2. (HW) Find the following limits:

(a) $\lim\limits_{n\to\infty} \dfrac{n^2+4n-11}{3n^3-4n^2+5n-2}$; (b) $\lim\limits_{n\to\infty} \dfrac{5n^4-6n+2}{-n^2+n-1}$; (c) $\lim\limits_{n\to\infty} \dfrac{\sqrt{3n^4+2n^3+5}}{n^2+7}$;

(d) $\lim\limits_{n\to\infty} \dfrac{1-n+2n^2}{2+4+\cdots+2n}$; (e) $\lim\limits_{n\to\infty} \dfrac{n^{10}-1}{1+n\cdot 1.1^n}$; (f) $\lim\limits_{n\to\infty} \left(\dfrac{-5n+4n^2-4}{n-5} - \dfrac{4n^2-3}{n+4}\right)$.

a) $$\lim_{n\to\infty} \frac{n^2+4n-11}{3n^3-4n^2+5n-2} = \lim_{n\to\infty} \frac{\frac{1}{n}+\frac{4}{n^2}-\frac{11}{n^3}}{3-\frac{4}{n}+\frac{5}{n^2}-\frac{2}{n^3}} = \frac{0+0-0}{3-0+0-0} = 0$$

b) $$\lim_{n\to\infty} \frac{5n^4-6n+2}{-n^2+n-1} = \lim_{n\to\infty} \frac{5n^2-\frac{6}{n}+\frac{2}{n^2}}{-1+\frac{1}{n}-\frac{1}{n^2}} = \frac{5\cdot\infty-0+0}{-1+0-0} = \frac{\infty}{-1} = -\infty$$

c) $$\lim_{n\to\infty} \frac{\sqrt{3n^4+2n^3+5}}{n^2+7} = \frac{\sqrt{3+\frac{2}{n}+\frac{5}{n^4}}}{1+\frac{7}{n^2}} = \frac{\sqrt{3+0+0}}{1+0} = \sqrt{3}$$

d) $$\lim_{n\to\infty} \frac{1-n+2n^2}{2+4+\ldots+2n} = \lim_{n\to\infty} \frac{1-n+2n^2}{n^2+n} = \lim_{n\to\infty} \frac{\frac{1}{n^2}-\frac{1}{n}+2}{1+\frac{1}{n}} = \frac{0-0+2}{1+0} = 2$$

e) $$\lim_{n\to\infty} \frac{n^{10}-1}{1+n\cdot 1.1^n} = 0$$

$\nabla\ n^{10} < n\cdot 1{,}1^n$ for $n\to\infty$, so:

$$\lim_{n\to\infty} \frac{\text{smaller number}}{\text{bigger number}} = 0$$

f) $$\lim_{n\to\infty} \left(\frac{-5n+4n^2-4}{n-5} - \frac{4n^2-3}{n+4}\right) = \lim_{n\to\infty} \frac{(-5n+4n^2-4)(n+4)-(4n^2-3)(n-5)}{(n-5)(n+4)} =$$

$-4n^3+3n+20n^2-15$

$$= \lim_{n\to\infty} \frac{-5n^2+4n^3-4n-20n+16n^2-16-(4n^3-3n-20n^2+15)}{n^2+4n-5n-20} =$$

$$= \lim_{n\to\infty} \frac{31n^2-21n-31}{n^2-n-20} = \lim_{n\to\infty} \frac{31-\frac{21}{n}-\frac{31}{n^2}}{1-\frac{1}{n}-\frac{20}{n^2}} = \frac{31-0-0}{1-0-0} = 31$$

Problem 3 ↓

3. (HW) Find the following limits:

$$\text{(a)} \lim_{n\to\infty} \frac{-5n^8+n-6}{\sqrt{6n^{16}+7n-6}+\sqrt{7n^8-3}}; \qquad \text{(b)} \lim_{n\to\infty} \sqrt[n]{\frac{n^2+4^n}{n+5^n}}.$$

a) $\lim\limits_{n\to\infty} \dfrac{-5n^8+n-6}{\sqrt{6n^{16}+7n-6}+\sqrt{7n^8-3}} = \lim\limits_{n\to\infty} \dfrac{-5+\frac{1}{n^7}-\frac{6}{n^8}}{\sqrt{6+\frac{7}{n^{15}}-\frac{6}{n^{16}}}+\sqrt{\frac{7}{n^8}-\frac{3}{n^{16}}}} =$

$= \lim\limits_{n\to\infty} \dfrac{-5+0-0}{\sqrt{6+0-0}+\sqrt{0+0}} = -\dfrac{5}{\sqrt{6}}$

b) $\lim\limits_{n\to\infty} \sqrt[n]{\dfrac{n^2+4^n}{n+5^n}} = ?$

∇ Let's look at a simpler limit:

$\lim\limits_{n\to\infty} \dfrac{n^2+4^n}{n+5^n}$; Obviously, 4^n and $5^n > n^2$ and n for $n\to\infty$, so we can rewrite it as:

$\lim\limits_{n\to\infty} \dfrac{4^n}{5^n} = \lim\limits_{n\to\infty} \left(\dfrac{4}{5}\right)^n = 0$

Knowing this, let's get back to the original limit:

$\lim\limits_{n\to\infty} \sqrt[n]{\dfrac{n^2+4^n}{n+5^n}}$; Let's rewrite it as: $\lim\limits_{n\to\infty} \sqrt[n]{\left(\dfrac{4}{5}\right)^n} = \lim\limits_{n\to\infty} \dfrac{4}{5}^{n\cdot\frac{1}{n}} = \dfrac{4}{5}$ $\blacktriangle$

4. (HW) Find the following limits:

$$\text{(a)} \lim_{n\to\infty} \frac{2\cdot 5^n - 5\cdot 4^n + 3}{4\cdot 5^n - 3\cdot 4^n + 2}; \qquad \text{(b)} \lim_{n\to\infty} \frac{2\cdot 6^{-n} + 5\cdot 5^{-n}}{4\cdot 5^{-n} - 3\cdot 6^{-n}}.$$

a) $\lim\limits_{n\to\infty} \frac{2\cdot 5^n - 5\cdot 4^n + 3}{4\cdot 5^n - 3\cdot 4^n + 2} = \frac{2}{4} = \frac{1}{2}$

$\triangledown$

(+3) in the numerator is negligable for $n\to\infty$, same goes for (+2) in the denominator. Let's rewrite!

$\lim\limits_{n\to\infty} \frac{2\cdot 5^n - 5\cdot 4^n}{4\cdot 5^n - 3\cdot 4^n}$; as $5^n > 4^n$ for $n\to\infty$, we can simplify further:

$$\lim_{n\to\infty} \frac{2\cdot 5^n}{4\cdot 5^n} = \lim_{n\to\infty} \frac{2}{4} = \frac{1}{2} \quad \blacktriangle$$

b) $\lim\limits_{n\to\infty} \frac{2\cdot 6^{-n} + 5\cdot 5^{-n}}{4\cdot 5^{-n} - 3\cdot 6^{-n}} = \frac{5}{4}$

$\triangledown$ Proof 1: the same idea as above we get:

$$\lim_{n\to\infty} \frac{5\cdot 5^{-n}}{4\cdot 5^{-n}} = \frac{5}{4} \quad \blacktriangle$$

$\triangledown$ Proof 2: Division by (5^{-n}):

$$\lim_{n\to\infty} \frac{2\cdot 6^{-n} + 5\cdot 5^{-n}}{4\cdot 5^{-n} - 3\cdot 6^{-n}} = \lim_{n\to\infty} \frac{2\cdot\left(\frac{6}{5}\right)^{-n} + 5}{4 - 3\cdot\left(\frac{6}{5}\right)^{-n}} = \frac{2\cdot 0 + 5}{4 - 3\cdot 0} = \frac{5}{4} \quad \blacktriangle$$

6. (HW) Find the following limits:

(a) $\lim_{n\to\infty} \sqrt{n}\left(\sqrt{n+2}-\sqrt{n-1}\right)$; (b) $\lim_{n\to\infty} \left(\sqrt{n^2+3n-1}-\sqrt{n^2-n}\right)$.

a) $\lim_{n\to\infty} \sqrt{n}\left(\sqrt{n+2}-\sqrt{n-1}\right) =$

$= \lim_{n\to\infty} \dfrac{\sqrt{n}\left(\sqrt{n+2}-\sqrt{n-1}\right)\left(\sqrt{n+2}+\sqrt{n-1}\right)}{\left(\sqrt{n+2}+\sqrt{n-1}\right)} =$

$= \lim_{n\to\infty} \dfrac{3\sqrt{n}}{\left(\sqrt{n+2}+\sqrt{n-1}\right)} = \lim_{n\to\infty} \dfrac{3}{\left(\sqrt{1+\frac{2}{n}}+\sqrt{1-\frac{1}{n}}\right)} =$

$= \dfrac{3}{\left(\sqrt{1+0}+\sqrt{1-0}\right)} = \dfrac{3}{2}$ ▲

b) $\lim_{n\to\infty} \left(\sqrt{n^2+3n-1}-\sqrt{n^2-n}\right) =$

$= \lim_{n\to\infty} \dfrac{\left(\sqrt{n^2+3n-1}-\sqrt{n^2-n}\right)\left(\sqrt{n^2+3n-1}+\sqrt{n^2-n}\right)}{\left(\sqrt{n^2+3n-1}+\sqrt{n^2-n}\right)} =$

$= \lim_{n\to\infty} \dfrac{(n^2+3n-1-n^2+n)}{\left(\sqrt{n^2+3n-1}+\sqrt{n^2-n}\right)} = \lim_{n\to\infty} \dfrac{4n-1}{\left(\sqrt{n^2+3n-1}+\sqrt{n^2-n}\right)} =$

$= \lim_{n\to\infty} \dfrac{4-\frac{1}{n}}{\left(\sqrt{1+\frac{3}{n}-\frac{1}{n^2}}+\sqrt{1-\frac{1}{n}}\right)} = \dfrac{4-0}{\sqrt{1+0-0}+\sqrt{1-0}} = \dfrac{4}{2} = 2$ ▲