1) Base step: $1^2 = \frac{1(1+1)(2+1)}{6}$

$$1 = \frac{2 \cdot 3}{6}$$

Confirmed!

Inductive step:

$$\sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

$$(n+1)^2 + \sum_{k=1}^{n} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

$$(n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6} - \sum_{k=1}^{n} k^2$$

$$(n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6} - \frac{n(n+1)(2n+1)}{6}$$

$$(n+1)^2 = \frac{(n+1)\left((n+2)(2n+3) - n(2n+1)\right)}{6}$$

$$(n+1)^2 = \frac{(n+1)\left(2n^2+3n+4n+6-2n^2-n\right)}{6}$$

$$(n+1)^2 = \frac{(n+1)(6n+6)}{6}$$

$$(n+1)^2 = (n+1)(n+1)$$

$(n+1)^2 = (n+1)^2$ Confirmed.

2) Base step:

n (number of cities) $= 2$

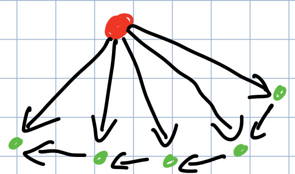
From green you can get to any city.

Inductive step:

Consider $n+1$ many cities.

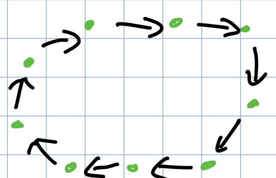
There are 2 cases:

Main city:



Only from the red city you can get to any other city.

Circle cities:



From any city you can get to any other city.

So, any $n+1$ cities will fall under these 2 categories. If there exists a city to which no other roads lead, then this is the city from which you can get to any other city. Otherwise you can get to any city from any city (circle example).

4) $99^0 \equiv 1 \pmod{100}$

$99^1 \equiv 99 \pmod{100}$

$99^2 \equiv 1 \pmod{100}$

$99^3 \equiv 99 \pmod{100}$

If the power is even, the remainder is 1.

$99^{1000} \equiv 1 \pmod{100}$

5) $\frac{a^3}{a-b} \equiv \frac{b^3}{a-b} \Longleftrightarrow a^3 \equiv b^3 \pmod{a-b}$

$\frac{\cancel{(a-b)}(a^2+ab+b^2)}{\cancel{(a-b)}} \equiv 0$ Q.E.D.

6) $11 | (5m+3n) \rightarrow 11 (9m+n)$:

Let $k \in \mathbb{Z}$.

$5m+3n = 11k$

$m = \frac{11k-3n}{5}$

$9m = \frac{9(11k-3n)}{5}$

$9m+n = \frac{9(11k-3n)+5n}{5}$

$9m+n = \frac{99k-22n}{5}$

$9m+n = 11\frac{(9k-2n)}{5}$

Thus, this is divisible by 11. divisible by 11

7) Let $M=100$

$[-100;99]$ $x=101$

$I(101) = remainder(101+100; 200) - 100$

$201 \equiv 1 \pmod{200}$

$I(101) = 1 - 100 = -99$

a) $I(x) = I(I(x))$

$r(x+M,2M) - M - (I(r(x+M,2M)-M) =$

$= r(x+M,2M) - M - (r(r(x+M,2M)-M+M,2M)-M) =$

$= r(x+M,2M) - M - r(r(x+M,2M),2M) + M =$

$= r(x+M,2M) - r(r(x+M,2M),2M) =$

$= I(x) + M - r(I(x)+M, 2M) = 0$

Let $x = 101$

$-99+100 - r(-99+100, 200) = 1 - r(1,200) = 0$

b) Consider 2 cases:

1. $x+y < M$, then it is obvious that we can use the idea from a) we can rewrite x as

$x+y$, then $I(x+y)=I(I(x)+I(y))$.

2. $x+y>M$

After operation $I(x)$, $I(y) \Rightarrow x+y<M$, then consider 1. and the equality will be true.

c) $I(xy)=I(I(x)\cdot I(y))$

Let $x=2$, $y=3$:

$6+100=I(102\cdot 103)$

$106=I(10506)$

$106=106$

Suppose x, y, then:

$I((x+1)(y+1))=I(I(x+1)\cdot I(y+1))$

$I((x+1)(y+1))=r(I(x+1)\cdot I(y+1)), 2M)-M$

If $I((x+1)(y+1))=I(x+1)\cdot I(y+1)$ then the whole equality is true, as proven from a).

$I((x+1)(y+1))=I(x+1)\cdot I(y+1)$

$r((x+1)(y+1)+M, 2M)-M=(r((x+1)+M, 2M)-M)(r((y+1)+M, 2M)-M)$

From the equation above it's clear that the remainder of a product is equal to the product of remainders.

3) $\underbrace{1+\frac{1}{2}+\frac{1}{3}+\ldots\frac{1}{k}}_{\sum_{m=1}^{k}\frac{1}{m}} \geq n$

Assume $\lim\limits_{k \to \infty} \sum\limits_{m=1}^{k} \frac{1}{m} = \infty$ because:

$\exists \varepsilon > 0 \ \forall N \in \mathbb{N} \ \exists k > N : \left| \sum\limits_{m=1}^{k} \frac{1}{m} \right| \geq \varepsilon$

That means that if we take a very large k, we will always be able to make $\sum\limits_{m=1}^{k} \frac{1}{m} \geq n$ ◢