2) Using $\int_0^1 \frac{dx}{1+x^2}$ prove $\lim_{n\to\infty} n\left(\frac{1}{n^2+1^2}+\frac{1}{n^2+2^2}+\dots \frac{1}{2n^2}\right)=\frac{\pi}{4}$

Proof: consider $f(x)=\frac{1}{1+x^2}$ on $[0,1]$ and a partian $[0,1]$ by points with mesh width $\frac{1}{n}$:

$$0=\frac{0}{n}<\frac{1}{n}<\frac{2}{n}<\dots<\frac{n}{n}\leq 1 \Rightarrow \varepsilon_i=\frac{i}{n}$$

$$f(\varepsilon_i)=\frac{1}{1+\left(\frac{i}{n}\right)^2}=\frac{n^2}{n^2+i^2} \Rightarrow \sum_{i=1}^{n} f(\varepsilon_i)\frac{1}{n}=$$

$$=\sum_{i=1}^{n}\frac{n}{n^2+i^2} \Rightarrow \int_0^1 \frac{dx}{1+x^2}=\arctan x\Big|_0^1=\frac{\pi}{4}$$

3) Find $\lim_{n\to\infty}\frac{1}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\dots+\sin\frac{(n-1)\pi}{n}\right)$

$$\lim_{n\to\infty}\sum_{i=1}^{n-1}\frac{1}{n}\left(\sin\frac{i\pi}{n}\right) \Rightarrow \Delta x=\frac{1}{n} \Rightarrow$$

$$\Rightarrow \int_0^1 \sin(x\pi)=-\frac{1}{\pi}\cos\pi x\Big|_0^1=\frac{2}{\pi}$$

5) Prove:

$$\frac{2}{3} < \int_0^1 \sqrt{x}\, e^x dx < e-1$$

Since $\quad x^{\frac{2}{3}} \le \sqrt{x}\, e^x \le e^x$

$$\int_0^1 \sqrt{x}\, dx < \int_0^1 \sqrt{x}\, e^x dx < \int_0^1 e^x dx$$

$$\frac{2}{3} < \int_0^1 \sqrt{x}\, e^x dx < e-1$$

Prove:

$$\ln 2 < \int_0^{3/4} \frac{2^x}{\sqrt{1+x^2}} dx < \frac{1}{\ln 2}$$

Since: $\quad \frac{1}{\sqrt{1+x^2}} \le \frac{2^x}{\sqrt{1+x^2}} \le 2^x$

$$\int_0^{3/4} \frac{dx}{\sqrt{1+x^2}} < \int_0^{3/4} \frac{2^x}{\sqrt{1+x^2}} dx < \int_0^{3/4} 2^x dx$$

$$\ln 2 < \int_0^{3/4} \frac{2^x}{\sqrt{1+x^2}} dx < \frac{2^{3/4}-1}{\ln 2} < \frac{1}{\ln 2}$$

$$\int_0^{\pi/2} \left(\frac{\sin 5x}{\sin x}\right)^2 dx = \int_0^{\pi/2} \frac{\sin^2 5x}{\sin^2 x} = 5x \Big|_0^{\pi/2} = \frac{5\pi}{2}$$