1. (1 point) Let $\begin{bmatrix} 1 & 2 & -2 \\ -2 & -2 & 6 \\ 2 & 1 & -2 \end{bmatrix}$. Find $adj(A)$.

Hint: A is invertible so you don't have to compute the cofactor matrix because $adj(A) = \det(A)A^{-1}$. Notice that you don't even need to compute $\det A$ separately because it will be naturally obtained in the process of finding A^{-1} via row transformations.

$$A^{-1} = \left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ -2 & -2 & 6 & 0 & 1 & 0 \\ 2 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[C_{3,1,-2}]{C_{2,1,2}} \left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & 2 & 2 & 2 & 1 & 0 \\ 0 & -3 & 2 & -2 & 0 & 1 \end{array}\right] \xrightarrow{d_{2,\frac{1}{2}}} \left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & \frac{1}{2} & 0 \\ 0 & -3 & 2 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{C_{3,2,3}} \left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 5 & 1 & \frac{3}{2} & 1 \end{array}\right] \xrightarrow{d_{3,\frac{1}{5}}} \left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{5} & \frac{3}{10} & \frac{1}{5} \end{array}\right] \xrightarrow[C_{1,3,2}]{C_{2,3,-1}} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{7}{5} & \frac{6}{10} & \frac{2}{5} \\ 0 & 1 & 0 & \frac{4}{5} & \frac{1}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & \frac{1}{5} & \frac{3}{10} & \frac{1}{5} \end{array}\right]$$

$$\xrightarrow{C_{1,2,-2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{5} & \frac{1}{5} & \frac{4}{5} \\ 0 & 1 & 0 & \frac{4}{5} & \frac{1}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & \frac{1}{5} & \frac{3}{10} & \frac{1}{5} \end{array}\right] = \begin{bmatrix} -\frac{1}{5} & \frac{1}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{1}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{3}{10} & \frac{1}{5} \end{bmatrix}$$

$$|A| = 1 \cdot 2 \cdot 5 = 10$$

$$10 \cdot \begin{bmatrix} -\frac{1}{5} & \frac{1}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{1}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{3}{10} & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} -2 & 2 & 8 \\ 8 & 2 & -2 \\ 2 & 3 & 2 \end{bmatrix} = adj(A)$$

2. (0.5 + 0.5 + 1 points) Let A be invertible $n \times n$ matrix. Find

(a) $adj(A^T)$; (b) $adj(\lambda A)$ for all $\lambda \in \mathbb{R}$; (c) $\det(adj(A))$.

a) $adj(A)^T$

b) $\lambda^{n-1} adj(A)$

c) $(\det(A))^{n-1}$

3. (2 points) Let A and B be *invertible* matrices. Prove that $adj(AB) = adj(B)adj(A)$.

Hint: $(AB)^{-1} = \ldots$.

Remark: it is true for non-invertible matrices as well but the proof of it is much harder and is not required.

1. $AB^{-1} = \dfrac{adj(AB)}{det(AB)}$

$Adj(AB) = det(AB) \cdot AB^{-1}$

2. $AB^{-1} = B^{-1} \cdot A^{-1}$

3. $det(AB) = det(A) \cdot det(B)$

4. $Adj(A) = det(A) \cdot A^{-1}$

$Adj(B) = det(B) \cdot B^{-1}$

5. $Adj(AB) = det(A) \cdot det(B) \cdot B^{-1} \cdot A^{-1}$

6. $Adj(AB) = det(B) \cdot det(A)$

4. (1 points) Find all $x \in \mathbb{R}$ such that $\det(A - x \cdot I_3) = 0$ for $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & -1 & 0 \end{bmatrix} - x \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{vmatrix} 1-x & 2 & 1 \\ 0 & 2-x & 1 \\ 0 & -1 & -x \end{vmatrix} = 0$$

$$(1-x) \cdot (-1)^2 \cdot \begin{vmatrix} 2-x & 1 \\ -1 & -x \end{vmatrix} = (1-x) \cdot (2x - x^2) - (1 \cdot (-1)) =$$

$$= -x^3 + 3x^2 + 1 - 3x = -3x(1-x) + 1 - x^3 =$$

$$= -3x(1-x) + (1-x)(1+x+x^2) = -(1-x)(3x - (1+x+x^2)) =$$

$$= -(1-x)(2x-1-x^2) = (1-x)(x^2 - 2x + 1) =$$

$$= (1-x)(x-1)^2 = 0 \implies \begin{cases} 1-x=0 \\ (x-1)^2 = 0 \end{cases} \implies x = 1$$

5. (2 points) For all $a_1, a_2, a_3, x \in \mathbb{R}$ compute $\det(A)$, where $A = \begin{bmatrix} x+a_1 & x & x \\ x & x+a_2 & x \\ x & x & x+a_3 \end{bmatrix}$.

Assume $x' = \begin{bmatrix} x \\ x \\ x \end{bmatrix}$ $a_1' = \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix}$ $a_2' = \begin{bmatrix} 0 \\ a_2 \\ 0 \end{bmatrix}$ $a_3' = \begin{bmatrix} 0 \\ 0 \\ a_3 \end{bmatrix}$

$$\det(A) = |x'+a_1',\ x'+a_2',\ x'+a_3'| = |x',\ x'+a_2',\ x'+a_3'| +$$

$$+ |a_1',\ x'+a_2',\ x'+a_3| = \underbrace{|x',\ x',\ x'+a_3'|}_{0} + |x',\ a_2',\ x'+a_3'| +$$

$$+ |a_1',\ a_2',\ x'+a_3'| + |a_1',\ x',\ x'+a_3'| =$$

$$= \underbrace{|x',\ a_2',\ x'|}_{0} + |x',\ a_2',\ a_3'| + |a_1',\ a_2',\ a_3'| + |a_1',\ a_2',\ x'| + |a_1',\ x',\ a_3'| +$$

$$+ \underbrace{|a_1',\ x',\ x'|}_{0} = |x',\ a_2',\ a_3'| + |a_1',\ a_2',\ a_3'| + |a_1',\ a_2',\ x'| + |a_1',\ x',\ a_3'| =$$

$$= \begin{vmatrix} x & 0 & 0 \\ x & a_2 & 0 \\ x & 0 & a_3 \end{vmatrix} + \begin{vmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{vmatrix} + \begin{vmatrix} a_1 & 0 & x \\ 0 & a_2 & x \\ 0 & 0 & x \end{vmatrix} + \begin{vmatrix} a_1 & x & 0 \\ 0 & x & 0 \\ 0 & x & a_3 \end{vmatrix} = (x \cdot a_2 \cdot a_3) + (a_1 \cdot a_2 \cdot a_3) +$$

$$+ (a_1 \cdot a_2 \cdot x) + (a_1 \cdot a_3 \cdot x)$$

$$\begin{vmatrix} a_1 & x & 0 \\ 0 & x & 0 \\ 0 & x & a_3 \end{vmatrix} = - \begin{vmatrix} a_1 & 0 & x \\ 0 & 0 & x \\ 0 & a_3 & x \end{vmatrix} = \begin{vmatrix} a_1 & 0 & x \\ 0 & a_3 & x \\ 0 & 0 & x \end{vmatrix} = a_1 \cdot a_3 \cdot x$$

6. (2 point) Let $Mat_n(\mathbb{Z})$ be the set of all $n \times n$ matrices with integer coefficients and $A \in Mat_n(\mathbb{Z})$. Prove that
(a) If A is invertible and $A^{-1} \in Mat_n(\mathbb{Z})$ then $\det(A) = \pm 1$; (b) If $\det(A) = \pm 1$ then $A^{-1} \in Mat_n(\mathbb{Z})$.

a) By def. $|A^{-1}| = \frac{1}{|A|}$, so $\det(A^{-1} \cdot A) = \det(I) = 1$. If $A \in Mat_n(\mathbb{Z})$ and $A^{-1} \in Mat_n(\mathbb{Z})$, the only possible values of $|A| = \pm 1$

b) $\det(A) = \pm 1$, then

$$\det(A) \cdot \det(A^{-1}) = \det(A \cdot A^{-1}) = \det(I) = 1$$

which is only possible if $\det(A) = \det(A^{-1}) = \pm 1 \Rightarrow$

$\Rightarrow$ if $A \in Mat_n(\mathbb{Z})$ then $A^{-1} \in Mat_n(\mathbb{Z})$