Applied Statistics

David Dalpiaz

2016-06-04

Contents

Chapter 1

Introduction

Welcome to Applied Statistics with `R`!

1.1 About This Book

This book was originally (and currently) designed for use with STAT 420, Methods of Applied Statistics, at the University of Illinois at Urbana-Champaign. It may certainly be used elsewhere, but you may find references to "this course" which is STAT 420.

This book is under active development. When possible, it would be best to always access the text online to be sure you are using the most up-to-date version. (Also the html version provides the most features such as changing text size, font, and colors.) If you are in need of a local copy, a pdf version is continously updated as well at http://daviddalpiaz.github.io/appliedstats/applied_statistics.pdf

Since this book is under active development you may encounter errors ranging from typos to broken code to poorly explained topics. If you do, please let us know! Simply send an email and we'll make the changes ASAP. Or if you know RMarkdown and are familiar with GitHub, make a pull request and fix an issue yourself!

TODO: Note about MathJax rendering.

1.2 Conventions

1.3 Acknowledgements

- Alex Stepanov
- David Unger
- James Balamuta

Chapter 2

Introduction to R

> "Measuring programming progress by lines of code is like measuring aircraft building progress by weight."
> - **Bill Gates**

R is both a programming language and software environment for statistical computing, which is *free* and *open-source*. To get started, you will need to install two pieces of software:

- R, the actual programming language, which can be installed from http://cran.r-project.org/
  - Chose your operating system, and select the most recent version. (As of writing, 3.3.0.)
- RStudio, an excellent IDE for working with R, which can be obtained from http://www.rstudio.com/ (Note, you must have R installed to use RStudio. RStudio is simply a way to interact with R.)

R's popularity is on the rise, and everyday it becomes a better tool for statistical analysis. It even generated this book! (A skill you will learn in this coruse.) There are many good resoruces for learning R. They are not necessary for this course, but you may find them useful if you would like a deeper understanding of R:

- Try R from Code School.
  - An interactive introduction to the basics of R. Could be very useful for getting up to speed on R's syntax.
- The Art of R Programming by Norman Matloff.
  - Gentle introduction to the programming side of R. (Whereas we will focus more on the data analysis side.) A free electronic version is available through the Illinois library.
- Advanced R by Hadley Wickham.
  - From the author of several extremely popular R packages. Good follow-up to The Art of R Programming. (And more up-to-date material.)
- The R Inferno by Patrick Burns.
  - Likens learning the tricks of R to descending through the levels of hell. Very advanced material, but may be important if R becomes a part of your everyday toolkit.

RStudio has a large number of useful keyboard shortcuts. A list of these can be found using a keyboard shortcut, the keyboard shortcut to rule them all:

- On Windows: `Option` + `Shift` + `K`
- On Mac: `Alt` + `Shift` + `K`

The RStudio team has developed a number of "cheatsheets" for working with both `R` and RStudio which can be found here or from the help menu inside of RStudio. This one for Base `R` in particular will summarize many of the concepts in this document.

2.1 Basic Calculations

To get started, we'll use `R` like a simple calculator. Note, in `R` the `#` symbol is used for comments. Lines which begin with two, `##` will indicate output.

- Addition, Subtraction, Multiplication and Division

```
3 + 2
```

```
## [1] 5
```

```
3 - 2
```

```
## [1] 1
```

```
3 * 2
```

```
## [1] 6
```

```
3 / 2
```

```
## [1] 1.5
```

- Exponents

```
3 ^ 2
```

```
## [1] 9
```

```
2 ^ (-3)
```

```
## [1] 0.125
```

```
100 ^ (1 / 2)
```

```
## [1] 10
```

```
sqrt(1 / 2)
```

```
## [1] 0.7071068
```

```
exp(1)
```

```
## [1] 2.718282
```

- Mathematical Constants

```
pi
```

```
## [1] 3.141593
```

```
exp(1)
```

```
## [1] 2.718282
```

- Logarithms

```
log(10) # natural log
```

```
## [1] 2.302585
```

```
log10(1000) # base 10 log
```

```
## [1] 3
```

```
log2(8) # base 2 log
```

```
## [1] 3
```

```
log(16, base = 4) # base 4 log
```

```
## [1] 2
```

- Trigonometry

```
sin(pi / 2)
```

```
## [1] 1
```

```
cos(0)
```

```
## [1] 1
```

2.2 Getting Help

In using `R` as a calculator, we have seen a number of functions. `sqrt()`, `exp()`, `log()` and `sin()` are all `R` functions. To get documentation about a function in `R`, simply put a question mark in front of the function name and RStudio will display the documentation, for example:

```
?log
?sin
?paste
?lm
```

Frequently one of the most difficult things to do when learning R is asking for help. First, you need to decide to ask for help, then you need to know how to ask for help. Your very first line of defense should be to google your error message or a short description of your issue. (The ability to solve problems using this method is quickly becoming an extremely valuable skill.) If that fails, and it eventually will, you should ask for help There are a number of things you should include when emailing an instructor, or posting to a help website such as http://stats.stackexchange.com/.

- Describe what you expect the code to do.
- State the end goal you are trying to acheive. (Sometimes what you expect the code to do, is not what you want to actually do.)
- The full text of any errors you have recieved.
- Provide enough code to recreate the error. Often for the purpose of this course, you could simply email your entire `.R` or `.Rmd` file.
- Sometimes it is also helpful to include a screenshot of your entire RStudio window when the error occurs.

If you follow these steps, you will get your issue resolved much quicker, and possibly learn more in the process.

2.3 Vectors

Many operations in R make heavy use of vectors. Vectors in R are indexed starting at 1. That is what the `[1]` in the output is indicating, that the first element of the row being displayed is the first element of the vector. Larger vectors will start additional rows with `[*]` where `*` is the index of the first element of the row.

Possibly the most common way to create a vector in R is using the `c()` function, which is short for combine. As the name suggests, it combines a list of numbers separated by commas.

```
c(1, 3, 5, 7, 8, 9)
```

```
## [1] 1 3 5 7 8 9
```

Here R simply outputs this vector. If we would like to store this vector in a **variable** we can do so with the assignment operator `=`. In this case the variable `x` now holds the vector we just created, and we can access the vector by typing `x`.

```
x = c(1, 3, 5, 7, 8, 9)
x
```

```
## [1] 1 3 5 7 8 9
```

As an aside, there is a long history of the assignment operator in R. For simplicity we will use `=`, but know that often you will see `<-` as the assignment operator. The pros and cons of these two are well beyond the scope of this book, but know that for our purposes you will have no issue if you simply use `=`.

Frequently you may wish to create a vector based on a sequence of numbers. The quickest and easiest way to do this is with the `:` operator, which creates a sequence of integers between two specified integers.

```
(y = 1:100)
```

```
##   [1]   1   2   3   4   5   6   7   8   9  10  11  12  13  14  15
##  [16]  16  17  18  19  20  21  22  23  24  25  26  27  28  29  30
##  [31]  31  32  33  34  35  36  37  38  39  40  41  42  43  44  45
##  [46]  46  47  48  49  50  51  52  53  54  55  56  57  58  59  60
##  [61]  61  62  63  64  65  66  67  68  69  70  71  72  73  74  75
##  [76]  76  77  78  79  80  81  82  83  84  85  86  87  88  89  90
##  [91]  91  92  93  94  95  96  97  98  99 100
```

Here we see `R` labeling the rows after the first since this is a large vector. Also, we see that by putting parentheses around the assignment, `R` both stores the vector in a variable called `y` and automatically outputs `y` to the console.

If we want to create a sequence that isn't limit to

TODO: fine control sequence TODO: function and arguemnts, etc. where to put?

```
(z = seq(from = 1, to = 2, by = 0.1))
```

```
##  [1] 1.0 1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8 1.9 2.0
```

TODO: add `rep`

```
rep(0.5, times = 10)
```

```
##  [1] 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5 0.5
```

```
rep(x, 3)
```

```
##  [1] 1 3 5 7 8 9 1 3 5 7 8 9 1 3 5 7 8 9
```

TODO: Accessing elements

```
x
```

```
## [1] 1 3 5 7 8 9
```

```
x[3]
```

```
## [1] 5
```

```
x[1:3]
```

```
## [1] 1 3 5
```

```
x[-2]
```

```
## [1] 1 5 7 8 9
```

One of the biggest strengths of `R` is its use of vectorized operations. (Frequently the lack of understanding of this concept leads of a belief that `R` is *slow*. `R` isn't the fastest language, but it has a reputation for being slower than it really is.)

```
x = 1:10
x + 1
```

```
##  [1]  2  3  4  5  6  7  8  9 10 11
```

```
2 * x
```

```
##  [1]  2  4  6  8 10 12 14 16 18 20
```

```
2 ^ x
```

```
##  [1]    2    4    8   16   32   64  128  256  512 1024
```

```
sqrt(x)
```

```
##  [1] 1.000000 1.414214 1.732051 2.000000 2.236068 2.449490
##  [7] 2.645751 2.828427 3.000000 3.162278
```

```
log(x)
```

```
##  [1] 0.0000000 0.6931472 1.0986123 1.3862944 1.6094379 1.7917595
##  [7] 1.9459101 2.0794415 2.1972246 2.3025851
```

We see that when a function like `log()` is called on a vector `x`, a vector is returned which has applied the function to each element of the vector `x`.

2.4 Matrix Calculations

`R` can also be used for matrix calculations. Matrices can be created using the `matrix` function.

TODO: matrix all same "data"" type. "order matters". has rows and columns

By default the `matrix` function reorders a vector into columns, but we can also tell `R` to use rows instead.

```
x = 1:9
x
```

```
## [1] 1 2 3 4 5 6 7 8 9
```

```
X = matrix(x, nrow = 3, ncol = 3)
X
```

```
##      [,1] [,2] [,3]
## [1,]    1    4    7
## [2,]    2    5    8
## [3,]    3    6    9
```

```
Y = matrix(x, nrow = 3, ncol = 3, byrow = TRUE)
Y
```

```
##      [,1] [,2] [,3]
## [1,]    1    2    3
## [2,]    4    5    6
## [3,]    7    8    9
```

```
Z = matrix(0, 2, 4)
Z
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    0    0    0    0
## [2,]    0    0    0    0
```

```
X
```

```
##      [,1] [,2] [,3]
## [1,]    1    4    7
## [2,]    2    5    8
## [3,]    3    6    9
```

```
X[1, 2]
```

```
## [1] 4
```

```
X[1, ]
```

```
## [1] 1 4 7
```

```
X[, 2]
```

```
## [1] 4 5 6
```

```
X[2, c(1, 3)]
```

```
## [1] 2 8
```

Matrices can also be created by combining vectors as columns, using `cbind` or combining vectors as rows using `rbind`.

```
x = 1:9
rev(x)
```

```
## [1] 9 8 7 6 5 4 3 2 1
```

```
rep(1, 9)
```

```
## [1] 1 1 1 1 1 1 1 1 1
```

```
cbind(x, rev(x), rep(1, 9))
```

```
##       x
##  [1,] 1 9 1
##  [2,] 2 8 1
##  [3,] 3 7 1
##  [4,] 4 6 1
##  [5,] 5 5 1
##  [6,] 6 4 1
##  [7,] 7 3 1
##  [8,] 8 2 1
##  [9,] 9 1 1
```

```
rbind(x, rev(x), rep(1, 9))
```

```
##   [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9]
## x    1    2    3    4    5    6    7    8    9
##      9    8    7    6    5    4    3    2    1
##      1    1    1    1    1    1    1    1    1
```

R can then be used to perform matrix calculations.

```
x = 1:9
y = 9:1
X = matrix(x, 3, 3)
Y = matrix(y, 3, 3)
X
```

```
##      [,1] [,2] [,3]
## [1,]    1    4    7
## [2,]    2    5    8
## [3,]    3    6    9
```

```
Y
```

```
##      [,1] [,2] [,3]
## [1,]    9    6    3
## [2,]    8    5    2
## [3,]    7    4    1
```

```
X + Y
```

```
##      [,1] [,2] [,3]
## [1,]   10   10   10
## [2,]   10   10   10
## [3,]   10   10   10
```

```
X - Y
```

```
##      [,1] [,2] [,3]
## [1,]   -8   -2    4
## [2,]   -6    0    6
## [3,]   -4    2    8
```

```
X * Y
```

```
##      [,1] [,2] [,3]
## [1,]    9   24   21
## [2,]   16   25   16
## [3,]   21   24    9
```

```
X / Y
```

```
##           [,1]      [,2]     [,3]
## [1,] 0.1111111 0.6666667 2.333333
## [2,] 0.2500000 1.0000000 4.000000
## [3,] 0.4285714 1.5000000 9.000000
```

Note that `X * Y` is not matrix multiplication. It is element by element multiplication. (Same for `X / Y`). Instead, matrix multiplication uses `%*%`. `t()` gives the transpose of a matrix, and `solve()` returns the inverse of a matrix.

```
X %*% Y
```

```
##      [,1] [,2] [,3]
## [1,]   90   54   18
## [2,]  114   69   24
## [3,]  138   84   30
```

```
t(X)
```

```
##      [,1] [,2] [,3]
## [1,]    1    2    3
## [2,]    4    5    6
## [3,]    7    8    9
```

```
Z = matrix(c(9, 2, -3, 2, 4, -2, -3, -2, 16), 3, byrow = T)
Z
```

```
##      [,1] [,2] [,3]
## [1,]    9    2   -3
## [2,]    2    4   -2
## [3,]   -3   -2   16
```

```
solve(Z)
```

```
##             [,1]        [,2]       [,3]
## [1,]  0.12931034 -0.05603448 0.01724138
## [2,] -0.05603448  0.29094828 0.02586207
## [3,]  0.01724138  0.02586207 0.06896552
```

```
X = matrix(1:6, 2, 3)
X
```

```
##      [,1] [,2] [,3]
## [1,]    1    3    5
## [2,]    2    4    6
```

```
dim(X)
```

```
## [1] 2 3
```

```
rowSums(X)
```

```
## [1]  9 12
```

```
colSums(X)
```

```
## [1]  3  7 11
```

```
rowMeans(X)
```

```
## [1] 3 4
```

```
colMeans(X)
```

```
## [1] 1.5 3.5 5.5
```

```
diag(Z)
```

```
## [1]  9  4 16
```

```
diag(1:5)
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    1    0    0    0    0
## [2,]    0    2    0    0    0
## [3,]    0    0    3    0    0
## [4,]    0    0    0    4    0
## [5,]    0    0    0    0    5
```

```
diag(5)
```

```
##      [,1] [,2] [,3] [,4] [,5]
## [1,]    1    0    0    0    0
## [2,]    0    1    0    0    0
## [3,]    0    0    1    0    0
## [4,]    0    0    0    1    0
## [5,]    0    0    0    0    1
```

2.5 Distributions

When working with different statistical distributions, we often want to make probabilistic statements based on the distribution.

We typically want to know one of four things:

- The density (pdf) value at a particular value of `x`.
- The distribution (cdf) value at a particular value of `x`.
- The quantile `x` value corresponding to a particular probability.
- A random value from a particular distribution.

This used to be done with statistical tables printed in the back of textbooks. Now, R has functions for obtaining density, distribution, quantile and random values.

The general naming structure of the relevant R functions is:

- `dname` calculates density (pdf) value at input `x`.
- `pname` calculates distribution (cdf) value at input `x`.
- `qname` calculates quantile `x` value at input probability.
- `rname` generates a random value from a particular distribution.

Note that `name` represents the name of the given distribution.

For example, to calculate the value of the pdf for a $N(2, 25)$ for `x = 3`, use:

```
dnorm(3, mean = 2, sd = 5)
```

```
## [1] 0.07820854
```

Or, to calculate the value of the cdf for a $N(2, 25)$ for `x = 3`, use:

```
pnorm(3, mean = 2, sd = 5)
```

```
## [1] 0.5792597
```

Or, to calculate the quantile for probability 0.975, use:

```
qnorm(0.975, mean = 2, sd = 5)
```

```
## [1] 11.79982
```

Lastly, to generate a random sample of size `n = 10`, use:

```
rnorm(10, mean = 2, sd = 5)
```

```
##  [1]  1.308002 -3.049038  3.010636 13.158754 -1.783378  2.956742
##  [7] -3.823563 -4.651155  3.569132  6.091301
```

These functions exist for many other distributions, including but not limited to:

Command	Distribution
*binom	Binomial
*t	t
*pois	Poisson
*f	F
*chisq	Chi-Squared

Where * can be d, p, q, and r.

2.6 Programming Basics

2.6.1 Logical Operators

Operator	Summary
<	Less than
>	Greater than
<=	Less than or equal to
>=	Greater than or equal to
==	Equal to
!=	Not equal to
!x	NOT x
x | y	x OR y
x & y	x AND y

In R, logical operators are vectorized.

```
heights = c(110, 120, 115, 136, 205, 156, 175)
weights = c(64, 67, 62, 60, 77, 70, 66)
heights < 121 | heights == 156
```

```
## [1]  TRUE  TRUE  TRUE FALSE FALSE  TRUE FALSE
```

```
weights[heights > 150]
```

```
## [1] 77 70 66
```

In R, the if/else syntax is:

```
if (...) {
  some R code
} else {
  more R code
}
```

For example,

```
x = 1
y = 3
if (x > y) {
  z = x * y
  print("x is larger than y")
} else {
  z = x + 5 * y
  print("x is less than or equal to y")
}
```

```
## [1] "x is less than or equal to y"
```

```
z
```

```
## [1] 16
```

TODO: ifelse

Now a `for` loop example,

```
x = 11:15
for (i in 1:5) {
  x[i] = x[i] + 1
}

x
```

```
## [1] 12 13 14 15 16
```

Note that this `for` loop is very normal in many programming languages, but not in `R`. In `R` we would not use a loop, instead we would simply use a vectorized operation:

```
x = 11:15
x = x + 1
x
```

```
## [1] 12 13 14 15 16
```

Lastly, we can write our own functions in `R`. For example,

```
standardize = function(x) {
  m = mean(x)
  std = sd(x)
  result = (x - m) / std
  result
}

x = rnorm(10, 2, 25)
standardize(x)
```

```
##  [1]  0.33043093  0.11299771 -0.63450375  0.56576632  0.58951659
##  [6]  0.66962731  0.03556724 -2.50433731  1.02488054 -0.18994559
```

TODO: function with arguments, control flow, if based return, how return works

```
get_sd = function(x, biased = FALSE) {
  n = length(x)
  if (biased) {
    std = sqrt((1 / n) * sum((x - mean(x)) ^ 2))
  } else {
    std = sqrt((1 / (n - 1)) * sum((x - mean(x)) ^ 2))
  }
  std
}
```

```
get_sd = function(x, biased = FALSE) {
  n = length(x) - 1 * biased
  sqrt((1 / n) * sum((x - mean(x)) ^ 2))
}
```

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x - \bar{x})^2}$$

$$\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x - \bar{x})^2}$$

TODO: Potentially save this for the SLR document, since it is already somewhat there.

2.7 Data Frames

TODO: data frames. have observations and variables

2.8 Importing Data

TODO: read() or RStudio

2.9 Scatter Plots

2.10 Hypothesis Tests in R

2.10.1 One Sample t-Test: Review

Suppose $x_i \sim \text{N}(\mu, \sigma^2)$ and we want to test $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$.

Assuming σ is unknown, use the one-sample Student's t test statistic:

$$T = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \sim t_{n-1}$$

where $\bar{x} = \dfrac{\sum_{i=1}^{n} x_i}{n}$ and $s = \sqrt{\dfrac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}}$

A $100(1-\alpha)\%$ CI for μ is given by

$$\bar{x} \pm t_{n-1}^{(\alpha/2)} \frac{s}{\sqrt{n}}$$

where $t_{n-1}^{(\alpha/2)}$ is the critical value such that $P\left(T > t_{n-1}^{(\alpha/2)}\right) = \alpha/2$ for $n-1$ degrees of freedom.

2.10.2 One Sample t-Test: Example

A store sells "16-ounce" boxes of *Captain Crisp* cereal. A random sample of 9 boxes was taken and weighed. The results were

$$15.5 \quad 16.2 \quad 16.1 \quad 15.8 \quad 15.6 \quad 16.0 \quad 15.8 \quad 15.9 \quad 16.2$$

ounces. Assume the weight of cereal in a box is normally distributed.

a) Compute the sample mean $\bar{x}$ and the sample standard deviation s.

$$\begin{aligned}
\bar{x} &= \frac{1}{n}\sum_{i=1}^{n} x_i = (1/9)(15.5 + \cdots + 16.2) = (1/9)(143.1) = \mathbf{15.9} \\
s^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right] \\
&= (1/8)\left[2275.79 - 9(15.9^2)\right] = (1/8)(0.5) = 0.0625 \\
s &= \sqrt{0.0625} = \mathbf{0.25}
\end{aligned}$$

```
x = c(15.5, 16.2, 16.1, 15.8, 15.6, 16.0, 15.8, 15.9, 16.2)
mean(x)
```

```
## [1] 15.9
```

```
sd(x)
```

```
## [1] 0.25
```

b) Construct a 95% confidence interval for the overall average weight of boxes of *Captain Crisp* cereal.

$t_{n-1}^{(\alpha/2)} = t_8^{(0.025)} = 2.306$, so the 95% CI for the average weight of a cereal box is:

$$15.9 \pm 2.306\sqrt{\frac{0.0625}{9}} = [15.708, 16.092]$$

Or, in R:

```
t.test(x, alternative = c("two.sided"), conf.level = 0.95)
```

```
##
##  One Sample t-test
##
## data:  x
## t = 190.8, df = 8, p-value = 6.372e-16
## alternative hypothesis: true mean is not equal to 0
## 95 percent confidence interval:
##  15.70783 16.09217
## sample estimates:
## mean of x
##      15.9
```

Or if we only wanted to display the interval:

```
tt = t.test(x, alternative = c("two.sided"), conf.level = 0.95)
tt$conf.int
```

```
## [1] 15.70783 16.09217
## attr(,"conf.level")
## [1] 0.95
```

Or, we could calculate it "by hand" in R.

```
qt(0.975, 8)
```

```
## [1] 2.306004
```

```
c(mean(x) - qt(0.975, 8) * sd(x) / sqrt(9),
  mean(x) + qt(0.975, 8) * sd(x) / sqrt(9))
```

```
## [1] 15.70783 16.09217
```

c) The company that makes *Captain Crisp* cereal claims that the average weight of its box is at least 16 ounces. Use a 0.05 level of significance to test the company's claim. What is the p-value of this test?

To test $H_0 : \mu \geq 16$ versus $H_1 : \mu < 16$, the test statistic is

$$T = \frac{15.9 - 16}{\sqrt{0.0625/9}} = -1.2$$

We know that $T \sim t_8$, so the rejection reject is $T < -t_{n-1}^{(\alpha)} = -t_8^{(0.05)} = -1.860$.

Therefore, we **do NOT reject the null hypothesis** at the $\alpha = .05$ level. We could have also bounded the p-value of the test using the t table.

```
t.test(x, mu = 16, alternative = c("less"), conf.level = 0.95)
```

```
##
##  One Sample t-test
##
## data:  x
## t = -1.2, df = 8, p-value = 0.1322
## alternative hypothesis: true mean is less than 16
## 95 percent confidence interval:
##      -Inf 16.05496
## sample estimates:
## mean of x
##      15.9
```

2.10.3 Two Sample t-Test: Review

Suppose $x_i \sim \text{N}(\mu_x, \sigma^2)$ and $y_i \sim \text{N}(\mu_y, \sigma^2)$.

Want to test $H_0 : \mu_x - \mu_y = \mu_0$ versus $H_1 : \mu_x - \mu_y \neq \mu_0$.

Assuming σ is unknown, use the two-sample Student's t test statistic:

$$T = \frac{(\bar{x} - \bar{y}) - \mu_0}{s_p \sqrt{\frac{1}{n} + \frac{1}{m}}} \sim t_{n+m-2}$$

where $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$, $\bar{y} = \frac{\sum_{i=1}^{m} y_i}{m}$, and $s_p^2 = \frac{(n-1)s_1^2 + (m-1)s_2^2}{n+m-2}$

A $100(1-\alpha)\%$ CI for $\mu_x - \mu_y$ is given by

$$(\bar{x} - \bar{y}) \pm t_{n+m-2}^{(\alpha/2)} \left(s_p \sqrt{\frac{1}{n} + \frac{1}{m}} \right)$$

where $t_{n+m-2}^{(\alpha/2)}$ is critical t_{n+m-2} value such that $P\left(T > t_{n+m-2}^{(\alpha/2)}\right) = \alpha/2$.

2.10.4 Two Sample t-Test: Example

Assume that the distributions of X and Y are $\text{N}(\mu_1, \sigma^2)$ and $\text{N}(\mu_2, \sigma^2)$, respectively. Given the $n = 6$ observations of X,

$$70, \quad 82, \quad 78, \quad 74, \quad 94, \quad 82$$

and the $m = 8$ observations of Y,

$$64, \quad 72, \quad 60, \quad 76, \quad 72, \quad 80, \quad 84, \quad 68$$

find the p-value for the test $H_0 : \mu_1 = \mu_2$ versus $H_1 : \mu_1 > \mu_2$.

First, note that the sample means and variances are given by

$$\bar{x} = (1/6)\sum_{i=1}^{6} x_i = (1/6)480 = 80$$
$$\bar{y} = (1/8)\sum_{i=1}^{8} y_i = (1/8)576 = 72$$
$$s_x^2 = (1/5)\sum_{i=1}^{6}(x_i - \bar{x})^2 = (1/5)344 = 68.8$$
$$s_y^2 = (1/7)\sum_{i=1}^{8}(y_i - \bar{y})^2 = (1/7)448 = 64$$

which implies that the pooled variance estimate is given by

$$\begin{aligned} s_p^2 &= \frac{(n-1)s_x^2 + (m-1)s_y^2}{n+m-2} \\ &= \frac{344+448}{12} \\ &= 66 \end{aligned}$$

Thus, the relevant t test statistic is given by

$$\begin{aligned} T &= \frac{(\bar{x}-\bar{y}) - \mu_0}{s_p\sqrt{\frac{1}{n}+\frac{1}{m}}} \\ &= \frac{(80-72)-0}{\sqrt{66}\sqrt{\frac{1}{6}+\frac{1}{8}}} \\ &= 1.82337 \end{aligned}$$

Note that $T \sim t_{12}$, so

$$0.025 < p - value < 0.05$$

since

$$t_{12}^{(0.025)} = 1.782 < 1.82337 < t_{12}^{(0.05)} = 2.179.$$

```
x = c(70, 82, 78, 74, 94, 82)
y = c(64, 72, 60, 76, 72, 80, 84, 68)
t.test(x, y, alternative = c("greater"), var.equal = TRUE)
```

```
##
##  Two Sample t-test
##
## data:  x and y
## t = 1.8234, df = 12, p-value = 0.04662
## alternative hypothesis: true difference in means is greater than 0
## 95 percent confidence interval:
##  0.1802451       Inf
## sample estimates:
## mean of x mean of y
##        80        72
```

Or, performing the calculations `by hand' in`R‘:

```
sPooled2 = ((6 - 1) * var(x) + (8 - 1) * var(y)) / (6 + 8 - 2)
sPooled2
```

```
## [1] 66
```

```
test_stat = (mean(x) - mean(y)) / sqrt(sPooled2 * (1 / 6 + 1 / 8))
test_stat
```

```
## [1] 1.823369
```

```
1 - pt(test_stat, 6 + 8 - 2)
```

```
## [1] 0.04661961
```

2.11 Simulation in R

2.11.1 Paired Differences

Consider the model:

$$X_{11}, X_{12}, \ldots, X_{1n} \sim N(\mu_1, \sigma^2)$$
$$X_{21}, X_{22}, \ldots, X_{2n} \sim N(\mu_2, \sigma^2)$$

Assume that $\mu_1 = 6$, $\mu_2 = 5$, $\sigma^2 = 4$ and $n = 25$.

Let $\bar{X}_1 = \frac{1}{n}\sum_{i=1}^{n} X_{1i}$, $\bar{X}_2 = \frac{1}{n}\sum_{i=1}^{n} X_{2i}$ and $D = \bar{X}_1 - \bar{X}_2$.

Find $P(0 < D < 2)$**.**

$$D = \bar{X}_1 - \bar{X}_2 \sim N\left(\mu_1 - \mu_2, \frac{\sigma^2}{n} + \frac{\sigma^2}{n}\right) = N\left(6 - 5, \frac{4}{25} + \frac{4}{25}\right)$$

So,

$$D \sim N(1, 0.32)$$

Thus,

$$P(0 < D < 2) = P(-1.77 < Z < 1.77) = 0.9616 - 0.0384 = 0.9232.$$

```
z = 1 / sqrt(0.32)
pnorm(z) - pnorm(-z)
```

```
## [1] 0.9229001
```

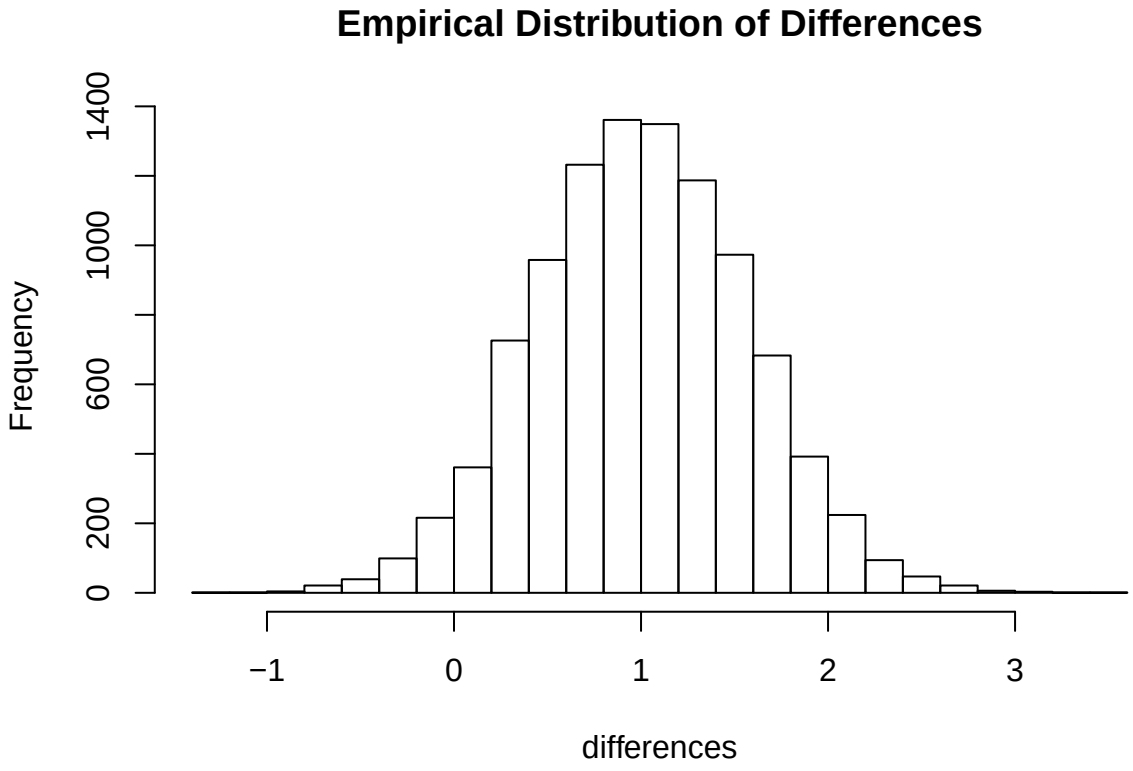
Empirical distribution of D

Generate $S = 1000$ datasets for each of group 1 and group 2. For each of the $s = 1 : 1000$ datasets, compute $d_s = \bar{x}_{1s} - \bar{x}_{2s}$. Make a histogram for the 1000 values of d. What is the proportion of values of d (among the 1000 values of d generated) that are between 0 and 2?

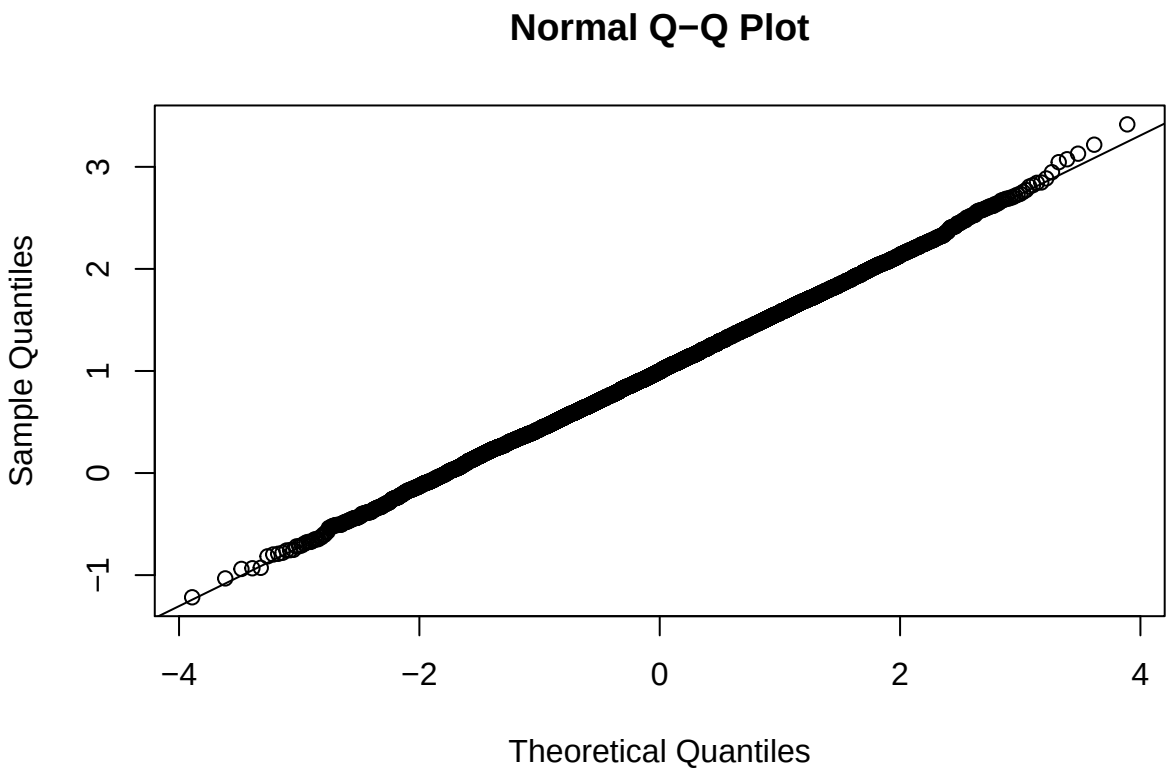
```
set.seed(42)
sample_size = 25
mu1 = 6
mu2 = 5
std = 2

samples = 10000
count = 0
differences = rep(0, samples)

for (i in 1:samples) {
  x1 = rnorm(sample_size, mu1, std)
  x2 = rnorm(sample_size, mu2, std)
  differences[i] = mean(x1) - mean(x2)
}

mean(0 < differences & differences < 2)
```

```
## [1] 0.9222
```

```
hist(differences, breaks = 20, main = "Empirical Distribution of Differences")
```

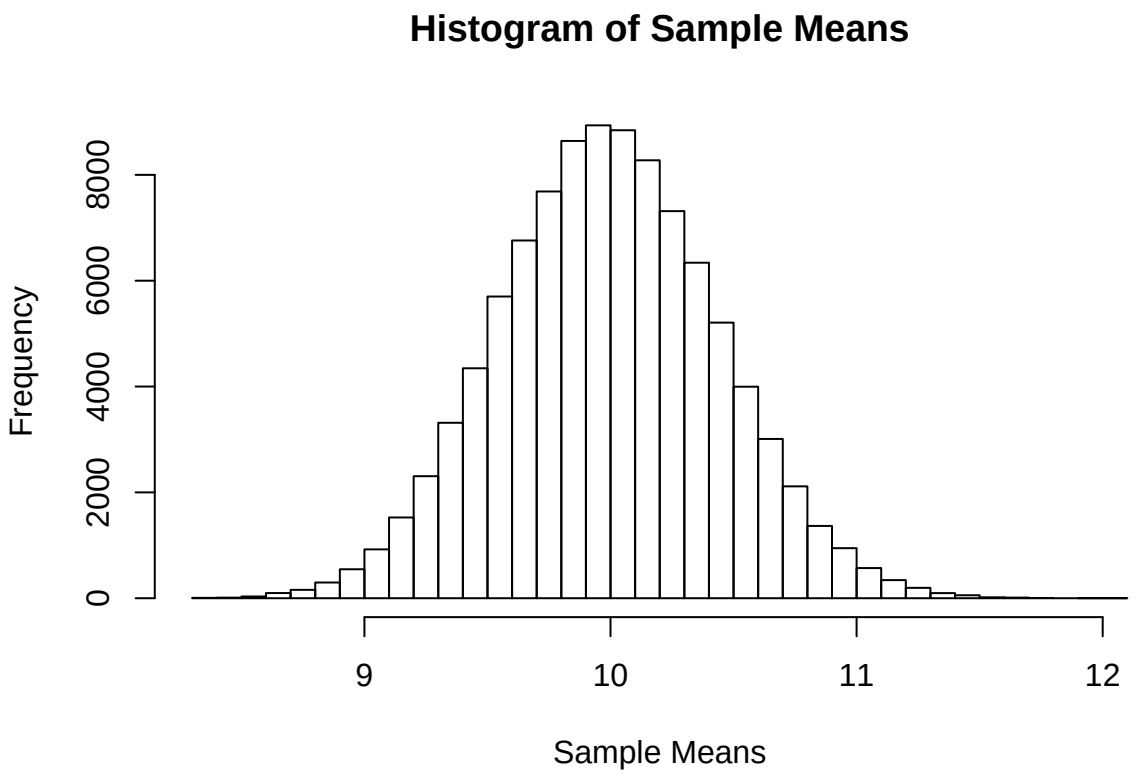


```
qqnorm(differences)
qqline(differences)
```



2.11.2 Distribution of a Sample Mean

```
set.seed(42)
sample_size = 50
mu = 10
samples = 100000
x_bar = rep(0, samples)

for(i in 1:samples){
  x_bar[i] = mean(rpois(sample_size, lambda = mu))
}

x_bar_hist = hist(x_bar, breaks = 50,
                  main = "Histogram of Sample Means",
                  xlab = "Sample Means")
```




```
c(mean(x_bar), mu)
```

```
## [1] 10.00009 10.00000
```

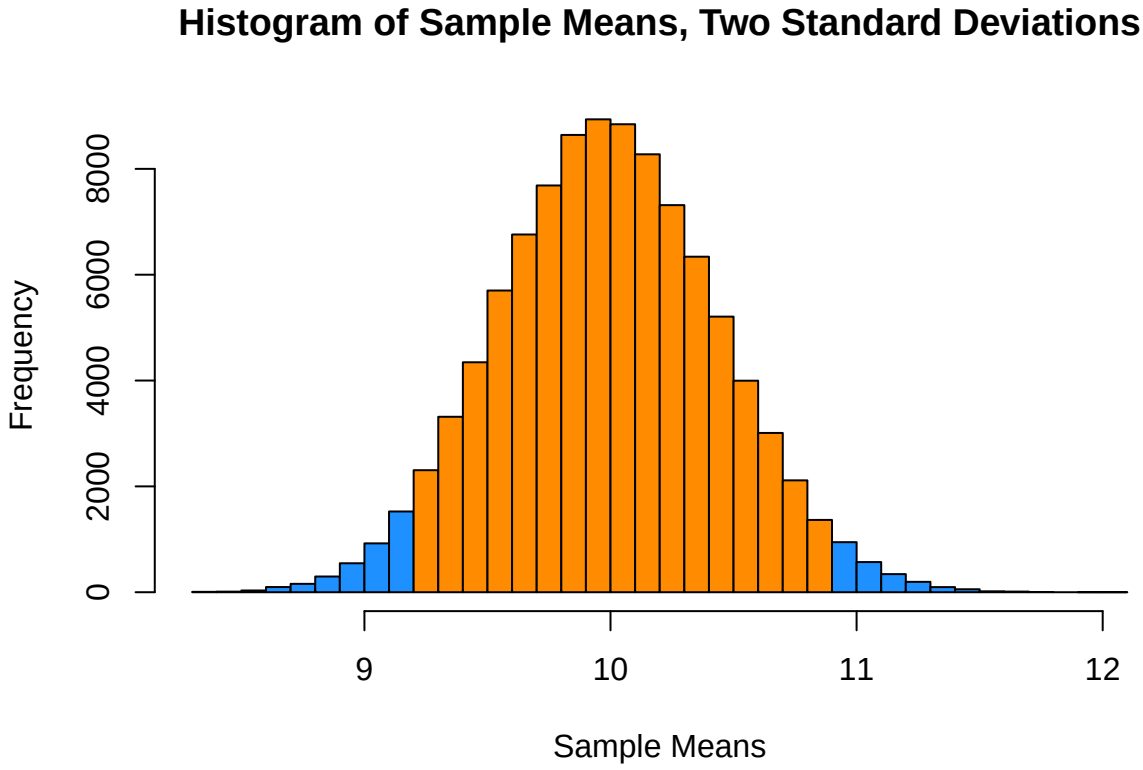
```
c(sd(x_bar), sqrt(mu) / sqrt(sample_size))
```

```
## [1] 0.4454965 0.4472136
```

```
mean(x_bar > mu - 2 * sqrt(mu) / sqrt(sample_size) &
       x_bar < mu + 2 * sqrt(mu) / sqrt(sample_size))
```

```
## [1] 0.95459
```

```
shading = ifelse(x_bar_hist$breaks > mu - 2 * sqrt(mu) / sqrt(sample_size) &
                   x_bar_hist$breaks < mu + 2 * sqrt(mu) / sqrt(sample_size),
                 "darkorange", "dodgerblue")

x_bar_hist = hist(x_bar, breaks = 50, col = shading,
                  main = "Histogram of Sample Means, Two Standard Deviations",
                  xlab = "Sample Means")
```


Histogram of Sample Means, Two Standard Deviations
Frequency
0
2000
4000
6000
8000
9
10
11
12
Sample Means

Chapter 3

Simple Linear Regression

"All models are wrong, but some are useful."
- **George E. P. Box**

3.1 Motivating Example

Suppose you are the owner of the *Momma Leona's Pizza*, a restaurant chain located near several college campuses. You currently own 10 stores and have data on the size of the student population as well as quarterly sales for each. Your data is summarized in the table below.

Restaurant	Student Population	Quarterly Sales
1	2	58
2	6	105
3	8	88
4	8	118
5	12	117
6	16	137
7	20	157
8	20	169
9	22	149
10	26	202

Here,

- Restaurant, i
- Student Population, x_i, in 1000s
- Quarterly Sales, y_i, in \$1000s

How can you use this data to

- Explain relationship
- Predict

One tool will do both, LINEAR REGRESSION

3.2 History

3.3 Model

$$y = f(x) + \epsilon$$

$$y = \beta_0 + \beta_1 x + \epsilon$$

3.4 What is the Best Line?

TODO: picture with lines

$$\underset{\beta_0,\beta_1}{\text{argmin}} \max |y_i - (\beta_0 + \beta_1 x_i)|$$

$$\underset{\beta_0,\beta_1}{\text{argmin}} \sum_{i=1}^{n} |y_i - (\beta_0 + \beta_1 x_i)|$$

$$\underset{\beta_0,\beta_1}{\text{argmin}} \sum_{i=1}^{n} (y_i - (\beta_0 + \beta_1 x_i))^2$$

historical, easy math

3.5 Least Squares Approach

Funciton to minimize.

$$f(\beta_0, \beta_1) = \sum_{i=1}^{n} (y_i - (\beta_0 + \beta_1 x_i))^2 = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2$$

Take derivatives.

$$\frac{df}{d\beta_0} = -2 \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)$$
$$\frac{df}{d\beta_1} = -2 \sum_{i=1}^{n} (x_i)(y_i - \beta_0 - \beta_1 x_i)$$

Set equal to zero.

$$\sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i) = 0$$
$$\sum_{i=1}^{n} (x_i)(y_i - \beta_0 - \beta_1 x_i) = 0$$

Rearrange, **normal equations**.

$$\sum_{i=1}^{n} y_i = n\beta_0 + \beta_1 \sum_{i=1}^{n} x_i$$
$$\sum_{i=1}^{n} x_i y_i = \beta_0 \sum_{i=1}^{n} x_i + \beta_1 \sum_{i=1}^{n} x_i^2$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} x_i y_i - \frac{(\sum_{i=1}^{n} x_i)(\sum_{i=1}^{n} y_i)}{n}}{\sum_{i=1}^{n} x_i^2 - \frac{(\sum_{i=1}^{n} x_i)^2}{n}} = \frac{SXY}{SXX}$$
$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

$$SXY = \sum_{i=1}^{n} x_i y_i - \frac{(\sum_{i=1}^{n} x_i)(\sum_{i=1}^{n} y_i)}{n} = \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})$$
$$SXX = \sum_{i=1}^{n} x_i^2 - \frac{(\sum_{i=1}^{n} x_i)^2}{n} = \sum_{i=1}^{n} (x_i - \bar{x})^2$$
$$SYY = \sum_{i=1}^{n} y_i^2 - \frac{(\sum_{i=1}^{n} y_i)^2}{n} = \sum_{i=1}^{n} (y_i - \bar{y})^2$$

$$\hat{\beta}_1 = \frac{SXY}{SXX} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

3.6 MLE Approach

3.7 Example

3.8 R

3.9 Decomposition of Variation

3.10 Coefficient of Determination

R^2

$$\sum_{i=1}^{n} (y_i - \bar{y})^2 = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 + \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2$$

Decomposing variation.

Chapter 4

Inference for Simple Linear Regression

"There are three types of lies: lies, damn lies, and statistics."
- **Benjamin Disraeli**

Bibliography