UTILITY MAXIMIZATION (A Calculus Approach)

Let's start by restating our consumer choice problem

Suppose George buys two goods, x and y, where prices per unit are P_x and P_y.

George has a utility function over x and y of $u(x, y)$.

His income is I.

George's goal is to

Maximize $u(x, y)$

Subject to: $P_x x + P_y y = I$

How should we solve this problem?

- One option: find optimal ratio using tangency ($\frac{MU_x}{MU_y} = \frac{P_x}{P_y}$), then plug into the constraint.
- A second (potentially better) option: use the Lagrange multiplier method.

LAGRANGE MULTIPLIER METHOD

Technique to maximize or minimize a function subject to one or more constraints.

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Maximize $u(x, y)$

Subject to: $P_x x + P_y y = I$

Step 1: Form the Lagrangian

L = objective function - λ(constraint solve for zero)

Function we are seeking to maximize (or minimize).

$u(x, y)$

Budget line, solved to be equal to zero (rather than I).

$P_x x + P_y y - I$

$$\boldsymbol{\mathcal{L} = u(x, y) - \lambda(P_x x + P_y y - I)}$$

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1. Lagrangian:

$$\mathcal{L} = u(x, y) - \lambda(P_x x + P_y y - I)$$

Step 2: Differentiate the Lagrangian

(first order conditions for a maximum)

$$\frac{\partial \mathcal{L}}{\partial x} =$$

$$\frac{\partial \mathcal{L}}{\partial y} =$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} =$$

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1. Lagrangian

$$\mathcal{L} = u(x, y) - \lambda(P_x x + P_y y - I)$$

2. Derivatives

(1) $\frac{\partial \mathcal{L}}{\partial x} = MU_x - \lambda P_x = 0$

(2) $\frac{\partial \mathcal{L}}{\partial y} = MU_y - \lambda P_y = 0$

(3) $\frac{\partial \mathcal{L}}{\partial \lambda} = -(P_x x + P_y y - I) = 0$

Step 3: Solve the resulting equations

WHAT DOES λ TELL US?

Represents the extra utility generated when the budget constraint is relaxed…
…the **marginal utility of income**, also referred to as the **"shadow price"**.

From previous work we know that…

$$\lambda = \frac{MU_x}{P_x} = \frac{MU_y}{P_y}$$

So, we could say that…

$$MU_I = \lambda = \frac{MU_x}{P_x} = \frac{MU_y}{P_y}$$

A derivation of this result can be found on Canvas

I will never ask you about this derivation.

AN EXAMPLE USING THE LAGRANGE

Suppose that a consumer's utility function for two goods (x and y) is

$$u(x, y) = 10x^{0.5} + 2y$$

$P_x = \$5$

$P_y = \$10$

The consumer has total income of $175

Goal: Determine the optimal basket

1. **Write the statement of the constrained optimization problem.**

AN EXAMPLE USING THE LAGRANGE

Maximize $10x^{0.5} + 2y$

$s.t.\ 5x + 10y = 175$

2. **Use the Lagrangian to solve the utility maximization problem.**

$$\mathcal{L} = 10x^{0.5} + 2y - \lambda(5x + 10y - 175)$$

$\frac{\partial \mathcal{L}}{\partial x} =$

$\frac{\partial \mathcal{L}}{\partial y} =$

$\frac{\partial \mathcal{L}}{\partial \lambda} =$

AN EXAMPLE USING THE LAGRANGE

Suppose that a consumer's utility function for two goods (x and y) is

$$u(x, y) = 10x^{0.5} + 2y$$

$P_x = \$5$

$P_y = \$10$

The consumer has total income of $175

$\lambda = \frac{1}{5},\ x = 25,\ y = 5$

More ways to find MU_I

$MU_x = \frac{5}{\sqrt{x}} = \frac{5}{\sqrt{25}} = 1$ $\quad \frac{MU_x}{P_x} = \frac{1}{5}$

$MU_y = 2$ $\quad \frac{MU_y}{P_y} = \frac{2}{10} = \frac{1}{5}$

3. **Interpret the Lagrange multiplier.**

4. **What type of utility function is this?**