Calculus Notes

Avid David

July 3, 2025

Contents

Part I
Integral Calculus in One Variable

1 Indefinite Integrals

1.1 Integral of a function

The integral is the inverse of the derivative operator. Given a function $f(x)$, we can find a function $F(x)$ such that $F'(x) = f(x)$. The integral of a function is not unique, as for each $f(x)$, $F(x) + C$ is also an integral of that function.

Concept

Both the derivative and the integral are linear operators, so we have:

$$\int [\alpha f(x) + \beta g(x)] \mathrm{d}x = \alpha \int f(x) \mathrm{d}x + \beta \int g(x) \mathrm{d}x$$

1.2 Ways to calculate indefinite integrals

Expansion

We use the linearity rule to turn a complicated integral into the sum of many simpler ones, then calculate one by one.

Changing the differential expression (u-substitution)

If $\displaystyle\int f(x)\mathrm{d}x = F(x) + C$ then $\displaystyle\int f(u)\mathrm{d}x = F(u) + C$, where $u = u(x)$ is a continuously differentiable function. Then, we can change the integrand $g(x)\mathrm{d}x$ into:

$$g(x) = f(u(x))u'(x)\mathrm{d}x$$

Then the integral turns into:

$$\int g(x)\mathrm{d}x = \int f(u(x))u'(x)\mathrm{d}x = \int f(u(x))\mathrm{d}u = F(u) + C$$

Insight

In the simple case $u = ax + b$, we have $\mathrm{d}u = a\mathrm{d}x$. Then if $\displaystyle\int f(x)\mathrm{d}x = F(x) + C$ then:

$$\int f(ax+b)\mathrm{d}x = \frac{1}{a}F(ax+b) + C$$

Change of variables

Consider the integral $I = \displaystyle\int f(x)\mathrm{d}x$, where $f(x)$ is a continuous function. We can change $f(x)$ such that we work with functions with known or easier antiderivatives:

1. **Change of variables type 1:**

   Let $x = \varphi(t)$, where $\varphi(t)$ is a monotonic and continuously differentiable function. Then:

   $$I = \int f(x)\mathrm{d}x = \int f[\varphi(t)]\varphi'(t)\mathrm{d}t$$

   Denote the antiderivative of $g(t) = f[\varphi(t)]\varphi'(t)$ as $G(t)$ and $h(x)$ as the inverse of $x = \varphi(t)$, we then have:

   $$\int g(t)\mathrm{d}t = G(t) + C \Rightarrow I = G[h(x)] + C$$

2. **Change of variables type 2:**

   Let $t = \psi(x)$, where $\psi(x)$ is a continuously differentiable function and we can write $f(x) = g[\psi(x)]\psi'(x)$. Then:

   $$I = \int f(x)\mathrm{d}x = \int g[\psi(x)]\psi'(x)\mathrm{d}x$$

   Denote the antiderivative of $g(t)$ as $G(t)$, then:

   $$I = G[\psi(x)] + C$$

   > **Important**
   >
   > Remember to change back to the original variable!

Integration by parts

Let $u = u(x)$ and $v = v(x)$ be continuously differentiable functions. We know:

$$\mathrm{d}(uv) = u\mathrm{d}v + v\mathrm{d}u \Rightarrow \int \mathrm{d}(uv) = \int u\mathrm{d}v + v\mathrm{d}u$$

Then we have the following formula:

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u$$

Consider the integral $I = \displaystyle\int f(x)\mathrm{d}x$ We need to express:

$$f(x)\mathrm{d}x = [g(x)h(x)]\mathrm{d}x = g(x)[h(x)\mathrm{d}x] = u\mathrm{d}v$$

then applie the integration by parts formula to $u = g(x)$, $v = h(x)\mathrm{d}x$

1.3 Integral of rational functions

A rational function is one with the form $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x. If $\deg P(x) < \deg Q(x)$ then it's called a *true rational function.*

Using polynomial division, we can rewrite any rational function:

$$f(x) = H(x) + \frac{r(x)}{Q(x)}$$

The integral of $H(x)$ can easily be computed. As for the true rational function $\dfrac{r(x)}{Q(x)}$, we will use partial fractions to decompose it into four simpler types of functions. First, using the method of Undetermined Coefficients, we can write $Q(x)$ as the product of linear polynomials and quadratic polynomials with no real roots:

$$Q(x) = \prod_{k=1}^{n}(x - a_k)^{\alpha_k} \prod_{k=1}^{n}(x^2 + p_k x + q_k)^{\beta_k}$$

Note that this can always be achieved as per the Fundamental Theorem of Algebra. Then, we just need to compute the following four types of integrals:

Insight

1. $\displaystyle\int \frac{A}{x-a}\mathrm{d}x$
2. $\displaystyle\int \frac{A}{(x-a)^k}\mathrm{d}x \quad (k \geq 2)$
3. $\displaystyle\int \frac{Mx+N}{x^2+px+q}\mathrm{d}x$
4. $\displaystyle\int \frac{Mx+N}{(x^2+px+q)^m}\mathrm{d}x \quad (m \geq 2)$

These integrals cover all cases because any partial fraction decomposition over $\mathbb{R}$ involves repeated linear factors and irreducible quadratic factors, possibly raised to powers. The first two ones are simple:

1. $\displaystyle\int \frac{A}{x-a}\mathrm{d}x = A \ln|x-a| + C$
2. $\displaystyle\int \frac{A}{(x-a)^k}\mathrm{d}x = A\int (x-a)^{-k} = \frac{-A}{(k-1)(x-a)^{k-1}} \quad (k \geq 2)$

The third one is a bit more complicated: we need to use a clever substitution, inspired by completing the square, to bring it back to familiar forms:

Let $a = \sqrt{q - \frac{p^2}{4}}$ and $t = x + \frac{p}{2}$, the integral becomes:

$$\begin{aligned}\int \frac{Mx+N}{x^2+px+q}\mathrm{d}x &= \int \frac{Mt + (N - \frac{Mp}{2})}{t^2+a^2}\mathrm{d}t \\ &= M\int \frac{t}{t^2+a^2}\mathrm{d}t + \left(N - \frac{Mp}{2}\right)\int \frac{1}{t^2+a^2}\mathrm{d}t\end{aligned}$$

For the first integral, substitute in $u = t^2$ and we have a logarithm. For the second one, divide both the numerator and the denominator by a^2 then substitute $u = \frac{t}{a}$, we yield the inverse tangent. Then the integral is simply:

$$\int \frac{Mx+N}{x^2+px+q}\mathrm{d}x = \frac{M}{2}\ln(x^2+px+q) + \frac{2N-Mp}{\sqrt{4q-p^2}}\arctan\frac{2x+p}{\sqrt{4q-p^2}} + C$$