3. (HW) Find the following limits:

(a) $\lim_{x\to+\infty}\left(\frac{x+1}{x+2}\right)^{1-x}$; (b) $\lim_{x\to 0}\left(\frac{2-3x}{5-4x}\right)^{\frac{1}{x^2}}$; (c) $\lim_{x\to 0}\left(\frac{1+2x}{1-3x}\right)^{\frac{3}{\sin 2x}}$.

a) $=\lim_{x\to\infty}\left(1-\frac{1}{x+2}\right)^{1-x}=\lim_{x\to\infty}\left(\left(1+\frac{1}{-x-2}\right)^{-x-2}\right)^{\frac{x-1}{x+2}}=\lim_{x\to\infty}\left(\left(1+\frac{1}{y}\right)^{y}\right)^{\frac{-y-3}{y}}=$

$-x-2=y$

$=\lim_{x\to\infty} e^{\frac{-y-3}{y}}=\lim_{x\to\infty} e^{\frac{x-1}{x+2}}=e$ $\quad \to =1$

b) $=\lim_{x\to 0}\left(1+\frac{x-3}{5-4x}\right)^{\frac{1}{x^2}}=\lim_{x\to 0}\left(\left(1+\frac{x-3}{5-4x}\right)^{\frac{5-4x}{x-3}}\right)^{\frac{x-3}{x^2(5-4x)}}=$

$=e^{\lim_{x\to 0}\frac{x-3}{x^2(5-4x)}}=e^{\lim_{x\to 0}\frac{0-3}{0(5-4\cdot 0)}}=e^{-\infty}=0$

c) $=\lim_{x\to 0}\left(\left(1+\frac{5x}{1-3x}\right)^{\frac{1-3x}{5x}}\right)^{\frac{15x}{\sin 2x(1-3x)}}=e^{\lim_{x\to 0}\left(\frac{15x}{\sin 2x(1-3x)}\right)}=$

$=e^{\lim_{x\to 0}\left(\frac{15}{2\cdot(1-0)}\right)}=e^{\frac{15}{2}}$

4. (HW) Find the limit $\lim_{x\to+\infty}(3x-2)\Big(\ln(5x+1)-\ln(5x-7)\Big)$.

$$=\lim_{x\to+\infty}\ln\left(\frac{5x+1}{5x-7}\right)^{3x-2}=\lim_{x\to+\infty}\ln\left(\left(1+\frac{8}{5x-7}\right)^{\frac{5x-7}{8}}\right)^{\frac{8(3x-2)}{5x-7}}=$$

$$=\lim_{x\to+\infty}\ln e^{\frac{8(3x-2)}{5x-7}}=\lim_{x\to+\infty}\frac{8(3x-2)}{5x-7}=\frac{24}{5}$$

5. (HW) Compute $\lim_{x\to a}\left(\frac{1+x}{2+x}\right)^{\frac{1-\sqrt{x}}{1-x}}$ if $a=0$, $a=1$, and $a=+\infty$.

$a=0$:
$$=\left(\frac{1+0}{2+0}\right)^{\frac{1-0}{1-0}}=\frac{1}{2}$$

$a=1$:
$$=\lim_{x\to1}\left(1+\frac{-1}{2+x}\right)^{\frac{1-\sqrt{x}}{1-x}}=\lim_{x\to1}\left(1+\frac{-1}{2+x}\right)^{\frac{1}{1+\sqrt{x}}}=\left(1-\frac{1}{3}\right)^{\frac{1}{2}}=$$
$$=\sqrt{\frac{2}{3}}=\frac{\sqrt{6}}{3}$$

$a=+\infty$:
$$=\lim_{x\to+\infty}\left(\left(1+\frac{-1}{2+x}\right)^{\frac{2+x}{-1}}\right)^{\frac{1-\sqrt{x}}{x^2+x-2}}=e^{\lim_{x\to+\infty}\frac{1-\sqrt{x}}{x^2+x-2}}=$$
$$=e^{\lim_{x\to+\infty}\frac{\frac{1}{x^2}-\frac{\sqrt{x}}{x^2}}{1+\frac{1}{x}-\frac{2}{x^2}}}=e^0=1$$

7. (HW) Find vertical and oblique asymptotes of the following functions:

(a) $f(x) = \dfrac{x+1}{x^2+3x-4}$; (b) $f(x) = \sqrt{\dfrac{x^3}{x-2}}$; (c) $f(x) = \sqrt{x^2-1} - x$;

(d) $f(x) = \dfrac{\sqrt{4x^4+1}}{|x|}$; (e) $f(x) = 2x + \operatorname{arccot} x$.

a) $x^2+3x-4 = (x+4)(x-1)$

$$\lim_{x\to -4-} \frac{-4+1}{16-12-4} = -\frac{3}{0} = -\infty$$

$$\lim_{x\to -4+} \frac{x+1}{x^2+3x-4} = \frac{-3}{-0} = +\infty$$

$$\lim_{x\to 1-} \frac{1+1}{1+3-4} = \frac{2}{-0} = -\infty$$

$$\lim_{x\to 1+} \frac{1+1}{1+3-4} = \frac{2}{0} = +\infty$$

So, -4 and 1 are vert. asymp.

$$k = \lim_{x\to\infty} \frac{x+1}{x^3+3x^2-4x} = \lim_{x\to\infty} \frac{\frac{1}{x^2}+\frac{1}{x^3}}{1+\frac{3}{x}-\frac{4}{x^2}} = \frac{0}{1} = 0$$

$$b = \lim_{x\to\infty} \frac{x+1}{x^2+3x-4} = \frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{3}{x}-\frac{4}{x^2}} = \frac{0}{1} = 0$$

$y = 0$

b) $f(x)=\sqrt{\frac{x^3}{x-2}}$

$$\lim_{x\to 2+}\sqrt{\frac{8}{2-2}} = \sqrt{\frac{8}{0}} = +\infty; \quad \lim_{x\to 2-} = \sqrt{\frac{8}{-0}} = -\infty$$

$$k_+ = \lim_{x\to+\infty}\sqrt{\frac{x^3}{x-2}}\cdot\frac{1}{x} = \lim_{x\to\infty}\sqrt{\frac{x^3}{x-2}}\cdot\sqrt{\frac{1}{x^2}} = \lim_{x\to\infty}\sqrt{\frac{x^3}{x^3-2x^2}} =$$

Ass. $x>0$

$$= \lim_{x\to+\infty}\sqrt{\frac{1}{1-0}} = 1$$

$$b_+ = \lim_{x\to+\infty}\left(\sqrt{\frac{x^3}{x-2}} - x\right) = \lim_{x\to+\infty}\frac{2x^2}{(x-2)\left(\sqrt{\frac{x^3}{x-2}}+x\right)} =$$

$$= \lim_{x\to+\infty}\frac{2}{\sqrt{1+\frac{2}{x-2}}+1} = \frac{2}{2} = 1$$

$y = x+1$

$$k_- = \lim_{x\to-\infty}\sqrt{\frac{x^3}{x-2}}\cdot\left(-\sqrt{\frac{1}{x^2}}\right) = \lim_{x\to-\infty}\sqrt{\frac{x^3}{-x^3+2x}} = \sqrt{\frac{1}{-1+0}} = -1$$

$$b_- = \lim_{x\to-\infty}\left(\sqrt{\frac{x^3}{x-2}}+x\right) = -\frac{2}{2} = -1 \quad (\text{From } b_+)$$

$y = -x+1$

c) $f(x)=\sqrt{x^2-1}-x$

The domain is $(-\infty;-1)\cup(1;+\infty)$, so there's no vert. asymp.

$$k_+ = \lim_{x\to+\infty}\left(\frac{\sqrt{x^2-1}-x}{x}\right) = \lim_{x\to+\infty}\left(\frac{x\sqrt{1-\frac{1}{x^2}}-x}{x}\right) =$$

$$= \lim_{x \to +\infty} \left(\frac{\sqrt{1 - \frac{1}{x^2}} - 1}{1} \right) = \frac{0}{1} = 0$$

$$b_+ = \lim_{x \to +\infty} \left(\sqrt{x^2 - 1} - x \right) = \lim_{x \to +\infty} \frac{x^2 - 1 - x^2}{\sqrt{x^2 - 1} + x} = \lim_{x \to +\infty} \frac{-1}{\sqrt{x^2 - 1} + x} =$$

$$= \lim_{x \to +\infty} \left(\frac{-\frac{1}{x}}{\left(\sqrt{1 - \frac{1}{x^2}} + 1\right)} = \frac{-0}{\sqrt{1 - 0} + 1} = 0\right.$$

$$y = 0 \cdot x + 0 \Rightarrow y = 0$$

$$k_- = \lim_{x \to -\infty} \left(\frac{\sqrt{x^2 - 1} - x}{x} \right) = \lim_{x \to -\infty} \left(-\sqrt{1 - \frac{1}{x^2}} - 1 \right) = -2$$

$$b_- = \lim_{x \to -\infty} \left(\sqrt{x^2 - 1} - x + 2x \right) = \lim_{x \to -\infty} \left(\sqrt{x^2 - 1} + x \right) = \text{mult. by conj. etc.}$$

$$= \lim_{x \to -\infty} \frac{-\frac{1}{x}}{-\sqrt{1 - \frac{1}{x^2}} - 1} = \frac{-0}{-1 - 1} = 0 \Rightarrow y = -2$$

d) $F(x) = \sqrt{\frac{4x^4 + 1}{|x|}}$

$$\lim_{x \to 0+} \sqrt{\frac{4 \cdot 0 + 1}{0}} = +\infty$$

$$\lim_{x \to 0-} \sqrt{\frac{4 \cdot 0 + 1}{|-0|}} = +\infty$$

$$k_+ = \lim_{x \to +\infty} \sqrt{\frac{4x^4 + 1}{|x|}} \cdot \frac{1}{x} \underset{\text{ass. } x > 0}{=} \lim_{x \to +\infty} \frac{\sqrt{4x^4 + 1}}{x^2} = \lim_{x \to +\infty} \frac{\sqrt{4 + \frac{1}{x^4}}}{1} = 2$$

$$b_- = \lim_{x \to +\infty} \frac{\sqrt{4x^4 + 1}}{|x|} - 2x = \lim_{x \to +\infty} \frac{x\sqrt{4x^2 + \frac{1}{x^2}} - 2x^2}{x} =$$

$$= \lim_{x \to +\infty} \sqrt{4x^2 + \frac{1}{x^2}} - 2x = \lim_{x \to +\infty} 2x - 2x + \frac{1}{x} = 0$$

$y = 2x$

$$k_- = \lim_{x \to -\infty} \frac{\sqrt{4x^4+1}}{-x^2} = \lim_{x \to -\infty} \frac{\sqrt{4+\frac{1}{x^4}}}{-1} = -2$$

$b_- = 0$

$y = -2x$

e) $F(x) = 2x + \operatorname{arccot}(x)$

Domain: $(-\infty; +\infty) \Rightarrow$ no vert. asymp.

$$k_+ = k_- = \lim_{x \to \infty} \frac{2x + \operatorname{arccot}(x)}{x} = \lim_{x \to \infty} \left(2 + \frac{\operatorname{arccot}(x)}{x}\right) = 2$$

$$b_+ = \lim_{x \to +\infty} 2x + \operatorname{arccot}(x) - 2x = 0 \Rightarrow y = 2x + 0$$

$$b_- = \lim_{x \to -\infty} 2x + \operatorname{arccot}(x) - 2x = \pi \quad y = 2x + \pi$$