1) $f(x) = -2ix^4 + (2+2i)x^3 + (2-i)x^2 + (1+2i)x + 3 - i$

$g(x) = -x^2 + (1-i)x - i$

$$\begin{array}{l|l} -2ix^4+(2+2i)x^3+(2-i)x^2+(1+2i)x+3-i & -x^2+(1-i)x-i \\ \hline -2ix^4+(2+2i)x^3+2x^2 & 2ix^2+i \end{array}$$

$$\cancel{-2x^2}+\cancel{2x^2}-ix^2+(1+2i)x+3-i$$

$$-ix^2+(1+i)x+1$$

$$\cancel{-x}-ix\cancel{+x}+2ix+3-i-1=$$

$$= ix-i+2 = i(x-1)+2$$

$$f(x) = \underbrace{(-x^2+(1-i)x-i)}_{g(x)}\underbrace{(2ix^2+i)}_{q(x)} + \underbrace{i(x-1)+2}_{r(x)}$$

$\deg q(x) > \deg r(x)$

2) $x^6+1 = (x^2+1)(x^4-x^2+1) = (x^2+1)(x^4+2x^2-3x^2+1) =$

$= (x^2+1)(x^2+1)(x^2-\sqrt{3}x+1)(x^2+\sqrt{3}x+1)$

3) $f(x) = x^{2n} - nx^{n+1} + nx^{n-1} - 1,\ n \in \mathbb{N} \Rightarrow f(1) = 0$

$f'(x) = 2nx^{2n-1} - n^2x^n - nx^n + n^2x^{n-2} - nx^{n-2} \Rightarrow f'(1) = 0$

$f''(x) = 4n^2x^{2n-2} - 2nx^{2n-2} - n^3x^{n-1} - n^2x^{n-1} + n^3x^{n-3} - 3n^2x^{n-3} + 2nx^{n-3} \Rightarrow$

$\Rightarrow f''(1) = 0$

$f'''(x) = 8n^3x^{2n-3} - 12n^2x^{2n-3} + 4nx^{2n-3} - n^4x^{n-2} + n^2x^{n-2} - 6n^3x^{n-4} +$

$+ 9n^2x^{n-4} + 2nx^{n-4} - 6x^{n-4} \Rightarrow \underline{8n^3}\ \underline{-12n^2} + 4n - n^4\ \underline{+n^2}\ \underline{-6n^3} +$

$+ \underline{9n^2} + 2n - 6 = 2n^3 - 2n^2 - n^4 + 6n - 6 \neq 0$

1 has a multiplicity of 3 for $n > 1$

For $n = 1$, multiplicity $= 0$

4) $x_0=-1$ $x_0=0$ $x_0=1$ $x_0=3$

$p(-1)=6$ $p(0)=5$ $p(1)=0$ $p(3)=2$

$$l_0=\frac{(x-0)(x-1)(x-2)}{(-1-0)(-1-1)(-1-2)}$$

$$l_1=\frac{(x+1)(x-1)(x-2)}{(0+1)(0-1)(0-2)}$$

$$l_2=\frac{(x+1)(x-0)(x-2)}{(-1+1)(1-0)(1-2)}$$

$$l_3=\frac{(x+1)(x-0)(x-1)}{(-3)\cdot(-2)\cdot 3}$$

$$p(x)=6l_0+5l_1+0l_2+3l_3$$

5) a) $x^{2022}+x^{1994}+x^{1500}+10$

$x+1$

$p(x)=q(x)(x+1)+r$

$p(-1)=q(-1)\cdot 0+r$

$r=(-1)^{2022}+(-1)^{1994}+(-1)^{1500}+10=11$

b) $x^{2000}+x^{1900}+x^{1500}+10$

x^2+1

$p(x)=q(x)(x-i)(x+i)+r$

$p(-i)=p(i)=q(x)\cdot 0+r$

$r=(-i)^{2022}+(-i)^{1900}+(-i)^{1500}+10=13$

6. (1 points) Consider three numbers $a_0 = 1$, $a_1 = 2$, $a_2 = 3$ and let $l_0(x), l_1(x), l_2(x)$ be the corresponding Lagrange basis polynomials, i.e. polynomials of degree two such that $l_i(a_j) = 1$ for $i = j$ and zero otherwise (see the seminar notes for more details). Find numbers a, b, c such that the following equality holds

$$2x^2 + 3x + 4 = a \cdot l_0(x) + b \cdot l_1(x) + c \cdot l_2(x).$$

Let $2x^2 + 3x + 4 = f(x)$

$f(1) = 9$

$f(2) = 18$

$f(3) = 31$

$l_0 = \frac{(x-2)(x-3)}{(1-2)(1-3)} \cdot 9 = a$

$l_1 = \frac{(x-1)(x-3)}{(2-1)(2-3)} \cdot 18 = b$

$l_2 = \frac{(x-1)(x-2)}{(3-1)(3-2)} \cdot 31 = c$

7. (1 point) Consider a set of numbers numbers $a_0 = 0$, $a_1 = 2, \ldots, a_{50} = 100$ and let $l_0(x), l_1(x), \ldots, l_{50}(x)$ be the corresponding Lagrange basis polynomials, i.e. polynomials of degree 50 such that $l_i(a_j) = 1$ for $i = j$ and zero otherwise (see the seminar notes for more details). For

$$f(x) = 0 \cdot l_0(x) + 2 \cdot l_1(x) + \ldots + 100 \cdot l_{50}(x) \quad \text{compute} \quad f(7).$$

$f(7) = 0 + 2 \cdot 0 \ldots 0 \cdot l_7(7) + \ldots 0 \cdot l_{50}(7) = 0$

cus $i \neq j$

8. (1 point) Let a, b, c be the roots of $x^3 + x^2 - 2x - 1$. Find a polynomial $f(x)$ with roots a^2, b^2, c^2.

$x^3 + x^2 - 2x - 1 = 0$

$a + b + c = -1$

$ab + bc + cd = -2$

$abc = 1$

$(a+b+c)^2=1$

$a^2+b^2+c^2+2(ab+bc+cd)=1$

$a^2+b^2+c^2-4=1$

$\underline{a^2+b^2+c^2=5}$

$(ab+bc+cd)^2=4$

$a^2b^2+b^2c^2+c^2d^2+2a^2bc+2ab^2c+2abc^2=4$

$a^2b^2+b^2c^2+c^2d^2+2abc(a+b+c)=4$

$a^2b^2+b^2c^2+c^2d^2-2=4$

$\underline{a^2b^2+b^2c^2+c^2d^2=6}$

$\underline{a^2b^2c^2=1}$

$\Downarrow$

$x^3-5x^2+6x-1=0$