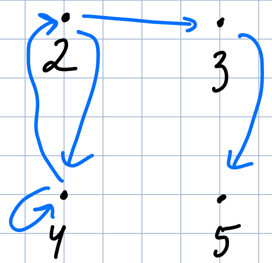
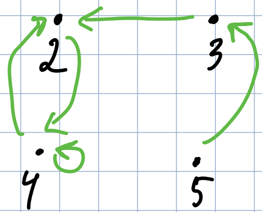
1. Let $R = \{(2,3),(3,5),(2,4),(4,4),(4,2)\}$. Draw 'arrow diagrams' for the relations R, R^{-1} and $R \circ R$. What are the sets $\mathrm{dom}(R \circ R \circ R)$ and $\mathrm{rng}(R \circ R \circ R)$? (Name every element of these sets.)

R



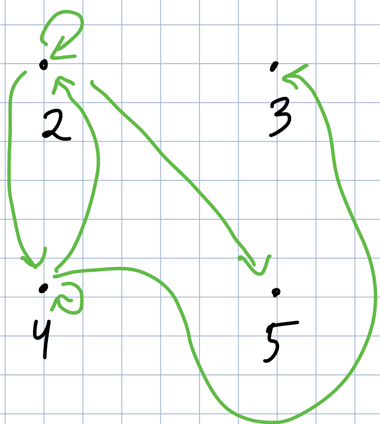
R^{-1}



$(2,3)$ $(3,5)$ $(2,4)$ $(4,4)$ $(4,2)$

$(2,3)$ $(3,5)$ $(2,4)$ $(4,4)$ $(4,2)$

$R \circ R = \{(2,5),(2,4),(2,2),(4,4),(4,2),(4,3)\}$



$R \circ R \circ R = \{(2,4),(2,2),(2,3),(4,3),(4,5),(4,4),(4,2)\}$

$\mathrm{dom}(R \circ R \circ R) = \{2,4\}$

$\mathrm{rng}(R \circ R \circ R) = \{4,2,3,5\}$

2. Let $A = \{1, 2, 3\}$. What is the relation $\subseteq \circ \subseteq$ on $\mathcal{P}(A)$? (Name every element of this relation.)

$$\mathcal{P}(A) = \{\{\emptyset\}, \{1\}, \{2\}, \{3\}, \{2,3\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}$$

$$(a,b) \in \subseteq \circ \subseteq \Leftrightarrow \exists c\,(a \subseteq c \wedge c \subseteq b)$$

$$\begin{aligned}\subseteq \circ \subseteq = \{&(\{\emptyset\},\{\emptyset\}), (\{\emptyset\},\{1\}), (\{\emptyset\},\{2\}), (\{\emptyset\},\{3\}),\\ &(\{\emptyset\},\{2,3\}), (\{\emptyset\},\{1,2\}), (\{\emptyset\},\{1,3\}), (\{\emptyset\},\{1,2,3\}),\\ &(\{1\},\{1\}), (\{1\},\{1,2\}), (\{1\},\{1,3\}), (\{1\},\{1,2,3\}),\\ &(\{2\},\{2\}), (\{2\},\{1,2\}), (\{2\},\{2,3\}), (\{2\},\{1,2,3\}),\\ &(\{3\},\{3\}), (\{3\},\{1,3\}), (\{3\},\{2,3\}), (\{3\},\{1,2,3\}),\\ &(\{2,3\},\{2,3\}), (\{2,3\},\{1,2,3\}),\\ &(\{1,3\},\{1,3\}), (\{1,3\},\{1,2,3\}),\\ &(\{1,2\},\{1,2\}), (\{1,2\},\{1,2,3\}), (\{1,2,3\},\{1,2,3\})\}\end{aligned}$$

3. What is the set $R^{-1}[\{12, 15, 42\}]$, where R is the divisibility relation $|$ on the set $\mathbb{Z}$?

$$R^{-1}[\{12,15,42\}] = \{a \in \operatorname{dom} R \mid \exists b \in \{12,15,42\}\ aRb\}$$

$$R^{-1}[\{12,15,42\}] = \{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm 5, \pm 15, \pm 7,$$

$\pm 14, \pm 21, \pm 42\}$

4. Prove that $R \circ (P \cup Q) = (R \circ P) \cup (R \circ Q)$ for every relations P, Q, R.

$R \circ (P \cup Q) = ((R \circ (P \circ Q))^{-1})^{-1} =$

$(R \circ (P \cup Q))^{-1} = (P \cup Q)^{-1} \circ R^{-1} = (P^{-1} \cup Q^{-1}) \circ R^{-1} =$

$= (P^{-1} \circ R^{-1}) \cup (Q^{-1} \circ R^{-1}) = (R \circ P)^{-1} \cup (R \circ Q)^{-1} = ((R \circ P) \cup (R \circ Q))^{-1}$

$= ((R \circ P) \cup (R \circ Q))^{-1})^{-1} = (R \circ P) \cup (R \circ Q)$

5. Does the inclusion $(R \circ P) \cap (R \circ Q) \subseteq R \circ (P \cap Q)$ hold for every relations P, Q, R?

$R = \{(2,0), (1,0)\}$

$P = \{(0,1)\}$ $\Rightarrow \{(0,0)\} \subseteq \emptyset$ $\perp$

$Q = \{(0,2)\}$

So, no.

6. Does the inclusion $R[X] \cap R[Y] \subseteq R[X \cap Y]$ hold for every relation R and sets X and Y?

$R = \{(0,2), (1,2)\}$

$X = \{0\}$ $\Rightarrow \{2\} \subseteq R[\emptyset] = \emptyset$ $\perp$

$Y = \{1\}$

So, no.

7. Does the identity $(R \cup Q)[X] = R[X] \cup Q[X]$ hold for every relations R, Q and set X?

$b \in (R \cup Q)[X] \Leftrightarrow \exists a \, (a \in X \wedge (a,b) \in R \cup Q) \Leftrightarrow$

$\Leftrightarrow \exists a (a \in X \wedge ((a,b) \in R \vee (a,b) \in Q)) \Leftrightarrow$

$\Leftrightarrow \exists a ((a \in X \wedge (a,b) \in R) \vee (a \in X \wedge (a,b) \in Q)) \Leftrightarrow$

$\Leftrightarrow \exists a (a \in X \wedge (a,b) \in R) \vee \exists a (a \in X \wedge (a,b) \in Q) \Leftrightarrow$

$\Leftrightarrow b \in R[X] \vee b \in Q[X] \Leftrightarrow b \in R[X] \cup Q[X]$