9) $a > 1$, $a \% 2 = 0$, $a \% 4 \neq 0$:

$a = 2 \cdot p_1^{\alpha_1} \cdot \ldots p_s^{\alpha_s}$

Number of divisors $= (1+1)(\alpha_1+1)\ldots(\alpha_s+1)$

E.g. $NoD(6) = 2^{1} \cdot 3^{1} = (1+1)(1+1) = 4$

Number of odd divisors $= (\alpha_1+1)\ldots(\alpha_s+1)$

Because $a \% 4 \neq 0$, the number x (2^{x}) is always $= 1$, otherwise 2^x $(x > 1)$ is divisible by 4.

Now we have two formulas, NoD and NoOD, the only difference between them is that $NoOD = \frac{1}{2} NoD$

$(\alpha_1+1)\ldots(\alpha_s+1) = \frac{(1+1)}{2}(\alpha_1+1)\ldots(\alpha_s+1)$

So, if the number of odd divisors is equal to $\frac{1}{2}$ of the number of all divisors, "a" has the same amount of even and odd divisors.

10) We need to prove that $3 | x$ and 9 doesn't.

Because if $x = y^2$: $y = 3 \cdot p_1 \ldots p_s$

$y^2 = 3^2 p_1 \ldots p_s$

$3^2 \to 9 \Rightarrow 9 | y^2$

Add all digits of x: $100 \cdot 0 + 100 \cdot 1 + 100 \cdot 2 = 300 \Rightarrow$

$\Rightarrow 3 | x$, 9 doesn't divide 300 $\Rightarrow$ 9 doesn't divide x. $\Rightarrow$ then $x \neq y^2$

11) Notice that $6k+5$ produces only odd numbers. All primes >2 are odd, so $6k+5$ has a chance to produce primes. Indeed, $6\cdot 0+5=5$, $6+5=11$ – all primes so far. Now let's assume that $\exists k\ \forall x>k: 6x+5$ doesn't produce any primes. Let's put all the primes we've created before in a set $A=\{p_1 \dots p_s\}$. Now consider a new integer $p' = \underbrace{6}\,\underbrace{(p_1 \dots p_s - 1)}_{k} + \underbrace{5}$

$$\boxed{6\cdot k+5}$$

p' is not divisible by 2 and 3, because $6\cdot p_1 \dots p_s - 1$ is an even number, adding 5 to it makes it odd $\Rightarrow$ $\Rightarrow 2 \nmid p'$, also $6k+5 \equiv 2 \pmod 3 \Rightarrow 3 \nmid p'$.

Notice that another formula, $6k+1$, also produces only odd numbers and the product of two integers of this form produces an integer of the same form.

Suppose all primes dividing p' are of the form $6k+1$, then by using the "multiplicative" property, p' also has to be of the form $6k+1$ which contradicts with the initial statement $(p' = 6(p_1 \dots p_s - 1)+5)$

Thus, p' has a prime divisor p of the form

$6k+5$. By assumption, $p \in \{p_1 \dots p_s\}$ and $p | p_1 \dots p_s$. Also $p | p' \Rightarrow p | 6(p_1 \dots p_s) - p'$ (because p divides both numbers individually $\Rightarrow p$ divides their difference)

But $6p_1 \dots p_s - (6(p_1 \dots p_s - 1) + 5) = 6p_1 \dots p_s - 6p_1 \dots p_s + 6 - 5 =$
$= 1$ which contradicts the statement that p is prime. So, $6k+5$ produces infinitely many primes. Q.E.D.

13) $p > 3$, $24 | (p^2 - 1)$

$(p^2 - 1) = (p-1)(p+1)$

As p is prime:

case 1) $p \equiv 1 \pmod 3 \Rightarrow p-1 \equiv 0 \pmod 3$

case 2) $p \equiv -1 \pmod 3 \Rightarrow p+1 \equiv 0 \pmod 3$

Hence, $p^2 - 1$ is divisible by 3.

Consider $p \bmod 4$.

case 1) $p \equiv 1 \pmod 4 \Rightarrow p-1 \equiv 0 \pmod 4$

case 2) $p \equiv -1 \pmod 4 \Rightarrow p-1 \equiv 2 \pmod 4$ then p is divisible by 2. $p+1 \equiv 0 \pmod 4$

Hence, either $p-1$ or $p+1$ is divisible by 4 and the other one is divisible by 2 $\Rightarrow$ their product is divisible by 8.

Also $p^2 - 1$ is divisible by 3, then $8 \cdot 3 = 24 \Rightarrow$
$\Rightarrow p^2 - 1$ is always divisible by 24. Q.E.D.

14) $\neg\exists \{a_k\}_{k\in\mathbb{N}}$: $a_1 \dots a_k$ are pairwise coprime.

Consider a set $\{a_k\}_{k\in\mathbb{N}}$, where $|a_n - a_{n+1}| = d$.

Find an element of a set $|a_s| > 1$, then

$a_s \cdot d + a_s = a_k$ $\longrightarrow$

$GCD(a_s, a_k) = |a_s|$ Q.E.D.

Note: an element of an arith. prog. is: $a' = a_s + d \cdot x$

if $x = a_s$, then $a_s | a_s + d \cdot a_s$,

so, the remainder is 0 and a_s is $gcd(a_s, a_s + d \cdot a_s)$

15) $\frac{n^2 - n + 1}{n^2 + 1}$ is irreducible.

By Euclidian algorithm:

$(n^2 - n + 1) = 1(n^2 + 1) - n$

$(n^2 + 1) = (-n) \cdot (-n) + 1$

$-n = 1 \cdot (-n) + 0 \implies gcd(1, 0) = 1 \implies gcd(n^2 - n + 1, n^2 + 1) = 1$

It's also possible to say that $\frac{n^2 - n + 1}{n^2 + 1} = 1 - \frac{n}{n^2 + 1}$

And $\frac{n}{n^2 + 1}$ is irreducible for any n, because if $n = p$ is prime, only p^2 would reduce it (e.g. $\frac{p}{p^2} = \frac{1}{p}$).

$n \neq p$: By FTA, $n = p_1^{\alpha_1} \dots p_s^{\alpha_s}$, $n^2 + 1 = p_1^{\beta_1} \dots p_s^{\beta_1}$ $\Big\}$ don't share any primes $\implies gcd = 1$

12) Let $x = p_1^{x_1} \cdots p_s^{x_s}$

$y = p_1^{y_1} \cdots p_s^{y_s}$

$z = p_1^{z_1} \cdots p_s^{z_s}$

$$\operatorname{lcm}(x,y,z) = \operatorname{lcm}(\operatorname{lcm}(x,y),z) =$$

$$= p_1^{\max(\max(x_1,y_1),z_1)} \cdots p_s^{\max(\max(x_s,y_s),z_s)} =$$

$$= p_1^{\max(x_1,y_1,z_1)} \cdots p_s^{\max(x_s,y_s,z_s)}$$ (same for gcd with min)

$$\operatorname{lcm}(x,y,z) = \frac{x \cdot y \cdot z \cdot \gcd(x,y,z)}{\gcd(x,y) \cdot \gcd(x,z) \cdot \gcd(y,z)}$$

$$p_1^{\max(x_1,y_1,z_1)} \cdots p_s^{\max(x_s,y_s,z_s)} = \frac{\underbrace{xyz} \cdot p_1^{\min(x_1,y_1,z_1)} \cdots p_s^{\min(x_1,y_1,z_1)}}{p_1^{\min(x_1,y_1)+\min(x_1,z_1)+\min(y_1,z_1)} \cdots p_s^{\min(x_s,y_s)+\min(x_s,z_s)+\min(y_s,z_s)}}$$

$$p_1^{\max(x_1,y_1,z_1)} \cdots p_s^{\max(x_s,y_s,z_s)} = \frac{p_1^{x_1+y_1+z_1+\min(x_1,y_1,z_1)} \cdots p_s^{x_s+y_s+z_s+\min(x_s,y_s,z_s)}}{p_1^{\min(x_1,y_1)+\min(x_1,z_1)+\min(y_1,z_1)} \cdots p_s^{\min(x_s,y_s)+\min(x_s,z_s)+\min(y_s,z_s)}}$$

After the powers substract from the second part of the equation, we get:

$\max(x_i, y_i, z_i) = x_i+y_i+z_i+\min(x_i,y_i,z_i)-\min(x_i;y_i) -$
$-\min(x_i; z_i)-\min(y_i,z_i)$

$x<y<z$:

$z_i = x_i+y_i+z_i+x_i-x_i-x_i-y_i$

$z_i=z_i$

$x<z<y$:

$y_i = x_i+y_i+z_i+x_i-x_i-x_i-z_i$

$y_i=y_i$

$y<z<x$:

$x_i=x_i+y_i+z_i+y_i-y_i-z_i-y_i$

$x_i=x_i$

$y<x<z$

$z_i=x_i+y_i+z_i+y_i-y_i-x_i-y_i$

$z_i=z_i$

$z<x<y$

$y_i=x_i+y_i+z_i+z_i-x_i-z_i-z_i$

$$\frac{y_i = y_i}{z < y < x}$$

$$x_i = x_i + y_i + z_i + z_i - y_i - z_i - z_i$$

$$x_i = x_i \quad \text{Q.E.D.}$$