1°. (2 points). Let $\varphi : \mathbb{R}[x,n] \to \mathbb{R}[x,n]$ defined as $\varphi(p(x)) = x^2 \cdot (p(x-1) - p(x+1))'$.

(a) Find $Spec(\varphi)$ (b) For each $\lambda \in Spec(\varphi)$ find its algebraic and geometric multiplicity.

The matrix is upper-triangular in the standard basis (explain why). Avoid doing unnecessary calculations.

a) $1 \to 0 = v_1$

$x \to x^2((x-1)-(x+1))' = 0 = v_2$

$x^2 \to x^2((x-1)^2-(x+1)^2)' = -4x^2 = v_3$

$x^3 \to x^2((x-1)^3-(x+1)^3)' = -12x^3 = v_4$

$x^4 \to x^2((x-1)^4-(x+1)^4)' = -24x^4 - 8x^2 = v_5$

$\vdots$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & -8 \\ 0 & 0 & 0 & -12 & 0 \\ 0 & 0 & 0 & 0 & -24 \end{bmatrix}$$

The matrix is upper triangular, as operator cannot increase the maximum power of the input. Therefore, spec consists of numbers from the diagonal.

$spec = \{-2(n-1)(n-2),\ n = \dim(\mathbb{R}[x,n])\}$

b) a. m. of 0 is 2.

g. m. of 0 is 2.

a. m. and g.m. of $-2(n-1)(n-2)$ $n \geq 3$ is 1, as it is only encountered once and the rank of $(A - (-2(n-1)(n-2))I) = n-(n-1) = 1$

2°. (1 point). Let $A = [a_1, \ldots, a_n]$ (at least one number is nonzero) and let $B = A^T A$. Consider $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, $\varphi(v) = Bv$.

(a) Find $Spec(\varphi)$ (b) For each $\lambda \in Spec(\varphi)$ find its algebraic and geometric multiplicity.

> What is the image of φ? What is the dimension of the kernel? The smart solution takes about two lines.

$\varphi: \mathbb{R}^n \to \mathbb{R} \to \mathbb{R}^n$, hence the image is $\left\langle \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right\rangle$

$\dim \ker \varphi = n-1$

a) $$A^T A \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A^T [a_1 x_1 + \ldots + a_n x_n] = \begin{bmatrix} a_1 \sum_{k=1}^n a_k x_k \\ \vdots \quad\quad \vdots \\ a_n \sum_{k=1}^n a_k x_k \end{bmatrix} =$$

$$= \sum_{k=1}^n a_k x_k \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$spec\, \varphi = \{0, \sum_{k=1}^n a_k^2\}$$

b) a.m. $(0) = \dim \ker \varphi = n-1$

g.m. $(0) = n - rk\varphi = n-(n-1) = 1$

a.m. $\sum_{k=1}^n a_k^2 = 1$

g.m. $\sum_{k=1}^n a_k^2 = 1$

3°. (0.4 points per item). For each matrix A find its characteristic $\chi_A(x)$ and minimal $m_A(x)$ polynomials.

(a) $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ (e) $\begin{bmatrix} -3 & 2 & 4 & 0 \\ -7 & 5 & 5 & 0 \\ -3 & 1 & 5 & 0 \\ -3 & 1 & 2 & 3 \end{bmatrix}$

Writing down $m_A(x)$ without proof of minimality will not be accepted.

> Recall that $\chi_A(x)$ and $m_A(x)$ have the same roots.

a) $\chi_A(x) = (x-2)^3(x-3)$

$$m_A(x) = (x-2)^2(x-3) \Rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^2 \cdot \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 0$$

$(x-2)(x-3)$ doesn't work ☹

b) $\chi_A(x) = (x-2)^3(x-3)$

$$m_A(x) = (x-2)^2(x-3) \Rightarrow \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1 \end{bmatrix} \cdot \begin{bmatrix} -1&1&0&0\\ 0&-1&1&0\\ 0&0&-1&0\\ 0&0&0&0 \end{bmatrix} = 0$$

$(x-2)(x-3)$ doesn't work either

c) $\chi_A(x) = (x-2)^2(x-3)^2$

$$m_A(x) = (x-2)(x-3) \Rightarrow \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&1&1\\ 0&0&0&1 \end{bmatrix} \cdot \begin{bmatrix} -1&0&0&0\\ 0&-1&0&0\\ 0&0&0&1\\ 0&0&0&0 \end{bmatrix}^2 = 0$$

only dropped one power

d) $\chi_A(x) = (x-2)^2(x-3)^2$

$$m_A(x) = (x-2)^2(x-3)^2 \Rightarrow \begin{bmatrix} 0&1&0&0\\ 0&0&0&0\\ 0&0&1&1\\ 0&0&0&1 \end{bmatrix}^2 \cdot \begin{bmatrix} -1&1&0&0\\ 0&-1&0&0\\ 0&0&0&1\\ 0&0&0&0 \end{bmatrix}^2 = 0$$

It didn't drop a single power

e) $$\begin{vmatrix} -3-\lambda & 2 & 4 & 0\\ -7 & 5-\lambda & 5 & 0\\ -3 & 1 & 5-\lambda & 0\\ -3 & 1 & 2 & 3-\lambda \end{vmatrix} = (\lambda-3)^2(\lambda-2)^2$$

$$m_A(x) = (x-3)(x-2)^2 \Rightarrow \begin{bmatrix} -6&2&4&0\\ -7&2&5&0\\ -3&1&2&0\\ -3&1&2&0 \end{bmatrix} \cdot \begin{bmatrix} -5&2&4&0\\ -7&3&5&0\\ -3&1&3&0\\ -3&1&2&1 \end{bmatrix}^2 = 0$$

No emotions...

4°. (1 point). For $A = \begin{bmatrix} 4 & -1 & -1 \\ 2 & 1 & -1 \\ 3 & -1 & 0 \end{bmatrix}$ find $a, b, c \in \mathbb{R}$ such that $A^{-1} = aA^2 + bA + cI$.

$$\chi_A(x) = -(x-2)^2(x-1) = -x^3 + 5x - 8x + 4$$

$$-A^3 + 5A^2 - 8A + 4 = 0$$

$$A \cdot \frac{1}{4}(A^2 - 5A + 8 \cdot I) = I$$

$$\frac{1}{4}\begin{bmatrix} 4 & -1 & -1 \\ 2 & 1 & -1 \\ 3 & -1 & 0 \end{bmatrix}^2 - \frac{5}{4}\begin{bmatrix} 4 & -1 & -1 \\ 2 & 1 & -1 \\ 3 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = A^{-1}$$

$$a = \frac{1}{4}, \quad b = -\frac{5}{4}, \quad c = 2$$

5. (1 point). Does there exist a 10×10 matrix such that $A^{100} = 0$ and $A^{99} \neq 0$?

$\dim = 10 \cdot 10 = 100$

$m_\varphi(x) = t^k, \; k \leq n$

k is 100 in our case $\quad 100 \leq 100$

Hence, it does exist.

6. (1 point). Find an example of a matrix A such that its minimal polynomial is equal to $(x+2)^5$ (the size of A is up to you).

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} = A \Rightarrow m_\varphi(x) = (x+2)^5$$

7. (2 points). Suppose that there are three types of pokemons: blue, red, and green pokemons.

It is known that in one day time:

1. a blue pokemon evolves into one red pokemon;
2. a red pokemon evolves into two green pokemons;
3. a green pokemon evolves into two blue and one red pokemons.

For example: if you start with one blue and one green pokemons, then, *in one day* you will have two blue and two red pokemons ($B \to R$ and $G \to 2B + R$) and *in two days* you will have two red and four green pokemons ($2B \to 2R$ and $2R \to 4G$).

Suppose you start with five blue pokemons, then, how many blue, red, and green pokemons will you have in 60 days?

> You will need to find the remainder of x^{60} when divided by a certain polynomial $\chi(x)$ with complex roots. The remainder has real coefficients but you will need to use complex numbers in the intermediate steps.

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \xrightarrow{day} \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad T(day, pokemons) = \begin{bmatrix}0&0&2\\1&0&1\\0&2&0\end{bmatrix} = A$$

$$\begin{bmatrix}0\\1\\0\end{bmatrix} \xrightarrow{day} \begin{bmatrix}0\\0\\2\end{bmatrix} \qquad \chi_{day}(x) = -x^3+2x+4$$

$$x^{60} = q(x)\cdot(-x^3+2x+4)+r(x)$$

$$\begin{bmatrix}0\\0\\1\end{bmatrix} \xrightarrow{day} \begin{bmatrix}2\\1\\0\end{bmatrix} \qquad \begin{cases} r(2) = 2^{60} \\ r(-1+i) = \left(\left(\left(-1+i\right)^2\right)^2\right)^{15} = -2^{30} \\ r(-1-i) = (-1-i)^{60} = -2^{30} \end{cases}$$

$$r(x) = 2^{60}\frac{(x+1-i)(x+1+i)}{(2+1-i)(2+1+i)} - 2^{30}\frac{(x-2)(x+1+i)}{(-1+i-2)(-1+i+1+i)} -$$

$$-2^{30}\frac{(x-2)(x+1-i)}{(-1-i-2)(-1-i+1-i)} =$$

$$= 2^{60}\left(\frac{x^2+2x+2}{10}\right) - 2^{30}\left(\frac{-x^2-2x+8}{20} + \frac{3x^2-4x-4}{20}i\right) + 2^{30}\left(\frac{x^2+2x-8}{20} + \frac{3x^2-4x-4}{20}i\right) =$$

$$= \frac{2^{60}x^2+2^{61}x+2^{61}}{10} - \frac{-2^{30}x^2-2^{31}x+2^{33}}{20} + \frac{2^{30}x^2+2^{31}x-2^{33}}{20} =$$

$$= \frac{2^{60}x^2+2^{61}x+2^{61}}{10} + \frac{2^{31}x^2+2^{32}x-2^{34}}{20} =$$

$$= \frac{2^{61}x^2+2^{31}x^2+2^{62}x+2^{32}x+2^{62}-2^{34}}{20} =$$

$$= \frac{(2^{61}+2^{31})}{20}x^2 + \frac{(2^{62}+2^{32})}{20}x + \frac{(2^{62}-2^{34})}{20}$$

$$A^{60} = \frac{(2^{61}+2^{31})}{20}A^2 + \frac{(2^{62}+2^{32})}{20}A + \frac{(2^{62}-2^{34})}{20}I$$

$$A^{60} = \frac{2^{61}+2^{31}}{20} \cdot \begin{bmatrix} 0 & 4 & 0 \\ 0 & 2 & 2 \\ 2 & 0 & 2 \end{bmatrix} + \frac{2^{62}+2^{32}}{20} \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} + \frac{2^{62}-2^{34}}{20} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & \frac{2^{61}+2^{31}}{5} & 0 \\ 0 & \frac{2^{61}+2^{31}}{10} & \frac{2^{61}+2^{31}}{10} \\ \frac{2^{61}+2^{31}}{10} & 0 & \frac{2^{61}+2^{31}}{10} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{2^{62}+2^{32}}{10} \\ \frac{2^{62}+2^{32}}{20} & 0 & \frac{2^{62}+2^{32}}{20} \\ 0 & \frac{2^{62}+2^{32}}{10} & 0 \end{bmatrix} + \begin{bmatrix} \frac{2^{62}-2^{34}}{20} & 0 & 0 \\ 0 & \frac{2^{62}-2^{34}}{20} & 0 \\ 0 & 0 & \frac{2^{62}-2^{34}}{20} \end{bmatrix} =$$

$$\Rightarrow \begin{bmatrix} \frac{2^{62}-2^{34}}{20} & \frac{2^{61}+2^{30}}{5} & \frac{2^{62}+2^{32}}{10} \\ \frac{2^{62}+2^{32}}{20} & \frac{2^{63}+2^{32}-2^{34}}{20} & \frac{2^{63}+2^{33}}{20} \\ \frac{2^{61}+2^{31}}{10} & \frac{2^{62}+2^{32}}{10} & \frac{2^{63}+2^{32}-2^{34}}{20} \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{2^{62}-2^{34}}{4} \\ \frac{2^{62}+2^{32}}{4} \\ \frac{3^{61}+2^{31}}{2} \end{bmatrix}$$

blue

green

red

That's a lot of fucking pockemons.