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DM 1

n° 2.12

(a) A slab is the intersection of two halfspaces. Since halfspaces are convex, and since the intersection of convex sets is convex, we can deduce that a slab is a convex set.
Moreover, since it is the intersection of a finite number of halfspaces, it is a polyhedron.

(b) With the same reasoning as (a), we have that a rectangle is a convex set (and also a polyhedron) since $\{x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i,\ i = 1, \dots, n\}$ is the intersection of a finite number of halfspaces.

(c) A wedge is the intersection of the two following halfspaces:
$\{x \in \mathbb{R}^n,\ a_1^T x \leq b_1\}$ and $\{x \in \mathbb{R}^n,\ a_2^T x \leq b_2\}$.
Again, it is a convex set and a polyhedron.

(d) We have: $\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\}$

$$= \bigcap_{y \in S} \{x \mid \|x - x_0\|_2 \leq \|x - y\|_2\}$$

For y fixed, the set $\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2\}$ is an halfspace so a convex set (see the proof at question (g) - case $\theta = 1$).

Then, $\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\}$ is a convex set, as the intersection of convex sets.

α (e) This set is not convex.

(f) We have : $\{x \mid x + S_2 \subseteq S_1\} = \bigcap_{y \in S_2} \{x \mid x + y \in S_1\}$

$= \bigcap_{y \in S_2} f_y^{-1}(S_1)$, where $f_y : x \mapsto x + y$.

For each y, since f_y is an affine function and S_1 is convex, we have that $f_y^{-1}(S_1)$ is convex.

Thus, the intersection $\bigcap_{y \in S_2} f_y^{-1}(S_1) = \{x \mid x + S_2 \subseteq S_1\}$ is a convex set.

(g) We have : $\{x \mid \|x - a\|_2 \leq \theta \|x - b\|_2\} = \{x \mid \|x - a\|_2^2 \leq \theta^2 \|x - b\|_2^2\}$

$= \{x \mid \langle x - a, x - a\rangle - \theta^2 \langle x - b, x - b\rangle \leq 0\}$

$= \{x \mid \langle x, x\rangle - 2\langle a, x\rangle + \langle a, a\rangle - \theta^2 \langle x, x\rangle + 2\theta^2 \langle b, x\rangle - \theta^2 \langle b, b\rangle \leq 0\}$

$= \{x \mid x^T x - 2a^T x + a^T a - \theta^2 x^T x + 2\theta^2 b^T x - \theta^2 b^T b \leq 0\}$

$= \{x \mid (1 - \theta^2) x^T x + (2\theta^2 b^T - 2a^T) x + a^T a - \theta^2 b^T b \leq 0\}$

- <u>if $\theta = 1$</u> : $\{x \mid \|x - a\|_2 \leq \theta \|x - b\|_2\}$

$= \{x \mid 2(b^T - a^T) x + a^T a - b^T b \leq 0\}$

$= \{x \mid (b - a)^T x \leq \frac{1}{2}(b^T b - a^T a)\}$

$= \{x \mid \tilde{a}^T x \leq \tilde{b}\}$ where $\tilde{a} := b - a$ ($\neq 0$ since $b \neq a$) and $\tilde{b} := \frac{1}{2}(b^T b - a^T a)$.

This is an halfspace, so a convex set.

- if $\theta < 1$: $\{x \mid \|x-a\|_2 \leq \theta \|x-b\|_2\} = \{x \mid \|x - x_0\|^T \leq R^2\}$

$$\text{where } x_0 := \frac{a - \theta^2 b}{1-\theta^2} \quad \text{and} \quad R := \sqrt{\frac{\theta^2 \|b\|_2^2 - \|a\|_2^2}{1-\theta^2} - \|x_0\|_2^2}.$$

This is the euclidian ball with center x_0 and radius R, so it's a convex set.

N°: 3.21

(a) $f(x) = \max_{i=1,\dots,k} \|A^{(i)}x - b^{(i)}\|$.

For i fixed, the function $f_i : x \mapsto \|A^{(i)}x - b^{(i)}\|$ can be written as following:

$f_i(x) = g(h_i(x))$ where $h_i : x \mapsto A^{(i)}x - b^{(i)}$ is an affine function, so a convex fct°
$g : u \mapsto \|u\|$ is a norm, so a convex function.

We then have that f_i is the composition of two convex functions,
so f_i is a convex function.

Thus, since $f_1, \dots, f_k$ are convex, then their pointwise maximum
$f(x) = \max_{i=1,\dots,k} f_i(x)$ is also convex.

(b) $f(x) = \sum_{i=1}^{r} |x|_{[i]}$ We can write f as:

$$f(x) = \max\{x_{i_1} + x_{i_2} + \dots + x_{i_r} \mid 1 \leq i_1 < i_2 < \dots < i_r \leq n\},$$

which is a pointwise maximum of affine function of the coefficients.
Thus, f is convex.

n°: 3.32

(a) We denote $h(x) := f(x) \times g(x)$. We want to show that:

$$h(\theta x + (1-\theta)y) \leq \theta h(x) + (1-\theta) h(y) \quad \forall\, x, y \in \operatorname{dom} h,\ 0 \leq \theta \leq 1.$$

Let set $x, y \in \operatorname{dom} h$, $0 \leq \theta \leq 1$. Since f and g are positive and convex,

$$\begin{aligned} h(\theta x + (1-\theta)y) &= f(\theta x + (1-\theta)y)\, g(\theta x + (1-\theta)y) \\ &\leq (\theta f(x) + (1-\theta) f(y))(\theta g(x) + (1-\theta) g(y)) \\ &\leq \underbrace{\theta f(x) g(x) + (1-\theta) f(y) g(y)}_{= \theta h(x) + (1-\theta) h(y)} + \underbrace{\theta(1-\theta)}_{\geq 0} \underbrace{(f(y) - f(x))(g(x) - g(y))}_{\leq 0} \end{aligned}$$

Since f and g are monotonic, $(f(y) - f(x))(g(x) - g(y)) \leq 0$.

Then, $h(\theta x + (1-\theta)y) \leq \theta h(x) + (1-\theta) h(y)$.

(b) With the same reasoning as (a), we have:

$$\begin{aligned} h(\theta x + (1-\theta) y) &\geq \theta f(x) g(x) + (1-\theta) f(y) g(y) + \underbrace{\theta(1-\theta)}_{\geq 0} \underbrace{(f(y) - f(x))(g(x) - g(y))}_{\geq 0} \\ &\geq \theta f(x) g(x) + (1-\theta) f(y) g(y) = \theta h(x) + (1-\theta) h(y). \end{aligned}$$

Thus, h is concave.

(c) We have that $\frac{1}{g}$ is convex, nondecreasing and positive, and then we are in the case (a).

We deduce that $f \times \frac{1}{g} = \frac{f}{g}$ is convex.

n° = 3.36

(a) $f(x) = \max\limits_{i=1,\dots,n} x_i$

$$f^*(y) = \sup_{x \in \mathbb{R}^n} \left(x^T y - \max_{i=1,\dots,n} x_i\right) \quad \forall y \in \operatorname{dom} f^*$$

We first look for the y such that the function $x \mapsto x^T y - f(x)$ is bounded.

* Assume first that there is k such that $y_k \leq 0$. Then, if we take the vector x such that $x_k = -t$ and $x_i = 0 \ \forall i \neq k$, we have:

$$x^T y - \max_i x_i = -t y_k \xrightarrow[t \to +\infty]{} +\infty \quad \} \text{ unbounded above}$$

So we need $y \geq 0$.

* If we have $y \geq 0$ but $\mathbb{1}^T y > 1$, and we choose $x = t\mathbb{1}$, then:

$$x^T y - \max_i x_i = t\mathbb{1}^T y - t = t\underbrace{(\mathbb{1}^T y - 1)}_{>0} \xrightarrow[t \to +\infty]{} +\infty \quad \} \text{ unbounded above.}$$

We do the same for $\mathbb{1}^T y < 1$ (and $x = -t\mathbb{1}$).

* Then, we necessarily have $y \geq 0$ and $\mathbb{1}^T y = 1$, and then:

$$x^T y = \sum_{i=1}^n x_i y_i \leq \max_{i=1,\dots,n} x_i \times \sum_{i=1}^n y_i = \max x_i \times \mathbb{1}^T y = \max x_i$$

$$\Rightarrow x^T y - \max_{i=1,\dots,n} x_i \leq 0.$$

For $x = 0$, we have $x^T y - \max\limits_{i=1,\dots,n} x_i = 0$.

Then, $f^*(y) = \begin{cases} 0 & \text{if } y \geq 0 \text{ and } \mathbb{1}^T y = 1 \\ \infty & \text{otherwise} \end{cases}$

(b) $f(x) = \sum_{i=1}^{r} x_{[i]}$

$$f^*(y) = x^T y - \sum_{i=1}^{r} x_{[i]}$$

* Assume there is k such that $y_k < 0$. Then, if we take the vector x such that $x_k = -t$ and $x_i = 0 \ \forall i \neq k$, we have

$$x^T y - f(x) = -t y_k + t \underset{t \to +\infty}{\longrightarrow} \infty.$$

So we need $y \geq 0$.

* Assume there is k such that $y_k > 1$. Then, if we take the vector x such that $x_k = t$ and $x_i = 0 \ \forall i \neq k$, we have

$$x^T y - f(x) = t y_k - t \underset{t \to +\infty}{\longrightarrow} +\infty$$

So we need $y \leq 1$.

* Assume $\mathbb{1}^T y \neq r$. We take $x = t\mathbb{1}$ ($\Rightarrow f(x) = tr$). We then have:

$$x^T y - f(x) = t(\underbrace{\mathbb{1}^T y - r}_{\neq 0})$$

* Thus, we take $0 \leq y \leq 1$ and $\mathbb{1}^T y = r$ and then $x^T y \leq f(x)$, with equality for $x = 0$

We deduce $f^*(y) = \begin{cases} 0 & \text{if } 0 \leq y \leq 1 \text{ and } \mathbb{1}^T y = r \\ \infty & \text{otherwise} \end{cases}$

(c) $f(x) = \max_{i=1,\dots,n} (a_i x + b_i)$

$$f^*(y) = \sup_{x \in \mathbb{R}} (xy - \max_{i=1,\dots,m} (a_i x + b_i))$$

* We notice that: if $y > a_m$, then for $x = t$ we have:

$$xy - f(x) > a_m t - f(t) = a_m t - a_k t - b_k = t(\underbrace{a_m - a_k}_{\geq 0 \text{ because } a_1 \leq \dots \leq a_m}) - b_k \underset{t \to +\infty}{\longrightarrow} +\infty$$

(where $k = k(t)$ is such that $f(t) = a_k t + b_k$).

Same reasoning for $y \leq a_1$.

* we take $y \in [a_1, a_m]$.

(d) $f(x) = x^p$ on $\mathbb{R}_{++}$

$$f^*(y) = \sup_{x \in \mathbb{R}_{++}} (xy - x^p)$$

<u>$p > 1$</u>: we denote $h(x) := xy - x^p$; h is convex as the sum of convex functions so h admits a max.

$$h'(x) = y - px^{p-1}$$

$h'(x) = 0 \iff y = px^{p-1} \iff x^{p-1} = \frac{y}{p} \iff x = \left(\frac{y}{p}\right)^{\frac{1}{p-1}}$

and that's working for $y > 0$ (since $x \in \mathbb{R}_{++}$).

$$\Rightarrow f^*(y) = \left(\frac{y}{p}\right)^{\frac{1}{p-1}} y - \left(\frac{y}{p}\right)^{\frac{p}{p-1}} \quad \text{for } y > 0.$$

For $y \leq 0$, since $x > 0$, we have $h(x) \leq 0$, so the maximum is reached at $x = 0$.

Then we have : $f^*(y) = \begin{cases} 0 & \text{if } y \leq 0 \\ y\left(\frac{y}{p}\right)^{\frac{1}{p-1}} - \left(\frac{y}{p}\right)^{\frac{p}{p-1}} & \text{if } y > 0 \end{cases}$

(e) $f(x) = -\left(\prod x_i\right)^{1/n}$ on $\mathbb{R}^n_{++}$; $f^*(y) = \sup_{x \in \mathbb{R}^n_{++}} \left(x^T y + \left(\prod(x_i)\right)^{1/n}\right)$

* Assume there is k such that $y_k > 0$. Then, if we take the vector x such that $x_k = t$ and $x_i = 1 \ \forall i \neq k$, we have:

$$x^T y - f(x) = t y_k + \sum_{i \neq k} y_i + t^{1/n} \underset{t \to +\infty}{\longrightarrow} +\infty.$$

So we need $y \leq 0$.

* Assume $y \leq 0$ and $\left(\prod(-y_i)\right)^{1/n} < \frac{1}{n}$. Then, if we take the vector x such that $x_i = -\frac{t}{y_i}$, we have:

$$\begin{aligned} x^T y - f(x) &= \sum_{i=1}^{n} -\frac{t}{\cancel{y_i}} \times \cancel{y_i} + \left(\prod_i \left(-\frac{t}{y_i}\right)\right)^{1/n} \\ &= -tn + t\left(\prod_i \left(-\frac{1}{y_i}\right)\right)^{1/n} \\ &= t\Big(\underbrace{\frac{1}{\left(\prod_i(-y_i)\right)^{1/n}}}_{>n} - n\Big) \underset{t \to +\infty}{\longrightarrow} +\infty \end{aligned}$$

So we need $\left(\prod(-y_i)\right)^{1/n} \geq \frac{1}{n}$.

* We take y such that $y \leq 0$ and $\left(\prod(-y_i)\right)^{1/n} \geq \frac{1}{n}$.

According to the Arithmetic Mean - Geometric Mean inequality (since $x \geq 0$), we have: $\frac{1}{n} \sum_i -x_i y_i \geq \left(\prod_i (-x_i y_i)\right)^{1/n}$

with $\left(\prod_i(-x_i y_i)\right)^{1/n} = \left(\prod_i x_i\right)^{1/n} \left(\prod_i (-y_i)\right)^{1/n} \geq \frac{1}{n}\left(\prod_i x_i\right)^{1/n}$

$\Rightarrow -\frac{1}{n} \sum_i x_i y_i \geq \frac{1}{n}\left(\prod_i x_i\right)^{1/n}$

$\Rightarrow x^T y \leq -\left(\prod_i x_i\right)^{1/n}$ with equality for $x_i = -\frac{1}{y_i}$

We deduce: $f^*(y) = \begin{cases} 0 & \text{if } y \leq 0 \text{ and } \left(\prod(-y_i)\right)^{1/n} \geq \frac{1}{n} \\ \infty & \text{otherwise} \end{cases}$