1°. (1 point) Find a basis for $\mathbb{W} = \{p(x) \in \mathbb{R}[x,4] \mid p(1) = p'(1) = 0\}$.

Let $V = \{p(x) \in \mathbb{R}[x,4] \mid p(1) = 0\}$

Clearly, the basis is $\langle (x-1), (x-1)x, (x-1)x^2, (x-1)x^3 \rangle$

Let $Z = \{p(x) \in \mathbb{R}[x,4] \mid p'(x) = 0\}$

The basis is $\langle (x-1)x, (x-1)x^2, (x-1)x^3 \rangle$

Thus, the basis of $V \cap Z$ is the basis for W, which is $\langle (x-1)x, (x-1)x^2, (x-1)x^3 \rangle$.

2°. (1+3 points) Let $\mathbb{U}_1 = \left\langle \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\rangle$ and $\mathbb{U}_2 = \left\langle \begin{bmatrix} -3 \\ -5 \\ 4 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix} \right\rangle$. Then find a basis for

(a) $\mathbb{U}_1 + \mathbb{U}_2$ (b) $\mathbb{U}_1 \cap \mathbb{U}_2$

a) Let v_1, v_2, v_3 be vectors of U_1.

v_1 can be expressed as $v_2 + (-1)v_3$:

$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix} = v_1 \Rightarrow U_1 = \left\langle \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\rangle \text{ - basis of } U_1$$

Let w_1, w_2, w_3 be vectors of U_2.

$w_3 = -w_1 - w_2$

$$\begin{bmatrix} 3 \\ 5 \\ -4 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ -3 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix} = w_3 \Rightarrow U_2 = \left\langle \begin{bmatrix} -3 \\ -5 \\ 4 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -2 \\ 3 \end{bmatrix} \right\rangle \text{ - basis of } U_2.$$

$$\mathcal{U}_1+\mathcal{U}_2=\left\langle \begin{bmatrix}0\\1\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\-1\\2\\0\end{bmatrix}, \begin{bmatrix}-3\\-5\\4\\-4\end{bmatrix}, \begin{bmatrix}2\\3\\-2\\3\end{bmatrix} \right\rangle$$

$$\begin{bmatrix}0&-1&-3&2\\1&-1&-5&3\\1&2&4&-2\\2&0&-4&3\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}1&0&-2&0\\0&1&3&0\\0&0&0&1\\0&0&0&0\end{bmatrix} \Rightarrow$$ third vector is expressed as a linear combination of other vectors. Thus, the basis is:

$$\mathcal{U}_1+\mathcal{U}_2=\left\langle \begin{bmatrix}0\\1\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\-1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\3\\-2\\3\end{bmatrix} \right\rangle$$

b) Using Zassenhaus algorithm and bases found in (a) we get:

$$\begin{bmatrix}0&1&1&2&0&1&1&2\\-1&-1&2&0&-1&-1&2&0\\-3&-5&4&-4&0&0&0&0\\2&3&-2&3&0&0&0&0\end{bmatrix} \xrightarrow{REF} \begin{bmatrix}-1&-1&2&0&&&&\\0&1&1&2&&&&\\0&0&1&1&&&&\\0&0&0&0&\underbrace{3\;\;5\;\;-4\;\;4}&&&\end{bmatrix} \Rightarrow$$

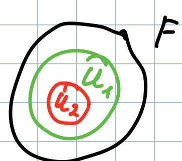
don't care what is here

$$\Rightarrow \left\langle \begin{bmatrix}3\\5\\-4\\4\end{bmatrix} \right\rangle$$ is the basis for $\mathcal{U}_1 \cap \mathcal{U}_2$

3°. (1 point) Let $\mathbb{V}$ be a vector space over a field $\mathbb{F}$ and $\dim \mathbb{V} = 33$. Let $\mathbb{U}_1, \mathbb{U}_2$ be subspaces of $\mathbb{V}$ with $\dim \mathbb{U}_1 = 25$, $\dim \mathbb{U}_2 = 14$. Find all possible values for $\dim(\mathbb{U}_1 + \mathbb{U}_2)$ and $\dim(\mathbb{U}_1 \cap \mathbb{U}_2)$.

$$\dim(\mathcal{U}_1+\mathcal{U}_2)=\dim(\mathcal{U}_1)+\dim(\mathcal{U}_2)-\dim(\mathcal{U}_1\cap\mathcal{U}_2)$$

Suppose $\mathcal{U}_2 \subset \mathcal{U}_1$:

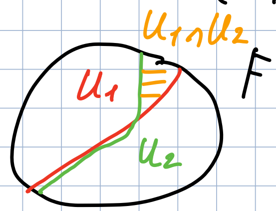


Then, $\dim(U_1 \cap U_2) = \dim(U_2) = 14$

$\dim(U_1 + U_2) = 25 + 14 - 14 = 25$ – min value for the sum.

Suppose that $\dim(U_1 \cap U_2)$ is minimum.

that is:



The minimum intersection is $\dim(U_2) - (\dim(F) - \dim(U_1)) =$
$= 14 - 8 = 6$

Then, $\dim(U_1 + U_2) = 25 + 14 - 6 = 33$ – max value for the sum.

Answer: $25 \leq \dim(U_1 + U_2) \leq 33$

$6 \leq \dim(U_1 \cap U_2) \leq 14$

4. (1 point) Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. Let $\mathbb{U}_1, \mathbb{U}_2$ be subspaces of $\mathbb{V}$. Prove that $\mathbb{U}_1 \cup \mathbb{U}_2$ is not a vector subspace unless $\mathbb{U}_1 \subset \mathbb{U}_2$ or $\mathbb{U}_2 \subset \mathbb{U}_1$.

Assume $U_1 \cup U_2$ is a subspace of V.

Assume for a contradiction that $U_1 \not\subset U_2$ and $U_2 \not\subset U_1$. Then there exists $u_1 \in U_1 \setminus U_2$ and $u_2 \in U_1 \setminus U_2$.

Clearly, u_1 and $u_2 \in U_1 \cup U_2 \Rightarrow u_1 + u_2 \in U_1 \cup U_2$, since the latter is a subspace of V. Then we either have $u_1 + u_2 \in U_1$ or U_2. In the first case $u_2 = (u_1 + u_2) - u_1 \in U_1$ which is $\perp$ that $u_2 \in U_2 \setminus U_1$. The proof for u_1 is similar.

Thus, $\mathcal{U}_1 \subset \mathcal{U}_2$ or $\mathcal{U}_2 \subset \mathcal{U}_1$

5. (2 points) Let $\varphi_A : \mathbb{R}^5 \to \mathbb{R}, A = [1,1,1,1,1]$ and $\mathbb{W} = \ker \varphi_A$. Find two subspaces $\mathbb{U}_1, \mathbb{U}_2$ of $\mathbb{W}$ such that $\dim \mathbb{U}_1 = \dim \mathbb{U}_2 = 2$ and $\mathbb{U}_1 \cap \mathbb{U}_2 = 0$.

Consider $\mathcal{U}_1 = \left\langle \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\rangle$, $\mathcal{U}_2 = \left\langle \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\rangle$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \qquad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} = 0 \qquad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = 0$$

Clearly, $\dim(\mathcal{U}_1) = \dim(\mathcal{U}_2) = 2$.

Using Zassenhaus algorithm, let's prove $\mathcal{U}_1 \cap \mathcal{U}_2 = 0$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & \boxed{0 \; 0 \; 0 \; 0 \; 0} \end{bmatrix}$$

no intersection vector.

Thus, $\mathcal{U}_1 \cap \mathcal{U}_2 = 0$

6. (1 points) Is it true that

$$\dim(\mathbb{U}_1+\mathbb{U}_2+\mathbb{U}_3) = \dim(\mathbb{U}_1)+\dim(\mathbb{U}_2)+\dim(\mathbb{U}_3)-\dim(\mathbb{U}_1\cap\mathbb{U}_2)-\dim(\mathbb{U}_1\cap\mathbb{U}_3)-\dim(\mathbb{U}_2\cap\mathbb{U}_3)+\dim(\mathbb{U}_1\cap\mathbb{U}_2\cap\mathbb{U}_3)$$

for all $\mathbb{U}_1, \mathbb{U}_2, \mathbb{U}_3 \subset \mathbb{V}$?

Hint: it is not true, you can find a counterexamle in $\mathbb{V} = \mathbb{R}^2$.

Consider distinct U_1, U_2, U_3 in $\mathbb{R}^2$.
Then, $\dim(U_1+U_2+U_3)=2$, as any vector from U_3 can be expressed as a linear combination of vectors from U_1 and U_2 since they are distinct.
While $\dim(U_1)+\dim(U_2)+\dim(U_3)-\dim(U_1\cap U_2)-$
$-\dim(U_1\cap U_3)-\dim(U_2\cap U_3)+\dim(U_1\cap U_2\cap U_3)=$
$=1+1+1-0-0-0+0=3$
So, $2\neq 3$. $\Rightarrow \perp$.