INPUTS AND PRODUCTION FUNCTIONS

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THEORY OF THE FIRM

Explanation of how a firm makes cost minimizing production decisions and how its cost varies with its output.

Three steps to describing firm behavior

- Production Technology
- Cost constraints
- Input choices

2

FACTORS OF PRODUCTION

Inputs into the production process…

Labor
- Skilled workers
- Unskilled workers
- Entrepreneurial efforts

Capital
- Land
- Buildings
- Machinery
- Inventories

Materials
- Steel
- Plastics
- Electricity
- Water

3

THE PRODUCTION FUNCTION

Function showing the maximum quantity of output that a firm can produce for every specified combination of inputs

Relates the quantity of output (Q) to the quantities of two inputs, labor (L) and capital (K).

$$Q = f(L, K)$$

Describes how many units can be produced when the firm operates **efficiently** given a **fixed technology**.

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TOTAL PRODUCT CURVE (or hill)

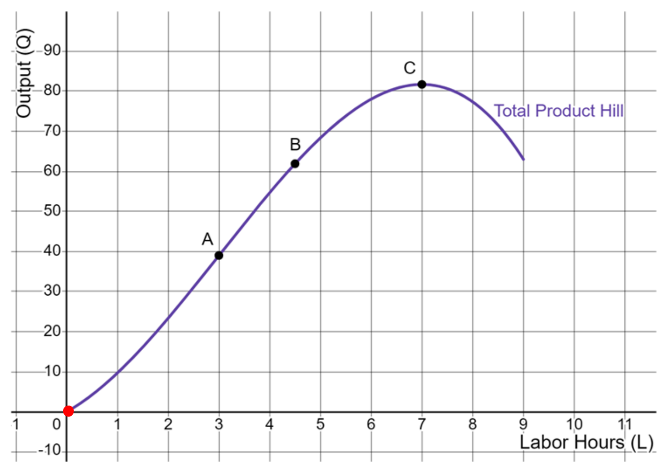
Shows how total output depends on the level of a single input.

For Example…

$$Q(L) = 7L + 3L^2 - \frac{L^3}{3}$$

At $L = 0$ output also equals 0.

No handmade pottery can be produced without using some labor



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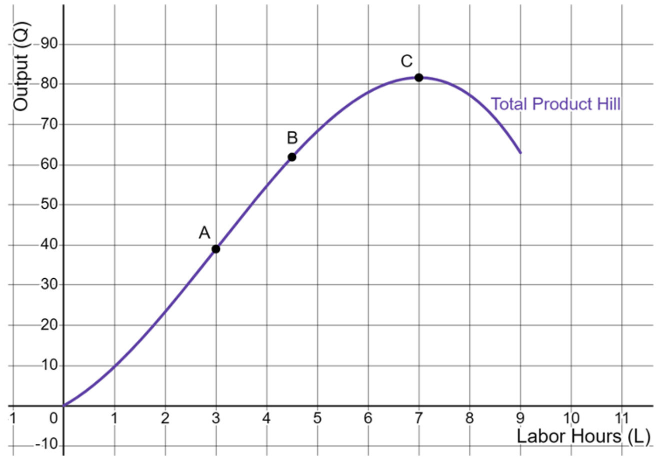

From 0 to A output rises with additional labor at an increasing rate.

Increasing Marginal Returns to Labor

$$\frac{dQ}{dL} > 0, \qquad \frac{d^2Q}{dL^2} > 0$$

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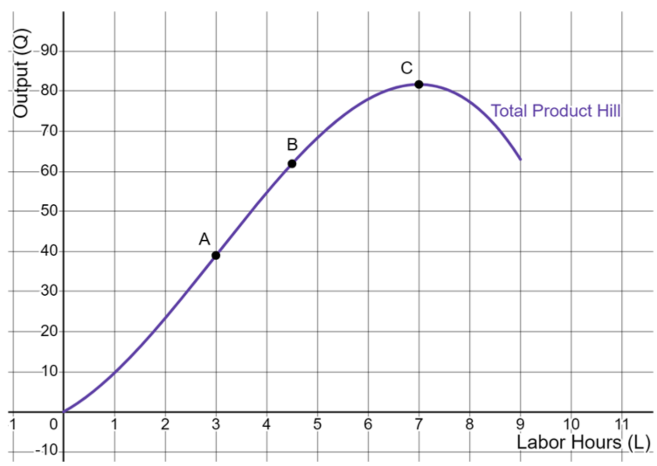

From A to C output rises with additional labor at a decreasing rate.

Diminishing Marginal Returns to Labor

$$\frac{dQ}{dL} > 0, \qquad \frac{d^2Q}{dL^2} < 0$$

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TOTAL PRODUCT FUNCTION (or hill)

Shows how total output depends on the level of a single input.

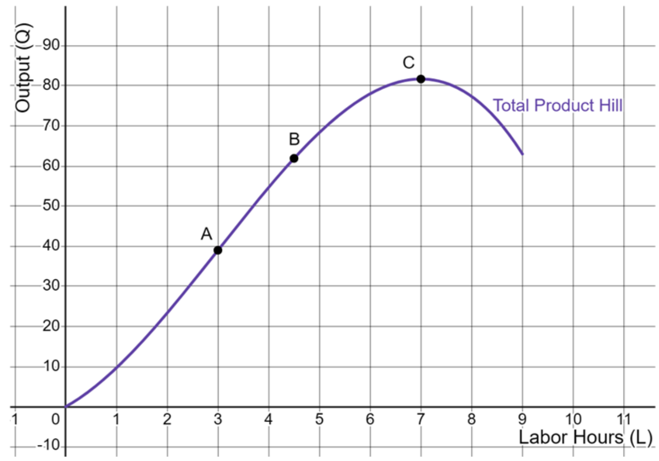
For Example…

$$Q(L) = 7L + 3L^2 - \frac{L^3}{3}$$

After C output falls with additional labor.

Diminishing Total Returns to Labor

$$\frac{dQ}{dL} < 0$$



A QUICK DEMONSTRATION

Consider a different production function…

$$Q(L) = 45L^2 - 3L^3$$

…solve for the range of labor hours under which the firm will experience

1. Increasing marginal returns to labor, $\boxed{\text{If } L < 5}$
2. Diminishing marginal returns to labor, and
3. Diminishing total returns to labor.

Compute $\frac{dQ}{dL}$ and $\frac{d^2Q}{dL^2}$

$\frac{dQ}{dL} = 90L - 9L^2$ $\frac{d^2Q}{dL^2} = 90 - 18L$

Increasing Marginal Returns if…

$$\frac{dQ}{dL} = 90L - 9L^2 > 0$$

$90L > 9L^2$

$10 > L$

and…

$$\frac{d^2Q}{dL^2} = 90 - 18L > 0$$

$90 > 18L$

$5 > L$

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Diminishing Total Returns if…

$$\frac{dQ}{dL} = 90L - 9L^2 < 0$$

$90L < 9L^2$

$10 < L$

AVERAGE AND MARGINAL PRODUCT OF LABOR

Average product

Output per unit of a particular input

$$AP_L = \frac{Q}{L}$$

Marginal Product

Additional output produced as an input is increased by one unit

$$MP_L = \frac{\Delta Q}{\Delta L} = \frac{dQ}{dL}$$

For Example…

$$Q(L) = 7L + 3L^2 - \frac{L^3}{3}$$

$\boldsymbol{AP_L =}$

$\boldsymbol{MP_L =}$

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RELATIONSHIP BETWEEN AVERAGE AND MARGINAL PRODUCTS

When $MP_L > AP_L$

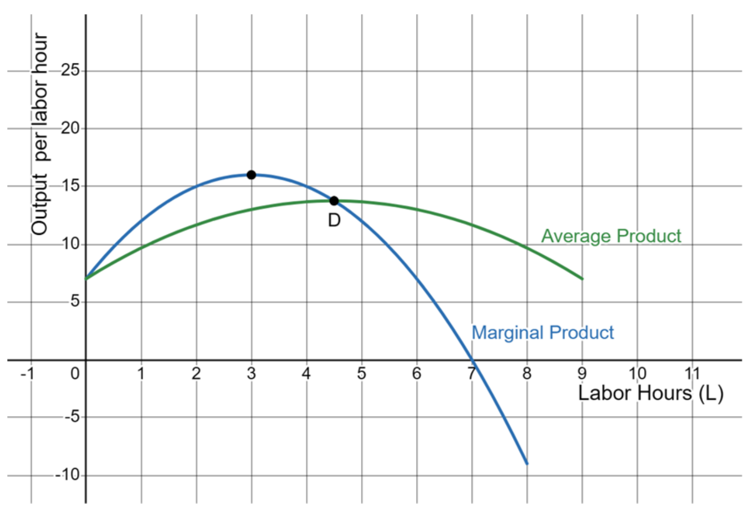
AP_L is increasing

When $MP_L < AP_L$

AP_L is decreasing

When $MP_L = AP_L$

AP_L is at its maximum value



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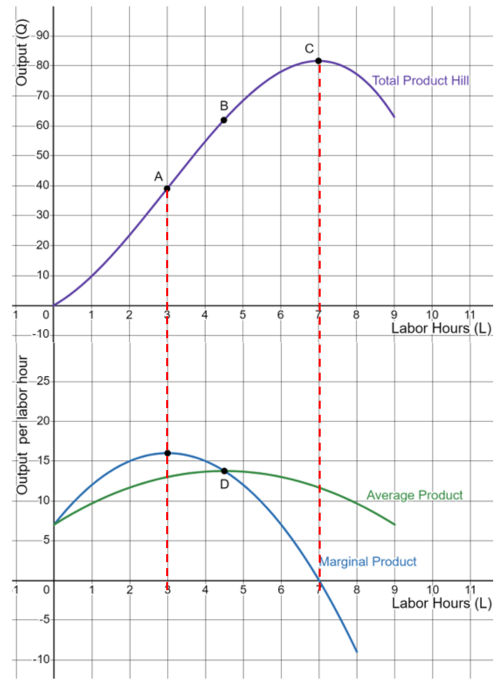
RELATIONSHIP BETWEEN PRODUCTION FUNCTION AND MARGINAL PRODUCT

Marginal product curve is rising (positive slope) when there are increasing marginal returns

Marginal product curve is falling (negative slope) when there are diminishing marginal returns or diminishing total returns

Marginal product is negative when there are diminishing total returns



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LAW OF DIMINISHING MARGINAL RETURNS

…As the use of an input increases with other inputs fixed, the marginal product of that variable input will eventually begin to decrease.

When labor input is small, extra labor increases output significantly as workers begin to specialize.

At some point there are too many workers and some workers become ineffective and marginal product of labor falls.

It may even become negative beyond a certain amount of labor.

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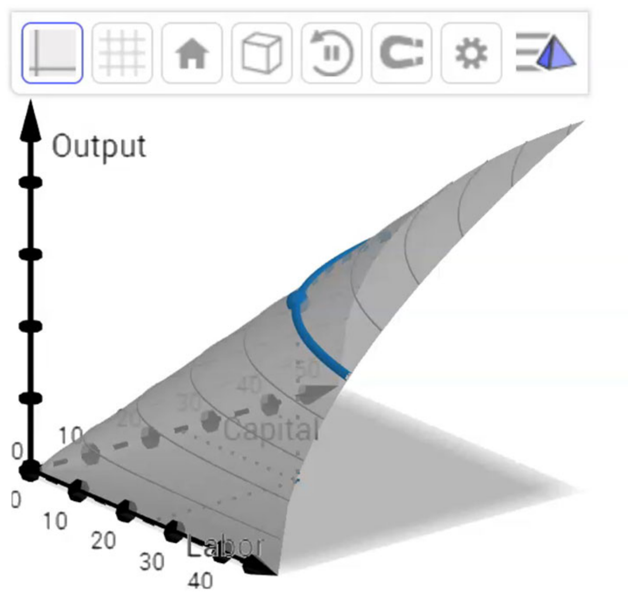
PRODUCTION WITH TWO INPUTS

TOTAL PRODUCT HILL

A three-dimensional graph of a production function.

Example…

$$Q = L^{0.5} K^{0.5}$$



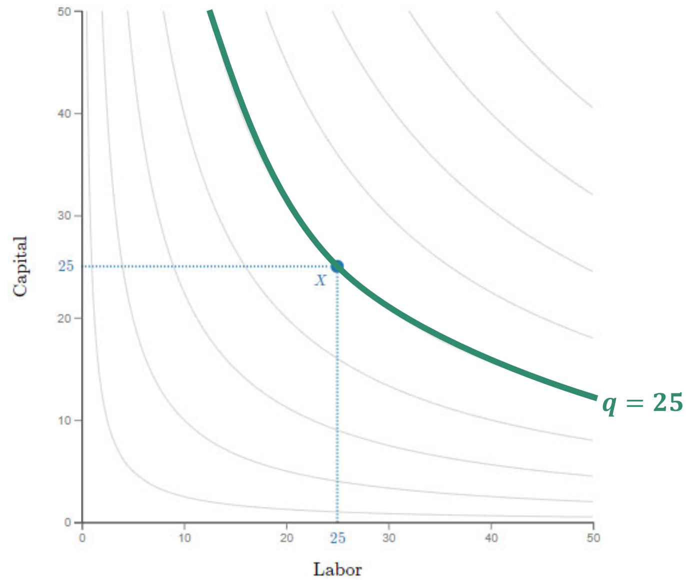
ISOQUANTS

Curve showing all possible combinations of inputs that yield the same output

Example…

$$Q = L^{0.5}K^{0.5}$$

Find the isoquant for $q = 25$



MARGINAL PRODUCTS (of labor and capital)

Marginal product of labor

Measures how productivity (output) changes as labor increases, holding capital constant

$$MP_L = \left.\frac{\Delta Q}{\Delta L}\right|_{K\ is\ constant} = \frac{\partial Q(L,K)}{\partial L}$$

Marginal product of capital

Measures how productivity (output) changes as capital increases, holding labor constant

$$MP_K = \left.\frac{\Delta Q}{\Delta K}\right|_{L\ is\ constant} = \frac{\partial Q(L,K)}{\partial K}$$

Marginal products should be

1. positive to reflect that output is increasing in L or K
2. diminishing as more L or K is employed.

MARGINAL PRODUCTS (of labor and capital)

Example…

$$Q = L^{0.5}K^{0.5}$$

1. Find the MP_L and MP_K
2. Demonstrate that this production function exhibits diminishing marginal returns in both inputs.

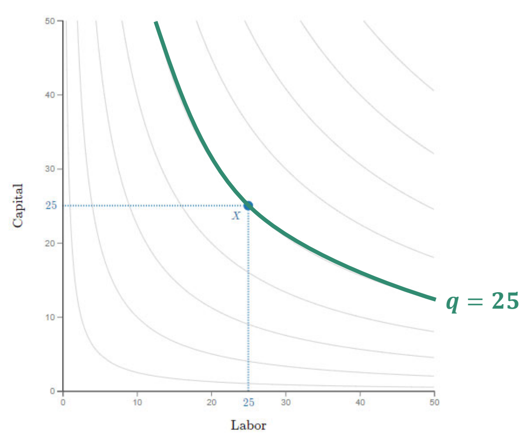
MARGINAL RATE OF TECHNICAL SUBST. (of L for K)

Amount by which the quantity of capital can be reduced when one extra unit of labor is used, holding the quantity of output constant.

Equal to the magnitude of the slope of the isoquant…

$$MRTS_{L,K} = -\frac{\Delta K}{\Delta L} = -\frac{dK}{dL}$$

$MRTS_{L,K}$ is diminishing…slope flattens as more L is hired and less K is employed.



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How do we find the MRTS??

$$MRTS_{L,K} = -\frac{dK}{dL} = \frac{MP_L}{MP_K}$$

As the firms gives up K to hire more L…

MP_L falls and MP_K increases.

So, the **MRTS is diminishing**.

MARGINAL PRODUCT AND $MRTS_{L,K}$

Example…

$$Q = L^{0.5}K^{0.5}$$

$$MP_L = 0.5K^{0.5}L^{-0.5}$$

$$MP_K = 0.5L^{0.5}K^{-0.5}$$

1. Find the $MRTS_{L,K}$.
2. Demonstrate that this production function exhibits diminishing $MRTS_{L,K}$.

SPECIAL PRODUCTION FUNCTIONS

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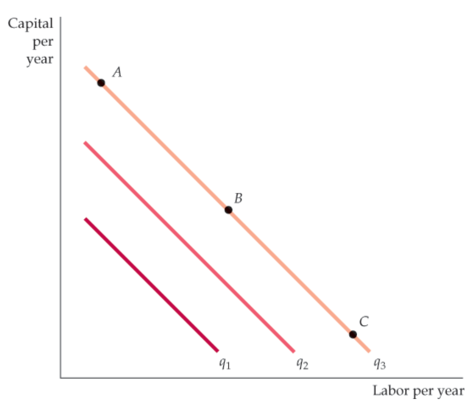
LINEAR PRODUCTION FUNCTIONS

Perfect substitutes



Properties of note…

1. MP_L and MP_K are positive and constant.
2. $MRTS_{L,K}$ is constant
3. Weakly convex isoquants

Example:

$$Q = \alpha L + \beta K$$

Where α and β are positive constants

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FIXED-PROPORTION PRODUCTION FUNCTION

Properties of note…

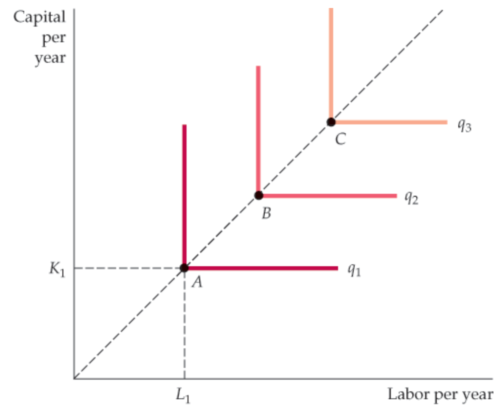
1. $MP_L = 0$ if $\alpha L > \beta K$
2. $MP_K = 0$ if $\beta K > \alpha L$
3. $MRTS_{L,K}$ is either 0 or ∞.
4. Weakly convex isoquants

Example:

$$Q = \min\{\alpha L, \beta K\}$$

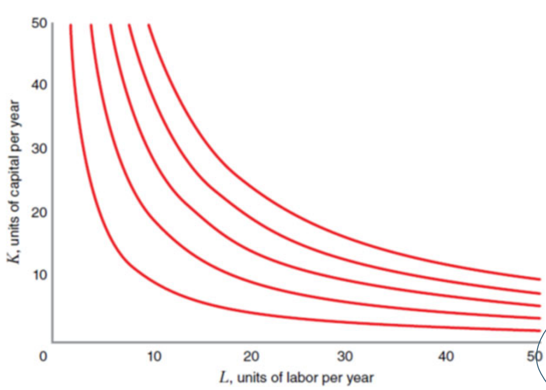
Where α and β are positive constants

Perfect Complements



COBB-DOUGLAS PRODUCTION FUNCTION

Cobb-Douglas



Properties of note…

1. MP_L and MP_K are positive and diminishing
2. $MRTS_{L,K}$ is diminishing
3. Strictly convex isoquants

Example:

$$Q = AK^{\alpha}L^{\beta}$$

Where A, α, and β are positive constants

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RETURNS TO SCALE

Rate at which output increases as inputs are increased proportionally

Increasing returns to scale

- When all inputs are scaled up by some amount $\lambda > 1$, output scales by more than λ.

Constant returns to scale

- When all inputs are scaled up by some amount $\lambda > 1$, output scales by λ.

Decreasing returns to scale

- When all inputs are scaled up by some amount $t\lambda > 1$, output scales by less than λ.

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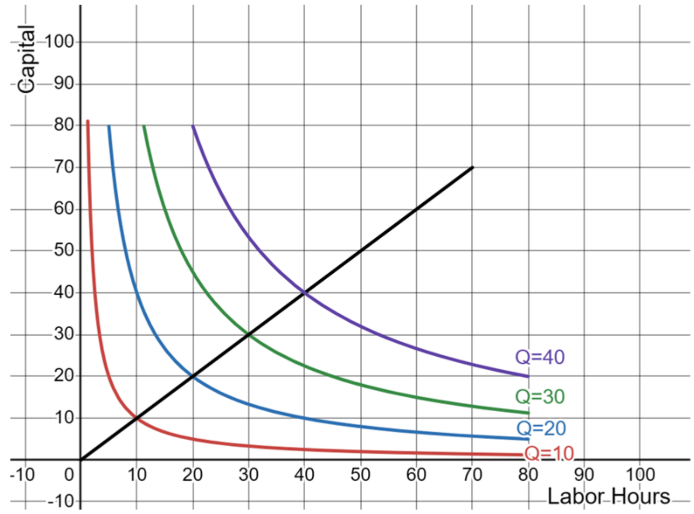
RETURNS TO SCALE (graphically)

Constant

If we **double** the inputs, then output doubles

if we **triple** the inputs, the output triples

Isoquants are equally spaced



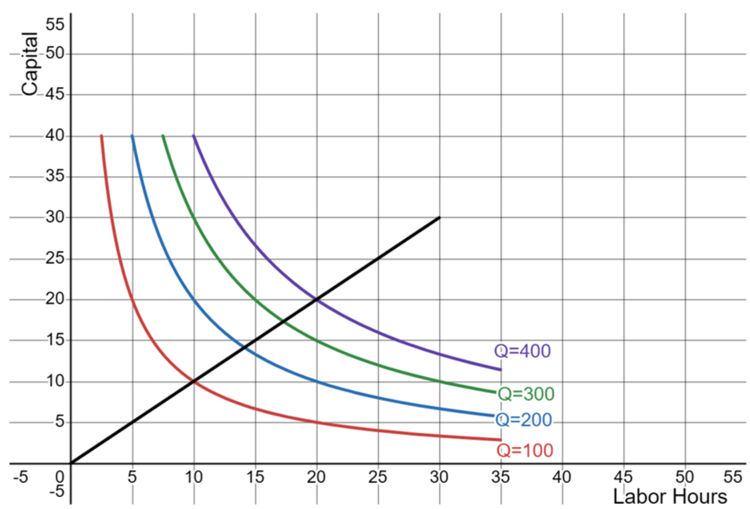
RETURNS TO SCALE (graphically)

Increasing

If we **double** the inputs, then output more than doubles.

In this case we only need to increase the inputs from 10 to 14 to double the output.

Isoquants are getting closer together as Q increases



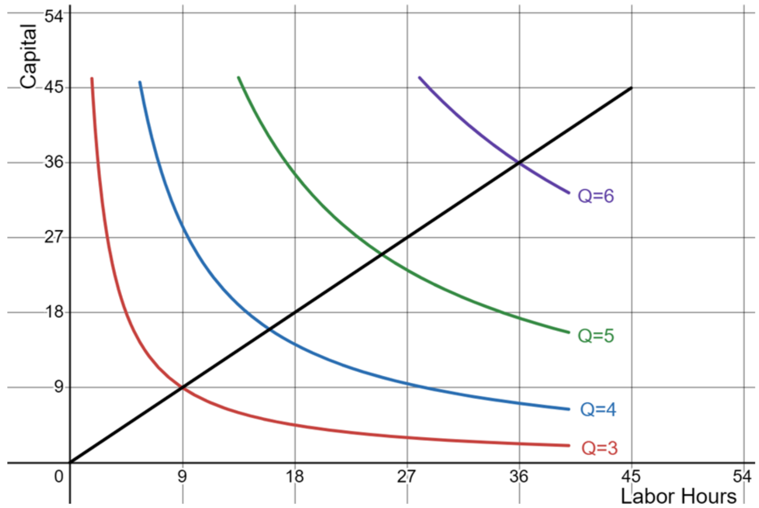
RETURNS TO SCALE (graphically)

Decreasing

If we **double** the inputs, then output less than doubles.

In this case we need to <u>quadruple</u> the inputs, to <u>double</u> the outputs.

Isoquants are getting farther apart as Q increases



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RETURNS TO SCALE (mathematically)

Determine whether the following production function exhibits increasing, decreasing, or constant returns to scale.

1. $Q = 15K^{0.5}L^{0.4}$

Define: $\boldsymbol{Q_1 = 15K_1^{0.5}L_1^{0.4}}$

Suppose: $K_2 = \lambda K_1$ and $L_2 = \lambda L_1$ $(\lambda > 1)$

How does Q_1 compare to Q_2?

$Q_2 = 15K_2^{0.5}L_2^{0.4}$

$Q_2 = 15(\lambda K_1)^{0.5}(\lambda L_1)^{0.4}$

$Q_2 = 15\lambda^{0.5}K_1^{0.5}\lambda^{0.4}L_1^{0.4}$

$Q_2 = \lambda^{0.9}\underbrace{15K_1^{0.5}L_1^{0.4}}_{Q_1}$

$\boldsymbol{Q_2 = \lambda^{0.9}Q_1 < \lambda Q_1}$

Scaled both inputs by λ, output scaled by less than λ
decreasing returns to scale

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RETURNS TO SCALE (mathematically)

Determine whether the following production function exhibits increasing, decreasing, or constant returns to scale.

1. $Q = 15K^{0.5}L^{0.4}$
2. $Q = min\{3K, 4L\}$

Suppose that…

$$L_1 = 1, \qquad K_1 = 1$$
$$Q_1 = min\{3 \times 1, 4 \times 1\}$$
$$Q_1 = \min\{3, 4\} = 3$$

Now double the inputs and observe how output changes…

$$L_2 = 2, \qquad K_2 = 2$$
$$Q_2 = \min\{3 \times 2, 4 \times 2\}$$
$$Q_2 = \min\{6, 8\} = 6$$

Doubled both inputs, output doubled in response
constant returns to scale