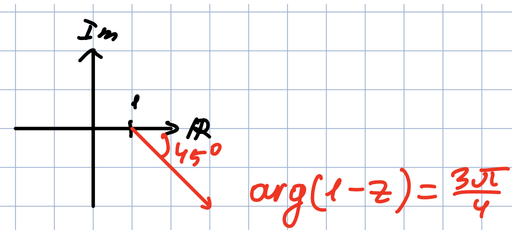
$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}^2 + \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\begin{cases} x_1^2 + x_2 \cdot x_3 + x_1 + 1 = 0 \\ x_1 x_2 + x_2 x_4 + x_2 + 0 = 0 \\ x_3 x_1 + x_4 x_3 + x_3 + 0 = 0 \\ x_3 x_2 + x_4^2 + x_4 + 1 = 0 \end{cases}$ $\begin{matrix} x_1 = 0 \\ x_2 = 1 \\ x_3 = -1 \\ x_4 = -1 \end{matrix}$

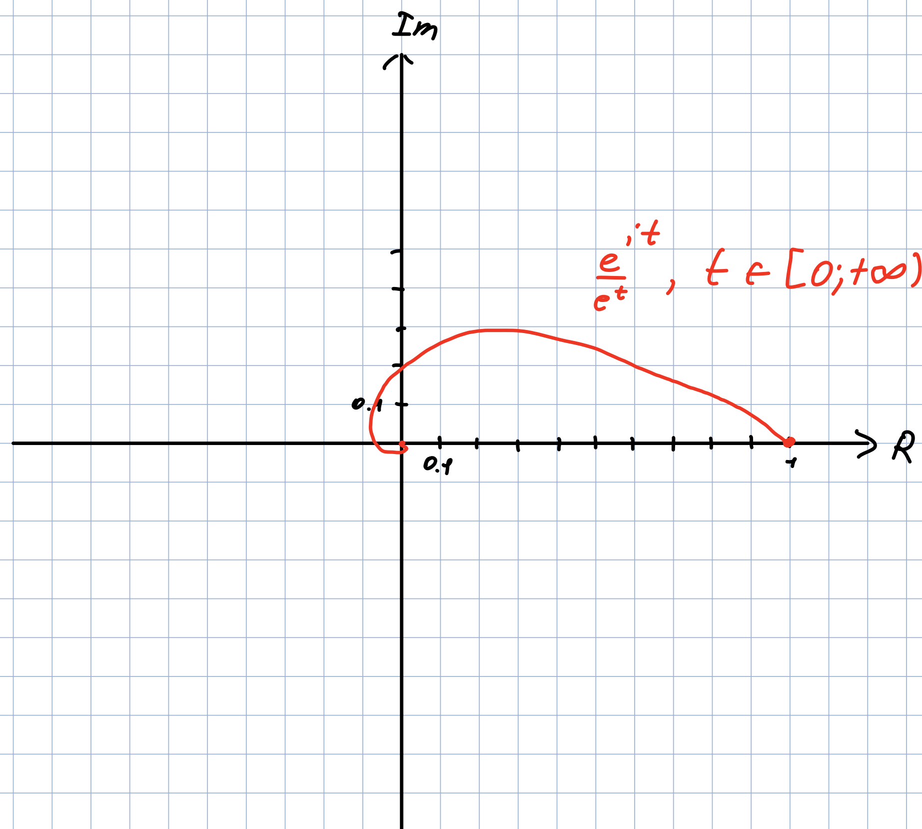
6. (2 points) Draw the following sets

(a) $arg(1-z) = \frac{3\pi}{4}$ (b) $\frac{e^{it}}{e^t}$, $t \in [0, +\infty)$.

a)



b)



$$\frac{2e^{i\frac{2\pi}{3}} - i\frac{7\pi}{4}}{\sqrt{2}e^{i\frac{\pi}{4}}}$$

$$e^{i\left(\frac{2\pi}{3}^{(4} - \frac{7\pi}{4}^{(3}\right)}$$

$$\frac{8\pi}{12} - \frac{21}{12}\pi = \left(-\frac{13\pi}{12}\right)^{50}$$