1. (2 points) Let

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 0 & 2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 4 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}.$$

Then, calculate

$$A(-B + 2 \cdot C)D.$$

1) $-B = \begin{bmatrix} -7 & -4 & 1 \\ -1 & -1 & -1 \\ -1 & -2 & -3 \end{bmatrix} \quad 2C = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

2) $-B + 2C = \begin{bmatrix} -5 & -2 & 1 \\ -1 & 1 & 1 \\ -1 & -2 & -1 \end{bmatrix} = E$

3) $E \cdot D = \begin{bmatrix} -5 & -2 & 1 \\ -1 & 1 & 1 \\ -1 & -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -16 \\ -3 \\ -4 \end{bmatrix} = F$

4) $AF = \begin{bmatrix} 1 & -1 & 4 \\ 0 & 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} -16 \\ -3 \\ -4 \end{bmatrix} = \begin{bmatrix} -29 \\ 2 \end{bmatrix}$

Answer: $\begin{bmatrix} -29 \\ 2 \end{bmatrix}$

2. (1 point per item) Let

$$A = \begin{bmatrix} -2 & 1 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 2 \\ -2 & 1 & 2 \\ 3 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & 1 \\ 3 & 0 & 1 \end{bmatrix}.$$

Then:

(a) find the number of multiplications required to calculate $(AB)C$ and $A(BC)$;

(b) calculate ABC.

a) $(AB)C$, $A = 1\times4$, $B = 4\times3$, $C = 3\times3$, $AB = 1\times3$

multiplications: $(1\cdot4\cdot3) + (1\cdot3\cdot3) = 21$

$A(BC)$, $A = 1\times 4$, $B = 4\times 3$, $C = 3\times 3$, $BC = 4\times 3$

multiplications: $(4\cdot 3\cdot 3) + (1\cdot 4\cdot 3) = 48$

b)

$$AB = \begin{bmatrix} -2 & 1 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 2 \\ -2 & 1 & 2 \\ 3 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 1 & 1 \end{bmatrix}$$

$$(AB)C = \begin{bmatrix} -7 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & 1 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 3 & 16 \end{bmatrix}$$

Answer: a) 21 and 48 mults respectively

b) $\begin{bmatrix} -2 & 3 & 16 \end{bmatrix}$

3. (2 points) Let

$$A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}.$$

Then, find all matrices which commute with A (that is, find the set $C(A) = \{B \in \mathrm{M}_2(\mathbb{R}) \mid AB = BA\}$).

[**hint:** see Problem 5 in Seminar 1.]

$$\left.\begin{aligned} AB &= \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4a+3c & 4b+3d \\ a+2c & b+2d \end{bmatrix} \\ BA &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4a+b & 3a+2b \\ 4c+d & 3c+2d \end{bmatrix} \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} 4a+3c & 4b+3d \\ a+2c & b+2d \end{bmatrix} = \begin{bmatrix} 4a+b & 3a+2b \\ 4c+d & 3c+2d \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} 4a+3c = 4a+b \\ 4b+3d = 3a+2b \\ a+2c = 4c+d \\ b+2d = 3c+2d \end{cases} \Leftrightarrow \begin{cases} 3c = b \\ 2b = 3(a-d) \\ 2c = a-d \\ b = 3c \end{cases} \Leftrightarrow \begin{cases} c = \frac{1}{3}b \\ 2b = 3(a-d) \\ \frac{2}{3}b = a-d \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} c = \frac{1}{3}b \\ 2b = 3\cdot\frac{2}{3}b \\ \frac{2}{3}b = a-d \end{cases} \Leftrightarrow \begin{cases} c = \frac{1}{3}b \\ b = b \\ a = \frac{2}{3}b+d \end{cases} \Rightarrow \begin{bmatrix} \frac{2}{3}b+d & b \\ \frac{1}{3}b & d \end{bmatrix}$$

Answer: $\begin{bmatrix} \frac{2}{3}b+d & b \\ \frac{1}{3}b & d \end{bmatrix}$ commutes with A.

4. (2 points) Let

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Then, find $I_2 + A + A^2 + \cdots + A^{2077}$.[1]

[**hint:** calculate A^2, A^3, and (maybe) A^4; guess a formula for A^n; using mathematical induction, prove the formula; use the fact that matrix addition is componentwise.]

1) $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\underbrace{A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, \quad A^3 = \begin{bmatrix} -1 & 3 \\ 0 & -1 \end{bmatrix}, \quad A^4 = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}}$

Pattern: The main diagonale will change the values of 1s depending whether the value of the power is odd or even. If it is odd, then the diagonale has 1s, and vice versa.

Also a_{12} = power. Its value also depends whether the power is odd or even. If it is odd, then a_{12} is positive, and vice versa.

Then, $I_2 + A + A^2 \ldots + A^{2077} = \begin{bmatrix} x_1 & x_2 \\ 0 & x_1 \end{bmatrix}$, where $x_1 = 1-1+1-1+1\ldots-1 = 0$

and $x_2 = 0+1-2+3-4\ldots+2077 = \frac{2077}{2} = 1039$

Answer: $\begin{bmatrix} 0 & 1039 \\ 0 & 0 \end{bmatrix}$

5. (2 points) Find all matrices $X \in \mathrm{M}_2(\mathbb{R})$[2] such that X commutes with *every* matrix $A \in \mathrm{M}_2(\mathbb{R})$.

[**hint:** you need to find all X such that $AX = XA$ for every matrix A in $\mathrm{M}_2(\mathbb{R})$; let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an arbitrary matrix from $\mathrm{M}_2(\mathbb{R})$, let $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ be an unknown matrix, then, calculate both AX and XA; when does the equality $AX = XA$ holds true for $a = 1$, $b = 0$, $c = 0$, and $d = 0$; what about $a = 0$, $b = 1$, $c = 0$, and $d = 0$?]

$$\left.\begin{aligned} AX &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_3 & ax_2 + bx_4 \\ cx_1 + dx_3 & cx_2 + dx_4 \end{bmatrix} \\ XA &= \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ax_1 + cx_2 & bx_1 + dx_2 \\ ax_3 + cx_4 & bx_3 + dx_4 \end{bmatrix} \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} ax_1+bx_3 & ax_2+bx_4 \\ cx_1+dx_3 & cx_2+dx_4 \end{bmatrix} = \begin{bmatrix} ax_1+cx_2 & bx_1+dx_2 \\ ax_3+cx_4 & bx_3+dx_4 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} ax_1+bx_3 = ax_1+cx_2 \\ ax_2+bx_4 = bx_1+dx_2 \\ cx_1+dx_3 = ax_3+cx_4 \\ cx_2+dx_4 = bx_3+dx_4 \end{cases} \Leftrightarrow \begin{cases} bx_3 = cx_2 \\ ax_2+bx_4 = bx_1+dx_2 \\ cx_1+dx_3 = ax_3+cx_4 \\ cx_2 = bx_3 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \underbrace{\begin{cases} bx_3 = cx_2 \\ ax_2+bx_4 = bx_1+dx_2 \\ cx_1+dx_3 = ax_3+cx_4 \end{cases}}_{\text{general formula for } AX=XA}$$

If $\begin{cases} a=1 \\ b,c,d=0 \end{cases}$; using the formula $\begin{cases} 0\cdot x_3 = 0\cdot x_2 \\ x_2+0\cdot x_4 = 0\cdot x_1+0\cdot x_2 \\ 0\cdot x_1+0\cdot x_3 = x_3+0\cdot x_4 \end{cases} \Leftrightarrow$

$$\Leftrightarrow \begin{cases} x_2, x_3 = 0 \\ x_1, x_4 \in \mathbb{R} \end{cases}$$

If $\begin{cases} b=1 \\ a,c,d=0 \end{cases} \Rightarrow \begin{cases} x_3 = 0\cdot x_2 \\ 0\cdot x_2+1\cdot x_4 = 1\cdot x_1+0\cdot x_2 \\ 0\cdot x_1+0\cdot x_3 = 0\cdot x_3+0\cdot x_4 \end{cases} \Leftrightarrow$

$$\Leftrightarrow \begin{cases} x_3 = 0 \\ x_4 = x_1 \\ x_2 \in \mathbb{R} \end{cases}$$

Combining answeres from two examples above we get:

$$\begin{cases} x_4 = x_1 \\ x_3, x_2 = 0 \end{cases}$$

Then, $X = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $AX = XA$ always!

-Zinkin Zakhar

tg: Avgustine