2) a) $\int_0^3 \frac{x+2}{\sqrt{x+1}}\,dx =$

$$\int \frac{x\,dx}{\sqrt{x+1}} \Big|\begin{matrix} u=x+1 \\ du=dx \end{matrix} = \int \frac{u-1}{\sqrt{u}}\,du = \int \frac{u}{\sqrt{u}}\,du - \int \frac{du}{\sqrt{u}} =$$

$$= \int u^{1/2}\,du - 2\sqrt{u} + C = \frac{u^{3/2}}{3/2} - 2\sqrt{u} + C = \frac{2}{3}(x+1)^{3/2} - 2\sqrt{x+1} + C$$

$$\int \frac{2}{\sqrt{x+1}}\,dx \Big|\begin{matrix} u=x+1 \\ du=dx \end{matrix} = 2\int \frac{du}{\sqrt{u}} = 4\sqrt{u} + C = 4\sqrt{x+1} + C$$

$$= \left(\frac{2}{3}(x+1)^{3/2} + 2\sqrt{x+1}\right)\Big|_0^3 = \frac{2}{3}\cdot 4^{3/2} + 4 - \frac{2}{3} - 2 = \frac{20}{3}$$

b) $\int_0^2 x^2\sqrt{4-x^2}\,dx =$

$$\int x^2\sqrt{4-x^2}\,dx \Big|\begin{matrix} x=2\sin t \\ dx=2\cos t\,dt \end{matrix} = \int 4\sin^2 t\,\sqrt{4-4\sin^2 t}\cdot 2\cos t\,dt =$$

$$= 16\int \sin^2 t\,\sqrt{1-\sin^2 t}\,\cos t\,dt = 16\int \sin^2 t\cos^2 t\,dt =$$

$$= 16\int (\sin t\cos t)^2\,dt = 4\int (2\sin t\cos t)^2\,dt =$$

$$= 4\int \sin^2 2t\,dt = 2\int (1-\cos 4t)\,dt = 2t - \int \cos 4t\,dt + C =$$

$$= 2t - \frac{1}{4}\sin 4t + C \Big| t = \arcsin\frac{x}{2} =$$

$$\sin 4x = 2\sin 2x\cos 2x = 4\sin x\cos x(\cos^2 x - \sin^2 x)$$

$$= 2\arcsin\frac{x}{2} - \frac{1}{4}\left(\frac{1}{2}\left(2\sqrt{2-x}\cdot x\cdot\sqrt{x+2} - \sqrt{2-x}\cdot x^3\cdot\sqrt{x+2}\right)\right) + C =$$

$$= 2\arcsin\frac{x}{2} - \frac{x\sqrt{4-x^2}}{4} + \frac{x^3\sqrt{4-x^2}}{8} + C$$

$$\ominus \left(2\arcsin\frac{x}{2} - \frac{x\sqrt{4-x^2}}{4} + \frac{x^3\sqrt{4-x^2}}{8}\right)\Big|_0^2 = \pi$$

c) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{6-5\sin x+\sin^2 x}\,dx \ominus$

$$\int \frac{\cos x}{6-5\sin x+\sin^2 x}\,dx \,\Bigg|\begin{matrix} t=\sin x \\ dt=\cos x\,dx\end{matrix} = \int\frac{dt}{6-5t+t^2} = \int\frac{dt}{(t-2)(t-3)} =$$

$$= -\int\frac{dt}{t-2} + \int\frac{1}{t-3} = -\ln|t-2| + \ln|t-3| + C =$$

$$= -\ln|\sin x-2| + \ln|\sin x-3| + C$$

$$\ominus \left(-\ln|\sin x-2| + \ln|\sin x-3|\right)\Big|_0^{\pi/2} = -\ln 1 + \ln 2 + \ln 2 - \ln 3 =$$

$$= \ln\frac{4}{3}$$

d) $\displaystyle\int_0^{\pi/2} \frac{1-\cos x}{1+\cos x}\,dx \ominus$

$$\int\frac{1-\cos x}{1+\cos x}\,dx\,\Bigg|\begin{matrix} t=\tan\frac{x}{2} \\ \cos x=\frac{1-t^2}{1+t^2} \\ dx=\frac{2dt}{1+t^2}\end{matrix} = \int\frac{1-\frac{1-t^2}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}\cdot\frac{2dt}{1+t^2} = \int\frac{2t^2dt}{1+t^2} =$$

$$= 2\int\frac{t^2+1-1}{1+t^2} = 2t - 2\int\frac{dt}{1+t^2} + C = 2t - 2\arctan t + C =$$

$$= 2\tan\frac{x}{2} - 2\arctan\left(\tan\frac{x}{2}\right) + C$$

$\ominus 2\tan\frac{x}{2} - 2\arctan(\tan\frac{x}{2})\Big|_0^{\pi/2} =$

$= 2 - \frac{\pi}{2}$

e) $\int_1^3 \arctan\sqrt{x}\,dx \ominus$

$\int \arctan\sqrt{x}\,dx \Big| \begin{matrix} u = \sqrt{x} \\ du = \frac{dx}{2\sqrt{x}} \end{matrix} = 2\int \arctan u \cdot u \cdot du \Big| \begin{matrix} t = \arctan u & dw = u \\ dt = \frac{du}{1+u^2} & w = \frac{u^2}{2} \end{matrix} =$

$= 2\left(\arctan\sqrt{x} \cdot \frac{x}{2} - \frac{1}{2}\int \frac{u^2 du}{1+u^2}\right) = \arctan\sqrt{x} \cdot x - \sqrt{x} + \arctan\sqrt{x} + C$

$\ominus (\arctan\sqrt{x} \cdot x - \sqrt{x} + \arctan\sqrt{x})\Big|_1^3 = \pi - \sqrt{3} + \frac{\pi}{3} - \frac{\pi}{4} + 1 - \frac{\pi}{4} =$

$= \frac{5\pi}{6} - \sqrt{3} + 1$

f) $\int_{1/e}^{e} |\ln x|\,dx = \int_{1/e}^{1} -\ln x\,dx + \int_1^e \ln x\,dx = (x\ln x - x)\Big|_{1/e}^{1} + (x\ln x - x)\Big|_1^e =$

$\int \ln x\,dx = x\ln x - x$

$= \left(1 - \frac{1}{e} - \frac{1}{e}\right) + (e - e + 1) = 2 - \frac{2}{e}$

5. (HW) Prove that if $f(x)$ is integrable on $[a,b]$ and $\int_a^b f(x)\,dx > 1$, then there exists a point c in (a,b) such that $f(c) > \frac{1}{b-a}$.

$f(x)$ is integrable $\Rightarrow$ it has max val. M and min value m:

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

$$f(c) = \frac{1}{b-a} \underbrace{\int_a^b f(x)dx}_{>1 \text{ (given)}} \Rightarrow f(c) > \frac{1}{b-a}$$

6. (HW) Prove that if $f(x)$ is integrable and continuous over $[a,b]$ and if $\int_\alpha^\beta f(x)\,dx \geqslant 0$ for any subinterval $[\alpha,\beta]$ of (a,b), then $f(x) \geqslant 0$ in $[a,b]$.

Suppose it's not, then $\exists c \; f(c) < 0$

$f(x)$ is continuous $\Rightarrow \forall \varepsilon > 0 \; \exists \delta > 0 \; |x - x_0| < \delta \Rightarrow$

$\Rightarrow |f(x) - f(x_0)| < \varepsilon$

Let $\alpha = c - \frac{\delta}{2}$, $\beta = c + \frac{\delta}{2} \Rightarrow \forall x \in [c - \frac{\delta}{2}, c + \frac{\delta}{2}] \; f(x) < 0 \Rightarrow$

$\Rightarrow \forall x \in [\alpha, \beta] \; \int_\alpha^\beta f(x)dx < 0$ which is a contradiction

to $\int_\alpha^\beta f(x)dx \geq 0$