ENGINEERING MATHEMATICS-II
UE25MA141B

Department of Science and Humanities

ENGINEERING MATHEMATICS-II

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UNIT 1: INTEGRAL CALCULUS

CLASS 1: DOUBLE INTEGRAL: INTRODUCTION & EVALUATION

TOPICS COVERED

1. DOUBLE INTEGRAL: INTRODUCTION AND EVALUATION
2. APPLICATION OF DOUBLE INTEGRAL: AREA, VOLUME OF THE SOLID AND AVERAGE VALUE OF THE FUNCTION.
3. JACOBIAN, CHANGE OF VARIABLES IN DOUBLE INTEGRAL
4. PROBLEMS ON CHANGE OF VARIABLES
5. CHANGE OF ORDER OF INTEGRATION
6. TRIPLE INTEGRAL: INTRODUCTION AND EVALUATION
7. APPLICATION OF TRIPLE INTEGRAL: VOLUME
8. FINDING AVERAGE VALUE USING TRIPLE INTEGRAL
9. CHANGE OF VARIABLES IN TRIPLE INTEGRAL: CYLINDRICAL COORDINATES
10. CHANGE OF VARIABLES IN TRIPLE INTEGRAL: SPHERICAL COORDINATES
11. APPLICATIONS: CENTER OF MASS
12. APPLICATIONS: MOMENT OF INERTIA

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INTRODUCTION TO DOUBLE INTEGRAL

The definite integral can be extended to functions of more than one variable. Consider, for example, a function of two variables $z = f(x, y)$. The double integral of function $f(x, y)$ is denoted by

$$\iint\limits_{R} f(x, y)\, dA,$$

where R is the region of integration in the xy-plane.

If the definite integral $\int\limits_{a}^{b} f(x)\, dx$ of a function of one variable $f(x) \geq 0$ is the area under the curve $f(x)$ from $x = a$ to $x = b$, then the double integral is equal to the volume under the surface $z = f(x, y)$ and above the xy-plane in the region of integration R (Figure 1).

DOUBLE INTEGRAL DEFINITION

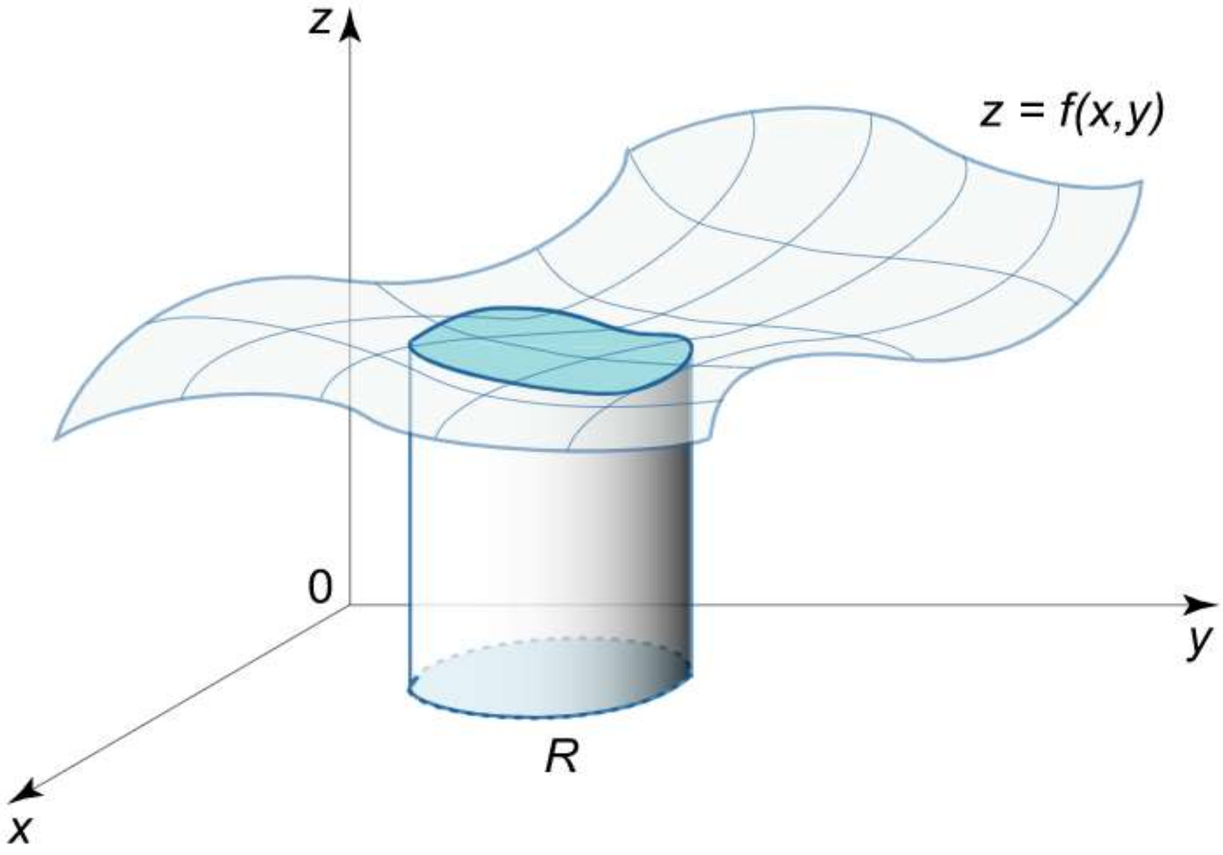


Figure 1.

DOUBLE INTEGRAL DEFINITION CONTINUED…

As in the case of integral of a function of one variable, a double integral is defined as a limit of a Riemann sum.

If the region R is a rectangle $[a,b] \times [c,d]$ (Figure 2), we can subdivide $[a,b]$ into small intervals with a set of numbers $\{x_0, x_1, \ldots, x_m\}$ so that

$$a = x_0 < x_1 < x_2 < \ldots < x_i < \ldots < x_{m-1} < x_m = b.$$

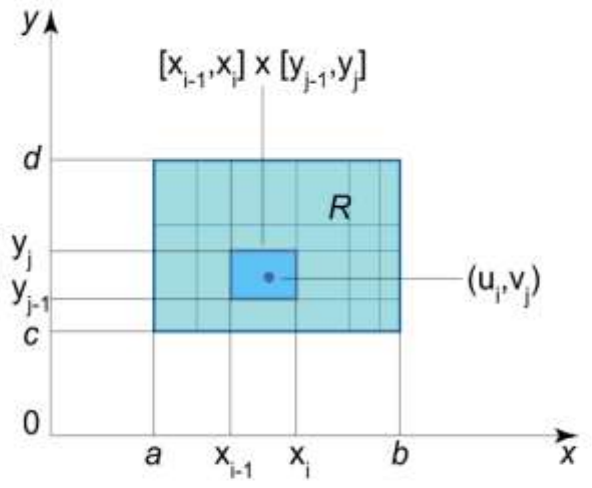


Figure 2.

INTERPRETATION OF DOUBLE INTEGRAL

Similarly, a set of numbers $\{y_0, y_1, \ldots, y_n\}$ is said to be a partition of $[c, d]$ along the y-axis, if

$$c = y_0 < y_1 < y_2 < \ldots < y_j < \ldots < y_{n-1} < y_n = d.$$

The Riemann sum of a function $f(x,y)$ over this partition of $[a, b] \times [c, d]$ is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f\left(u_i, v_j\right) \Delta x_i \Delta y_j,$$

where $\left(u_i, v_j\right)$ is some point in the rectangle $\left(x_{i-1}, x_i\right) \times \left(y_{j-1}, y_j\right)$ and $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$.

DEFINITION OF DOUBLE INTEGRAL

Definition. We call **the double integral** of $f(x, y)$ under the region R the value

$$\iint_R f(x, y)\, dA = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^\star, y_k^\star)\Delta A_k.$$

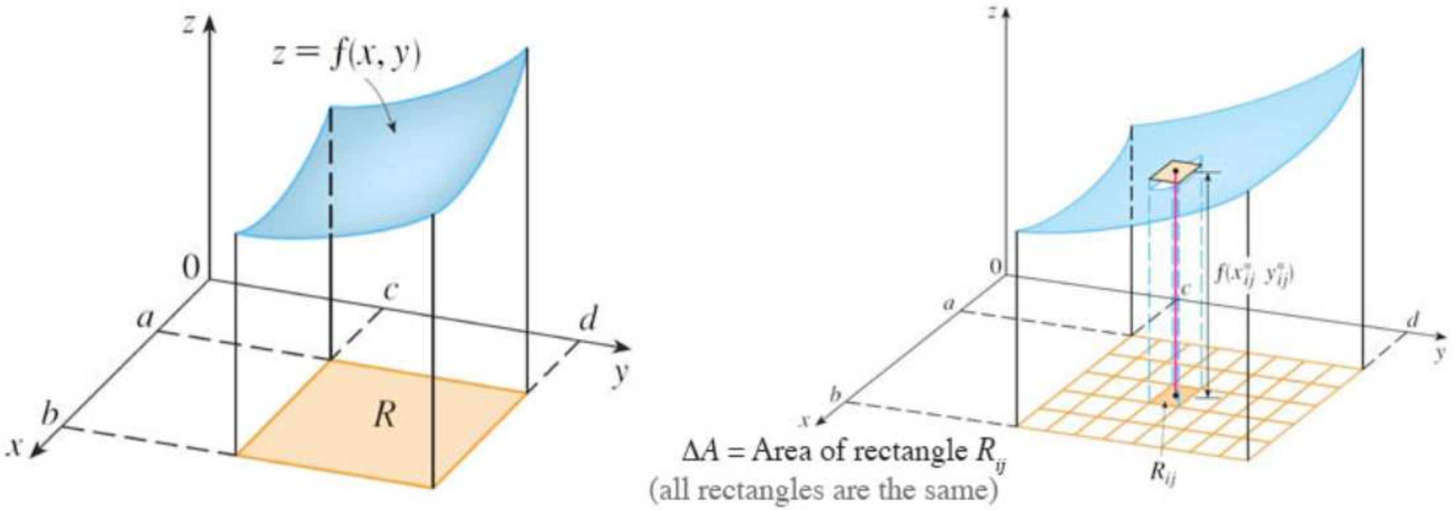
Therefore, the volume V that we intended to determine is given by

$$V = \iint_R f(x, y)\, dA = \iint_R f(x, y)\, dx\, dy.$$

FUBINI'S THEOREM FOR RECTANGULAR REGION

Let R be the rectangle defined by $a \le x \le b,\ c \le y \le d$. If $f(x, y)$ is continuous on that rectangle, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx$$



FUBINI'S THEOREM FOR RECTANGULAR REGION

Use Fubini's theorem to compute the double integral $\iint_R f(x,y)\,dA$ where $f(x,y)=x$ and $R=[0,2]\times[0,1]$.

Solution

Fubini's theorem offers an easier way to evaluate the double integral by the use of an iterated integral. Note how the boundary values of the region R become the upper and lower limits of integration.

$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy$$

$$= \int_{y=0}^{y=1}\int_{x=0}^{x=2} x\,dx\,dy$$

$$= \int_{y=0}^{y=1}\left[\frac{x^2}{2}\Big|_{x=0}^{x=2}\right]dy$$

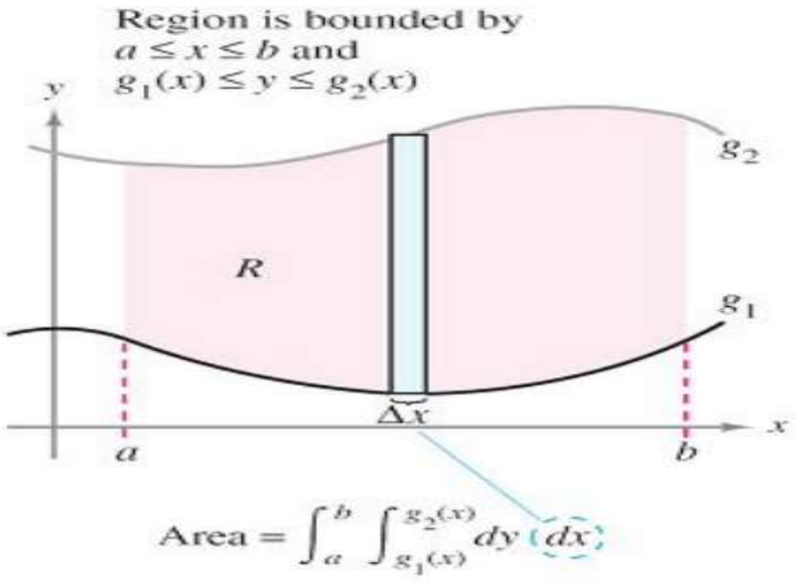
$$= \int_{y=0}^{y=1} 2\,dy = 2y\Big|_{y=0}^{y=1} = 2$$

FUBINI'S THEOREM FOR NON RECTANGULAR REGION

If R is not a rectangular region, then we have to distinguish between two types of regions: • **Regions type I**

$$R = \left\{ (x, y) \in \mathbb{R}^2 : a \le x \le b,\ g_1(x) \le y \le g_2(x) \right\}$$

such that $g_1(x)$ and $g_2(x)$ are continuous in $[a, b]$.



Vertically simple region

FUBINI'S THEOREM FOR NON RECTANGULAR REGIONS

- **Regions type II**

$$R = \left\{(x, y) \in \mathbb{R}^2 : h_1(y) \le x \le h_2(y),\ c \le y \le d\right\}$$

such that $h_1(y)$ and $h_2(y)$ are continuous in $[c, d]$.

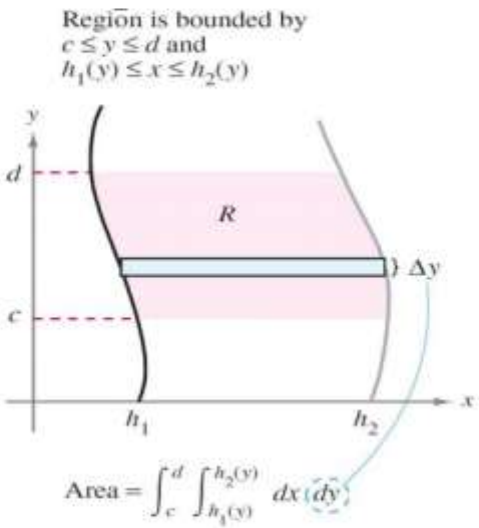
Let $f(x, y)$ be continuous in the region R.

- *If R is a region of type I, then*

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

- *If R is a region of type II, then*

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$



Horizontally simple region

PROPERTIES OF DOUBLE INTEGRALS

Let f and g be continuous over a closed, bounded plane region R, and let c be a constant.

1. $$\iint_R cf(x, y)\, dA = c\iint_R f(x, y)\, dA$$

2. $$\iint_R [f(x, y) \pm g(x, y)]\, dA = \iint_R f(x, y)\, dA \pm \iint_R g(x, y)\, dA$$

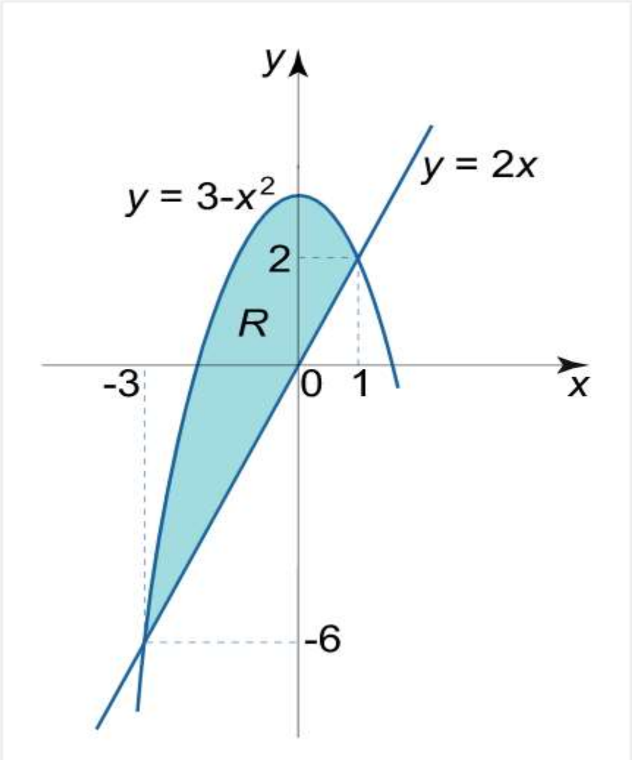
3. $$\iint_R f(x, y)\, dA \geq 0, \qquad \text{if } f(x, y) \geq 0$$

4. $$\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA, \qquad \text{if } f(x, y) \geq g(x, y)$$

5. $$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA,$$ where R is the union of two nonoverlapping subregions R_1 and R_2.

PROPERTIES OF DOUBLE INTEGRALS

Find the integral $\iint\limits_R y\,dy\,dx$, where R is bounded by the straight line $y = 2x$ and parabola $y = 3 - x^2$.



PROPERTIES OF DOUBLE INTEGRALS

Find the average value of $f(x, y) = \frac{1}{2}xy$ over the region R, where R is a rectangle with vertices (0, 0), (4, 0), (4, 3), and (0, 3).

The average value is given by

$$\frac{1}{A}\int_R\int f(x, y)\, dA = \frac{1}{12}\int_0^4\int_0^3 \frac{1}{2}xy\, dy\, dx$$

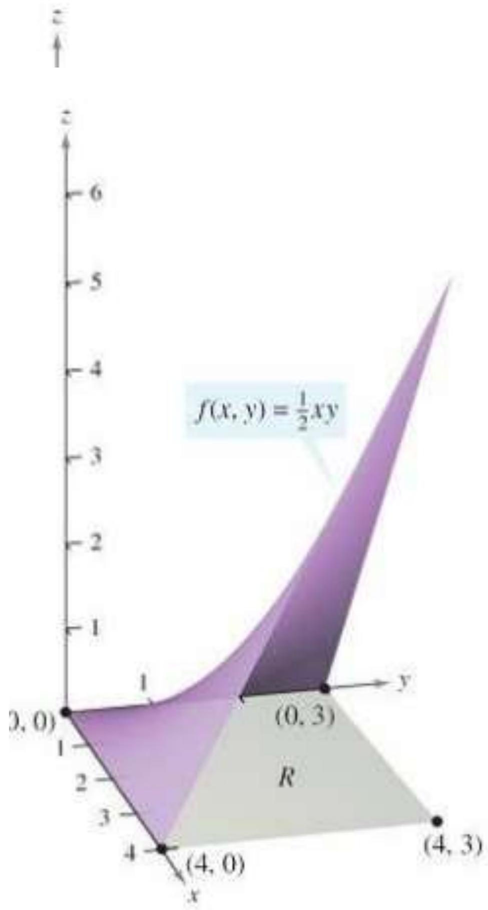
$$= \frac{1}{12}\int_0^4 \left[\frac{1}{4}xy^2\right]_0^3 dx$$

$$= \left(\frac{1}{12}\right)\left(\frac{9}{4}\right)\int_0^4 x\, dx$$

$$= \frac{3}{16}\left[\frac{1}{2}x^2\right]_0^4$$

$$= \left(\frac{3}{16}\right)(8)$$

$$= \frac{3}{2}.$$



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UNIT 1 : INTEGRAL CALCULUS
CLASS 3: CHANGE OF ORDER OF INTEGRATION

CHANGE OF ORDER OF INTEGRATION

If, in the given double integral the integration is w.r.t. x and then w.r.t y, the process of converting the order of integration is called change of order of integration. Change of order of integration changes the limits of integration.

Consider a double integral $\int\int_R f(x, y)dxdy$ where R is region.

Assume that R lies between the lines $x = x_0, x = x_1$ and curves $y = f_1(x)$ and $y = f_2(x)$. For points of R, x lies in the interval $[x_0, x_1]$, y varies between $f_1(x)$ and $f_2(x)$, where $f_1(x)$ and $f_2(x)$ are the ordinates of the points at which the boundary of R is intersected by line through (x, y) and parallel to y-axis.

DOUBLE INTEGRAL

$$\iint_R f(x,y)\,dxdy = \int_{x=x_0}^{x_1}\left[\int_{y=f_1(x)}^{f_2(x)} f(x,y)\,dy\right]dx$$

By change of order of integration, limits of y will be constants y_0, y_1 and x varies between $g_1(y)$ and $g_2(y)$ which the boundary is intersected by the line through (x,y) and parallel to x-axis.

$$\iint_R f(x,y)\,dxdy = \int_{y=y_0}^{y_1}\left[\int_{x=g_1(y)}^{g_2(y)} f(x,y)\,dx\right]dy.$$

PROCEDURE FOR CHANGE OF ORDER OF INTEGRATION

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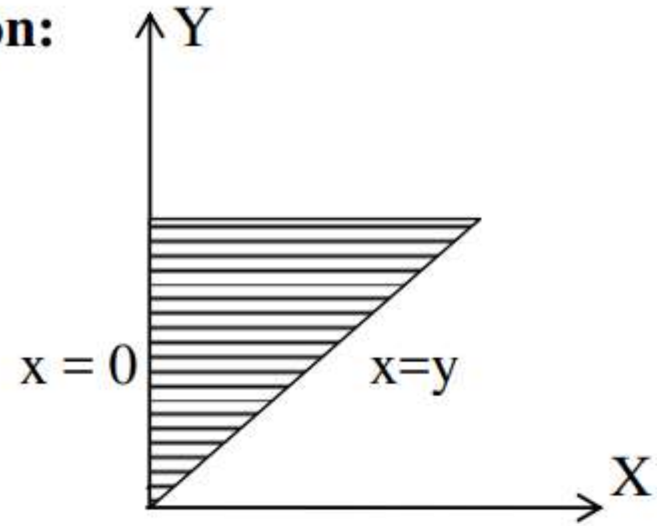
1. Identify the variables for the limits.
2. Trace the curve.
3. If we are evaluating with respect to y first, then take strip parallel to y-axis.
   If the evaluation is with respect to x first, then take strip parallel to x-axis.
4. Rotate the strip to 90^0 in anti clock wise direction and identify the starting and ending points of the strip, which will be below and upper units of that variable.
5. Identify the limits for other variables for the region of consideration.
6. Evaluate the double integral with new order of integration.

EXAMPLES OF CHANGE OF ORDER OF INTEGRATION

Evaluate $\int_0^\infty \int_0^y ye^{-\frac{y^2}{x}} dxdy$ by changing the order of integration.

Solution:



Given x=0, x = y, y = 0, y = ∞ .

By changing the order of integration y: x to ∞, x : 0 to ∞

2D JACOBIAN

$$\int_0^\infty \int_0^y y e^{-\frac{y^2}{x}} dx dy = \int_0^\infty \int_x^\infty y e^{-\frac{y^2}{x}} dy dx$$

$$= \int_0^\infty \int_x^\infty y e^{-\frac{y^2}{x}} d\left(\frac{y^2}{2}\right) dx$$

$$= \frac{1}{2}\int_0^\infty \left[\frac{e^{-\frac{y^2}{x}}}{-1/x}\right]_x^\infty dx \; = \frac{1}{2}\int_0^\infty x e^{-x} dx$$

$Take\, u = x, dv = e^{-x} dx \quad implies\, du = dx, v = -e^{-x},$

by integration by parts,

$$= \frac{1}{2}\left[x\left(\frac{e^{-x}}{-1}\right) - e^{-x}\right]_0^\infty = \frac{1}{2}$$

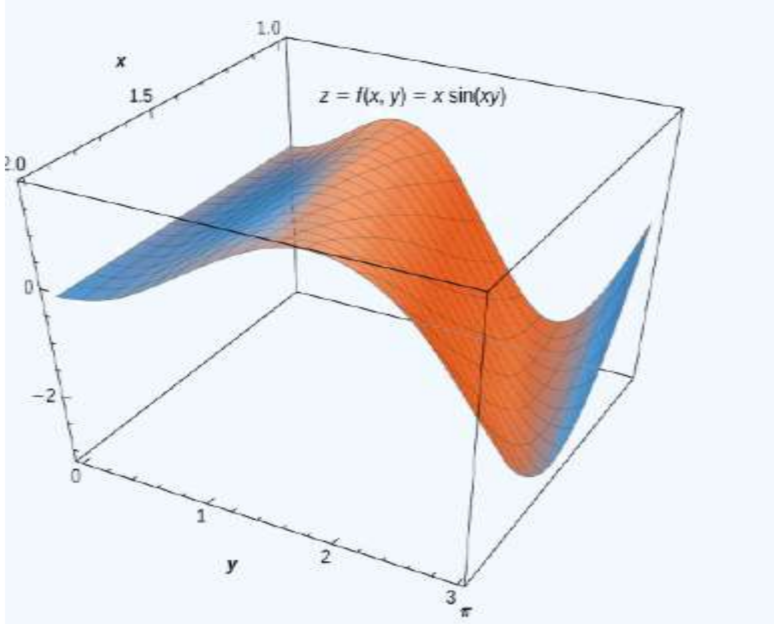
FUBINI'S THEOREM FOR RECTANGULAR REGION

Consider the double integral $\iint_R x \sin(xy)\, dA$ over the region $R = \{(x, y) \mid 0 \leq x \leq \pi, 1 \leq y \leq 2\}$

a. Express the double integral in two different ways.

b. Analyze whether evaluating the double integral in one way is easier than the other and why.

c. Evaluate the integral.



FUBINI'S THEOREM FOR NON RECTANGULAR REGION

a. We can express $\iint_R x\sin(xy)\,dA$ in the following two ways: first by integrating with respect to y and then with respect to x; second by integrating with respect to x and then with respect to y.

$$\iint_R x\sin(xy)\,dA = \int_{x=0}^{x=\pi}\int_{y=1}^{y=2} x\sin(xy)\,dy\,dx$$

Integrate first with respect to y.

$$= \int_{y=1}^{y=2}\int_{x=0}^{x=\pi} x\sin(xy)\,dx\,dy$$

Integrate first with respect to x.

FUBINI’S THEOREM FOR NON RECTANGULAR REGIONS

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b. If we want to integrate with respect to y first and then integrate with respect to x, we see that we can use the substitution $u = xy$, which gives $du = x\,dy$. Hence the inner integral is simply $\int \sin u\,du$ and we can change the limits to be functions of x,

$$\iint_R x\sin(xy)\,dA = \int_{x=0}^{x=\pi}\int_{y=1}^{y=2} x\sin(xy)\,dy\,dx = \int_{x=0}^{x=\pi}\left[\int_{u=x}^{u=2x}\sin(u)\,du\right]dx.$$

However, integrating with respect to x first and then integrating with respect to y requires integration by parts for the inner integral, with $u = x$ and $dv = \sin(xy)dx$

Then $du = dx$ and $v = -\frac{\cos(xy)}{y}$, so

$$\iint_R x\sin(xy)\,dA = \int_{y=1}^{y=2}\int_{x=0}^{x=\pi} x\sin(xy)\,dx\,dy = \int_{y=1}^{y=2}\left[-\frac{x\cos(xy)}{y}\Big|_{x=0}^{x=\pi} + \frac{1}{y}\int_{x=0}^{x=\pi}\cos(xy)\,dx\right]dy.$$

Since the evaluation is getting complicated, we will only do the computation that is easier to do, which is clearly the first method.

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c. Evaluate the double integral using the easier way.

$$\iint_R x\,\sin(xy)\,dA = \int_{x=0}^{x=\pi}\int_{y=1}^{y=2} x\,\sin(xy)\,dy\,dx$$

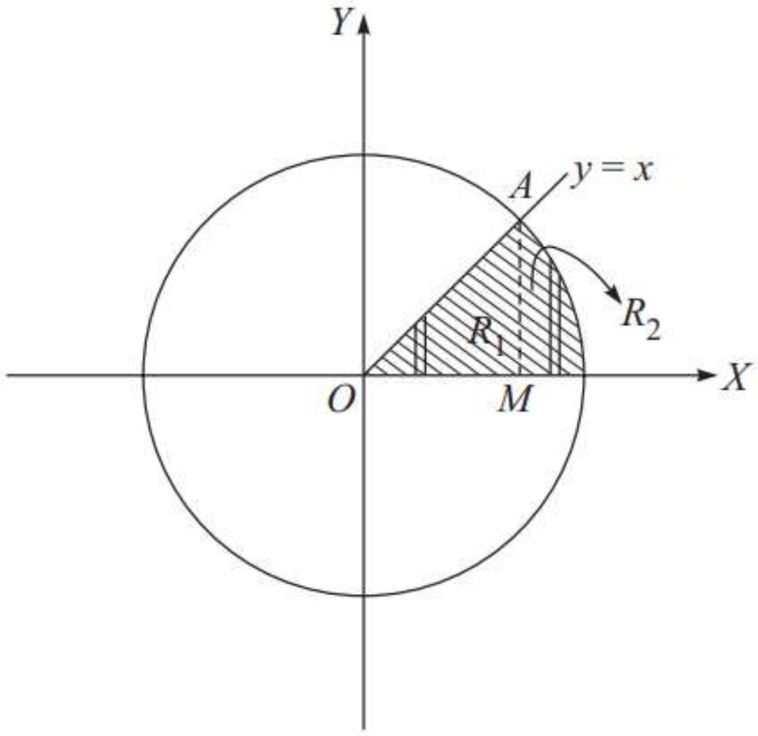
$$= \int_{x=0}^{x=\pi}\left[\int_{u=x}^{u=2x}\sin(u)\,du\right]dx = \int_{x=0}^{x=\pi}\left[-\cos u\Big|_{u=x}^{u=2x}\right]dx$$

$$= \int_{x=0}^{x=\pi}(-\cos 2x + \cos x)\,dx$$

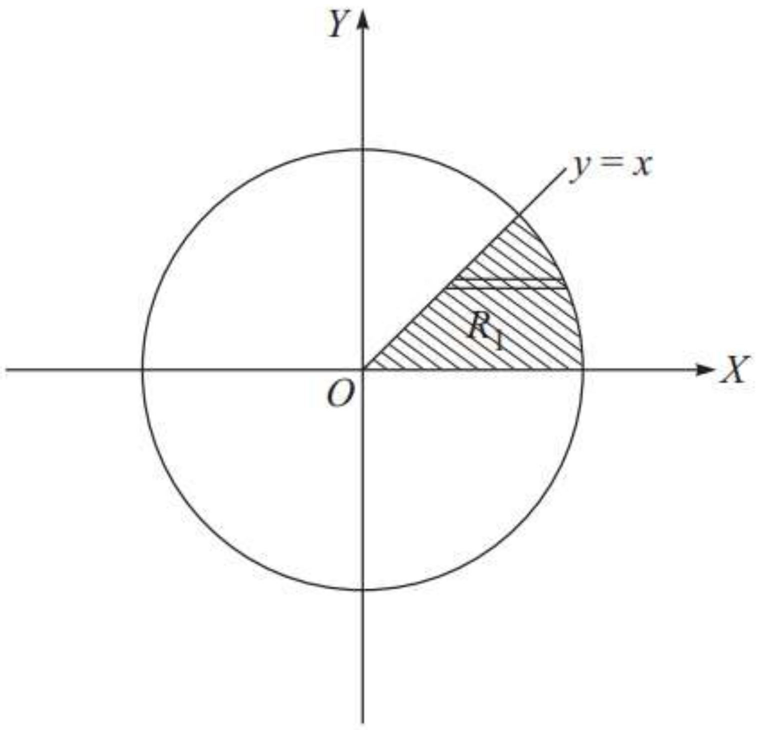
$$= \left(-\frac{1}{2}\sin 2x + \sin x\right)\Bigg|_{x=0}^{x=\pi} = 0.$$

PROPERTIES OF DOUBLE INTEGRALS

Express as a single integral $\int_{0}^{a/\sqrt{2}} \int_{0}^{x} x \, dy \, dx + \int_{a/\sqrt{2}}^{a} \int_{0}^{\sqrt{a^2-x^2}} x \, dy \, dx$ and evaluate it.



(i)



(ii)

PROPERTIES OF DOUBLE INTEGRALS

Solution: Let $I_1 = \int_0^{a/\sqrt{2}} \int_0^{x} x\,dy\,dx$ and $I_2 = \int_{a/\sqrt{2}}^{a} \int_0^{\sqrt{a^2-x^2}} x\,dy\,dx$

Let R_1 and R_2 be the regions over which I_1 and I_2 are being integrated, respectively and are shown by the shaded region in Fig. 4.4(i).

Also from Fig. 4.4(ii), it is clear that

$$R = R_1 + R_2$$

$$\therefore \qquad I = I_1 + I_2 = \iint_R x\,dx\,dy$$

For evaluating I, change the order of integration and then take an elementary strip parallel to the x-axis from $y = x$ to $y = \sqrt{a^2 - x^2}$, i.e., the circle $x^2 + y^2 = a^2$. Thus,

$$I = \int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} x\,dx\,dy = \int_0^{a/\sqrt{2}} \left[\int_y^{\sqrt{a^2-y^2}} x\,dx \right] dy$$

$$= \int_0^{a/\sqrt{2}} \frac{x^2}{2} \Bigg|_y^{\sqrt{a^2-y^2}} dy = \frac{1}{2} \int_0^{a/\sqrt{2}} (a^2 - y^2 - y^2)\,dy = \frac{1}{2} \left[a^2 y - \frac{2y^3}{3} \right]_0^{a/\sqrt{2}} = \frac{a^3}{3\sqrt{2}}$$

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UNIT 1: INTEGRAL CALCULUS
CLASS 3: JACOBIANS

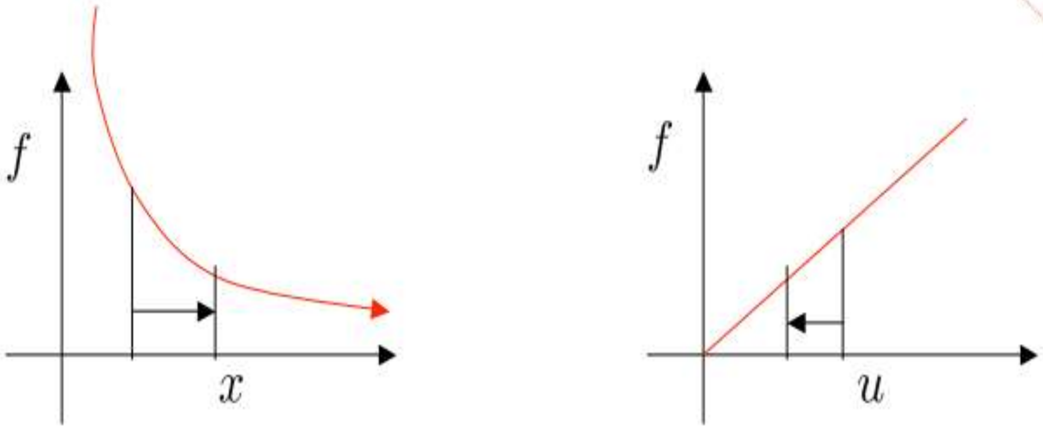
The Jacobian, named after the German mathematician

Carl Gustav Jacobi (1804-1851)

JACOBIAN

- In 1D problems we are used to a simple change of variables, e.g. from *x* to *u*

$$\int_a^b f(x)\mathrm{d}x = \int_\alpha^\beta f(x(u))\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u$$



1D Jacobian

maps strips of width d*x* to strips of width d*u*

- Example: $\int_1^2 \frac{1}{x}\mathrm{d}x = \ln(2)$ Substitute $x = u^{-1} \rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = -u^{-2}$

$$= -\int_1^{\frac{1}{2}} \frac{u}{u^2}\mathrm{d}u = [\ln u]_{\frac{1}{2}}^1 = \ln(2)$$

2D JACOBIAN

- For a continuous 1-to-1 transformation from (*x*,*y*) to (*u*,*v*)
- Then $$x = x(u,v) \text{ and } y = y(u,v)$$
- Where Region (in the *xy* plane) maps onto region R in the *uv* plane R'

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

2D Jacobian

maps areas dxdy to areas dudv

- Hereafter call such terms x_u

$$= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u$$

JACOBIAN IN 3D

Suppose that x, y and z are three independent variables which can be expressed in term of three other independent variables u, v and w by the formula

$$x = x(u, v, w), y = y(u, v, w), z = z(u, v, w)$$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

PROPERTY OF JACOBIAN

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- The Jacobian matrix $\frac{\partial(x,y)}{\partial(u,v)}$ is the ***inverse matrix*** of $\frac{\partial(u,v)}{\partial(x,y)}$ i.e.,

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Because (and similarly for d*y*)

$$\mathrm{d}x = x_u \mathrm{d}u + x_v \mathrm{d}v = x_u \mathrm{d}u + x_v (v_x \mathrm{d}x + v_y \mathrm{d}y)$$

$$\begin{aligned} x \text{ constant } &\Rightarrow \mathrm{d}x = 0 \Rightarrow 0 = x_u u_y + x_v v_y \\ y \text{ constant } &\Rightarrow \mathrm{d}y = 0 \Rightarrow 1 = x_u u_x + x_v v_x \end{aligned}$$

- This makes sense because Jacobians measure the relative areas of d*x*d*y* and d*u*d*v*, i.e

$$\det(AB) = \det(A)\det(B) = 1 \Rightarrow \det(A) = \frac{1}{\det B}$$

- So

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1}$$

CHANGE OF VARIABLES

You will recall in one-dimensional calculus, when given an integral of the form $\int g'(u) f(g(u))\, du$, we performed the change of variable $x = g(u)$ which gave us $dx = g'(u)\, du$ and thus

$$\int g'(u) f(g(u))\, du = \int f(x)\, dx$$

We can write this slightly differently as follows. Since $x = g(u)$, $\dfrac{dx}{du} = g'(u)$ hence, we have

$$\int f(x)\, dx = \int f(g(u)) \frac{dx}{du} du$$

The Jacobian is what generalizes $\dfrac{dx}{du}$ in the above formula.

CHANGE OF VARIABLES

A change of variable is usually described as a transformation T from uv-plane to the xy-plane given by $T(u, v) = (x, y)$
where x and y are given by

$$\begin{aligned} x &= x(u, v) \\ y &= y(u, v) \end{aligned}$$

We assume that the first order partials of $x(u, v)$ and $y(u, v)$ are continuous. When it is the case we say that T is a C^1 **transformation**. Such a transformation will map a region S in the uv-plane into another region R into the xy-plane

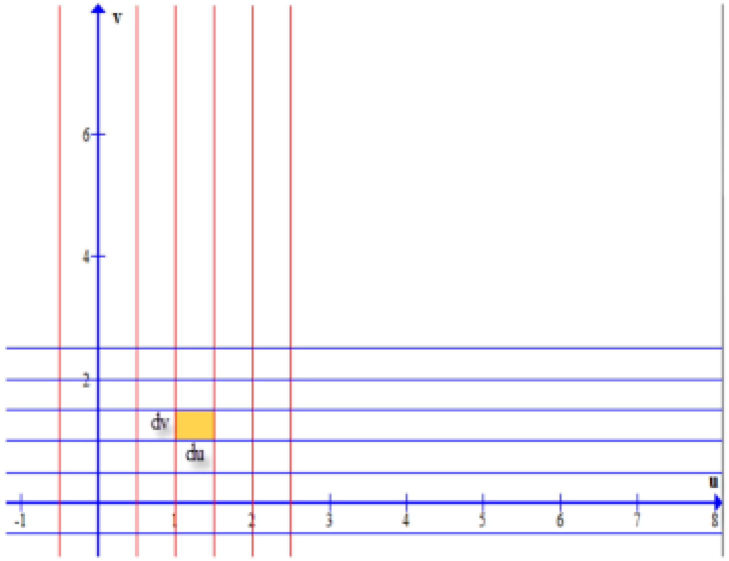
In double integral, we divided the region of integration into small regions.

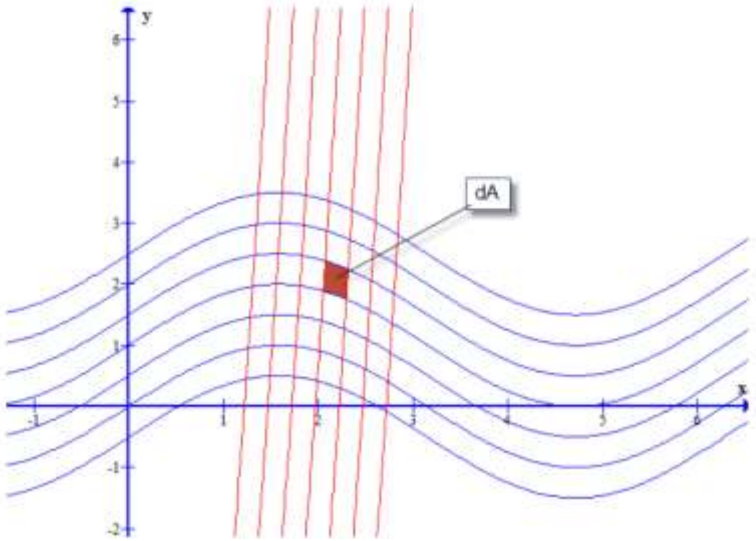
The region of integration was then approximated by rectangles of area

$$dA = dudv.$$

When we apply the transformation T,
uv-plane and the grid are transformed as well as the region of integration.

TRANSFORMATION FROM XY PLANE TO UV PLANE





small region created by the grid are no longer rectangles their area dA is not $dxdy$.

INTERPRETATION OF JACOBIAN

From the differential formula, we see that if $x = x(u, v)$ then

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$$

Similarly, if $y = y(u, v)$ then

$$dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv$$

We can write this in matrix form as follows:

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \begin{bmatrix} du \\ dv \end{bmatrix}$$

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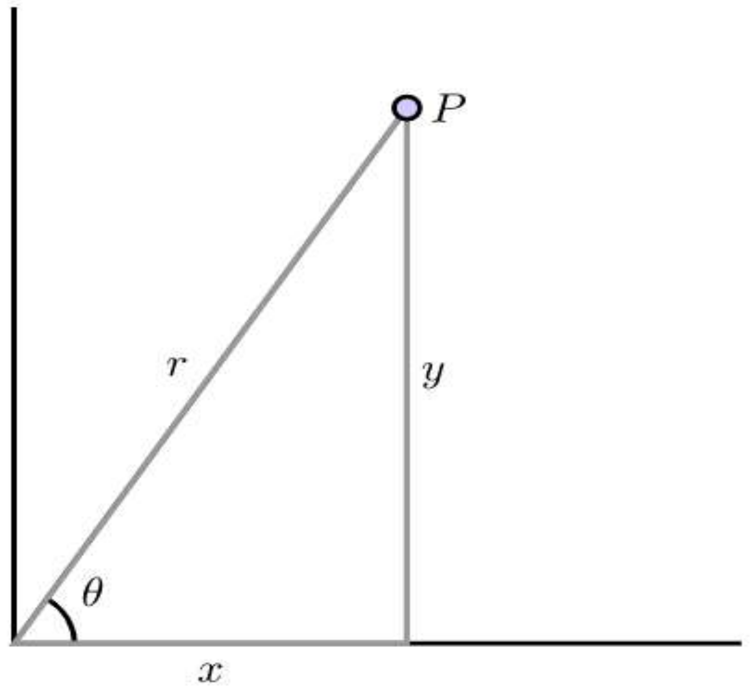
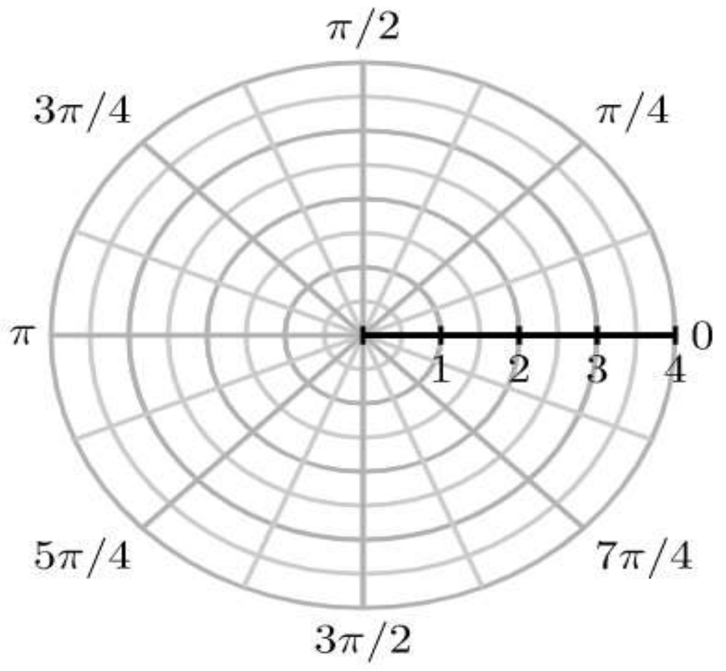
UNIT 1: INTEGRAL CALCULUS

CLASS 4: PROBLEMS ON CHANGE OF VARIABLES IN DOUBLE INTEGRAL

POLAR COORDINATES

The rectangular coordinate system is best suited for graphs and regions that are naturally considered over a rectangular grid. The polar coordinate system is an alternative that offers good options for functions and domains that have more circular characteristics. A point P in rectangular coordinates that is described by an ordered pair (x, y), where x is the displacement from P to the y-axis and y is the displacement from P to the x-axis





CARTESIAN TO POLAR COORDINATES

Cartesian to polar coordinates, we have $x = r\cos\theta$ *and* $y = r\sin\theta$.

The Jacobian of x *and* y *with respect to* r *and* θ *is*

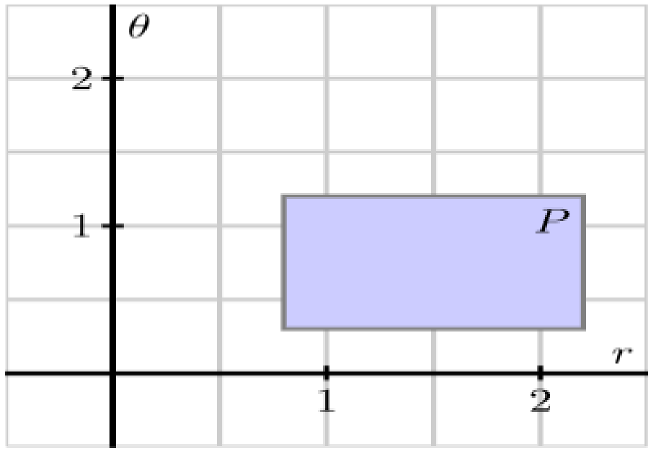
$$\begin{aligned}
\frac{\partial(x,y)}{\partial(r,\theta)} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \\
&= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\
&= r\cos^2\theta + r\sin^2\theta \\
\frac{\partial(x,y)}{\partial(r,\theta)} &= r
\end{aligned}$$

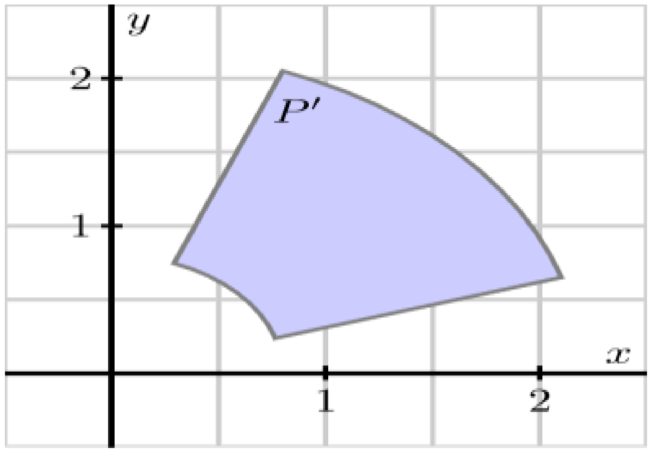
CARTESIAN TO POLAR COORDINATES

we have

$$\begin{aligned}\iint\limits_R f(x,y)\,dxdy &= \iint\limits_S f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| drd\theta \\ &= \iint\limits_S f(r\cos\theta, r\sin\theta)\, r\,drd\theta\end{aligned}$$

where S is the region in the $r\theta$-plane corresponding to R in the xy-plane.





WHEN TO CONVERT FROM CARTESIAN TO POLAR

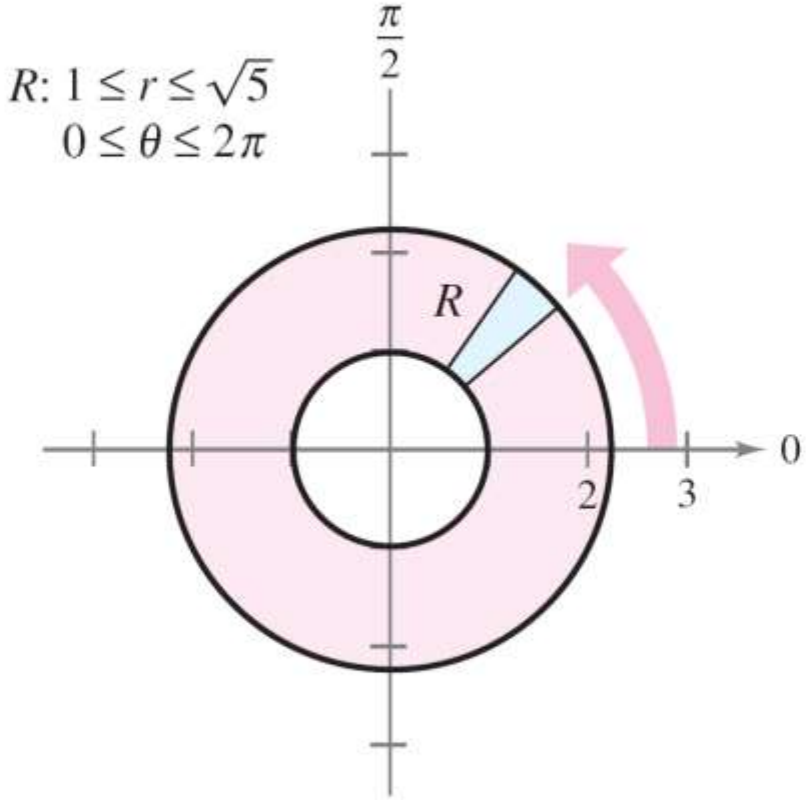
While there is no firm rule for when polar coordinates can or should be used, they are a natural alternative anytime the domain of integration may be expressed simply in polar form, and/or when the integrand involves expressions such as $\sqrt{x^2 + y^2}$.

PROBLEMS ON DOUBLE INTEGRAL IN POLAR COORDINATES

Let R be the annular region lying between the two circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 5$. Evaluate the integral $\int_R\int (x^2 + y)\, dA$.

Solution:

The polar boundaries are $1 \leq r \leq \sqrt{5}$ and $0 \leq \theta \leq 2\pi$,



PROBLEMS ON POLAR COORDINATES CONTD…

Furthermore, $x^2 = (r\cos\theta)^2$ and $y = r\sin\theta.$

So, you have

$$\iint_R (x^2 + y)\, dA = \int_0^{2\pi}\int_1^{\sqrt{5}} (r^2\cos^2\theta + r\sin\theta) r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_1^{\sqrt{5}} (r^3\cos^2\theta + r^2\sin\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{r^4}{4}\cos^2\theta + \frac{r^3}{3}\sin\theta\right)\Big]_1^{\sqrt{5}} d\theta$$

$$= \int_0^{2\pi} \left(6\cos^2\theta + \frac{5\sqrt{5}-1}{3}\sin\theta\right) d\theta$$

$$= \int_0^{2\pi} \left(3 + 3\cos 2\theta + \frac{5\sqrt{5}-1}{3}\sin\theta\right) d\theta$$

$$= \left(3\theta + \frac{3\sin 2\theta}{2} - \frac{5\sqrt{5}-1}{3}\cos\theta\right)\Big]_0^{2\pi}$$

$$= 6\pi.$$

EXAMPLES OF CHANGE OF VARIABLES IN DOUBLE INTEGRAL

Calculate the double integral $\iint\limits_R \sin\sqrt{x^2+y^2}\,dxdy$ by transforming to polar coordinates. The region R is the disk $x^2+y^2 \le \pi^2$.

Solution.

The region R is presented in Figure 8.

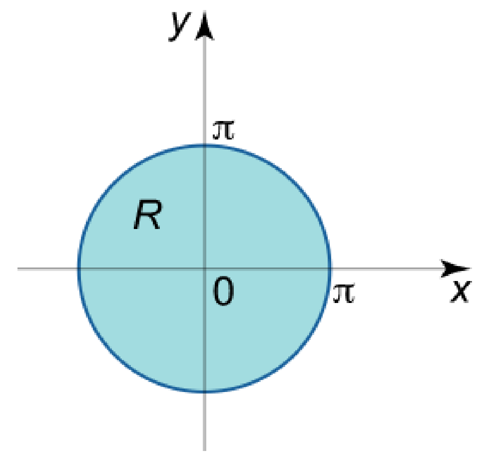


Figure 8.

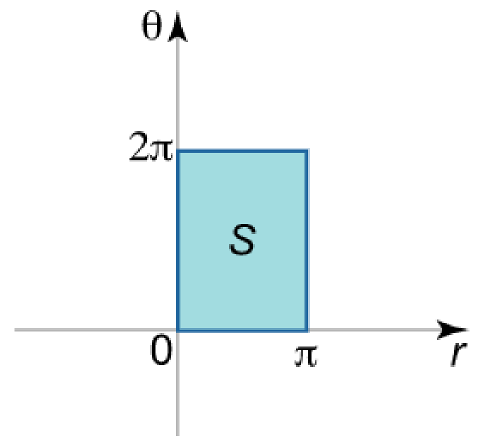


Figure 9.

The image S of the initial region R is defined by the set

$$\{S = (r,\theta) \mid 0 \le r \le \pi,\ 0 \le \theta \le 2\pi\}.$$

EXAMPLE CONTD…

and is shown in Figure 9. The double integral in polar coordinates becomes

$$I = \iint\limits_R \sin\sqrt{x^2+y^2}\,dxdy = \iint\limits_S r\sin r\,dr d\theta = \int\limits_0^{2\pi} d\theta \int\limits_0^{\pi} r\sin r\,dr = 2\pi\int\limits_0^{\pi} r\sin r\,dr.$$

We compute this integral using integration by parts:

$$\int\limits_a^b u\,dv = \left.(uv)\right|_a^b - \int\limits_a^b v\,du.$$

Let $u = r$, $dv = \sin r\,dr$. Then $du = dr$, $v = \int \sin r\,dr = -\cos r$. Hence,

$$I = 2\pi\int\limits_0^{\pi} r\sin r\,dr = 2\pi\left[\left.(-r\cos r)\right|_0^{\pi} - \int\limits_0^{\pi}(-\cos r)\,dr\right]$$

$$= 2\pi\left[\left.(-r\cos r)\right|_0^{\pi} + \int\limits_0^{\pi}\cos r\,dr\right] = 2\pi\left[\left.(-r\cos r)\right|_0^{\pi} + \left.(\sin r)\right|_0^{\pi}\right]$$

$$= 2\pi\left.(\sin r - r\cos r)\right|_0^{\pi} = 2\pi\left[(\sin\pi - \pi\cos\pi) - (\sin 0 - 0\cdot\cos 0)\right] = 2\pi\cdot\pi = 2\pi^2.$$

EXAMPLE ON POLAR COORDINATES

PES UNIVERSITY

Let $f(x,y) = e^{x^2+y^2}$ on the disk $D = \{(x,y) : x^2+y^2 \le 1\}$. evaluate $\iint_D f(x,y)\,dA$.

In rectangular coordinates the double integral $\iint_D f(x,y)\,dA$ can be written as the iterated integral

$$\iint_D f(x,y)\,dA = \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx.$$

We cannot evaluate this iterated integral, because $e^{x^2+y^2}$ does not have an elementary antiderivative with respect to either x or y. However, since $r^2 = x^2+y^2$ and the region D is circular, it is natural to wonder whether converting to polar coordinates will allow us to evaluate the new integral. To do so, we replace x with $r\cos(\theta)$, y with $r\sin(\theta)$, and $dy\,dx$ with $r\,dr\,d\theta$ to obtain

$$\iint_D f(x,y)\,dA = \iint_D e^{r^2}\,r\,dr\,d\theta.$$

EXAMPLE CONTD…

The disc D is described in polar coordinates by the constraints $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. Therefore, it follows that

$$\iint_D e^{r^2}\, r\, dr\, d\theta = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} e^{r^2}\, r\, dr\, d\theta.$$

We can evaluate the resulting iterated polar integral as follows:

$$\begin{aligned}
\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} e^{r^2}\, r\, dr\, d\theta &= \int_{\theta=0}^{2\pi} \left(\frac{1}{2} e^{r^2} \Big|_{r=0}^{r=1} \right) d\theta \\
&= \frac{1}{2} \int_{\theta=0}^{\theta=2\pi} (e-1)\, d\theta \\
&= \frac{1}{2}(e-1) \int_{\theta=0}^{\theta=2\pi} d\theta \\
&= \frac{1}{2}(e-1)\, [\theta] \Big|_{\theta=0}^{\theta=2\pi} \\
&= \pi(e-1).
\end{aligned}$$

THANK YOU

ENGINEERING MATHEMATICS-II

Department of Science and Humanities

ENGINEERING MATHEMATICS-II

UNIT 1: INTEGRAL CALCULUS

CLASS 9: CHANGE OF VARIABLES IN TRIPLE INTEGRAL, CYLINDRICAL COORDINATES

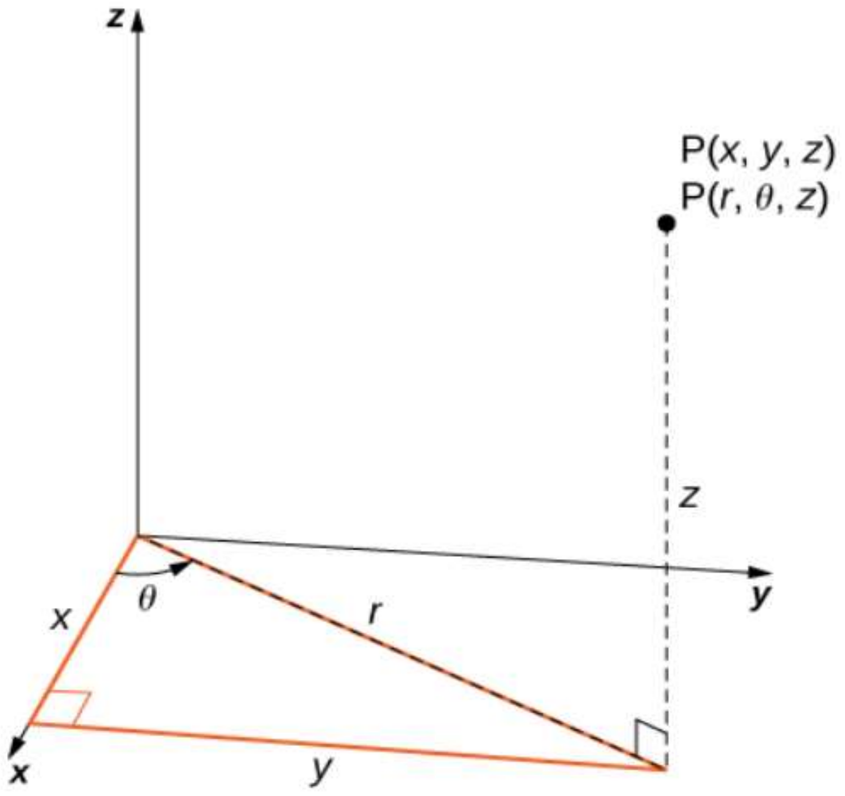
TRIPLE INTEGRAL IN CYLINDRICAL COORDINATES: MOTIVATING QUESTIONS

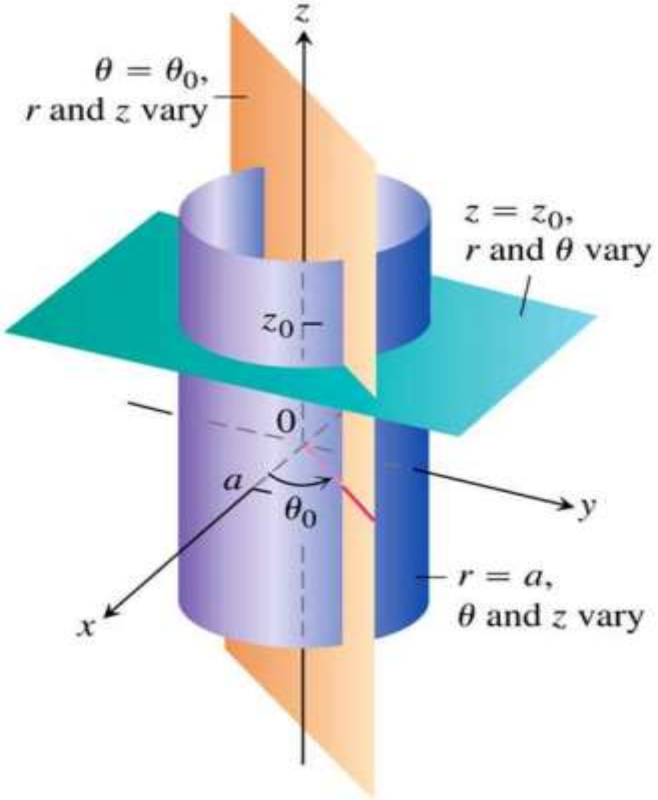
- What are the cylindrical coordinates of a point, and how are they related to Cartesian coordinates?
- What is the volume element in cylindrical coordinates? How does this inform us about evaluating a triple integral as an iterated integral in cylindrical coordinates?

We have encountered two different coordinate systems in $\mathbb{R}^2$ — the rectangular and polar coordinates systems — and seen how in certain situations, polar coordinates form a convenient alternative. In a similar way, there are two additional natural coordinate systems in $\mathbb{R}^3$. Given that we are already familiar with the Cartesian coordinate system for $\mathbb{R}^3$, we next investigate the cylindrical and spherical coordinate systems (each of which builds upon polar coordinates in $\mathbb{R}^2$). In what follows, we will see how to convert among the different coordinate systems, how to evaluate triple integrals using them, and some situations in which these other coordinate systems prove advantageous.

TRIPLE INTEGRAL IN CYLINDRICAL COORDINATES:

In three-dimensional space $\mathbb{R}^3$ a point with rectangular coordinates (x, y, z) can be identified with cylindrical coordinates (r, θ, z) and vice versa.





To convert from rectangular to cylindrical coordinates, we use the conversion

- $x = r\cos\theta$
- $y = r\sin\theta$
- $z = z$

To convert from cylindrical to rectangular coordinates, we use

- $r^2 = x^2 + y^2$ and
- $\theta = \tan^{-1}\left(\frac{y}{x}\right)$
- $z = z$

Note that that z-coordinate remains the same in both cases.

For cylindrical coordinates, the change of variables function is

$$(x, y, z) = \mathbf{T}(r, \theta, z)$$

where the components of $\mathbf{T}$ are given by

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z. \end{aligned}$$

We can compute that

$$\begin{aligned} D\mathbf{T}(\rho, \theta, \phi) &= \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial z} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial z} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial z} \end{vmatrix} \\ &= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= r\cos^2\theta + r\sin^2\theta = r. \end{aligned}$$

Note that if $g(x, y, z)$ is the function in rectangular coordinates and the box B is expressed in rectangular coordinates, then the triple integral

$$\iiint_B g(x, y, z)dV$$

is equal to the triple integral

$$\iiint_B g(r\cos\theta, r\sin\theta, z)r\,dr\,d\theta\,dz$$

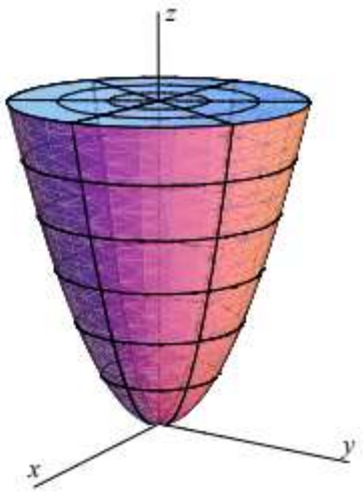
and we have

$$\iiint_B g(x, y, z)dV = \iiint_B g(r\cos\theta, r\sin\theta, z)r\,dr\,d\theta\,dz = \iiint_B f(r, \theta\, z)r\,dr\,d\theta\,dz.$$

	Circular cylinder	Circular cone	Sphere	Paraboloid
Rectangular	$x^2+y^2=c^2$	$z^2=c^2\left(x^2+y^2\right)$	$x^2+y^2+z^2=c^2$	$z=c\left(x^2+y^2\right)$
Cylindrical	$r=c$	$z=cr$	$r^2+z^2=c^2$	$z=cr^2$

EXAMPLE 1: TRIPLE INTEGRAL IN CYLINDRICAL COORDINATES

Let $\mathcal{U}$ be the solid enclosed by $z = x^2 + y^2$ and $z = 9$. Rewrite the triple integral $\iiint_{\mathcal{U}} x \, dV$ as an iterated integral. (You need not evaluate, but can you guess what the answer is?)



, the given triple integral is equal to the iterated integral

$$\boxed{\int_0^9 \int_0^{2\pi} \int_0^{\sqrt{z}} r \cos\theta \cdot r \; dr \; d\theta \; dz} \;=\; \int_0^9 \int_0^{2\pi} \left(\frac{1}{3} r^3 \cos\theta \Big|_{r=0}^{r=\sqrt{z}} \right) dr \; d\theta \; dz$$

$$= \int_0^9 \int_0^{2\pi} \frac{1}{3} z^{3/2} \cos\theta \; d\theta \; dz$$

$$= \int_0^9 \left(\frac{1}{3} z^{3/2} \sin\theta \Big|_{\theta=0}^{\theta=2\pi} \right) dz$$

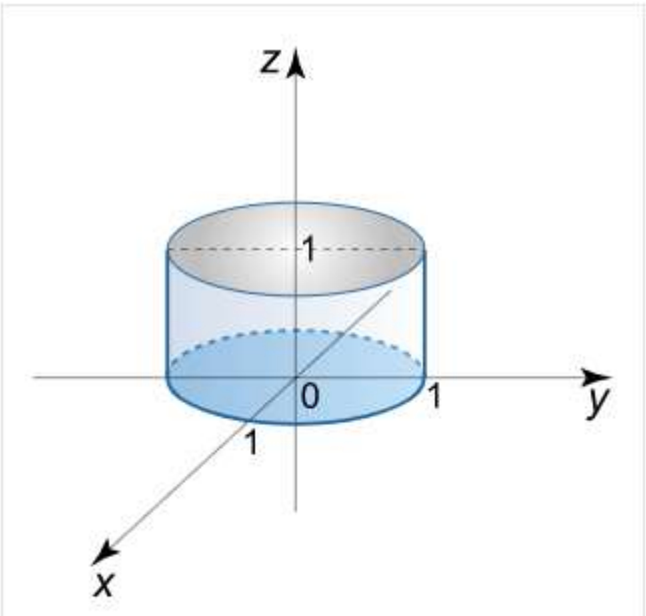
$$= \boxed{0}$$

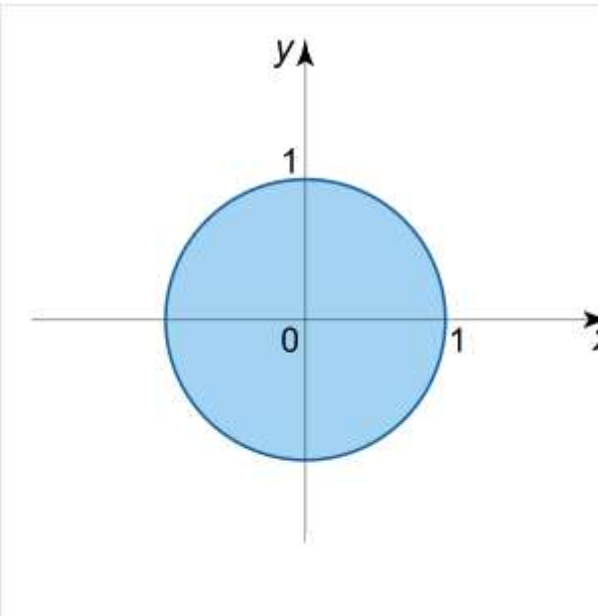
PROBLEMS

Evaluate the integral

$$\iiint\limits_U \left(x^4 + 2x^2y^2 + y^4\right) dxdydz,$$

where the region U is bounded by the surface $x^2 + y^2 \leq 1$ and the planes $z = 0$, $z = 1$



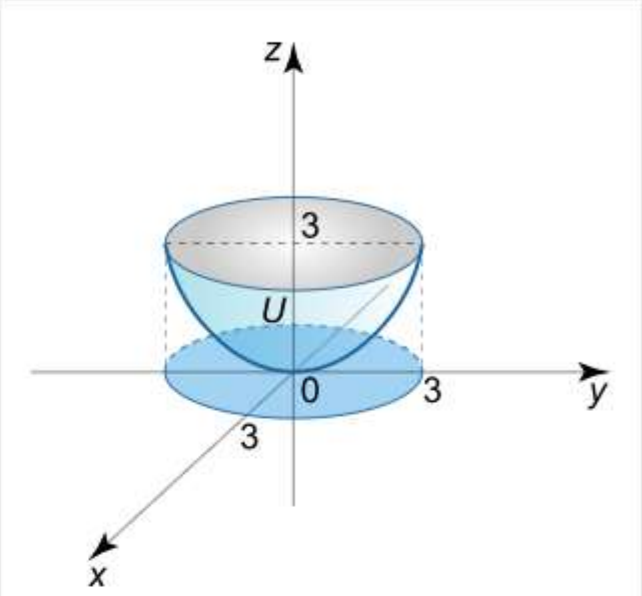


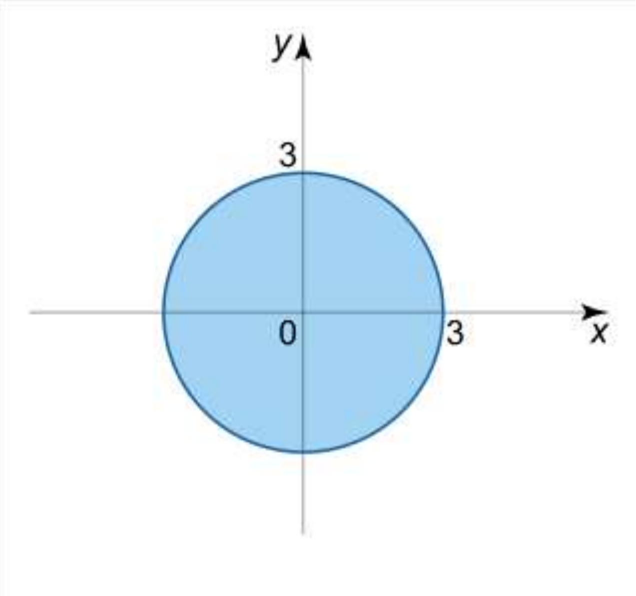
Problems

Find the integral

$$\iiint\limits_U (x^2 + y^2)\, dxdydz,$$

where the region U is bounded by the surfaces $x^2 + y^2 = 3z$, $z = 3$.





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ENGINEERING MATHEMATICS-II

Department of Science and Humanities

ENGINEERING MATHEMATICS-II

UNIT 1: INTEGRAL CALCULUS

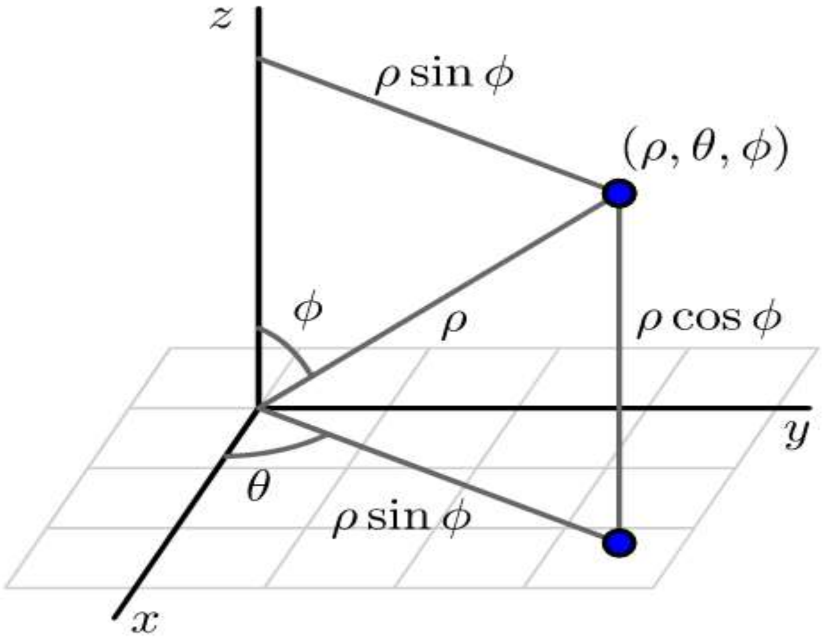
CLASS 10: CHANGE OF VARIABLES IN TRIPLE INTEGRAL, SPHERICAL COORDINATES

TRIPLE INTEGRAL IN SPHERICAL COORDINATES: MOTIVATING QUESTIONS

- What are the spherical coordinates of a point, and how are they related to Cartesian coordinates?
- What is the volume element in spherical coordinates? How does this inform us about evaluating a triple integral as an iterated integral in spherical coordinates?

TRIPLE INTEGRAL IN SPHERICAL COORDINATES:

the spherical coordinates of a point in 3- space have the form (ρ, θ, ϕ),
where ρ is the distance from the point to the origin, θ has the same meaning as in polar coordinates, and ϕ is the angle between the positive z axis and the vector from the origin to the point.



TRIPLE INTEGRAL IN SPHERICAL COORDINATES:

- The spherical coordinates of a point P in 3-space are ρ (rho), θ, and ϕ (phi), where ρ is the distance from P to the origin, θ is the angle that the projection of P onto the xy-plane makes with the positive x-axis, and ϕ is the angle between the positive z axis and the vector from the origin to P. When P has Cartesian coordinates (x, y, z), the spherical coordinates are given by

$$\rho^2 = x^2 + y^2 + z^2, \qquad \tan(\theta) = \frac{y}{x}, \qquad \cos(\phi) = \frac{z}{\rho}.$$

  Given the point P in spherical coordinates (ρ, ϕ, θ), its rectangular coordinates are

$$x = \rho \sin(\phi) \cos(\theta), \qquad y = \rho \sin(\phi) \sin(\theta), \qquad z = \rho \cos(\phi).$$

- The volume element dV in spherical coordinates is $dV = \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi$. Thus, a triple integral $\iiint_S f(x, y, z)\, dA$ can be evaluated as the iterated integral

$$\iiint_S f(\rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), \rho \cos(\phi))\, \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi.$$

TRIPLE INTEGRAL IN SPHERICAL COORDINATES:

For spherical coordinates, the change of variables function is

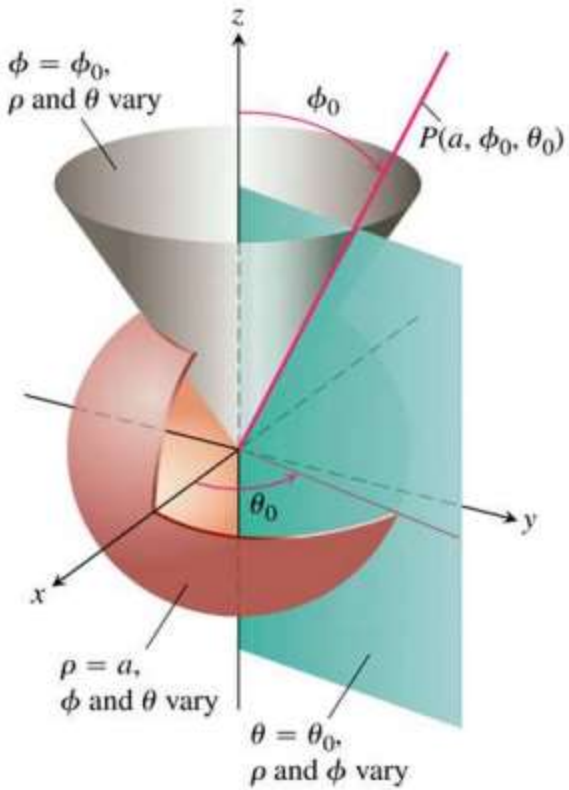
$$(x, y, z) = \mathbf{T}(\rho, \theta, \phi)$$

where the components of $\mathbf{T}$ are given by

$$\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi. \end{aligned}$$

We can compute that

$$\begin{aligned} D\mathbf{T}(\rho, \theta, \phi) &= \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} \\ &= \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} \\ &= -\rho^2 \sin\phi. \end{aligned}$$



The change of variable factor is the absolute value of the determinant

$$|D\mathbf{T}(\rho, \theta, \phi)| = \rho^2 \sin\phi.$$

VOLUME USING TRIPLE INTEGRAL IN SPHERICAL COORDINATES:

The Volume Element in Spherical Coordinates

$$dV = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

Finding the Volume of l'Hemisphèric

Hot air balloons

EXAMPLE 1: VOLUME USING TRIPLE INTEGRAL IN SPHERICAL COORDINATES

PES UNIVERSITY

Finding the Volume of an Ellipsoid

Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

We again use symmetry and evaluate the volume of the ellipsoid using spherical coordinates.

As before, we use the first octant $x \geq 0, y \geq 0,$ and $z \geq 0$ and then multiply the result by $8.$

In this case the ranges of the variables are

$$0 \leq \varphi \leq \frac{\pi}{2}, 0 \leq \rho \leq \frac{\pi}{2}, 0 \leq \rho \leq 1, \text{ and } 0 \leq \theta \leq \frac{\pi}{2}.$$

EXAMPLE 1: VOLUME USING TRIPLE INTEGRAL IN SPJERICAL COORDINATES

PES UNIVERSITY

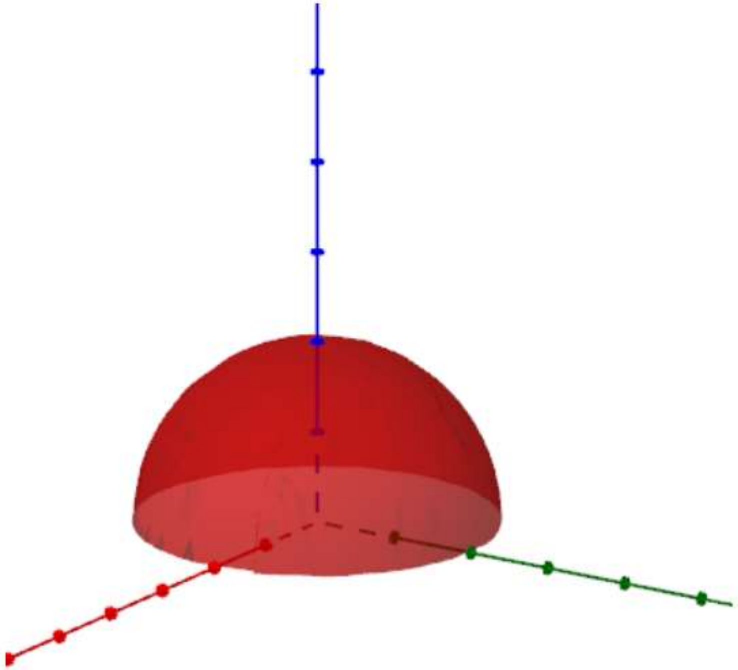
Then the volume of the ellipsoid becomes

$$\begin{aligned}
V &= \iiint\limits_D dx\,dy\,dz \\
&= 8 \int\limits_{\theta=0}^{\theta=\pi/2} \int\limits_{\rho=0}^{\rho=1} \int\limits_{\varphi=0}^{\varphi=\pi/2} abc\rho^2 \sin\theta\, d\varphi\, d\rho\, d\theta \\
&= 8abc \int\limits_{\varphi=0}^{\varphi=\pi/2} d\varphi \int\limits_{\rho=0}^{\rho=1} \rho^2 d\rho \int\limits_{\theta=0}^{\theta=\pi/2} \sin\theta\, d\theta \\
&= 8abc \left(\tfrac{\pi}{2}\right)\left(\tfrac{1}{3}\right)(1) \\
&= \tfrac{4}{3}\pi abc.
\end{aligned}$$

EXAMPLE 2: EVALUATION TRIPLE INTEGRAL IN SPHERICAL COORDINATES

Evaluate $\displaystyle\iiint_E e^{(x^2+y^2+z^2)^{3/2}}\, dV$ where E is below $x^2+y^2+z^2 \le 1$ (often referred to as top half of the unit ball) and above $z = 0$.

1. A sketch will be helpful here.



4

$$E = \begin{cases} 0 \le \rho \le 1 \\ 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi/2 \end{cases}$$

EXAMPLE 2: EVALUATION TRIPLE INTEGRAL IN SPHERICAL COORDINATES

2. Rewrite the function into spherical coordinates

$$e^{(x^2+y^2+y^2)^{3/2}} = e^{(}(\rho^2)^{3/2}) = e^{\rho^3}$$

3. Set up the integral

$$\int_0^1 \int_0^{2\pi} \int_0^{\pi/2} e^{\rho^3} \cdot \rho^2 \sin(\phi)\, d\phi\, d\theta\, d\rho$$

4. Note that the integrand is a product of functions of ϕ, ρ, and θ. This means we can rewrite the integral as

$$\int_0^1 \rho^2 e^{\rho^3}\, d\rho \cdot \int_0^{2\pi} 1\, d\theta \cdot \int_0^{\pi/2} \sin(\phi)\, d\phi$$

EXAMPLE 2: EVALUATION TRIPLE INTEGRAL IN SPHERICAL COORDINATES

(a) $\displaystyle\int_0^1 \rho^2 e^{\rho^3}\, d\rho$

i. Let $u = \rho^3$

ii. $du = 3\rho^2\, d\rho \Rightarrow \dfrac{1}{3}\, du = \rho^2\, d\rho$

iii. If $\rho = 0$, $u = (0)^3 = 0$

iv. If $\rho = 1$, $u = (1)^3 = 1$

v. Substitute

$$\int_0^1 \frac{1}{3} e^u\, du = \frac{1}{3} e^u \Big|_0^1 = \frac{1}{3} e - \frac{1}{3}$$

EXAMPLE 2: EVALUATION TRIPLE INTEGRAL IN SPHERICAL COORDINATES

(b) $\int_0^{2\pi} 1 \, d\theta \quad = \int_0^{2\pi} 1 \, d\theta = \theta\Big|_0^{2\pi} = 2\pi$

(c) $\int_0^{\pi/2} \sin(\phi) \, d\phi \quad = \int_0^{2\pi} 1 \, d\theta = \theta\Big|_0^{2\pi} = 2\pi$

(d) Finally, multiply the three integral values together

$$\int_0^1 \int_0^{2\pi} \int_0^{\pi/2} e^{\rho^3} \cdot \rho^2 \sin(\phi) \, d\phi \, d\theta \, d\rho = \left(\frac{1}{3}e - \frac{1}{3}\right) \cdot 2\pi \cdot 1 = \frac{2}{3}\pi e - \frac{2}{3}\pi$$

EXAMPLE 3: AVERAGE TEMPERATURE USING TRIPLE INTEGRAL IN SPHERICAL COORDINATES

PES UNIVERSITY

Suppose the temperature at (x, y, z) is $T = 1/(1 + x^2 + y^2 + z^2)$.

Find the average temperature in the unit sphere centered at the origin.

In two dimensions we add up the temperature at "each" point and divide by the area; here we add up the temperatures and divide by the volume, $(4/3)\pi$:

$$\frac{3}{4\pi}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}}\frac{1}{1+x^2+y^2+z^2}\,dz\,dy\,dx$$

EXAMPLE 3: AVERAGE TEMPERATURE USING TRIPLE INTEGRAL IN SPHERICAL COORDINATES

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This looks quite messy; since everything in the problem is closely related to a sphere, we'll convert to spherical coordinates.

$$\frac{3}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{1}\frac{1}{1+\rho^2}\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\frac{3}{4\pi}(4\pi-\pi^2)=3-\frac{3\pi}{4}.$$

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UNIT 1: INTEGRAL CALCULUS

CLASS 7:APPLICATIONS OF TRIPLE INTEGRAL AND PROBLEMS ON FINDING VOLUME USING TRIPLE INTEGRAL

APPLICATIONS OF TRIPLE INTEGRAL

- The triple integral

$$V(S) = \iiint_S 1 \, dV$$

represents the *volume* of the solid S.

- The *average value* of the function $f = f(x, y, x)$ over a solid domain S is given by

$$f_{\text{AVG}(S)} = \left(\frac{1}{V(S)} \right) \iiint_S f(x, y, z) \, dV,$$

where $V(S)$ is the volume of the solid S.

APPLICATIONS OF TRIPLE INTEGRAL

- The *center of mass* of the solid S with density $\delta = \delta(x, y, z)$ is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{\iiint_S x\,\delta(x, y, z)\,dV}{M},$$
$$\bar{y} = \frac{\iiint_S y\,\delta(x, y, z)\,dV}{M},$$
$$\bar{z} = \frac{\iiint_S z\,\delta(x, y, z)\,dV}{M},$$

and $M = \iiint_S \delta(x, y, z)\,dV$ is the mass of the solid S.

APPLICATIONS OF TRIPLE INTEGRAL

The **second moments** (or **moments of inertia**) about the and x-, y-, and z-axes are as follows.

$$I_x = \iiint_Q (y^2 + z^2)\rho(x, y, z)\, dV \qquad \text{Moment of inertia about } x\text{-axis}$$

$$I_y = \iiint_Q (x^2 + z^2)\rho(x, y, z)\, dV \qquad \text{Moment of inertia about } y\text{-axis}$$

$$I_z = \iiint_Q (x^2 + y^2)\rho(x, y, z)\, dV \qquad \text{Moment of inertia about } z\text{-axis}$$

VOLUME USING TRIPLE INTEGRAL: DEFINITION

The simplest application of triple integrals is to compute volumes in an alternate way.

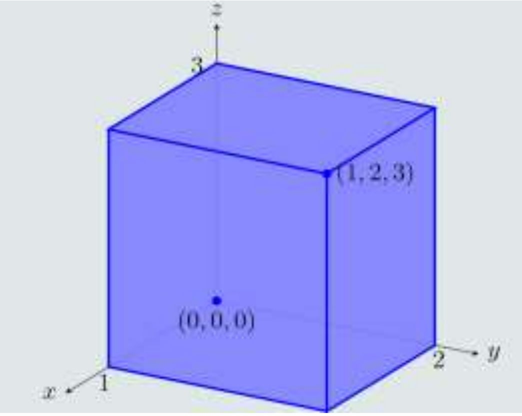
If the three-variable function f is the constant 1, then the triple integral $\iiint_S dV$ evaluates to the volume of the closed bounded region S.

If the three-variable function f is the constant 1 and S is bounded by constants, then we are simply computing the volume of a rectangular box.

VOLUME USING TRIPLE INTEGRAL: EXAMPLE:1

Compute the volume of the box with opposite corners at $(0,0,0)$ and $(1,2,3)$.

$$0 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 3.$$



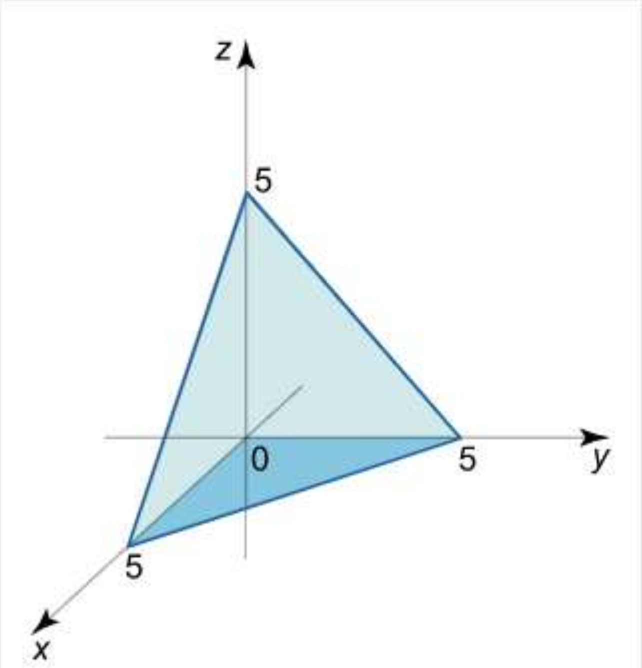
$$\begin{aligned}\int_0^1\int_0^2\int_0^3 dz\,dy\,dx &= \int_0^1\int_0^2 z\Big|_0^3\,dy\,dx\\ &= \int_0^1\int_0^2 3\,dy\,dx\\ &= \int_0^1 3y\Big|_0^2\,dx\\ &= \int_0^1 6\,dx = 6.\end{aligned}$$

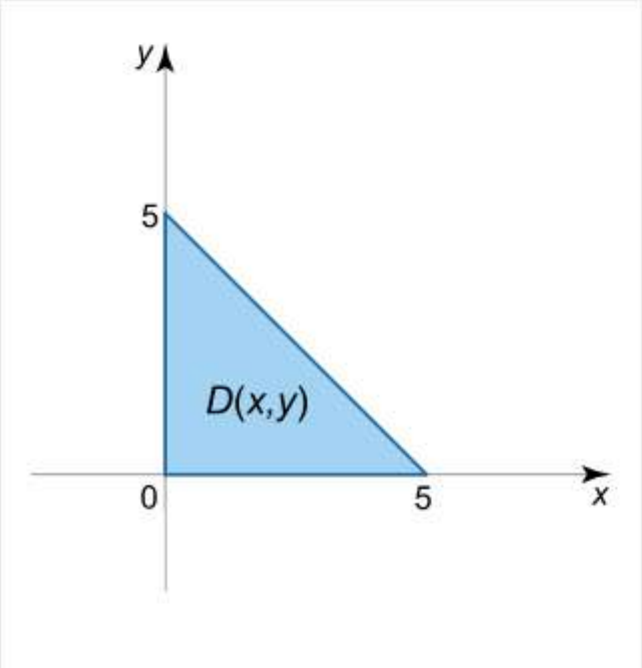
$$\int_0^3\int_0^2\int_0^1 dx\,dy\,dz,\quad \int_0^3\int_0^2\int_0^1 dx\,dy\,dz,\quad \int_0^3\int_0^1\int_0^2 dy\,dx\,dz,$$

$$\int_0^1\int_0^3\int_0^2 dy\,dz\,dx,\quad \int_0^2\int_0^1\int_0^3 dz\,dx\,dy, \text{ or } \int_0^1\int_0^2\int_0^3 dz\,dy\,dx.$$

VOLUME USING TRIPLE INTEGRAL: EXAMPLE:2

Find the volume of the tetrahedron bounded by the planes $x + y + z = 5$, $x = 0$, $y = 0$,





VOLUME USING TRIPLE INTEGRAL: EXAMPLE:2 contd.....

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Representing the triple integral as an iterated integral, we can find the volume of the tetrahedron:

$$V = \iiint\limits_U dx\,dy\,dz = \int\limits_0^5 dx \int\limits_0^{5-x} dy \int\limits_0^{5-x-y} dz = \int\limits_0^5 dx \int\limits_0^{5-x} dy \cdot \left[z|_0^{5-x-y} \right]$$

$$= \int\limits_0^5 dx \int\limits_0^{5-x} (5-x-y)\,dy = \int\limits_0^5 dx \left[\left(5y - xy - \frac{y^2}{2} \right) \Big|_{y=0}^{y=5-x} \right]$$

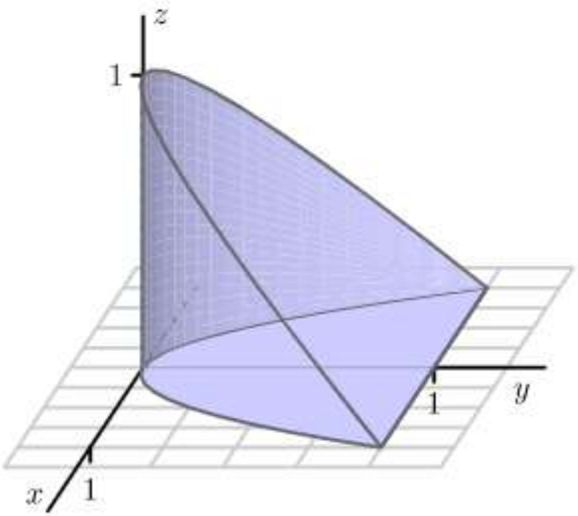
$$= \int\limits_0^5 \left[5(5-x) - x(5-x) - \frac{(5-x)^2}{2} \right] dx$$

$$= \int\limits_0^5 \left(25 - 5x - 5x + x^2 - \frac{25 - 10x + x^2}{2} \right) dx = \frac{1}{2} \int\limits_0^5 \left(25 - 10x + x^2 \right) dx$$

$$= \frac{1}{2} \left[\left(25x - \frac{10x^2}{2} + \frac{x^3}{3} \right) \Big|_0^5 \right] = \frac{1}{2} \left(125 - 5 \cdot 25 + \frac{125}{3} \right) = \frac{125}{6}.$$

VOLUME USING TRIPLE INTEGRAL: EXAMPLE:3

Consider the solid S that is bounded by the parabolic cylinder $y = x^2$ and the planes $z = 0$ and $z = 1 - y$



$z: 0 \to 1 - y$

$y: 0 \to x^2$

$x: -1 \to 1$

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UNIT 1: INTEGRAL CALCULUS

CLASS 8: APPLICATIONS OF TRIPLE INTEGRAL ON FINDING AVERAGE VALUE USING TRIPLE INTEGRAL

AVERAGE VALUE OF A FUNCTION OF THREE VARIABLES

Recall that we found the average value of a function of two variables

by evaluating the double integral over a region on the plane and then

dividing by the area of volume of the solid.

If $f(x, y, z)$ is integrable over a solid bounded region E with positive volume *V(E)*

$$f_{ave} = \frac{1}{V(E)} \iiint_E f(x, y, z)\, dV$$

AVERAGE VALUE OF A FUNCTION OF THREE VARIABLES: EXAMPLE 1

Note that the volume is $V(E) = \iiint_E 1\,\mathrm{d}V$

Suppose the temperature at a point is given by $T = xyz$. Find the average temperature in the cube with opposite corners at (0,0,0) and (2,2,2).

In two dimensions:

1. Add up the temperature at each point in a region
2. Divide by the area.

In three dimensions:

1. Add up the temperature at each point in space.
2. Divide by the volume.

AVERAGE VALUE OF A FUNCTION OF THREE VARIABLES: EXAMPLE 1 CONTD…

Therefore, the average temperature in the cube is

$$\frac{1}{8}\int_0^2\int_0^2\int_0^2 xyz\,dx\,dy\,dz = \frac{1}{8}\int_0^2 x\,dx\int_0^2 y\,dy\int_0^2 z\,dz$$

AVERAGE VALUE OF A FUNCTION OF THREE VARIABLES: EXAMPLE 2

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The temperature at a point (x, y, z) of a solid E, bounded by the coordinate planes and the plane $x + y + z = 1$ is $T(x, y, z) \cdot= (xy + 8z + 20)^0 c$. Find the average temperature over the solid.

Use the theorem given above and the triple integral to find the numerator and the denominator. Then do the division.

Notice that the plane $x + y + z = 1$ has intercepts (1,0,0),(0,1,0) and (0,0,1).

AVERAGE VALUE OF A FUNCTION OF THREE VARIABLES: EXAMPLE 2 contd....

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The region E looks like

$$E = \{(x_1 y, z) | 0 \le x \le 1, 0 \le y \le 1 - x, 0 \le z \le$$

$$\iiint_E f(x,y,z)\,dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} (xy + 8z + 20)\,dz\,dy\,dx = \frac{147}{40}.$$

The volume integral is

$$V(E) = \iiint_E 1\,dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} 1\,dz\,dy\,dx = \frac{1}{6}.$$

Hence the average value is $$T_{ave} = \frac{147/40}{1/6} = \frac{6(147)}{40} = \frac{441}{20}\ {}^\circ\mathrm{C}$$

AVERAGE VALUE USING TRIPLE INTEGRAL: EXAMPLE:1

Find the average value of the function $f(x, y, z) = ye^{-xy}$ over the rectangular prism $0 \leq x \leq 2, 0 \leq y \leq 3, 0 \leq z \leq 2$

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