17) $a^{10}+b^{10}+c^{10}+d^{10}+e^{10}+f^{10} \equiv 0 \ (11)$

if $x \equiv 0 \ (11) \Rightarrow \left[\begin{array}{l} x^{10} \equiv 0 \ (11) \\ x^{10} \equiv 1 \ (11) \end{array}\right.$

The sum of the remainders is divisible by 11 when all of them are zeros. Thus, a, b, c, d, e, f is divisible by 11, and their product is divisible by 11^6.

18) $19x + 22y = -21$

$\gcd(19, 22) = 1 \Rightarrow 1 \mid -21 \Rightarrow$ blue has infinitely many solutions. Let's find one of them.

By Euclidean algorithm:

$19 = 22 \cdot 0 + 19$

$22 = 19 \cdot 1 + 3$

$19 = 3 \cdot 6 + 1$

$3 = 1 \cdot 3 + 0$

$1 = 19 - 3 \cdot 6$

$1 = 19 - 6(22 - 19)$

$1 = 19 - 6 \cdot 22 + 6 \cdot 19$

$1 = 7 \cdot 19 - 6 \cdot 22$

From this $x = 7$, $y = -6$, but we're not done

$19 \cdot (7) + 22 \cdot (-6) = 1 \mid \cdot (-21)$

$19 \cdot (-147) + 22 \cdot (126) = -21$

$$\left.\begin{array}{l} x = -147 \\ y = 126 \end{array}\right\} \Rightarrow \begin{array}{l} x = -147 - 22r \\ y = 126 + 19r \end{array} \quad r \in \mathbb{Z}$$

19) $39x \equiv 104\ (221)$

$d = \gcd(39, 221) = 13 \mid 104 \Rightarrow 13$ distinct solutions.

I didn't read the task properly and proceeded to look for solutions. Turned out I didn't have to...

Ass. x_0 is a solution:

$$39x_0 - 104 = 221y_0,\ y_0 \in \mathbb{Z}$$

$$39x_0 - 221y_0 = 104 \mid 13$$

$$3x_0 - 17y_0 = 8$$

By Euc.:

$$3 = -17 \cdot 0 + 3$$

$$-17 = 3 \cdot (-6) + 1$$

$$3 = 1 \cdot 3 + 0$$

$$1 = -17 + 3 \cdot 6$$

$$3x' - 17y' = 1$$

$$3 \cdot (6) - 17 \cdot (1) = 1$$

$$x' = 6,\ y' = 1$$

$$3x_0 - 17x_0 = 8$$

$$3(6 \cdot 8) - 17(1 \cdot 8) = 8$$

$$3 \cdot \underbrace{48}_{x_0} - 17 \cdot \underbrace{8}_{y_0} = 8$$

So, one of the solutions is 48.

20) $\begin{cases} x \equiv -14 \ (12) \\ x \equiv 6 \ (11) \\ x \equiv 19 \ (5) \end{cases} \iff \begin{cases} x \equiv 10 \ (12) \\ x \equiv 6 \ (11) \\ x \equiv 4 \ (5) \end{cases}$

$12u + 11v = 1 \Rightarrow u = 1, \ v = -1$

$x_1 = 6 \cdot 12 \cdot 1 + 10 \cdot 11 \cdot (-1) = -38$

$-38 \equiv 226 \ (11 \cdot 12 = 264)$

$\begin{cases} x \equiv 94 \ (132) \\ x \equiv 4 \ (5) \end{cases}$

$132u + 5v = 1$

$132 = 5 \cdot 26 + 2$

$\underline{5 = 2 \cdot 2 + 1}$

$1 = 5 - 2 \cdot 2$

$1 = 5 - 2(132 - 5 \cdot 26)$

$1 = 5 - 2 \cdot 132 + 10 \cdot 26$

$1 = -2 \cdot 132 + 53 \cdot 5$

$> u = -2, \ v = 53$

$x_2 = 4 \cdot 132 \cdot (-2) + 94 \cdot 5 \cdot 53 = -1056 + 24910 =$

$23854 \equiv 94 \ (132 \cdot 5 = 660)$

21) $\gcd(3^{168} - 1, 3^{140} - 1) \overset{def}{=} 3^{\gcd(168, 140)} - 1 = 3^{28} - 1$

$\gcd(3^{168} - 3^{140}, 3^{140} - 1) = \gcd(3^{28} - 1, 3^{140} - 1) = 3^{28} - 1$

22) $3^{3^{\cdots}} = ? \ (46)$

$3^{3} \equiv 27 \ (46)$ 1

$3^{27} \equiv 13 \ (46)$ 2

$3^{13} \equiv 9 \ (46)$ 3

$3^{9} \equiv 41 \ (46)$ 4

$3^{41} \equiv 29 \ (46)$ 5

$3^{29} \equiv 25 \ (46)$ 6

$3^{25} \equiv 27 \ (46)$ 7

length of the cycle is 7.

$2020 - 7 \cdot 288 = 4 \Rightarrow 3^{3^{\cdots}} \equiv 41 \ (46)$