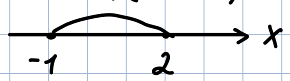
2. (HW) Find the domain and the range for the following functions:

$$\text{(a)}\ \ f(x) = \sqrt{2 + x - x^2}; \qquad \text{(b)}\ \ f(x) = \begin{cases} x + 1, & -3 \leqslant x < -1 \\ 2 - x^2, & x \geqslant -1 \end{cases}$$

a) $2 + x - x^2 \geq 0$

$x^2 - x - 2 \leq 0$

$(x+1)(x-2) \leq 0$



$D(f) = [-1; 2]$

$y = \sqrt{2 + x - x^2}$ $\quad y \geq 0!$

$y^2 = 2 + x - x^2$

$-x^2 + x + 2 - y^2 = 0$

$x^2 - x - 2 + y^2 = 0$

$x = \dfrac{1 \pm \sqrt{1 - 4(-2 + y^2)}}{2}$

$x = \dfrac{1 \pm \sqrt{9 - 4y^2}}{2}$

$9 - 4y^2 \geq 0$

$-4y^2 \geq -9$

$4y^2 \leq 9$

$y^2 \leq \frac{9}{4}$

$-\frac{3}{2} \leq 0 \leq y \leq \frac{3}{2}$

$R(f) = [0; \frac{3}{2}]$

b) $D(f) = [-3; +$ infinity$)$

$y = x + 1, \ x \in [-3; 1)$

$-2 \leq y < 2$

$y = 2 - x^2, \ x \in [-1; +\infty)$

$y \geq 1$

$R(f) = [-2; +\infty)$

4. (HW) Use the $\varepsilon - \delta$ definition of limit to prove that $\lim_{x\to 9}\sqrt{x} = 3$. Find δ such that $|\sqrt{x} - 3| < 0.01$ whenever $0 < |x - 9| < \delta$.

$\forall \varepsilon > 0 \ \exists \delta > 0$ for $\forall x$, $0 < |x - x_0| < \delta$: $|f(x) - A| < \varepsilon$

$\triangledown$ Given $\varepsilon > 0$, choose $\delta = \varepsilon$

Suppose $0 < |x - 9| < \delta$

Check $|\sqrt{x} - 3| = \left|\frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(\sqrt{x}+3)}\right| = \left|\frac{x-9}{(\sqrt{x}+3)}\right| = \frac{|x-9|}{\underbrace{\sqrt{x}+3}_{>3}} < |x-9| < \delta = \varepsilon$ ▲

Because $|\sqrt{x} - 3| < \varepsilon = 0.01$ and $\varepsilon > \delta$ from above:

$0 < |x - 9| < 0.01$

6. (HW) Use negation to the Heine definition of limit to show that $\lim_{x\to 0}\frac{x}{|x|}$ does not exist.

$\triangledown$

$\exists \{x_n\}, \{x_n'\}$, $x_n, x_n' \in \mathring{U}(x_0)$, $x_n \to x_0$, $x_n' \to x_0$ but $\lim_{n\to\infty} F(x_n) \neq \lim_{n\to\infty} f(x_n')$

For $f(x) = \frac{x}{|x|}$ consider $x_n = \frac{x}{x}$, $x_n' = -\frac{x}{x}$

$x_n \to 0$, $x_n' \to 0$ as $n \to \infty$, $f(x_n) = 1$, $f(x_n') = -1$:

$\lim_{x\to 0}\frac{x}{x} = 1$, $\lim_{x\to 0} -\frac{x}{x} = -1$ $\Rightarrow \lim_{x\to\infty}\frac{x}{|x|}$ doesn't exist. ▲

8. (HW) Find the following limits:

(a) $\lim_{x\to a}\frac{x^2-1}{2x^2-x-1}$ for $a = 0$; $a = 1$; $a = \infty$; (b) $\lim_{x\to 0}\frac{(1+x)(1+2x)(1+3x)-1}{x}$.

a) $\lim_{x\to 0}\frac{0-1}{2\cdot 0 - 0 - 1} = \frac{-1}{-1} = 1$

$\lim_{x\to 1}\frac{(x-1)(x+1)}{2(x+\frac{1}{2})(x-1)} = \frac{1+1}{2(1+\frac{1}{2})} = \frac{2}{3}$

$\lim_{x\to\infty}\frac{1-\frac{1}{x^2}}{2-\frac{1}{x}-\frac{1}{x^2}} = \frac{1-0}{2-0-0} = \frac{1}{2}$

b) $\lim\limits_{x\to 0} \frac{(6x^3+11x^2+6x+1)-1}{x} = \lim\limits_{x\to 0} \frac{x(6x^2+11x+6)}{x} = 6\cdot 0+11\cdot 0+6=6$

9. (HW) Find the following limits:

(a) $\lim\limits_{x\to\infty}\left(\sqrt{x^2+3x-1}-\sqrt{x^2+7}\right)$ (b) $\lim\limits_{x\to 3}\frac{\sqrt{6+x}-x}{\sqrt{28-x}-5}$.

a) $\lim\limits_{x\to\infty} \frac{(\sqrt{x^2+3x-1}-\sqrt{x^2+7})(\sqrt{x^2+3x-1}+\sqrt{x^2+7})}{(\sqrt{x^2+3x-1}+\sqrt{x^2+7})} =$

$= \lim\limits_{x\to\infty} \frac{x^2+3x-1-x^2-7}{(\sqrt{x^2+3x-1}+\sqrt{x^2+7})} = \lim\limits_{x\to\infty} \frac{3x-8}{x\sqrt{1+\frac{3}{x}-\frac{1}{x^2}}+x\sqrt{1+\frac{7}{x^2}}} =$

$= \lim\limits_{x\to\infty} \frac{x(3-\frac{8}{x})}{x(\sqrt{1+\frac{3}{x}-\frac{1}{x^2}}+\sqrt{1+\frac{7}{x^2}})} = \frac{3-0}{1+0-0+1+0} = \frac{3}{2}$

b) $\lim\limits_{x\to 3} \frac{(\sqrt{6+x}-x)(\sqrt{6+x}+x)(\sqrt{28-x}+5)}{(\sqrt{28-x}-5)(\sqrt{6+x}+x)(\sqrt{28-x}+5)} = \lim\limits_{x\to 3} \frac{(6+x-x^2)(\sqrt{28-x}+5)}{(28-x-25)(\sqrt{6+x}+x)} =$

$= \lim\limits_{x\to 3} \frac{(x-3)(x+2)(\sqrt{28-x}+5)}{(x-3)(\sqrt{6+x}+x)} = \frac{5(\sqrt{25}+5)}{\sqrt{9}+3} = \frac{50}{6} = \frac{25}{3}$

10. (HW) Find the following limits:

(a) $\lim\limits_{x\to+\infty}\frac{3\cdot 2^x-7\cdot 3^x}{2^{x+2}+5\cdot 3^x}$; (b) $\lim\limits_{x\to-\infty}\frac{3\cdot 2^x-7\cdot 3^x}{2^{x+2}+5\cdot 3^x}$.

a)

$\lim\limits_{x\to+\infty} \frac{3\cdot\left(\frac{2}{3}\right)^x-7}{4\cdot\left(\frac{2}{3}\right)^x+5} = \frac{3\cdot 0-7}{4\cdot 0+5} = -\frac{7}{5}$

b)

$\lim\limits_{x\to-\infty} \frac{3-7\cdot\left(\frac{3}{2}\right)^x}{4+5\cdot\left(\frac{3}{2}\right)^x} = \frac{3-7\cdot 0}{4+5\cdot 0} = \frac{3}{4}$