1. (2 points) Prove that $C(\mathcal{A},\mathcal{B})C(\mathcal{B},\mathcal{A}) = I_n$.

$C(B,A) = C(A,B)^{-1}$ by th. 15.1.3

Also, $A \cdot A^{-1} = I_n$

Thus, $C(A,B) \cdot C(A,B)^{-1} = I_n$

2. (2 points) Find $C(\mathcal{A},\mathcal{B})$ and $C(\mathcal{B},\mathcal{A})$for

(a) $\mathcal{A} = \big((x-2)(x-3),(x-1)(x-2),(x-1)(x-3)\big)$, $\mathcal{B} = \big(1,x,x^2\big)$ in $\mathbb{R}[x,2]$;

(b) $\mathcal{A} = \big(1,(x-1),(x-1)^2\big)$, $\mathcal{B} = \big(1,x,x^2\big)$ in $\mathbb{R}[x,2]$.

a) if $B = (1, x, x^2) \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A = \big((6-5x+x^2), (2-3x+x^2), (3-4x+x^2)\big)$

$\Downarrow$

$$\begin{bmatrix} 6 & 2 & 3 \\ -5 & -3 & -4 \\ 1 & 1 & 1 \end{bmatrix} = C(A,B)$$

$$C(B,A) = C(A,B)^{-1} \Rightarrow \begin{bmatrix} 6 & 2 & 3 \\ -5 & -3 & -4 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & 3/2 & 9/2 \\ -1 & -2 & -4 \end{bmatrix} = C(B,A)$$

b) $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \Leftrightarrow C(A,B)$

$$C(B,A) = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

3. (2 points) Prove that

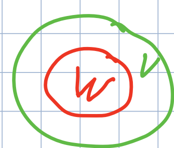
(a) If φ is injective then $\operatorname{Ker} \varphi_A = 0$ (b) If $\operatorname{Ker} \varphi_A = 0$ then φ_A is injective.

a) If φ is injective, the inverse image of every element consists of a single element. Thus, only one input to the function gives 0 as a result $\Rightarrow \ker \varphi_A = 0$

b) $f(x) = f(y)$ iff $f(y-x) = 0$ iff $y - x \in \operatorname{Ker} \varphi_A \Rightarrow y = x \Rightarrow$
$\Rightarrow \varphi_A$ is injective.

4. (2 points) Prove that if $m > n$ then φ_A is not surjective.

Suppose a function $T: V \to W$, where
V are polynomials of at most degree 2.
W are $p'(v)$.
Clearly, when taking a derivative we won't be able to get a polynomial with $\deg = 2$, so our V is going to look like this:



Let's prove it:
T is surjective when basis of $T =$
$=$ basis of W

basis of $T = (1, x, x^2) \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

basis of $W = (1, x) \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Thus, range of $T \neq W$

5. (2 points) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ be three *distinct nonzero* vectors. Suppose there exists a 3×3 matrix A and *distinct* numbers $\lambda_1, \lambda_2, \lambda_3$ such that $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for $i = 1, 2, 3$.

(a) Prove that $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is a basis;

(b) Let $\mathcal{A} = (\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$ be the standard basis, i.e. $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and so on. Prove that

$$C(\mathcal{B}, \mathcal{A}) A C(\mathcal{A}, \mathcal{B}) = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

a) 1) $\dim(\mathcal{B}) = 3 \leq \dim(\mathbb{R}^3) = 3$

2) Suppose v_1 and v_2 are LD, then $v_2 = \alpha \cdot v_1$:
$\alpha \neq 0$

$Av_2 = \lambda_2 v_2 = \lambda_2 \alpha v_1$

Also: $Av_2 = A(\alpha v_1) = \alpha A v_1 = \alpha \lambda_1 v_1$.

Since $\lambda_1 \neq \lambda_2$ $\perp$. $\Rightarrow$ v_1, v_2 are LI.

Suppose v_1, v_2, v_3 are LD:

$V_3 = \alpha_1 V_1 + \alpha_2 V_2$, $\alpha_1 \neq 0$
$\alpha_2 \neq 0$

$A V_3 = \lambda_3 V_3 = \lambda_3(\alpha_1 V_1 + \alpha_2 V_2) = \alpha_1 \lambda_3 V_1 + \alpha_2 \lambda_3 V_2$

$A V_3 = A(\alpha_1 V_1 + \alpha_2 V_2) = \alpha_1 A V_1 + \alpha_2 A V_2 = \alpha_1 \lambda_1 V_1 + \alpha_2 \lambda_2 V_2$

$\alpha_1 \lambda_3 V_1 + \alpha_2 \lambda_3 V_2 = \alpha_1 \lambda_1 V_1 + \alpha_2 \lambda_2 V_2$

$\alpha_1 \lambda_3 V_1 - \alpha_1 \lambda_1 V_1 + \alpha_2 \lambda_3 V_2 - \alpha_2 \lambda_2 V_2 = 0$

$\alpha_1 V_1 (\lambda_3 - \lambda_1) + \alpha_2 V_2 (\lambda_3 - \lambda_2) = 0$

Since $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq 0$, $V_1 \neq V_2$ $\Rightarrow$ $\alpha_1 = \alpha_2 = 0$ $\perp$

Thus, V_1, V_2, V_3 are LI.

b) $C(\mathcal{A},\mathcal{B}) = \mathcal{B}$ as $\mathcal{A} = I_3$.

$C(\mathcal{B},\mathcal{A}) = C(\mathcal{A},\mathcal{B})^{-1} = \mathcal{B}^{-1}$

Thus,

$$B^{-1} \cdot A \cdot B = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$AB = A[v_1, v_2, v_3] = [Av_1, Av_2, Av_3] = [\lambda_1 v_1, \lambda_2 v_2, \lambda_3 v_3]$$

$$B^{-1}(AB) = B^{-1}[\lambda_1 v_1, \lambda_2 v_2, \lambda_3 v_3] = [\lambda_1(B^{-1}v_1), \lambda_2(B^{-1}v_2), \lambda_3(B^{-1}v_3)]$$

Also $B^{-1}v_i = e_i$ since $B^{-1} = C(B,A)$

Hence:

$$[\lambda_1 e_1, \lambda_2 e_2, \lambda_3 e_3] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \text{Q.E.D.}$$