1. (2 points) Let  $V = \mathbb{R}^2$  and operations  $\oplus : V \times V \to V$ ,  $\odot : \mathbb{R} \times V \to V$  be defined as

$$(u, v) \oplus (x, y) = (u + x + 1, v + y - 3);$$
  
 $\lambda \odot (x, y) = (\lambda x + \lambda - 1, \lambda y - 3\lambda + 3),$ 

for every (u, v),  $(x, y) \in V$  and  $\lambda \in \mathbb{R}$ . Then, <u>using Definition 12.1</u> (see the lecture notes), prove that  $(V, \oplus, \odot)$  is a vector space over the field of real numbers.

Hint: since  $(0,0) \oplus (x, y) = (0 + x + 1, 0 + y - 3) = (x + 1, y - 3) \neq (x, y)$ , it ought to be clear that (0,0) is not the zero vector.

$$= (\lambda_{1} \times + \lambda_{1} - 1 + \lambda_{2} \times + \lambda_{2} - 1 + 1), \ \lambda_{1} y - 3\lambda_{1} + 3 + \lambda_{2} y - 3\lambda_{2} + 3 - 3) =$$

$$= ((\lambda_{1} + \lambda_{2}) \times + (\lambda_{1} + \lambda_{2}) - 1, (\lambda_{1} + \lambda_{2}) y - 3(\lambda_{1} + \lambda_{2}) + 3)$$

$$?) (\lambda_{1} \lambda_{2}) \cdot (X, y) = (\lambda_{1} \lambda_{2} \times + \lambda_{1} \lambda_{2} - 1, \lambda_{1} \lambda_{2} y - 3\lambda_{1} \lambda_{2} + 3)$$

$$\lambda_{1} \cdot (\lambda_{2} \cdot (X, y)) = \lambda_{1} \cdot (\lambda_{2} \times + \lambda_{2} - 1, \lambda_{1} \lambda_{2} y - 3\lambda_{2} + 3) =$$

$$= (\lambda_{1} \lambda_{2} \times + \lambda_{2} - 1) + \lambda_{1} - 1, \lambda_{1} (\lambda_{2} y - 3\lambda_{2} + 3) - 3\lambda_{1} + 3) =$$

$$= (\lambda_{1} \lambda_{2} \times + \lambda_{1} \lambda_{2} - \lambda_{1} + \lambda_{1} - 1, \lambda_{1} y_{2} y - 3\lambda_{1} \lambda_{2} + 3\lambda_{1} - 3\lambda_{1} + 3) =$$

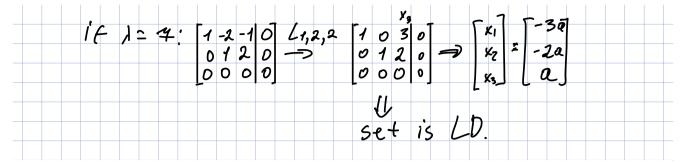
$$= (\lambda_{1} \lambda_{2} \times + \lambda_{1} \lambda_{2} - 1, \lambda_{1} \lambda_{2} y - 3\lambda_{1} \lambda_{2} + 3)$$

$$8) \quad 1 \cdot (X, y) = (X + 1 - 1, y - 3 + 3) = (X, y)$$

**2.** (2 points) Find all  $\lambda \in \mathbb{R}$  such that the set of vectors

$$S = \left\{ \begin{bmatrix} -1\\2\\1 \end{bmatrix}, \begin{bmatrix} 2\\-3\\2 \end{bmatrix}, \begin{bmatrix} 1\\0\\\lambda \end{bmatrix} \right\}$$

is linearly dependent.



**3.** (2 points) Find all  $\lambda \in \mathbb{R}$  such that the vector  $p(x) = x^2 + \lambda x - 3$  belongs to the linear span  $\langle x^2 - 1, x + 1, x^2 + 3x + 2 \rangle$ .

$$X^{2}-1 = X^{2}+\lambda \times -3 \implies \lambda = \frac{2}{X} \implies x^{2}+\frac{2}{X} \cdot x -3 = x^{2}-4$$

$$X^{2}+3 \times +2 = x^{2}+\lambda \times -3$$

$$3 \times +2 = \lambda \times -3$$

$$3 \times +2 +3 -\lambda \times =0$$

$$3 \times +5 -\lambda \times =0$$

$$3 \times +5 -(3+\frac{5}{X}) =0$$

$$3 \times +5 -3 \times -5 =0$$

$$\lambda = (3+\frac{5}{X})$$

$$X+1=x^{2}+\lambda \times -3$$

$$\lambda = (-x+\frac{1}{X}+1)$$

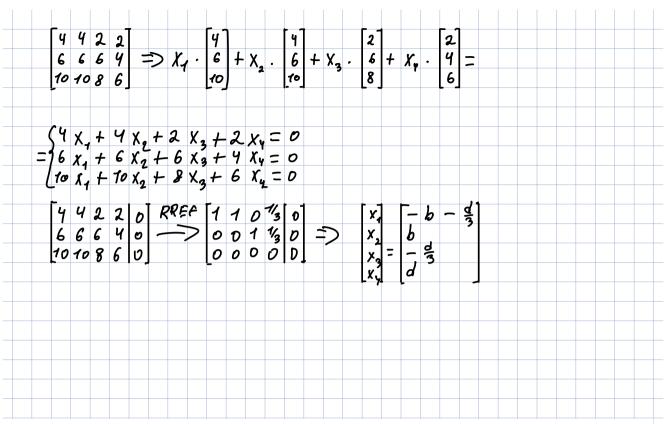
**4.** (2 points) Let 
$$A = \begin{bmatrix} 4 & 4 & 2 & 2 \\ 6 & 6 & 6 & 4 \\ 10 & 10 & 8 & 6 \end{bmatrix}$$
 and  $\varphi : \mathbb{R}^4 \to \mathbb{R}^3, \quad \varphi(\vec{v}) = A\vec{v}$ .

(a) Find two vectors  $\vec{x}_1, \vec{x}_2$  such that  $\text{Ker } \varphi = \langle \vec{x}_1, \vec{x}_2 \rangle$  (the choice is not unique, of course);

 $Hint: Recall \ \mathbf{6} \ from \ HW4 \ (group \ 232).$ 

(b) Find two vectors  $\vec{y}_1, \vec{y}_2$  such that  $\text{Im } \varphi = \langle \vec{y}_1, \vec{y}_2 \rangle$  (the choice is not unique, of course).

Hint: see 6 and 8 from seminar 12, take any two linearly independent columns of A and show that they span Im  $\varphi$ .



**5.** (1 point) Let  $\mathbb{V}$  be a vector space over  $\mathbb{F}$ ,  $\langle v_1, \dots v_n \rangle \subset \mathbb{V}$ . Suppose  $v_n = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_{n-1} v_{n-1}$ ,  $\lambda_i \in \mathbb{F}$  and at least one of  $\lambda_i$  is non-zero. Prove that  $\langle v_1, \dots v_n \rangle = \langle v_1, \dots v_{n-1} \rangle$ .

**6.** (1 point) Let 
$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$
,  $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^4$ . Find  $v_3, v_4 \in \mathbb{R}^4$  such that  $\mathbb{R}^4 = \langle v_1, v_2, v_3, v_4 \rangle$ .

Hint:

Notice that  $\mathbb{R}^4 = \langle e_1, e_2, e_3, e_4 \rangle$ , where  $e_i$  are the vectors with 1 at the *i*-th row and 0 at the other 3 rows. Therefore we have

$$\mathbb{R}^4 = \langle e_1, e_2, e_3, e_4 \rangle = \langle e_1, e_2, e_3, e_4, v_1, v_2 \rangle,$$

and now we can use 5 and throw out two vectors.

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