1. (2 points) Let $V = \mathbb{R}^2$ and operations $\oplus \colon V \times V \to V$, $\odot \colon \mathbb{R} \times V \to V$ be defined as

$$(u,\, v) \oplus (x,\, y) = (u + x + 1,\, v + y - 3);$$
$$\lambda \odot (x,\, y) = (\lambda x + \lambda - 1,\, \lambda y - 3\lambda + 3),$$

for every $(u,\, v)$, $(x,\, y) \in V$ and $\lambda \in \mathbb{R}$. Then, using Definition 12.1 (see the lecture notes), prove that $(V, \oplus, \odot)$ is a vector space over the field of real numbers.

Hint: since $(0,0) \oplus (x,\, y) = (0 + x + 1,\, 0 + y - 3) = (x + 1, y - 3) \neq (x, y)$, it ought to be clear that $(0,0)$ is not the zero vector.

1) $(u,v) \oplus (x,y) = (u+x+1,\ v+y-3)$

$(x,y) \oplus (u,v) = (x+u+1,\ y-3+v)$

2) $((u,v) \oplus (x,y)) \oplus (p,m) = (u+x+1,\ v+y-3) \oplus (p,m) =$

$= (u+x+1+p+1,\ v+y-3+m-3)$

$(u,v) \oplus ((x,y) \oplus (p,m)) = (u,v) \oplus (x+p+1,\ y+m-3) =$

$= (u+x+p+1+1,\ v+y+m-3-3)$

3) $\exists\, 0 \in V$ s.t. $0 + x = x$

$(-1,3) + (x,y) = (-1+x+1,\ 3+y-3) = (x,y)$

4) $(-x-1, -y+3) + (x,y) = (-x-1+x+1,\ -y+3+y-3) = (0,0)$

5) $\lambda \cdot ((x,y) + (c,k)) = \lambda \cdot (x+c+1, y+k-3) =$

$= (\lambda(x+c+1) + \lambda - 1,\ \lambda(y+k-3) - 3\lambda + 3)$

$\lambda(x,y) + \lambda(c,k) = (\lambda x + \lambda - 1,\ \lambda y - 3\lambda + 3) + (\lambda c + \lambda - 1,\ \lambda k - 3k + 3) =$

$= (\lambda x + \lambda - 1 + \lambda c + \lambda - 1 + 1,\ \lambda y - 3\lambda + 3 + \lambda k - 3k + 3 - 3) =$

$= (\lambda(x+c+1) + \lambda - 1,\ \lambda(y+k-3) - 3\lambda + 3)$

6) $(\lambda_1 + \lambda_2) \cdot (x,y) = ((\lambda_1 + \lambda_2)x + (\lambda_1 + \lambda_2) - 1,\ (\lambda_1 + \lambda_2)y - 3(\lambda_1 + \lambda_2) + 3)$

$\lambda_1 \cdot (x,y) + \lambda_2 \cdot (x,y) = (\lambda_1 x + \lambda_1 - 1,\ \lambda_1 y - 3\lambda_1 + 3) +$

$+ (\lambda_2 x + \lambda_2 - 1,\ \lambda_2 y - 3\lambda_2 + 3) =$

$= (\lambda_1 x + \lambda_1 - 1 + \lambda_2 x + \lambda_2 \cancel{-1} \cancel{+1},\ \lambda_1 y - 3\lambda_1 + 3 + \lambda_2 y - 3\lambda_2 \cancel{+3} \cancel{-3}) =$

$= ((\lambda_1+\lambda_2)x + (\lambda_1+\lambda_2) - 1,\ (\lambda_1+\lambda_2)y - 3(\lambda_1+\lambda_2)+3)$

7) $(\lambda_1\lambda_2)\cdot(x,y) = (\lambda_1\lambda_2 x + \lambda_1\lambda_2 - 1,\ \lambda_1\lambda_2 y - 3\lambda_1\lambda_2 + 3)$

$\lambda_1\cdot(\lambda_2\cdot(x,y)) = \lambda_1\cdot(\lambda_2 x+\lambda_2-1,\ \lambda_2 y - 3\lambda_2+3) =$

$=(\lambda_1(\lambda_2 x+\lambda_2-1)+\lambda_1-1,\ \lambda_1(\lambda_2 y-3\lambda_2+3)-3\lambda_1+3)=$

$=(\lambda_1\lambda_2 x+\lambda_1\lambda_2-\lambda_1+\lambda_1-1,\ \lambda_1\lambda_2 y-3\lambda_1\lambda_2+3\lambda_1-3\lambda_1+3)=$

$=(\lambda_1\lambda_2 x+\lambda_1\lambda_2-1,\ \lambda_1\lambda_2 y-3\lambda_1\lambda_2+3)$

8) $1\cdot(x,y)=(x+1-1,\ y-3+3)=(x,y)$

2. (2 points) Find all $\lambda \in \mathbb{R}$ such that the set of vectors

$$S = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ \lambda \end{bmatrix} \right\}$$

is linearly dependent.

$$x_1 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 2 & -3 & 0 & 0 \\ 1 & 2 & \lambda & 0 \end{array}\right] \end{array} \xrightarrow{d_{1,-1}} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 2 & -3 & 0 & 0 \\ 1 & 2 & \lambda & 0 \end{array}\right] \xrightarrow[L_{3,1,-1}]{L_{2,1,-2}} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 4 & \lambda+1 & 0 \end{array}\right] \xrightarrow{L_{3,2,-4}} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & \lambda-7 & 0 \end{array}\right] \text{ if } \lambda \neq 7$$

$$\xrightarrow{d_{3,\frac{1}{\lambda-7}}} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[L_{1,3,1}]{L_{2,3,-2}} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{L_{1,2,2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

if $\lambda \neq 7 \Rightarrow$ the set is LI.

if $\lambda = 4$: $\begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{L_{1,2,2}} \begin{array}{cccc} & & x_3 & \\ \end{array} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3a \\ -2a \\ a \end{bmatrix}$

$\Downarrow$

set is LD.

3. (2 points) Find all $\lambda \in \mathbb{R}$ such that the vector $p(x) = x^2 + \lambda x - 3$ belongs to the linear span

$$\langle x^2 - 1,\ x + 1,\ x^2 + 3x + 2 \rangle.$$

$x^2 - 1 = x^2 + \lambda x - 3 \Rightarrow \lambda = \frac{2}{x} \Rightarrow x^2 + \frac{2}{x} \cdot x - 3 = x^2 - 1$

$x^2 + 3x + 2 = x^2 + \lambda x - 3$

$3x + 2 = \lambda x - 3$

$3x + 2 + 3 - \lambda x = 0$

$3x + 5 - \lambda x = 0$

$3x + 5 - \left(3 + \frac{5}{x}\right)x = 0$

$3x + 5 - 3x - 5 = 0$

$\lambda = \left(3 + \frac{5}{x}\right)$

$x + 1 = x^2 + \lambda x - 3$

$\lambda = \left(-x + \frac{4}{x} + 1\right)$

4. (2 points) Let $A = \begin{bmatrix} 4 & 4 & 2 & 2 \\ 6 & 6 & 6 & 4 \\ 10 & 10 & 8 & 6 \end{bmatrix}$ and $\varphi : \mathbb{R}^4 \to \mathbb{R}^3$, $\varphi(\vec{v}) = A\vec{v}$.

(a) Find two vectors $\vec{x}_1, \vec{x}_2$ such that $\operatorname{Ker} \varphi = \langle \vec{x}_1, \vec{x}_2 \rangle$ (the choice is not unique, of course);

Hint: Recall **6** from HW4 (group 232).

(b) Find two vectors $\vec{y}_1, \vec{y}_2$ such that $\operatorname{Im} \varphi = \langle \vec{y}_1, \vec{y}_2 \rangle$ (the choice is not unique, of course).

Hint: see **6** and **8** from seminar 12, take any two linearly independent columns of A and show that they span $\operatorname{Im} \varphi$.

$$\begin{bmatrix} 4 & 4 & 2 & 2 \\ 6 & 6 & 6 & 4 \\ 10 & 10 & 8 & 6 \end{bmatrix} \Rightarrow x_1 \cdot \begin{bmatrix} 4 \\ 6 \\ 10 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 4 \\ 6 \\ 10 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix} + x_4 \cdot \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} =$$

$$= \begin{cases} 4x_1 + 4x_2 + 2x_3 + 2x_4 = 0 \\ 6x_1 + 6x_2 + 6x_3 + 4x_4 = 0 \\ 10x_1 + 10x_2 + 8x_3 + 6x_4 = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 4 & 4 & 2 & 2 & 0 \\ 6 & 6 & 6 & 4 & 0 \\ 10 & 10 & 8 & 6 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -b - \frac{d}{3} \\ b \\ -\frac{d}{3} \\ d \end{bmatrix}$$

5. (1 point) Let $\mathbb{V}$ be a vector space over $\mathbb{F}$, $\langle v_1, \ldots v_n \rangle \subset \mathbb{V}$. Suppose $v_n = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_{n-1} v_{n-1}$, $\lambda_i \in \mathbb{F}$ and at least one of λ_i is non-zero. Prove that $\langle v_1, \ldots v_n \rangle = \langle v_1, \ldots v_{n-1} \rangle$.

$\langle v_1 \ldots v_n \rangle = \langle v_1 \ldots v_{n-1} \rangle$

We know that v_n can be made up of all vectors befor v_n. Then v_n can be thrown out.

6. (1 point) Let $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^4$. Find $v_3, v_4 \in \mathbb{R}^4$ such that $\mathbb{R}^4 = \langle v_1, v_2, v_3, v_4 \rangle$.

Hint:

Notice that $\mathbb{R}^4 = \langle e_1, e_2, e_3, e_4 \rangle$, where e_i are the vectors with 1 at the i-th row and 0 at the other 3 rows. Therefore we have

$$\mathbb{R}^4 = \langle e_1, e_2, e_3, e_4 \rangle = \langle e_1, e_2, e_3, e_4, v_1, v_2 \rangle,$$

and now we can use **5** and throw out two vectors.

$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ $v_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ These v_i, $i \in \{1, 2, 3, 4\}$ can create $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$ it spans the whole field.