Simple Type Theory (Simplified)

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1 Introduction

In this document we describe typed lambda calculus with sums, except that we explicitly keep track of contexts and all other parts of the theory using inference rules. This is the simple type theory described in [4]. This is a formalization of the type theory described in [1], but so that well formed contexts and types are generated following explicit inference rules, like in the appendix A2 of [2]. The terminology is mostly taken from [3].

2 Basics

A **term** is a value of a **type**. Some terms are **variables** (as we explain later). Each term t has a set $FV(t)$ of **free variables** (as we explain later).

There are six kinds of expressions:

1. A **typing declaration** $x : A$ says that x is a term of type A.
2. A **universal declaration** A_type says that A is a type.
3. An **equality declaration** $x \equiv y : A$ says values x and y of type A are equal.
4. A **context** Γ is a list of typing declarations. We write $\Gamma :: \Delta$ to denote the concatenation (joining) of lists.
5. A **context declaration** $\Gamma \quad \text{ctx}$ is a declaration that the context Γ is "well formed" (the meaning will be clear later from the rules).
6. A **judgment** is something of the form $\Gamma \vdash d$ where Γ is a context, and d is either a typing declaration or a universal declaration or an equality declaration. Sometimes we call d the **declaration** of the judgment $\Gamma \vdash d$.

A **rule** is something of the form

$$\frac{J_1 \quad J_2 \quad \dots \quad J_n}{K}$$

where $J_1, J_2, \ldots, J_n$ and K are all judgments. The meaning of the rule is that if each judgment in $J_1, J_2, \ldots, J_n$ can be derived in the type theory then judgment K may also be derived. Judgments can be stacked to make proof trees. An axiom is a rule

$$\frac{}{K}$$

with no prerequisites.

In addition to the assumed rules (which we name)

3 Forming base types

We write . to denote the empty context. The fact that the empty context is well formed is formalized by the rule:

$$\frac{}{.\quad \text{ctx}}\ \text{Empty Context} \tag{1}$$

The next rule allows a well formed context to be extended by introducing a **base type** A:

$$\frac{\Gamma \quad \text{ctx}}{\Gamma \vdash A\_\text{type}}\ \text{Base Type Formation} \tag{2}$$

The base type A must not appear in the context Γ. Here we assume we have some list of base types [1]. If we are trying to model a particular system we may have specific base types ready, but for now let us just think of base types as variable types (although in this document we reserve the phrase "variable" for terms). So it is fine for A to be any type new to the context.

4 Forming Other Types

This rule lets us form the unit type

$$\frac{\Gamma \quad \text{ctx}}{\Gamma \vdash 1\_\text{type}}\ \text{Unit Formation} \tag{3}$$

Next, product types

$$\frac{\Gamma \vdash A\_\text{type} \quad \Gamma \vdash B\_\text{type}}{\Gamma \vdash A \times B\_\text{type}}\ \text{Product Formation} \tag{4}$$

Next the empty type

$$\frac{\Gamma \quad \text{ctx}}{\Gamma \vdash 0\_\text{type}}\ \text{Empty Formation} \tag{5}$$

Next, sum types

$$\frac{\Gamma \vdash A\_\text{type} \quad \Gamma \vdash B\_\text{type}}{\Gamma \vdash A + B\_\text{type}} \text{ Sum Formation} \tag{6}$$

Next function types

$$\frac{\Gamma \vdash A\_\text{type} \quad \Gamma \vdash B\_\text{type}}{\Gamma \vdash A \to B\_\text{type}} \text{ Function Formation} \tag{7}$$

5 Forming Variables

We make a variable x of type A using the following rule

$$\frac{\Gamma \vdash A\_\text{type}}{\Gamma :: (x : A) \quad \text{ctx}} \text{ Context Extension} \tag{8}$$

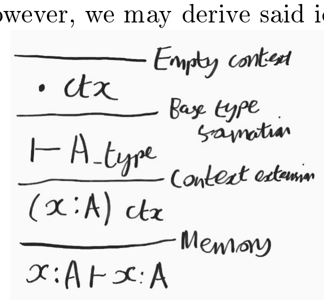
Here the variable x must not appear in the context Γ.

The set of free variables of this newly made x is $FV(x) = \{x\}$. Also $BV(x) = \{\}$ where $BV(t)$ denotes the set of bound variables of a term t, and $\{\}$ denotes the empty set.

Judgments about variables can be formed with the following rule

$$\frac{\Gamma :: (x : A) :: \Delta \quad \text{ctx}}{\Gamma :: (x : A) :: \Delta \vdash x : A} \text{ Memory} \tag{9}$$

The Memory rule appears as Vble1 on page 554 of [2]. The rule also appears in [1]. However [10] uses a different rule which they call the identity rule instead. However, we may derive said identity rule as follows:



In category theory Memory could be thought of as projection.

5.1 About Substitution

For a term u and a variable x and a term a of the same type as x we write $u[a/x]$ to denote the result of taking term u and replacing all free occurrences of x with term a. In such a case [5], we say that a is free for x in u if and only if no free variables of a become bound in $u[a/x]$ (in other words, if the intersection of $FV(a)$ and $BV(u[a/x])$ is empty). Recall that the possible bindings that happen in our system just consist of three cases (1) in $(\lambda x : A).t$ we have that each variable x in term t binds to λ, and (2) in $\text{match}\,(s, x.u, y.v)$ we have that each variable x in u becomes bound, and (3) we have that in match $(s, x.u, y.v)$ we we have that each variable y in v becomes bound.

To keep track of what is required within a computer, rather than just doing $(FV(m), FV(n)) \mapsto FV(m) \cup FV(n)$ etc. when terms are combined, we can form an expression tree for a term. We can also form corresponding expression trees for the free and bound variables, and when we construct $u\,[a/x]$ we replace the leaf x in u with the tree for a. Also, when we have binding operators, they 'color' the corresponding variables which might appear within the appropriate places deeper within the expression tree, and and we should make sure that after substitution, so we want to gaurd against previously free variables of the appropriate names, from a, being there , when we replace x with a in u so that the newly added subtree ends up lying within a colored region, involving new bindings of said variables.

Maybe we should use De-Brujin Notation. !!!

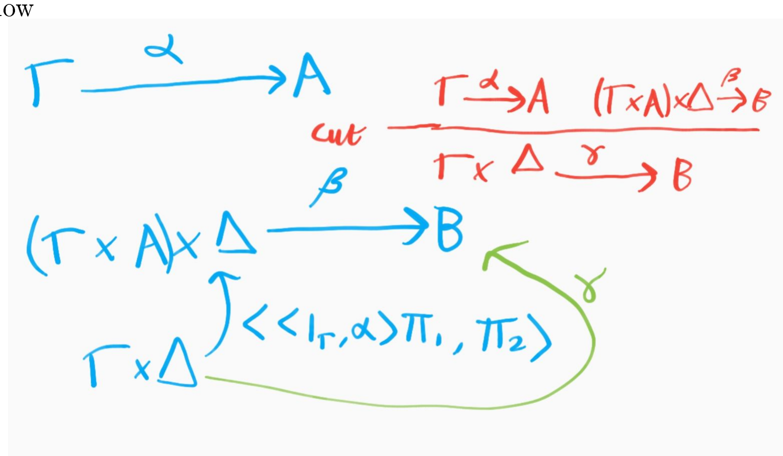
6 Other Structural Rules

Our next rules are taken from [10].

$$\frac{\Gamma \vdash a : A \quad \Gamma :: (x : A) :: \Delta \vdash b : B}{\Gamma :: \Delta \vdash b[a/x] : B}\ \text{Cut} \tag{10}$$

where x is free for a in b (is this required ? need to check about variables of chapter 1 of [10]).

The way the cut rule can be vizualized in terms of category theory as pictured below



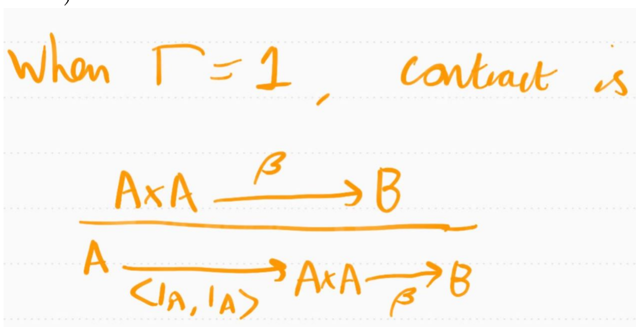
Our next rule is weakening

$$\frac{\Gamma :: \Delta \vdash b : B \quad \Gamma \vdash A\_\text{type}}{\Gamma :: (x : A) :: \Delta \vdash b : B}\ \text{Weakening} \tag{11}$$

In terms of category theory, this just corresponds to doing the product of the source arrow with some other arrow, and then doing a projection before composing with the arrow. Note after the weakening rule has been applied, some may consider the variable x to occur within b invisably.

$$\frac{\Gamma :: (x : A, y : A) :: \Delta \vdash b : B}{\Gamma :: (x : A) :: \Delta \vdash b[x/y] : B} \text{ Contract} \tag{12}$$

The picture below should how to think of contract categorically (discounting Γ and Δ).



Our final structural rule is

$$\frac{\Gamma :: (x : A, y : A) :: \Delta \vdash b : B}{\Gamma :: (y : A, x : A) :: \Delta \vdash b : B} \text{ Exchange} \tag{13}$$

This allows us to think of the context as a set. Categorically exchange can be thought of as corresponding to permuting the order of objects in a product.

I wonder if cut and exchange are required.

7 The Other Typing Rules

We write $FV(t)$ to denote the set of free variables of a term t. We also write $BV(t)$ to denote the set of bound variables of a term t (the meaning of which will become clear later).

7.1 Products

We describe products as negative types [7].

$$\frac{\Gamma \quad \text{ctx}}{\Gamma \vdash * : 1} \text{ Unit-Intro} \tag{14}$$

Here $*$ has the empty set $FV(*) = \{\}$ of free variables. Also $BV(*) = \{\}$ is the empty set.

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B}{\Gamma \vdash \langle a, b \rangle : A \times B} \text{ Product-Intro} \tag{15}$$

Here $FV(\langle a, b \rangle) = FV(a) \cup FV(b)$ where $\cup$ denotes the set theoretic union. Also $BV(\langle a, b \rangle) = BV(a) \cup BV(b)$

$$\frac{\Gamma \vdash p : A \times B}{\Gamma \vdash \text{fst}(p) : A} \text{ Product-Elim1} \tag{16}$$

$FV(\text{fst}(p)) = FV(p)$. Also $BV(\text{fst}(p)) = BV(p)$.

$$\frac{\Gamma \vdash p : A \times B}{\Gamma \vdash \text{snd}(p) : B} \text{ Product-Elim2} \tag{17}$$

$FV(\text{snd}(p)) = FV(p)$. Also $BV(\text{snd}(p)) = BV(p)$.

7.2 Sums

We describe sums as positive types.

The following elimination rule for the empty type is like that described in [4]

$$\frac{\Gamma \vdash A\_\text{type} \quad \Gamma \vdash e : 0}{\Gamma \vdash \text{abort}_A(e) : A} \text{ Empty-Elim} \tag{18}$$

$FV(\text{abort}_A(e)) = FV(e)$ Also $BV(\text{abort}_A(e)) = BV(e)$

Next, we have introduction rules for sum types

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash B\_\text{type}}{\Gamma \vdash \text{inl}_{A+B}(a) : A + B} \text{ Sum-Intro1} \tag{19}$$

$FV(\text{inl}_{A+B}(a)) = FV(a)$ Also $BV(\text{inl}_{A+B}(a)) = BV(a)$

$$\frac{\Gamma \vdash A\_\text{type} \quad \Gamma \vdash b : B}{\Gamma \vdash \text{inr}_{A+B}(b) : A + B} \text{ Sum-Intro2} \tag{20}$$

$FV(\text{inr}_{A+B}(b)) = FV(b)$ Also $BV(\text{inr}_{A+B}(b)) = BV(b)$

$$\frac{\Gamma \vdash s : A + B \quad \Gamma, x : A \vdash u : C \quad \Gamma, y : B \vdash v : C}{\Gamma \vdash \text{match}\,(s, x.u, y.v) : C} \text{ Sum-Elim} \tag{21}$$

$FV\,(\text{match}\,(s, x.u, y.v)) = FV(s) \cup FV(u) \cup FV(v) - [\{x\} \cup \{y\}]$ where $L - R$ denotes the set of members of set A that are not in set B. Also $BV\,(\text{match}\,(s, x.u, y.v)) = BV(s) \cup BV(u) \cup BV(v) \cup \{x\} \cup \{y\}$.

Here $x.u$ denotes that variable x is bound to u in $\text{match}\,(s, x.u, y.v)$. And so $\text{match}\,(s, x.u, y.v)$ binds free occurrences of x and y.

7.3 Functions

We describe functions as negative types.

$$\frac{\Gamma :: (x : A) \vdash b : B}{\Gamma \vdash (\lambda x : A)\,.b : A \to B} \text{ Function-Intro} \tag{22}$$

$FV\,((\lambda x : A)\,.b) = FV(b) - \{x\}$. Also $BV\,((\lambda x : A)\,.b) = BV(b) \cup \{x\}$.

Here the variable x is to bound in $(\lambda x : A)\,.b$.

$$\frac{\Gamma \vdash f : A \to B \quad \Gamma \vdash a : A}{\Gamma \vdash f(a) : B} \text{ Function-Elim} \tag{23}$$

$FV(f(a)) = FV(f) \cup FV(a)$. Also $BV(f(a)) = BV(f) \cup BV(a)$.

8 Equational Theory

In this section we give the rules for making equality declarations

8.1 Equivalence Relation

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash a \equiv a : A} \text{ Reflexive} \tag{24}$$

$$\frac{\Gamma \vdash a \equiv a' : A}{\Gamma \vdash a' \equiv a : A} \text{ Symmetric} \tag{25}$$

$$\frac{\Gamma \vdash a \equiv a' : A \quad a' \equiv a'' : A}{\Gamma \vdash a \equiv a'' : A} \text{ Transitive} \tag{26}$$

8.2 Products

The uniqueness principle (η conversion rule) for the unit type is

$$\frac{\Gamma \vdash v : 1}{\Gamma \vdash v \equiv * : 1} \text{ Unit-Uniqueness} \tag{27}$$

The computation rules (β reduction rules) for the product type are

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B}{\Gamma \vdash \mathrm{fst}\,(\langle a, b \rangle) \equiv a} \text{ Product-Computation1} \tag{28}$$

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B}{\Gamma \vdash \mathrm{snd}\,(\langle a, b \rangle) \equiv b} \text{ Product-Computation2} \tag{29}$$

The uniqueness principle (η conversion rule) for the product types is

$$\frac{\Gamma \vdash p : A \times B}{\Gamma \vdash p \equiv \langle \mathrm{fst}(p), \mathrm{snd}(p) \rangle : A \times B} \text{ Product-Uniqueness} \tag{30}$$

8.3 Sums

The uniqueness principle (η conversion rule) of the empty type [6] is

$$\frac{\Gamma \vdash e : 0 \quad \Gamma \vdash A\_\text{type} \quad \Gamma \vdash x : A}{\Gamma \vdash \text{abort}_A(e) \equiv x : A} \text{ Empty-Uniqueness} \tag{31}$$

The computation rules (β reduction rules) for sum types are

$$\frac{\Gamma \vdash a : A \quad \Gamma :: (x : A) \vdash u : C \quad \Gamma :: (y : B) \vdash v : C}{\Gamma \vdash \text{match}\,(\text{inl}_{A+B}(a), x.u, y.v) \equiv u\,[a/x] : C} \text{ Sum-Computation1} \tag{32}$$

provided that a is free for x in u.

$$\frac{\Gamma \vdash b : B \quad \Gamma :: (x : A) \vdash u : C \quad \Gamma :: (y : B) \vdash v : C}{\Gamma \vdash \text{match}\,(\text{inr}_{A+B}(b), x.u, y.v) \equiv v\,[b/y] : C} \text{ Sum-Computation2} \tag{33}$$

provided that b is free for y in v.

(I am guessing these "freeness" conditions are required in each rule involving substitution, just like as in the internal language of a topos, described in [5]. Different variables can be used within the substitution if there are problems, as discussed in Bell's book on Local Set Theory).

The uniqueness principle (η conversion rule) for the sum types is

$$\frac{\Gamma \vdash s : A + B \quad \Gamma, h : A + B \vdash m : C}{\Gamma \vdash \text{match}\,(s, x.m\,[\text{inl}_{A+B}(x)/h]\,, y.m\,[\text{inr}_{A+B}(y)/h]) \equiv m\,[s/h] : C} \text{ Sum-Uniqueness} \tag{34}$$

provided that $s, \text{inl}_{A+B}(x)$ and $\text{inr}_{A+B}(y)$ all each free for h in m.

8.4 Functions

The computation rule (β reduction rule) for function types is

$$\frac{\Gamma :: (x : A) \vdash m : C \quad \Gamma \vdash a : A}{\Gamma \vdash ((\lambda x : A)\,.m)\,(a) \equiv m\,[a/x] : C} \text{ Function-Computation} \tag{35}$$

provided that a is free for x in u.

The uniqueness rule (η conversion rule) for function types is

$$\frac{\Gamma \vdash f : A \to B}{\Gamma \vdash f \equiv (\lambda x : A)\,.\,(f(x)) : A \to B} \text{ Function-Uniqueness} \tag{36}$$

Provided that x is not a free variable of f (that is, provided $x \notin FV(f)$).

We also require that the same function operating on equal inputs gives equal outputs, that is

$$\frac{\Gamma \vdash f : A \to B \quad \Gamma \vdash a \equiv a' : A}{\Gamma \vdash f(a) \equiv f(a') : B} \text{ Function-Similar Inputs} \tag{37}$$

We also require that operating equal functions on the same input give equal outputs, that is

$$\frac{\Gamma :: (x : A) \vdash b \equiv b' : B}{\Gamma \vdash (\lambda x : A) . b \equiv (\lambda x : A) . b' : A \to B} \text{ Function-Similar Functions 1} \tag{38}$$

Finally, if there is an α conversion from $(\lambda x : A) . b$ to $(\lambda y : A) . b'$ then we consider $(\lambda x : A) . b \equiv (\lambda y : A) . b'$, however α conversion is somewhat technical to implement, and I presume there is no need for α conversion if the variable names are chosen well. I guess the official rule for α conversion is

$$\frac{\Gamma :: (x : A) \vdash b : B \quad \Gamma :: (y : A) \vdash b' : B \quad \Gamma \vdash b\,[y/x] \equiv b' : B}{\Gamma \vdash (\lambda x : A) . b \equiv (\lambda y : A) . b' : A \to B} \tag{39}$$

provided that y is free for x in b. But I am not sure this rule properly defines alpha conversion (see [8]).

I guess the α conversion rules corresponding to the sum type are

$$\frac{\Gamma \vdash s : A + B \quad \Gamma :: (x : A) \vdash u : C \quad \Gamma :: (y : B) \vdash v : C \quad \Gamma :: (x' : A) \vdash u' : C \quad \Gamma \vdash u\,[x'/x] \equiv u' : C}{\Gamma \vdash \text{match}\,(s, x.u, y.v) \equiv \text{match}\,(s, x'.u', y.v) : C} \tag{40}$$

provided x' is free for x in u.

$$\frac{\Gamma \vdash s : A + B \quad \Gamma :: (x : A) \vdash u : C \quad \Gamma :: (y : B) \vdash v : C \quad \Gamma :: (y' : B) \vdash v' : C \quad \Gamma \vdash v\,[y'/y] \equiv v' : C}{\Gamma \vdash \text{match}\,(s, x.u, y.v) \equiv \text{match}\,(s, x.u, y'.v') : C} \tag{41}$$

provided y' is free for y in v.

9 Equality implies type declaration

I propose that the following additional rule be added

$$\frac{\Gamma \vdash a \equiv a' : A}{\Gamma \vdash a : A} \text{ Equal Existence} \tag{42}$$

This rule seems to make sense. Note that now we could simplify the type theory by replacing $a : A$ with $a \equiv a : A$ throughout the theory if we want.

10 Simplified Handling of Types

Adding the following rules seems sensible to simplify statements of theorems

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash A\_\text{type}} \text{ Has Type} \tag{43}$$

$$\frac{\Gamma \vdash A \times B\_\mathrm{type}}{\Gamma \vdash A\_\mathrm{type}} \text{ Unform Product 1} \tag{44}$$

$$\frac{\Gamma \vdash A \times B\_\mathrm{type}}{\Gamma \vdash B\_\mathrm{type}} \text{ Unform Product 2} \tag{45}$$

$$\frac{\Gamma \vdash A + B\_\mathrm{type}}{\Gamma \vdash A\_\mathrm{type}} \text{ Unform Sum 1} \tag{46}$$

$$\frac{\Gamma \vdash A + B\_\mathrm{type}}{\Gamma \vdash B\_\mathrm{type}} \text{ Unform Sum 2} \tag{47}$$

$$\frac{\Gamma \vdash A \to B\_\mathrm{type}}{\Gamma \vdash A\_\mathrm{type}} \text{ Unform Function 1} \tag{48}$$

$$\frac{\Gamma \vdash A \to B\_\mathrm{type}}{\Gamma \vdash B\_\mathrm{type}} \text{ Unform Function 2} \tag{49}$$

These rules can be expressed more concisely by using Mclarty's double line notation.

The following optional rule may be useful in some circumstances (similar rules could be introducted to extract parts of terms involved in products and sums).

$$\frac{\Gamma \vdash (\lambda x : A)\,.b : A \to B}{\Gamma \vdash b : B} \text{ Function Extract} \tag{50}$$

I am not sure if the above rule can be derived in this theory.

In a similar way we may also wish to include the following rules

$$\frac{\Gamma \vdash a : A}{\Gamma \quad \mathrm{ctx}} \text{ Context Extract1} \tag{51}$$

$$\frac{\Gamma \vdash A\_\mathrm{type}}{\Gamma \quad \mathrm{ctx}} \text{ Context Extract2} \tag{52}$$

Context Extract 1 can be derived using Has Type followed by Context Extract 2.

Maybe there is no need for Context Extract 2 because we have Context Extension.

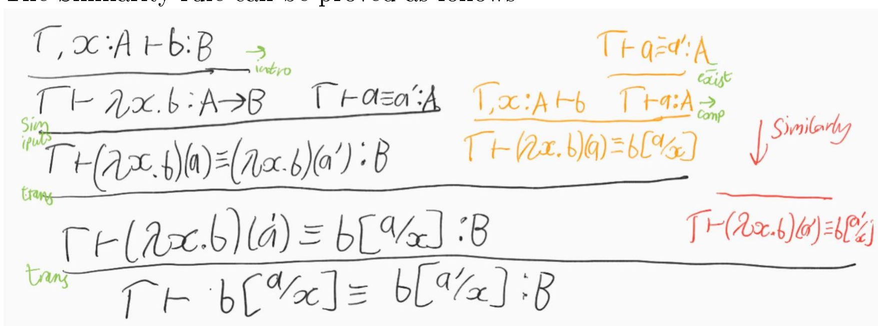
11 Consequences

11.1 Similarity

Using Equal Existence (to simplify) we may derive the following rule

$$\frac{\Gamma : a \equiv a' : A \quad \Gamma, x : A \vdash b : B}{\Gamma \vdash b[a/x] \equiv b[a'/x] : B} \text{ Similarity} \tag{53}$$

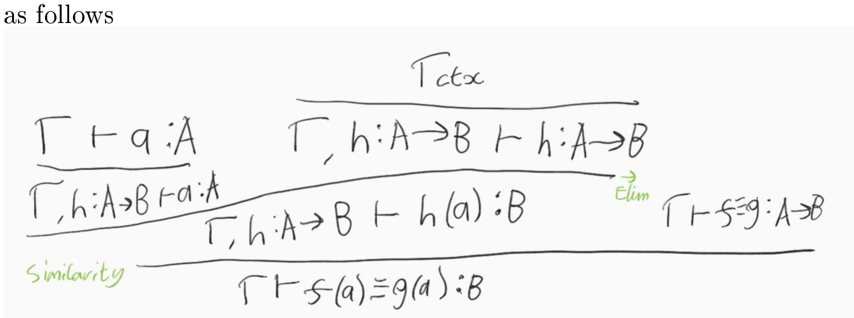
where x is free for a and a' in b.
The Similarity rule can be proved as follows



Using the Similarity rule one may derive the following rule

$$\frac{\Gamma \vdash f \equiv g : A \to B \quad \Gamma \vdash a : A}{\Gamma \vdash f(a) \equiv g(a) : B} \text{ Function-Similar Functions 2} \tag{54}$$

as follows



11.2 Transposition With Function Types

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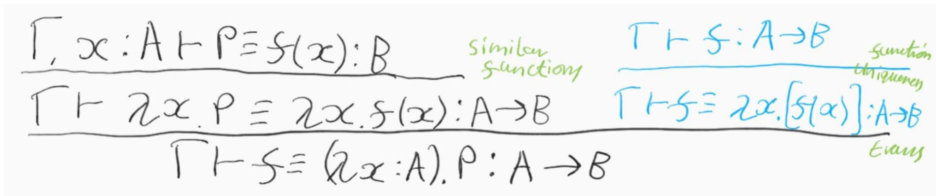
Using Similar Functions 2 (and other previous results) we can now obtain the follows two important results about how function types work

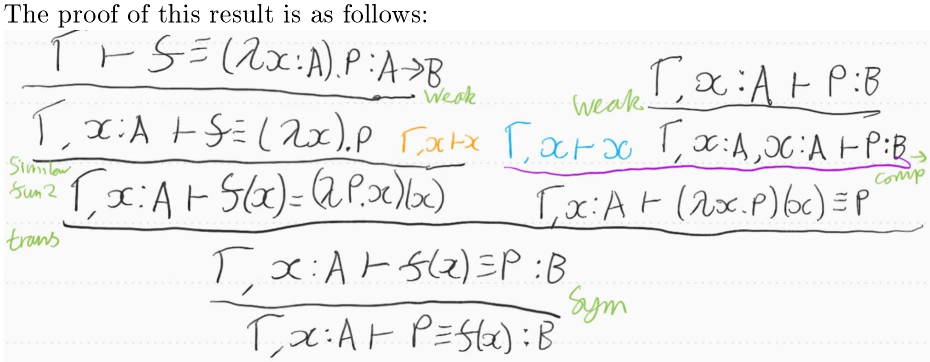
$$\frac{\Gamma \vdash f : A \to B \quad \Gamma :: (x : A) \vdash p \equiv f(x) : B}{\Gamma \vdash f \equiv (\lambda x : A) . p : A \to B} \text{ Curry} \tag{55}$$

The proof of this result is as follows:



$$\frac{\Gamma :: (x : A) \vdash p : B \quad \Gamma \vdash f \equiv (\lambda x : A)\,.p : A \to B}{\Gamma :: (x : A) \vdash p \equiv f(x) : B}\ \text{Uncurry} \tag{56}$$

The proof of this result is as follows:



11.3 Componentwise equal terms in products

We propose

$$\frac{\Gamma \vdash p : A \times B \quad \Gamma \vdash a \equiv \mathrm{fst}(p) : A \quad \Gamma \vdash b \equiv \mathrm{snd}(p) : B}{\Gamma \vdash \langle a, b \rangle \equiv p : A \times B}\ \text{Equal pair} \tag{57}$$

The proof is as follows

$$
\dfrac{
\dfrac{
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\dfrac{\dfrac{\Gamma \vdash a : A}{\Gamma, y \vdash a : A}\,\text{weak} \quad \dfrac{}{\Gamma, y \vdash y : B}}{\Gamma, y : B \vdash \langle a, y \rangle : A \times B}\,\times\text{ intro} \quad \Gamma \vdash b \equiv \mathrm{snd}(p)
}{\Gamma \vdash \langle a, b \rangle \equiv \langle a, \mathrm{snd}(p) \rangle}\,\text{sim}
\quad
\dfrac{
\dfrac{\dfrac{}{\Gamma, x : A \vdash x} \quad \dfrac{\dfrac{\Gamma \vdash b \equiv \mathrm{snd}(p) : B}{\Gamma \vdash \mathrm{snd}(p) : B}\,\equiv\text{exist}}{\Gamma, x \vdash \mathrm{snd}(p) : B}\,\text{weak}}{\Gamma, x : A \vdash \langle x, \mathrm{snd}(p) \rangle}\,\times\text{ intro} \quad \Gamma \vdash a \equiv \mathrm{fst}(p)
}{\Gamma \vdash \langle a, \mathrm{snd}(p) \rangle \equiv \langle \mathrm{fst}(p), \mathrm{snd}(p) \rangle}\,\text{sim}
}{\Gamma \vdash \langle a, b \rangle \equiv \langle \mathrm{fst}(p), \mathrm{snd}(p) \rangle}\,\text{trans}
\quad
\dfrac{\Gamma \vdash p : A \times B}{\Gamma \vdash p \equiv \langle \mathrm{fst}(p), \mathrm{snd}(p) \rangle}\,\times\text{ uniq}
}{\Gamma \vdash \langle a, b \rangle \equiv p : A \times B}\,\text{trans}
$$

We also propose

$$
\frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B \quad \Gamma \vdash \langle a, b \rangle \equiv p : A \times B}{\Gamma \vdash a \equiv \mathrm{fst}(p) : A} \text{ Equal first} \tag{58}
$$

The proof is as follows

$$
\dfrac{
\dfrac{\Gamma \vdash a : A \quad \Gamma \vdash b : B}{\Gamma \vdash a \equiv \mathrm{fst}(\langle a, b \rangle)}\,\times\text{ comp 1}
\quad
\dfrac{\dfrac{\dfrac{}{\Gamma, z : A \times B \vdash z : A \times B}}{\Gamma, z : A \times B \vdash \mathrm{fst}(z) : A}\,\times\text{ elim 1} \quad \Gamma \vdash \langle a, b \rangle \equiv p}{\Gamma \vdash \mathrm{fst}(\langle a, b \rangle) \equiv \mathrm{fst}(p)}\,\text{sim}
}{\Gamma \vdash a \equiv \mathrm{fst}(p) : A}\,\text{trans}
$$

We also propose

$$
\frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B \quad \Gamma \vdash \langle a, b \rangle \equiv p : A \times B}{\Gamma \vdash b \equiv \mathrm{snd}(p) : B} \text{ Equal second} \tag{59}
$$

The proof of this is similar to that of Equal first.

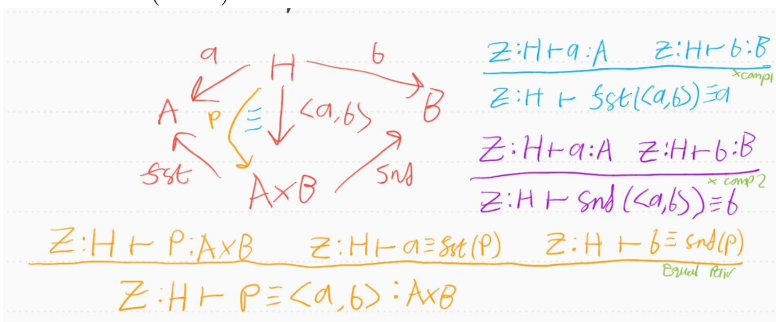
11.4 Universality of Product

The first projection arrows is given by using product elimination rule 1

$$
\dfrac{\dfrac{}{v : A \times B \vdash v : A \times B}}{v : A \times B \vdash \mathrm{fst}(v) : A}\,\times\text{ Elim1}
$$

the second projection arrow is given similarly.

The notion that pairing and projection undo each other to make the red part of the diagram commute can be obtained by using product computation rules 1 and 2 with $\Gamma = (z : H)$

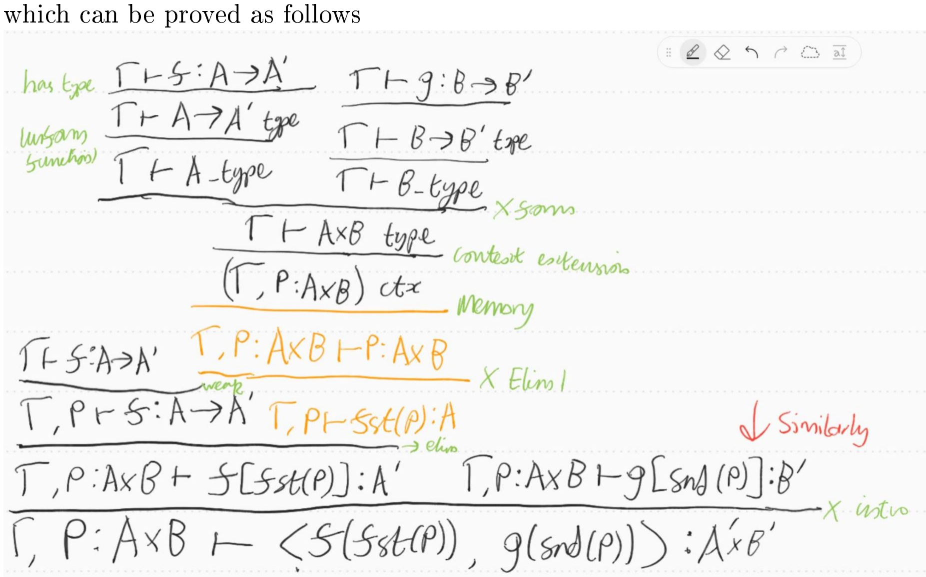


The uniqueness of $\langle a, b\rangle$ (that is, the fact that any $H \xrightarrow{p} A \times B$ such that $\mathrm{fst} \circ p = a$ and $\mathrm{snd} \circ p = b$ must be $p = \langle a, b\rangle$) can be established using Equal Pair with $\Gamma = (z : H)$.

We can form the product $(f \times g) : A \times B \to A' \times B'$ using the rule

$$\frac{\Gamma \vdash f : A \to A' \quad \Gamma \vdash g : B \to B'}{\Gamma, p : A \times B \vdash \langle f\,(\mathrm{fst}(p))\,, g\,(\mathrm{snd}(p))\rangle : A' \times B'} \text{ Multiply functions} \tag{60}$$

which can be proved as follows



11.5 The Many Forms of Judgements

The notation $x_1 : A_1, x_2 : A_2, \ldots, x_n : A_n \vdash d : D$ can always be considered to have several equivalent meanings. We illustraite this with the following rules for the $n = 2$ case:

$$\frac{\Gamma, x : A, y : B \vdash d : D}{\Gamma \vdash (\lambda x : A)\,[(\lambda y : B)\,.d] : A \to (B \to D)} \text{ Judgement to functions} \tag{61}$$

proof

$$\dfrac{\dfrac{\Gamma, x:A, y:B \vdash d:D}{\Gamma, x:A \vdash (\lambda y:B).d : B \to D}\rightarrow \text{intro}}{\Gamma \vdash (\lambda x:A).[(\lambda y:B).d] : A \to (B \to C)}\rightarrow \text{intro}$$

$$\frac{\Gamma \vdash k : A \to (B \to D)}{\Gamma, x : A, y : B \vdash (k(x))\,(y) : D} \text{ Functions to judgements} \tag{62}$$

proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash k : A \to (B \to D)}{\Gamma, x \vdash k : A \to (B \to D)}\text{weak} \quad \dfrac{}{\Gamma, x:A \vdash x:A}}{\Gamma, x \vdash k(x) : B \to D}\rightarrow \text{Elim}}{\Gamma, x:A, y:B \vdash k(x) : B \to D}\text{weak} \quad \dfrac{}{\Gamma, x:A, y:B \vdash y:B}}{\Gamma, x:A, y:B \vdash (k(x))(y) : D}\rightarrow \text{elim}$$

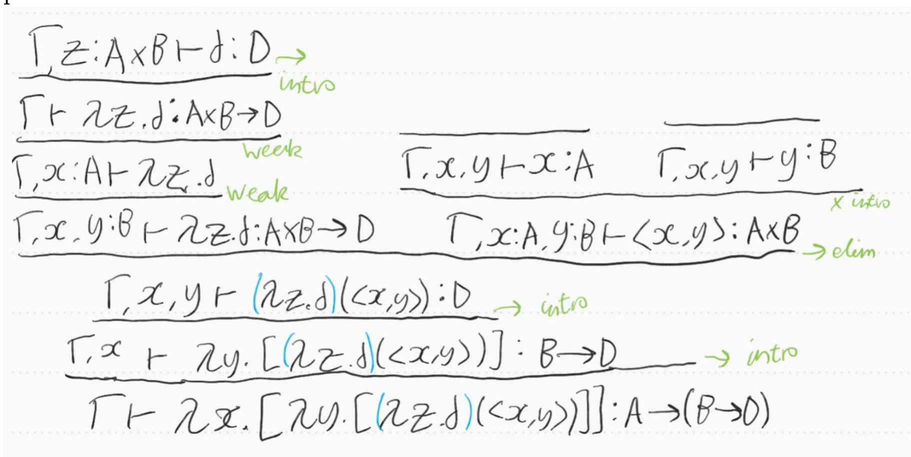
$$\frac{\Gamma \vdash k : A \to (B \to D)}{\Gamma, z : A \times B \vdash (k\,[\mathrm{fst}(z)])\,(\mathrm{snd}(z)) : D} \text{ Functions to pair function} \tag{63}$$

proof

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, z \vdash z : A \times B}}{\Gamma, z \vdash \mathrm{fst}(z) : A}\times\text{elim1} \quad \dfrac{\Gamma \vdash k : A \to (B \to D)}{\Gamma, z \vdash k : A \to (B \to D)}\text{weak}}{\Gamma, z \vdash k[\mathrm{fst}(z)] : B \to D}\rightarrow\text{elim} \quad \dfrac{\dfrac{}{\Gamma, z \vdash z}}{\Gamma, z \vdash \mathrm{snd}(z) : B}\times\text{elim2}}{\Gamma, z \vdash (k[\mathrm{fst}(z)])(\mathrm{snd}(z)) : D}\rightarrow\text{elim}$$

$$\frac{\Gamma, z : A \times B \vdash d : D}{\Gamma \vdash (\lambda x : A)\,.\,[(\lambda y : B)\,.\,[((\lambda z : A \times B)\,.d)\,(\langle x, y\rangle)]] : A \to (B \to D)} \text{ Pair function to functions} \tag{64}$$

proof



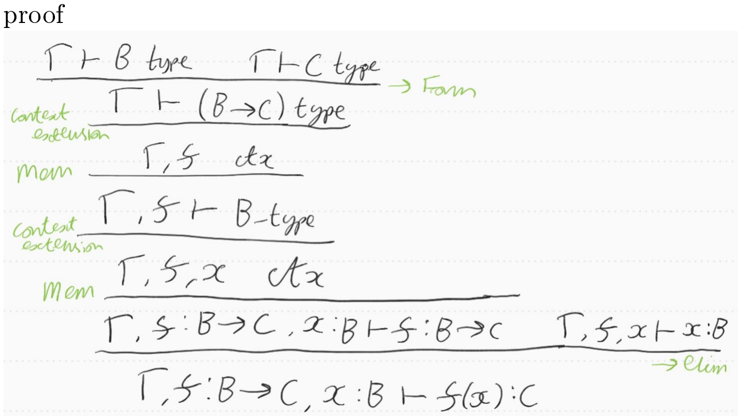
Note that many of these conversions could also be performed by inline functions (for example the isomorphism from $\left(D^B\right)^A$ and $D^{A\times B}$).

11.6 Properties of Functions

Note $B \to C$ can also be written as C^B. The evaluation arrow $C^B \times B \xrightarrow{e} C$ can be expressed via the rule

$$\frac{\Gamma \vdash B\_\mathrm{type} \quad \Gamma \vdash C\_\mathrm{type}}{\Gamma, f : B \to C, x : B \vdash f(x) : C}\ \text{Evaluation} \tag{65}$$

proof



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The proofs are as follows***

Identity function

Identity law

An alternative interpretation of entailment notation

Function Composition

Converting between $A \to (B \to C)$ and $(A \times B) \to C$.
Universality of function types
univerality of initial, terminal and coproduct
points. terminal objects and elements
raising arrows to powers
Lawvere's lambda

12 Further Directions

1. It would be good to show that we can prove all the categorical machinery (terminal object, products, initial object, coproducts, exponential objects) can be produced, and that rules expressing their universal properties can be derived.

2. If we code up the above theory, we just seem to have to keep track of dependencies (like how $\langle a, b \rangle$ depends on a and b), and some label, and the lists of free and bound variables, and which type a term has. We could imagine all this as part of the data structure of the term, and then terms start to look a lot more like objects in a category of presheaves, or like algebras (W types). This suggests we could abstract this theory, and open up the possibility of categorical analysis of the meta theory.

3. We may want to add natural number types, like in [9].

4. We may want to add equalizers, subobject classifiers and reference to monomorphisms, so we can do topos theory.

5. The theory above allows us to model any bicartesian closed category [1]. We can model **Cat** by adding appropriate axioms, as described in [5].

Do example of making an element of 1 times A

read how substitution is treated

prove exponential properties

Maybe we should use De-Brujin Notation. !!!

Should we also add the idea that equal things can replace reach other, like we can get from Mclarty/Bell ?

Await answer

Add proofs

Finish exponentials

Note cut seems optional

Add equalizer game

Show product universality

Show exponential universality

References

[1] *Extensional Normalisation and Type-Directed Partial Evaluation for Typed Lambda Calculus with Sums* Vincent Balat, Roberto Di Cosmo, Marcelo Fiore

[2] Homotopy Type Theory Book

[3] ncatlab type theory page ncatlab type theory

[4] Video https://www.youtube.com/watch?v=KHteAK7GSRY

[5] *Elementary categories, elementary toposes* Colin McLarty

[6] stack exchange website math stack exchange

[7] ncatlab products product types

[8] FUNCTIONAL PEARLS alpha-conversion is easy THORSTEN ALTENKIRCH

[9] *Introduction to higher order categorical logic* Lambek and Scott

[10] Practical Foundations of Mathematics, Chapter 2, Paul Taylor