1. (1.5 points). Let $\{0\} \to \mathbb{V} \xrightarrow{\alpha} \mathbb{U} \xrightarrow{\beta} \mathbb{W} \to \{0\}$ be an exact sequence. Prove that

(a) $\beta\alpha v = 0$ for all $v \in \mathbb{V}$; (b) α is injective and β is surjective (c) $\dim \mathbb{U} = \dim \mathbb{V} + \dim \mathbb{W}$ (if they all are finite).

a) $\beta\alpha V = \beta(\alpha V)$; αV sends some vector v to $\mathcal{U}$.
$\beta(\alpha V)$ sends that vector to 0, because $\operatorname{Ker}\beta = \operatorname{Im}\alpha$.

b) Because $\{0\} \to V \xrightarrow{\alpha} \mathcal{U}$ is exact $\Rightarrow$ α is injective.
$\mathcal{U} \xrightarrow{\beta} W \to \{0\}$ is exact $\Rightarrow$ β is surjective.

c) $\dim(\mathcal{U}) = \operatorname{Ker}(\mathcal{U}) + \operatorname{Im}(\mathcal{U}) = \dim(V) + \dim(W)$

2°. (1.5 points). Let $\alpha : Skew_n \to Mat_n(\mathbb{R})$, $\alpha(X) = X$ and $\beta : Mat_n(\mathbb{R}) \to Sym_n$, $\beta(X) = X + X^T$. Check that $\{0\} \to Skew_n \xrightarrow{\alpha} Mat_n(\mathbb{R}) \xrightarrow{\beta} Sym_n \to \{0\}$ is an exact sequence.

The $\operatorname{Ker}\beta$ is a set of all skew-sym. matrices
(proven in previous hw)
The $\operatorname{Im}\alpha$ are all skew-sym. matrices.
Hence, it's an exact sequence.

3. (2 points). Let $\{0\} \xrightarrow{\varphi_0} \mathbb{V}_1 \xrightarrow{\varphi_1} \mathbb{V}_2 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_{n-1}} \mathbb{V}_n \to \{0\}$ be an exact sequence and $\dim \mathbb{V}_i < \infty$ for all i.

Compute $\dim \mathbb{V}_1 - \dim \mathbb{V}_2 + \dim \mathbb{V}_3 - \dim \mathbb{V}_4 + \ldots + (-1)^{n+1} \dim \mathbb{V}_n$.

We have: $\sum_{i=1}^{n} (-1)^{i+1} \dim V_i \overset{?}{=} 0$

We know: $\sum_{i=1}^{n} (-1)^{i+1} \operatorname{Ker}\varphi_{i-1} + \sum_{i=1}^{n} (-1)^{i+1} \operatorname{Im}\varphi_i = 0.$

Hence, $\forall n \in \mathbb{N}$ the sum is 0.

4. (2 points). Find an exact sequence $\{0\} \to \mathbb{V} \xrightarrow{\alpha} \mathbb{U} \xrightarrow{\beta} \mathbb{W} \to \{0\}$ such that $\dim \mathbb{U} = \infty$, $\dim \mathbb{V} = \infty$, $\dim \mathbb{W} < \infty$.
Hint: in the easiest example $\mathbb{W} = \mathbb{R}$.

$\{0\} \to R[x] \mid p(1)=0 \xrightarrow{\alpha} R[x] \xrightarrow{\beta} R \to \{0\}$

$\alpha(\bar{v}) = \bar{v}$, doing nothing
$\beta(\bar{u}) = u(0)$, plugging 0 to a polynomial.

5. (2 points). Does there exist a matrix $A \in Mat_n(\mathbb{R})$ such that linear transformation

$$\varphi : Mat_n(\mathbb{R}) \to Mat_n(\mathbb{R}),\ \varphi(X) = [A, X]$$

is surjective?

No, φ is never surjective.
Proof: suppose such A exists and φ is surjective.
Clearly, A is not $[0]_n$, as $\varphi(X) = 0$ in all cases.
Otherwise for any A we have 2 cases:
1) $X = A \Rightarrow A^2 - A^2 = 0$
2) $X = [0]_n \Rightarrow A \cdot 0 - A \cdot 0 = 0$

Hence, $\operatorname{Ker} \varphi \geq 2$ and $\dim(\operatorname{Im} \varphi) = n^2 - \operatorname{Ker} \varphi \leq \dim \varphi = n^2$.
Hence, φ is not surjective.

6^o. (1 point). Let $U \in Mat_n(\mathbb{R})$ be invertible. Is it true that

$$\varphi : Mat_n(\mathbb{R}) \to Mat_n(\mathbb{R}),\ \varphi(X) = UXU^{-1}$$

is bijective?

φ is invertible $\varphi^{-1} = U^{-1} X U \Rightarrow$ bijective.

Alternatively, let $X \in \operatorname{Ker}\varphi$:

$$U X U^{-1} = 0 \quad | : U^{-1} U$$

$$X = 0$$

Let $A \in \mathrm{Mat}_n \mathbb{R}$:

$U A U^{-1}$ is transferring to A.

Hence, any matrix has a unique output and φ is bijective.