(b) (HW) $\lim_{n\to\infty} \left(\sqrt{n^2+1} - \sqrt{n^2-1}\right) \cdot \sin(n^2+3)$;

1) Clearly, $\lim_{n\to\infty} \sin(n^2+3)$ doesn't exist, so if we prove that $\lim_{n\to\infty}(\sqrt{n^2+1}-\sqrt{n^2-1})=0$ then the original seq. is also an infinitesimal.

2) Find $\lim_{n\to\infty}(\sqrt{n^2+1}-\sqrt{n^2-1})$:

$$\lim_{n\to\infty}\frac{(\sqrt{n^2+1}-\sqrt{n^2-1})(\sqrt{n^2+1}+\sqrt{n^2-1})}{\sqrt{n^2+1}+\sqrt{n^2-1}} = \lim_{n\to\infty}\frac{n^2+1-n^2+1}{\sqrt{n^2+1}+\sqrt{n^2-1}} = \lim_{n\to\infty}\frac{2}{\sqrt{n^2+1}+\sqrt{n^2-1}} =$$

$$= \lim_{n\to\infty}\frac{\frac{2}{n}}{\sqrt{1+\frac{1}{n^2}}+\sqrt{1-\frac{1}{n^2}}} = \frac{0}{1+0+1-0} = 0 \quad \text{Q.E.D.}$$

(d) (HW) $\lim_{n\to\infty} \frac{(4\cos n - 3n)^2(2n^5 - n^3 + 1)}{(6n^3 + 5n\sin n)(n+2)^4}$.

1) $(4\cos n - 3n)^2 = n^2(\frac{4\cos n}{n} - 3)^2$

2) $(2n^5 - n^3 + 1) = n^5(2 - \frac{1}{n^2} + \frac{1}{n^5})$

3) $(6n^3 + 5n\sin n) = n^3(6 + \frac{5\sin n}{n^2})$

4) $(n+2)^4 = n^4(1+\frac{2}{n})^4$

5) $\lim_{n\to\infty}\frac{n^7(\frac{4\cos n}{n}-3)^2(2-\frac{1}{n^2}+\frac{1}{n^5})}{n^7(6+\frac{5\sin n}{n^2})(1+\frac{2}{n})^4} = \lim_{n\to\infty}\frac{-3^2\cdot 2}{6\cdot 1^4} = \lim_{n\to\infty}\frac{18}{6} = \lim_{n\to\infty} 3 = 3$

4. (HW) Prove that the sequence $\{x_n\}$, given by $x_{n+1} = \sqrt{2+x_n}$, $x_1 = \sqrt{2}$, is convergent and find the limit of this sequence.

1) Prove $\{x_n\}$ is bounded:

Obviously, lower bound is 0, as $\sqrt{smth} > 0$

Assume $x_n < 2$:
base step: $\sqrt{2} < 2$
inductive step: $x_{n+1} = \sqrt{2+x_n} < \sqrt{2+2} = 2$

Thus, upper bound is $2 \Rightarrow \{x_n\}$ is bounded.

2) Prove $\{x_n\}$ is $\uparrow$:

$$x_{n+1}^2 - x_n^2 = 2 + x_n - x_n^2 = -1(x_n+1)(x_n-2) = \underbrace{(x_n+1)}_{>0}\underbrace{(2-x_n)}_{>0}$$

3) By Weierstrass th.:

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2+x_n} = \sqrt{2+\lim_{n\to\infty} x_n}$$

$$\lim_{n\to\infty} x_n = A \Rightarrow A = \sqrt{2+A}$$

$$A^2 = 2+A$$

$$A^2 - A - 2 = 0$$

$A_1 = -1$ $\quad$ $\{x_n\} > 0$, so -1 doesn't work.

$$A_2 = 2$$

So, $\lim_{n\to\infty} x_n = 2$

7. (HW) Prove that the sequence $\{x_n\}$, given by $x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{x_n}\right)$, is convergent and find the limit of this sequence if $x_0 = 2$. (*Hint.* Use inequality $\frac{x+y}{2} \geqslant \sqrt{xy}$ for $x \geqslant 0$, $y \geqslant 0$)

1) Prove $\{x_n\}$ is bounded:

Obviously, $\sup\{x_n\} = 2$, as $x_n \downarrow$ which is proven later.

Assume $x_n > 1$:

Base step: $2 > 1$

Inductive step: $x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{x_n}\right) > \frac{1}{2}(1+1) = \frac{2}{2} = 1$

$\inf\{x_n\} = 1$

2) Prove $\{x_n\}$ is $\downarrow$:

$$x_{n+1} - x_n = \frac{1}{2}\left(x_n + \frac{1}{x_n}\right) - x_n = \frac{x_n^2+1}{2x_n} - x_n = \frac{-x_n^2+1}{2x_n} < 0 \text{ for } \{x_n\}$$

3) By Weierstrass th:

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(x_n + \frac{1}{x_n}\right) = \lim_{n\to\infty} \frac{x_n^2+1}{2x_n} = \frac{\left(\lim_{n\to\infty} x_n^2\right)+1}{2\lim_{n\to\infty} x_n}$$

$$\lim_{n\to\infty} x_n = A$$

$$A = \frac{A^2+1}{2A}$$

$$\frac{A^2+1}{2A} - \frac{2A^2}{2A} = 0$$

$$\frac{-A^2+1}{2A} = 0$$

$A_1 = -1$ $\{x_n\} > 0$, so -1 doesn't work.

$A_2 = 1$

So, $\lim_{n\to\infty} x_n = 1$