1. (1 + 0.5 points). Let $\mathbb{V}$ be a vector space and $\varphi \in \mathcal{L}(\mathbb{V}, \mathbb{V})$.

(a) Let $\dim \mathbb{V} < \infty$. Prove that φ is injective if and only if it's surjective;

(b) Is it true if $\dim \mathbb{V} = \infty$?

a) φ is injective $\Leftrightarrow$ $\ker\varphi = 0 \Leftrightarrow \dim(\ker\varphi) = 0 \Leftrightarrow$

$\Leftrightarrow \dim(\operatorname{Im}\varphi) = \dim V \Leftrightarrow \operatorname{Im}\varphi = V \Leftrightarrow$ φ is surjective.

b) No, consider $R^\infty \to R^\infty$, $\varphi([x_1, x_2 \dots]) = [0, x_1 \dots]$

It's clearly linear (proved in hw 18), injective and not surjective.

2°. (2 points). Let $\varphi \in \mathcal{L}(\mathbb{R}^4, \mathbb{R}^3)$, $T(\varphi, \mathcal{E}, \mathcal{E}) = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 5 & 3 & 2 \\ 1 & 3 & 2 & 1 \end{bmatrix}$. Find

(a) Bases $\mathcal{A}, \mathcal{B}$ such that $T(\varphi, \mathcal{A}, \mathcal{B}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (b) Matrices C_1, C_2 such that $C_1 T(\varphi, \mathcal{E}, \mathcal{E}) C_2 = T(\varphi, \mathcal{A}, \mathcal{B})$.

a) $\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 5 & 3 & 2 \\ 1 & 3 & 2 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a-b \\ -a \\ a \\ b \end{bmatrix}$

$\ker\varphi = \left\langle \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\rangle$ $\quad \operatorname{Im}\varphi = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\rangle$

$A = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle$

φ $\quad 0$

$B = \left\langle \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle$

b) $T(\varphi, A, B) = \underbrace{C(B, \mathcal{E})}_{= C(\mathcal{E}, B)^{-1}} T(\varphi, \mathcal{E}, \mathcal{E}) C(\mathcal{E}, A)$

$$\begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&0 \end{bmatrix} = \begin{bmatrix} 5&-2&0\\ -2&1&0\\ 1&-1&1 \end{bmatrix} \begin{bmatrix} 1&2&1&1\\ 2&5&3&2\\ 1&3&2&1 \end{bmatrix} \begin{bmatrix} 1&0&1&-1\\ 0&1&-1&0\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix}$$

3°. (2 points). Let $\varphi \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^4)$, $T(\varphi, \mathcal{E}, \mathcal{E}) = \begin{bmatrix} 1&-1&1\\ 1&1&-1\\ 1&2&1\\ 3&1&1 \end{bmatrix}$. Find

(a) Bases $\mathcal{A}, \mathcal{B}$ such that $T(\varphi, \mathcal{A}, \mathcal{B}) = \begin{bmatrix} 1&0&0\\ 0&1&0\\ 0&0&1\\ 0&0&0 \end{bmatrix}$ (b) Matrices C_1, C_2 such that $C_1 T(\varphi, \mathcal{E}, \mathcal{E}) C_2 = T(\varphi, \mathcal{A}, \mathcal{B})$.

a) $\begin{bmatrix} 1&-1&1\\ 1&1&-1\\ 1&2&1\\ 3&1&1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1&0&0\\ 0&1&0\\ 0&0&1\\ 0&0&0 \end{bmatrix}$ $\quad Ker\,\varphi = 0 \quad Im\,\varphi = \langle e_1, e_2, e_3 \rangle$

$A = \langle e_1, e_2, e_3 \rangle$

$B = \left\langle \begin{bmatrix} 1\\1\\1\\3 \end{bmatrix}, \begin{bmatrix} -1\\1\\2\\1 \end{bmatrix}, \begin{bmatrix} 1\\-1\\1\\1 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\1 \end{bmatrix} \right\rangle$

b) $T(\varphi, A, B) = C(B, \mathcal{E})\, T(\varphi, \mathcal{E}, \mathcal{E})\, C(\mathcal{E}, A)$

$$C(\mathcal{E}, B)^{-1} = \begin{bmatrix} 1&-1&1&0\\ 1&1&-1&0\\ 1&2&1&0\\ 3&1&1&1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2}&\frac{1}{2}&0&0\\ -\frac{1}{3}&0&\frac{1}{3}&0\\ \frac{1}{6}&-\frac{1}{2}&\frac{1}{3}&0\\ -\frac{4}{3}&-1&-\frac{2}{3}&1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} & 0 \\ \frac{1}{6} & -\frac{1}{2} & \frac{1}{3} & 0 \\ -\frac{4}{3} & -1 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

4^o. (2 points). Let $\mathcal{A} = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right)$, $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right)$, $\varphi \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^4)$, $T(\varphi, \mathcal{A}, \mathcal{B}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Find (a) $T(\varphi, \mathcal{E}, \mathcal{E})$ (b) matrices C_1, C_2 such that $C_1 T(\varphi, \mathcal{A}, \mathcal{B}) C_2 = T(\varphi, \mathcal{E}, \mathcal{E})$.

a and b)

$$T(\varphi, \mathcal{E}, \mathcal{E}) = C(\mathcal{E}, \mathcal{B}) T(\varphi, \mathcal{A}, \mathcal{B}) C(\mathcal{A}, \mathcal{E})$$

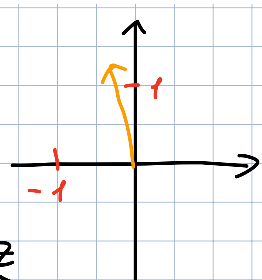
$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -3 & -1 & 5 \\ 2 & 1 & -3 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & -1 \\ -1 & 0 & 2 \\ -2 & -1 & 3 \end{bmatrix}$$

5^o. (1 point). Let $\varphi \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$, $R = T(\varphi, \mathcal{E}, \mathcal{E}) = \begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} & 0 \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and let $v = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $w = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(a) Draw (sketch) vectors Rv and Rw; (b) Find $\ker\varphi$ and $\operatorname{im}\varphi$.

a)

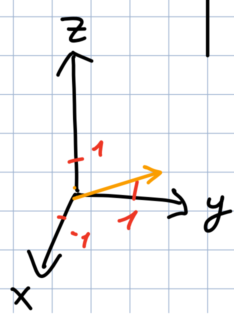
$$Rv = \begin{bmatrix} \cos\frac{\pi}{3} - \sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} + \cos\frac{\pi}{3} \\ 0 \end{bmatrix}$$



b) $\ker\varphi = 0$

$\operatorname{im}\varphi = \langle e_1, e_2, e_3 \rangle$

$$Rw = \begin{bmatrix} \cos\frac{\pi}{3} - \sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} + \cos\frac{\pi}{3} \\ 1 \end{bmatrix}$$



Consider the following relations on the set $Mat(n, m, \mathbb{R})$.

(1) $A \sim_1 B$ if there exist invertible matrices C_1, C_2 such that $C_1 A C_2 = B$ (2) $A \sim_2 B$ if $RREF(A) = RREF(B)$.

6. (1.5 points).

(a) Prove that both $\sim_1, \sim_2$ are equivalence relations;

(b) Is it true that if $A \sim_1 B$ then $A \sim_2 B$?

(c) Is it true that if $A \sim_2 B$ then $A \sim_1 B$?

a) Reflexivity: $A \sim_1 A$

$\forall A \quad I_n A \cdot I_n = A.$

Symmetry: $A \sim_1 B \Rightarrow B \sim_1 A$

$C_1 A C_2 = B \Rightarrow C_1^{-1} B C_2^{-1} = A$

Transitivity: $A \sim_1 B$ and $B \sim_1 F \Rightarrow A \sim_1 F$

Suppose $A \sim_1 B$ and $B \sim_1 F$, thus,

$C_1 A C_2 = B$ and $C_3 B C_4 = F$, then $C_5 A C_6 = F$:

$\Downarrow$

$C_3(C_1 A C_2) C_4 = F \Rightarrow \underbrace{(C_3 C_1)}_{C_5} A \underbrace{(C_2 C_4)}_{C_6.} = F$

Reflexivity: $A \sim_2 B$

$rref(A) = rref(A)$

Symmetry: $A \sim_2 B$

$rref(A) = rref(B) \Rightarrow rref(B) = rref(A)$

Transitivity: $A \sim_2 B$ and $B \sim_2 C \Rightarrow A \sim_2 C$

$rref(A) = rref(B)$; $rref(B) = rref(C) \Rightarrow$

$\Rightarrow rref(A) = rref(C)$

b) No, look at (2)

c) Yes, because rref is unique, there's always a matrix G: $GA = rref(A) = rref(B)$

So, consider $C_1 A C_2 = B$

Let $C_2 = I_n$

$C_1 A = B$ which is exactly what we need.