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DM 2

n° 1

1) We first rewrite (P) in a standard form: $\min_x c^T x$ s.t. $-x \leq 0$, $b - Ax = 0$

Lagrangian: $L(x, \lambda, \nu) = c^T x + \lambda^T(-x) + \nu^T(b - Ax)$
$= c^T x - \lambda^T x + \nu^T b - \nu^T A x$
$= (c - \lambda - A^T \nu)^T x + \nu^T b$

Lagrange dual function: $g(\lambda, \nu) = \inf_x L(x, \lambda, \nu) = \inf_x (c - \lambda - A^T\nu)^T x + \nu^T b$

Since L is linear in x, we have $g(\lambda, \nu) = \begin{cases} \nu^T b & \text{if } c - A^T\nu - \lambda = 0 \\ -\infty & \text{otherwise} \end{cases}$

Dual problem: $\max_{\lambda \geq 0} g(\lambda, \nu)$ ie $\max b^T \nu$ s.t. $\lambda \geq 0$, $c - A^T\nu - \lambda = 0$

or more simply $\max b^T \nu$ s.t. $A^T \nu \leq c$. This is the problem (D).

2) We first rewrite (D) in a standard form: $\min_y -b^T y$ s.t. $A^T y - c \leq 0$

Lagrangian: $L(y, \lambda) = -b^T y + \lambda^T(A^T y - c)$
$= (A\lambda - b)^T y - \lambda^T c$

Lagrange dual function: $g(\lambda) = \inf_y L(y, \lambda) = \inf_y (A\lambda - b)^T y - \lambda^T c$.

Again, L is linear in y and then $g(\lambda) = \begin{cases} -\lambda^T c & \text{if } A\lambda - b = 0 \\ -\infty & \text{otherwise} \end{cases}$

Dual problem: $\max_{\lambda \geq 0} g(\lambda)$ ie $\min c^T \lambda$ s.t. $\lambda \geq 0$, $A\lambda - b = 0$. This is the problem (P).

3) Let rewrite the problem in a standard form: $\min_{x,y} \; c^T x - b^T y$

$$\text{s.t} \quad -x \leq 0$$
$$A^T y - c \leq 0$$
$$b - Ax = 0$$

Lagrangian: $L(x,y,\lambda,\mu,\nu) = c^T x - b^T y + \lambda^T(-x) + \mu^T(A^T y - c) + \nu^T(b - Ax)$

$$= c^T x - \lambda^T x - \nu^T A x - b^T y + (A\mu)^T y - \mu^T c + \nu^T b$$

$$= (c - \lambda - A^T \nu)^T x + (A\mu - b)^T y - \mu^T c + \nu^T b.$$

Lagrange dual function: $g(\lambda,\mu,\nu) = \inf_{x,y} L(x,y,\lambda,\mu,\nu)$

$$= \inf_{x,y} \; (c - \lambda - A^T\nu)^T x + (A\mu - b)^T y - \mu^T c + \nu^T b$$

The minimum over x is bounded below if and only if $c - \lambda - A^T \nu = 0$.

The minimum over y is bounded below if and only if $A\mu - b = 0$.

Then, $g(\lambda,\mu,\nu) = \begin{cases} -c^T\mu + b^T\nu & \text{if } c - A^T\nu - \lambda = 0 \text{ and } A\mu - b = 0 \\ -\infty & \text{otherwise} \end{cases}$

Dual problem: $\max_{\substack{\lambda \geq 0 \\ \mu \geq 0}} g(\lambda,\mu,\nu) = \max \; b^T\nu - c^T\mu$

$$\text{s.t} \quad \lambda \geq 0, \; \mu \geq 0$$
$$c - A^T\nu - \lambda = 0$$
$$A\mu - b = 0$$

We can rewrite the constraints as: $\mu \geq 0$, $A\mu = b$, $A^T\nu \leq c$

Then, the dual problem is $\quad \min \; c^T\mu - b^T\nu$

$$\text{s.t} \quad A\mu = b$$
$$\mu \geq 0$$
$$A^T\nu \leq c$$

Then, this problem is self-dual.

4) * If we take x such that $Ax=b$ and $x \geq 0$, then (x, y^*) is feasible for the problem (Self_Dual). We have $c^T x^* - b^T y^* \leq c^T x - b^T y^*$

$$\Rightarrow c^T x^* \leq c^T x \quad \text{(for all } x \text{ feasible for (P))}$$

We deduce that x^* solves (P).

* If we take y such that $A^T y \leq c$, then (x^*, y) is feasible for the problem (Self_Dual). We have $c^T x^* - b^T y^* \leq c^T x^* - b^T y$

$$\Rightarrow b^T y^* \leq b^T y \quad \text{(for all } y \text{ feasible for (D))}.$$

We deduce that y^* solves (D).

* (D) is the dual of (P). x^* is optimal for (P) and y^* is optimal for (D). By strong duality of linear programs, we then have:

$$c^T x^* = b^T y^* \quad \Rightarrow c^T x^* - b^T y^* = 0.$$

Since $\min_{x,y} c^T x - b^T y = c^T x^* - b^T y^*$, the optimal value of (Self-Dual) is 0.
s.t $Ax = b$
$x \geq 0$
$A^T y \leq c$

n° 2

1) We denote $g : x \mapsto \|x\|_1$. $g^*(y) = \sup_x (y^T x - g(x)) = \sup_x (y^T x - \|x\|_1)$.

* Let take a norm $\|.\|$. The corresponding dual norm is defined as $\|y\|_* = \sup_{\|x\| \leq 1} y^T x$

↳ if $\|y\|_* \leq 1$, then $y^T x \leq \|x\|$ for all x (by definition of the dual norm) with equality for $x=0$.

↳ if $\|y\|_* > 1$, there exists u such that $\|u\| \leq 1$ and $u^T y = \|y\|_* > 1$.

If we choose $x = tu$, we have:

$$y^T x - \|x\| = y^T tu - t\|u\| = t(y^T u - \|u\|) = t(\|y\|_* - \|u\|) \xrightarrow[t \to +\infty]{} +\infty$$

Then, $\sup_x (y^T x - \|x\|) = \begin{cases} 0 & \text{if } \|y\|_* \leq 1 \\ +\infty & \text{otherwise} \end{cases}$

* Since the dual norm of the l_1-norm is the l_∞-norm, we then have:

$$g^*(y) = \begin{cases} 0 & \text{if } \|y\|_\infty \leq 1 \\ +\infty & \text{otherwise} \end{cases}.$$

2) We rewrite the optimization problem as following: $\min_{x,y} \|y\|_2^2 + \|x\|_1$ s.t $y = Ax - b$

Lagrangian: $L(x,y,\nu) = \|y\|_2^2 + \|x\|_1 + \nu^T(y - Ax + b)$

Lagrange dual function: $g(\nu) = \inf_{x,y} L(x,y,\nu)$

$$= \inf_{x,y} \|y\|_2^2 + \|x\|_1 + \nu^T(y - Ax + b)$$

$$= \inf_x (\|x\|_1 - A^T\nu x) + \inf_y (\|y\|_2^2 + \nu^T y) + b^T\nu$$

↳ Since $\sup_x (y^T x - \|x\|_1) = \begin{cases} 0 & \text{if } \|y\|_\infty \leq 1 \\ +\infty & \text{otherwise} \end{cases}$, we have

$$\inf_x (\|x\|_1 - A^T\nu x) = \begin{cases} 0 & \text{if } \|A^T\nu\|_\infty \leq 1 \\ -\infty & \text{otherwise} \end{cases}$$

↳ $\inf_y (\|y\|_2^2 + \nu^T y)$ By setting the gradient at 0, we have:

$$2y - \nu = 0 \iff y = \frac{\nu}{2}$$

$$\Rightarrow \inf_y (\|y\|_2^2 + \nu^T y) = \|\frac{\nu}{2}\|_2^2 + \nu^T\left(\frac{\nu}{2}\right) = \frac{1}{4}\|\nu\|_2^2 + \frac{1}{2}\|\nu\|_2^2 = \frac{3}{4}\|\nu\|_2^2.$$

Then, $g(\nu) = \begin{cases} \frac{3}{4}\|\nu\|_2^2 + b^T\nu & \text{if } \|A^T\nu\|_\infty \leq 1 \\ -\infty & \text{otherwise} \end{cases}$

Dual problem: $\max \frac{3}{4}\|\nu\|_2^2 + b^T\nu$

s.t $\|A^T\nu\|_\infty \leq 1$

n°3

1) $\min_{\omega} \frac{1}{n} \sum_{i=1}^{n} \mathcal{L}(\omega, x_i, y_i) + \frac{\tau}{2} \|\omega\|_2^2 = \min_{\omega} \frac{1}{n\tau} \sum_{i=1}^{n} \mathcal{L}(\omega, x_i, y_i) + \frac{1}{2} \|\omega\|_2^2$

We denote $z_i := \max\{0, 1 - y_i \omega^T x_i\}$. We call them the "slack variables".

If (x_i, y_i) is well classified, we have $y_i \omega^T x_i \geq 1$ and $z_i = 0$

$\Rightarrow y_i \omega^T x_i \geq 1 - z_i$

Otherwise, we have $y_i \omega^T x_i \leq 1 \Rightarrow z_i = 1 - y_i \omega^T x_i > 0$.

By definition, we have $z_i \geq 0$ and $z_i \geq 1 - y_i \omega^T x_i$.

Thus, we can rewrite the problem as $\min_{\omega, z} \frac{1}{n\tau} \sum_{i=1}^{n} z_i + \frac{1}{2} \|\omega\|_2^2$

s.t $z_i \geq 1 - y_i \omega^T x_i \quad \forall i = 1, \dots, n$

$z \geq 0$

i.e $\min_{\omega, z} \frac{1}{n\tau} \mathbb{1}^T z + \frac{1}{2} \|\omega\|_2^2$

s.t $z_i \geq 1 - y_i \omega^T x_i \quad \forall i = 1, \dots, n$

$z \geq 0$

2) Lagrangian : $L(\omega, z, \lambda, \pi) = \frac{1}{n\tau} \mathbb{1}^T z + \frac{1}{2} \|\omega\|_2^2 + \sum_{i=1}^{n} \lambda_i (1 - y_i (\omega^T x_i) - z_i) - \pi^T z$

$= \left(\frac{1}{2} \|\omega\|_2^2 - \sum_{i=1}^{n} \lambda_i y_i (\omega^T x_i) \right) + \left(\frac{1}{n\tau} \mathbb{1} - \pi - \lambda \right)^T z + \sum_{i=1}^{n} \lambda_i$

Lagrange dual function : $g(\lambda, \pi) = \inf_{\omega, z} L(\omega, z, \lambda, \pi)$

$= \inf_{\omega} \frac{1}{2} \|\omega\|_2^2 - \sum_{i=1}^{n} \lambda_i y_i (\omega^T x_i) + \inf_{z} \left(\frac{1}{n\tau} \mathbb{1} - \pi - \lambda \right)^T z + \sum_{i=1}^{n} \lambda_i$

↳ $\inf_{z} \left(\frac{1}{n\tau} \mathbb{1} - \pi - \lambda \right)^T z = \begin{cases} 0 & \text{if } \frac{1}{n\tau} \mathbb{1} - \pi - \lambda = 0 \\ -\infty & \text{otherwise} \end{cases}$

↳ $\inf_{\omega} \frac{1}{2} \|\omega\|_2^2 - \sum_{i=1}^{n} \lambda_i y_i (\omega^T x_i)$. By setting the gradient at 0, we have

$\frac{1}{2} \times 2\omega - \sum_{i=1}^{n} \lambda_i y_i x_i = 0 \iff \omega = \sum_{i=1}^{n} \lambda_i y_i x_i$

$\inf_{\omega} \frac{1}{2} \|\omega\|_2^2 - \sum_{i=1}^{n} \lambda_i y_i (\omega^T x_i) = \frac{1}{2} \left\| \sum_{i=1}^{n} \lambda_i y_i x_i \right\|_2^2 - \left(\sum_{i=1}^{n} \lambda_i y_i x_i \right)^T \sum_{j=1}^{n} \lambda_j y_j x_j$

$$\Rightarrow \inf_{w} \frac{1}{2}\|w\|_2^2 - \sum_{i=1}^{n} \lambda_i y_i (w^T x_i) = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \lambda_j y_i y_j x_i^T x_j$$

Then, $g(\lambda,\pi) = \begin{cases} \sum_{i=1}^{n} \lambda_i - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i\lambda_j y_i y_j x_i^T x_j & \text{if } \frac{1}{n\tau}\mathbb{1} - \pi - \lambda = 0 \\ -\infty & \text{otherwise} \end{cases}$

Dual problem: $\max_{\lambda \geq 0, \pi \geq 0} g(\lambda,\pi) = \max \sum_{i=1}^{n} \lambda_i - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i\lambda_j y_i y_j x_i^T x_j$

s.t $0 \leq \pi \leq \frac{1}{n\tau}\mathbb{1}$

$(\lambda = \frac{1}{n\tau}\mathbb{1} - \pi \geq 0 \Rightarrow \pi \leq \frac{1}{n\tau}\mathbb{1})$

n°5

1) Lagrangian: $L(x,\lambda,\nu) = c^T x + \lambda^T(Ax - b) - \nu^T x + x^T \mathrm{diag}(\nu) x$

$= x^T \mathrm{diag}(\nu) x + (c + A^T\lambda - b)^T x - b^T\lambda.$

Lagrange dual function: $g(\lambda,\nu) = \inf_x L(x,\lambda,\nu)$

$= \begin{cases} -\frac{1}{4}\sum_{i=1}^{n} (c_i + A_i^T\mu - \nu_i)^2/\nu_i - b^T\lambda & \text{if } \nu \geq 0 \\ -\infty & \text{otherwise} \end{cases}$

where we denote A_i the i-th column of A; and with the convention $\frac{z^2}{0} = \infty$.

Lagrange dual problem: $\max_{\lambda\geq 0} g(\lambda,\nu) = \max -\frac{1}{4}\sum_{i=1}^{n}(c_i + A_i^T\lambda - \nu_i)^2/\nu_i - b^T\nu$

s.t $\lambda \geq 0$, $\nu \geq 0$

By using the hint, we have $\sup_{\nu_i \geq 0}\left(-\frac{(c_i + A_i^T\lambda - \nu_i)^2}{\nu_i}\right) = 4\min\{0, (c_i + A_i^T\lambda)\}$.

We then rewrite the dual problem:

$$\max_{\lambda\geq 0} \sum_{i=1}^{n} \min\{0, (c_i + A_i^T\lambda)\} - b^T\lambda$$

2) Lagrangien : $L(x,\lambda,\mu,\nu) = c^T x + \lambda^T(Ax - b) - \mu^T x + \nu^T(x - \mathbb{1})$

$$= (c + A^T\lambda - \mu + \nu)^T x - b^T\lambda - \mathbb{1}^T\nu$$

Lagrange dual function : $g(\lambda,\mu,\nu) = \begin{cases} -b^T\lambda - \mathbb{1}^T\nu & \text{if } A^T\lambda - \mu + c + \nu = 0 \\ -\infty & \text{otherwise} \end{cases}$

Dual problem : $\max \; -b^T\lambda - \mathbb{1}^T\nu$

$$\text{s.t } \lambda \geq 0, \mu \geq 0, \nu \geq 0$$
$$A^T\lambda - \mu + c + \nu = 0$$

This problem is equivalent to the problem $\max \sum_{i=1}^{m} \{0, (c_i + A_i^T\lambda)\} - b^T\lambda$.

$$\text{s.t } \lambda \geq 0$$

Then, we obtain the same lower bound with Lagrangian and LP relaxation.