1. (2 points) By forming the augmented matrix and transforming it into reduced row echelon form, find the general solution to the following system of linear equations:

$$\begin{cases} x_1 + 4x_2 + 5x_3 + 2x_4 + 3x_5 = 1, \\ -x_1 + 6x_2 + x_3 + 3x_4 + 2x_5 = 2, \\ 3x_1 + 2x_2 - 2x_3 + x_4 = -7, \\ -x_1 + 2x_2 + 4x_3 + x_4 + 2x_5 = 3. \end{cases}$$

$$\left[\begin{array}{ccccc|c} 1 & 4 & 5 & 2 & 3 & 1 \\ -1 & 6 & 1 & 3 & 2 & 2 \\ 3 & 2 & -2 & 1 & 0 & -7 \\ -1 & 2 & 4 & 1 & 2 & 3 \end{array}\right] \xrightarrow{\substack{L_{2,1,1} \\ L_{3,1,-3} \\ L_{4,1,1}}} \left[\begin{array}{ccccc|c} 1 & 4 & 5 & 2 & 3 & 1 \\ 0 & 10 & 6 & 5 & 5 & 3 \\ 0 & -10 & -17 & -5 & -9 & -10 \\ 0 & 6 & 9 & 3 & 5 & 4 \end{array}\right] \xrightarrow{d_{2,\frac{1}{10}}} \left[\begin{array}{ccccc|c} 1 & 4 & 5 & 2 & 3 & 1 \\ 0 & 1 & \frac{3}{5} & \frac{1}{2} & \frac{1}{2} & \frac{3}{10} \\ 0 & -10 & -17 & -5 & -9 & -10 \\ 0 & 6 & 9 & 3 & 5 & 4 \end{array}\right] \xrightarrow{L_{1,2,-4}} \left[\begin{array}{ccccc|c} 1 & 0 & \frac{13}{5} & 0 & 1 & -\frac{1}{5} \\ 0 & 1 & \frac{3}{5} & \frac{1}{2} & \frac{1}{2} & \frac{3}{10} \\ 0 & -10 & -17 & -5 & -9 & -10 \\ 0 & 6 & 9 & 3 & 5 & 4 \end{array}\right]$$

$$\xrightarrow{L_{3,2,10}} \left[\begin{array}{ccccc|c} 1 & 0 & \frac{13}{5} & 0 & 1 & -\frac{1}{5} \\ 0 & 1 & \frac{3}{5} & \frac{1}{2} & \frac{1}{2} & \frac{3}{10} \\ 0 & 0 & -11 & 0 & -4 & -7 \\ 0 & 6 & 9 & 3 & 5 & 4 \end{array}\right] \xrightarrow{L_{4,2,-6}} \left[\begin{array}{ccccc|c} 1 & 0 & \frac{13}{5} & 0 & 1 & -\frac{1}{5} \\ 0 & 1 & \frac{3}{5} & \frac{1}{2} & \frac{1}{2} & \frac{3}{10} \\ 0 & 0 & -11 & 0 & -4 & -7 \\ 0 & 0 & \frac{27}{5} & 0 & 2 & \frac{11}{5} \end{array}\right] \xrightarrow{d_{3,-\frac{1}{11}}} \left[\begin{array}{ccccc|c} 1 & 0 & \frac{13}{5} & 0 & 1 & -\frac{1}{5} \\ 0 & 1 & \frac{3}{5} & \frac{1}{2} & \frac{1}{2} & \frac{3}{10} \\ 0 & 0 & 1 & 0 & \frac{4}{11} & \frac{7}{11} \\ 0 & 0 & \frac{27}{5} & 0 & 2 & \frac{11}{5} \end{array}\right] \longrightarrow$$

$$\xrightarrow{L_{1,3,-\frac{13}{5}}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & \frac{3}{55} & -\frac{102}{55} \\ 0 & 1 & \frac{3}{5} & \frac{1}{2} & \frac{1}{2} & \frac{3}{10} \\ 0 & 0 & 1 & 0 & \frac{4}{11} & \frac{7}{11} \\ 0 & 0 & \frac{27}{5} & 0 & 2 & \frac{11}{5} \end{array}\right] \xrightarrow{L_{2,3,-\frac{3}{5}}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & \frac{3}{55} & -\frac{102}{55} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{31}{100} & -\frac{9}{110} \\ 0 & 0 & 1 & 0 & \frac{4}{11} & \frac{7}{11} \\ 0 & 0 & \frac{27}{5} & 0 & 2 & \frac{11}{5} \end{array}\right] \xrightarrow{L_{4,3,-\frac{27}{5}}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & \frac{3}{55} & -\frac{102}{55} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{31}{100} & -\frac{9}{110} \\ 0 & 0 & 1 & 0 & \frac{4}{11} & \frac{7}{11} \\ 0 & 0 & 0 & 0 & \frac{2}{55} & -\frac{68}{55} \end{array}\right] \longrightarrow$$

$$\xrightarrow{d_{4,\frac{55}{2}}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & \frac{3}{55} & -\frac{102}{55} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{31}{100} & -\frac{9}{110} \\ 0 & 0 & 1 & 0 & \frac{4}{11} & \frac{7}{11} \\ 0 & 0 & 0 & 0 & 1 & -34 \end{array}\right] \xrightarrow{L_{1,4,-\frac{3}{55}}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{31}{100} & -\frac{9}{110} \\ 0 & 0 & 1 & 0 & \frac{4}{11} & \frac{7}{11} \\ 0 & 0 & 0 & 0 & 1 & -34 \end{array}\right] \xrightarrow{L_{2,4,-\frac{31}{110}}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{19}{2} \\ 0 & 0 & 1 & 0 & \frac{4}{11} & \frac{7}{11} \\ 0 & 0 & 0 & 0 & 1 & -34 \end{array}\right] \longrightarrow$$

$$\xrightarrow{L_{3,4,-\frac{4}{11}}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{19}{2} \\ 0 & 0 & 1 & 0 & 0 & 13 \\ 0 & 0 & 0 & 0 & 1 & -34 \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{19}{2} - \frac{1}{2}a \\ 13 \\ a \in \mathbb{R} \\ -34 \end{bmatrix}$$

2. (2 points) For any $\lambda \in \mathbb{R}$, solve the following system of linear equations:

$$\left[\begin{array}{ccc|c} \lambda & 0 & 3 & -2 \\ 0 & -2 & 2 & 3 \\ -2 & 2 & 1 & 1 \end{array}\right].$$

[**hint**: you want λ to be in the (3,3)-th position of the augmented matrix; if you swap some columns then do not forget to pay attention to the names of the variables.]

$$\begin{array}{ccc} x_3 & x_2 & x_1 \end{array}$$
$$\left[\begin{array}{ccc|c} 3 & 0 & \lambda & -2 \\ 2 & -2 & 0 & 3 \\ 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow{t_{1,3}} \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 2 & -2 & 0 & 3 \\ 3 & 0 & \lambda & -2 \end{array}\right] \xrightarrow[l_{3,1,-3}]{l_{2,1,-2}} \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & -6 & 4 & 1 \\ 0 & -6 & \lambda+6 & -5 \end{array}\right] \xrightarrow{l_{3,2,-1}} \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & -6 & 4 & 1 \\ 0 & 0 & \lambda+2 & -6 \end{array}\right]$$

if $\lambda = -2 \rightarrow$ no solutions.

if $\lambda \neq -2$:

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & -6 & 4 & 1 \\ 0 & 0 & \lambda+2 & -6 \end{array}\right] \xrightarrow{d_{3,\frac{1}{\lambda+2}}} \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & -6 & 4 & 1 \\ 0 & 0 & 1 & \frac{-6}{\lambda+2} \end{array}\right] \xrightarrow[l_{1,3,2}]{l_{2,3,-4}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1-\frac{24}{\lambda+2} \\ 0 & -6 & 0 & 1+\frac{24}{\lambda+2} \\ 0 & 0 & 1 & \frac{-6}{\lambda+2} \end{array}\right] \rightarrow$$

$$\xrightarrow{d_{2,-\frac{1}{6}}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1-\frac{24}{\lambda+2} \\ 0 & 1 & 0 & -\frac{\lambda+26}{6\lambda+12} \\ 0 & 0 & 1 & \frac{-6}{\lambda+2} \end{array}\right] \xrightarrow{l_{1,2,-2}} \begin{array}{c} \begin{array}{ccc} x_3 & x_2 & x_1 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{4\lambda-4}{3\lambda+6} \\ 0 & 1 & 0 & -\frac{\lambda+26}{6\lambda+12} \\ 0 & 0 & 1 & \frac{-6}{\lambda+2} \end{array}\right] \end{array} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-6}{\lambda+2} \\ -\frac{\lambda+26}{6\lambda+12} \\ \frac{4\lambda-4}{3\lambda+6} \end{bmatrix}$$

Answer: if $\lambda = -2 \rightarrow$ no solutions.

if $\lambda \neq 2$: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-6}{\lambda+2} \\ -\frac{\lambda+26}{6\lambda+12} \\ \frac{4\lambda-4}{3\lambda+6} \end{bmatrix}$

3. (1 point) Find quadratic polynomials (that is, polynomials of the form $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$) $f_1(x)$, $f_2(x)$, $f_3(x)$ such that

$$\begin{bmatrix} f_1(1) & f_1(2) & f_1(3) \\ f_2(1) & f_2(2) & f_2(3) \\ f_3(1) & f_3(2) & f_3(3) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 4 \\ 4 & 1 & 2 \\ 0 & 3 & 2 \end{bmatrix}.$$

[**hint**: form and then solve 3 systems of linear equations.]

1. $\begin{cases} a+b+c=2 \\ 4a+2b+c=0 \\ 9a+4b+c=4 \end{cases}$

$$\begin{array}{ccc} a & b & c \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 4 & 2 & 0 & 6 \\ 9 & 4 & 0 & 4 \end{array}\right] \xrightarrow[L_{3,1,-9}]{L_{2,1,-4}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & -4 & -8 \\ 0 & -5 & -9 & -14 \end{array}\right] \xrightarrow{d_{2,-\frac{1}{2}}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & -5 & -9 & -14 \end{array}\right] \xrightarrow[L_{1,2,-1}]{L_{3,2,5}}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 6 \end{array}\right] \xrightarrow[L_{1,3,1}]{L_{2,3,-2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 6 \end{array}\right]$$
$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \\ 6 \end{bmatrix}$$

2. $\begin{cases} a+b+c=4 \\ 4a+2b+c=1 \\ 9a+4b+c=2 \end{cases}$

$$\begin{array}{ccc} a & b & c \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 4 & 2 & 0 & 1 \\ 9 & 4 & 0 & 2 \end{array}\right] \xrightarrow[L_{3,1,-9}]{L_{2,1,-4}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & -4 & -15 \\ 0 & -5 & -9 & -34 \end{array}\right] \xrightarrow{d_{2,-\frac{1}{2}}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & \frac{15}{2} \\ 0 & -5 & -9 & -34 \end{array}\right] \xrightarrow[L_{1,2,-1}]{L_{3,2,5}}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -\frac{7}{2} \\ 0 & 1 & 2 & \frac{15}{2} \\ 0 & 0 & 1 & 7/2 \end{array}\right] \xrightarrow[L_{1,3,1}]{L_{2,3,-2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{7}{2} \end{array}\right]$$
$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{7}{2} \end{bmatrix}$$

3. $\begin{cases} a+b+c=0 \\ 4a+2b+c=3 \\ 9a+4b+c=2 \end{cases}$

$$\begin{array}{ccc} a & b & c \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 4 & 2 & 0 & 3 \\ 9 & 4 & 0 & 2 \end{array}\right] \xrightarrow[L_{3,1,-9}]{L_{2,1,-4}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & -4 & 3 \\ 0 & -5 & -9 & 2 \end{array}\right] \xrightarrow{d_{2,-\frac{1}{2}}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -\frac{3}{2} \\ 0 & -5 & -9 & 2 \end{array}\right] \xrightarrow[L_{1,2,-1}]{L_{3,2,5}}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & \frac{3}{2} \\ 0 & 1 & 2 & -\frac{3}{2} \\ 0 & 0 & 1 & -\frac{11}{2} \end{array}\right] \xrightarrow[L_{1,3,1}]{L_{2,3,-2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & \frac{19}{2} \\ 0 & 0 & 1 & -\frac{11}{2} \end{array}\right]$$
$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -4 \\ \frac{19}{2} \\ -\frac{11}{2} \end{bmatrix}$$

4. (1 point) Find numbers $a, b, c \in \mathbb{R}$ such that the following equality holds true

$$\frac{x(5+x)}{(1-x)(2+x^2)} = \frac{a}{1-x} + \frac{b+cx}{2+x^2}.$$

$$\frac{x(5+x)}{(1-x)(2+x^2)} = \frac{a(2+x^2)+b(1-x)+c(x-x^2)}{(1-x)(2+x^2)}$$

$$\frac{x^2+5x}{(1-x)(2+x^2)} = \frac{2a+ax^2+b-bx+cx-cx^2}{(1-x)(2+x^2)}$$

$$\frac{1\cdot x^2+5x+0}{(1-x)(2+x^2)} = \frac{(a-c)x^2+(c-b)x+2a+b}{(1-x)(2+x^2)}$$

$$\begin{cases} a-c=1 \\ c-b=5 \\ 2a+b=0 \end{cases} \iff \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 5 \\ 2 & 1 & 0 & 0 \end{array}\right] \xrightarrow{L_{3,1,-2}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 5 \\ 0 & 1 & 2 & -2 \end{array}\right] \xrightarrow{d_{2,-1}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & -5 \\ 0 & 1 & 2 & -2 \end{array}\right] \xrightarrow{L_{3,2,-1}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

$$\xrightarrow{d_{3,\frac{1}{3}}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[L_{1,3,1}]{L_{2,3,1}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$$

5. (1 point) Let $X \in \mathrm{M}_3(\mathbb{R})$. By finding the inverse of the first matrix, find the solution to the following matrix equation:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} X = \begin{bmatrix} 2 & 4 & 0 \\ 0 & 1 & 3 \\ 4 & 2 & 2 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & 9 & 0 & 0 & 1 \end{array}\right] \xrightarrow[L_{3,1,-1}]{L_{2,1,-1}} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 2 & 8 & -1 & 0 & 1 \end{array}\right] \xrightarrow{d_{3,\frac{1}{2}}} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 1 & 4 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right] \xrightarrow[L_{3,2,-1}]{L_{1,2,-1}}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 2 & -1 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right] \xrightarrow[L_{2,3,-3}]{L_{1,3,2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -\frac{5}{2} & 4 & -\frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right] \Rightarrow \begin{bmatrix} 3 & -3 & 1 \\ -\frac{5}{2} & 4 & -\frac{3}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 0 \\ 0 & 1 & 3 \\ 4 & 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & -3 & 1 \\ -\frac{5}{2} & 4 & -\frac{3}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -4 & 10 & -4 \\ -1 & 1 & 0 \\ 8 & -6 & 2 \end{bmatrix} = X$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \cdot \begin{bmatrix} -4 & 10 & -4 \\ -1 & 1 & 0 \\ 8 & -6 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 0 \\ 0 & 1 & 3 \\ 4 & 2 & 2 \end{bmatrix}$$

6. (3 point) Let $X \in \mathrm{M}_3(\mathbb{R})$. Without finding the inverse of the first matrix, find the solution to the following matrix equation:

$$\begin{bmatrix} -4 & 1 & -1 \\ 4 & 7 & 6 \\ 1 & 1 & 1 \end{bmatrix} X = \begin{bmatrix} 5 & 2 & 1 \\ 3 & -4 & -1 \\ 2 & -2 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} -4 & 1 & -1 \\ 4 & 7 & 6 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} X_1 & X_4 & X_7 \\ X_2 & X_5 & X_8 \\ X_3 & X_6 & X_9 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 1 \\ 3 & -4 & -1 \\ 2 & -2 & 0 \end{bmatrix}$$

$$\begin{cases} -4X_1 + X_2 - X_3 = 5 \\ 4X_1 + 7X_2 + 6X_3 = 3 \\ X_1 + X_2 + X_3 = 2 \end{cases} \qquad \begin{cases} -4X_4 + X_5 - X_6 = 2 \\ 4X_4 + 7X_5 + 6X_6 = -4 \\ X_4 + X_5 + X_6 = -2 \end{cases} \qquad \begin{cases} -4X_7 + X_8 - X_9 = 1 \\ 4X_7 + 7X_8 + 6X_9 = -1 \\ X_7 + X_8 + X_9 = 0 \end{cases}$$

$$\left[\begin{array}{ccc|c} -4 & 1 & -1 & 5 \\ 4 & 7 & 6 & 3 \\ 1 & 1 & 1 & 2 \end{array}\right] \xrightarrow{t_{1,3}} \qquad \left[\begin{array}{ccc|c} -4 & 1 & -1 & 2 \\ 4 & 7 & 6 & -4 \\ 1 & 1 & 1 & -2 \end{array}\right] \xrightarrow{t_{1,3}} \qquad \left[\begin{array}{ccc|c} -4 & 1 & -1 & 1 \\ 4 & 7 & 6 & -1 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{t_{1,3}}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 4 & 7 & 6 & 3 \\ -4 & 1 & -1 & 5 \end{array}\right] \xrightarrow[L_{3,1,4}]{L_{2,1,-4}} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 4 & 7 & 6 & -4 \\ -4 & 1 & -1 & 2 \end{array}\right] \xrightarrow[L_{3,1,4}]{L_{2,1,-4}} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 4 & 7 & 6 & -1 \\ -4 & 1 & -1 & 1 \end{array}\right] \xrightarrow[L_{3,1,4}]{L_{2,1,-4}}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 3 & 2 & -5 \\ 0 & 5 & 3 & 13 \end{array}\right] \xrightarrow{d_{2,\frac{1}{3}}} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & 3 & 2 & 4 \\ 0 & 5 & 3 & -6 \end{array}\right] \xrightarrow{d_{2,\frac{1}{3}}} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 3 & 2 & -1 \\ 0 & 5 & 3 & 1 \end{array}\right] \xrightarrow{d_{2,\frac{1}{3}}}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & \frac{2}{3} & -\frac{5}{3} \\ 0 & 5 & 3 & 13 \end{array}\right] \xrightarrow[L_{3,2,-5}]{L_{1,2,-1}} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & 1 & \frac{2}{3} & \frac{4}{3} \\ 0 & 5 & 3 & -6 \end{array}\right] \xrightarrow[L_{3,2,-5}]{L_{1,2,-1}} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 5 & 3 & 1 \end{array}\right] \xrightarrow[L_{3,2,-5}]{L_{1,2,-1}}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & \frac{11}{3} \\ 0 & 1 & \frac{2}{3} & -\frac{5}{3} \\ 0 & 0 & -\frac{1}{3} & \frac{64}{3} \end{array}\right] \xrightarrow{d_{3,-\frac{1}{3}}} \qquad \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & -\frac{10}{3} \\ 0 & 1 & \frac{2}{3} & \frac{4}{3} \\ 0 & 0 & -\frac{1}{3} & -\frac{38}{3} \end{array}\right] \xrightarrow{d_{3,-\frac{1}{3}}} \qquad \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & -\frac{1}{3} & \frac{8}{3} \end{array}\right] \xrightarrow{d_{3,-\frac{1}{3}}}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & \frac{11}{3} \\ 0 & 1 & \frac{2}{3} & -\frac{5}{3} \\ 0 & 0 & 1 & -64 \end{array}\right] \xrightarrow[L_{2,3,-\frac{1}{3}}]{L_{2,3,-\frac{2}{3}}} \qquad \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & -\frac{10}{3} \\ 0 & 1 & \frac{2}{3} & \frac{4}{3} \\ 0 & 0 & 1 & 38 \end{array}\right] \xrightarrow[L_{2,3,-\frac{1}{3}}]{L_{2,3,-\frac{2}{3}}} \qquad \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -8 \end{array}\right] \xrightarrow[L_{2,3,-\frac{1}{3}}]{L_{2,3,-\frac{2}{3}}}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & 25 \\ 0 & 1 & \frac{2}{3} & 41 \\ 0 & 0 & 1 & -64 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & -16 \\ 0 & 1 & \frac{2}{3} & -24 \\ 0 & 0 & 1 & 38 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & 3 \\ 0 & 1 & \frac{2}{3} & 5 \\ 0 & 0 & 1 & -8 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 25 \\ 41 \\ -64 \end{bmatrix} \qquad \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -16 \\ -24 \\ 38 \end{bmatrix} \qquad \begin{bmatrix} x_7 \\ x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -8 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 1 & -1 \\ 4 & 7 & 6 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 25 & -16 & 3 \\ 41 & -24 & 5 \\ -64 & 38 & -8 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 1 \\ 3 & -4 & -1 \\ 2 & -2 & 0 \end{bmatrix}$$