

## OR CHRONICLE

# LP Modeling for Asset-Liability Management: A Survey of Choices and Simplifications

ManMohan S. Sodhi

Cass Business School, City University London, Northampton Square, London EC 1V 0HB, United Kingdom,  
m.sodhi@city.ac.uk

Dynamic linear programming (LP) models for asset-liability management (ALM) are quite powerful and flexible but face two challenges: (1) many modeling choices, not all consistent with one another or with finance theory, and (2) solution difficulties due to the large number of scenarios obtained from standard interest-rate models. We first survey these modeling choices with a view to help researchers make self-consistent choices. Next, we review how the dynamic LP model for ALM and the representation of uncertainty therein have been simplified in the past to motivate new or hybrid models emphasizing tractability. To this end, we review existing static LP models as extreme modeling simplifications and aggregation as a simplification of uncertainty.

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## 1. Introduction

Asset-liability management (ALM) refers to the buying and selling of securities (assets) to meet current and future payments (liabilities) under uncertainty associated usually but not exclusively with interest-rate movements. It is practiced for pension-fund management and in the banking and the insurance industries primarily. Although linear programming (LP) is a flexible and powerful way to model ALM using stochastic programming with recourse, there are two challenges with these “dynamic” models. The first challenge is having to choose from a plethora of modeling choices with not all choices being consistent with each other or with finance theory. The second challenge is that the models are difficult to solve due to the large number of scenarios that grows exponentially with the number of time periods when using standard interest-rate models or stochastic processes. As such, we first survey modeling choices with a view to help researchers make self-consistent choices. Next, we show that existing “static” models are extremely simplified forms of dynamic models regarding both uncertainty and recourse variables, so there is plenty of scope to simplify dynamic models to gain solution tractability. Indeed, dynamic models that have been reported are simplifications of some sort due to solution difficulties, so our goal here is to provide researchers with simplification choices.

These modeling choices and solution difficulties may be part of the reason for modeling missteps in the past. Many operations research (OR)-based ALM models use prices of securities different from those in the market or allow spurious arbitrage opportunities. Rules of thumb, e.g., adding

an “options-adjusted” spread, are sometimes incorporated to otherwise sophisticated models. At times, the focus has wavered with some researchers proposing simplifications to take advantage of a particular parallel computer architecture or a particular algorithm. Yet others have proposed simplistic models to appeal to users. Researchers have not always built upon earlier successful work, e.g., the dynamic model by Bradley and Crane (1972). Happily, research is evolving in a way that builds on Bradley and Crane’s work and also incorporates finance theory although the above two challenges remain.

Despite these challenges, it is worthwhile to consider LP for at least four reasons. First, LP models can make specific recommendations at the individual asset level to actively leverage assets to enhance net worth (objective) rather than simply evaluating a static recommendation such as simulation approaches or providing only broad recommendations in terms of asset classes such as stochastic control methods. Second, meeting regulatory requirements and liquidity requirements (constraints) and using the output of auxiliary interest-rate and cash-flow models as input is straightforward with LP. Third, LP can incorporate “imperfections” including taxes, transaction costs, and regulatory or institution-specific requirements over time more flexibly than other approaches. Fourth, recent advances in LP solver technology and in stochastic programming also help make the case for LP.

These modeling advantages have not translated into widespread use of dynamic LP models in practice despite well-reported successes in the OR literature. According to a *Financial Times* columnist, practitioners view dynamic

models as “excessively complex” with difficulties of explaining “the downside risk of even an optimal solution” so their consensus is that “much more work remains to be done” [before widespread adoption] (Riley 2002). Banks have used advanced financial models for pricing derivatives and there are also reports of static LP-based models having been used in the banking sector (Dembo 1990, Ruffel 1986). Smaller banks’ needs are more modest as studied by Moynihan et al. (2002), who present a decision-support system for credit unions. As industry emphasis is on managing downside risk, it is simulation models that hold sway in industry emphasizing scenario generation, reporting, and workflow management. Some OR researchers have also moved in this direction (e.g., Mulvey et al. 2000). The insurance industry appears to lag in ALM, and hence LP modeling, as evident from a relatively recent Swiss Reinsurance Company report exhorting the life and property/casualty industry to adopt ALM practices (Swiss Reinsurance Company 2000). Dynamic models have solution tractability challenges as well requiring the use of parallel computers with as few as 4,000 scenarios (Mulvey and Shetty 2004). Although Dert (1999) shows that a multistage dynamic model provides better solutions than a myopic or a simple static model, others have raised questions on the benefit of dynamic models over more sophisticated static models (Zenios 1995).

Therefore, for a number of reasons, it is worthwhile to explore how to simplify dynamic LP models (as done, e.g., by Gaivoronski and de Lange 2000 using fix-mix or by Mulvey et al. 1999 using hybrid securities) before we can integrate them into packaged ALM software as hybrids with simulation (as done, e.g., by Boender 1997 and by Seshadri et al. 1999). The current state-of-the-art (e.g., Mulvey and Shetty 2004) using parallel and distributed computers may not fit the computing environment in most companies. Dembo and Rosen (1999) mention a growing interest in static methods as well as “semi-dynamic” strategies in traditional areas of finance; they claim that static methods may be adequate when enterprisewide risk (subsuming ALM) is concerned with as many as 400 risk factors. This paper seeks to help researchers understand simplification choices. Targeting the packaged ALM software market is not a bad idea: the 400 largest financial institutions reportedly spent \$613 million on ALM software globally in 1998 (O’Connell 1999). Following a slowdown in spending in the 1999–2003 period, analysts expect similar spending of about \$600 million globally in 2004—expected to grow to \$850 million in 2009 (Keppler 2004).

As limitations of this work, we do not cover non-LP modeling approaches and refer the reader to Ziemba and Mulvey (1999) and to Zenios and Ziemba (2004). The models in this paper are stripped down to essential constraints for matching cash flows rather than industry-specific or regulatory ones, although we provide some references in the next section that are useful to understand industry-specific liabilities. We also do not provide a definitive solution in terms of a model, but present this survey in the hope that

researchers will use it to that end. Finally, we have limited the discussion to interest-rate risk as has most of the OR literature, even though credit risk (of bad loans) is quite important for banks as is exchange-rate risk. The shortfalls in pension funds of many large companies following the stock market downfall at the turn of the millennium points to the importance of market risk as well, but we do not address this either except to refer to researchers who create joint scenarios over multiple risk factors. Still, much of the discussion here is relevant regardless of whether the risk is limited to interest rates or stems from other factors as well.

## 2. Background

The Financial Accounting Standards Board in the United States defines assets as probable future economic benefits and liabilities as probable future sacrifices of economic benefits. ALM entails leveraging these assets and liabilities to enhance the net worth of the institution while quantifying the various associated risks, managing liquidity, and streamlining the management of regulatory requirements (Ong 1998). ALM has evolved from an accounting focus in the 1960s and 1970s to a centralized, “integrated” risk-management function applied to the entire balance sheet in the mid 1990s (Jarrow and van Deventer 1998) and different companies are at different “rungs” of this evolution (Mulvey et al. 1997, Mulvey and Ziemba 1999).

In the banking industry, liabilities are customer accounts, while assets are mortgages and personal loans. Cash flows depend on interest-rate movement, e.g., people prepay loans when interest rates drop. The failure of many banks in the early 1980s provided a compelling need to manage interest-rate risk better. As such, investors reward more stable high-quality earnings by higher stock prices for financial institutions (Giarla 1991). Ong (1998) and Bitner and Goddard (1992) discuss ALM practice in the banking industry. Cohen and Gibson (1978) focus on banking with Part V devoted to bond portfolio management and interest-rate risk. Adamidou et al. (1993) and Ben-Dov et al. (1992) describe an optimal portfolio system at Prudential-Bache. Seshadri et al. (1999) use simulation and quadratic optimization to “formulate, test, and refine” the Federal Home Loan Bank of New York’s ALM policies.

In the insurance industry, assets are “fixed-income” securities such as treasuries, corporate bonds, and mortgage-backed instruments. Liabilities are customer-related insurance payouts that need not be tied to interest rates. Babbal and Staking (1991) report that exposure to interest-rate risk is the strongest predictor among tested variables to explain insolvency for property/liability insurers. Cariño et al. (1994), Cariño and Ziemba (1998), and Cariño et al. (1998) describe the need for ALM in insurance companies and the decision-support systems they built for the Japanese insurance company Yasuda Kasai. Worzel et al. (1994) describe managing a fixed-income portfolio at Metropolitan Life Insurance using an index as the “liability.” Holmer (1994, 1999) describes an application at Federal National

Mortgage Association in reference to mortgage insurance and mortgage investment. Sweeney et al. (1999) outline Falcon Asset Management's ALM approach for global insurance companies where the assets include cash and equities in multiple currencies. An industry research report describes the need for ALM in the life insurance and property/casualty industry (Swiss Reinsurance Company 2000), while Gaivoronski and de Lange (2000) provide a model for casualty insurers. Hoyland and Wallace (2001b) analyze a legal regulation in the Norwegian life insurance business using a dynamic ALM model.

Another growing area is pension funds. Liabilities, e.g., defined benefits, are payouts to workers who are already retired or who will retire in the future. Assets usually include both fixed-income securities and equities. Mulvey (1996) describes an approach to generate scenarios—interest rates, pension withdrawals, and economic scenarios—for pension-fund management at Towers Perrin; Mulvey et al. (2000) describe the overall system as additionally having a nonlinear “optimization simulation” model. Boender (1997), Dert (1999), and Drijver et al. (2000) consider Dutch pension funds, while Gondzio and Kouwenberg (2001) provide a decomposition-based algorithm for these implemented on a particular parallel computer. Kouwenberg (2001) discusses liability scenarios in this context as well.

To give a sense of relative enthusiasm for ALM software in these three industries, the global breakdown of spending was 79% in the banking sector, 4% in the insurance sector, and 17% in the pension funds sector (Keppler 2004). Some researchers have looked specifically at software issues in ALM; for instance, Dempster and Ireland (1991) and Sodhi (1996) describe object-oriented implementations.

### 3. The Modeling Challenge and Choices

The large number of modeling choices are a challenge because of the need to main consistency and because of trade-offs between model completeness (i.e., capturing essential constraints and decision variables to get a meaningful solution) and tractability implied by these choices. The choices pertain to (1) the representation of uncertainty, (2) the use of market or computed prices and avoiding arbitrage, (3) the representation of time periods and the decision horizon, (3) the objective function, (4) constraints, and (5) transaction costs and taxes.

#### 3.1. An Example Model

Below is a model similar to the ones by Bradley and Crane (1972), Klaassen (1994, 1998), Consigli and Dempster (1998), Zenios (1995), and Dert (1999), among others. Dynamic stochastic models are not new and appear in an older collection of articles edited by Ziemba and Vickson (1975). An important difference from Bradley and Crane's model is our requirements on auxiliary models to ensure *external consistency*, i.e., computed prices are the same

as market prices at time 0, and *internal consistency*, i.e., the absence of any arbitrage opportunities. In the following model,  $\zeta_t$  is a scenario that ends at time  $t$  and  $\zeta_{t-1}$  refers to its “parent” scenario with which it shares all interest-rate states at times  $0, \dots, t-1$ . From scenario  $\zeta_t$  emerge two child scenarios  $\zeta_{t+1}$  with equal probability. This implies that for any  $t$ , the probabilities of all scenarios are equal, so the probabilities for the terminal scenarios at the horizon are  $2^{-T}$  each.

#### The Scenario-Based Model, $P_{\text{scen}}$

##### Decision Variables

$x_{i,\zeta_t}$ : amount purchased of (original) principal of security  $i$ .

$y_{i,\zeta_t}$ : amount sold of principal.

$h_{i,\zeta_t}$ : holdings of principal *after* trades.

$l_{\zeta_t}$ : single-period lending at the current short rate.

$b_{\zeta_t}$ : single-period borrowed at the current short rate (plus the premium  $D$ ).

##### Inputs Computed from Auxiliary Security Pricing Models

$\kappa_{i,\zeta_t}$ : total cash flow generated owing to interest and principal payback per unit holding of principal.

$\pi_{i,\zeta_t}$ : ex-dividend computed price at time  $t$  for principal (same as market price when  $t=0$ ).

$\rho_{\zeta_t}$ : single-period interest rate; \$1 at time  $t \equiv (1 + \rho_{\zeta_t})$  in  $t+1$ .

$\delta_{\zeta_t}$ : present value of a cash flow of \$1 at time period  $t$  in scenario  $\zeta_t$ .

##### User-Provided Input

$L_{\zeta_t}$ : expected liability.

$D$ : premium paid over the short rate for single-period borrowing; strictly positive.

$T_i$ : proportionality constant for transaction costs, for security  $i$ ; strictly positive.

$T$ : horizon (in number of months).

$l_{-1}$ : single-period lending due in current time period ( $t=0$ ).

$b_{-1}$ : single-period borrowing from last period due in current time period.

$h_{i,-1}$ : holding of security  $i$  at beginning of current time period.

##### Objective Function

$$\max 2^{-T} \sum_{\zeta_T} \delta_{\zeta_T} \left\{ \sum_i [\pi_{i,\zeta_T} h_{i,\zeta_T}] + l_{\zeta_T} - b_{\zeta_T} \right\}.$$

##### Constraints

$$\begin{aligned} \sum_i \kappa_{i,\zeta_t} h_{i,\zeta_{t-1}} + l_{\zeta_{t-1}} (1 + \rho_{\zeta_{t-1}}) + b_{\zeta_t} + \sum_i (1 - T_i) \pi_{i,\zeta_t} y_{i,\zeta_t} \\ - \sum_i (1 + T_i) \pi_{i,\zeta_t} x_{i,\zeta_t} - l_{\zeta_t} - b_{\zeta_{t-1}} (1 + \rho_{\zeta_{t-1}} + D) = L_{\zeta_t} \quad \forall \zeta_t, \end{aligned}$$

$$h_{i,\zeta_t} + y_{i,\zeta_t} - x_{i,\zeta_t} - h_{i,\zeta_{t-1}} = 0 \quad \forall i, \forall \zeta_t,$$

$$x_{i,\zeta_t}, y_{i,\zeta_t}, h_{i,\zeta_t}, l_{\zeta_t}, b_{\zeta_t} \geq 0 \quad \forall i, \forall \zeta_t$$

for all  $t = 0, \dots, T$ .

Let us summarize the design choices for this model.

**Representation of Uncertainty.** This model relies on an auxiliary interest-rate model to provide interest-rate scenarios that evolve in the form of a binary tree with any pair of child nodes being equiprobable. The auxiliary interest-rate model comprises only the single-period or “short” risk-free interest rate. Black et al. (1990) and Ho and Lee (1986) take binary trees with adjacent nodes with identical interest-rate values so that there are only  $t + 1$  distinct interest-rate states for any time period or only  $T(T + 1)/2$  distinct states in all rather than  $2^{T+1}$  states. We call such a binary tree a *binomial tree*, as we can represent the number of heads in sequential coin tosses by such a tree. The number of distinct scenarios or paths remain unchanged from a binary tree. Again,  $\zeta_{t-1}$  is the “parent” scenario that shares all interest-rate states at times  $0, \dots, t - 1$  with scenario  $\zeta_t$ . There are  $2^t$  scenarios at time  $t$  each with 2 children scenarios in time  $t + 1$ , *all of which are used in the LP model*. This is why we multiply  $2^{-T}$  into the objective function for the probability of each terminal scenario, but this is a constant and can be ignored.

These probabilities are such that the market price of any security under consideration is equal to the expected present value of cash flows in all the scenarios. Moreover, the expected single-period rate of return of any security in any scenario is equal to the risk-free rate for that scenario. This means that there is no premium for risk, so we term these probabilities “risk neutral.” These probabilities have nothing to do with the real world in the sense of representing the likelihood of the interest rate going up or down in the next period: Instead, these are computational artifacts to make it easy for us to match market prices to computed ones. Justification for their use comes from the fact that we can completely hedge against the risk of any security when markets are complete by creating a dynamic portfolio that returns the risk-free rate in any future scenario. So all investors, whether risk averse or risk neutral, value the portfolio equally. Rather than create such portfolios for every security, it is computationally easier to assume that all investors are risk neutral and then use risk neutral probabilities that match market prices and the expectation of future cash flows using these probabilities (see, for example, Hull 2003, pp. 203–205).

Theoretically, these probabilities should be unique, but in practice their value depends on the interest-rate model and on the benchmark securities selected to create the interest-rate tree. We use the Black et al. (1990) model (say, with months as periods) that takes these probabilities to be  $1/2$  for each “up” or “down” movement of the single-period interest rate. It is the values of the interest rate on the tree nodes that are then computed by matching the market values of U.S. treasuries and options on them to their respective computed expected present values.

**External Consistency.** Security prices in the model are those from the market. As explained above, first the interest-rate tree is generated using a selection of benchmark securities, typically U.S. (or other) treasuries and options on them. Next, for any security under consideration, we use an auxiliary security-specific cash-flow model to generate cash flows for that security for all interest-rate scenarios. For instance, for a tranche of a mortgage-backed offering, we would need a model for the cash flows (and prices) of that security, given any interest-rate scenario, to generate its cash flows for the entire set of scenarios. Finally, we use backward and forward computations on the scenario tree to adjust these cash flows (and prices) to ensure that the expected present value of these cash flows is equal to the security’s market price. Refer to Black et al. (1990), Sodhi (1996), and the similarly motivated *implied tree model* for stocks described by Hull (2003, pp. 460–461) for examples.

**Internal Consistency.** The auxiliary cash-flow models for each security as explained above ensure that at any state on the interest-rate tree, all securities have the same expected single-period return over all scenarios passing through that state. This return is the same as the single-period interest rate at that state. As discussed later and in the appendix, this prevents spurious arbitrage opportunities in the model that can lead to unbounded solutions.

**Time Periods and Decision Horizon.** The model takes as given some decision horizon  $T$  as well as the size of the time intervals. Using the above mentioned interest-rate models, the total number of scenarios and hence the number of variables and constraints increases as  $2^T$ .

**Decision Variables.** The model has scenario-specific decision variables  $x, y, h, l$ , and  $b$  corresponding to the current scenario  $\zeta_0$  at  $t = 0$  and to the future scenarios  $\zeta_t$ ,  $0 < t \leq T$ .

**Objective Function.** The model maximizes the expected *present* value of the portfolio’s value at the horizon after trading has taken place. Recall that we are using “risk-neutral” probabilities in our model, so the use of this linear objective function is valid only if the portfolio manager is actually risk neutral and if his subjective probabilities for the terminal scenarios are the same as the risk neutral ones. A slight relaxation of this is his risk aversion only to excessive borrowing, so that his utility function is the same as the expected present value minus any penalties for any borrowings in excess of a threshold value or a series of such thresholds with increasingly higher penalties. We still need to assume that his subjective probabilities are the same as the risk-neutral probabilities. A more general alternative is to incorporate the manager’s subjective probabilities (as distinct from the risk-neutral ones in the interest-rate tree) for the terminal scenarios and apply his utility function. Klaassen (1997) uses an objective function this way with any utility function that satisfies the expected value condition  $E_{\zeta} u(x_1, x_2, \dots, x_{\zeta}, \dots) = \sum_{\zeta} p_{\zeta} u(x_{\zeta})$ , where  $p_{\zeta}$  is the subjective probability.

**Constraints.** There are two sets of constraints besides the nonnegativity ones. The first set represents cash balance and the second set maintains the inventory of holdings for each security. Nonnegativity constraints in this model rule out short selling. Additional constraints can be added to this model. A limit on borrowing, say  $B$ , could incorporate a penalty variable  $b_{\zeta_t}^p$  for excessive borrowing. As before,  $b_{\zeta_t}$  would be borrowed at  $(1 + \rho_{\zeta_t} + D)$  with  $b_{\zeta_t} \leq B$  as a constraint, but  $b_{\zeta_t}^p$  could be borrowed only at the higher cost of  $(1 + \rho_{\zeta_t} + \gamma D)$ ,  $\gamma > 1$ . As explained above, this is useful if we can assume that the portfolio manager is risk neutral except when losses in any period requiring any borrowing above this threshold value. So the borrowing limit  $B$  and the penalty factor  $\gamma$  can be used to generate “risk-return” trade-off curves. The cost of excessive borrowing then reflects a risk aversion decrement to the otherwise risk-neutral utility function that is the expected present value. There is no reason to have just one limit: we could have any number of thresholds  $B_1, B_2$ , etc. with strictly increasing borrowing costs for borrowing exceeding these thresholds. Another constraint could be to require a certain amount of liquidity that translates into a lower bound for single-period lending  $l_{\zeta_t}$  in all scenarios.

**Market “Imperfections.”** The model assumes a transaction cost proportional to the size of the transaction stemming from the bid-ask price spread. Borrowing for a single period is strictly more expensive than lending as reflected by the premium on borrowing. This particular model does not take taxes into account for simplicity even though there are situations when modeling these is a must.

We can characterize the above model with

REMARK 1. The model  $\mathbf{P}_{\text{scen}}$  is externally and internally consistent, i.e., the computed price (expected net present value) for each security matches its market price and there are no opportunities for arbitrage.

REMARK 2. The model  $\mathbf{P}_{\text{scen}}$  is feasible and bounded.

The first remark follows from the use of the auxiliary models for interest-rate scenario creation and for interest-rate specific cash flows of different securities. Feasibility is checked by setting  $x_{i,\zeta_t}$ ,  $y_{i,\zeta_t}$ , and  $h_{i,\zeta_t}$  to zero for all scenarios  $\zeta_t$  (for all  $t$ ) and by funding the liabilities with single-period borrowing. To prove boundedness, we can show the dual model is feasible (see the appendix).

The additional constraints described above do not take away feasibility or boundedness from  $\mathbf{P}_{\text{scen}}$ . Next, we review modeling choices made by different researchers.

### 3.2. Representation of Uncertainty

To model uncertainty, we need to choose the factors underlying this uncertainty and then use an appropriate auxiliary model (i.e., a stochastic process) to generate scenarios over which we can optimize. Many researchers assume that all the considered assets and liabilities are derivatives of a single factor of uncertainty, the *short rate*, the single-period

interest rate in the context of discrete-time models. Evidence for a single factor in explaining the prices of bonds and other fixed-income securities is somewhat reassuring (e.g., Dattatreya and Fabozzi 1995, Chapter 1). An additional factor, the so-called *long rate*, adds to this assurance, e.g., Mulvey (1996) uses the two-factor model of Brennan and Schwartz to generate a sample of interest-rate scenarios. Hull (2003, Chapters 23 and 24) discusses interest-rate models and valuation of such derivative securities as bond options. Kusy and Ziemba (1986) focus on the uncertainty of deposit flows in a bank instead of interest rates.

Using two or more factors is fine for simulation, but is problematic in any LP-based model that requires all scenarios. One could use Bradley and Crane’s (1972) model as different paths of events on an  $n$ -ary tree. However, tractability, which is a problem even with single-factor models, becomes a much bigger problem when all scenarios from the  $n$ -ary tree need to be included in the LP model. As a result, researchers take only a sample of scenarios, e.g., Mulvey and Thorlacius (1999) sample antithetic scenarios for interest rates, exchange rates, stock returns, and inflation for their static method.

Therefore, many researchers restrict themselves to a single factor pertaining to interest rates for LP models and use an auxiliary model, a stochastic process, whose parameters are determined by matching computed and market prices of such benchmark securities as treasury bonds and treasury options. Then, other securities’ market and computed prices are matched using such a model. The simplest representation of a stochastic process in discrete time is a binary tree, but if we take this tree to be binomial, there would be only  $T(T + 1)/2$  states in all rather than  $2^{T+1}$ . Not all models use binomial trees (e.g., Heath et al. 1990). Other models may allow negative interest rates (Ho and Lee 1986) that are quite uncommon in practice.

As mentioned before, many interest-rate models use a computing artifice called “risk neutrality.” The idea is that all investors regardless of their risk profile would pay the same amount for a security whose value is contingent on some underlying factor when other assets in the market can be combined dynamically to replicate the cash flow for this security and hedge its uncertainty (Trigeorgis and Mason 1987, Varian 1987). To avoid the computation of creating such portfolios, we can therefore choose to compute security prices (or match to market prices) as if all the investor were risk neutral. The interest-rate models such as the ones mentioned above use this idea because the tree is generated to ensure that the market price of a set of securities, typically treasuries and options on these, equals their expected present value of future cash flows. Recall that an implication of risk neutrality is that all securities have the same expected return under risk-neutral probabilities as the riskless rate in any period (Cox and Ross 1976, Harrison and Kreps 1979, Klaassen 1998).

“Risk neutrality” does not mean that the investor is actually risk neutral or that his or her portfolio’s risk measures

are not affected by risk—nor are risk-neutral probabilities real in the sense of likelihood of events. The only reality associated with “risk neutrality” is the market prices of securities and the scenario-specific cash flows. As such, an objective function that maximizes the expected net present value using these probabilities is valid only for a risk-neutral investor whose subjective probabilities are the same as these risk-neutral ones.

Not everyone is convinced about using risk-neutral interest-rate models or even a single factor. Kouwenberg (2001) as well as Gaivoronski and de Lange (2000) create event trees with a few asset *classes* (stocks, bonds, real estate, etc.) to match the first so many moments of the joint continuous probabilities (obtained from history or future expectations) of returns with those obtainable from the tree as recommended by Hoyland and Wallace (2001a). These moments include the means and the covariance matrix of these asset and even liability classes and matching these requires solving a nonlinear problem that penalizes deviations. Consigli and Dempster (1998) use United Kingdom data from 1924–1991 comprising third-order autoregressive equations to generate annual returns (and hence probability distributions) for similar asset classes: ordinary shares, fixed-interest irredeemable bonds, bank deposits, index-linked securities, and real estate. Dert (1999) follows a similar approach for Dutch pension funds with a “vector” autoregressive model for wage inflation, price inflation, cash, stocks, property, bonds, and GNP. Pflug et al. (2000) go further in this direction by using principal component analysis on historical data to extract factors of uncertainty that drive asset returns and interest rates and use these to create scenario trees by matching statistical properties. These researchers also use risk-averse (concave) or risk-neutral (linear) objective functions.

Mulvey and Shetty (2004) describe the overall context for generating scenarios for economic factors and project asset returns and projected liabilities based on these economic scenarios. They stress robustness in that the solution should not vary much if the chosen scenarios and their probabilities were to be perturbed.

### 3.3. External Consistency

While it may seem obvious that model-computed and market prices should match, this is not always the case in the literature. When they do not, an LP model seeking to maximize expected net present value would recommend buying more of a security that appears cheap given its cash flows regardless of how well it meets the liabilities.

### 3.4. Internal Consistency

If a security were to offer a higher return than all other securities in a particular interest-rate state and the same in others, an LP model maximizing expected net present value would seek to invest more in that security. The solution could even be unbounded if the return for this security is

more than the borrowing rate in any state. Throwing in bounds or other constraints can mask this problem, but the solution would be biased towards such securities. Many models in the literature have this problem either due to not ensuring that all securities have the same return using “risk-neutral” probabilities or by taking only a sample of scenarios for the LP model (Klaassen 1997, 2002) instead of all of them as advocated here.

### 3.5. Time Periods and the Decision Horizon

Time periods should reflect (potential) trading dates but they may actually depend on those in the auxiliary interest-rate models. Either way, the time-interval size is a challenge. A two-year calendar horizon can be represented as  $T = 24$  periods of one month each or as  $T = 8$  periods with one quarter each. If the time intervals are small, we have more scenarios, constraints, and variables. For instance, even with monthly intervals ( $T = 24$ ) with  $n = 50$  securities, we have an LP model with more than 1.7 billion constraints and 5.1 billion decision variables! If time intervals are too big, creating parameters for the auxiliary interest-rate model is problematic. Such models use the prices and cash flows of selected benchmark securities, e.g., U.S. treasuries and options on these, to determine the interest-rate process. Another problem is aggregating monthly cash flows for a single time interval of one year or an even longer period. Finally, the consequent fewer trading dates bring into question the quality of the solution. Still, to reduce  $T$ , researchers take successive time periods of increasing length (e.g., Cariño et al. 1994, Cariño and Ziemba 1998, Cariño et al. 1998).

One could avoid this trade-off altogether by taking a short calendar horizon. But doing so can result in poor solutions because short horizons do not capture the risk characteristics of many fixed-income securities. For example, treasury bonds are less risky in contrast to equities in the long run due to termination. An open question is how to deal with very long decision horizons, say a few decades, such as those that could be pertinent to pension funds.

### 3.6. Decision Variables

As in any multiperiod model, there are *current* or operational variables— $x_{i,\xi_0}$ ,  $y_{i,\xi_0}$ ,  $b_{\xi_0}$ , and  $l_{\xi_0}$  for decisions to be made now ( $t = 0$ ) and *future* or recourse variables— $x_{i,\xi_t}$ ,  $y_{i,\xi_t}$ ,  $b_{\xi_t}$ , and  $l_{\xi_t}$  for the future ( $1 \leq t \leq T$ ) that are needed to estimate future costs contingent on interest-rate movements. There are also inventory variables  $h_{i,\xi_t}$  to keep track of asset holdings over time.

Ideally, to avoid the higher-priced odd lots, both current and future variables should be restricted to be integer, e.g., either buy or do not buy the entire amount of a particular mortgage-backed security (or tranche). However, researchers ignore this restriction to avoid the computing burden. One justification for this relaxation is that the decision-support system recommendations are likely to be

used only as a guideline by the portfolio manager. However, using a dynamic programming model with two hypothetical assets, Edirisinghe et al. (1993) suggest that the cost of an optimal portfolio to meet a single terminal period liability can be much higher when integral lot size restrictions are imposed on both current and future transactions.

The computational burden associated with the future variables can be reduced in at least three ways although we need an empirical understanding of the impact of doing so on the optimal values of the *current* variables: (1) We can aggregate securities into “classes” to reduce their number in the model (e.g., Mulvey 1994), but doing so raises difficult questions: Is purchasing such a “security” tantamount to requiring the proportional purchase of all assets in this class and if so, what proportions should be used? How will maturing assets be replaced in the class? (2) We can employ aggregation in the future time periods (e.g., Klaassen 1998) to reduce the number of decision variables. (3) We can consider a shorter horizon for recourse  $T_1 < T$  to reduce the number of decision variables, the extreme case being that of static models with  $T_1 = 0$ .

Additional variables can be used for industry-specific constraints. Dert (1999) uses binary variables for a Dutch pension-fund problem to model chance constraints limiting the probability of underfunding at any node on the scenario tree, proposing a heuristic for solving the resulting MILP model. Drijver et al. (2000) add conditional constraints with binary variables to Dert’s model.

### 3.7. Objective Function

Horizon wealth for any terminal-period scenario  $\zeta_T$  is defined as

$$w_{\zeta_T} = \left\{ \sum_i [\pi_{i,\zeta_T} h_{i,\zeta_T}] + l_{\zeta_T} - b_{\zeta_T} \right\}$$

with probability  $p_{\zeta_T}$ . Researchers have used it in different objective functions:

*Maximizing expected net present value* as in model  $\mathbf{P}_{\text{scen}}$ ,

$$\max \sum_{\zeta_T} \delta_{\zeta_T} p_{\zeta_T} w_{\zeta_T}$$

is consistent with maximizing the value of the firm if the decision maker was risk neutral and if the probabilities  $p_{\zeta_T}$  were subjective rather than risk neutral. Kusy and Ziemba (1986) use this objective function with subjective probabilities implying therefore that the portfolio manager is actually risk neutral. To capture risk aversion, they include penalties, or “shortfall” (as well as “surplus”) costs in the objective function corresponding to the violation of deposit flow and policy constraints.

*Minimizing initial investment*

$$\min \sum_i [\pi_{i,\zeta_0} x_{i,\zeta_0}] + l_{\zeta_0} - b_{\zeta_0}$$

has been used in many models (e.g., Dahl 1993, Dahl et al. 1993) but is “too conservative” because it does not seek to

maximize returns (Ambachtsheer 1987). This can be used with risk-neutral probabilities. A similar objective function arises for defined-benefit pension-fund management where one seeks to minimize the expected costs of funding (Dert 1999).

*Maximizing expected horizon return or value*

$$\max \sum_{\zeta_T} p_{\zeta_T} w_{\zeta_T}$$

banks on the popularity of horizon return as a metric for portfolio managers. This is not meaningful with risk-neutral probabilities but Mulvey (1996) maximizes expected horizon value in weighted combination with minimizing risk measured by different criteria. Bradley and Crane (1972) do so using subjective scenarios, implicitly assuming that the user is risk neutral. To adjust for risk aversion, they add constraints to limit the amount of loss. Horizon return, the annualized ratio of the expected horizon value of the portfolio and its present market value, is a popular measure in fixed-income practice (e.g., Dattatreya and Fabozzi 1995, Chapter 4), so maximizing horizon return is attractive for decision-support developers (e.g., Adamidou et al. 1993, Worzel et al. 1994). Mean-variance models adapted for ALM also maximize horizon return (see the next section).

*Maximizing risk-averse utility*  $u(\cdot)$  as a function of the discrete *subjective* probability distribution  $f(\cdot)$  of wealth over (typically user-specified) terminal scenarios,

$$\max u(f(w_{\zeta_T}))$$

is another objective function that researchers have used. A simplified version is the expected utility function  $\sum_{\zeta_T} p_{\zeta_T} u_1(w_{\zeta_T})$  used, for instance, by Klaassen (1997). Any utility-based objective entails nonlinear functions, e.g., expected log value of excess horizon return (Worzel et al. 1994). Mulvey (1994) maximizes the expected utility of different wealth profiles during the time between the starting date and the selected horizon. Kallberg and Ziemba (1983) discuss the relevance of different forms of concave utility functions and show that they all give similar results if the average risk aversion is the same. Others attempt to avoid nonlinearity by adding constraints to model risk aversion (Kusy and Ziemba 1986, Cariño et al. 1994, Cariño and Ziemba 1998, Cariño et al. 1998).

### 3.8. Constraints

As in model  $\mathbf{P}_{\text{scen}}$ , it is necessary to have at least two sets of constraints for each scenario, one to balance the cash including meeting liabilities and the other to balance the security holdings after trading in each time period. We may also add a limit on single-period borrowing (as a soft constraint) and also require a minimum level of single-period lending for reasons of liquidity.

Diversification constraints imposing user-provided restrictions on security holdings (or purchases), e.g.,

$$L_i \leq h_{i,\zeta_i} \leq U_i,$$

are not only unnecessary in a good stochastic model, they are also undesirable as they detract from optimality (Birge and Louveaux 1997). *They also mask problems where the LP model would have otherwise resulted in an unbounded solution due to internal or external consistency.* Still, context-specific constraints such as those pertaining to legality and policy listed by Kusy and Ziemba (1986) can be selectively imposed.

Limits to borrowing (e.g.,  $b_{\zeta_t} \leq B$ ) and to short selling (e.g.,  $x_{i,\zeta_t} \geq 0$ ) vary in models. Finance theory models generally ignore such restrictions while OR models impose them. Still, Dahl et al. (1993) have no nonnegativity constraints on the investment variables to allow unlimited short selling. On the other hand, Klaassen's (1998) unaggregated model like  $\mathbf{P}_{\text{scen}}$  disallows short selling; he also imposes hard constraints on borrowing. As mentioned earlier, a soft limit to borrowing with  $b_{\zeta_t}^p$  as excessive borrowing at cost  $(1 + \rho_{\zeta_t} + \gamma D)$ ,  $\gamma > 1$ , can be used to generate the risk-return trade-off if excessive borrowing is viewed as risky. This limit reduces the (linear) objective function by more than the normal borrowing cost of  $(1 + \rho_{\zeta_t} + D)$ .

Additional constraints can come from incorporating risk aversion to avoid using concave utility functions in the objective. For example, Bradley and Crane (1972) impose a lower limit on the maximum capital loss in any sequence of events. Dahl et al. (1993) constrain the scenario-specific present value of the portfolio to exceed the scenario-specific present value of liabilities in all generated scenarios. Adamidou et al. (1993) also require a minimum scenario-specific return under each scenario. Worzel et al. (1994) constrain the horizon return to exceed the index return less  $\varepsilon$  under all scenarios.

As already mentioned, some researchers, e.g., Dert (1999) and Drijver et al. (2000), use chance constraints to limit the probability of not meeting the liabilities (underfunding in pension-fund management). These replace the constraints based on the capital adequacy formula of Chambers and Charnes (1961). However, according to Kusy and Ziemba (1986), such models have theoretical problems in multiperiod situations.

### 3.9. Transaction Costs and Taxes

Keeping transactions costs low by making transaction costs and taxes explicit is a motivation for any dynamic model. As in  $\mathbf{P}_{\text{scen}}$ , most researchers take transaction costs to be proportional to the market value of a security at the time the transaction is made, with the proportionality constant being specific to the security category (Kusy and Ziemba 1986, Klaassen 1998). Buying at the asked price and selling at the bid price is the same as proportional transaction costs (Bradley and Crane 1972). It is certainly worth modeling transaction costs because simulation shows that not including these results in poorer performance (Logue and Maloney 1988, Mulvey 1993). While transactions beyond the decision horizon and their costs are usually ignored,

Mulvey and Vladimirov (1992) explicitly include transaction costs beyond the horizon in their model.

Taxes can be either explicitly modeled or simply ignored for simplicity depending on the context and on the purpose of the model. Modeling taxes is justified when there are infrequently traded securities requiring keeping track of the different capital and income tax payments. Thus, many researchers have specifically modeled taxes by tracking when securities are bought and sold in the model and making the tax payments explicit (Bradley and Crane 1972, Kusy and Ziemba 1986, Cariño et al. 1994, Cariño and Ziemba 1998, Cariño et al. 1998). However, doing so means adding many more variables relative to a model such as  $\mathbf{P}_{\text{scen}}$  that ignores taxes. Ignoring taxes is justified in a frequently trading environment when the same (marginal) income tax applies to all profits and losses. Returns on tax-exempt instruments such as municipal bonds are simply adjusted upwards to allow their use in the model as other taxed securities (e.g., Sodhi 1996).

## 4. Simplifying the Model

Ignoring recourse decisions and even uncertainty to obtain so-called static models is one way to simplify the model. Although these models may not seem interesting in light of their flaws and the contrasting benefits of dynamic models, viewing them as modeling simplifications of dynamic models can motivate the development of hybrid static-dynamic models. By not taking recourse decisions into account, static models can possibly result in higher-cost portfolios in the long run (e.g., Dert 1999). They also may not have any notion of external and internal consistency possibly resulting in unbounded solutions unless diversification bounds are artificially imposed. Hiller and Schaak (1990) survey static models.

### 4.1. Cash-Flow Matching

Consider the following set of modeling simplifications to  $\mathbf{P}_{\text{scen}}$ : let the scenarios collapse into a single linear path or scenario so that  $\zeta_t \equiv t$  for all  $t$  and allow no trading, borrowing, and lending in the future. The resulting model without uncertainty or recourse is called cash-flow matching:

$$\begin{aligned} (CF_1) \quad & \min \sum_{i=1}^N \pi_i x_i \\ \text{s.t.} \quad & \sum_i \kappa_{i,t} x_i \geq L_t \quad \forall t, \\ & x_i \geq 0 \quad \forall i, \end{aligned}$$

with the equality in the first set of constraints in  $\mathbf{P}_{\text{scen}}$  being relaxed to an inequality here. As before,  $L_t$  is the liability amount in period  $t$  and for security  $i$  from a pool of  $N$  securities,  $x_i$  is the amount purchased now ( $t = 0$ ) with  $\pi_i$  its market price and  $\kappa_{i,t}$  the cash flow generated in period  $t$ . Note that there are  $T$  constraints. The single-terminal scenario can be based on the current treasury yield curve.



Hiller and Schaak (1990) add multiple scenarios in which borrowing and lending is allowed in future periods but not selling or buying of assets. We restate their model using evolving scenarios  $\zeta_t$  in all periods ( $0 \leq t \leq T$ ) as we did for  $\mathbf{P}_{\text{scen}}$  rather than the terminal scenarios  $\zeta_T$  they used. This is because the latter require *nonanticipatory constraints* that they missed (all terminal scenarios with an identical history till time  $t$  must have the same decisions until this period to avoid using the perfect terminal-scenario-specific knowledge of the future).

$$\begin{aligned}
 (CF_2) \quad & \min \sum_{i=1}^N \pi_i x_i + (b_0 - l_0) \\
 \text{s.t.} \quad & \sum_i \kappa_{i,\zeta_t} x_i + l_{\zeta_{t-1}} (1 + \rho_{\zeta_{t-1}}) + b_{\zeta_t} \\
 & - l_{\zeta_t} - b_{\zeta_{t-1}} (1 + \rho_{\zeta_{t-1}} + D) = L_{\zeta_t} \quad \forall \zeta_t, \forall t, \\
 & x_i \geq 0 \quad \forall i, \\
 & l_{\zeta_t}, b_{\zeta_t} \geq 0 \quad \forall \zeta_t, \forall t.
 \end{aligned}$$

This model is a static-dynamic hybrid with uncertainty and with future single-period borrowing and lending. The number of constraints increases from  $T$  in  $(CF_1)$  to  $2^{T+1} - 1$  in  $(CF_2)$ . But there are fewer variables than  $\mathbf{P}_{\text{scen}}$  because there are no recourse decisions. Hiller and Schaak (1990) only sample scenarios, which is part of the reason they obtain unbounded solutions that could be avoided by attending to internal and external consistency. We could also add recourse decisions for the first so many periods as well to make the model more dynamic.

## 4.2. Duration Matching

The  $T$  constraints in the deterministic model  $(CF_1)$  can be relaxed further by aggregating them into a single constraint. If the weights used for aggregation are  $\delta_t$  (the present value of \$1 paid at time  $t$  in the single scenario corresponding the current yield curve), we get what we call *the present-value matching model*

$$\begin{aligned}
 (PM) \quad & \min \sum_{i=1}^N \pi_i x_i \\
 \text{s.t.} \quad & \sum_{i=1}^N P_i x_i \geq P_L, \\
 & x_i \geq 0 \quad \forall i,
 \end{aligned}$$

with a single constraint requiring that the present value of future cash flows  $P_i = \sum_t \kappa_{i,t} \delta_t$  exceeds the present value of liabilities. When no attention is paid to external consistency in  $(CF_1)$ ,  $P_i$  and  $\pi_i$  need not be equal, which biases the solution towards securities that appear cheaper. Some researchers have used this problem as a scenario-specific subproblem with scenario-specific price  $P_i$  that is indeed different from the current market price  $\pi_i$  (see the next subsection).

More constraints can be added to this using the first derivative of  $P_i$  relative to interest rates called *duration*  $D_i$  and the second derivative called *convexity*  $C_i$ . The motivation is that a small change in interest rates would leave the present values for assets and liabilities matched when their durations are equal. However, when interest rates do change, so do the durations, thus requiring the convexities to be equal as well to ensure that the durations remain the same; some researchers and practitioners have suggested adding a third derivative as well. The model is

$$\begin{aligned}
 (DM) \quad & \min \sum_{i=1}^N \pi_i x_i \\
 \text{s.t.} \quad & \sum_{i=1}^N P_i x_i \geq P_L, \\
 & \sum_{i=1}^N D_i x_i = D_L, \\
 & \sum_{i=1}^N C_i x_i = C_L, \\
 & x_i \geq 0 \quad \forall i,
 \end{aligned}$$

where  $L$  denotes liability. Although cash-flow matching may appear to result in a more expensive portfolio relative to duration matching because of more constraints, Maloney and Logue (1989) provide empirical evidence suggesting similar yields in the long run; Zipkin (1992) also compares the two.

While present value is linear over component cash flows, the definitions of duration and convexity chosen from the many existing ones need to ensure that these are linear as well. The above derivative-based definition using  $\delta_t = (1 + y)^{-t}$  where  $y$  is the yield of the security in question is called *dollar duration*  $= dP_i/dy$ . An older definition of duration for security  $i$  is

$$D_i = \frac{\sum_t t \kappa_{i,t} \delta_t}{\sum_t \kappa_{i,t} \delta_t}$$

and is owed to Macaulay (1938). It is therefore sometimes called *Macaulay duration*. For a small parallel shift in the entire yield curve, this is proportional to the percentage change in the asset's value (Hicks 1939). Other definitions of duration have been proposed keeping this property in mind. One is to replace  $\delta_t$  by  $e^{-yt}$ ,  $y$  being the continuously compounded yield value, but this is incorrect unless the yield curve is flat (Weil 1973, Ingersoll et al. 1978).

Researchers have created variants that essentially add more constraints or redefine duration. These include adding diversification constraints on the percentage composition of the portfolio (Adamidou et al. 1993); segmenting liabilities along time segments, different sectors, and credit ratings and doing duration matching separately for each (Dahl et al. 1993). Aggregation can be done separately for

different asset classes to obtain more constraints, but as we add more constraints we get closer (roughly speaking) to the unaggregated model ( $CF_1$ ) resulting in portfolios that have higher optimal cost but should require less trading in future periods. Regarding redefinitions of duration, Dahl (1993) uses factor analysis to obtain multiple factors to explain the yield curve and matches the first derivative of value (“duration”) with respect to each of these factors. Zenios (1995) suggests using “option-adjusted” duration and convexity for any security that is riskier than a treasury.

Duration matching, being (only roughly) an aggregation of cash-flow matching ( $CF_1$ ), is easier to solve but adds to the flaws of ( $CF_1$ ) with one operational and one theoretical problem. The operational problem is that trading has to be done frequently as duration itself changes with changing interest rates possibly resulting in higher transaction costs than ( $CF_1$ ). Bowden (1997) avoids the problem of parallel shift by using directional derivatives in which the entire yield curve  $y(t)$  is shifted by a function  $h(t)$  that has the worst impact on the assets and liabilities combined, but doing so need not prevent arbitrage opportunities. The theoretical problem is that parallel shifts implicitly assumed in the definition of duration are inconsistent with absence of arbitrage, and therefore, “traditional duration measures give, at times, grossly misleading measurements of risk” (Ingersoll et al. 1978) and have “the hidden property of maximizing uncontrolled risk” (Dahl et al. 1993).

Still, the ease of computing duration and of solving duration matching could be attractive in a hybrid dynamic method that uses duration matching for scenario-based subproblems. Another hybrid could be motivated by targeting ( $CF_2$ ) for aggregation to reduce the large number of constraints just as duration matching reduces the number of constraints in ( $CF_1$ ) by “aggregation.”

### 4.3. Stochastic Present-Value/Horizon-Return Matching

We can bring in uncertainty explicitly to present-value matching ( $PM$ ) discussed above. Some models implicitly assume that the present value of the assets matches those of the liabilities under all the different interest-rate scenarios; then we do not need to match duration. These models are also static-dynamic hybrids in that uncertainty is explicitly modeled although recourse variables are not. Two models, one by Dembo (1993) and another by Hiller and Eckstein (1993), may be viewed in this light. Adaptations of the mean-variance portfolio selection for equities can be seen as matching horizon return.

Dembo (1993) determines the lowest-cost portfolio for each terminal scenario  $\zeta_T$  separately as scenario-specific subproblems and then uses a “coordination” model to determine an overall portfolio. Scenario generation is left to the user. For each scenario the present value of the optimal scenario-specific portfolio is required to match the

scenario-specific present value  $P_{L, \zeta_T}$  of the liabilities as follows:

$$\begin{aligned} (\text{Scenario-}\zeta_T) \quad & \min \quad v_{\zeta_T} = \sum_{i=1}^N \pi_{i, \zeta_T} x_{i, \zeta_T} \\ \text{s.t.} \quad & \sum_{i=1}^N P_{i, \zeta_T} x_i = P_{L, \zeta_T}, \\ & l_i \leq x_{i, \zeta_T} \leq u_i \quad \forall i, \end{aligned}$$

where  $P_{i, \zeta_T}$  is the scenario-specific present value of the purchased security  $i$  and  $P_{L, \zeta_T}$  is that for the liability under scenario  $\zeta_T$ . We need an auxiliary model to figure out how to compute the scenario-specific price  $\pi_{i, \zeta_T}$  that may be quite different from the actual market price  $\pi_i$ . For each scenario, the solution cannot be diverse because if the bounds are large enough, each scenario subproblem would recommend only one security that appears to be the cheapest relative to the present value of its cash flows. We could simply order securities  $i$  by their ratio  $P_{i, \zeta_T} / \pi_{i, \zeta_T}$  and order the maximum possible of each in turn until the liabilities’ present value is matched. These solutions are then combined to create a diversified portfolio in a “coordination” model:

(Coordination)

$$\begin{aligned} \min \quad & \sum_{\zeta_T} p_{\zeta_T} \left\{ \|(\pi x - v_{\zeta_T}^*)\| + \left\| \sum_{i=1}^N P_{i, \zeta_T} x_i - P_{L, \zeta_T} \right\| \right\} \\ \text{s.t.} \quad & \sum_i \pi_i x_i \leq C, \\ & l \leq x \leq u, \end{aligned}$$

where  $\|\cdot\|$  is any selected norm,  $v_{\zeta_T}^*$  is the scenario-specific optimal value,  $p_{\zeta_T}$  is the probability of realizing scenario  $\zeta_T$ , and  $C$  is the budget. If the chosen norm is the absolute value  $|\cdot|$ , then the coordination model can be converted to an LP model. Note the need for users to specify diversification bounds in both of the above models.

Hiller and Eckstein (1993) use Benders decomposition with scenario-specific subproblems that they solve on a Connection Machine (CM-2). They require the present value of assets plus any shortfall to exceed the present value of liabilities in all the interest-rate scenarios that they sample from the Black et al. (1990) interest-rate model. Again, using a computed price in the objective function that is different from the market price in a budget constraint leads to biased solutions as does sampling scenarios.

Instead of the present value of assets and liabilities, we could match the horizon value or horizon return. To control risk, we can require that, under all (terminal) scenarios, the scenario-specific portfolio horizon return exceed the mean horizon return less a penalty variable. We also

need the expected portfolio return to exceed the “return” of the liabilities  $\lambda_L$  as follows:

$$\begin{aligned}
 (MV) \quad & \max \quad \eta \sum_i \rho_i x_i - (1 - \eta) \sum_{\zeta_T} d_{\zeta_T} \\
 \text{s.t.} \quad & \sum_{i=1}^N (R_{i, \zeta_T} - \rho_i) x_i + d_{\zeta_T} \geq 0 \quad \forall \zeta_T, \\
 & \sum_i x_i = 1, \\
 & \sum_i \rho_i x_i \geq \lambda_L, \\
 & x_i \geq 0 \quad \forall i,
 \end{aligned}$$

where for the  $i$ th security,  $\rho_i$  is the expected horizon return across all scenarios and  $R_{i, \zeta_T}$  the scenario-specific return.  $d_{\zeta_T}$  is the scenario-specific “downside” deviation of the scenario-specific portfolio return from the expected return and is penalized in the objective function with a weight that complements that on the expected return. This model turns out to be a variant of Markowitz’ (1991) *mean-variance model* for equity portfolios except that the risk is measured here by the downside deviation instead of by variance. If we add a symmetric upside deviation (penalizing it differently), the measure of risk becomes *mean absolute deviation* (Zenios 1995). Mean absolute deviation as a proxy for variance has been used as early as Wagner (1959) for regression to allow the use of LP and more recently for stock portfolios by Konno and Yamazaki (1991); see also Mulvey (2001). Correlations can also be used in more sophisticated versions; Mulvey and Zenios (1994) show how to compute these. Leippold et al. (2004) extend the mean variance model in the ALM context to multiple periods and derive closed-form solutions for maximizing the surplus between assets and liabilities under restrictive assumptions of i.i.d. returns.

## 5. Simplifying the Representation of Uncertainty

The focus of many researchers has been on how to solve variants of the Bradley and Crane (1972) model that have an  $n$ -ary scenario tree, paths on which are scenarios. True,  $n$  is only 2 for auxiliary interest-rate models such as those by Ho and Lee (1986) or Black et al. (1990), but the number of scenarios still grows exponentially with the number of time periods. Due to the consequent difficulty of solving such a large problem, researchers have used decomposition, or have approximated uncertainty in different ways, for instance by taking only a (random) sample of scenarios as in some of the stochastic extensions of static models. These approaches have limitations or problems so there is need for alternatives.

Using decomposition, Bradley and Crane obtain subproblems that can be solved efficiently. Kusy and Ziemba (1986) use the algorithm in Wets (1984) for a stochastic LP

with fixed recourse, restricting uncertainty to that of deposit flows. Birge (1982) provides a solution method to tackle the large size of multistage stochastic LP’s in general. Building upon the work of Kusy and Ziemba and that of Birge, Cariño et al. (1994), Cariño and Ziemba (1998), and Cariño et al. (1998) all use Bender’s decomposition to solve problems that have up to 6 periods, 7 asset classes, and 256 economic scenarios for a Japanese insurance company. Mulvey and Vladimirou (1992) adopt a generalized network structure for a multiperiod asset allocation problem with different classes of bonds, equities, and real estate. They solve problems up to 8 periods, 15 asset classes, and 100 scenarios using the progressive hedging algorithm of Rockafellar and Wets (1991). Consigli and Dempster (1998) note the efficacy of nested Benders decomposition and report their experience with different LP and quadratic programming solvers using data with up to 10 periods and 2,688 terminal scenarios. Gondzio and Kouwenberg (2001) use Benders decomposition and an efficient model generator to solve problems with as many as  $13^6$  or more than 4.8 million scenarios for a Dutch pension-fund application on a parallel computer with 16 processors. On the other hand, Mulvey and Shetty (2004) describe challenges in solving for even a modest number of scenarios (4,096) on a 128-processor machine; they describe the use of interior-point and parallel Cholesky methods as well as new ways to reduce the number of floating point operations. Dert (1999) refers to an iterative heuristic for a chance-constrained ALM model that tackles only one or two time periods in each iteration. Thus, despite the merits of decomposition, the large number of scenarios means that researchers have to approximate uncertainty in some way for an optimal solution.

### 5.1. Aggregation

Aggregation appears to be a reasonable way to approximate the uncertainty itself, but not every aggregation is externally and internally consistent (Klaassen 1997, 2002). We already have useful results in systematic aggregation for the general stochastic linear program (Wright 1994) as well as for the ALM model (Klaassen 1998). The latter involves aggregating nodes on the scenario tree by aggregating states and/or time periods although this causes a mismatch of computed and market prices. Thus, part of the challenge is retaining consistency with finance theory after any such approximation.

Researchers have approximated uncertainty by taking only a sample of scenarios generated from an arbitrage-free stochastic process. But the result is unbounded or biased solutions resulting from spurious arbitrage opportunities within the model (Klassen 1997, 2002) or simply poor quality solutions (Kouwenberg 2001) depending on how the scenarios were generated. Also, the solution in each run can be quite different from the previous one with the same input because the number of randomly-generated scenarios is only a tiny percentage of a large population of highly varying scenarios. That means we need to handle

all scenarios somehow. On the other hand, Gaivoronski and de Lange (2000) suggest that it is better to simplify decision variables, e.g., by using simple proportions of different assets after the first one or two periods rather than to aggregate scenarios due to worse solutions in the latter case.

## 5.2. Aggregating Scenarios as Sub-Filtrations

Wright (1994) views aggregation as coarsening the information structure. His work pertains to the aggregation for the general stochastic LP model in measure-theoretic terms. We specialize part of his work to aggregating ALM models such as  $\mathbf{P}_{\text{scen}}$  or Klaassen's (1998) unaggregated ALM model and derive error bounds for this specialized aggregation.

For exposition, we use partitions of  $\Omega$ , the set of terminal scenarios  $\zeta_T$ , rather than the concept of *filtration*—a sequence of sets  $F_0 \subset F_1 \subset F_2 \subset \dots \subset F_T$  each closed under set operations with elements that are subsets of  $\Omega$ —as Wright does. At time  $t = 0$ , there is one set with all the  $2^T$  terminal scenarios that are possible when using an interest rate stochastic model such as that by Black et al. (1990) or by Ho and Lee (1986). At the final time period, there is only one terminal scenario and the set of possibilities is partitioned into  $2^T$  singleton sets of one scenario each. In the previous period  $t = T - 1$ , two final-period scenarios are possible so the set of possibilities is partitioned into sets of two final-period scenarios each and so on. Thus, each set of possibilities starting from  $t = 0$  is “refined” into finer-grained partitions going from  $t = 0$  to  $t = T$ .

Aggregation in this context means making the above information structure for  $\mathbf{P}_{\text{scen}}$  coarser in different ways. For example, aggregating the two scenarios in the set of possibilities in period  $T - 1$  can reduce the final-period scenarios from  $2^T$  to  $2^{T-1}$ . However, successive refinement needs to be maintained; not all aggregations map to partitions that are successive refinements. Also, not every successive partition may make sense if internal and external consistency requirements are violated as a result of the aggregation; these requirements correspond to the aggregated problem being primal and dual feasible just like the original problem.

Wright's work generalizes that of Birge (1985), who determines error bounds for the expected value problem obtained by aggregating the rows and columns using a weighting function corresponding to the probability distribution on the random variables. Birge's results apply when the right-hand side is stochastic and not when the coefficient matrix is also stochastic as in the case  $\mathbf{P}_{\text{scen}}$ . Earlier, Zipkin (1980a, b) tackled row aggregation and column aggregation separately for the general LP deriving error bounds. In our case, the stochastic LP has a finite number of discrete scenarios and is therefore an ordinary LP. So we can think of aggregating scenarios also in terms of first aggregating columns (Zipkin 1980a) and then either aggregating rows (Zipkin 1980b) or aggregating columns in the dual.

To derive error bounds using Wright's work, we need primal and dual feasibility of both the original and the aggregated problem as a condition; recall that  $\mathbf{P}_{\text{scen}}$  itself meets this requirement. We state these bounds as

**PROPOSITION 1.** *Assume that the aggregated primal problem  $\mathbf{P}_{\text{agg}}$  is feasible and bounded as is the original problem  $\mathbf{P}_{\text{scen}}$ . Let  $z_{\text{agg}}^*$  and  $z_{\text{scen}}^*$  denote their optimal values, respectively. Suppose that  $\xi$  is an upper bound on the feasible solutions of  $\mathbf{P}_{\text{scen}}$  and that there are  $n_t$  decision variables corresponding to each time period  $t$ . Let  $A$  denote the entire constraint coefficient matrix and  $c$  the objective function coefficient vector. If  $d$  is the disaggregated vector corresponding to a dual feasible solution of  $\mathbf{P}_{\text{agg}}$ , then*

$$z_{\text{agg}}^* \leq z_{\text{scen}}^* \leq z_{\text{agg}}^* - \psi(d),$$

where

$$\psi(d) = \sum_{t=0}^T \sum_{j=1}^{n_t} \xi_{tj} \cdot \left[ c_{tj} - \sum_{\tau=0}^T [A_{\tau t}]_j d_{\tau} \right]^{-} < 0.$$

**PROOF.** The proof follows from Theorems 6 and 7 of Wright (1994) and by observing that we can apply strong duality because  $\mathbf{P}_{\text{scen}}$  is an ordinary LP after all. Unlike the general stochastic LP as tackled by Wright, we do not have either  $\xi$  or  $\eta$  as a random vector because our stochastic LP is an ordinary LP as explained earlier.  $\square$

## 5.3. Aggregating States and Time Periods

Wright's (1994) results and the error bounds computed above for arbitrary successive partitions hold when the aggregated problem is bounded and feasible, but any arbitrary aggregation even with successive partitioning need not maintain feasibility and boundedness. Klaassen (1998) proposes aggregating states for a binomial-tree interest-rate model like that of Ho and Lee (1986) or Black et al. (1990) to decrease the number of scenarios. By aggregating states, he aggregates scenarios in a manner consistent with Wright while showing that the arbitrage-free relationships from the unaggregated problem are retained. The proof for dual feasibility of the aggregated problem then follows the same lines as that for the unaggregated problem (see the appendix). The aggregated problem is also primal feasible when the hard borrowing constraint in Klaassen's original model is replaced by a soft one. Thus, the aggregated problem obtained by aggregating states as per Klaassen meets the conditions for the error bound computed above.

Klaassen also aggregates time periods on the interest-rate state tree to aggregate scenarios. This can also be considered as a special case of Wright's approach if we consider the aggregated time period as a “do-nothing” state where no securities are bought or sold and the partition of possible end-period scenarios is not sub-partitioned further. Klaassen shows that internal consistency is not violated as a result of this aggregation. Using both state and

time aggregation, he also proposes a successive aggregation and disaggregation approach to find increasingly accurate solutions, with a focus on stabilizing the optimal values of the current variables.

Klaassen's approach maintains internal consistency by keeping the prices of securities arbitrage free but can violate external consistency as computed prices for certain securities (e.g., an option on a treasury) may not equal their market price upon aggregation. Aggregating time periods also causes inconsistencies between market prices and computed prices but for a different reason. As Klaassen himself observes in a different context, the "calculated model price of a security that is not included in the benchmark [securities] may vary significantly as a function of the chosen number of [time periods]" (1998, p. 37).

A possible solution is post-aggregation adjustment of cash flows in future time periods, and hence computed prices, to ensure the match between the market price and computed price for all securities, but doing so needs to be empirically verified.

## 6. Conclusion

We have reviewed LP-based modeling choices for ALM, the older static methods as simplifications of an LP model, and aggregation as a simplification of representing uncertainty in this model. Static models are extreme simplifications of dynamic models such as  $\mathbf{P}_{\text{scen}}$  in terms of both modeling elements and the representation of uncertainty. Given that static models are easy to solve and dynamic models are near impossible for any reasonable number of time periods, there appears to be plenty of scope to simplify dynamic models to hybrid static-dynamic ones that use aggregation to simplify uncertainty. The modeling choices presented show that such models must ensure primal- and dual-feasibility that are consistent with using market prices and avoiding spurious arbitrage opportunities.

However, much empirical work needs to be done to look for modeling simplifications. For instance, we could reduce the model size by including recourse variables only up to a certain time period  $T'$  short of the decision horizon  $T$ . That way we can have fine-grained time periods (e.g., calendar months) while reducing the number of decision variables by a factor of  $2^{T-T'}$ . Or, we could aggregate uncertainty scenarios to a single horizon after  $T'$ —this would be quite useful if  $T$  spans a few decades—and use static models beyond  $T'$  with a dynamic model representing the time  $0, \dots, T' - 1$ . But we need to understand the impact of doing so on the optimal solution in the long run.

There is scope for theoretical research as well. One topic is to practically incorporate investors' risk appetite and subjective beliefs while enjoying the benefits of "risk-neutral" computation (see, e.g., Klaassen 1998). Alternatively, we need to understand the relative merits and shortcomings of using historical data to generate future scenarios without risk neutrality. Expanding models with one factor of uncertainty to two or more factors of uncertainty is also quite

desirable for practical situations such as pension funds with uncertainty surrounding both assets of different types and liabilities. In general, to manage risk we need to extend this framework to include other sources of uncertainty including equities, foreign exchange rates, gold prices, and natural calamities such as earthquakes and hurricanes. Aggregation also needs to be thought through when multiple factors are to be used and consistency needs to be maintained.

Another direction of research for simplifying the ALM LP could be to explore the artificial intelligence-rooted approaches of neuro-dynamic programming (Bertsekas and Tsitsiklis 1996) and approximate dynamic programming (e.g., Bertsimas and Demir 2002, de Farias and van Roy 2003). After all, the curse of dimensionality that we experience in the explosion of the total number of scenarios is common to many dynamic programming problems. To solve such multistage problems, researchers have proposed approximating at any stage the objective function for later stages. However, creating an approximate LP for the ALM problem by sampling constraints as proposed by de Farias and van Roy (2003) could result in the same problems as sampling scenarios. It may be simpler to aggregate future states as described earlier to reduce the number of variables than it would be with approximate dynamic programming because no approximation functions would need to be obtained. The large number of securities to consider at any stage can also be daunting in coming up with approximation functions.

Further work is also needed on auxiliary models for generating cash flows of securities given interest-rate (or other) scenarios. Any ALM model needs the output of such models for cash flow for different types of assets and liabilities, for instance, prepayment models for mortgage-backed securities. Even hospital bills and other accounts payable have been securitized and we need cash flow models for them. We also need to be able to model cash flows due to liabilities that may depend on other risk factors.

One outcome of this survey is that much remains to be done to further LP modeling for ALM algorithmic advancements. While ALM modeling is nowhere near infancy, it is nowhere near maturity either. Indeed, the dearth of empirical findings, the absence of consensus on modeling choices, the use of ad-hoc solution techniques, and the paucity of modeling break-throughs in the literature in the past few years (1998–2003) suggest more of a mid-life crisis. Seeking hybrid methods that use aggregation and are compatible with finance theory may be a way forward.

## Appendix

The dual of model  $\mathbf{P}_{\text{scen}}$  has unrestricted decision variables  $u_{\xi_t}$  and  $v_{i, \xi_t}$ :

### Scenario-Based Dual Model $\mathbf{D}_{\text{scen}}$

$$\max \left\{ L_{\xi_0} - \sum_i \kappa_i h_{i,-1} - l_{-1}(1 + \rho_{-1}) + b_{-1} \right\} u_{\xi_0} + \sum_{t=1}^T \sum_{\xi_t} L_{\xi_t} u_{\xi_t}$$

subject to

$$\begin{aligned} -(1+T_i)\pi_{i,\zeta_t}u_{\zeta_t} - v_{i,\zeta_t} &\leq 0 \quad \forall i, \forall \zeta_t, \\ (1-T_i)\pi_{i,\zeta_t}u_{\zeta_t} + v_{i,\zeta_t} &\leq 0 \quad \forall i, \forall \zeta_t, \\ v_{i,\zeta_t} + \kappa_{i,\zeta_{t+1}}^{(1)}u_{\zeta_{t+1}}^{(1)} - v_{i,\zeta_{t+1}}^{(1)} + \kappa_{i,\zeta_{t+1}}^{(2)}u_{\zeta_{t+1}}^{(2)} - v_{i,\zeta_{t+1}}^{(2)} &\leq 0 \quad \forall i, \forall \zeta_t, \\ -u_{\zeta_t} + (1+\rho_{\zeta_t})u_{\zeta_{t+1}}^{(1)} + (1+\rho_{\zeta_t})u_{\zeta_{t+1}}^{(2)} &\leq 0 \quad \forall \zeta_t, \\ u_{\zeta_t} - (1+\rho_{\zeta_t}+D)u_{\zeta_{t+1}}^{(1)} - (1+\rho_{\zeta_t}+D)u_{\zeta_{t+1}}^{(2)} &\leq 0 \quad \forall \zeta_t \end{aligned}$$

for  $t = 0, \dots, T-1$ . For  $t = T$ , we require

$$\begin{aligned} -(1+T_i)\pi_{i,\zeta_T}u_{\zeta_T} - v_{i,\zeta_T} &\leq 0 \quad \forall i, \forall \zeta_T, \\ (1-T_i)\pi_{i,\zeta_T}u_{\zeta_T} + v_{i,\zeta_T} &\leq 0 \quad \forall i, \forall \zeta_T, \\ v_{i,\zeta_T} &\leq -\delta_{\zeta_T}\pi_{i,\zeta_T} \quad \forall i, \forall \zeta_T, \\ -u_{\zeta_T} &\leq -\delta_{\zeta_T} \quad \forall \zeta_T, \\ u_{\zeta_T} &\leq \delta_{\zeta_T} \quad \forall \zeta_T. \end{aligned}$$

Scenarios labeled  $\zeta_{t+1}^{(1)}$  and  $\zeta_{t+1}^{(2)}$  refer to the two child scenarios of any scenario  $\zeta_t$  ( $t < T$ ).

The model  $\mathbf{D}_{\text{scen}}$  is feasible. First, consider the constraints for  $t = T$ . The values

$$u_{\zeta_T} = \delta_{\zeta_T} > 0, \quad v_{i,\zeta_T} = -\pi_{i,\zeta_T}\delta_{\zeta_T} < 0$$

can be easily seen to satisfy all the constraints. Next, consider the constraints for  $t < T$ . The values

$$u_{\zeta_t} = 2^{T-t}\delta_{\zeta_t} > 0, \quad v_{i,\zeta_t} = -\pi_{i,\zeta_t}2^{T-t}\delta_{\zeta_t} < 0$$

can be shown to satisfy all the constraints as well. Doing so requires using the relationship between the present value of \$1 for child scenarios to that of their parent

$$\delta_{\zeta_{t+1}}^{(1)} = \delta_{\zeta_{t+1}}^{(2)} = \frac{\delta_{\zeta_t}}{1+\rho_{\zeta_t}},$$

as well as the relationship between the prices

$$\pi_{i,\zeta_t} = \frac{(\kappa_{i,\zeta_{t+1}}^{(1)} + \pi_{\zeta_{t+1}}^{(1)}) + (\kappa_{i,\zeta_{t+1}}^{(2)} + \pi_{\zeta_{t+1}}^{(2)})}{2(1+\rho_{\zeta_t})},$$

which follows from the arbitrage-free requirement, given that the two child scenarios have a conditional probability of 1/2 each.

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