1^o. **(0.5 points per item).** For each matrix determine whether or not it is diagonalizable.

If A is diagonalizable then find e^A. If moreover $Spec(A)$ is positive find X such that $X^2 = A$.

(a) $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} -2 & 0 & 3 \\ -3 & 1 & 3 \\ -6 & 0 & 7 \end{bmatrix}$

If A is not diagonalizable, you need to explain why. Recall that matrix X is not unique but it's sufficient to find just one.

a) $$\det[A-\lambda I] = \begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & 1-\lambda & -1 \\ 2 & -1 & -\lambda \end{vmatrix} =$$

$$= (1-\lambda)(1-\lambda)(-\lambda) + 2 - 1 - 2(1-\lambda) - (1-\lambda) - \lambda = (1-\lambda)(\lambda+1)(\lambda-2) = 0$$

$$\ker \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 2 & -1 & -1 \end{bmatrix} = \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle = E(1)$$

$$\ker \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix} = \left\langle \begin{bmatrix} -1/5 \\ 3/5 \\ 1 \end{bmatrix} \right\rangle = E(-1)$$

$$\ker \begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 2 & -1 & -2 \end{bmatrix} = \left\langle \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\rangle = E(2)$$

$$B = \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/5 \\ 3/5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\rangle \Rightarrow T(\varphi, B) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$e^A = \begin{bmatrix} 1 & -1/5 & 1 \\ 1 & 3/5 & 0 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{-1} & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e^2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1/5 & 1 \\ 1 & 3/5 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2e}+\frac{e}{6}+\frac{e^2}{3} & \frac{1}{e}-e^2 & -\frac{1}{2e}-\frac{e}{6}+\frac{2e^2}{3} \\ \frac{1}{2e}-\frac{e}{2} & \frac{1}{e} & -\frac{1}{2e}+\frac{e}{2} \\ \frac{1}{2e}-\frac{5e}{6}+\frac{e^2}{3} & \frac{1}{e}-e^2 & -\frac{1}{2e}+\frac{5e}{6}+\frac{2e^2}{3} \end{bmatrix}$$

b) $$E(2) = \left\langle \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \right\rangle, \quad E(-1) = \begin{bmatrix} 0 \\ -1/2 \\ 1 \end{bmatrix}, \quad E(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad T(\varphi, B) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e^A = \begin{bmatrix} 3 & 0 & 1 \\ -2 & -1/2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^2 & 0 & 0 \\ 0 & e^{-1} & 0 \\ 0 & 0 & e^0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 1 \\ -2 & -1/2 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{e^2+1}{2} & -e^2+1 & \frac{-e^2+1}{2} \\ -\frac{e^2}{3}+\frac{1}{3e} & \frac{2e^2}{3}+\frac{1}{3e} & \frac{e^2}{3}-\frac{1}{3e} \\ \frac{e^2}{6}-\frac{2}{3e}+\frac{1}{2} & -\frac{e^2}{3}-\frac{2}{3e}+1 & -\frac{e^2}{6}+\frac{2}{3e}+\frac{1}{2} \end{bmatrix}$$

c) Eigenvalue 1, $E(1) = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rangle$ Not enough for diagonalization.

d) $$E(4) = \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}, \quad E(1) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$T(\varphi, \mathcal{B}) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^A = \begin{bmatrix} 1/2 & 0 & 1 \\ 1/2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^4 & 0 & 0 \\ 0 & e^1 & 0 \\ 0 & 0 & e^1 \end{bmatrix} \cdot \begin{bmatrix} 1/2 & 0 & 1 \\ 1/2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -e^4+2e & 0 & e^4-e \\ -e^4+e & e & e^4-e \\ -2e^4+2e & 0 & 2e^4-e \end{bmatrix}$$

$$X = \begin{bmatrix} 1/2 & 0 & 1 \\ 1/2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{4} & 0 & 0 \\ 0 & \sqrt{1} & 0 \\ 0 & 0 & \sqrt{1} \end{bmatrix} \cdot \begin{bmatrix} 1/2 & 0 & 1 \\ 1/2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -2 & 0 & 3 \end{bmatrix} \Rightarrow X^2 = A$$

2. (1 + 0.5 points). Let $A, B \in Mat_n(\mathbb{R})$

(a) Prove that if $AB = BA$ then $e^A e^B = e^{A+B}$;

> Since $AB = BA$ we have $(A+B)^k = \sum \binom{k}{j} A^k B^{j-k}$; $e^A e^B = (I + A + \frac{A^2}{2!} + \ldots) \cdot (I + B + \frac{B^2}{2!} + \ldots) = \ldots$
> $e^{A+B} = I + (A+B) + \frac{(A+B)^2}{2!} + \ldots$; Multiply the LHS and use binomial on the RHS.

(b) Find an example showing that in general $e^A e^B \neq e^{A+B}$ (2×2 case is the easiest, obviously).

> You need to explicitly compute $e^A e^B$ and e^{A+B} and show that they are different.

a) $$e^A e^B = \sum_{n=0}^{\infty} \frac{A^n}{n!} \sum_{n=0}^{\infty} \frac{B^n}{n!} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{A^m B^n}{m!\,n!} = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{A^m B^{l-m}}{m!(l-m)!} =$$

$$= \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{m=0}^{l} \frac{l!}{m!(l-m)} A^m B^{l-m} = \sum_{l=0}^{\infty} \frac{(A+B)^l}{l!} = e^{A+B}$$

b) $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ $\quad e^{A} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^5 & 0 \\ 0 & e^{-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{e^5}{3}+\frac{2}{3e} & \frac{2e^5}{3}-\frac{2}{3e} \\ \frac{e^5}{3}-\frac{1}{3e} & \frac{2e^5}{3}+\frac{1}{3e} \end{bmatrix}$

$B = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$ $\quad e^{B} = \begin{bmatrix} 1/2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^2 & 0 \\ 0 & e \end{bmatrix} \cdot \begin{bmatrix} 1/2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -e^2+2e & e^2-e \\ -2e^2+2e & 2e^2-e \end{bmatrix}$

$A + B = \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix}$ $\quad e^{A+B} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} e^6 & 0 \\ 0 & e \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} e & e^6 - e \\ 0 & e^6 \end{bmatrix}$

Clearly, $e^A e^B \neq e^{A+B}$

3. (0.5 points). An operator $\varphi \in \mathcal{L}(\mathbb{V})$ is called *nilpotent* of degree k if $\varphi^k = 0$ and $\varphi^{k-1} \neq 0$. Prove that a nilpotent of degree $k > 1$ is not diagonalizable.

If it was diagonalizable what exactly would be on the diagonal?

Suppose A is diagonalizable and $A \neq 0$.

$A = CDC^{-1} \Rightarrow A^n = CD^nC^{-1} \Rightarrow D^n = 0 \Rightarrow A = 0 \ \lightning$.

The only eigenvalue of A is 0.

4°. (2 points). Let $N = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Compute (a) e^N (b) e^A, where $A = I_5 + N$.

(a) N is nilpotent so $\sum \frac{N^k}{k!}$ is not infinite; (b) $I_5 N = N I_5$ hence use problem **2**.

$$e^N := \sum_{k=0}^{\infty} \frac{1}{k!} N^k$$

$$e^N = I_5 + N + \frac{N^2}{2} + \frac{N^3}{6} + \frac{N^4}{24} + 0 \ldots + 0 \quad \text{as } N^j = 0 \quad j > 4.$$

a) $e^N = \begin{bmatrix} 1 & 1 & 1/2 & 1/6 & 1/24 \\ 0 & 1 & 1 & 1/2 & 1/6 \\ 0 & 0 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

b) $e^A = e^N \cdot e^{I_5} = \begin{bmatrix} e & e & e/2 & e/6 & e/24 \\ 0 & e & e & e/2 & e/6 \\ 0 & 0 & e & e & e/2 \\ 0 & 0 & 0 & e & e \\ 0 & 0 & 0 & 0 & e \end{bmatrix}$

6°. (1 point). Let $A = \begin{bmatrix} -15 & 4 & 2 \\ -49 & 13 & 7 \\ -17 & 4 & 4 \end{bmatrix}$ and $B = e^A$. Find (a) $\operatorname{Tr} B$ (b) $\det B$.

You don't need to compute B, all you need to know is $Spec(A)$.

$\operatorname{spec} A = \{2, 1, -1\}$

a) $\operatorname{tr} B = e^2 + e + e^{-1} = e^2 + e + \frac{1}{e}$

b) $\det B = e^2 \cdot e \cdot \frac{1}{e} = e^2$

5. (1.5 points). Find a function $x(t)$ which satisfies $x'' - 4x' + 3x = 0$ and such that $x(0) = 0, x'(0) = 1$. Show the correctness of your answer by substitution (substitute your $x(t)$ to the equation and check that it holds).

$x'' - 4x' + 3x = 0$

$\begin{cases} x' = y \\ y' - 4y + 3x = 0 \end{cases} \Rightarrow y' = 4y - 3x$

$\bar{r}' = A \cdot r$ $\quad A = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1/3 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/3 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$

$x(0) = 0$

$x'(0) = 1$

$$e^{At} = \begin{bmatrix} 1/3 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{t} \end{bmatrix} \cdot \begin{bmatrix} 1/3 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{e^{3t}}{2}+\frac{3e^{t}}{2} & \frac{e^{3t}}{2}-\frac{e^{t}}{2} \\ -\frac{3e^{3t}}{2}+\frac{3e^{t}}{2} & \frac{3e^{3t}}{2}-\frac{e^{t}}{2} \end{bmatrix}$$

$$e^{At} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{e^{3t}-e^{t}}{2} \\ \frac{3e^{t}-e^{t}}{2} \end{bmatrix} \Rightarrow x(t) = \frac{e^{3t}-3^{t}}{2}$$

7^o. (1.5 points). Let $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}, n > 1$. Find a formula for F_n using diagonalization of a suitable matrix (in this HW any other methods will not be accepted).

Fibonacci matrix is defined as:

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \cdot \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix}$$

$$E_A\left(\frac{1-\sqrt{5}}{2}\right) = \left\langle \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} \right\rangle$$

$$E_A\left(\frac{1+\sqrt{5}}{2}\right) = \left\langle \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} \right\rangle$$

$$A^n = \begin{bmatrix} \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}^n \cdot \begin{bmatrix} \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$