1°. (1 point). Compute $\mathrm{rk}(A)$, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}^{2020}$.

$A^2 = \begin{bmatrix} 1 & 10 & 31 \\ 0 & 16 & 50 \\ 0 & 0 & 36 \end{bmatrix}$, $A^3 = \begin{bmatrix} 1 & 42 & 239 \\ 0 & 64 & 380 \\ 0 & 0 & 216 \end{bmatrix} \dots$

$\mathrm{rk}(A) = 3$. First powers didn't change A^n rank $\Rightarrow$
$\Rightarrow \mathrm{rk}(A^{2020}) = 3$.

2°. (2 points). For all λ compute $\mathrm{rk}(AB)$, where $A = \begin{bmatrix} 1 & -\lambda & 0 & 0 \\ \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & -1 & \lambda \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & 1 & 3 & 4 & 3 & 5 \\ 2 & 2 & 5 & 3 & 0 & 7 & 5 \\ 2 & 4 & 1 & 0 & 0 & 2 & 5 \\ 2 & 1 & 2 & 0 & 0 & 1 & 0 \end{bmatrix}$.

$A \xrightarrow{t_{3,4}} \begin{bmatrix} 1 & -\lambda & 0 & 0 \\ \lambda & 1 & 0 & 0 \\ 0 & 0 & -1 & \lambda \\ 0 & 0 & \lambda & 1 \end{bmatrix} \xrightarrow{d_{3,-1}} \begin{bmatrix} 1 & -\lambda & 0 & 0 \\ \lambda & 1 & 0 & 0 \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & \lambda & 1 \end{bmatrix}$ From here, it's always possible to get RREF through 4 elementary operations $\Rightarrow \mathrm{rk}(A) = 4$ for any λ.

$B \xrightarrow{RREF} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & X \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \end{array}\right]$

$\mathrm{rk}(B) = 4$

So, $\mathrm{rk}(A \cdot B) = \mathrm{rk}(I_4 \cdot B) = \mathrm{rk}(B) = 4$

4°. (2 points). Find a rank factorization of $\begin{bmatrix} 1 & 1 & 4 & 3 & 5 & 2 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 1 & 0 & 4 & 2 & 4 & 3 \\ 1 & 0 & 5 & 3 & 4 & 4 \end{bmatrix}$.

$$\xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & -2 & 4 & -1 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = A, \quad rk(A) = 3$$

$$C = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 0 & 5 \end{bmatrix}; \quad F = \begin{bmatrix} 1 & 0 & 0 & -2 & 4 & -1 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$A = CF.$$

5. (1 + 2 points). Let $A = \begin{bmatrix} 1 & 2 & 3 & 1 & 5 & 6 \\ 1 & 1 & 1 & 0 & 2 & 2 \\ 1 & 2 & 3 & 1 & 5 & 6 \\ 1 & 1 & 1 & 0 & 2 & 2 \end{bmatrix}$. Find matrices $A_1, \ldots, A_k$ such that $\mathrm{rk}(A_i) = 1$ for all i and

(a) $A = A_1 + \ldots + A_k$; (b) find the smallest possible k in (a).

a) $$A \xrightarrow{RREF} \underbrace{\begin{bmatrix} 1 & 0 & -1 & -1 & -1 & -2 \\ 0 & 1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{= F}; \quad A_i = CF_{(i)}$$

$$A_1 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & -1 & -1 & -2 \\ 1 & 0 & -1 & -1 & -1 & -2 \\ 1 & 0 & -1 & -1 & -1 & -2 \\ 1 & 0 & -1 & -1 & -1 & -2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 4 & 2 & 6 & 8 \\ 0 & 1 & 2 & 1 & 3 & 4 \\ 0 & 2 & 4 & 2 & 6 & 8 \\ 0 & 1 & 2 & 1 & 3 & 4 \end{bmatrix}$$

$$A = A_1 + A_2$$

Also it's ones)) that as A_1 you can take row 1 and 3. A_2 would be row 2 and 4, but that solution is not based, so I'd rather reupload with a proper one.

b) $k = rk(A) = 2$. Suppose $k = 1 \Rightarrow A_1 = A$ but $rk(A) \neq 1 \perp$.

$k = 0 \Rightarrow A_1 = null \neq A \perp$.

Thus, 2 is minimum.

3^o. (1 point). For any $\lambda \in \mathbb{R}$ compute $\mathrm{rk}(A - \lambda \cdot I_3)$, where $A = \begin{bmatrix} -7 & -4 & 16 \\ 2 & 2 & -4 \\ -4 & -2 & 9 \end{bmatrix}$.

Hint: it's 3 for almost all λ; determinant might be useful.

$$A - \lambda \cdot I_3 = \begin{bmatrix} -7-\lambda & -4 & 16 \\ 2 & 2-\lambda & -4 \\ -4 & -2 & 9-\lambda \end{bmatrix} =$$

$$= (-7-\lambda)(2-\lambda)(9-\lambda) + (-4)(-4)(-4) + (16)(2)(-2) - (16)(2-\lambda)(-4) - (-4)(2) \cdot$$
$$\cdot (9-\lambda) - (-7-\lambda)(-4)(-2) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = 0$$

$$\Rightarrow \lambda_1 = 1, \ \lambda_2 = 2$$

for λ_1:

$$\begin{bmatrix} -8 & -4 & 16 \\ 2 & 1 & -4 \\ -4 & -2 & 8 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} -8 & -4 & 16 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow rk = 1$$

For λ_2:

$$\begin{bmatrix} -9 & -4 & 16 \\ 2 & 0 & -4 \\ -4 & -2 & 7 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} -9 & -4 & 16 \\ 0 & -8/9 & -4/9 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow rk = 2$$

6. (1 point). Let A be a square matrix and $\varphi : \mathbb{R}^n \to \mathbb{R}^n, \varphi(v) = Av$. Prove that it is bijective if and only if $\mathrm{rk}(A) = n$.

Consider rank factorization of A:

If $rk(A) = n$:

$$A = A \cdot I_n$$

φ is bijective iff it's invertible, meaning its rank factorization has I_n as one of the elements.

If it doesn't have I_n, then it's not invertible $\Rightarrow$

$\Rightarrow$ not bijective.