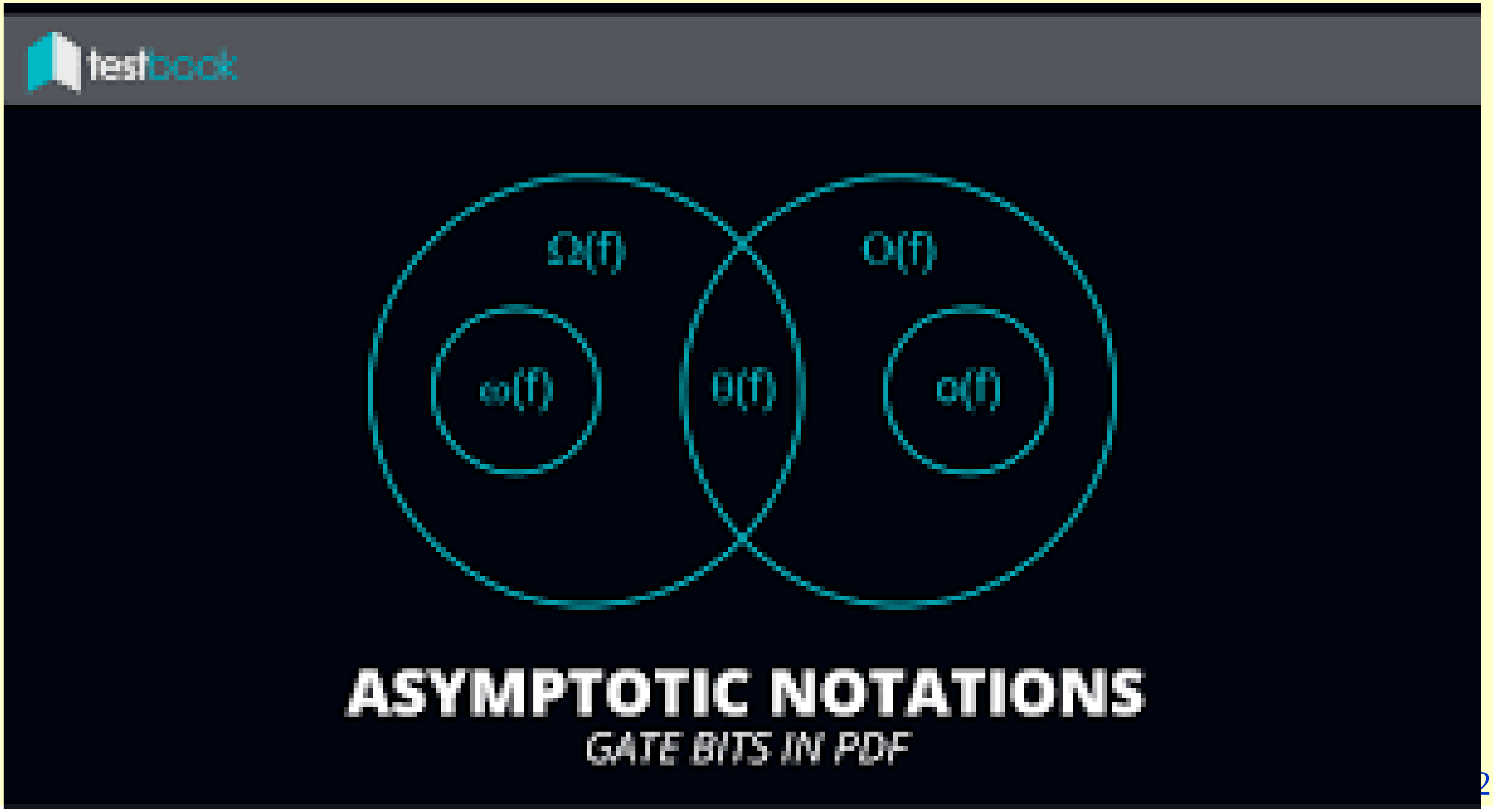
Asymptotic Complexity & Notations

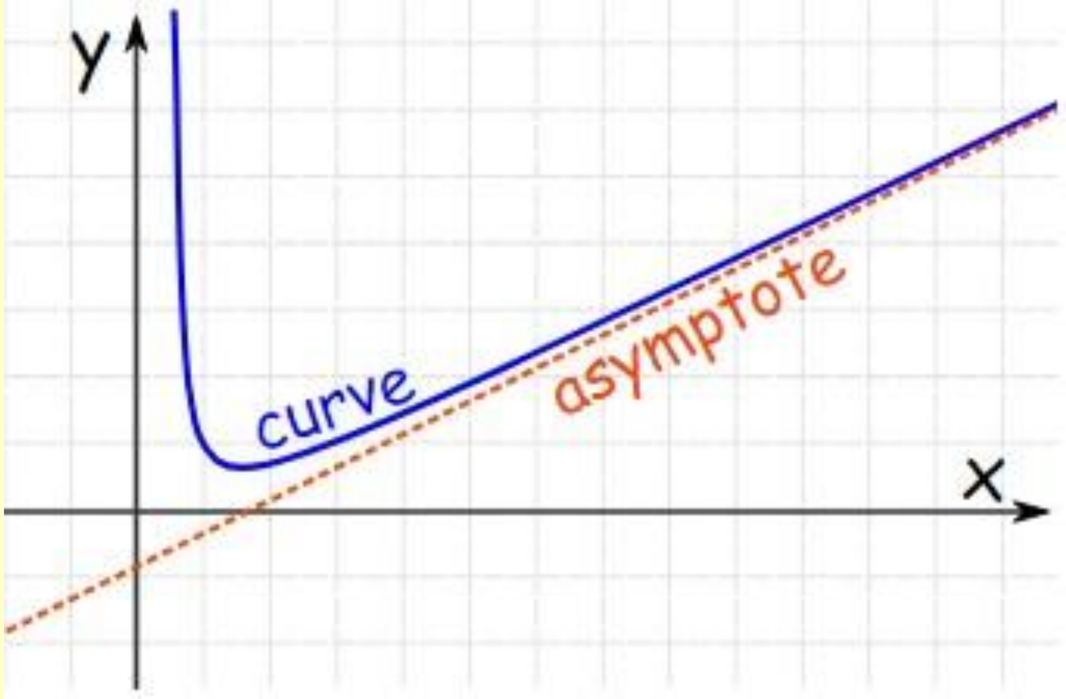
Asymptotic Analysis?

Asymptotic analysis is analyzing what happens to the run time (or other performance metric) as the input size n goes to infinity.



What is an asymptote?

- An asymptote is a **line** that a curve approaches, as it heads towards infinity:



Asymptotic analysis

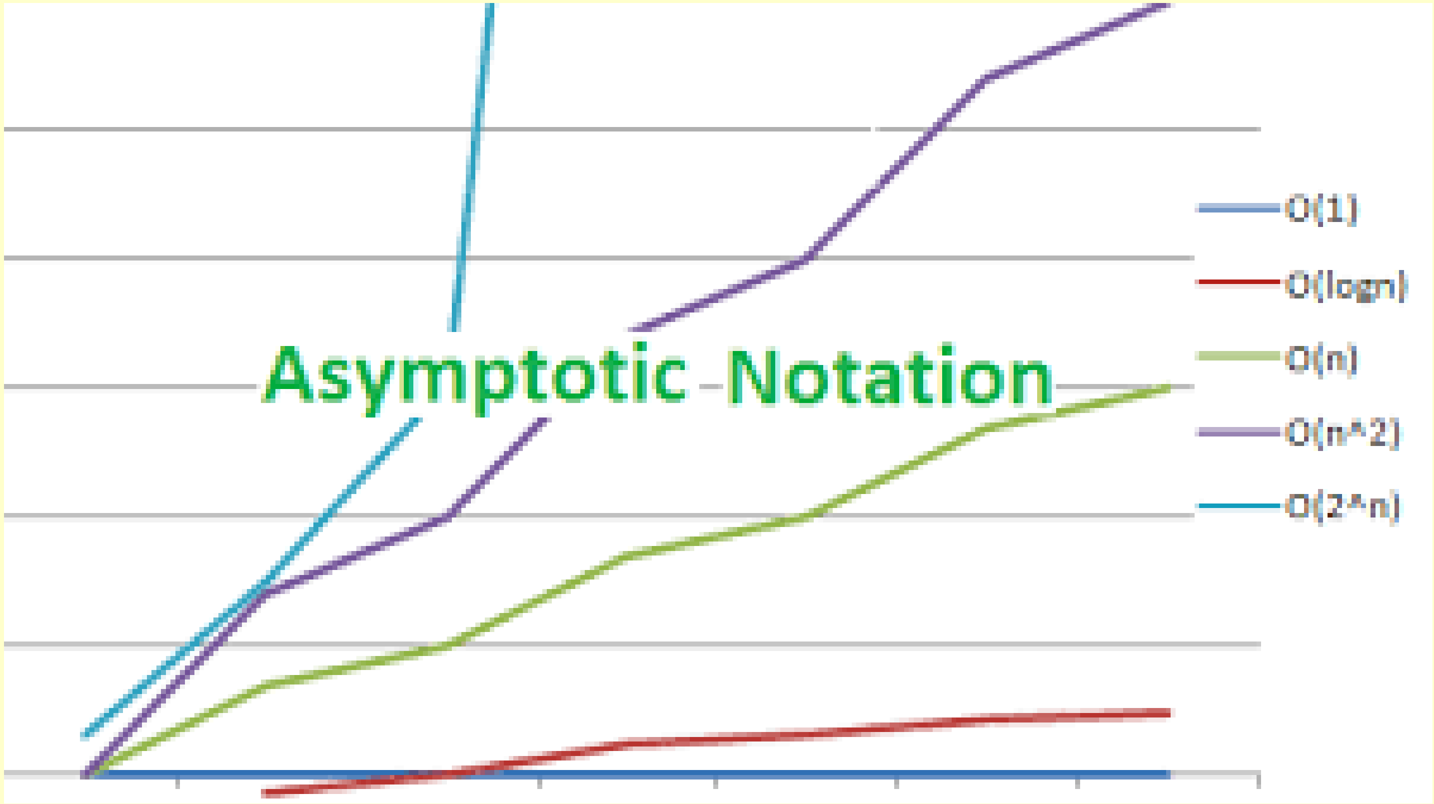
- Asymptotic analysis of an algorithm refers to defining the mathematical boundary/framing of its run-time performance.
- is a mathematical representation of its complexity.
- Use only the **most significant terms** in the complexity of that algorithm and ignore least significant terms in the complexity of that algorithm

Quick Sort

Insertion Sort

Asymptotic Complexity

- Running time of an algorithm as a function of input size n **for large *n***.
- Expressed using only the **highest-order term** in the expression for the exact running time.
  - Describes behavior of function in the limit.
  - Written using ***Asymptotic Notations***.


Asymptotic Notation
O(1)
O(logn)
O(n)
O(n^2)
O(2^n)

Asymptotic Notations

- **Asymptotic notations: Θ, O, Ω**
- Defined for functions over the natural numbers.

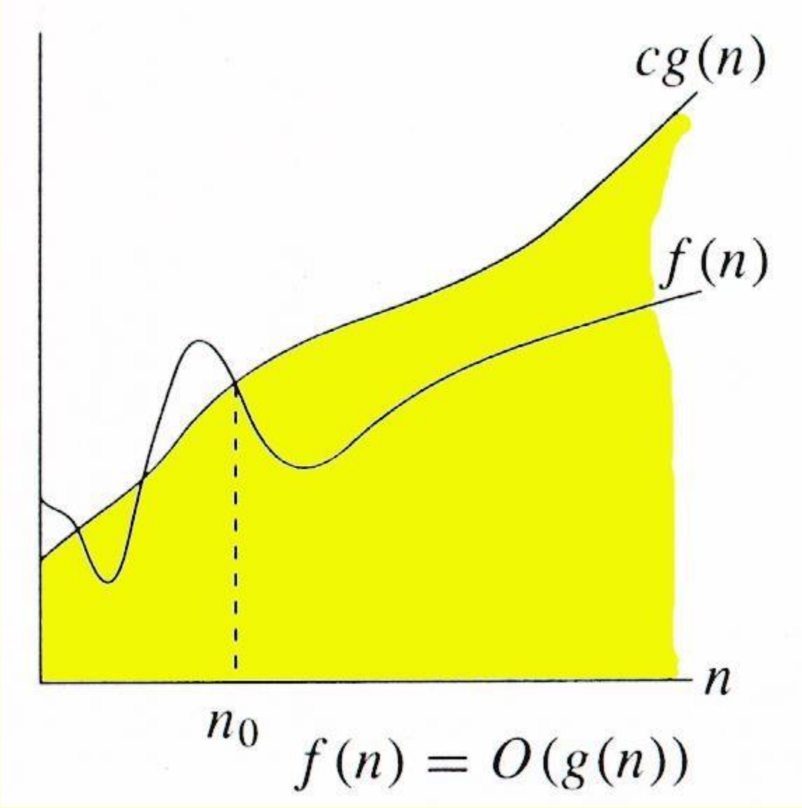
  Example: $f(n) = \Theta(n^2)$.

  Describes how $f(n)$ grows in comparison to n^2.
- Define a ***set*** of functions; in practice used to compare two function sizes.
- The notations describe different rate-of-growth relations between the defining function and the defined set of functions.

O-notation

For function $g(n)$, we define $O(g(n))$, big-O of n, as the set:

$O(g(n)) = \{f(n)$:
$\exists$ positive constants c and n_0,
such that $\forall n \geq n_0$,
we have $0 \leq f(n) \leq cg(n)$ $\}$



Intuitively: Set of all functions whose *rate of growth* is the same as or lower than that of $g(n)$.

$g(n)$ is an *asymptotic upper bound* for $f(n)$.

Asymptotic Notations

Prove that $3n+2 \in O(n)$

Solution :

$t(n) = 3n+2$ —— (1)

$g(n) = n$ —— (2)

To prove, $3n+2 \in O(n)$, the following condition is to be satisfied.

$t(n) \leq c.g(n), \; n \geq n_0$

(i e), in this case,

$$\boxed{3n+2 \leq c.n, \; n \geq n_0}$$

hence find c, constant & n_0, non-negative integer

Asymptotic Notations

Assign $c=1, n_0=1$, then $n = 1, 2, 3, \ldots$

$n=1 \Rightarrow 3(1)+2 \le 1 \cdot 1 = 5 \le 1$ × false, will be false for all 'n's

Assign $c=2, n_0=1$, then $n = 1, 2, 3 \ldots$

$n=1 \Rightarrow 3(1)+2 \le 2 \cdot 1 = 5 \le 2$ × false, will be false for all 'n's

Assign $c=3, n_0=1$, then $n=1, 2, 3 \ldots$

$n=1 \Rightarrow 3(1)+2 \le 3 \cdot 1 = 5 \le 3$ × false, will be false for all n's.

Assign $c=4, n_0=1$, then $n=1, 2, 3 \ldots$

Asymptotic Notations

$n=1 \Rightarrow 3(1)+2 \le 4\cdot 1 = 5 \le 4$ × false.

$n=2 \Rightarrow 3(2)+2 \le 4\cdot 2 = 8 \le 8$ ✓

$n=3 \Rightarrow 3(3)+2 \le 4\cdot 3 = 11 \le 12$ ✓

$n=4 \Rightarrow 3(4)+2 \le 4\cdot 4 = 14 \le 16$ ✓

→ when $c=4$, and $n_0=2$, the

→ condition, $3n+2 \le 4n$, for $n \ge 2$

→ Since the condition is satisfied,

$\boxed{3n+2 \in O(n)}$ proved.

O-notation

- define the **upper bound** of an algorithm in terms of Time Complexity.
- Big - Oh notation always indicates the **maximum time** required by an algorithm for all input values

Examples

$O(g(n)) = \{f(n)$: $\exists$ positive constants c and n_0, such that $\forall n \geq n_0$, we have $0 \leq f(n) \leq cg(n)$ $\}$

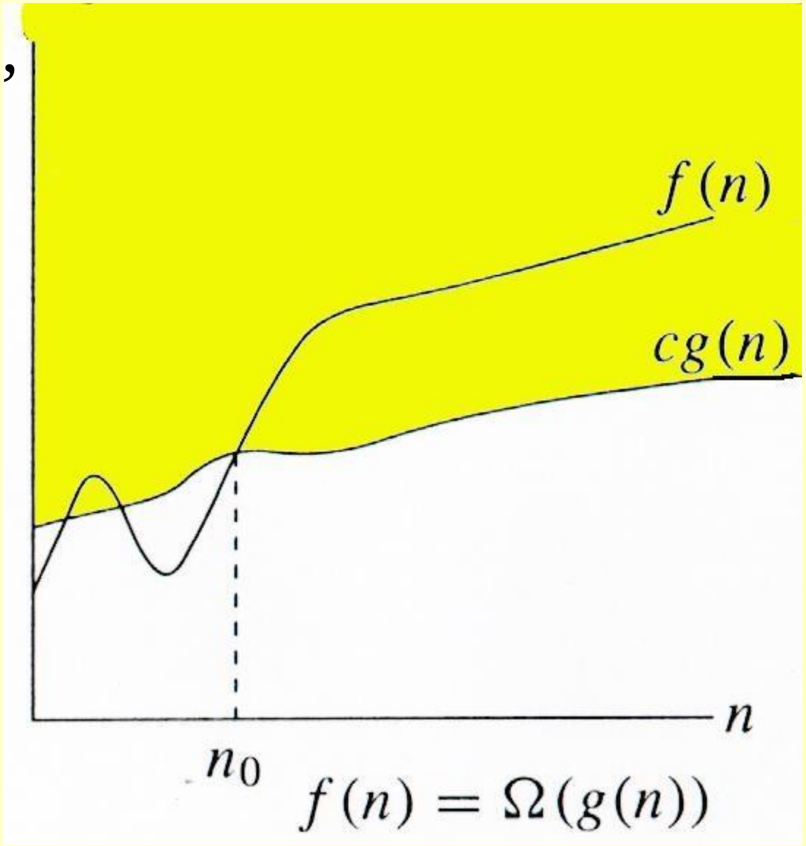
- $3n+2=O(n)$ /* $3n+2 \leq 4n$ for $n \geq 2$ */
- $3n+3=O(n)$ /* $3n+3 \leq 4n$ for $n \geq 3$ */
- $100n+6=O(n)$ /* $100n+6 \leq 101n$ for $n \geq 10$ */
- $10n^2+4n+2=O(n^2)$ /* $10n^2+4n+2 \leq 11n^2$ for $n \geq 5$ */
- $6*2^n+n^2=O(2^n)$ /* $6*2^n+n^2 \leq 7*2^n$ for $n \geq 4$ */

Ω -notation

For function $g(n)$, we define $\Omega(g(n))$, big-Omega of n, as the set:

$\Omega(g(n)) = \{f(n) :$
$\exists$ positive constants c and n_0,
such that $\forall n \geq n_0$,
we have $0 \leq cg(n) \leq f(n)\}$

Intuitively: Set of all functions whose *rate of growth* is the same as or higher than that of $g(n)$.



$g(n)$ is an *asymptotic lower bound* for $f(n)$.

$$f(n) = \Theta(g(n)) \Rightarrow f(n) = \Omega(g(n)).$$

$$\Theta(g(n)) \subset \Omega(g(n))$$

Example

$\Omega(g(n)) = \{f(n)$: $\exists$ positive constants c and n_0, such that $\forall n \geq n_0$, we have $0 \leq cg(n) \leq f(n)\}$

- $\sqrt{n} = \Omega(\lg n)$.

Choose c and n_0.

Example

Prove that $3n+2 \in \Omega(n)$

$t(n) = 3n+2$

$g(n) = n$.

Solution :

To prove $3n+2 \in \Omega(n)$, we need to satisfy the following condition,

$$t(n) \geq c\,g(n),\ n \geq n_0$$

Hence find c, constant & no, nonnegative integer.

Example

Assign $c = 1$, $n_0 = 1$, $n = 1, 2, 3 \cdots$

$n = 1 \Rightarrow 3(1) + 2 \geq 1 \cdot 1 = 5 \geq 1$ ✓ true

$n = 2 \Rightarrow 3(2) + 2 \geq 1 \cdot 2 = 8 \geq 2$ ✓ true

{will be true for all values of n}

Assign $c = 2$, $n_0 = 1$, $n = 1, 2, 3 \cdots$

$n = 1 \Rightarrow 3(1) + 2 \geq 2 \cdot 1 = 5 \geq 2$ ✓

$n = 2 \Rightarrow 3(2) + 2 \geq 2 \cdot 2 = 8 \geq 4$ ✓

} will be true for all values of 'n' . . .

Example

Assign $c=3$, $n_0=1$, $n=1,2,3\ldots$

$n=1 \Rightarrow 3(1)+2 \geq 3\cdot1 = 5 \geq 3$
$n=2 \Rightarrow 3(2)+2 \geq 3\cdot2 = 8 \geq 6$

} will be true for all values of 'n'

Assign $c=4$, $n_0=1$, $n=1,2,3\ldots$

$n=1 \Rightarrow 3(1)+2 \geq 4\cdot1 = 5 \geq 4$ ✓
$n=2 \Rightarrow 3(2)+2 \geq 4\cdot2 = 8 \geq 8$ ✓

Example

$n=3 \Rightarrow 3(3)+2 \geq 4\cdot 3 = 11 \geq 12$ ✗ false.

when we assign $c=4$, we end up with a false condition, hence, $c=3$ & $n_0=1$, well satisfy the condion,

$\underline{t(n) \geq c\cdot g(n)}.$

so,

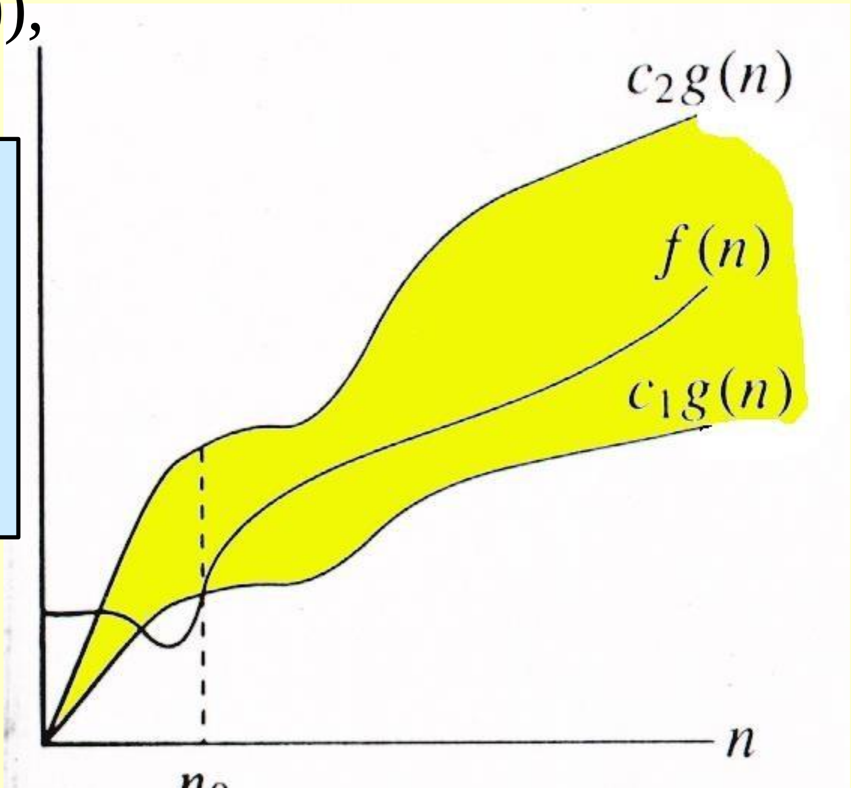
$3n+2 \geq 3n, \; n \geq 1$

Hence $\boxed{3n+2 \in \Omega(n)}$

Θ-notation

For function $g(n)$, we define $\Theta(g(n))$, big-Theta of n, as the set:

$\Theta(g(n)) = \{f(n)$:
$\exists$ positive constants c_1, c_2, and n_0,
such that $\forall n \geq n_0$,
we have $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)\}$



$f(n) = \Theta(g(n))$

Intuitively: Set of all functions that have the same *rate of growth* as $g(n)$.

***g*(*n*) is an *asymptotically tight bound* for *f*(*n*).**

$$f(n) = \Theta(g(n)) \Rightarrow f(n) = O(g(n)).$$
$$\Theta(g(n)) \subset O(g(n)).$$

Example

Prove that $3n+2 \in \Theta(n)$

Solution:

$$t(n) = 3n+2$$

$$g(n) = n$$

To prove $3n+2 \in \Theta(n)$, we need to satisfy the following condition,

$$c_2 \, g(n) \le t(n) \le c_1 \, g(n), \text{ for all } n \ge n_0$$

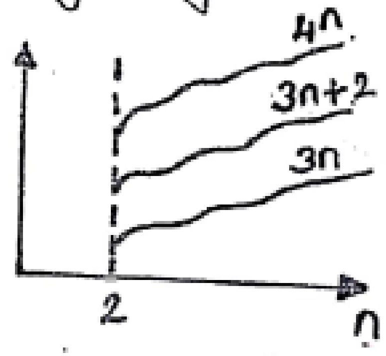
Hence find c, constant & n_0, non negative integer,

Example

From the result of Big Oh and Big Omega,

i.e., Big Oh $\Rightarrow$ $3n+2 \leq 4n$, $\forall\ n \geq 2$

Big Omega $\Rightarrow$ $3n+2 \geq 3n$, $\forall\ n \geq 1$,



we get, $c_1 = 4$ and $c_2 = 3$, $n_0 = 2$

Example

Assign $c_2 = 3$, $c_1 = 4$, $n_0 = 2$, $n = 1, 2, 3 \ldots$

$$3n \le 3n+2 \le 4n$$

$n = 2$, $\Rightarrow$ $3(2) \le 3(2)+2 \le 4(2) = 6 \le 8 \le 8$ (True)

$n = 3$, $\Rightarrow$ $3(3) \le 3(3)+2 \le 4(3) = 9 \le 11 \le 12$ (True)

(will be true for all Values of 'n'

↳ when $c_2 = 3$, $c_1 = 4$ & $n_0 = 2$,

↳ Condition, $3n \le 3n+2 \le 4n$, for $n \ge 2$ is satisfied.

Hence $\boxed{3n+2 \in \Theta(n)}$ proved.

Example

Assign $c_2 = 3$, $c_1 = 4$, $n_0 = 2$, $n = 1, 2, 3 \ldots$

$$3n \le 3n + 2 \le 4n$$

$n = 2$, $\Rightarrow$ $3(2) \le 3(2) + 2 \le 4(2) = 6 \le 8 \le 8$ (True)

$n = 3$, $\Rightarrow$ $3(3) \le 3(3) + 2 \le 4(3) = 9 \le 11 \le 12$ (True)

(will ! true for all values of 'n'

↳ when $c_2 = 3$, $c_1 = 4$ & $n_0 = 2$,

↳ Condition, $3n \le 3n + 2 \le 4n$, for $n \ge 2$ is satisfied.

Hence $\boxed{3n + 2 \in \Theta(n)}$ proved.

Example-2

Show that $\frac{n(n-1)}{2} \in \Omega(n^2)$.

$$t(n) = \frac{n(n-1)}{2} \; .\!\!= \frac{n^2}{2} - \frac{n}{2}$$

$$g(n) = n^2.$$

To prove that $\frac{n^2}{2} - \frac{n}{2} \in \Omega(n^2)$, the following condition is to be satisfied,

$$t(n) \geq c \cdot g(n) \quad \text{for all } n \geq n_0.$$

i.e., in this case,

$$\frac{n^2}{2} - \frac{n}{2} \leq c \cdot n^2.$$

Example

Assign $\boxed{c = \frac{1}{2}}$.

$\underline{\underline{n=1}}$

$$\frac{n^2}{2} - \frac{n}{2} \geq c \cdot n^2$$

$$\frac{1}{2} - \frac{1}{2} \geq \frac{1}{2}(1)$$

$0 \geq \frac{1}{2}$ ———— False.

$\underline{\underline{n=2}}$

$$\frac{\cancel{4}^2}{\cancel{2}} - \frac{\cancel{2}^1}{\cancel{2}} \geq \frac{1}{\cancel{2}} \times \cancel{2} \times 2$$

$1 \geq 2$ ———— False.

$\underline{\underline{n=10}}$

$$\frac{(10)^2}{2} - \frac{10}{2} \geq \frac{1}{2} \times \cancel{10}^{5} \times 10$$

$50 - 10 \geq 50$ ———— False $\Big\{$ this will be false for all

Example

So, Assign $\boxed{c = \frac{1}{4}}$.

n=1

$$\frac{n^2}{2} - \frac{n}{2} \geq c \cdot n^2$$

$$\frac{1}{2} - \frac{1}{2} \geq \frac{1}{4}(1)$$

$0 \geq \frac{1}{4}$ —— False.

Example

$n=2$

$$\frac{n^2}{2} - \frac{n}{2} \geq c.n^2$$

$$\frac{\cancel{4}^{2}}{\cancel{2}} - \frac{\cancel{2}^{1}}{\cancel{2}} \geq \frac{1}{4} \times 2 \times 2$$

$1 \geq 1$ —— True.

$n=3$

$$\frac{n^2}{2} - \frac{n}{2} \geq c.n^2$$

$$\frac{9}{2} - \frac{3}{2} \geq \frac{1}{4} \times 3 \times 3$$

$$\frac{6}{2} \geq \frac{9}{4}.$$

$3 \geq 2.25$ —— True.

$n=100$

$$\frac{(100)^2}{2} - \frac{100}{2} \geq \frac{1}{\cancel{4}_1} \times \cancel{100}^{25} \times 100$$

$$\frac{10000}{2} - 50 \geq 2500$$

$$5000 - 50 \geq 2500$$

$$4950 \geq 2500 \quad \text{———— True.}$$

[This will be true for all values of n]

Thus we assign, $\boxed{c = \frac{1}{4},\ n_0 = 2.}$ 28

$$\frac{n^2}{2} - \frac{n}{2} \geq \frac{1}{4} n^2$$

$$\Rightarrow \frac{n(n-1)}{2} \geq \frac{1}{4} n^2, \quad \text{for } n \geq 2$$

$$\frac{n(n-1)}{2} \in \Omega(n^2)$$

Examples

$\Theta(g(n)) = \{f(n) :$ $\exists$ positive constants c_1, c_2, and n_0, such that $\forall n \geq n_0$, $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)\}$

1. $3n+2 = \Theta(n)$

For n>=2, c1=3 and c2=4

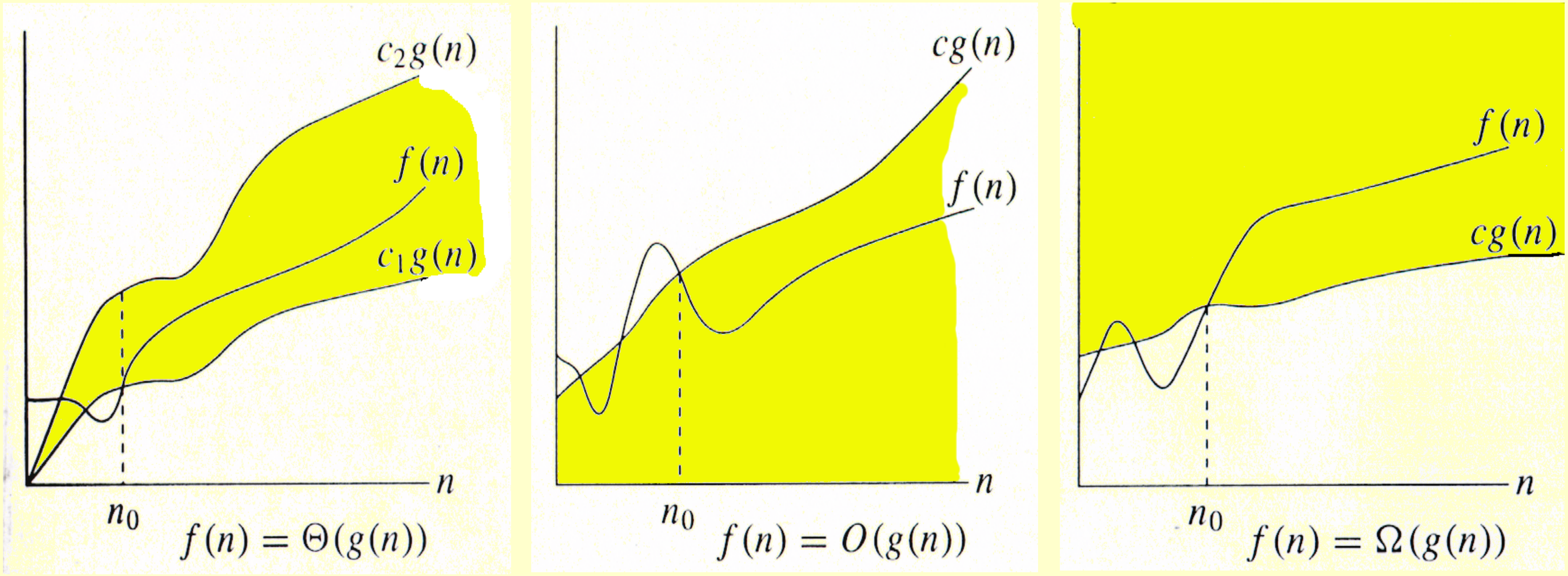
2.10$n^2 + 4n + 2 >= \Theta(n^2)$

For n=?, c1=10 and c2=11

10$n^2 + 4n + 2 >= 10n^2$

10$n^2 + 4n + 2 <= 11n^2$

Relations Between Θ, *O*, Ω



Relations Between Θ, O, Ω

Theorem : For any two functions $g(n)$ and $f(n)$,

$f(n) = \Theta(g(n))$ iff

$f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

- i.e., $\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$
- In practice, asymptotically tight bounds are obtained from asymptotic upper and lower bounds.

o-notation

For a given function $g(n)$, the set little-o:

$$o(g(n)) = \{f(n): \underline{\forall\, c > 0}, \exists\, n_0 > 0 \text{ such that } \forall\, n \geq n_0, \text{ we have } 0 \leq f(n) < cg(n)\}.$$

$f(n)$ becomes insignificant relative to $g(n)$ as n approaches infinity:

$$\lim_{n \to \infty} [f(n) / g(n)] = 0$$

$2n = O(n)$ but $2n \neq o(gn)$

$2n < c \cdot n \;\; \forall\, n \geq n_0$. $\boxed{\text{for some } c}$

$\forall\, c > 0$

if $c = 1$

$\underline{2n < n} \rightarrow$ Not true

$f(n) = o(g(n)$

Condition must be satisfied.

$2n = o(n^2)$

$2n < cn^2 \;\forall\, c > 0 \;\forall\, n \geq n_0.$

$2 < c \cdot n$ [/by n]

$2 < 1 \cdot n$ [$n \geq 3$]

if $C = 0.1$

$2 < 0.1 * n \;\forall\, n \geq n_0.$

↳ Value must be 21.

Little ω-notation

Let f(n) and g(n) be functions that map positive integers to positive real numbers

$(g(n))$ = $\{f(n)$: $\underline{\forall\ c > 0}$, $\exists\ n_0 > 0$ such that

$\forall\ n \geq n_0$, we have $0 \leq c.f(n) < g(n)\}$.

$$f(n) = o(g(n))$$

iff $\displaystyle\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0.$

$$2n = o(n^2)$$

$$\lim_{n \to \infty} \frac{2n}{n^2} = \lim_{n \to \infty} \frac{2}{n} = \frac{2}{\infty} = 0.$$

$$\lim_{n \to \infty} \frac{2n}{n} = 2 \quad \therefore \quad 2n \neq o(n)$$

—— x ——

Small omega (ω)

$f(n) = \omega(g(n))$ iff

$g(n) \equiv o(f(n))$

$\boxed{2n = o(n^2)}$ $\Rightarrow$ seen already

iff $\forall\, c > 0\ \exists\, n_0 > 0$ such that $0 \leq c \cdot g(n) < f(n)$

$\forall\, n \geq n_0$

$f(n) = n^2 \quad g(n) = \log n$

$$\lim_{n\to\infty} \frac{n^2}{\log n} \quad \frac{\infty^2}{\log \infty} = \frac{\infty}{\infty} =$$

Indefinite form

$$\cancel{\lim} \frac{2n}{\frac{1}{n}} = \lim_{n\to\infty} 2n^2 = 2 \times \infty^2 = 2\infty = \infty.$$

$f(n) = \omega\, g(n)$

$n^2 = \omega \log(n)$

Bigoh - O $\leq$

Big Omega - Ω $\geq$

Big Theta - Θ $=$

Small Oh - o $<$

Small Omega - ω $>$

While calculating $\lim_{n\to\infty} \frac{t(n)}{g(n)}$ if we get in indeterminant form, like $\frac{\infty}{\infty}$ or $\frac{0}{0}$, then apply L'Hopital's Rule and substitute limits.

L'Hopital's Rule:

$$\lim_{n\to\infty} \frac{t(n)}{g(n)} = \lim_{n\to\infty} \frac{t'(n)}{g'(n)}$$

Ex:1 $\frac{1}{2}n(n-1)$ and n^2

$t(n) = \frac{1}{2}n(n-1)$

$g(n) = n^2$.

$$\lim_{n\to\infty} \frac{\frac{1}{2}n(n-1)}{n^2} = \frac{1}{2}\lim_{n\to\infty} \frac{n^2-n}{n^2}$$

$$= \frac{1}{2}\lim_{n\to\infty} 1 - \frac{1}{n} = \frac{1}{2}\left[1 - \frac{1}{\infty}\right]$$

$$= \frac{1}{2}(1) = \frac{1}{2} > 0$$

$$\therefore \boxed{\frac{1}{2}n(n-1) \in \Theta(n^2)}$$

Ex:2 $\log_2 n$ & $\sqrt{n}$

$t(n) = \log_2 n$

$g(n) = \sqrt{n}$.

$$\lim_{n\to\infty} \frac{\log_2 n}{\sqrt{n}} = \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{\frac{1}{2\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{1}{\cancel{n}\sqrt{n}} \times \frac{2\sqrt{n}}{1}$$

$$= \lim_{n\to\infty} \frac{2}{\sqrt{n}} = \frac{2}{\infty} = 0$$

$$\therefore \boxed{\log_2 n \in O(\sqrt{n})}$$

Common Time Complexities

BETTER	$O(1)$	constant time
	$O(\log n)$	log time
	$O(n)$	linear time
	$O(n \log n)$	log linear time
	$O(n^2)$	quadratic time
	$O(n^2 \log n)$	log quadratic time
	$O(n^3)$	cubic time
	$O(2^n)$	exponential time
WORSE	$O(n^n)$	exponential time