

Calculus Notes

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Part I

Integral Calculus in One Variable

1 Indefinite Integrals

1.1 Integral of a function

The integral is the inverse of the derivative operator. Given a function $f(x)$, we can find a function $F(x)$ such that $F'(x) = f(x)$. The integral of a function is not unique, as for each $f(x)$, $F(x) + C$ is also an integral of that function.

Concept

Both the derivative and the integral are linear operators, so we have:

$$\int [\alpha f(x) + \beta g(x)] dx = \alpha \int f(x) dx + \beta \int g(x) dx$$

1.2 Ways to calculate indefinite integrals

Expansion

We use the linearity rule to turn a complicated integral into the sum of many simpler ones, then calculate one by one.

Changing the differential expression (u -substitution)

If $\int f(x) dx = F(x) + C$ then $\int f(u) dx = F(u) + C$, where $u = u(x)$ is a continuously differentiable function. Then, we can change the integrand $g(x) dx$ into:

$$g(x) = f(u(x))u'(x) dx$$

Then the integral turns into:

$$\int g(x) dx = \int f(u(x))u'(x) dx = \int f(u(x)) du = F(u) + C$$

Insight

In the simple case $u = ax + b$, we have $du = a dx$. Then if $\int f(x) dx = F(x) + C$ then:

$$\int f(ax + b) dx = \frac{1}{a} F(ax + b) + C$$

Change of variables

Consider the integral $I = \int f(x)dx$, where $f(x)$ is a continuous function. We can change $f(x)$ such that we work with functions with known or easier antiderivatives:

1. Change of variables type 1:

Let $x = \varphi(t)$, where $\varphi(t)$ is a monotonic and continuously differentiable function. Then:

$$I = \int f(x)dx = \int f[\varphi(t)]\varphi'(t)dt$$

Denote the antiderivative of $g(t) = f[\varphi(t)]\varphi'(t)$ as $G(t)$ and $h(x)$ as the inverse of $x = \varphi(t)$, we then have:

$$\int g(t)dt = G(t) + C \Rightarrow I = G[h(x)] + C$$

2. Change of variables type 2:

Let $t = \psi(x)$, where $\psi(x)$ is a continuously differentiable function and we can write $f(x) = g[\psi(x)]\psi'(x)$. Then:

$$I = \int f(x)dx = \int g[\psi(x)]\psi'(x)dx$$

Denote the antiderivative of $g(t)$ as $G(t)$, then:

$$I = G[\psi(x)] + C$$

Important

Remember to change back to the original variable!

Integration by parts

Let $u = u(x)$ and $v = v(x)$ be continuously differentiable functions. We know:

$$d(uv) = u dv + v du \Rightarrow \int d(uv) = \int u dv + v du$$

Then we have the following formula:

$$\int u dv = uv - \int v du$$

Consider the integral $I = \int f(x)dx$ We need to express:

$$f(x)dx = [g(x)h(x)]dx = g(x)[h(x)dx] = u dv$$

then apply the integration by parts formula to $u = g(x)$, $v = h(x)dx$

1.3 Integral of rational functions