1°. (1 point) Let Sym_n and $Skew_n$ be the spaces of $n \times n$ symmetric and skew-symmertic matrices respectively. Prove that $Mat_n(\mathbb{R}) = Sym_n \oplus Skew_n$ and for any matrix $A \in Mat_n(\mathbb{R})$ find formulas for $pr_{Sym_n}(A)$, $pr_{Skew_n}(A)$.

Clearly, the $\dim(Mat_n(\mathbb{R})) = n^2$, as we need n^2 matrices with a single 1 and 0 everywhere else.

Skew-symmetric matrices have elements on one side with respect to the diagonal, so the $\dim(Skew_n(\mathbb{R})) = \frac{(n^2-n)}{2}$, where we subtract n to account for the diagonal.

The same works for symmetric matrices, but with $+n$: $\frac{n^2+n}{2} = \dim(Sym_n(\mathbb{R}))$

$$\dim(Mat_n(\mathbb{R})) = \dim(Skew_n(\mathbb{R})) + \dim(Sym_n(\mathbb{R}))$$

$$n^2 = \frac{n^2-n}{2} + \frac{n^2+n}{2}$$

$$n^2 = \frac{n^2+n^2}{2}$$

$n^2 = n^2 \Rightarrow$ the sum is direct, there's no intersection.

As any square matrix can be expressed as:

$$A = \underbrace{\frac{A+A^T}{2}}_{pr_{Sym_n}(A)} + \underbrace{\frac{A-A^T}{2}}_{pr_{Skew_n}(A)}$$

2^o. (1 point) Let $\mathbb{R}^{\mathbb{R}}$ be the vector space of all functions $f : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}$ and $\mathbb{A} = \{f \in \mathbb{R}^{\mathbb{R}} \mid f(-x) = f(x)\}$, $\mathbb{B} = \{f \in \mathbb{R}^{\mathbb{R}} \mid f(-x) = -f(x)\}$ be the subspaces of even and odd functions.

(a) Prove that $\mathbb{R}^{\mathbb{R}} = \mathbb{A} \oplus \mathbb{B}$ (b) compute $pr_{\mathbb{A}}(e^x)$, $pr_{\mathbb{A}}(e^x)$ and sketch their graphs.

a) def of even function: $f(x) = f(-x)$

def of odd function: $-f(x) = f(-x)$

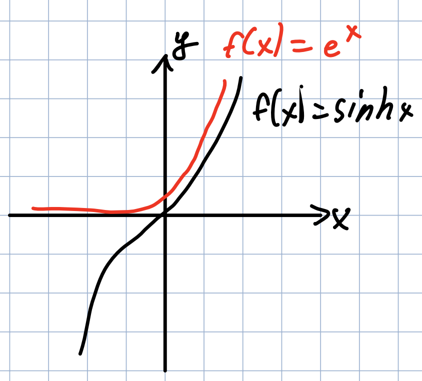
Hence, $\dfrac{f(x) - f(-x)}{2}$ is odd. $\dfrac{f(-x) + f(x)}{2}$ is even.

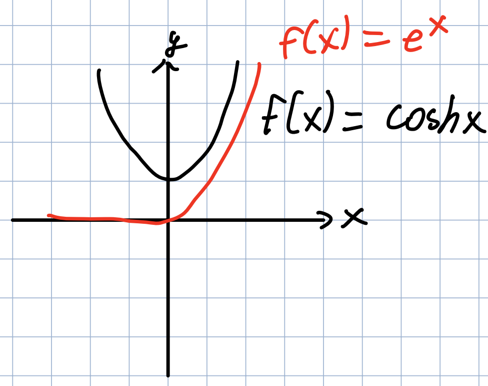
Also, $f(x) = \underbrace{\dfrac{f(x) + f(-x)}{2}}_{\text{even}} + \underbrace{\dfrac{f(x) - f(-x)}{2}}_{\text{odd}}$

Thus, $\mathbb{R}^{\mathbb{R}} = A \oplus B$

b) From (a) any $f(x)$ can be written as:

$$e^x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = \underbrace{\cosh x}_{= pr_A(e^x)} + \underbrace{\sinh x}_{= pr_B(e^x)}$$





3. (2 + 0.5 + 0.5 points) Let $A = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 2 \\ -4 & -7 & 0 & -5 & 0 & -2 \\ 0 & 1 & 2 & 1 & 0 & -2 \\ 4 & 11 & 0 & 9 & 0 & 2 \\ 4 & 3 & -8 & 3 & -2 & 6 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$. For $\lambda \in \mathbb{R}$ let $\mathbb{V}_\lambda = \{v \in \mathbb{R}^6 \mid Av = \lambda v\}$.

(a) For each $\lambda \in \{2, -2, 4\}$ find a basis in $\mathbb{V}_\lambda$;

Hint: $Av = \lambda v \Rightarrow (A - \lambda \cdot I_6)v = 0$ so you just need to find a basis in the space of general solutions of $(A - \lambda \cdot I_6)v = 0$;

(b) Prove that $\mathbb{R}^6 = \mathbb{V}_{-2} \oplus \mathbb{V}_2 \oplus \mathbb{V}_4$

Hint: use the criterion that involves dimension, it's a one line exercise after (a) is solved;

(c) For each $\lambda \in \{2, -2, 4\}$ find $pr_{\mathbb{V}_\lambda}(x)$ for $x = [1, 2, 3, 4, 5, 6]^T$.

a) For $\lambda = 2$: $(A - 2 \cdot I_6)v$

$$= \begin{bmatrix} 0 & -1 & 0 & -1 & 0 & 2 \\ -4 & -9 & 0 & -5 & 0 & -2 \\ 0 & 1 & 0 & 1 & 0 & -2 \\ 4 & 11 & 0 & 7 & 0 & 2 \\ 4 & 3 & -8 & 3 & -4 & 6 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix} \cdot v = 0$$

$$\left[\begin{array}{cccccc|c} 0 & -1 & 0 & -1 & 0 & 2 & 0 \\ -4 & -9 & 0 & -5 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 & -2 & 0 \\ 4 & 11 & 0 & 7 & 0 & 2 & 0 \\ 4 & 3 & -8 & 3 & -4 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cccccc|c} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} \beta \\ -\beta \\ \frac{1}{2}\beta - \frac{1}{2}\alpha \\ \beta \\ \alpha \\ 0 \end{bmatrix}$$

$$\mathbb{V}_2 = \left\langle \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ \frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle$$

$\lambda = -2$:

$$\left[\begin{array}{cccccc|c} 4 & -1 & 0 & -1 & 0 & 2 & 0 \\ -4 & -5 & 0 & -5 & 0 & -2 & 0 \\ 0 & 1 & 4 & 1 & 0 & -2 & 0 \\ 4 & 11 & 0 & 11 & 0 & 2 & 0 \\ 4 & 3 & -8 & 3 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\varphi \\ -\alpha \\ \frac{1}{2}\varphi \\ \alpha \\ \beta \\ \varphi \end{bmatrix}$$

$$\mathbb{V}_{-2} = \left\langle \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$

$\lambda = 4$:

$$\left[\begin{array}{cccccc|c} -2 & -1 & 0 & -1 & 0 & 2 & 0 \\ -4 & -11 & 0 & -5 & 0 & -2 & 0 \\ 0 & 1 & -2 & 1 & 0 & -2 & 0 \\ 4 & 11 & 0 & 5 & 0 & 2 & 0 \\ 4 & 3 & -8 & 3 & -6 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} \alpha \\ \alpha \\ -\alpha \\ -3\alpha \\ \alpha \\ 0 \end{bmatrix}$$

$$\mathbb{V}_4 = \left\langle \begin{bmatrix} 1 \\ 1 \\ -1 \\ -3 \\ 1 \\ 0 \end{bmatrix} \right\rangle$$

b) $\dim(\mathbb{R}^6) = \dim(V_{-2}) + \dim(V_2) + \dim(V_4)$

$6 = 2 + 3 + 1$

$6 = 6$

c) Let's take $V_{-2} \oplus V_2 \oplus V_4$ as basis for $\mathbb{R}^6$.

Let's find $[x]_{V_{-2} \oplus V_2 \oplus V_4}$

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & -\frac{1}{2} & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & -1 & 2 \\ -1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 3 \\ -3 & 1 & 0 & 0 & 0 & 1 & 4 \\ 1 & 0 & 1 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 6 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 & 0 & -12 \\ 0 & 0 & 1 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 0 & 13 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 \end{array}\right]$$

$$\underbrace{-3\begin{bmatrix} 1 \\ 1 \\ -1 \\ -3 \\ 1 \\ 0 \end{bmatrix}}_{= pr_{V_4}(x)} \underbrace{- 12\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - 5\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{pr_{V_{-2}}(x)} + \underbrace{13\begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix} + 7\begin{bmatrix} 1 \\ -1 \\ \frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{= pr_{V_2}(x)} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$$

4^o. (1 point) Let $\mathbb{U} = \{f(x) \in \mathbb{R}[x, 100] \mid f(5) = 0\}$. Find a direct complement of $\mathbb{U}$.

Hint: if you use one of the seminar problems it becomes a one-line exercise.

Let $\mathbb{V} = \{f(x) \in \mathbb{R}[x, 100]$.

$\mathcal{U} = \langle (x-5), (x-5)x \dots (x-5)x^{99} \rangle$.

A direct compliment would be just some constant, e.g. $1 \Rightarrow \mathbb{V} = \langle 1 \rangle$

$R[x, 100] = \mathcal{U} \oplus \mathbb{V}$

5. (1 + 1 point) Let $D = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, i.e. for example $Dx^2 = 2x^2$ or $Dx^2y = 2x^2y + x^2y$ and so on.

(a) Show that for some $f(x, y) \in \mathbb{R}[x, y; 3]$ we have $Df(x, y) = \lambda f(x, y)$ and find all possible values of such λ;

(b) Let $\mathbb{V}_\lambda = \{f(x, y) \in \mathbb{R}[x, y; 3] \mid Df(x, y) = \lambda f(x, y)\}$. For all possible values λ found in (a) compute $\dim \mathbb{V}_\lambda$;

(c) Prove that $\mathbb{R}[x, y; 3] = \mathbb{V}_{\lambda_1} \oplus \mathbb{V}_{\lambda_2} \oplus \ldots \oplus \mathbb{V}_{\lambda_k}$, where $\lambda_1, \ldots, \lambda_k$ are the values found in (a).

a) Consider different cases:

$D(x^0, y^0) = 0 \Rightarrow \lambda = 0$

$D(x, y^0) = x \Rightarrow \lambda = 1$

$D(x^2, y^0 = 2x^2 \Rightarrow \lambda = 2$

$D(x^3, y^0) = 3x^3 \Rightarrow \lambda = 3$

$D(x, y) = xy + xy = 2xy \Rightarrow \lambda = 2$

$D(x^2, y) = 3x^2y \Rightarrow \lambda = 3$

From this, $\lambda = \deg(x, y) \in \{0, 1, 2, 3\}$

b) $\mathbb{V}_0 = \langle 1 \rangle$

$\mathbb{V}_1 = \langle x, y \rangle$

$\mathbb{V}_2 = \langle x^2, y^2, xy \rangle$

$\mathbb{V}_3 = \langle x^3, y^3, x^2y, xy^2 \rangle$

c) all vectors in (b) are LI $\Rightarrow$ there's no intersec-tion. $\Rightarrow \mathbb{R}[x, y; 3] = \mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \mathbb{V}_2 \oplus \mathbb{V}_3$.

6°. (2 points) For each $\lambda \in \mathbb{R}$ compute the rank of the following matrix.

$$A = \begin{bmatrix} 1 & 2 & \lambda \\ -1 & 1 & -3 \\ 1 & -3 & 1 \end{bmatrix}$$

In this problem using a computer is not allowed because of the parameter. You need to show steps.

Hint: The above theorem allows you to use not only row but also column operations.

$$\begin{bmatrix} 1 & 2 & \lambda \\ -1 & 1 & -3 \\ 1 & -3 & 1 \end{bmatrix} \xrightarrow{t_{3,1}} \begin{bmatrix} 1 & -3 & 1 \\ -1 & 1 & -3 \\ 1 & 2 & \lambda \end{bmatrix} \xrightarrow[L_{3,1,-1}]{L_{2,1,1}} \begin{bmatrix} 1 & -3 & 1 \\ 0 & -2 & -2 \\ 0 & 5 & \lambda-1 \end{bmatrix} \xrightarrow{d_{2,-\frac{1}{2}}} \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 1 \\ 0 & 5 & \lambda-1 \end{bmatrix} \xrightarrow{L_{3,2,-5}} \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \lambda-6 \end{bmatrix}$$

1) $\lambda = 6$:

$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{L_{1,2,-1}} \begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = A, \quad rk(A) = 2$$

2) $\lambda \neq 6$

$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \lambda-6 \end{bmatrix} = A, \quad \det(A) = 1 \cdot 1 \cdot (\lambda - 6) \Rightarrow$$

$\Rightarrow$ when $\lambda \neq 6$, $rk(A) = 3$.