

Calculus Notes

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Part I

Integral Calculus in One Variable

1 Indefinite Integrals

1.1 Integral of a function

The integral is the inverse of the derivative operator. Given a function $f(x)$, we can find a function $F(x)$ such that $F'(x) = f(x)$. The integral of a function is not unique, as for each $f(x)$, $F(x) + C$ is also an integral of that function.

Concept

Both the derivative and the integral are linear operators, so we have:

$$\int [\alpha f(x) + \beta g(x)] dx = \alpha \int f(x) dx + \beta \int g(x) dx$$

1.2 Ways to calculate indefinite integrals

1. Expansion

We use the linearity rule to turn a complicated integral into the sum of many simpler ones, then calculate one by one.

2. Changing the differential expression (u -substitution)

If $\int f(x) dx = F(x) + C$ then $\int f(u) dx = F(u) + C$, where $u = u(x)$ is a continuously differentiable function. Then, we can change the integrand $g(x) dx$ into:

$$g(x) dx = f(u(x)) u'(x) dx$$

Then the integral turns into:

$$\int g(x) dx = \int f(u(x)) u'(x) dx = \int f(u(x)) du = F(u) + C$$

Insight

In the simple case $u = ax + b$, we have $du = a dx$. Then if $\int f(x) dx = F(x) + C$ then:

$$\int f(ax + b) dx = \frac{1}{a} F(ax + b) + C$$

3. **Change of variables** Consider the integral $I = \int f(x)dx$, where $f(x)$ is a continuous function. We can change $f(x)$ such that we work with functions with known or easier antiderivatives:

(a) **Change of variables type 1:**

Let $x = \varphi(t)$, where $\varphi(t)$ is a monotonic and continuously differentiable function. Then:

$$I = \int f(x)dx = \int f[\varphi(t)]\varphi'(t)dt$$

Denote the antiderivative of $g(t) = f[\varphi(t)]\varphi'(t)$ as $G(t)$ and $h(x)$ as the inverse of $x = \varphi(t)$, we then have:

$$\int g(t)dt = G(t) + C \Rightarrow I = G[h(x)] + C$$

(b) **Change of variables type 2:**

Let $t = \psi(x)$, where $\psi(x)$ is a continuously differentiable function and we can write $f(x) = g[\psi(x)]\psi'(x)$. Then:

$$I = \int f(x)dx = \int g[\psi(x)]\psi'(x)dx$$

Denote the antiderivative of $g(t)$ as $G(t)$, then:

$$I = G[\psi(x)] + C$$

Important

Remember to change back to the original variable!

4. **Integration by parts** Let $u = u(x)$ and $v = v(x)$ be continuously differentiable functions. We know:

$$d(uv) = u dv + v du \Rightarrow \int d(uv) = \int u dv + v du$$

Then we have the following formula:

$$\int u dv = uv - \int v du$$

Consider the integral $I = \int f(x)dx$ We need to express:

$$f(x)dx = [g(x)h(x)]dx = g(x)[h(x)dx] = u dv$$

then apply the integration by parts formula to $u = g(x)$, $v = h(x)dx$

1.3 Integral of rational functions

A rational function is one with the form $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x . If $\deg P(x) < \deg Q(x)$ then it's called a *true rational function*.

Using polynomial division, we can rewrite any rational function:

$$f(x) = H(x) + \frac{r(x)}{Q(x)}$$

The integral of $H(x)$ can easily be computed. As for the true rational function $\frac{r(x)}{Q(x)}$, we will use partial fractions to decompose it into four simpler types of functions. First, using the method of Undetermined Coefficients, we can write $Q(x)$ as the product of linear polynomials and quadratic polynomials with no real roots:

$$Q(x) = \prod_{k=1}^n (x - a_k)^{\alpha_k} \prod_{k=1}^n (x^2 + p_k x + q_k)^{\beta_k}$$

Note that this can always be achieved as per the Fundamental Theorem of Algebra. Then, we just need to compute the following four types of integrals:

Insight

1. $\int \frac{A}{x - a} dx$
2. $\int \frac{A}{(x - a)^k} dx \quad (k \geq 2)$
3. $\int \frac{Mx + N}{x^2 + px + q} dx$
4. $\int \frac{Mx + N}{(x^2 + px + q)^m} dx \quad (m \geq 2)$

These integrals cover all cases because any partial fraction decomposition over \mathbb{R} involves repeated linear factors and irreducible quadratic factors, possibly raised to powers. The first two ones are simple:

1. $\int \frac{A}{x - a} dx = A \ln |x - a| + C$
2. $\int \frac{A}{(x - a)^k} dx = A \int (x - a)^{-k} = \frac{-A}{(k - 1)(x - a)^{k-1}} \quad (k \geq 2)$

The third one is a bit more complicated: we need to use a clever substitution, inspired by completing the square, to bring it back to familiar forms:

Let $a = \sqrt{q - \frac{p^2}{4}}$ and $t = x + \frac{p}{2}$, the integral becomes:

$$\begin{aligned}\int \frac{Mx + N}{x^2 + px + q} dx &= \int \frac{Mt + (N - \frac{Mp}{2})}{t^2 + a^2} dt \\ &= M \int \frac{t}{t^2 + a^2} dt + \left(N - \frac{Mp}{2}\right) \int \frac{1}{t^2 + a^2} dt\end{aligned}$$

For the first integral, substitute in $u = t^2$ and we have a logarithm. For the second one, divide both the numerator and the denominator by a^2 then substitute $u = \frac{t}{a}$, we yield the inverse tangent. Then the integral is simply:

$$\int \frac{Mx + N}{x^2 + px + q} dx = \frac{M}{2} \ln(x^2 + px + q) + \frac{2N - Mp}{\sqrt{4q - p^2}} \arctan \frac{2x + p}{\sqrt{4q - p^2}} + C$$

Before moving on to solve the last integral, let's review a bit of complex numbers:

Concept

Some important (and beautiful) formulas::

Euler's Formula: $e^{ix} = \cos x + i \sin x$

De Moivre's Formula: $(\cos x + i \sin x)^n = \cos nx + i \sin nx$

We will prove it using ODEs. Let $f(x) = e^x$ and $g(x) = \cos x + i \sin x$. We know:

$$f'(x) = ie^{ix} = if(x)$$

$$g'(x) = -\sin x + i \cos x = ig(x)$$

And also the initial values:

$$f(0) = g(0) = 1$$

By uniqueness of solutions to first-order linear ODEs, Euler's formula is proven. By substituting $f(nx) = g(nx)$, De Moivre's formula is proven. Then we have:

Insight

- $\cos x = \Re(e^{ix}) = \frac{e^{ix} + e^{-ix}}{2}$

- $\sin x = \Im(e^{ix}) = \frac{e^{ix} - e^{-ix}}{2i}$

From this, we can derive a general formula for powers of the sine and cosine:

$$\cos^n x = \left(\frac{e^{ix} + e^{-ix}}{2} \right)^n = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} e^{i(n-2k)x}$$

Now if we group k and $n - k$ in pairs to get $\cos(n - 2k)$, we get:

$$\cos^n x = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} \cos((n - 2k)x)$$

Analogously, we also have:

$$\sin^n x = \frac{1}{(2i)^n} \sum_{k=0}^n \binom{n}{k} (-1)^k \cos\left(\frac{\pi n}{2} - (n - 2k)x\right)$$

Now, back to the fourth integral, we first do the same thing: change variables $a = \sqrt{q - \frac{p^2}{4}}$ and $t = x + \frac{p}{2}$. This yields:

$$\begin{aligned} \int \frac{Mx + N}{(x^2 + px + q)^m} dx &= \int \frac{Mt + (N - \frac{Mp}{2})}{(t^2 + a^2)^m} dt \\ &= M \int \frac{t}{(t^2 + a^2)^m} dt + \left(N - \frac{Mp}{2}\right) \int \frac{1}{(t^2 + a^2)^m} dt \end{aligned}$$

Again, to compute the first integral, we use the substitution $u = t^2$. Then it is:

$$M \int \frac{t}{(t^2 + a^2)^m} dt = \frac{-M}{2(m - 1)(t^2 + a^2)^{m-1}} + C$$

For the second integral however, this time we need a different substitution. Let $t = a \tan z$. Then: $t^2 + a^2 = a^2 \sec^2 z$, and $dt = a \sec^2 z dz$. Then the integral turns into:

$$\int \frac{1}{(t^2 + a^2)^m} dt = a^{2m-1} \int \cos^{2m-2} z dz$$

Let $I(z) = \int \cos^{2m-2} z dz$ and now, we can take advantage of the linearity rule and the formula for powers of cosine we just derived. The integral turns into:

$$\begin{aligned} I(z) &= \int \frac{1}{2^{2m-2}} \sum_{k=0}^{2m-2} \binom{2m-2}{k} \cos(2(m - k - 1)z) dz \\ &= \frac{1}{2^{2m-2}} \sum_{k=0}^{2m-2} \binom{2m-2}{k} \frac{1}{2m - 2 - 2k} \sin(2(m - k - 1)z) + C \\ &= \frac{1}{2^{2m-2}} \left[\binom{2m-2}{m-1} z + \sum_{k=0, k \neq m-1}^{2m-2} \binom{2m-2}{k} \frac{1}{2m - 2 - 2k} \sin(2(m - k - 1)z) \right] + C \end{aligned}$$

Then the integral, as a whole, evaluates to:

$$\int \frac{Mx + N}{(x^2 + px + q)^m} dx = \frac{-M}{2(m - 1)(x^2 + px + q)^{m-1}} + \left(N - \frac{Mp}{2}\right) \left(\sqrt{q - \frac{p^2}{4}}\right)^{2m-1} \cdot I(z) + C$$

1.4 Integral of trigonometric functions

1. The general method

Consider the integral $\int f(\sin x, \cos x)dx$, where the integrand is a rational function in terms of $\sin x$ and $\cos x$. We can use the "universal trigonometric substitution" $x = \tan \frac{t}{2}$. Then:

$$\sin x = \frac{2t}{1+t^2}; \quad \cos x = \frac{1-t^2}{1+t^2}; \quad \tan x = \frac{2t}{1-t^2}; \quad dt = \frac{2dt}{1+t^2}$$

The integrand turns into a rational function in terms of t .

2. Integrals of the form $\int \sin^m x \cos^n x dx$, where m, n are positive integers

- If m is odd, we let $t = \cos x$
- If n is odd, we let $t = \sin x$
- If both m and n are even, we use power-reduction formulae:

$$\sin^2 x = \frac{1 - \cos 2x}{2}; \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Then we'll have a similar integral with the form $\int \sin^k 2x \cos^l 2x dx$

3. Special forms of $\int f(\sin x, \cos x)dx$

- Let $t = \cos x$ if $f(-\sin x, \cos x) = -f(\sin x, \cos x)$
- Let $t = \sin x$ if $f(\sin x, -\cos x) = -f(\sin x, \cos x)$
- Let $t = \tan x$ if $f(-\sin x, -\cos x) = f(\sin x, \cos x)$

1.5 Integral of irrational expressions

There are two main ways to solve these integrals: using trigonometric substitution, and using the Euler substitution. The trig-sub is very intuitive, but the Euler substitution is also very nice:

Concept

Let $t = x + \sqrt{x^2 + a}$ for the integral $\int f(x, \sqrt{x^2 + a})dx$. Then:

$$dt = 1 + \frac{x}{\sqrt{x^2 + a}}dx = \frac{x + \sqrt{x^2 + a}}{\sqrt{x^2 + a}}dx \Rightarrow \frac{dt}{t} = \frac{dx}{\sqrt{x^2 + a}}$$

2 Definite Integrals

2.1 Definition

Say $f(x)$ is defined and bounded on $[a, b]$. Partition $[a, b]$ into n subintervals $[x_i, x_{i+1}]$ where $a = x_0 < x_1 < \dots < x_n = b$. In each interval $[x_i, x_{i+1}]$, we choose a point $\xi \in [x_i, x_{i+1}]$ and form the expression

$$S_n = \sum_{i=0}^{n-1} f(\xi_i) \Delta x_i$$

where $\Delta x_i = x_{i+1} - x_i$. Here, S_n is the Riemann sum. Denote $\lambda = \max_{1 \leq i \leq n} \Delta x_i$. If there exists the limit $I = \lim_{\lambda \rightarrow 0} S_n$ that doesn't depend on how we partition $[a, b]$ and how we choose ξ_i

then I is called the definite integral of the function $f(x)$ on $[a, b]$, denoted $\int_a^b f(x) dx$. Then we say $f(x)$ is integrable on $[a, b]$.

We then have defined the definite integral for all $a < b$. We can then define, if $b < a$, $\int_a^b f(x) dx = - \int_b^a f(x) dx$ and when $a = b$, $\int_a^b f(x) dx = 0$.

2.2 Riemann Integrability

The sufficient and necessary condition for a bounded function $f(x)$ to be integrable on $[a, b]$ is $\lim_{\lambda \rightarrow 0} (S - s) = 0$, where:

$$S = \sum_{i=1}^{n+1} M_i \Delta x_i \quad s = \sum_{i=1}^{n+1} m_i \Delta x_i$$

$$M_i = \sup_{x \in [x_i, x_{i+1}]} f(x) \quad m_i = \inf_{x \in [x_i, x_{i+1}]} f(x)$$

From there, we have a few crucial theorems:

Concept

- If $f(x)$ is continuous on $[a, b]$ then it is integrable on $[a, b]$.
- If $f(x)$ is bounded on $[a, b]$ and has discontinuities on $[a, b]$ then it is integrable on $[a, b]$
- If $f(x)$ is bounded and monotonic on $[a, b]$ then it is integrable on $[a, b]$

2.3 Properties of the Definite Integral

1. Property 1 (Linearity:)

$$\int_a^b [\alpha f(x) + \beta g(x)] dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx$$

2. **Property 2:** Given three closed intervals $[a, b]$, $[b, c]$, $[a, c]$, if $f(x)$ is integrable on the longest interval then it is also integrable on the other two intervals, and:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

3. **Property 3:** Suppose $a < b$. Then:

- (a) If $f(x) \geq 0 \forall x \in [a, b]$ then $\int_a^b f(x)dx \geq 0$
- (b) If $f(x) \geq g(x) \forall x \in [a, b]$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
- (c) If $f(x)$ is integrable on $[a, b]$ then $|f(x)|$ is integrable on $[a, b]$ and:

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx$$

- (d) If $m \leq f(x) \leq M \forall x \in [a, b]$ then:

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

4. **Property 4 (First Mean Value Theorem):**

Suppose $f(x)$ is integrable on $[a, b]$ and $m \leq f(x) \leq M \forall x \in [a, b]$, then there exists μ such that:

$$\int_a^b f(x)dx = \mu(b-a), \quad m < \mu < M$$

If $f(x)$ is continuous on $[a, b]$ then there exists $c \in [a, b]$ such that:

$$\int_a^b f(x)dx = f(c)(b-a)$$

5. **Property 5 (Second Mean Value Theorem):** If we have these three conditions:

- (a) $f(x)$ and $f(x)g(x)$ are integrable on $[a, b]$
- (b) $m \leq f(x) \leq M \forall x \in [a, b]$
- (c) $g(x)$ does not change signs on $[a, b]$

Then there exists μ such that:

$$\int_a^b f(x)g(x)dx = \mu \int_a^b g(x)dx, \quad m < \mu < M$$

If $f(x)$ is continuous on $[a, b]$ then there exists $c \in [a, b]$ such that:

$$\int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx$$

2.4 Integral Functions

Suppose $f(x)$ is an integrable function on $[a, b]$, then for all $x \in [a, b]$, f is also integrable on $[a, x]$. We can then define the function $F(x) = \int_a^x f(t)dt$. We then have some very important foundational theorems:

Concept

- If $f(x)$ is integrable on $[a, b]$ then $F(x)$ is continuous on $[a, b]$
- If f is continuous at $x_0 \in [a, b]$ then $F(x)$ is differentiable at x_0 and

$$F'(x_0) = f(x_0)$$

- If $f(x)$ is continuous on the closed interval $[a, b]$ and $F(x)$ is an integral of $f(x)$ then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

2.5 Ways to Calculate Definite Integrals

1. Integration by parts:

Suppose $u(x), v(x)$ are continuously differentiable functions on $[a, b]$. Then:

$$\int_a^b u dv = uv \Big|_a^b - \int_a^b v du$$

2. Change of variables:

(a) Substitute $x = \varphi(t)$

Consider $I = \int_a^b f(x)dx$ with $f(x)$ being continuous on $[a, b]$. Substitute $x = \varphi(t)$ with the following three conditions:

- $\varphi(t)$ has a continuous derivative on $[a, b]$
- $\varphi(a) = \alpha$; $\varphi(b) = \beta$
- When t changes from α to β in $[\alpha, \beta]$ then $x = \varphi(t)$ continuously changes from a to b

Then we have the following formula:

$$\int_a^b f(x)dx = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt$$

(b) Substitute $t = \varphi(x)$

Suppose the integral we are trying to solve has the form $I = \int_a^b f[\varphi(x)]\varphi'(x)dx$, where $\varphi(x)$ is monotonic and is continuously differentiable on $[a, b]$. Then:

$$\int_a^b f[\varphi(x)]\varphi'(x)dx = \int_{\varphi(a)}^{\varphi(b)} f(t)dt$$

3. Recursion or Induction

We can look at an example. Calculate:

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x \cos nx dx$$

We use integration by parts: Let $u = \cos^n x$ and $dv = \cos nx dx$. Then $v = \frac{1}{n} \sin nx$. Applying the integration by parts formula, we have:

$$\begin{aligned} I_n &= \frac{1}{n} \int_0^{\frac{\pi}{2}} \cos^n x \cos nx dx \\ &= \frac{1}{n} \cos^n x \sin nx \Big|_0^{\frac{\pi}{2}} + \frac{1}{n} \int_0^{\frac{\pi}{2}} n \cos^{n-1} x \sin x \sin nx dx \\ &= \int_0^{\frac{\pi}{2}} \cos^{n-1} x \sin x \sin nx dx \end{aligned}$$

Then we see:

$$\begin{aligned} 2I_n &= \int_0^{\frac{\pi}{2}} \cos^n x \cos nx dx + \int_0^{\frac{\pi}{2}} \cos^{n-1} x \sin x \sin nx dx \\ &= \int_0^{\frac{\pi}{2}} \cos^{n-1} x (\cos x \cos nx + \sin x \sin nx) dx \\ &= \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos(n-1)x dx \\ &= I_{n-1} \end{aligned}$$

It's easy to compute $I_0 = \frac{\pi}{2}$. Then this is a geometric series, and $I_n = \frac{\pi}{2^{n+1}}$.

2.6 Important Results