1*. Suppose that $A \cap B = \varnothing$. Then $C^{A \cup B} \sim C^A \times C^B$.

Consider a bijection $f: A \cup B \to C^A \times C^B$ defined the following way: if a number c came from A: $C^A \times C^B = (1, c) \times (0, c)$. If it was from B: $C^A \times C^B = (0, c) \times (1, c)$. Hence, the first value acts as an indicator function and $C^{A \cup B} \sim C^A \times C^B$

2. Let $\mathcal{P}_1(A)$ be the set of all subsets of A of the form $\{x\}$. Prove that $\mathcal{P}_1(A) \sim A$ for any A.

Let A have n many elements. Then $\mathcal{P}_1(A)$ has n many singletons from A. Hence, $\mathcal{P}_1(A) \sim A$.

3. Using indicator functions, prove the following statements for arbitrary A, B, C:

a) $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$;

b) $(A \setminus B) \cup B = A \iff B \subseteq A$.

$$\mathbb{1}_{(A \cup B) \setminus C} = \mathbb{1}_{\bar{C}} = 1 - \mathbb{1}_C$$

$$\mathbb{1}_{(A \setminus C) \cup (B \setminus C)} = \mathbb{1}_{(\bar{C} \cap B) \cup (\bar{C} \cap A)} =$$

$$= \mathbb{1}_{(\bar{C} \cap B)} + \mathbb{1}_{(\bar{C} \cap A)} - \mathbb{1}_{(\bar{C} \cap B)} \cdot \mathbb{1}_{(\bar{C} \cap A)} =$$

$$= (1 - \mathbb{1}_C)\mathbb{1}_B + (1 - \mathbb{1}_C)\mathbb{1}_A - (1 - \mathbb{1}_C)\mathbb{1}_B(1 - \mathbb{1}_C)\mathbb{1}_A =$$

$$= \mathbb{1}_B - \mathbb{1}_C\mathbb{1}_B + \mathbb{1}_A - \mathbb{1}_C\mathbb{1}_A - (\mathbb{1}_B - \mathbb{1}_C\mathbb{1}_B)(\mathbb{1}_A - \mathbb{1}_C\mathbb{1}_A) =$$

$= \mathbb{1}_B - \mathbb{1}_C\mathbb{1}_B + \mathbb{1}_A - \mathbb{1}_C\mathbb{1}_A - (\mathbb{1}_B\mathbb{1}_A - \mathbb{1}_C\mathbb{1}_A\mathbb{1}_B - \mathbb{1}_A\mathbb{1}_B\mathbb{1}_C +$

$+ \underbrace{\mathbb{1}_C^2}_{=\mathbb{1}_C} \mathbb{1}_A\mathbb{1}_B) = \mathbb{1}_B - \mathbb{1}_C\mathbb{1}_B + \mathbb{1}_A - \mathbb{1}_C\mathbb{1}_A - \mathbb{1}_B\mathbb{1}_A - \mathbb{1}_C\mathbb{1}_A\mathbb{1}_B =$

$= (A \cup B) \setminus C$

b) $\mathbb{1}_{(A\setminus B)\cup B} = \mathbb{1}_A \Rightarrow \mathbb{1}_A \cdot \mathbb{1}_B + \mathbb{1}_B - \mathbb{1}_A \cdot \mathbb{1}_{\bar{B}} \mathbb{1}_B =$

$= \mathbb{1}_A \cdot \mathbb{1}_B + \mathbb{1}_B = \mathbb{1}_A(1 - \mathbb{1}_B) + \mathbb{1}_B = \mathbb{1}_A - \mathbb{1}_B\mathbb{1}_A + \mathbb{1}_B =$

$= \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_A\mathbb{1}_B \Rightarrow A \cup B = A \Longleftrightarrow B \subseteq A$

4. Applying Cantor–Bernstein–Schröder theorem (if you need), prove that:

a) $\mathbb{N}^{\mathbb{N}\times\mathbb{Q}} \times \mathbb{N} \sim \mathbb{R}^{\mathbb{Q}}$;

b) $\underline{5}^{\mathbb{N}} \sim \underline{3}^{\mathbb{N}}$;

c) any square (with the interior) and disc (the interior of a circle) in the plane are equivalent to each other; (*Hint: think of the motions of the plane.*)

d) the set of all possible triangles in the plane is equivalent to $\mathbb{R}$.

a) $\mathbb{N}^{\mathbb{N}\times\mathbb{Q}} \times \mathbb{N} \sim \mathbb{R}^{\mathbb{Q}} \times \mathbb{N} \precsim \mathbb{R}^{\mathbb{R}} \times \mathbb{R} \sim \mathbb{R}^3 \sim \mathbb{R} \precsim \mathbb{R}^{\mathbb{Q}} \precsim \mathbb{R}^2 \sim$

$\sim \mathbb{R} \precsim \mathbb{R}^{\mathbb{Q}}$

Hence, $\mathbb{N}^{\mathbb{N}\times\mathbb{Q}} \times \mathbb{N} \sim \mathbb{R}^{\mathbb{Q}}$

b) $\underline{5}^{\mathbb{N}} \precsim \mathbb{N}^{\mathbb{N}} \sim \mathbb{R} \sim \underline{2}^{\mathbb{N}} \precsim \underline{3}^{\mathbb{N}} \precsim \underline{5}^{\mathbb{N}}$

Hence, $\underline{5}^{\mathbb{N}} \sim \underline{3}^{\mathbb{N}}$

c) square on a plane has 4 points x_1, x_2, y_1, y_2
Hence, it is $\mathbb{R}^4$

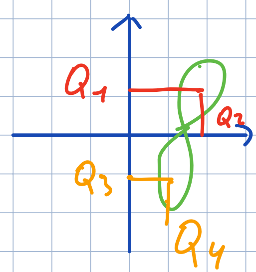
disk on a plane has 3 points x, y, r
Hence, it is $\mathbb{R}^3$
We know $\mathbb{R}^4 \sim \mathbb{R} \sim \mathbb{R}^3 \Rightarrow \mathbb{R}^4 \sim \mathbb{R}^3$.

d) triangle on a plane has 3 points x, y, z
Hence, it is $\mathbb{R}^3 \sim \mathbb{R}$.

5*. Let X be some set of pairwise disjoint figures-of-eight in the plane. Prove that $X \lesssim \mathbb{N}$, that is, there are no more than countably many such figures in X.

Let $8 \in X$, then $f: 8 \Rightarrow Q^2 \times Q^2$, where Q^2 is a point inside the loop of 8.

If any two figures of eight intersect, they must share an ordered pair $Q^2 \times Q^2$, which is a $\perp$.



Hence, $X \lesssim Q^2 \times Q^2 \sim Q^4 \sim \mathbb{N}^4 \sim \mathbb{N}$

6*. Prove that there exists a set $S \subseteq \mathcal{P}(\mathbb{R})$ such that all of the following hold: (a) $S \sim \mathbb{R}$; (b) if $X, Y \in S$ and $X \neq Y$, then $X \cap Y = \varnothing$; (c) if $X \in S$, then $X \sim \mathbb{R}$. (*Hint: try to find a similar set $S' \subseteq \mathcal{P}(\mathbb{R}^2)$; then apply a bijection.*)

Let f be a bijection: $\mathbb{R}^2 \Rightarrow \mathbb{R}$ and $S = \{ f(\{x\} \times \mathbb{R}) \mid x \in \mathbb{R} \}$
(b) holds

Let $f(\{c\}\times\mathbb{R})\in S$, then $g:\mathbb{R}\to f(\{c\}\times\mathbb{R})$

$g(x)=f(c,x)$ is a bijection, hence $f(\{c\}\times\mathbb{R})\sim\mathbb{R}$

(c) holds

Let $h:\mathbb{R}\to S$ as $h(x)=f(\{x\}\times\mathbb{R})$ is a bijection and $S\sim\mathbb{R}$.

(a) holds

7*. Let $C=\{f\in\mathbb{R}^{\mathbb{R}}\mid$ the function f is continuous$\}$. Prove that $C\sim\mathbb{R}$. (*Heine's (sequential) definition of continuity might be helpful.*)

From Heine's definition:

$\forall x\in\mathbb{R}$ exists a sequence $q_n, n\in\mathbb{N}$ of rationals converging to x.

f is continuous $\Rightarrow f(x)=\lim\limits_{n\to\infty} f(q_n)$

So, for any value of x there's a sequence,

hence, $\mathbb{R}^{\mathbb{R}}\sim\mathbb{Q}^{\mathbb{N}}\sim\mathbb{R}$