1. Give formal definitions for the following notions. Construct the negation to each of them.

(a) $\lim_{x\to a+0} f(x) = -\infty$; (b) (HW) $\lim_{x\to a-0} f(x) = L$; (c) (HW) $\lim_{x\to -\infty} f(x) = +\infty$.

b) $\forall \varepsilon>0\ \exists \delta>0\ \forall x\in X,\ a-\delta<x<x_0 : |f(x)-A|<\varepsilon$

$\exists \varepsilon>0\ \forall \delta>0\ \exists x\in X,\ a-\delta<x<x_0 : |f(x)-A|>\varepsilon$

c) $\forall \varepsilon>0\ \exists \delta>0\ \forall x\in X,\ |x|>\delta : |f(x)-A|<\varepsilon$

$\exists \varepsilon>0\ \forall \delta>0\ \exists x\in X,\ |x|>\delta : |f(x)-A|>\varepsilon$

5. (HW) Find the following one-sided limits:

(a) $\lim_{x\to 7+0} \frac{|x-7|}{x^2-5x-14}$; (b) $\lim_{x\to 7-0} \frac{|x-7|}{x^2-5x-14}$.

a) $\lim_{x\to 7+0} \frac{(x-7)}{(x-7)(x+2)} = \lim_{x\to 7+0} \frac{1}{x+2} = \frac{1}{9}$

b) $\lim_{x\to 7-0} \frac{-(x-7)}{(x-7)(x+2)} = \lim_{x\to 7-0} \frac{-1}{x+2} = -\frac{1}{9}$

6. (HW) Find the following one-sided limits:

(a) $\lim_{x\to -1+0} \frac{(\sin x)+1}{x+1}$; (b) $\lim_{x\to -1-0} \frac{(\sin x)+1}{x+1}$.

a) $= \frac{\sin(-1)+1}{-1+0+1} = \frac{\sin(-1)+1}{+0} = +\infty$

b) $= \frac{\sin(-1)+1}{-1-0+1} = \frac{\sin(-1)+1}{-0} = -\infty$

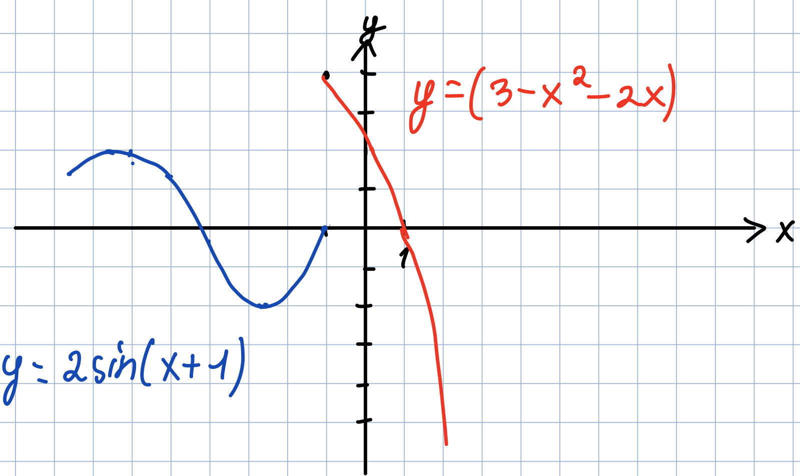
7. (HW) Suppose that $f(x) = \begin{cases} 2\sin(x+1), & x \leqslant -1 \\ 3 - x^2 - 2x, & x > -1 \end{cases}$. Sketch the graph of this piecewise defined function. Find $\lim_{x\to -1-} f(x)$, $\lim_{x\to -1+} f(x)$. Does $\lim_{x\to -1} f(x)$ exist?

$$\lim_{x \to -1-0} (2\sin(x+1)) = 2 \cdot 0 = 0$$

$$\lim_{x \to -1+0} (3 - x^2 - 2x) = 3 - 1 + 2 = 4$$

$\lim\limits_{x \to -1} f(x)$ doesn't exist. Proof by Heine's definition:

$\exists \{x_n\}, \{x_n'\}, \; x_n, x_n' \in \mathring{U}(x_0), \; x_n \to x_0, \; x_n' \to x_0$ but $\lim\limits_{n\to\infty} f(x_n) \neq \lim\limits_{n\to\infty} f(x_n')$



8. (HW) Find a_i and b_i such that

$$\text{(a)} \lim_{x\to+\infty} \left(\sqrt{x^2 - x + 1} - a_1 x - b_1\right) = 0; \quad \text{(b)} \lim_{x\to-\infty} \left(\sqrt{x^2 - x + 1} - a_2 x - b_2\right) = 0.$$

a) $\lim\limits_{x \to +\infty} \sqrt{x^2 - x + 1} - a_1 x = b_1$

$$\lim_{x \to +\infty} \sqrt{x^2 - x + 1} - a_1 x \cdot \frac{\sqrt{x^2 - x + 1} + a_1 x}{\sqrt{x^2 - x + 1} + a_1 x} = b_1$$

$$\lim_{x \to +\infty} \frac{x^2 - x + 1 - a_1^2 x^2}{\sqrt{x^2 - x + 1} + a_1 x} = b_1$$

Suppose $a = 1$:

$$\lim_{x\to-\infty} \frac{-x+1}{\sqrt{x^2-x+1}+x} = b_1$$

$$\lim_{x\to+\infty} \frac{x\left(-1+\frac{1}{x}\right)}{x\left(\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}+1\right)} = b_1$$

$$\frac{-1+0}{\sqrt{1-0+0}+1} = b_1$$

$$\frac{-1}{2} = -b_1$$

$$b_1 = -0.5$$

$$a_1 = 1$$

b) $\lim_{x\to-\infty} \sqrt{x^2-x+1} - a_1 x = b_2$ suppose $a=-1$, as $a=1$ leads to $\lim_{x\to-\infty} f(x) = +\infty$

$$\lim_{x\to-\infty} \left(\sqrt{x^2-x+1}+x\right)\cdot\frac{\left(\sqrt{x^2-x+1}-x\right)}{\left(\sqrt{x^2-x+1}-x\right)} = b_2$$

$$\lim_{x\to-\infty} \frac{x^2-x+1-x^2}{\sqrt{x^2-x+1}-x} = b_2$$

$$\lim_{x\to-\infty} \frac{-x+1}{\sqrt{x^2-x+1}-x} = b_2$$

$$\frac{-1+0}{-\sqrt{1}+\frac{1}{x}} = b_2$$

$$\lim_{x\to-\infty} \frac{}{-\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}-1} = b_2$$

$$b_2 = \frac{1}{2}$$

$$a = -1$$

10. (HW) Evaluate the following limits:

(a) $\lim\limits_{x\to 0} \dfrac{\tan 5x}{\sin 2x}$; (b) $\lim\limits_{x\to \frac{\pi}{2}} \left(\cos x \cdot \sin \dfrac{2}{2x-\pi}\right)$; (c) $\lim\limits_{x\to 0} \dfrac{1-\cos 8x}{5x^2}$;

(d) $\lim\limits_{x\to 0} \dfrac{2x-5x^2+x^3}{\sin 3x}$; (e) $\lim\limits_{x\to 0} x\cdot\cot(5x)$; (f*) $\lim\limits_{x\to 0} \dfrac{x^2}{\sqrt{1+x\sin x}-\sqrt{\cos x}}$.

a) $\lim\limits_{x\to 0} \dfrac{\sin 5x}{\cos 5x} \cdot \dfrac{1}{\sin 2x} = \dfrac{5}{2} \lim\limits_{x\to 0} \dfrac{1}{\cos 5x} = \dfrac{5}{2}\cdot 1 = \dfrac{5}{2}$

b) $\lim\limits_{x\to \frac{\pi}{2}} \cos x \cdot \lim\limits_{x\to\frac{\pi}{2}} \sin\dfrac{2}{2x-\pi} = 0\cdot(\text{bounded seq}) = 0$

$\left|\sin\dfrac{2}{2x-\pi}\right| < 1$

c) $\lim\limits_{x\to 0} \dfrac{1-(1-2\sin^2 4x)}{5x^2} = \lim\limits_{x\to 0} \dfrac{2\sin^2 4x}{5x^2} = 2\cdot \lim\limits_{x\to 0}\dfrac{\sin^2 4x}{5x^2} =$

$= 2\cdot \lim\limits_{x\to 0}\left(\dfrac{\sin 4x}{\sqrt{5}x}\right)^2 = 2\cdot\left(\dfrac{4}{\sqrt{5}}\right)^2 = 2\cdot\dfrac{16}{5} = \dfrac{32}{5}$

d) $\lim\limits_{x\to 0} \dfrac{x(2-5x+x^2)}{\sin 3x} = \dfrac{1}{3}\cdot\lim\limits_{x\to 0}(2-5x+x^2) = \dfrac{1}{3}\cdot(2-5\cdot 0+0) = \dfrac{2}{3}$

e) $\lim\limits_{x\to 0} x\cdot\dfrac{\cos 5x}{\sin 5x} = \dfrac{1}{5}\cdot\lim\limits_{x\to 0}\cos 5x = \dfrac{1}{5}\cdot 1 = \dfrac{1}{5}$

$$\lim_{x\to-\infty} \frac{-x+1}{\sqrt{x^2-x+1}-x} = b_2$$

$$a = \frac{x^2-x+1}{\sqrt{a}}$$

$$\lim_{x\to-\infty} \frac{-1+\frac{1}{x}}{-\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}-1} = b_2$$

$$y = -x$$

$$x = -10$$

$$\sqrt{\underset{-100}{100} + \underset{-100}{10} + 1}$$

$$\lim_{n\to\infty}\left(\left(-\frac{5n^3-4n^2-3n}{n+5} + \frac{5n^3-5n}{n-5}\right) \cancel{\sin\left(\frac{\pi}{n}\right)}\right)$$

$$\frac{48n^3+\dots}{n^2-25}$$

$$\frac{47n^2-10n+1}{n^2-5n} = \frac{47}{1}$$