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Abstract‡

The mean-variance method has been one of the popular methods used by most financial institutions in making the decision of asset allocation since the 1950s. This paper presents an alternative method for asset allocation. Instead of minimizing risk for a given expected return or maximizing expected return for a fixed level of risk, our approach considers *simultaneously* maximizing the rate of return of portfolio, minimizing the risk of obtaining lower return, and maximizing the possibility of reaching higher return. By using a triangular possibilistic distribution to describe the uncertainty of the return, we introduce a possibilistic linear programming model which we solve by a multiple objective linear programming technique with two control constraints. We present a solution algorithm that provides maximal flexibility for decision makers to effectively balance the portfolio's return and risk. Numerical examples show the efficiency of the algorithm.

Key words and phrases: *Mean-variance method; possibility distribution; multiple-objective, fuzzy sets.*

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1 Introduction

Asset allocation decisions are often reached through a three-step process: first, the risk and return characteristics of available and relevant investment opportunities are identified; second, the investor's risk tolerances or expected returns are parameterized; and finally, the risk-return trade-offs of the investor are combined with those observed in the market to produce an optimal asset allocation. A frequently used tool for asset allocation problems is the mean-variance optimization technique developed by Markowitz (1952).

Mean-variance optimization refers to a mathematical process to determine the security (or asset class) weights that provide a portfolio with the minimum risk for a given expected return or, conversely, the maximum expected return for a given level of risk. The inputs needed to conduct mean-variance optimization are security expected returns, expected standard deviations, and expected cross-security correlation. The Markowitz model has been one of the methods widely used by institutional investors, retail brokerage houses, and pension fund managers.

Another type of asset allocation strategy is dynamic asset allocation, which continually adjusts a portfolio's allocation in response to changing market conditions. The most popular use of these strategies is portfolio insurance, which attempts to remove the downside risk faced by a portfolio. A popular means of implementing portfolio insurance is to engage in a series of transactions that give the portfolio the return distribution of a call option. Black and Scholes (1973) show that under certain assumptions, the payoff of an option can be duplicated through a continuously revised combination of the underlying asset and a risk-free bond. Rubenstein and Leland (1981) extend this insight by showing that a dynamic strategy that increases the stock allocation of a portfolio in rising markets and reinvested the remaining portion in cash would replicate the payoffs to a call option on an index of stocks.

Portfolio insurance concentrates on only two assets, both of which are carefully predetermined. To the extent that its assumptions about the behavior of uninsured investors turn out to be less than 100% correct, however, the increasing volatility of risky assets could drive insured portfolios to sell or buy even more aggressively than they would have in the first place; see Sharpe (1992).

On the other hand, tactical asset allocation (active asset allocation) is the process of diverging from the strategic asset allocation when an investor's short-term forecasts deviate from the long-term forecasts used to formulate the strategic allocation. If the investor can make accurate short-term forecasts, tactical asset allocation has the potential to enhance returns. In practice, tactical asset allocators are the investors providing portfolio insurance; see Sharp and Perrold (1988).

In this study, instead of the traditional mean-variance approach, we describe the uncertainty of the rate of return by a triangular possibilistic distribution. A possibilistic linear programming model (see, Lai and Hwang (1992, Chapter 5)) is formulated and then solved by introducing two control constraints to the auxiliary multi-objective linear programming model. By selecting different values for the parameters in control constraints, our method can be applied in solving the following problems:

- Maximizing the most possible return and minimizing the risk of obtaining lower return as well as maximizing the possibility of obtaining higher return.
- Minimizing the risk of obtaining lower return and maximizing the possibility of obtaining higher return for a specified most possible return.
- Maximizing the most possible return and maximizing the possibility of obtaining higher return for a given risk tolerance.

2 Models

Let us consider the problem of allocating capital C among N asset classes, $S_1, S_2, \ldots, S_N$. In the mean-variance optimization method [Fong and Fabozzi (1992)], the rate of return, R_i, of asset S_i is assumed to be a random variable with μ_i and σ_i denoting the mean and standard deviation, respectively, of R_i for $i = 1, 2, \ldots, N$, and ρ_{ij} denoting the correlation between R_i and R_j for $i, j = 1, 2, \ldots, N$.

If the N assets are combined linearly to form a portfolio, where the allocation weight, x_i, for asset class S_i, is equal to the dollar value of the asset class relative to the dollar value of the portfolio, then the rate of return of the portfolio is

$$R_p = \sum_{i=1}^{N} x_i R_i$$

which is also a random variable. The expected return of the portfolio is

$$\mu_p = \sum_{i=1}^{N} x_i \mu_i$$

and the variance of the portfolio is

$$\sigma_p^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_i \sigma_j \rho_{ij}.$$

2.1 Mean-Variance Analysis

The mean-variance method for determining weights $x_1, x_2, \ldots, x_N$ is to fix the expected portfolio return μ_p to a desirable level μ and determine the allocation weights $x_1, x_2, \ldots, x_N$ that minimize the risk level σ_p^2 of the portfolio for the fixed μ. The following quadratic programming model (1) is employed to accomplish this goal:

Model 1:

$$\begin{array}{rl} \min & \sigma_p^2 = \sum_{i,j=1}^{N} x_i x_j \sigma_i \sigma_j \rho_{ij} \\ \text{subject to} & \sum_{i=1}^{N} x_i \mu_i = \mu \\ & \sum_{i=1}^{N} x_i = 1 \\ & l_i \le x_i \le \mu_i, \; i = 1, 2, \ldots, N \end{array}$$

where l_i is the lower bound and u_i is the upper bound on funds allocated to the i-th asset class, $i = 1, 2, \ldots, N$.

An equivalent approach is to fix the risk level σ_p^2 of a portfolio to a tolerable level σ^2 and determine the weights $x_1, x_2, ..., x_N$ that maximize the expected portfolio return μ_p for the fixed σ^2. The following quadratic programming model (2) is employed to accomplish this goal:

Model 2:

$$\begin{array}{rl} \max & \mu_p = \sum_{i=1}^{N} x_i \mu_i \\ \text{subject to} & \sum_{i,j=1}^{N} x_i x_j \sigma_i \sigma_j \rho_{ij} = \sigma^2 \\ & \sum_{i=1}^{N} x_i = 1 \\ & l_i \le x_i \le \mu_i, \quad i = 1, 2, \ldots, N. \end{array}$$

2.2 Mean-Variance-Skewness Analysis

The mean-variance method, which does not consider the skewness of the return random variable R_p, is frequently used in practice for asset allocation. In a continuous time model with asset prices following a diffusion process, Ito's differentiation rule[1] implies the higher moments are irrelevant to asset allocation decisions. In this case, the mean-variance method provides optimal portfolio selection. In a discrete model, however, Samuelson (1970) shows that the mean-variance efficiency becomes inadequate and the higher moments become relevant to the portfolio selection.

It has been shown empirically by Simkowitz and Beedles (1978) and Singleton and Wingender (1986) that stock return distributions are often positively skewed. Under asymmetrically distributed asset returns, it is important to take skewness into consideration in discrete models of portfolio selection. Arditti and Levy (1975) have illustrated the important role of skewness in the pricing of stocks. As shown by Arditti (1967), the investor's preference for more skewness is consistent with the notion of decreasing absolute risk aversion, because a positive-skewness asset return refers to a right-hand elongated tail of density function of asset return.

If the skewness of R_p, defined as $E[(R_p - \mu_p)^3]/\sigma_p^3$, is incorporated in the mean-variance method, model (1) then becomes a multiple objective nonlinear programming model:

Model 3:

$$\begin{aligned} \min \quad & \sigma_p^2 = \sum_{i,j=1}^{N} x_i x_j \sigma_i \sigma_j \rho_{ij} \\ \max \quad & E[(R_p - \mu_p)^3]/\sigma_p^3 \\ \text{subject to} \quad & \sum_{i=1}^{N} x_i \mu_i = \mu \end{aligned}$$

[1]Ito's differentiation rule states that if $f = f(X, t)$ and X follows

$$dX = \alpha\, dt + \sigma\, dW.$$

Then df can be written:

$$df = [\alpha f_x + \frac{1}{2}\sigma^2 f_{xx} + f_t]\, dt + \sigma f_x\, dW.$$

See Shimko (1992).

$$\sum_{i=1}^{N} x_i = 1$$
$$l_i \leq x_i \leq \mu_i \quad i = 1, 2, \ldots, N$$

while model (2) becomes the following:

Model 4:

$$\begin{aligned}
\max \quad & \mu_p = \sum_{i=1}^{N} x_i \mu_i \\
\max \quad & E[(R_p - \mu_p)^3]/\sigma_p^3 \\
\text{subject to} \quad & \sum_{i,j=1}^{N} x_i x_j \sigma_i \sigma_j \rho_{ij} = \sigma^2 \\
& \sum_{i=1}^{N} x_i = 1 \\
& l_i \leq x_i \leq \mu_i, \quad i = 1, 2, \ldots, N
\end{aligned}$$

where

$$E[(R_p - \mu_p)^3] = \sum_{i,j,k=1}^{N} x_i x_j x_k \sigma_{ijk}$$

and σ_{ijk} is defined as

$$E[(R_i - \mu_i)(R_j - \mu_j)(R_k - \mu_k)].$$

The skewness of a portfolio of securities is not simply a weighted average of the skewness of the component securities. Like variance, it depends on the joint movement of securities. This means that to measure the skewness on a portfolio, a great number of estimates of joint movement must be made. As indicated by Elton and Gruber (1995), for these estimates to be feasible, it requires the type of single indexed model or multiple indexed model development to calculate the correlation structure of security returns and some simple techniques for determining the three dimensional efficient frontier. This developmental work has not been done.

In this paper, instead of dealing with models (3) or (4) directly, we present a possibilistic linear programming model to implement the idea of maximizing the expected return, minimizing the risk, and maximize skewness simultaneously without estimating the third moments of securities. Possibility theory studies primarily imprecise phenomena. Possibilistic decision making models handle practical decision making problems where input data are imprecise. Applications of possibility theory to linear programming problems with imprecise coefficients have been discussed by Lai and Hwang (1992).

3 The Possibilistic Model

The possibility distribution π_X of an event X states the degree possibility of the occurrence of the event. To illustrate the difference between the possibility distribution and the probability distribution, we consider the following simple example due to Zadeh (1978, p. 8): Consider the statement "Hans ate X eggs for breakfast," where $X = \{1, 2, \ldots\}$. A possibility distribution as well as a probability distribution may be associated with X. The possibility distribution $\pi_X(u)$ can be interpreted as the degree of ease with which Hans can eat u eggs while the probability distribution $P_X(u)$ might have been determined by observing Hans at breakfast for 100 days. The values of $\pi_X(u)$ and $P_X(u)$ might be as shown in the following table: We observe that a

u	1	2	3	4	5	6	7	8
$\pi_X(u)$	1.0	1.0	1.0	1.0	0.8	0.6	0.4	0.2
$P_X(u)$	0.1	0.8	0.1	0.0	0.0	0.0	0.0	0.0

high degree of possibility does not imply a high degree of probability. If, however, an event is not possible, it is also improbable. Thus, in a way the possibility is an upper bound for the probability. For a more detailed discussion of possibility theory, readers are referred to Zimmermann (1991, Chapter 8) or Dubois and Prade (1988).

For our model, we use the possibility distribution to describe the uncertainty of the rate of return. Because uncertainty from the return of assets can be regarded as the nature of imprecision, possibility distributions are suitable for characterizing such kinds of uncertainty. Moreover, using the possibility distribution may also reduce the impact of the underlying structure of the asset market shifts.

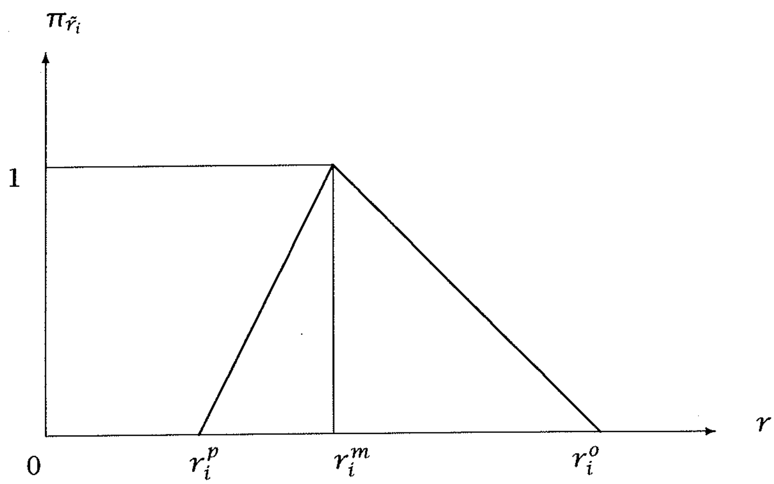
For the i-th asset $S_i, i = 1, 2, \ldots, N$, we describe the imprecise rate of return by $\tilde{r}_i = (r_i^p, r_i^m, r_i^o)$, where r_i^p, r_i^m, and r_i^o are the most pessimistic value, the most possible value, and the most optimistic value for the rate of return, respectively. Assume that the imprecise rate of return, $\tilde{r}_i = (r_i^p, r_i^m, r_i^o)$, with $r_i^p < r_i^m < r_i^o$, has the triangular possibility distribution $\pi_{\tilde{r}_i}$ defined as:

$$\pi_{\tilde{r}_i}(r) = \begin{cases} 0 & \text{for } r < r_i^p \text{ or } r > r_i^o \\ 1 & \text{for } r = r_i^m \\ (r - r_i^p)/(r_i^m - r_i^p) & \text{for } r_i^p \le r < r_i^m \\ (r_i^o - r)/(r_i^o - r_i^m) & \text{for } r_i^m \le r < r_i^o \end{cases} \tag{1}$$

and is displayed in Figure 1.

Figure 1

The Triangular Possibility Distribution of $\tilde{r}_i$



As shown in Figure 1, possibility distribution of the rate of return describes the possibility degree of occurrence of each possible rate of return. For example, if for i-th asset S_i, $\pi_{\tilde{r}_i}(0.10) = 0.8$, then the possibility degree of occurrence of $r_i = 10\%$ is 0.8.

Next, let x_i denote the allocation weight and $\tilde{r}_i = (r_i^p, r_i^m, r_i^o)$ denote the imprecise rate of return to asset S_i for $i = 1, 2, \ldots, N$. Then the imprecise rate of return for the portfolio is

$$\tilde{r} = \sum_{i=1}^{N} \tilde{r}_i x_i.$$

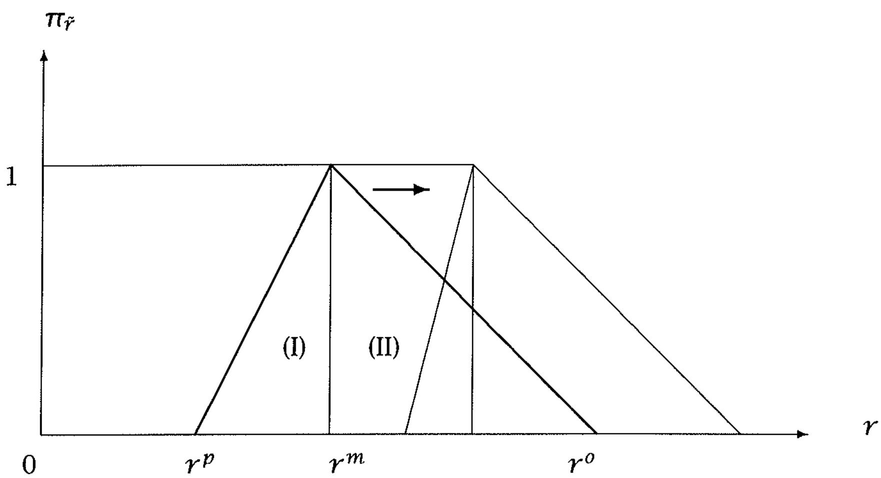
Linear combinations of triangular possibility distributions are also triangular possibility distribution, and $\tilde{r} = (r^p, r^m, r^o)$ is given by

$$r^p = \sum_{i=1}^{N} r_i^p x_i, \quad r^m = \sum_{i=1}^{N} r_i^m x_i, \quad r^o = \sum_{i=1}^{N} r_i^o x_i.$$

Notice that the triangular possibility distribution $\pi_{\tilde{r}}$ for $\tilde{r}$, as indicated by a bold triangle in Figure 2, is determined by r^p, r^m, and r^o according to the definition of $\pi_{\tilde{r}}$.

Figure 2

The Triangular Possibility Distribution of $\tilde{r}$



We now select the optimal portfolio that maximizes the portfolio return by solving the following possibilistic linear programming model:

Model 5:

$$\begin{aligned} \max \quad & \sum_{i=1}^{N} \tilde{r}_i x_i \\ \text{subject to} \quad & \sum_{i=1}^{N} x_i = 1 \\ & l_i \le x_i \le \mu_i, \quad i = 1, 2, \ldots, N. \end{aligned}$$

4 Solution Procedures

From Figure 2, we observe that r^m has the highest degree of possibility to be the rate of return for the portfolio; we therefore define portfolio return as r^m. We also notice that, in Figure 2, the larger the area of region (I) is, the more possible it is for the portfolio to obtain lower return. As the area of region (I) is $(r^m - r^p)/2$, we define portfolio risk as $(r^m - r^p)$. Similarly, $(r^o - r^m)/2$ is the area of region (II) in Figure 2. Larger values of $(r^o - r^m)/2$ indicate higher degrees of possibility for the portfolio to reach higher return. We define portfolio skewness as $(r^o - r^m)$.

In order to maximize the imprecise rate of portfolio return, $\tilde{r}$, we select the optimal portfolio in the sense of maximizing portfolio return, minimizing portfolio risk, and maximizing portfolio skewness. Therefore model (5) can be approximated by the multiple objective linear programming model (6):

Model 6:

$$
\begin{aligned}
\max \quad & z^{(1)} = \sum_{i=1}^{N} r_i^m x_i \\
\min \quad & z^{(2)} = \sum_{i=1}^{N} (r_i^m - r_i^p) x_i \\
\max \quad & z^{(3)} = \sum_{i=1}^{N} (r_i^o - r_i^m) x_i \\
\text{subject to} \quad & \beta_l \le \sum_{i=1}^{N} r_i^m x_i \le \beta_u \\
& \gamma_l \le \sum_{i=1}^{N} (r_i^m - r_i^p) x_i \le \gamma_u \\
& \sum_{i=1}^{N} x_i = 1 \\
& l_i \le x_i \le \mu_i, \quad i = 1, 2, \ldots, N.
\end{aligned}
$$

There are three objectives in model (6):

- The first objective is to maximize portfolio return;
- The second objective is minimizing portfolio risk; and
- The third objective is maximizing portfolio skewness.

This strategy is essentially analogous to maximizing mean return, minimizing variance, and maximizing skewness for a random rate of return. In Figure 2, the triangle made by the thin lines denotes the optimal triangular possibility distribution for the imprecise rate of return for the portfolio.

4.1 Selecting the Parameters

Model (6) has two control constrains:

$$\beta_l \leq \sum_{i=1}^{N} r_i^m x_i \leq \beta_u$$

and

$$\gamma_l \leq \sum_{i=1}^{N} (r_i^m - r_i^p) x_i \leq \gamma_u.$$

By selecting parameters β_l and β_u, the decision makers could use the first control constraint to assure portfolio return within the desirable range. On the other hand, by selecting parameters γ_l and γ_u, the second control constraint can be used to adjust portfolio risk to a tolerable range. In the following, we discuss three special cases for selecting parameters β_l, β_u, γ_l, and γ_u.

Case 1: If we set

$$\beta_l = \min_{i=1,2,\ldots,N} \{r_i^m\}, \quad \beta_u = \max_{i=1,2,\ldots,N} \{r_i^m\},$$

and

$$\gamma_l = \min_{i=1,2,\ldots,N} \{r_i^m - r_i^p\}, \quad \gamma_u = \max_{i=1,2,\ldots,N} \{r_i^m - r_i^p\},$$

both control constraints become inactive and model (6) is reduced to model (7), as proposed by Lia and Hwang (1992):

Model 7:

$$\max \quad z^{(1)} = \sum_{i=1}^{N} r_i^m x_i$$

$$\min \quad z^{(2)} = \sum_{i=1}^{N} (r_i^m - r_i^p) x_i$$

$$\max \quad z^{(3)} = \sum_{i=1}^{N} (r_i^o - r_i^m) x_i$$

$$\text{subject to} \quad \sum_{i=1}^{N} x_i = 1$$
$$l_i \le x_i \le \mu_i, \quad i = 1, 2, \ldots, N.$$

Case 2: If we set $\beta_l = \beta_u = \beta$, a constant,

$$y_l = \min_{i=1,2,\ldots,N} \{r_i^m - r_i^p\},$$

and

$$y_u = \max_{i=1,2,\ldots,N} \{r_i^m - r_i^p\},$$

then the first objective and the second control constraint in model (6) become inactive. In this case, model (6) becomes the following:

Model 8:

$$\min \quad z^{(2)} = \sum_{i=1}^{N} (r_i^m - r_i^p) x_i$$
$$\max \quad z^{(3)} = \sum_{i=1}^{N} (r_i^o - r_i^m) x_i$$
$$\text{subject to} \quad \sum_{i=1}^{N} r_i^m x_i = \beta$$
$$\sum_{i=1}^{N} x_i = 1$$
$$l_i \le x_i \le \mu_i, \quad i = 1, 2, \cdots, N.$$

It is easy to notice the similarity between models (3) and (8).
Case 3: If we set $y_l = y_u = y$, a constant,

$$\beta_l = \min_{i=1,2,\cdots,N} \{r_i^m\} \quad \text{and} \quad \beta_u = \max_{i=1,2,\cdots,N} \{r_i^m\},$$

then the second objective and the first control constraint in model (6) become inactive. In this case, model (6) becomes:

Model 9:

$$\max \quad z^{(1)} = \sum_{i=1}^{N} r_i^m x_i$$
$$\max \quad z^{(3)} = \sum_{i=1}^{N} (r_i^o - r_i^m) x_i$$

$$\text{subject to} \quad \sum_{i=1}^{N} (r_i^m - r_i^p) x_i = y$$
$$\sum_{i=1}^{N} x_i = 1$$
$$l_i \le x_i \le \mu_i, \quad i = 1, 2, \ldots, N..$$

Notice that model (9) is also analogous to model (4).

The selection of parameters β_l, β_u, γ_l, and γ_u may be based on either experience or managerial judgment. The examples given in Section 5 show the significance of our control constraints.

4.2 The Solution

In trying to find the solution to model (6), we must remember that model (6) has three simultaneous objectives: (i) maximizing portfolio return, (ii) maximizing portfolio skewness, and (iii) minimizing portfolio risk. With these multiple conflicting and competing objectives we cannot expect to achieve the best values for all objectives simultaneously. Therefore trade-offs among conflicting objectives are necessary.

There are various techniques to handle these trade-offs. Examples of such techniques include utility theory, goal programming, fuzzy programming, or iterative approaches. In this paper, we use Zimmermann's fuzzy programming method (1978) with a normalization process to solve the multiple objective linear programming model (6).

Let X denote the set of feasible solutions satisfying all the constraints in programming model (6). Next, for the objective function $z^{(1)}$ defined in model (6), we first calculate

$$z_{min}^{(1)} = \min_{x \in X} \sum_{i=1}^{N} r_i^m x_i$$

and

$$z_{max}^{(1)} = \max_{x \in X} \sum_{i=1}^{N} r_i^m x_i.$$

Then we define the linear membership function $\mu_{z^{(1)}}$ as

$$\mu_{z^{(1)}}(z) = \begin{cases} 1 & \text{if } z \ge z_{max}^{(1)}; \\ (z - z_{min}^{(1)})/(z_{max}^{(1)} - z_{min}^{(1)}) & \text{if } z_{min}^{(1)} < z < z_{max}^{(1)}; \\ 0 & \text{if } z \le z_{min}^{(1)}. \end{cases} \quad (2)$$

Now, for the second objective function $z^{(2)}$ of model (6), we calculate

$$z_{min}^{(2)} = \min_{x \in X} \sum_{i=1}^{N} (r_i^m - r_i^p)\, x_i$$

and

$$z_{max}^{(2)} = \max_{x \in X} \sum_{i=1}^{N} (r_i^m - r_i^p)\, x_i.$$

The corresponding linear membership function $\mu_{z^{(2)}}(z)$ is:

$$\mu_{z^{(2)}}(z) = \begin{cases} 1 & \text{if } z \le z_{min}^{(2)}; \\ \frac{z_{max}^{(2)} - z^{(2)}}{z_{max}^{(2)} - z_{min}^{(2)}} & \text{if } z_{min}^{(2)} < z < z_{max}^{(2)}; \\ 0 & \text{if } z \ge z_{max}^{(2)}. \end{cases} \tag{3}$$

Similarly, for the objective function $z^{(3)}$ of model (6), we compute

$$z_{min}^{(3)} = \min_{x \in X} \sum_{i=1}^{N} (r_i^o - r_i^m)\, x_i$$

and

$$z_{max}^{(3)} = \max_{x \in X} \sum_{i=1}^{N} (r_i^o - r_i^m)\, x_i$$

and the corresponding linear membership function $\mu_{z^{(3)}}(z)$

$$\mu_{z^{(3)}} = \begin{cases} 1 & \text{if } z \ge z_{max}^{(3)}; \\ \frac{z - z_{min}^{(3)}}{z_{max}^{(3)} - z_{min}^{(3)}} & \text{if } z_{min}^{(3)} < z < z_{max}^{(3)}; \\ 0 & \text{if } z \le z_{min}^{(3)}. \end{cases} \tag{4}$$

Finally, we solve the following max-min problem

$$Y = \max_{x \in X} \{\min_{x \in X} \, (\mu_{z^{(1)}}(x),\, \mu_{z^{(2)}}(x),\, \mu_{z^{(3)}}(x))\} \tag{5}$$

to obtain the optimal allocation weights $x_1, x_2, \ldots, x_N$.

By introducing a variable y, equation (5) is then equivalent to a single-objective linear programming problem:

Model 10:

$$
\begin{array}{ll}
\max & y \\
\text{subject to} & \mu_{z^{(1)}}(x) \geq y \\
& \mu_{z^{(2)}}(x) \geq y \\
& \mu_{z^{(3)}}(x) \geq y \\
& x \in X.
\end{array}
$$

The optimal solution of model (10) provides a satisfying solution under the strategy of maximizing portfolio return r^p, minimizing portfolio risk $(r^m - r^p)$, and maximizing portfolio skewness $(r^o - r^m)$.

Our algorithm for asset allocation is now summarized as follows:

Step 1: For each available asset S_i, estimate the most possible return rate r_i^m, the most pessimistic return r_i^p, and the most possible return rate r_i^o, $i = 1, 2, \ldots, N$.

Step 2: Determine the initial values for parameters $\beta_l, \beta_u, \gamma_l$, and γ_u calculated by:

$$\beta_l = \min_{i=1,2,\ldots,N} \{r_i^m\}, \quad \beta_u = \max_{i=1,2,\ldots,N} \{r_i^m\}$$

and

$$\gamma_l = \min_{i=1,2,\cdots,N} \{r_i^m - r_i^p\} \qquad \gamma_u = \max_{i=1,2,\cdots,N} \{r_i^m - r_i^p\}.$$

The parameters $\beta_l, \beta_u, \gamma_l$, and γ_u can also be determined by experience and managerial judgment.

Step 3: For each objective function $z^{(j)}$ $(j = 1, 2, 3)$ in model (6), use linear programming techniques to find its maximal value $z_{max}^{(j)}$ and its minimal value $z_{min}^{(j)}$ subjected to the four constraints in model (6).

Step 4: Solve the following linear programming model with $N + 1$ variables to determine allocation weights $x_1, x_2, \ldots, x_N$:

$$
\begin{array}{ll}
\max & y \\
\text{subject to} & \sum_{i=1}^{N} r_i^m x_i - (z_{max}^{(1)} - z_{min}^{(1)}) y \geq z_{min}^{(1)}
\end{array}
$$

$$\sum_{i=1}^{N} (r_i^m - r_i^p)x_i + (z_{max}^{(2)} - z_{min}^{(2)})\, y \le z_{max}^{(2)}$$
$$\sum_{i=1}^{N} (r_i^o - r_i^m)x_i + (z_{max}^{(3)} - z_{min}^{(3)})\, y \ge z_{min}^{(3)}.$$

Step 5: For the optimal solution $x_1^*, x_2^*, \ldots, x_N^*$, calculate

$$(r^p)^* = \sum_{i=1}^{N} r_i^p x_i^*, \quad (r^m)^* = \sum_{i=1}^{N} r_i^m x_i^*$$

and

$$(r^o)^* = \sum_{i=1}^{N} r_i^o x_i^*.$$

If $(r^m)^* - (r^p)^* \ge \xi$, where ξ is the risk tolerance bound, then decrease the value of γ_u and goto Step 2;

Else if $(r^m)^* \le \eta$, where η is the lower bound for the most possible rate of return, then increase β_l and goto Step 2;
else STOP! $x_1^*, x_2^*, \ldots, x_N^*$ is the optimal solution.

5 Numerical Examples

5.1 Data Used to Construct Examples

Assume there are six asset classes in the market and the i-th asset class has mean and standard deviation of μ_i and σ_i respectively, $i = 1, 2, \ldots, 6$. The values of μ_i and σ_i are taken from Fong and Fabozzi (1992, p. 145) and Lederman and Klein (1994, Chapter 2, p. 27). Next we define r_i^p, r_i^m, r_i^o by setting $r_i^p = \mu_i - 2\sigma_i$, $r_i^m = \mu_i$, and $r_i^o = \mu_i + 3\sigma_i$, $i = 1, 2, \ldots, 6$, with some adjustment. Table 1 displays the basic data used in the examples. The data are summarized in Table 1.

Example 1: We solve model (7) by setting $\beta_l = 0.05$, $\beta_u = 0.17$, $\gamma_l = 0.008$, and $\gamma_u = 0.4$. (See Case 1 of model (6).) The optimal allocation weights are $x_1 = 0.0061, x_2 = 0.5, x_3 = 0.0354, x_5 = 0.4584$, and $(r^p, r^m, r^o) = (-0.0881, 0.1078, 0.4307)$. The optimal allocation is almost a combination of the second most risky asset and the second most conservative asset.

Table 1
Data For Examples

	μ	σ	r^p	r^m	r^o
Stock 1	0.17	0.200	-0.230	0.17	0.800
Stock 2	0.15	0.185	-0.220	0.15	0.750
Bound 1	0.12	0.055	0.010	0.12	0.270
Bound 2	0.08	0.050	-0.020	0.08	0.200
Cash	0.06	0.005	0.050	0.06	0.090
T-bill	0.05	0.004	0.042	0.05	0.075

Example 2: We solve model (3.3) by fixing portfolio return at 22 different values. (See Case 2 of model (6).) The computational results are summarized in the following Tables 2 and 3.

The fifth column in Table 3 gives the set $\{r \mid \pi_{\tilde{r}_i}(r) \geq 0.85\}$, which contains all the possible values of the return rate whose degree of occurrence is at least 0.85. This interval is called the acceptable event with degree of occurrence at least 0.85. Similarly, the last column in Table 3 gives the acceptable event with degree of occurrence at least 0.95.

We observe that both portfolio risk $(r^m - r^p)$ and portfolio skewness $(r^o - r^m)$ increase as portfolio return r^m increases, which is consistent with the fact that as r^m is pushed higher, more weight should be allocated to higher risk assets. We also observe that when r^m increases gradually, the weights are adjusted gradually, showing that our numerical results are stable.

Example 3: We solve model (9) by fixing portfolio risk for 22 different values. (See Case 3 of model (6).) The computational results are summarized in Tables 4 and 5.

Example 4: We solve model (6) by adjusting β_l to control portfolio return. The computational results are summarized in Tables 6 and 7. From these tables we observe that β_l controls portfolio return effectively.

Table 2
Solutions for Different Values of Portfolio Return

No.	$(\beta_l, \beta_u, \gamma_l, \gamma_u)$	Optimal Solution X^*	$\tilde{r} = (r^p, r^m, r^o)$
1	(0.055, 0.055, 0.008, 0.4)	$x_5 = 0.5, x_6 = 0.5$	(0.046, 0.055, 0.0825)
2	(0.060, 0.060, 0.008, 0.4)	$x_2 = 0.0454, x_3 = 0.0152, x_5 = 0.4394, x_6 = 0.5$	(0.0331, 0.6, 0.1152)
3	(0.065, 0.065, 0.008, 0.4)	$x_2 = 0.0713, x_3 = 0.0597, x_5 = 0.3690, x_6 = 0.5$	(0.0244, 0.065, 0.1403)
4	(0.070, 0.070, 0.008, 0.4)	$x_2 = 0.0935, x_3 = 0.1098, x_5 = 0.2967, x_6 = 0.5$	(0.0164, 0.07, 0.164)
5	(0.075, 0.075, 0.008, 0.4)	$x_2 = 0.1157, x_3 = 0.1598, x_5 = 0.2245, x_6 = 0.5$	(0.0084, 0.075, 0.1876)
6	(0.080, 0.080, 0.008, 0.4)	$x_2 = 0.1381, x_3 = 0.2096, x_5 = 0.1524, x_6 = 0.5$	(0.0003, 0.08, 0.2113)
7	(0.085, 0.085, 0.008, 0.4)	$x_2 = 0.1605, x_3 = 0.2593, x_5 = 0.0802, x_6 = 0.5$	(-0.0077, 0.085, 0.2351)
8	(0.090, 0.090, 0.008, 0.4)	$x_2 = 0.1829, x_3 = 0.3089, x_5 = 0.0081, x_6 = 0.5$	(-0.0157, 0.09, 0.2588)
9	(0.095, 0.095, 0.008, 0.4)	$x_2 = 0.2310, x_3 = 0.3128, x_6 = 0.4562$	(-0.0285, 0.095, 0.2919)
10	(0.100, 0.100, 0.008, 0.4)	$x_2 = 0.2801, x_3 = 0.3142, x_6 = 0.4057$	(-0.0414, 0.1, 0.3253)
11	(0.105, 0.105, 0.008, 0.4)	$x_2 = 0.3250, x_3 = 0.3214, x_6 = 0.3536$	(-0.0534, 0.105, 0.357)
12	(0.110, 0.110, 0.008, 0.4)	$x_2 = 0.3701, x_3 = 0.3284, x_6 = 0.3015$	(-0.0655, 0.11, 0.3889)
13	(0.115, 0.115, 0.008, 0.4)	$x_2 = 0.4150, x_3 = 0.3357, x_6 = 0.2493$	(-0.0775, 0.115, 0.4206)
14	(0.120, 0.120, 0.008, 0.4)	$x_2 = 0.4601, x_3 = 0.3428, x_6 = 0.1972$	(-0.0895, 0.12, 0.4524)
15	(0.125, 0.125, 0.008, 0.4)	$x_1 = 0.0092, x_2 = 0.5, x_3 = 0.3415, x_6 = 0.1494$	(-0.1024, 0.125, 0.4858)
16	(0.130, 0.130, 0.008, 0.4)	$x_1 = 0.0504, x_2 = 0.5, x_3 = 0.3421, x_6 = 0.1074$	(-0.1137, 0.13, 0.5157)
17	(0.135, 0.135, 0.008, 0.4)	$x_1 = 0.0956, x_2 = 0.5, x_3 = 0.3361, x_6 = 0.0683$	(-0.1258, 0.135, 0.5473)
18	(0.140, 0.140, 0.008, 0.4)	$x_1 = 0.1410, x_2 = 0.5, x_3 = 0.3296, x_6 = 0.0293$	(-0.1379, 0.14, 0.579)
19	(0.145, 0.145, 0.008, 0.4)	$x_1 = 0.2204, x_2 = 0.4660, x_3 = 0.3136$	(-0.1501, 0.145, 0.6105)
20	(0.150, 0.150, 0.008, 0.4)	$x_1 = 0.3137, x_2 = 0.4771, x_3 = 0.2092$	(-0.175, 0.15, 0.6653)
21	(0.155, 0.155, 0.008, 0.4)	$x_1 = 0.4067, x_2 = 0.4888, x_3 = 0.1045$	(-0.2, 0.155, 0.7202)
22	(0.160, 0.160, 0.008, 0.4)	$x_1 = 0.5, x_2 = 0.5$	(-0.225, 0.16, 0.775)

Table 3
Optimal Portfolio Return and Risk Analysis Using Table 2

No.	r^m	$r^m - r^p$	$r^o - r^m$	85%	95%
1	0.055	0.0090	0.0275	(0.0537, 0.0591)	(0.0546, 0.0564)
2	0.060	0.0269	0.0552	(0.0560, 0.0683)	(0.0587, 0.0628)
3	0.065	0.0406	0.0753	(0.0590, 0.0763)	(0.0630, 0.0688)
4	0.070	0.0536	0.0940	(0.0620, 0.0841)	(0.0673,.0.0747)
5	0.075	0.0666	0.1126	(0.0650, 0.0919)	(0.0717, 0.0806)
6	0.080	0.0797	0.1313	(0.0680, 0.0997)	(0.0760, 0.0866)
7	0.085	0.0927	0.1501	(0.0711, 0.1075)	(0.0804, 0.0925)
8	0.090	0.1057	0.1688	(0.0741, 0.1153)	(0.0847, 0.0984)
9	0.095	0.1235	0.1969	(0.0765, 0.1245)	(0.0888, 0.1048)
10	0.100	0.1414	0.2253	(0.0788, 0.1338)	(0.0929, 0.1113)
11	0.105	0.1584	0.2520	(0.0812, 0.1433)	(0.0971, 0.1176)
12	0.110	0.1755	0.2789	(0.0837, 0.1518)	(0.1012, 0.1239)
13	0.115	0.1925	0.3056	(0.0861, 0.1608)	(0.1054, 0.1303)
14	0.120	0.2095	0.3324	(0.0886, 0.1699)	(0.1095, 0.1366)
15	0.125	0.2274	0.3608	(0.0909, 0.1791)	(0.1136, 0.1430)
16	0.130	0.2437	0.3857	(0.0934, 0.1879)	(0.1178, 0.1493)
17	0.135	0.2608	0.4123	(0.0959, 0.1968)	(0.1220, 0.1556)
18	0.140	0.2779	0.4390	(0.0983, 0.2059)	(0.1286, 0.1620)
19	0.145	0.2951	0.4655	(0.1007, 0.2148)	(0.1302, 0.1683)
20	0.150	0.3250	0.5153	(0.1013, 0.2273)	(0.1338, 0.1758)
21	0.155	0.3550	0.5647	(0.1018, 0.2397)	(0.1323, 0.1832)
22	0.160	0.3850	0.6150	(0.1023, 0.2523)	(0.1408, 0.1908)

Table 4
Solutions for Different Values of Portfolio Risk

No.	$(\beta_l, \beta_u, \gamma_l, \gamma_u)$	Optimal Solution X^*	$\tilde{r} = (r^p, r^m, r^o)$
1	(0.05, 0.17, 0.009, 0.009)	$x_5 = 0.5, x_6 = 0.5$	(0.046, 0.055, 0.0825)
2	(0.05, 0.17, 0.0269, 0.0269)	$x_2 = 0.0202, x_3 = 0.1038, x_5 = 0.5, x_6 = 0.376$	(0.0408, 0.0643, 0.1164)
3	(0.05, 0.17, 0.0406, 0.0406)	$x_2 = 0.0359, x_3 = 0.1824, x_5 = 0.5, x_6 = 0.2817$	(0.0308, 0.0714, 0.1423)
4	(0.05, 0.17, 0.0536, 0.0536)	$x_2 = 0.0506, x_3 = 0.2577, x_5 = 0.5, x_6 = 0.1917$	(0.0245, 0.0781, 0.1669)
5	(0.05, 0.17, 0.0666, 0.0666)	$x_2 = 0.0746, x_3 = 0.3001, x_5 = 0.5, x_6 = 0.1253$	(0.0169, 0.0835, 0.1914)
6	(0.05, 0.17, 0.0797, 0.0797)	$x_2 = 0.1061, x_3 = 0.3167, x_5 = 0.5, x_6 = 0.0773$	(0.0081, 0.0878, 0.2159)
7	(0.05, 0.17, 0.0927, 0.0927)	$x_2 = 0.1382, x_3 = 0.3301, x_5 = 0.5, x_6 = 0.0317$	(-0.0008, 0.0919, 0.2402)
8	(0.05, 0.17, 0.1057, 0.1057)	$x_2 = 0.1704, x_3 = 0.3435, x_5 = 0.4861$	(-0.0097, 0.0959, 0.2643)
9	(0.05, 0.17, 0.1235, 0.1235)	$x_2 = 0.2189, x_3 = 0.3469, x_5 = 0.4342$	(-0.0230, 0.1005, 0.2969)
10	(0.05, 0.17, 0.1414, 0.1414)	$x_2 = 0.2674, x_3 = 0.3514, x_5 = 0.3812$	(-0.0363, 0.1052, 0.3297)
11	(0.05, 0.17, 0.1584, 0.1584)	$x_2 = 0.3139, x_3 = 0.3538, x_5 = 0.3322$	(-0.0489, 0.1095, 0.3608)
12	(0.05, 0.17, 0.1755, 0.1755)	$x_2 = 0.3610, x_3 = 0.3552, x_5 = 0.2837$	(-0.0617, 0.1138, 0.3922)
13	(0.05, 0.17, 0.1925, 0.1925)	$x_2 = 0.4071, x_3 = 0.3596, x_5 = 0.2333$	(-0.0743, 0.1182, 0.4234)
14	(0.05, 0.17, 0.2095, 0.2095)	$x_2 = 0.4511, x_3 = 0.3710, x_5 = 0.1779$	(-0.0867, 0.1229, 0.4545)
15	(0.05, 0.17, 0.2274, 0.2274)	$x_2 = 0.4975, x_3 = 0.3831, x_5 = 0.1194$	(-0.0996, 0.1278, 0.4873)
16	(0.05, 0.17, 0.2437, 0.2437)	$x_1 = 0.0411, x_2 = 0.5, x_3 = 0.3765, x_5 = 0.0823$	(-0.1116, 0.1321, 0.5169)
17	(0.05, 0.17, 0.2608, 0.2608)	$x_1 = 0.0911, x_2 = 0.5, x_3 = 0.3529, x_5 = 0.0561$	(-0.1246, 0.1362, 0.5482)
18	(0.05, 0.17, 0.2779, 0.2779)	$x_1 = 0.1476, x_2 = 0.5, x_3 = 0.3034, x_5 = 0.049$	(-0.1385, 0.1394, 0.5794)
19	(0.05, 0.17, 0.2951, 0.2951)	$x_1 = 0.2042, x_2 = 0.5, x_3 = 0.2546, x_5 = 0.0412$	(-0.1524, 0.1427, 0.6108)
20	(0.05, 0.17, 0.325, 0.325)	$x_1 = 0.3025, x_2 = 0.5, x_3 = 0.1703, x_5 = 0.0272$	(-0.1765, 0.1485, 0.6654)
21	(0.05, 0.17, 0.355, 0.355)	$x_1 = 0.4013, x_2 = 0.5, x_3 = 0.0848, x_5 = 0.0139$	(-0.2008, 0.1542, 0.7202)
22	(0.05, 0.17, 0.385, 0.385)	$x_1 = 0.5, x_2 = 0.5$	(-0.225, 0.16, 0.775)

Table 5
Optimal Portfolio Return and Risk Analysis Using Table 4

No.	r^m	$r^m - r^p$	$r^o - r^m$	85%	95%
1	0.0550	0.0090	0.0275	(0.0537, 0.0541)	(0.0500, 0.0564)
2	0.0643	0.0235	0.0521	(0.0608, 0.0721)	(0.0631, 0.0669)
3	0.0714	0.0406	0.0709	(0.0653, 0.0820)	(0.0694, 0.0749)
4	0.0781	0.0536	0.0888	(0.0701, 0.0914)	(0.0754, 0.0825)
5	0.0835	0.0666	0.1079	(0.0735, 0.0997)	(0.0802, 0.0889)
6	0.0878	0.0797	0.1281	(0.0758, 0.1070)	(0.0838, 0.0942)
7	0.0919	0.0927	0.1483	(0.0780, 0.1142)	(0.0873, 0.0993)
8	0.0959	0.1056	0.1684	(0.0801, 0.1212)	(0.0906, 0.1043)
9	0.1005	0.1235	0.1964	(0.0820, 0.1300)	(0.0943, 0.1103)
10	0.1052	0.1414	0.2245	(0.0840, 0.1389)	(0.0981, 0.1164)
11	0.1095	0.1584	0.2513	(0.0857, 0.1472)	(0.1016, 0.1221)
12	0.1138	0.1755	0.2784	(0.0875, 0.1556)	(0.1050, 0.1277)
13	0.1182	0.1925	0.3052	(0.0893, 0.1640)	(0.1086, 0.1335)
14	0.1229	0.2096	0.3316	(0.0915, 0.1726)	(0.1124, 0.1395)
15	0.1278	0.2274	0.3595	(0.0937, 0.1817)	(0.1165, 0.1458)
16	0.1321	0.2437	0.3848	(0.0958, 0.1898)	(0.1199, 0.1513)
17	0.1362	0.2608	0.4120	(0.0971, 0.1980)	(0.1232, 0.1568)
18	0.1394	0.2779	0.4400	(0.0977, 0.2054)	(0.1255, 0.1614)
19	0.1427	0.2951	0.4681	(0.0984, 0.2129)	(0.1279, 0.1661)
20	0.1485	0.3250	0.5169	(0.0998, 0.2260)	(0.1323, 0.1743)
21	0.1542	0.3550	0.5660	(0.1100, 0.2391)	(0.1365, 0.1825)
22	0.160	0.3850	0.6150	(0.1023, 0.2523)	(0.1408, 0.1908)

6 Summary

We presented an asset allocation method using possibilistic programming techniques to characterize the imprecise nature of the rate of return. Unlike the traditional mean-variance method, our asset allocation method takes the portfolio's skewness into consideration. It provides two control constraints that permit maximal flexibility for decision makers to effectively balance the portfolio's return and the portfolio's risk. The optimal allocation decision is made by solving several linear programming problems. Software packages are available that can efficiently solve linear programming problems.

Table 6
Solutions for Different Values of β_l

No.	$(\beta_l, \beta_u, \gamma_l, \gamma_u)$	Optimal Solution X^*	$\tilde{r} = (r^p, r^m, r^o)$
1	(0.08, 0.17, 0.008, 0.4)	$x_2 = 0.4977, x_3 = 0.2554, x_5 = 0.2469$	(-0.0946, 0.1201, 0.4645)
2	(0.11, 0.17, 0.008, 0.4)	$x_1 = 0.0788, x_2 = 0.5, x_5 = 0.3596, x_6 = 0.0616$	(-0.1214, 0.1352, 0.5407)
3	(0.14, 0.17, 0.008, 0.4)	$x_1 = 0.4137, x_2 = 0.3113, x_3 = 0.275$	(-0.1609, 0.15, 0.6387)
4	(0.16, 0.17, 0.008, 0.4)	$x_1 = 0.5, x_2 = 0.5$	(-0.225, 0.16, 0.775)

Table 7
Optimal Portfolio Return and Portfolio Risk Analysis

No.	r^m	$r^m - r^p$	$r^o - r^m$	85%	95%
1	0.1201	0.2147	0.3444	(0.0879, 0.1718)	(0.0879, 0.1373)
2	0.1352	0.2566	0.4055	(0.0967, 0.196)	(0.1224, 0.1555)
3	0.15	0.3009	0.4887	(0.1049, 0.2233)	(0.135, 0.1744)
4	0.16	0.385	0.615	(0.1023, 0.2523)	(0.1408, 0.1908)

Some problems still remain to be solved. For example, instead of obtaining the most pessimistic value, the most probable value, and the most optimistic value for the rate of return from mean and standard deviation as shown in the examples, we could use simulation to generate the data directly from the historical resources. We could also use the possibility programming method to solve multistage asset allocation problems and asset/liability management problems.

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