Calculus Notes

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Contents

Part I
Integral Calculus in One Variable

1 Indefinite Integrals

1.1 Integral of a function

The integral is the inverse of the derivative operator. Given a function $f(x)$, we can find a function $F(x)$ such that $F'(x) = f(x)$. The integral of a function is not unique, as for each $f(x)$, $F(x) + C$ is also an integral of that function.

Concept

Both the derivative and the integral are linear operators, so we have:

$$\int [\alpha f(x) + \beta g(x)] \mathrm{d}x = \alpha \int f(x) \mathrm{d}x + \beta \int g(x) \mathrm{d}x$$

1.2 Ways to calculate indefinite integrals

1. **Expansion**

   We use the linearity rule to turn a complicated integral into the sum of many simpler ones, then calculate one by one.

2. **Changing the differential expression (u-substitution)**

   If $\displaystyle\int f(x)\mathrm{d}x = F(x) + C$ then $\displaystyle\int f(u)\mathrm{d}x = F(u) + C$, where $u = u(x)$ is a continuously differentiable function. Then, we can change the integrand $g(x)\mathrm{d}x$ into:

   $$g(x) = f(u(x))u'(x)\mathrm{d}x$$

   Then the integral turns into:

   $$\int g(x)\mathrm{d}x = \int f(u(x))u'(x)\mathrm{d}x = \int f(u(x))\mathrm{d}u = F(u) + C$$

   Insight

   In the simple case $u = ax + b$, we have $\mathrm{d}u = a\mathrm{d}x$. Then if $\displaystyle\int f(x)\mathrm{d}x = F(x) + C$ then:

   $$\int f(ax+b)\mathrm{d}x = \frac{1}{a}F(ax+b) + C$$

3. **Change of variables** Consider the integral $I = \int f(x)\mathrm{d}x$, where $f(x)$ is a continuous function. We can change $f(x)$ such that we work with functions with known or easier antiderivatives:

   (a) **Change of variables type 1:**
   Let $x = \varphi(t)$, where $\varphi(t)$ is a monotonic and continuously differentiable function. Then:

   $$I = \int f(x)\mathrm{d}x = \int f[\varphi(t)]\varphi'(t)\mathrm{d}t$$

   Denote the antiderivative of $g(t) = f[\varphi(t)]\varphi'(t)$ as $G(t)$ and $h(x)$ as the inverse of $x = \varphi(t)$, we then have:

   $$\int g(t)\mathrm{d}t = G(t) + C \Rightarrow I = G[h(x)] + C$$

   (b) **Change of variables type 2:**
   Let $t = \psi(x)$, where $\psi(x)$ is a continuously differentiable function and we can write $f(x) = g[\psi(x)]\psi'(x)$. Then:

   $$I = \int f(x)\mathrm{d}x = \int g[\psi(x)]\psi'(x)\mathrm{d}x$$

   Denote the antiderivative of $g(t)$ as $G(t)$, then:

   $$I = G[\psi(x)] + C$$

   > **Important**
   >
   > Remember to change back to the original variable!

4. **Integration by parts** Let $u = u(x)$ and $v = v(x)$ be continuously differentiable functions. We know:

   $$\mathrm{d}(uv) = u\mathrm{d}v + v\mathrm{d}u \Rightarrow \int \mathrm{d}(uv) = \int u\mathrm{d}v + v\mathrm{d}u$$

   Then we have the following formula:

   $$\int u\mathrm{d}v = uv - \int v\mathrm{d}u$$

   Consider the integral $I = \int f(x)\mathrm{d}x$ We need to express:

   $$f(x)\mathrm{d}x = [g(x)h(x)]\mathrm{d}x = g(x)[h(x)\mathrm{d}x] = u\mathrm{d}v$$

   then apply the integration by parts formula to $u = g(x)$, $v = h(x)\mathrm{d}x$

1.3 Integral of rational functions

A rational function is one with the form $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x. If $\deg P(x) < \deg Q(x)$ then it's called a *true rational function.*

Using polynomial division, we can rewrite any rational function:

$$f(x) = H(x) + \frac{r(x)}{Q(x)}$$

The integral of $H(x)$ can easily be computed. As for the true rational function $\dfrac{r(x)}{Q(x)}$, we will use partial fractions to decompose it into four simpler types of functions. First, using the method of Undetermined Coefficients, we can write $Q(x)$ as the product of linear polynomials and quadratic polynomials with no real roots:

$$Q(x) = \prod_{k=1}^{n}(x - a_k)^{\alpha_k} \prod_{k=1}^{n}(x^2 + p_k x + q_k)^{\beta_k}$$

Note that this can always be achieved as per the Fundamental Theorem of Algebra. Then, we just need to compute the following four types of integrals:

Insight

1. $\displaystyle\int \frac{A}{x-a}\mathrm{d}x$
2. $\displaystyle\int \frac{A}{(x-a)^k}\mathrm{d}x \quad (k \geq 2)$
3. $\displaystyle\int \frac{Mx+N}{x^2+px+q}\mathrm{d}x$
4. $\displaystyle\int \frac{Mx+N}{(x^2+px+q)^m}\mathrm{d}x \quad (m \geq 2)$

These integrals cover all cases because any partial fraction decomposition over $\mathbb{R}$ involves repeated linear factors and irreducible quadratic factors, possibly raised to powers. The first two ones are simple:

1. $\displaystyle\int \frac{A}{x-a}\mathrm{d}x = A\ln|x-a| + C$
2. $\displaystyle\int \frac{A}{(x-a)^k}\mathrm{d}x = A\int (x-a)^{-k} = \frac{-A}{(k-1)(x-a)^{k-1}} \quad (k \geq 2)$

The third one is a bit more complicated: we need to use a clever substitution, inspired by completing the square, to bring it back to familiar forms:

Let $a = \sqrt{q - \frac{p^2}{4}}$ and $t = x + \frac{p}{2}$, the integral becomes:

$$\begin{aligned}\int \frac{Mx+N}{x^2+px+q}\mathrm{d}x &= \int \frac{Mt + (N - \frac{Mp}{2})}{t^2+a^2}\mathrm{d}t \\ &= M\int \frac{t}{t^2+a^2}\mathrm{d}t + \left(N - \frac{Mp}{2}\right)\int \frac{1}{t^2+a^2}\mathrm{d}t\end{aligned}$$

For the first integral, substitute in $u = t^2$ and we have a logarithm. For the second one, divide both the numerator and the denominator by a^2 then substitute $u = \frac{t}{a}$, we yield the inverse tangent. Then the integral is simply:

$$\int \frac{Mx+N}{x^2+px+q}\mathrm{d}x = \frac{M}{2}\ln(x^2+px+q) + \frac{2N-Mp}{\sqrt{4q-p^2}}\arctan\frac{2x+p}{\sqrt{4q-p^2}} + C$$

Before moving on to solve the last integral, let's review a bit of complex numbers:

Concept

Some important (and beautiful) formulas::

$$\begin{aligned}\textbf{Euler's Formula:}\quad & e^{ix} = \cos x + i\sin x \\ \textbf{De Moivre's Formula:}\quad & (\cos x + i\sin x)^n = \cos nx + i\sin nx\end{aligned}$$

We will prove it using ODEs. Let $f(x) = e^x$ and $g(x) = \cos x + i\sin x$. We know:

$$\begin{aligned}f'(x) &= ie^{ix} = if(x) \\ g'(x) &= -\sin x + i\cos x = ig(x)\end{aligned}$$

And also the initial values:

$$f(0) = g(0) = 1$$

By uniqueness of solutions to first-order linear ODEs, Euler's formula is proven. By substituting $f(nx) = g(nx)$, De Moivre's formula is proven. Then we have:

Insight

- $\cos x = \Re(e^{ix}) = \dfrac{e^{ix}+e^{-ix}}{2}$
- $\sin x = \Im(e^{ix}) = \dfrac{e^{ix}+e^{-ix}}{2i}$

From this, we can derive a general formula for powers of the sine and cosine:

$$\cos^n x = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^n = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{i(n-2k)x}$$

Now if we group k and $n-k$ in pairs to get $\cos(n-2k)$, we get:

$$\cos^n x = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos((n-2k)x)$$

Analogously, we also have:

$$\sin^n x = \frac{1}{(2i)^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\cos(\frac{\pi n}{2}-(n-2k)x)$$

Now, back to the fourth integral, we first do the same thing: change variables $a = \sqrt{q-\frac{p^2}{4}}$ and $t = x+\frac{p}{2}$. This yields:

$$\begin{aligned}\int \frac{Mx+N}{(x^2+px+q)^m}\mathrm{d}t = & \int \frac{Mt+(N-\frac{Mp}{2})}{(t^2+a^2)^m}\mathrm{d}t \\ = & M\int\frac{t}{(t^2+a^2)^m}\mathrm{d}t + \left(N-\frac{Mp}{2}\right)\int\frac{1}{(t^2+a^2)^m}\mathrm{d}t\end{aligned}$$

Again, to compute the first integral, we use the substitution $u = t^2$. Then it is:

$$M\int\frac{t}{(t^2+a^2)^m}\mathrm{d}t = \frac{-M}{2(m-1)(t^2+a^2)^{m-1}}+C$$

For the second integral however, this time we need a different substitution. Let $t = a\tan z$. Then: $t^2+a^2 = a^2\sec^2 z$, and $\mathrm{d}t = a\sec^2\mathrm{d}z$. Then the integral turns into:

$$\int\frac{1}{(t^2+a^2)^m}\mathrm{d}t = a^{2m-1}\int\cos^{2m-2} z\mathrm{d}z$$

Let $I(z) = \int\cos^{2m-2} z\mathrm{d}z$and now, we can take advantage of the linearity rule and the formula for powers of cosine we just derived. The integral turns into:

$$\begin{aligned}I(z) = & \int\frac{1}{2^{2m-2}}\sum_{k=0}^{2m-2}\binom{2m-2}{k}\cos(2(m-k-1)z)\mathrm{d}z \\ = & \frac{1}{2^{2m-2}}\sum_{k=0}^{2m-2}\binom{2m-2}{k}\frac{1}{2m-2-2k}\sin(2(m-k-1)z)+C \\ = & \frac{1}{2^{2m-2}}\left[\binom{2m-2}{m-1}z+\sum_{k=0,k\neq m-1}^{2m-2}\binom{2m-2}{k}\frac{1}{2m-2-2k}\sin(2(m-k-1)z)\right]+C\end{aligned}$$

Then the integral, as a whole, evaluates to:

$$\int\frac{Mx+N}{(x^2+px+q)^m}\mathrm{d}x = \frac{-M}{2(m-1)(x^2+px+q)^{m-1}}+\left(N-\frac{Mp}{2}\right)\left(\sqrt{q-\frac{p^2}{4}}\right)^{2m-1}\cdot I(z)+C$$

1.4 Integral of trigonometric functions

1. **The general method**

   Consider the integral $\int f(\sin x, \cos x)\mathrm{d}x$, where the integrand is a rational function in terms of $\sin x$ and $\cos x$. We can use the "universal trigonometric substitution" $x = \tan\frac{t}{2}$. Then:

   $$\sin x = \frac{2t}{1+t^2}; \quad \cos x = \frac{1-t^2}{1+t^2}; \quad \tan x = \frac{2t}{1-t^2}; \mathrm{d}t = \frac{2\mathrm{d}t}{1+t^2}$$

   The integrand turns into a rational function in terms of t.

2. **Integrals of the form $\int \sin^m x \cos^n x \mathrm{d}x$, where m, n are positive integers**

   - If m is odd, we let $t = \cos x$
   - If n is odd, we let $t = \sin x$
   - If both m and n are even, we use power-reduction formulae:

     $$\sin^2 x = \frac{1-\cos 2x}{2}; \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

     Then we'll have a similar integral with the form $\int \sin^k 2x \cos^l 2x \mathrm{d}x$

3. **Special forms of $\int f(\sin x, \cos x)\mathrm{d}x$**

   - Let $t = \cos x$ if $f(-\sin x, \cos x) = -f(\sin x, \cos x)$
   - Let $t = \sin x$ if $f(\sin x, -\cos x) = -f(\sin x, \cos x)$
   - Let $t = \tan x$ if $f(-\sin x, -\cos x) = f(\sin x, \cos x)$

1.5 Integral of irrational expressions

There are two main ways to solve these integrals: using trigonometric substitution, and using the Euler substitution. The trig-sub is very intuitive, but the Euler substitution is also very nice:

Concept

Let $t = x + \sqrt{x^2+a}$ for the integral $\int f(x, \sqrt{x^2+a})\mathrm{d}x$. Then:

$$\mathrm{d}t = 1 + \frac{x}{\sqrt{x^2+a}}\mathrm{d}x = \frac{x+\sqrt{x^2+a}}{\sqrt{x^2+a}}\mathrm{d}x \Rightarrow \frac{\mathrm{d}t}{t} = \frac{\mathrm{d}x}{\sqrt{x^2+a}}$$

2 Definite Integrals

2.1 Definition

Say $f(x)$ is defined and bounded on $[a, b]$. Partition $[a, b]$ into n subintervals $[x_i, x_{i+1}]$ where $a = x_0 < x_1 < \ldots < x_n = b$. In each interval $[x_i, x_{i+1}]$, we choose a point $\xi \in [x_i, x_{i+1}]$ and form the expression

$$S_n = \sum_{i=0}^{n-1} f(\xi_i)\Delta x_i$$

where $\Delta x_i = x_{i+1} - x_i$. Here, S_n is the Riemann sum. Denote $\lambda = \max_{1 \le i \le n} \Delta x_i$. If there exists the limit $I = \lim_{\lambda \to 0} S_n$ that doesn't depend on how we partition $[a, b]$ and how we choose ξ_i then I is called the definite integral of the function $f(x)$ on $[a, b]$, denoted $\int_a^b f(x)\mathrm{d}x$. Then we say $f(x)$ is integrable on $[a, b]$.

We then have defined the definite integral for all $a < b$. We can then define, if $b < a$, $\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x$ and when $a = b$, $\int_a^b f(x)\mathrm{d}x = 0$.

2.2 Riemann Integrability

The sufficient and necessary condition for a bounded function $f(x)$ to be integrable on $[a, b]$ is $\lim_{\lambda \to 0}(S - s) = 0$, where:

$$S = \sum_{i=1}^{n+1} M_i \Delta x_i \qquad s = \sum_{i=1}^{n+1} m_i \Delta x_i$$

$$M_i = \sup_{x \in [x_i, x_{i+1}]} f(x) \qquad m_i = \inf_{x \in [x_i, x_{i+1}]} f(x)$$

From there, we have a few crucial theorems:

Concept

- If $f(x)$ is continuous on $[a, b]$ then it is integrable on $[a, b]$.
- If $f(x)$ is bounded on $[a, b]$ and has discontinuities on $[a, b]$ then it is integrable on $[a, b]$
- If $f(x)$ is bounded and monotonic on $[a, b]$ then it is integrable on $[a, b]$

2.3 Properties of the Definite Integral

1. **Property 1 (Linearity:)**

$$\int_a^b [\alpha f(x) + \beta g(x)]\mathrm{d}x = \alpha \int_a^b f(x)\mathrm{d}x + \beta \int_a^b g(x)\mathrm{d}x$$

2. **Property 2:** Given three closed intervals $[a,b],[b,c],[a,c]$, if $f(x)$ is integrable on the longest interval then it is also integrable on the other two integrals, and:

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x$$

3. **Property 3:** Suppose $a < b$. Then:

   (a) If $f(x) \geq 0\ \forall\ x \in [a,b]$ then $\displaystyle\int_a^b f(x)\mathrm{d}x \geq 0$

   (b) If $f(x) \geq g(x)\ \forall\ x \in [a,b]$ then $\displaystyle\int_a^b f(x)\mathrm{d}x \geq \int_a^b g(x)\mathrm{d}x$

   (c) If $f(x)$ is integrable on $[a,b]$ then $|f(x)|$ is integrable on $[a,b]$ and:

$$\left|\int_a^b f(x)\mathrm{d}x\right| \leq \int_a^b |f(x)|\mathrm{d}x$$

   (d) If $m \leq f(x) \leq M\ \forall\ x \in [a,b]$ then:

$$m(b-a) \leq \int_a^b f(x)\mathrm{d}x \leq M(b-a)$$

4. **Property 4 (First Mean Value Theorem):**

   Suppose $f(x)$ is integrable on $[a,b]$ and $m \leq f(x) \leq M\ \forall\ x \in [a,b]$, then there exists μ such that:

$$\int_a^b f(x)\mathrm{d}x = \mu(b-a), \qquad m < \mu < M$$

   If $f(x)$ is continuous on $[a,b]$ then there exists $c \in [a,b]$ such that:

$$\int_a^b f(x)\mathrm{d}x = f(c)(b-a)$$

5. **Property 5 (Second Mean Value Theorem):** If we have these three conditions:

   (a) $f(x)$ and $f(x)g(x)$ are integrable on $[a,b]$

   (b) $m \leq f(x) \leq M\ \forall\ x \in [a,b]$

   (c) $g(x)$ does not change signs on $[a,b]$

   Then there exists μ such that:

$$\int_a^b f(x)g(x)\mathrm{d}x = \mu\int_a^b g(x)\mathrm{d}x, \qquad m < \mu < M$$

   If $f(x)$ is continuous on $[a,b]$ then there exists $c \in [a,b]$ such that:

$$\int_a^b f(x)g(x)\mathrm{d}x = f(c)\int_a^b g(x)\mathrm{d}x$$

2.4 Integral Functions

Suppose $f(x)$ is an integrable function on $[a, b]$, then for all $x \in [a, b]$, f is also integrable on $[a, x]$. We can then define the function $F(x) = \int_a^x f(t)\mathrm{d}t$. We then have some very important foundational theorems:

Concept

- If $f(x)$ is integrable on $[a, b]$ then $F(x)$ is continuous on $[a, b]$
- If f is continuous at $x_0 \in [a, b]$ then $F(x)$ is differentiable at x_0 and

$$F'(x_0) = f(x_0)$$

- If $f(x)$ is continuous on the closed interval $[a, b]$ and $F(x)$ is an integral of $f(x)$ then:

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$$

2.5 Ways to Calculate Definite Integrals

1. **Integration by parts:**

   Suppose $u(x), v(x)$ are continuously differentiable functions on $[a, b]$. Then:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

2. **Change of variables:**

   (a) **Substitute** $x = \varphi(t)$

   Consider $I = \int_a^b f(x)\mathrm{d}x$ with $f(x)$ being continuous on $[a, b]$. Substitute $x = \varphi(t)$ with the following three conditions:

   - $\varphi(t)$ is has a continuous derivative on $[a, b]$
   - $\varphi(a) = \alpha;\ \varphi(b) = \beta$
   - When t changes from α to β in $[\alpha, \beta]$ then $x = \varphi(t)$ continuously changes from a to b

   Then we have the following formula:

$$\int_a^b f(x)\mathrm{d}x = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)\mathrm{d}t$$

   (b) **Substitute** $t = \varphi(x)$

Suppose the integral we are trying to solve has the form $I = \int_a^b f[\varphi(x)]\varphi'(x)\mathrm{d}x$, where $\varphi(x)$ is monotonic and is continuously differentiable on $[a, b]$. Then:

$$\int_a^b f[\varphi(x)]\varphi'(x)\mathrm{d}x = \int_{\varphi(a)}^{\varphi(b)} f(t)\mathrm{d}t$$

3. **Recursion or Induction**

   We can look at an example. Calculate:

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x \cos nx \mathrm{d}x$$

We use integration by parts: Let $u = \cos^n x$ and $\mathrm{d}v = \cos nx\mathrm{d}x$. Then $v = \frac{1}{n}\sin nx$. Applying the integration by parts formula, we have:

$$\begin{aligned}
I_n =& \frac{1}{n}\int_0^{\frac{\pi}{2}} \cos^n x \cos nx \mathrm{d}x \\
=& \frac{1}{n}\cos^n x \sin nx\Big|_0^{\frac{\pi}{2}} + \frac{1}{n}\int_0^{\frac{\pi}{2}} n\cos^{n-1} x \sin x \sin nx \mathrm{d}x \\
=& \int_0^{\frac{\pi}{2}} \cos^{n-1} x \sin x \sin nx \mathrm{d}x
\end{aligned}$$

Then we see:

$$\begin{aligned}
2I_n =& \int_0^{\frac{\pi}{2}} \cos^n x \cos nx \mathrm{d}x + \int_0^{\frac{\pi}{2}} \cos^{n-1} x \sin x \sin nx \mathrm{d}x \\
=& \int_0^{\frac{\pi}{2}} \cos^{n-1} x(\cos x \cos nx + \sin x \sin nx)\mathrm{d}x \\
=& \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos(n-1)x \mathrm{d}x \\
=& I_{n-1}
\end{aligned}$$

It's easy to compute $I_0 = \frac{\pi}{2}$. Then this is a geometric series, and $I_n = \frac{\pi}{2^{n+1}}$.

2.6 Important Results

1. **The Fundamental Theorem of Calculus**

$$\frac{\mathrm{d}y}{\mathrm{d}x}\int_a^x f(t)\mathrm{d}t = f(x)$$

And more generally:

$$\frac{\mathrm{d}y}{\mathrm{d}x}\int_a^{g(x)} f(t)\mathrm{d}t = f(g(x))g'(x)$$

2. **Riemann Sum**

   Recall the formula from earlier:

$$S_n = \sum_{i=0}^{n-1} f(\xi_i)\Delta x_i \qquad \Delta x_i \in [x_i, x_{i+1}]$$

   If we partition $[a, b]$ into n subintervals with equal length using $a = x_0 < x_1 < \ldots < x_n = b$, where $x_i = a + (b-a)\dfrac{i}{n}$ then:

$$S_n = \frac{b-a}{n} \sum_{i=0}^{n-1} f(\xi_i) \qquad \xi_i \in [x_i, x_{i+1}]$$

   If $f(x)$ is integrable on $[a, b]$ and choosing $\xi_i = x_i$, we have the left and right sums:

   Important

   Choosing $\xi_i = x_i$ yields:

$$\lim_{n\to\infty} \frac{b-a}{n} \left[\sum_{i=0}^{n-1} f\left(a + \frac{b-a}{n} i\right) \right] = \int_a^b f(x)\mathrm{d}x$$

   Choosing $\xi_i = x_{i+1}$ yields:

$$\lim_{n\to\infty} \frac{b-a}{n} \left[\sum_{i=1}^{n} f\left(a + \frac{b-a}{n} i\right) \right] = \int_a^b f(x)\mathrm{d}x$$

3. **Integral Equalities**

   - $\displaystyle\int_a^b f(x)\mathrm{d}x = \int_a^b f(a+b-x)\mathrm{d}x$
   - $\displaystyle\int_0^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x = \int_0^{\frac{\pi}{2}} f(\cos x)\mathrm{d}x$
   - $\displaystyle\int_{-a}^{a} \frac{f(x)}{1+b^x}\mathrm{d}x = \int_0^a f(x)\mathrm{d}x$
   - $\displaystyle\int_a^b x^m(a+b-x)^n\mathrm{d}x = \int_a^b x^n(a+b-x)^m\mathrm{d}x$

4. **Integral Inequalities**

   Important

   Cauchy-Schwarz Inequality:

$$\left(\int_a^b f(x)g(x)\mathrm{d}x\right)^2 \leq \left(\int_a^b f^2(x)\mathrm{d}x\right)\left(\int_a^b g^2(x)\mathrm{d}x\right)$$