1. Give an example of a binary relation $P \subseteq \mathbb{R} \times \mathbb{R}$ such that:

a) P is not functional, injective, not total, and surjective;

b) P is functional, not injective, total, and not surjective.

a) Let $P = \{(0,0), (0,1), (1,2)\}$

P is $\begin{cases} \text{not func: } 0R0 \text{ and } 0R1 \\ \text{inj} \\ \text{not total for the field } \{0,1,2\} \\ \text{surj for the field } \{0,1,2\} \end{cases}$

b) Let $P = \{(0,0), (1,0), (2,1)\}$

P is $\begin{cases} \text{Func} \\ \text{Not inj: } 0R0 \text{ and } 1R0 \\ \text{total for the field } \{0,1,2\} \\ \text{Not surj for the field } \{0,1,2\} \end{cases}$

2. Let a relation $R \subseteq A \times B$ be functional.

a) Then $R[R^{-1}[X]] \subseteq X$ for every set X.

b) Does the converse inclusion always hold?

a) By functionality, $\forall x \forall y \forall z\ (xRy \wedge xRz \Rightarrow y = z)$.

$c \in R[R^{-1}[X]] \Leftrightarrow \exists b\, (b \in R^{-1}[X] \wedge bRc) \Leftrightarrow$

$\Leftrightarrow \exists a \exists b\, (a \in X \wedge aR^{-1}b \wedge bRc) \Leftrightarrow$

$\Leftrightarrow \exists a \exists b\, (a \in X \wedge bRa \wedge bRc)$ by func. $a = c$.

And because $c = a$, $a \in X \Rightarrow c \in X \Rightarrow R[R^{-1}[X]] \subseteq X$

b) Consider $A = \{0, 1\}$, $B = \{0\}$ $X = \{1\}$

$R = \{(0,0), (1,0)\}$

$R^{-1}[X] = \{\emptyset\} \Rightarrow R[\{\emptyset\}] \subseteq X = \{1\}$ $\perp$

3. Suppose that $f\colon A \to B$ and $g\colon A \to B$. Prove that $f \cup g\colon A \to B$ iff $f = g$.

Assume $f \cup g : A \to B$. Let $(a,b) \in f$. Since $f \cup g$ is total, $\exists c \in B\ ((a,b) \in f \cup g \vee (a,c) \in f \cup g)$

As $f \cup g$ is functional, $b = c \Rightarrow (a,b) \in g$ and $f \subseteq g$. WLOG, $g \subseteq f$ in a simillar manner. $f = g$.

4. Suppose that $f\colon A \to B$ and $g\colon B \to C$. Prove that if $g \circ f$ is an injection, then f is an injection as well.

Assume $f(x) = f(y) \Rightarrow g(f(x)) = g(f(y)) \Rightarrow x = y$ because $g \circ f$ is injective $\Rightarrow f$ is inj.

6. Suppose that $f\colon A \to B$ and $f^{-1}\colon B \to A$. Then f is a bijection from A to B.

$$\left.\begin{array}{l} f \circ f^{-1} : B \to A \to B = B \to B = id_B \\ f^{-1} \circ f : A \to B \to A = A \to A = id_A \end{array}\right\} f \text{ is a bijection.}$$

7. Give an example element from the following sets:

a) $\mathbb{Q}^{\underline{3}}$;

b) $\mathbb{R}^{\mathbb{Q}}$;

c) $\mathbb{R}^{\mathbb{R}\times\mathbb{Z}}$.

a) $Q^{\underline{3}} : \underline{3} \to \mathbb{Q}$ $\{(0,0),(1,\frac{1}{2}),(2,\frac{1}{4})\}$

$\mathbb{R}^{\mathbb{Q}} : \mathbb{Q} \to \mathbb{R}$ $\{(1,1),(\frac{1}{2},2),(\frac{1}{3},3)\ldots\}$

$\mathbb{R}^{\mathbb{R}\times\mathbb{Z}} : \mathbb{R}\times\mathbb{Z} \to \mathbb{R}$ $\{((\frac{1}{2},0),\frac{1}{2}),$
$((\pi,0),\pi),$
$((2^{10},0),2^{10})\ldots\}$

5*. Prove that a function $f\colon A \to B$ is an injection iff for every set C and every functions $g, h\colon C \to A$, from $f \circ g = f \circ h$, it follows that $g = h$.

Suppose f is injective and $f(g(x)) = f(h(x))$

then $g(x) = h(x) \Rightarrow g = h$

Suppose f is not injective $\Rightarrow \exists a, b : f(a) = f(b)$
$a \neq b$.

Let $g : \underline{3} \to A$ $\quad g(0) = g(1) = \varepsilon$
$h : \underline{3} \to A$ $\quad h(0) = \varepsilon,\ h(1) = \delta$

Then $f(g(1)) \overset{\text{by assumption}}{=} f(h(1))$ but $g(1) = \varepsilon \neq h(1) = \delta$.