1. (0.5 point per item) Which of the following matrices are in reduced row echelon form, and which are not? For those that are not, show which conditions are not satisfied (if there are multiple such conditions, one ought to list all of them):

a) $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; b) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$;

c) $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$; d) $\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$;

e) $\begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$; f) $\begin{bmatrix} 1 & 2 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

a) Condition 4 is not satisfied.
b) Condition 4 is not satisfied.
c) Condition 1 is not satisfied.
d) Condition 3 is not satisfied.
e) Condition 2 and 4 are not satisfied.
f) RREF

2. (2 points) Using Gaussian elimination, find the reduced row echelon form of the following matrix (if you deem it convenient, you may deviate from the general algorithm, but, naturally, you can use only elementary row operations):

$$\begin{bmatrix} 1 & -3 & 1 & 0 & 3 & 0 \\ 1 & -3 & 1 & -3 & -6 & -6 \\ -5 & 17 & -1 & 0 & -7 & 0 \\ 2 & -4 & 6 & 3 & 23 & 6 \end{bmatrix}. \rightarrow$$

$$\xrightarrow[l_{4,1,-2}]{l_{2,1,-1},\ l_{3,1,5}} \begin{bmatrix} 1 & -3 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 & -9 & 0 \\ 0 & 2 & 4 & 0 & 8 & 0 \\ 0 & 2 & 4 & 3 & 17 & 0 \end{bmatrix} \xrightarrow{d_{3,\frac{1}{2}}} \begin{bmatrix} 1 & -3 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 & -9 & 0 \\ 0 & 1 & 2 & 0 & 4 & 0 \\ 0 & 2 & 4 & 3 & 17 & 0 \end{bmatrix} \xrightarrow{t_{2,3}} \begin{bmatrix} 1 & -3 & 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & -3 & -9 & 0 \\ 0 & 2 & 4 & 3 & 17 & 0 \end{bmatrix} \xrightarrow{l_{4,2,-2}}$$

$$\xrightarrow{l_{4,2,-2}} \begin{bmatrix} 1 & -3 & 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & -3 & -9 & 0 \\ 0 & 0 & 0 & 3 & -9 & 0 \end{bmatrix} \xrightarrow{l_{4,3,1}} \begin{bmatrix} 1 & -3 & 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & -3 & -9 & 0 \\ 0 & 0 & 0 & 0 & -18 & 0 \end{bmatrix} \xrightarrow[d_{4,-18}]{d_{3,-\frac{1}{3}}} \begin{bmatrix} 1 & -3 & 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \longrightarrow$$

$$\xrightarrow[d_{1,4,-3},\ d_{2,4,-4}]{d_{3,4,-3}} \begin{bmatrix} 1 & -3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{d_{1,2,3}} \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \text{ RREF}$$

3. (1 point) Using an algorithm from Seminar 3, find the inverse of the following matrix

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 4 & 3 & -2 \\ -1 & -1 & 1 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 0 & 0 & -1 & 1 & 0 & 0 \\ 4 & 3 & -2 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{t_{3,1}} \left[\begin{array}{ccc|ccc} -1 & -1 & 1 & 0 & 0 & 1 \\ 4 & 3 & -2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{d_{1,-1}} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 0 & -1 \\ 4 & 3 & -2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{array}\right] \longrightarrow$$

$$\xrightarrow{d_{2,1,-4}} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 0 & -1 \\ 0 & -1 & 2 & 0 & 1 & 4 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{array}\right] \xrightarrow[d_{3,-1}]{d_{2,-1}} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 0 & -1 \\ 0 & 1 & -2 & 0 & -1 & -4 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right] \xrightarrow[d_{1,3,1}]{d_{2,3,2}} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -2 & -1 & -4 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right] \xrightarrow{d_{1,2,-1}}$$

$$\xrightarrow{d_{1,2,-1}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & -2 & -1 & -4 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 1 & 3 \\ -2 & -1 & -4 \\ -1 & 0 & 0 \end{bmatrix}$$

4. (1 point) Let $A = \begin{bmatrix} 3 & 6 \\ 2 & 5 \end{bmatrix}$. Then, calculate $B = l_{2,1,-2} \circ d_{1,\frac{1}{3}}(A)$ and $C = d_{2,\frac{1}{3}} \circ l_{2,1,-2} \circ l_{1,2,-1}(A)$.

The operation $g \circ f(x)$ (which is called function composition) means that the function g is applied to the result of applying the function f to x.

Observation: note that both matrices B and C are in row echelon form; since $B \neq C$, it means that row echelon form may not be unique while reduced row echelon form is always unique (the proof of the last fact is (probably) beyond the scope of our course).

$$B = l_{2,1,-2} \circ d_{1,\frac{1}{3}}(A) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$l_{1,2,-1}(A) = \begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix}$$

$$l_{2,1,-2} \circ l_{1,2,-1}(A) = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$$

$$d_{2,\frac{1}{3}} \circ l_{2,1,-2} \circ l_{1,2,-1}(A) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$