1 °. (1.5 points). Which of the following functions are linear transformations?

(a) $\varphi : Mat_n(\mathbb{R}) \to \mathbb{R},\ \varphi(A) = \operatorname{Tr} A$; (b) $\varphi : Mat_n(\mathbb{R}) \to \mathbb{R},\ \varphi(A) = \det A$; (c) $\varphi : \mathbb{R}[x, n] \to \mathbb{R},\ \varphi(p) = p(1)$.

a) 1) $\varphi(\lambda A) = \lambda\varphi(A)$ i.e. $\begin{bmatrix}1&0\\0&1\end{bmatrix} = A$; $\varphi(2A) = 4 = 2\varphi(A)$

2) $\varphi(A+B) = \varphi(A) + \varphi(B)$ $\varphi\Big(\overset{A}{\overset{\shortparallel}{\begin{bmatrix}1&0\\0&1\end{bmatrix}}} + \overset{B}{\overset{\shortparallel}{\begin{bmatrix}2&0\\0&2\end{bmatrix}}}\Big) = \varphi(A) + \varphi(B)$

Linear.

b) $A = \begin{bmatrix}5&6\\2&4\end{bmatrix}$

$\varphi(A) = 8$

$\varphi(2A) = 32 \neq 2\cdot\varphi(A) = 2\cdot 4 = 16$

Not linear.

c) $p = 2x^2 + 3x + 1$ $p_1 = 4x^4 + 2$

$\varphi(p) = 2 + 3 + 1 = 6$

$\varphi(2p) = \varphi(4x^2 + 6x + 2) = 12 = 2\cdot\varphi(p)$

$\varphi(p + p_1) = 6 + 6 = 12 = \varphi(p) + \varphi(p_1)$

Linear.

2 °. (1 point). Let $\mathcal{B} = \{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$. Find a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$f([x, y]^T) \neq [[f(e_1)]_{\mathcal{B}}, f(e_2)]_{\mathcal{B}}] \begin{bmatrix} x \\ y \end{bmatrix}.$$

Let f be a function which adds 1 to every element of a matrix.

$f([x, y]^T) = \begin{bmatrix} x+1 \\ y+1 \end{bmatrix}$

Clearly, $[f(e_1)]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $[f(e_1)]_B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

So, $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+y \\ x+2y \end{bmatrix} \neq \begin{bmatrix} x+1 \\ y+1 \end{bmatrix}$

3 ^{o}. (1.5 points). For each linear transformation find its ker and Im

(a) $\varphi : Mat_n(\mathbb{R}) \to Mat_n(\mathbb{R}),\ \varphi(A) = A + A^T$;

(b) $\varphi : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}},\ \varphi(f) = f(x) - f(-x)$ ($\mathbb{R}^{\mathbb{R}}$ is the vector space of all functions $\mathbb{R} \to \mathbb{R}$);

(c) $\varphi : \mathbb{R}^{\infty} \to \mathbb{R}^{\infty},\ \varphi([x_1, x_2, \ldots]) = [0, x_1, x_2, \ldots]$ (i.e. it's a right shift of an infinite sequence).

a) $\ker \varphi = \{ A \in Mat_n(\mathbb{R}) \mid A^T = -A \}$ a.k.a all skew-sym.

$Im\, \varphi = \{ A \in Mat_n(\mathbb{R}) \mid a_{ij} = a_{ji},\ a \in Mat_n(\mathbb{R}) \}$
all sym. matrices.

b) $\ker \varphi = \{ f \in \mathbb{R}^{\mathbb{R}} \mid f(x) = f(-x) \}$

$Im\, \varphi = \{ f \in \mathbb{R}^{\mathbb{R}} \mid f(-x) = -f(x) \}$

c) $\ker \varphi = \{ \{x_n\}_{n=1}^{\infty} \mid x_1 = x_2 \ldots = 0 \}$

$Im\, \varphi = \{ \{x_n\}_{n=1}^{\infty} \mid x_1 = 0 \}$

4 ^{o}. (0.5 + 0.5 + 1 + 0.5 points). Does there exist a linear transformation $\varphi : \mathbb{V} \to \mathbb{W}$ such that

(a) $\mathbb{V} = \mathbb{W} = \mathbb{R}^2,\ \ker \varphi = \{[x, y]^T \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

(b) $\mathbb{V} = \mathbb{W} = \mathbb{R}[x, n],\ \ker \varphi = \{p(x) \in \mathbb{R}[x, n] \mid \deg p(x) = 3\}$;

(c) $\mathbb{V} = \mathbb{R}^4,\ \mathbb{W} = \mathbb{R},\ \ker \varphi = \langle e_1, e_2 \rangle$;

(d) $\mathbb{V} = C[-1, 1],\ \mathbb{W} = \mathbb{R},\ \ker \varphi = \{f(x) \in C[-1, 1] \mid \int\limits_{-1}^{0} f(x)dx = -\int\limits_{0}^{1} f(x)dx\}$?

($C[-1, 1]$ is the space of all continuous functions on $[-1, 1]$)

a) Suppose φ exists. $[0, 0]^T \in \ker \varphi$, but $0^2 + 0^2 \neq 1$.
Contradiction.

b) Suppose φ exists. Consider a condition $n < 3$.

By $\deg p(x) = 3$ there's no kernel, which contradicts the definition of a vector space.

c) The basis of V is $\langle e_1, e_2, e_3, e_4 \rangle$

W is $\langle 1 \rangle$

$\dim V = 4$

$\dim(W) = 1$

$\dim(\ker\varphi) = 2$

Theorem: $\dim(\operatorname{Im}\varphi) + \dim(\ker\varphi) = \dim V$

$1 + 2 \neq 4$

Such LT doesn't exist.

d) It does exist. A definite integral is a LT. But only even functions will be in the kernel:

e.g. $\int_{-1}^{0} f(x)\,dx = -\int_{0}^{1} f(x)\,dx$

$\int_{-1}^{0} \sin x\,dx = -0.45\ldots$

$+ \;= 0$

$\int_{0}^{1} \sin x\,dx = 0.45\ldots$

5. (1.5 points). Consider $\mathbb{V} = \langle e^x, xe^x, x^2e^x, x^3e^x \rangle \subset C(\mathbb{R})$.

(a) Prove that $\mathcal{B} = \{e^x, xe^x, x^2e^x, x^3e^x\}$ is a basis of $\mathbb{V}$;

(b) Find a coordinate representation of $\frac{d}{dx} : \mathbb{V} \to \mathbb{V}$, i.e. $T(\frac{d}{dx}, \mathcal{B}, \mathcal{B})$;

(c) Compute $\int x^2e^x dx$ and $\int x^3e^x dx$. *Hint:* use the inverse of the matrix from (b).

Let the power of x before e^x denote the position of 1 in the matrix, e.g.

$e^x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ then $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Clearly, it's LI and spans $\mathbb{V}$.

b) $T(\frac{d}{dx}, B, B) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, because $\frac{d}{dx} \cdot e^x = e^x$

$\frac{d}{dx} \cdot xe^x = xe^x + x$

$\frac{d}{dx} \cdot x^2e^x = x^2e^x + 2xe^x$

$\frac{d}{dx} \cdot x^3e^x = x^3e^x + 3x^2e^x$

c) $T^{-1} = \begin{bmatrix} 1 & -1 & 2 & -6 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$x^2 \cdot e^x = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$; $\int x^2e^x = \begin{bmatrix} 1 & -1 & 2 & -6 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} = 2e^x - 2xe^x + x^2e^x$

$x^3 \cdot e^x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$; $\int x^3 \cdot e^x = \begin{bmatrix} 1 & -1 & 2 & -6 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 6 \\ -3 \\ 1 \end{bmatrix} = -6e^x + 6xe^x - 3x^2e^x + x^3e^x$

6. (2 points). For $f(x) = e^{\sqrt{3}x}\sin x$ Compute $f^{(3032)}(0)$ (3032th derivative at 0) using linear transformations as follows:

1. Find a vector spaces $\mathbb{V}$ such that $f(x) \in \mathbb{V}$ and $f^{(n)} \in \mathbb{V}$ for all n; *Hint:* see the previous problem, $\dim \mathbb{V} = 2$.
2. Find a matrix representation T of $\frac{d}{dx}$;
3. Apply T 3032 times, i.e. compute T^{3032}; (*Hint:* take a look at **4** from HW9)
4. Substitute $x = 0$ to the result.

1. Let $\mathbb{V} = \langle \underset{\begin{bmatrix}1\\0\end{bmatrix}}{\underset{\parallel}{e^{\sqrt{3}x}\sin x}},\ \underset{\begin{bmatrix}0\\1\end{bmatrix}}{\underset{\parallel}{e^{\sqrt{3}x}\cos x}} \rangle$

2. $\frac{d}{dx} \cdot e^{\sqrt{3}x}\sin x = \sqrt{3}e^{\sqrt{3}x}\sin x + e^{\sqrt{3}x}\cos x = \begin{bmatrix}\sqrt{3}\\1\end{bmatrix}$

$\frac{d}{dx} \cdot e^{\sqrt{3}x}\cos x = -e^{\sqrt{3}x}\sin x + \sqrt{3}e^{\sqrt{3}x}\cos x = \begin{bmatrix}-1\\\sqrt{3}\end{bmatrix}$

3. $T = \begin{bmatrix}\sqrt{3} & -1\\1 & \sqrt{3}\end{bmatrix}^{3032} = 2^{3031} \cdot \begin{bmatrix}-1 & \sqrt{3}\\-\sqrt{3} & -1\end{bmatrix}$ based on hw 4 knowledge.

4. $2^{3031} \cdot \begin{bmatrix}-1 & \sqrt{3}\\-\sqrt{3} & -1\end{bmatrix} \cdot \begin{bmatrix}1\\0\end{bmatrix} = 2^{3031} \cdot \begin{bmatrix}-1\\-\sqrt{3}\end{bmatrix} =$

$= 2^{3031} \cdot (-e^{\sqrt{3}x}\sin x - \sqrt{3}e^{\sqrt{3}x}\cos x)$

$x = 0$:

$2^{3031} \cdot (0 - \sqrt{3}) = -\sqrt{3} \cdot 2^{3031}$