1. (1 point) Calculate:

$$\begin{bmatrix} -1 & -2 & 3 \\ -1 & 0 & 1 \end{bmatrix} \cdot \left(\operatorname{tr}\left(\begin{bmatrix} 1 & 6 & 2 \\ 2 & 7 & 1 \\ 9 & 8 & -6 \end{bmatrix} \right) \cdot \begin{bmatrix} -3 & 1 & -8 \\ 1 & 3 & 4 \\ 2 & 4 & 1 \end{bmatrix}^{\mathrm{T}} - \begin{bmatrix} 1 & 2 & -3 \\ 8 & -2 & 3 \\ -9 & 0 & 1 \end{bmatrix} \right).$$

1) $\operatorname{tr}\begin{bmatrix} 1 & 6 & 2 \\ 2 & 7 & 1 \\ 9 & 8 & -6 \end{bmatrix} = 2$

2) $\begin{bmatrix} -3 & 1 & -8 \\ 1 & 3 & 4 \\ 2 & 4 & 1 \end{bmatrix}^{T} = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 3 & 4 \\ -8 & 4 & 1 \end{bmatrix}$

3) $2 \cdot \begin{bmatrix} -3 & 1 & 2 \\ 1 & 3 & 4 \\ -8 & 4 & 1 \end{bmatrix} = \begin{bmatrix} -6 & 2 & 4 \\ 2 & 6 & 8 \\ -16 & 8 & 2 \end{bmatrix}$

3) $\begin{bmatrix} -6 & 2 & 4 \\ 2 & 6 & 8 \\ -16 & 8 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -3 \\ 8 & -2 & 3 \\ -9 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 0 & 7 \\ -6 & 8 & 5 \\ -7 & 8 & 1 \end{bmatrix}$

4) $\begin{bmatrix} -1 & -2 & 3 \\ -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -7 & 0 & 7 \\ -6 & 8 & 5 \\ -7 & 8 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 8 & -14 \\ 0 & 8 & -6 \end{bmatrix}$

Solution: $\begin{bmatrix} -2 & 8 & -14 \\ 0 & 8 & -6 \end{bmatrix}$

2. (1 point) Let A be an m-by-n matrix, then, prove that $B = AA^{\mathrm{T}}$ is symmetric.

To prove B is symmetric, let's look at B^T:

$B^T = (A \cdot A^T)^T = A^T \cdot A$ Q.E.D.

3. (1 point) Let $A = \begin{bmatrix} 2 & -6 & 7 \\ 1 & 0 & 3 \\ -4 & 1 & 1 \end{bmatrix}$. Then, find matrices B and C such that B is symmetric, C is skew-symmetric, and $A = B + C$.

Using Formulas, we get:

$$B = \frac{1}{2}(A + A^T) = \frac{1}{2}\left(\begin{bmatrix} 2 & -6 & 7 \\ 1 & 0 & 3 \\ -4 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 & -4 \\ -6 & 0 & 1 \\ 7 & 3 & 1 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} 4 & -5 & 3 \\ -5 & 0 & 4 \\ 3 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{5}{2} & \frac{3}{2} \\ -\frac{5}{2} & 0 & 2 \\ \frac{3}{2} & 2 & 1 \end{bmatrix}$$

$$C = \frac{1}{2}(A - A^T) = \frac{1}{2}\left(\begin{bmatrix} 2 & -6 & 7 \\ 1 & 0 & 3 \\ -4 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -4 \\ -6 & 0 & 1 \\ 7 & 3 & 1 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} 0 & -7 & 11 \\ 7 & 0 & 2 \\ -11 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{7}{2} & \frac{11}{2} \\ \frac{7}{2} & 0 & 1 \\ -\frac{11}{2} & -1 & 0 \end{bmatrix}$$

$$B + C = \begin{bmatrix} 2 & -\frac{5}{2} & \frac{3}{2} \\ -\frac{5}{2} & 0 & 2 \\ \frac{3}{2} & 2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{7}{2} & \frac{11}{2} \\ \frac{7}{2} & 0 & 1 \\ -\frac{11}{2} & -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -6 & 7 \\ 1 & 0 & 3 \\ -4 & 1 & 1 \end{bmatrix} = A$$

4. (2 points) Express $t_{r,s}$ as a sequence of elementary operations $d_{r,\lambda}$ and $l_{r,s,\lambda}$ (that is, swap rows r and s using only elementary row operations of first two types).

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{l_{2,1,-1}} \begin{bmatrix} a & b \\ c-a & d-b \end{bmatrix} \xrightarrow{l_{1,2,1}} \begin{bmatrix} a+c-a & b+d-b \\ c-a & d-b \end{bmatrix} = \begin{bmatrix} c & d \\ c-a & d-b \end{bmatrix} \xrightarrow{l_{2,1,-1}}$$

$$\xrightarrow{l_{2,1,-1}} \begin{bmatrix} c & d \\ c-a-c & d-b-d \end{bmatrix} = \begin{bmatrix} c & d \\ -a & -b \end{bmatrix} \xrightarrow{d_{2,-1}} \begin{bmatrix} c & d \\ a & b \end{bmatrix} = B$$

5. Let

$$A = \begin{bmatrix} 2 & 1 & -3 \\ -2 & 1 & 7 \\ 6 & 4 & -4 \end{bmatrix}.$$

Then

(a) (1 point) successively apply (starting from the left) the following operations to the matrix A:

$$l_{2,1,1};\ l_{3,1,-3};\ d_{2,1/2};\ l_{3,2,-1};\ d_{3,1/3};\ l_{2,3,-2};\ l_{1,3,3};\ l_{1,2,-1};\ d_{1,1/2};$$

(you should receive the identity matrix at the end.)

$$A=\begin{bmatrix}2&1&-3\\-2&1&7\\6&4&-4\end{bmatrix}\xrightarrow{L_{2,1,1}}\begin{bmatrix}2&1&-3\\0&2&4\\6&4&-4\end{bmatrix}\xrightarrow{L_{3,1,-3}}\begin{bmatrix}2&1&-3\\0&2&4\\0&1&5\end{bmatrix}\xrightarrow{d_{2,\frac{1}{2}}}\begin{bmatrix}2&1&-3\\0&1&2\\0&1&5\end{bmatrix}\xrightarrow{L_{3,2,-1}}\begin{bmatrix}2&1&-3\\0&1&2\\0&0&3\end{bmatrix}\rightarrow$$

$$\xrightarrow{d_{3,\frac{1}{3}}}\begin{bmatrix}2&1&-3\\0&1&2\\0&0&1\end{bmatrix}\xrightarrow{L_{2,3,-2}}\begin{bmatrix}2&1&-3\\0&1&0\\0&0&1\end{bmatrix}\xrightarrow{L_{1,3,3}}\begin{bmatrix}2&1&0\\0&1&0\\0&0&1\end{bmatrix}\xrightarrow{L_{1,2,-1}}\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}\xrightarrow{d_{1,\frac{1}{2}}}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

(b) (2 points) following the idea of Problem 4 from Seminar 2, find A^{-1}.

$$A=\begin{bmatrix}2&1&-3\\-2&1&7\\6&4&-4\end{bmatrix}$$

If $[A|I]\xrightarrow{ERO}[I|B]$ then $B=A^{-1}$

$$\left[\begin{array}{ccc|ccc}2&1&-3&1&0&0\\-2&1&7&0&1&0\\6&4&-4&0&0&1\end{array}\right]\xrightarrow{d_{3,\frac{1}{2}}}\left[\begin{array}{ccc|ccc}2&1&-3&1&0&0\\-2&1&7&0&1&0\\3&2&-2&0&0&\frac{1}{2}\end{array}\right]\xrightarrow{L_{2,1,-1}}\left[\begin{array}{ccc|ccc}2&1&-3&1&0&0\\-4&0&10&-1&1&0\\3&2&-2&0&0&\frac{1}{2}\end{array}\right]\longrightarrow$$

$$\xrightarrow{L_{3,1,-2}}\left[\begin{array}{ccc|ccc}2&1&-3&1&0&0\\-4&0&10&-1&1&0\\-1&0&4&-2&0&\frac{1}{2}\end{array}\right]\xrightarrow{d_{2,\frac{1}{2}}}\left[\begin{array}{ccc|ccc}2&1&-3&1&0&0\\-2&0&5&-\frac{1}{2}&\frac{1}{2}&0\\-1&0&4&-2&0&\frac{1}{2}\end{array}\right]\xrightarrow{L_{1,3,2}}\left[\begin{array}{ccc|ccc}0&1&5&-3&0&1\\-2&0&5&-\frac{1}{2}&\frac{1}{2}&0\\-1&0&4&-2&0&\frac{1}{2}\end{array}\right]\rightarrow$$

$$\xrightarrow{L_{2,3,-2}}\left[\begin{array}{ccc|ccc}0&1&5&-3&0&1\\0&0&-3&\frac{7}{2}&\frac{1}{2}&-1\\-1&0&4&-2&0&\frac{1}{2}\end{array}\right]\xrightarrow{d_{2,-\frac{1}{3}}}\left[\begin{array}{ccc|ccc}0&1&5&-3&0&1\\0&0&1&-\frac{7}{6}&-\frac{1}{6}&\frac{1}{3}\\-1&0&4&-2&0&\frac{1}{2}\end{array}\right]\xrightarrow{L_{1,2,-5}}\left[\begin{array}{ccc|ccc}0&1&0&\frac{17}{6}&\frac{5}{6}&-\frac{2}{3}\\0&0&1&-\frac{7}{6}&-\frac{1}{6}&\frac{1}{3}\\-1&0&4&-2&0&\frac{1}{2}\end{array}\right]\rightarrow$$

$$\xrightarrow{L_{3,2,-4}}\left[\begin{array}{ccc|ccc}0&1&0&\frac{17}{6}&\frac{5}{6}&-\frac{2}{3}\\0&0&1&-\frac{7}{6}&-\frac{1}{6}&\frac{1}{3}\\-1&0&0&\frac{8}{3}&\frac{2}{3}&-\frac{5}{6}\end{array}\right]\xrightarrow{d_{3,-1}}\left[\begin{array}{ccc|ccc}0&1&0&\frac{17}{6}&\frac{5}{6}&-\frac{2}{3}\\0&0&1&-\frac{7}{6}&-\frac{1}{6}&\frac{1}{3}\\1&0&0&-\frac{8}{3}&-\frac{2}{3}&\frac{5}{6}\end{array}\right]\xrightarrow{t_{3,1}}\left[\begin{array}{ccc|ccc}1&0&0&-\frac{8}{3}&-\frac{2}{3}&\frac{5}{6}\\0&0&1&-\frac{7}{6}&-\frac{1}{6}&\frac{1}{3}\\0&1&0&\frac{17}{6}&\frac{5}{6}&-\frac{2}{3}\end{array}\right]\rightarrow$$

$$\xrightarrow{t_{2,3}}\left[\begin{array}{ccc|ccc}1&0&0&-\frac{8}{3}&-\frac{2}{3}&\frac{5}{6}\\0&1&0&\frac{17}{6}&\frac{5}{6}&-\frac{2}{3}\\0&0&1&-\frac{7}{6}&-\frac{1}{6}&\frac{1}{3}\end{array}\right]$$

7. (1 point) Let A and B be two square matrices of size n, then, is it possible that $AB - BA = I_n$?
[**hint**: take a look at properties of the trace.]

As $tr(AB) = tr(BA)$: $AB - BA \neq I_n$, because the main diagonale will have zeros $\begin{bmatrix} 0 & a & b \\ d & 0 & c \\ e & f & 0 \end{bmatrix}$

Q.E.D.