03/01/24

Objective of Mathematical Models:-

i) Simplify the complex problem.
ii) Predict the outcome with help of scientific knowledge

Process of Mathematical Models:-

i) Find and identify problem
ii) Understanding problem.
iii) Data collection

Problem → We have ~~toy~~ toy company produces soldier & train toy each toy is produced in two stages → i) constructed in carpentary shop ii) send to finishing shop to make toy soldier cost ₹100 for raw materal & ₹140 for labour, ~~similarly for making train toy~~ it take 1 hr in carpentary & 2hr in finishing shop.

To make train toy ₹90 → raw material & ₹100 for labour it take 1hr in carpentary & 1 hr in finishing.

In one week 80hr of work can be done in carpentary shop whearas 100 hr work can be done in finishing.

(S) Toy soldier → ₹270
(t) train toy → ₹210
} sold at this price.

Due to increasing demand company can sold max '40' toy soldier and no limitation for train.

What is the best product that maximize profit.

→ $100s + 140s + 90t + 100t$ → total cost.

$1s + 1(t) \leq 80$

$2s + t \leq 100$

$0 \leq s \leq 40, \; T \geq 0$

Maximize

$210t + 270s - (100s + 140s + 90t + 100t) = (30s + 20$

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$x_1 \Rightarrow$ no. of soldier toy
$x_2 \Rightarrow$ no. of train toys

$Z = 30x_1 + 20x_2$

total hrs in carpentary
$x_1 + x_2 \leq 80$

total hrs in finishing
$2x_1 + x_2 \leq 100$

$(x_1 \leq 40)$ $x_1, x_2 \geq 0$

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Types of Mathematical Models :→

① Emperical models →
eg) Regression models & machine learning model based on data

② Stocastic models → This kind of models involves Randomness or uncertainity.
eg i) Browian motion
ii) Monte Carlo simulation
iii) Queing theory

③ Simulation Model

④ ~~Sampling) model~~
Statistical model → sampling is done.

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① Ordinary Differential Equation → having 2 variables and 1 variable is independent.
$f(x, y, y', y'', \ldots)$

② Partial differential Equation → having more than 2 variab[les] and atleast 2 independent variable.
$z = F(a, b)$

- Order & Degree of Differential Equation.

$y'' + ay' + by = x^1$ → $\boxed{\text{order} = 2}$ $\boxed{\text{degree} = 1}$

$y'' \to \frac{d^2y}{dx^2}$, $y' \to \frac{dy}{dx}$

order = highest derivative.

degree = power of highest derivative.

$[y'' + (y')^3 + ay^2 = x]$ → $\boxed{\text{Order} = 2}$ $\boxed{\text{degree} = 1}$

$[(y')^{3/2} + (y'')^0 + 3 = y]$ → $\begin{bmatrix} \text{order} = 1 \\ \text{degree} = \frac{3}{2} \end{bmatrix}$

[Remember → degree cannot be zero]

- Linear & Non linear differential Equation.

Linear → (i) degree must be 1 in independent variable & it derivative.

(ii) No term having product of dependent varia and its derivatives. $\left(y\frac{dy}{dx}\right)$

eg $\left(\frac{dy}{dx} + xy = \frac{1}{y^2}\right)$ → no linear.

System of ODEs

$$\begin{bmatrix} \frac{dy_1}{dt} \to f_1(y_1, y_2, \ldots y_n) \\ \frac{dy_2}{dt} \to f_2(y_1, y_2, \ldots y_n) \\ \vdots \\ \vdots \\ \frac{dy_n}{dt} \to f_n(y_1, y_2, \ldots y_n) \end{bmatrix}$$

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Compartmental Models :→

Includes process of incoming & outgoing over period of time.

$$\longrightarrow \boxed{\text{compartment}} \longrightarrow$$

$$\text{(burning activities)} \xrightarrow{CO_2} \boxed{\text{Atmosphere}} \xrightarrow{CO_2} \text{(plant respiration)}$$

→ Compartment model framework is extremly natural & valuable which formulate models for processes having inputs and/or output over time.

☆ Radioactive Decay Problems :→

Balance laws of Physics :→

Net rate change of any substance ⇒ {rate in} − {rate out}

↳ Act of emitting radiation

let $N(t)$ is the number of radioactive nuclei at time t and $kN(t)$

$$\boxed{\frac{dN}{dt} = -kN(t)} \qquad N(t_0) = N_0 \longrightarrow (1)$$

$$\boxed{\text{No. of } N(t)} \xrightarrow{\text{decay}} kN(t)$$

Suppose t is current time & Δt is time step.

$N(t) \rightarrow$ current particles

$N(t+\Delta t) \rightarrow N(t) - kN(t)\Delta t$

$N(t+\Delta t) - N(t) = -kN(t)\Delta t$

$$\lim_{\Delta t \to 0} \frac{N(t+\Delta t)-N(t)}{\Delta t} = \lim_{\Delta t \to 0} -kN(t)$$

$$\left[\frac{dN}{dt} = -kN(t)\right] \text{ — (i)}$$

$$[N(t_0) = N_0] \text{ — (ii)}$$

Seperation of variables :→

$$\int \frac{dN}{N} = \int -k \, dt$$

$$\log N = -kt + c$$

$$\boxed{N(t) = Ae^{-k\cancel{(t-t_0)}}} \rightarrow \text{(ii)} \quad \text{where } A = e^c$$

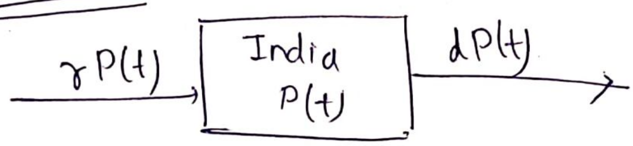
$$N(0) = N_0$$

$$N(0) = Ae^0$$

$$\boxed{N_0 = A} \rightarrow \text{substitute in eq}^n \text{ (ii)}$$

$$\boxed{N(t) = N_0 e^{-kt}}$$

★ Population Growth :→



$$\frac{dP}{dt} = rP(t) - dP(t)$$

$$\frac{dP}{dt} = (r-d)P(t)$$

$\boxed{P(0) = P_0} \rightarrow$ initial population [initial value problem] IVP

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$$\underbrace{\frac{1}{P}\frac{dP}{dt}}_{\text{per capita growth rate of P in unit time}} = r - d$$

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Books

Part I ① Mathematical Modelling with case studies by Belinda Barnes & Glenn Roberto Fulford

Part II ② Discrete event system simulation.

J. Banks, J. Carson, B.L. Nelson etc.

Half life

Let τ is half life of $N(t)$ $\left[N(t+\tau) = \frac{N(t)}{2}\right]$

$$\frac{N(t+\tau)}{N(t)} \Rightarrow \frac{\cancel{N_0} e^{-k(t+\tau-t_0)}}{\cancel{N_0} e^{-k(t-t_0)}} = \frac{1}{2}$$

$$\Rightarrow \frac{\cancel{e^{-k(t-t_0)}} \cdot e^{-k\tau}}{\cancel{e^{-k(t-t_0)}}} = \frac{1}{2}$$

$$\left[e^{-k\tau} = \frac{1}{2}\right]$$

$$-k\tau = \log 1 - \log 2$$

$$-k\tau = -\log 2$$

$$k\tau = \log 2$$

$$\boxed{\tau = \frac{\log(2)}{k}}$$

0.38

Uranium, C^{14}, Lead (210), Polonium P^{14}

(45 billion yrs) (22 yrs) (less than 1s)

5568 yrs

All paintings contains small amount of lead-210, lead white contains lead metal extracted from rocks and extremly small amount of radium 226 whose half life is 16000 yrs.

Let $N(t)$ be the amount of lead-210 then

$$\frac{dN}{dt} = -\lambda N + R(t) \; ; \; n(0) = N(t_0) \quad \text{--- (i)}$$

where $R(t)$ is rate of disintegration of radium 226, per minute/g of white lead. ($R = 0.8$, current rate of disintegration lead $210 = 8.5 \, m^{-1}g^{-1}$

→ we Assume that $R(t) = R$ since half life of radium is very large

$$\underline{N(t)} = \frac{R}{\lambda}\left(1 - e^{-\lambda(t-t_0)}\right) + n_0 e^{-\lambda(t-t_0)} \quad \text{--- (ii)}$$

Rearranging (ii)

$$\lambda n_0 = \lambda N e^{\lambda(t-t_0)} - R\left(e^{\lambda(t-t_0)} - 1\right)$$

λn_0 is disintegration rate from initial time.

Half life $\boxed{\lambda = \frac{\log(2)}{\tau}}$ $\left[2^{\frac{150}{11}} = e^{\lambda(t-t_0)}\right]$ $= \dfrac{e^{300\lambda}}{[\lambda n_0 = 98{,}000]}$

$\frac{0.693}{22 \times}$

> 30,000

So the painting is not old.

Modelling Pollution in Lakes:→

Mass incoming → | Mass of pollutant | → Mass outgoing.

Assumption → i) The lake has constant volume & it continuously will mix so that pollutant is uniform throughout.

ii) Let $c(t)$ be the concentration of pollutant in lake at time 't'. Let F be the rate at which water flows out of lake in m^3 per day since volume is constant

$$\left\{\begin{matrix}\text{Flow of mixture}\\ \text{into lake}\end{matrix}\right\} = \left\{\begin{matrix}\text{Flow of mixture}\\ \text{out of lake}\end{matrix}\right\} = F$$

$$\left\{\begin{matrix}\text{Rate change in}\\ \text{mass}\end{matrix}\right\} = \left\{\begin{matrix}\text{Rate}\\ \text{entering}\end{matrix}\right\} - \left\{\begin{matrix}\text{rate}\\ \text{leaving}\end{matrix}\right\}$$

Now we get problem of change in mass

$$m(t) = F(C_{in}) - F\frac{M(t)}{V} \longrightarrow (i)$$

we know that

$$m(t) = c(t).V$$

$$[m'(t) = c'(t)V]$$

derivative of (i)

$$c'(t)\,V = F c_{in} - \frac{F c(t) \cancel{V}}{\cancel{V}}$$

$$\left[\, c'(t) \cancel{V} = \frac{F c_{in}}{V} - \frac{F c(t)}{V} \,\right]$$

$\boxed{c(0)=c_0}$

$$\frac{dc(t)}{dt} = \frac{F c_{in}}{V} - \frac{F c(t)}{V} \quad \text{with } c(0)=c_0$$

$$\int \frac{1}{c_{in}-c}\,dc = \int \frac{F}{V}\,dt$$

$$-\log(c_{in}-c) = \frac{F}{V}t + K.$$

$$\left[\, c(t) = c_{in} - e^{-K} e^{-\frac{Ft}{V}} \,\right] \longrightarrow (i)$$

$$c(0) = c_{in} - e^{-K} e^{0}$$

$$\boxed{c_0 = c_{in} - e^{-K}}$$

$$\log c_0 = -K$$

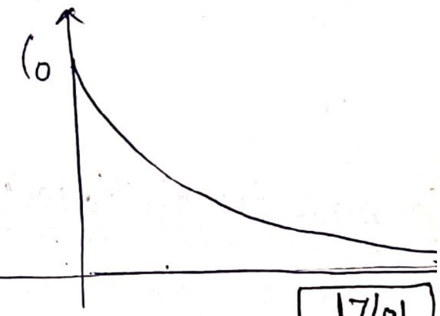
$$\cancel{\boxed{K = -\log c_0}}$$

$$\boxed{e^{-K} = c_{in} - c_0}$$

substitute in eq^n (i)

$$\cancel{c(t) = c_{in} - (c_{in}-c_0)\, e^{-\frac{Ft}{V}}}$$

$$\boxed{c(t) = c_{in} - (c_{in}-c_0)\, e^{-\frac{Ft}{V}}}$$



* Matlab for solving OP (OP 45 solvers)

$\boxed{17/01}$

Problem → How long it will take for a lake so pollution level to reach 5% of its initial level, if only fresh water flows into the lake.

$c_{in} = 0$ → initial level of pollution.

$$c(t) = c_0 e^{-\frac{Ft}{V}}$$

$$c(t) = 0.05\, c_0$$

$$0.005\, \cancel{c_0} = \cancel{c_0}\, e^{-\frac{Ft}{V}}$$

$$\ln(0.05) = -\frac{Ft}{V}$$

$$\boxed{t = -\frac{V}{F}\ln(0.05)}$$

Lake Erie in America :→

$V = 458 \times 10^9 \, m^3$

$F = 486 \times 10^6 \, m^3/day$

$t = -\frac{(458 \times 10^{9})}{480 \times 10^{6}} \ln(0.05)$

$\Rightarrow (0.954) \times (2.995)$

$\Rightarrow 2.857 \times 10^3$ day.

$\Rightarrow \frac{2857 \text{ days}}{364}$

$\Rightarrow \boxed{7.82 \text{ years}}$

Lake ontario →

$V = 1636 \times 10^9 \, m^3$

$f = 5.72 \times 10^6 \, m^3/days$

$(t = 23.8 \text{ years})$

Economic Growth Model :→

* Ibn khaldum, Arabian economic thinker.
* Classical Economic growth increases in population is available labour increases then production of good also thrives.
* Classical Economic growth model deals with economic growth considered increased stackes of capital good (production / output) that are dependent on available labour & investment of capital.
* Let us assume that output (or product) $y(t)$. stock capital $K(t)$, available labour $L(t)$ all at time t.

$$Y = F(K, L) \qquad \left[\frac{dk}{dt} = SY(t) = \frac{dk}{dt} = SF(K, L) \right]$$

* Let $L(t) = L_0 e^{rt}$ (labour increases exponentially over time)

$$\frac{dK}{dt} = SF(K, L_0 e^{rt}) \rightarrow ①$$

[Let assume that we have
$\left[P = \frac{K(t)}{L(t)}\right]$ ratio of capital as to labour at time t.]

$K(t) = P(t)L(t)$

$$\boxed{K(t) = P(t) L_0 e^{rt}}$$

Differentiate w.r.t time.

$$\frac{dK}{dt} = P(t) L_0 e^{rt} \cdot r + \frac{dP}{dt} L_0 e^{rt}$$

$$\frac{dK}{dt} = L_0 e^{rt} \left(rP(t) + \frac{dP}{dt}\right) \rightarrow ②$$

Equate ① & ②

$$\left(\frac{dP}{dt} + Pr\right) L_0 e^{rt} = SF(K, L_0 e^{rt}) = SL_0 e^{rt} F\left(\frac{K}{L_0 e^{rt}}, 1\right)$$

$$= S L_0 e^{rt} F(P)$$

$$\Rightarrow \left(\frac{dP}{dt} + Pr\right) = S F(p)$$

$$\boxed{\frac{dP}{dt} = SF(p) - Pr}$$

F(p) is output per worker as function of capital/worker

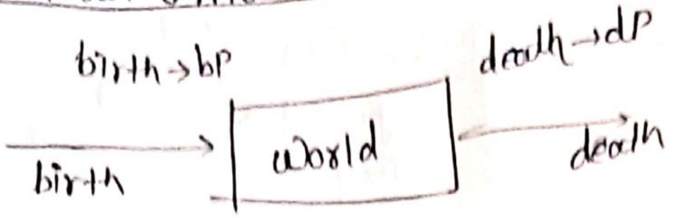
→ The rate change in capital labour (P) is the difference b/n increment of the capital and increment of labour.

cop-douglas $\left[\frac{dP}{dt} = S \cdot p^{a} - rP\right]$

$(f(P) = p^{a})$

$[F(L, K) = K^{a} L^{1-a}$ where $a < 1]$

Population Growth models



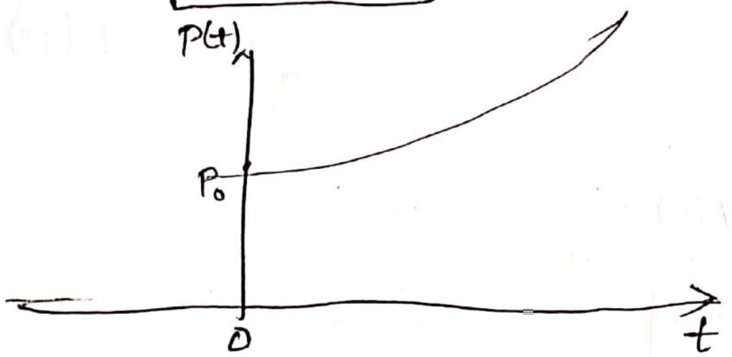
$$\left\{\begin{matrix}\text{rate change in}\\ \text{Population}\end{matrix}\right\} = \left\{\begin{matrix}\text{rate of}\\ \text{birth}\end{matrix}\right\} - \left\{\begin{matrix}\text{rate of}\\ \text{death}\end{matrix}\right\}$$

$$\frac{dP}{dt} = bP - dP$$
$$= (b-d)P$$
$$[P(0) = P_0]$$

$$r = b - d > 0 \rightarrow b > d$$

$$\rightarrow \frac{dP}{dt} = rP$$
↳ growth rate,

$$\boxed{P(t) = P_0 e^{rt}}$$



⇒ Scarcity of food (limited resources in compartment)

Per Capita — intraspecific competition

$$\{\text{Rate change}\} = \{\text{birth rate}\} - \{\text{death rate}\} - \{\alpha P\}$$

$$\frac{1}{P}\frac{dP}{dt} = b - d - \alpha P$$

$$\left[\frac{dP}{dt} = bP - dP - \alpha P^2\right]$$

$$\frac{dP}{dt} = rP - \alpha P^2]$$

$$\frac{dP}{dt} = rP\left[1 - \frac{\alpha}{r}P\right]$$

let $K = \frac{r}{\alpha}$

$$\frac{dP}{dt} = rP\left[1 - \frac{P}{K}\right]$$

[K = carrying capacity of environment]

$P(0) = P_0$

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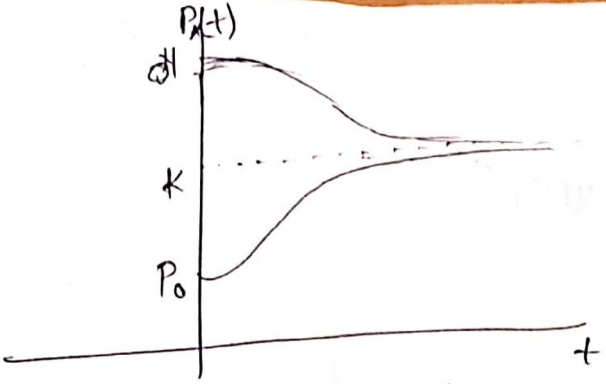

→ logistic growth model

r = growth rate

$P(t)$ = population ~~of~~ time t.

Double life (double population rate)

Let $P(t)$ is the population at time t

$$P(t+T) = 2P(t)$$

$$\frac{P(t+T)}{P(t)} = 2$$

$$P(t) = P_0 e^{r(t-t_0)}$$

$$(P(t_0) = P_0)$$

$$\frac{P(t+T)}{P(t)} = 2 = \frac{\cancel{P_0} e^{r(t+T-t_0)}}{\cancel{P_0} e^{r(t-t_0)}}$$

$$2 = e^{rT}$$

$$rT = \ln(2)$$

$$\left[T = \frac{\ln(2)}{r}\right]$$

n times of current population

$$\left[T_n = \frac{\ln(n)}{r}\right]$$

Que) In 1990, the growth rate of world was $= 0.017/yr$. $X_0 = 5.3b$

$$\Rightarrow \quad \frac{\ln(2)}{0.017} \Rightarrow \frac{0.69}{0.017} \Rightarrow 40\,yrs$$

$$\frac{0690}{17}$$

in $\boxed{2030}$

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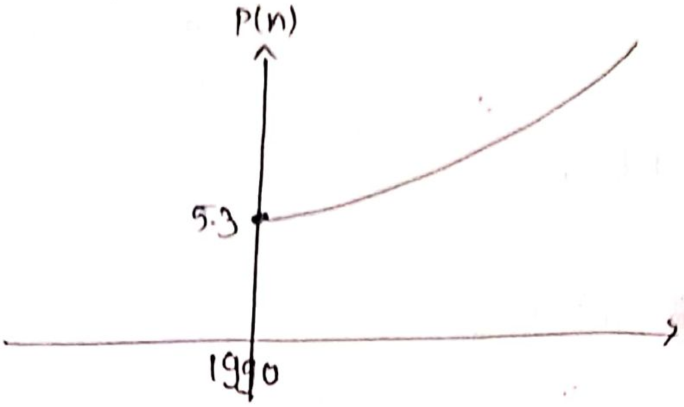
if rate $= \frac{\text{in } 2050}{0.017/yr}$

$\Rightarrow \frac{0.69}{0.01} \Rightarrow \underline{69 \text{ yrs}} \quad \Rightarrow \underline{2059}$



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Logistic Growth Model :→

$$\frac{dx}{dt} = r x\left(1 - \frac{x}{K}\right) = r x\left(\frac{K-x}{K}\right)$$

$x \rightarrow$ is the population of individual
$r \rightarrow$ Growth rate of population
$K \rightarrow$ carrying capacity of the Environment

Apply seperation of variable.

$$\frac{(K)\,dx}{x(K-x)} = \frac{r\,dt}{\cancel{K}}$$

$$\frac{K\,dx}{\cancel{x}K - x^2} = r\,dt$$

$$\frac{K\,\ln(\cancel{x}(K-x))}{K-2\cancel{x}} = rt$$

$$\int \frac{K-x+x}{x(K-x)}\,dx = rt + C$$

$$\int \left[\frac{\cancel{K-x}}{x(\cancel{K-x})} + \frac{\cancel{x}}{\cancel{x}(K-x)}\right] dx = rt + C$$

$$\ln(x) - \ln(K-x) = rt + C$$

$$\ln\frac{x}{K-x} = rt + C$$

$$\frac{x}{K-x} = e^{rt+C} \qquad C_1 = e^{C}$$

$$\boxed{\frac{x}{K-x} = C_1 e^{rt}}$$

$$X = (k-x)\, c_1 e^{rt} \rightarrow (i)$$

let initial condition $[X(0) = X_0]$

$$X_0 = (K - X_0)\, c_1$$

$$\left[c_1 = \frac{X_0}{k - X_0} \right] \longrightarrow \text{substitute this in (i)}$$

$$\boxed{X = (k-x)\left(\frac{X_0}{K - X_0}\right) e^{rt}}$$

$$X\left(1 + \left(\frac{X_0}{k - X_0}\right) e^{rt}\right) = K\left(\frac{X_0}{k - X_0}\right) e^{rt}$$

$$X(t) = \frac{K\left(\frac{X_0}{k - X_0}\right) e^{rt}}{\left(1 + \left(\frac{X_0}{k - X_0}\right) e^{rt}\right)}$$

$$= \frac{K}{\left(1 + \left(\frac{k - X_0}{X_0}\right) e^{-rt}\right)} \rightarrow \text{divide by } \left(\frac{X_0}{k - X_0}\right) \text{ in numerator \& denomenador}$$

$$\boxed{X(t) = \left[\frac{K}{1 + m e^{-rt}}\right]} \longrightarrow \text{where } m = \left[\frac{k}{X_0} - 1\right]$$

$t \rightarrow \infty$ $\boxed{X(t) = K}$

Case 1 $\rightarrow$ $0 < X_0 < K$

2 $\rightarrow$ $X_0 > K$

3 $\rightarrow$ $X_0 = K$ $\rightarrow$ always constant

Que) $\frac{dx}{dt} = rX\left(1 - \frac{X}{K}\right)$

$r = 1, \; K = 1000$

$\rightarrow$

Harvesting Models :→

24/01/24

$X \to$ no. of individuals.

$$\frac{dX}{dt} = rX\left(1-\frac{X}{k}\right) - H \to (1)$$

H: const rate of harvesting per unit time

Logistic Growth → Lake → Harvesting

$$\left[\frac{dX}{dt} = rX\left(1-\frac{X}{k}\right) - \alpha X\right] \to (2)$$

Find exact sol^n of model (1) & model (2)

$$\frac{dX}{dt} + \alpha X = rX\left(1-\frac{X}{k}\right)$$

$$\frac{1}{X}\left(\frac{k}{k-X}\right)\left(\frac{dX}{dt}+\alpha X\right) = r$$

$k\alpha X$

$$\frac{1}{X}\left(\frac{k}{k-X}\right)(dX + \alpha X\,dt) = r\,dt$$

$$\int \frac{1}{X}\left(\frac{k}{k-X}\right) dX = \int\left(r - \frac{k\,\alpha \cancel{X}}{\cancel{X}(k-X)}\right) dt$$

$$\int \frac{k-\cancel{X}+\cancel{X}}{X(k-X)}\,dX = \int r - \frac{k\alpha}{(k-X)}\,dt$$

$$\ln X - \ln(k-X) =$$

1) $$\frac{dX}{dt} = rX\left(1-\frac{X}{k}\right) - H.$$

$$\frac{dX}{dt} = \frac{r}{k}\left(kX - X^2 - \frac{kH}{r}\right)$$

$$\frac{dX}{\left(kX - X^2 - \frac{kH}{r}\right)} = \frac{r}{k}\,dt$$

→ partial differentiation

$$X^2 - kX + \frac{kH}{r} = (x-\alpha)(x-\beta)$$

$$A(x-\alpha) + B(x-\beta) = 1$$

$x=\alpha$ $\quad B = \frac{1}{\alpha-\beta}$, $A = \frac{1}{\beta-\alpha}$

$$\frac{dx}{x^2-kx \pm \frac{kH}{r}} = \frac{-r\,dt}{k}$$

$$\frac{dx}{(x-\alpha)(x-\beta)} = \frac{-r}{k}dt$$

$$A(x-\alpha)+B(x-\beta)=1$$

$$\left(A=\frac{1}{\beta-\alpha} \quad B=\frac{1}{\alpha-\beta}\right)$$

$$\frac{1}{(\beta-\alpha)}\left(\frac{1}{x-\alpha}\right)+\left(\frac{1}{\alpha-\beta}\right)\left(\frac{1}{x-\beta}\right)dx=-\frac{r}{k}dt$$

$$\frac{1}{\alpha-\beta}\left(\frac{1}{x-\alpha}-\frac{1}{x-\beta}\right)dx=-\frac{r}{k}dt$$

$$\ln\left(\frac{x-\alpha}{x-\beta}\right) = -\frac{r}{k}(\alpha-\beta)\,dt$$

$$\frac{x-\alpha}{x-\beta} = e^{-\frac{r}{k}(\alpha-\beta)dt}$$

$$\left[\alpha,\beta \Rightarrow \frac{k \pm \sqrt{k^2-\frac{4kH}{r}}}{2}\right]$$

---

② $$\frac{dx}{dt} = rx\left(1-\frac{x}{k}\right)-\alpha x$$

$$\Longrightarrow \frac{dx}{dt} = rx-\frac{rx^2}{k}-\alpha x \qquad (r-\alpha)x-\frac{rx^2}{k}$$

$$\frac{dx}{dt} \Rightarrow \frac{x}{k}\left[k(r-\alpha)-rx\right]$$

$$\frac{dx}{x\left[k(r-\alpha)-rx\right]} = \frac{1}{k}dt$$

$$1 = A\left(k(r-\alpha)-rx\right]+Bx$$

at $x=0$ $$\left[A=\frac{1}{k(r-\alpha)}\right] \longrightarrow (i)$$

at $\xrightarrow{k(\alpha-r)-rx=0}$ $x=\frac{k(r-\alpha)}{r}$ $$\left[B \Rightarrow \frac{r}{k(r-\alpha)}\right] \longrightarrow (ii)$$

$$\frac{A}{x}+\frac{B}{k(r-\alpha)-rx} = \frac{1}{k}dt$$

putting A & B from (i) & (ii)

$$\frac{1}{k(r-\alpha)}\left[\frac{1}{x}+\frac{r}{k(r-\alpha)-rx}\right] = \frac{1}{\cancel{k}}dt$$

$$\frac{1}{x}+\frac{r}{k(r-\alpha)-rx} = (r-\alpha)\,dt$$

$$\ln x + \frac{\ln\left(k(r-\alpha)-rx\right)}{-r} \cdot r = (r-\alpha)t$$

$$\ln x - \ln\left(k(r-\alpha)-rx\right) = (r-\alpha)t$$

$$\ln\left[\frac{x}{k(r-\alpha)-rx}\right) = (r-\alpha$$

$$\boxed{\frac{x}{k(r-\alpha)-rx} = c_1 e^{(r-\alpha}}$$

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Dynamical System :→

Dynamics is the ~~star~~ primarly the study of the time evolutionary system & corresponding system equation is known as dynamical system.

→ A system of n first order DE's in $\mathbb{R}^n$ space is called a dynamical system of dimension n which determines the time behaviour of the system.

Deterministic Process ⤵

A process is called deterministic process if its entire future and past are uniquely determined by its state of current / present time.

* Types of dynamical system →

A) Continuous type DS → DE

B) Discrete type DS ↓

Difference Equation

$$x(n+1) = f(x(n))$$

$$x^{0} = f(x)$$

→ A continuous DS mathematically can be written as

$$x^{0} = f(\vec{x}, t)$$

$$X = (x_1, x_2, \dots x_n) \in \mathbb{R}^n$$

$$F = (f_1, f_2, \dots f_n)$$

$$x_1^{0} = f_1(x_1, \cancel{x_2}, x_3 \dots x_n; t)$$

$$x_2^{0} = f_2(x_1, x_2, \cancel{x_3} \dots x_n, t)$$

ere X is known as state variable

F is a smooth function defined on the domain of the function

$$(\mathbb{R}^n \times \mathbb{R}) \to [\mathbb{R}^{n+1}]$$

→ $x^{0} = f(x)$ [time independent] → Autonomous System

* **Kirchoff's law**

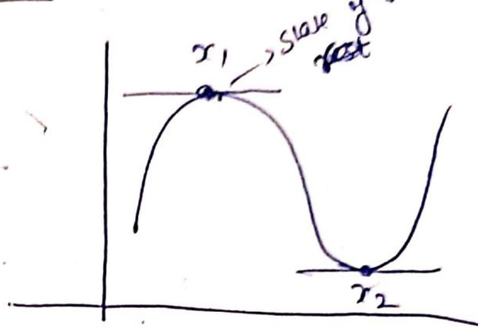
(i) $\frac{dV}{dt} = \frac{I}{C}$ and $\frac{dI}{dt} = -\frac{RI}{L} - \frac{V}{L}$

(ii) $\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right)$ → Autonomous.

$\frac{dN}{dt} = r(t)N\left(1-\frac{N}{K}\right)$ → Non Autonomous.

* **Fixed Points / Critical Point / Equilibrium Point :→**

$[\dot{x} = f(x) = 0]$ → x^* → critical point



Critical points are state of the rest.

* $\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right)$

$0 = rN\left(1-\frac{N}{K}\right)$

0, $1-\frac{N}{K} = 0$

[critical points ← $(N=K)$, 0]

Assignment

Critical points of all models.

~~(i) Compartment model :→~~

(i) Radioactive Decay

$\frac{dN}{dt} = -KN$

$0 = -KN$

[critical points, $N=0$]

$\boxed{\frac{d^2N}{dt^2} = -K}$ → $-ve$ so stable.

(ii) Population Growth

→ $\frac{dP}{dt} = (r-d)P$

$0 = (r-d)P$

critical points, $(P=0)$

$\boxed{r=d}$

$\boxed{\frac{d^2P}{dP^2} (r-d)}$ if $r>d$ → unstable

else → stable

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(iii) Pollution in Lake

$$\frac{dc}{dt} = \frac{Fc_{in}}{V} - \frac{Fc}{V}$$

$$0 = \frac{Fc_{in}}{V} - \frac{Fc}{V}$$

$$\frac{\cancel{F}c}{\cancel{V}} = \frac{\cancel{F}c_{in}}{\cancel{V}}$$

Critical point $\rightarrow \left[c = c_{in}\right]$

$$\frac{d^2c}{dt^2} = -\frac{F}{V}$$

~~if $c_{in} > 1$ +ve [unstable]~~

~~$c_{in} < 1$ stable~~

$-ve \rightarrow$ [so stable]

(iv) Economic Growth

$$\frac{dP}{dt} = Sp^a - \gamma P$$

rate in change of capital labour

$$0 = Sp^a - \gamma P$$

$$\gamma P = Sp^a \qquad p^{1-a} = \frac{S}{\gamma}$$

critical point $\rightarrow \left[\cancel{\gamma = Sp^{a-1}}\right] \rightarrow \left[P = \left(\frac{S}{\gamma}\right)^{\frac{1}{(1-a)}}\right]$

$$\frac{d^2P}{dt^2} = aSp^{a-1} - \gamma$$

~~if $aSp^{a-1} > \gamma$~~

$$= aS\left(\frac{S}{\gamma}\right)^{\frac{a-1}{1-a}} - \gamma$$

$$\Rightarrow a\cancel{S}\left(\frac{\gamma}{\cancel{S}}\right) - \gamma$$

$$= \gamma(a-1)$$

if $a > 1 \rightarrow$ unstable

$a < 1 \rightarrow$ stable.

(v) Limited Resources market

$$\frac{dP}{dt} = \gamma P\left[1 - \frac{\alpha}{\gamma}P\right]$$

$$0 = \gamma P\left[1 - \frac{\alpha}{\gamma}P\right]$$

$$\gamma P = 0 \quad \& \quad 1 - \frac{\alpha}{\gamma}P = 0$$

$$\left[P = \frac{\gamma}{\alpha}\right]$$

critical points $\longrightarrow P = 0 \quad \&$

$$\frac{d^2P}{dt^2} = \gamma P\left[0 - \frac{\alpha}{\gamma}\right] + \gamma\left[1 - \frac{\alpha}{\gamma}P\right]$$

at $P = 0 \Rightarrow \frac{d^2P}{dt^2} = \gamma$

+ve $\rightarrow$ unstable

at $P = \frac{\gamma}{\alpha} \Rightarrow \frac{d^2P}{dt^2} = \frac{\gamma^2}{\alpha}\left[0 - \frac{\alpha}{\gamma}\right]$

$$\Rightarrow \left[\frac{\gamma^2}{\alpha}\cdot\frac{\alpha}{\gamma}\right] \quad -\gamma$$

Always stable

~~if $\frac{\gamma^2}{\alpha} > \gamma \rightarrow$ unstable~~

~~if $\frac{\gamma^2}{\alpha} < \gamma \rightarrow$ stable~~

(vi) Logistic Growth Model

$$\frac{dx}{dt} = rX\left(1 - \frac{X}{K}\right)$$

$rX = 0 \qquad 1 - \frac{X}{K} = 0$

critical points $\rightarrow [X = 0 \;, \; X = K]$

$$\frac{d^2X}{dt^2} = rX\left(0 - \frac{1}{K}\right) + r\left(1 - \frac{X}{K}\right)$$

at $X = 0 \rightarrow \left(\frac{d^2X}{dt^2} \Rightarrow r\right) \rightarrow$ ~~always~~ unstable.

at $X = K \Rightarrow -r \rightarrow$ ~~always~~ stable.

(vii) Harvesting Model.

1) $$\frac{dx}{dt} = rX\left(1 - \frac{X}{K}\right) - H$$

$$H = rX\left(1 - \frac{X}{K}\right)$$

$$\frac{rX^2}{K} - rX + H = 0$$

$$X^2 - KX + \frac{KH}{r} = 0$$

$$x = \frac{K \pm \sqrt{K^2 - \frac{4KH}{r}}}{2}$$

$$x = \frac{K + \sqrt{K^2 - \frac{4HK}{r}}}{2}, \quad x = \frac{K - \sqrt{K^2 + \frac{4HK}{r}}}{2}$$

$$\frac{d^2x}{dt^2} = rX\left(0 - \frac{1}{K}\right) + r\left(1 - \frac{X}{K}\right).$$

at X

---

2) $$\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right) - \alpha X$$

$$\alpha \cancel{X} = r\cancel{X}\left(1 - \frac{X}{K}\right)$$

$$\alpha = r - \frac{Xr}{K}$$

$$\boxed{X = \frac{(r - \alpha)K}{r}}$$

and $x = 0$

$$\frac{d^2X}{dt^2} = rX\left(0 - \frac{1}{K}\right) + r\left(1 - \frac{X}{K}\right) - \alpha.$$

at $X = 0 \rightarrow \frac{d^2X}{dt^2} \Rightarrow r - \alpha$

if $r > \alpha \rightarrow$ unstable

$r < \alpha \rightarrow$ stable

at $X = \frac{(r-\alpha)K}{r} \Rightarrow \frac{d^2X}{dt^2} \Rightarrow$

$$-\cancel{r}\frac{(r-\alpha)\cancel{K}}{\cancel{r}}\left(\frac{1}{\cancel{K}}\right) + r\left(1 - \frac{(r-\alpha)}{r}\right) - \alpha$$

$\Rightarrow -(r-\alpha) + r - (r - \alpha) - \alpha$

$\Rightarrow -(r - \alpha)$

$\Rightarrow$ $r > \alpha \rightarrow$ stable

$r < \alpha \rightarrow$ unstable

$$~~x = \frac{K\alpha}{r}~~$$

02/02/24

→ Equilibrium Points ⌉

$$x^{\circ} = f(x) \rightarrow (1)$$

x_e is said to be equilibrium point of (1) if

$$[f(x_e) = 0]$$

* It is a constant solution of a dynamical system.

Stability of Equilibrium points ⌉

Stability refers to sustainability

→ Stability of equilibrium points determines wheather the solution of dynamical system remains near the equilibrium point after a small perturbation

Stable ⌉

An equilibrium point is stable if initial condition that starts nearby equilibrium point stay close to that equilibrium point

$x^{\circ} = f(x)$ $\quad$ $x(t) \rightarrow$ Solution of DS

$x_e \rightarrow$ Equilibrium point

$x(0) \rightarrow$ initial value.

The equilibrium point x_e is stable if for ~~a~~ $\epsilon > 0$ $\exists$ $\delta > 0$ such that

$$\| x(0) - x_e \| < \delta \Rightarrow \| x(t) - x_e \| < \epsilon \quad \forall t > 0$$

↑ distance

if → $\| x(t) - x_e \| = 0$ → asymptotically stable

How to find stability of dynamical system :⌉

i) 1D system $\quad x^{\circ} = f(x), \quad x(0) = x_0$

if $f'(x_e) < 0$, x_e is stable

if $f'(x_e) > 0$, x_e is unstable

$f'(x_e)=0$, x_e is a degenarate equilibrium point

Que → $\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$ $N_1 = 0$,
$N_2 = K$

→ $f'(N) = r - \frac{2rN}{K}$

$f'(N_1) = \boxed{f'(0) = r} > 0.$

N_1 → is unstable.

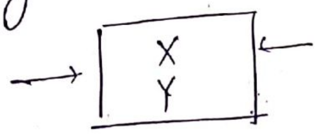
$f'(N_2) = f'(K) = r - 2r \Rightarrow -r < 0$

N_2 → is stable.

→ Jacobian Matrix

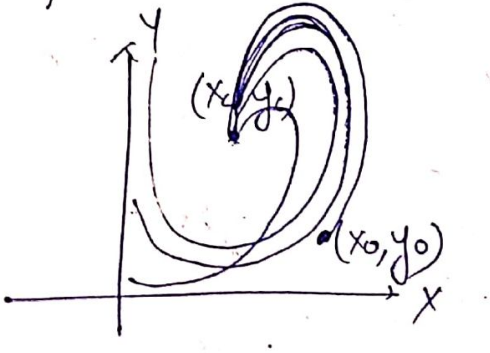
05/02/24

→ A dynamical system in 2D



$$\left.\begin{aligned}\frac{dx}{dt} &= F(x,y)\\ \frac{dy}{dt} &= G(x,y)\end{aligned}\right] \rightarrow (1)$$

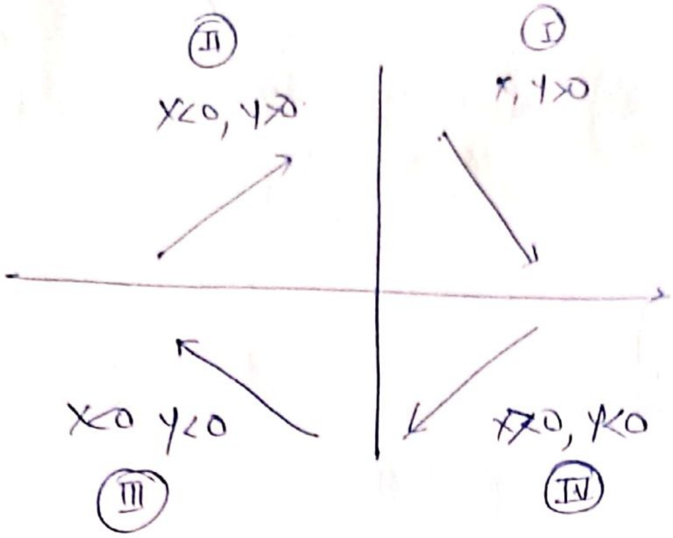
→ coupled dynamical systems.
→ system of differential equation.



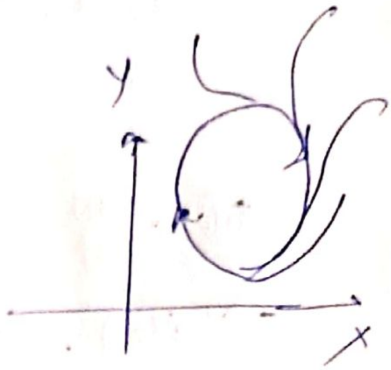
X-Y space → phase space

5/02/24

(Q10) $\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x$



(Clock wise)



<u>Chain Rule</u>

$$\Rightarrow \left(\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \right)$$

$$\left(\frac{dy}{dx} = \frac{-x}{y} \right)$$

$$y\, dy = -x\, dx$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + c$$

$$x^2 + y^2 = 2c$$

$$\boxed{x^2 + y^2 = k}$$

if $x(0) = 1$ & $y(0) = 1$

$$\boxed{x^2 + y^2 = 2}$$

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Linearization

5/02/24

$$\bar{X} = AX$$

$$\bar{X} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$AX = b$$

$AX = \lambda X$ → eigen vector corresponding to eigen value 'λ'

λ: eigen value of A

$(|A - \lambda I| = 0)$

$((A - I\lambda)X = 0)$

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$$\left.\begin{aligned} \frac{dx}{dt} &= F(x,y) \\ \frac{dy}{dt} &= G(x,y) \end{aligned}\right] \text{(1)}$$

let (x_e, y_e) is equilibrium point of (1)

from defi of equilibrium pt

$F(x_e, y_e) = 0$

$G(x_e, y_e) = 0$

Let us suppose we make very small perturbation.

$(x(t), y(t))$ is solution of system 2.

$x(t) = x_e + \xi(t)$ $\quad y(t) = y_e + \eta(t)$

Here $\xi(t)$ and $\eta(t)$ are small disturbation that goes to zero

where $(x(t), y(t)) \longrightarrow (x_e, y_e)$

$$\frac{dx}{dt} = \frac{d(x_e + \xi(t))}{dt} = F(x_e + \xi, y_e + \eta) = \frac{d\xi}{dt}$$

$$\frac{dy}{dt} = \frac{d(y_e + \eta(t))}{dt} = G(x_e + \xi, y_e + \eta) = \frac{d\eta}{dt}$$

$F(x_e+\xi, y_e+\eta)$ and $G(x_e+\xi, y_e+\eta)$ can be expanded with the help of Taylor series.

$$\frac{d\xi}{dt} = F(x_e, y_e) + F_x(x_e, y_e)\xi + F_y(x_e, y_e)\eta + O(h^2)$$

$$\frac{d\eta}{dt} = G(x_e, y_e) + G_x(x_e, y_e)\xi + G_y(x_e, y_e)\eta + O(h^2)$$

→ we ignored higher order terms.

$$\left.\begin{aligned} \frac{d\xi}{dt} &= F_x(x_e, y_e)\xi + F_y(x_e, y_e)\eta \\ \frac{d\eta}{dt} &= G_x(x_e, y_e)\xi + G_y(x_e, y_e)\eta \end{aligned}\right\} \text{(iii)}$$

$$\begin{vmatrix} x' \\ y' \end{vmatrix} = \begin{vmatrix} \xi' \\ \eta' \end{vmatrix} = \underbrace{\begin{bmatrix} F_x(x_e, y_e) & F_y(x_e, y_e) \\ G_x(x_e, y_e) & G_y(x_e, y_e) \end{bmatrix}}_{\text{Jacobian Matrix}} \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0$$

$$(a_{11}-\lambda)(a_{22}-\lambda) - (a_{12})(a_{21}) = 0$$

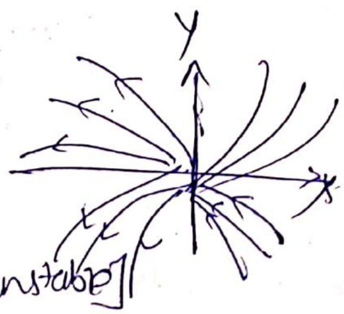
$$\lambda^2 - a_{11}\lambda - a_{22}\lambda + a_{11}a_{22} - a_{12}a_{21} = 0$$

$$\lambda^2 - \lambda(a_{11}+a_{22}) + a_{11}a_{22} - a_{12}a_{21} = 0.$$

Case (I) $\boxed{\text{Real eigen Values}}$

↳ λ_1, λ_2 → both are -ve. [System is stable]

$\lambda_1 < 0 \quad \lambda_2 < 0.$



Case (II) Saddle behaviour

λ_1, λ_1 → both have opposite sign.

[System is unstable]

$\lambda_1 \lambda_2 < 0$

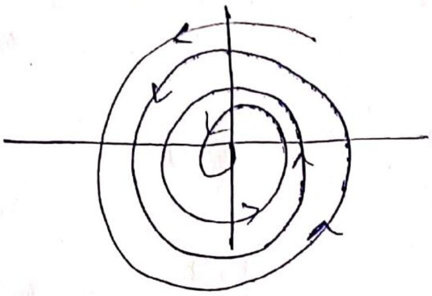
$\lambda_1 > 0 \quad \lambda_2 > 0$ ——→ system is unstable.

Imaginary eigen values :

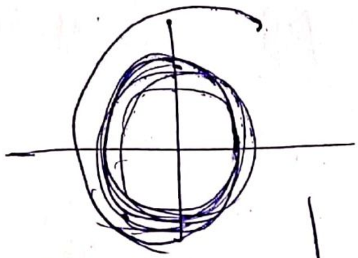
Case (I) : $\lambda_1, \lambda_2 = \alpha \pm i\beta$

α → real part ; β → imaginary part

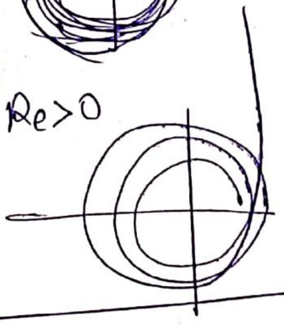
if $Re < 0$



Case (II) → if $Re = 0$ → never come to equilibrium point only it will oscillate.



Case (III) → $Re > 0$ unstable.



7/02/.

$$\lambda^2 - \underbrace{(a_{11} + a_{22})}_{\text{Trace of matrix}}\lambda + \underbrace{(a_{11}a_{22} - a_{12}a_{21})}_{\text{determinant of matrix}} = 0.$$

Trace of matrix → p ; determinant of matrix → q

Δ (discriminant) $= p^2 - 4q$

*) Both eigen value will be real → $(p^2 > 4q)$ $(\Delta > 0)$

(1) If trace

Δ	p	q	Equilibrium pt.
$\Delta > 0$	$p < 0$	$q > 0$	$\Rightarrow$ stable node $\lambda_1 < 0$ $\lambda_2 < 0$
$\Delta > 0$	$p > 0$	$q > 0$	$\Rightarrow$ unstable node $\lambda_1 > 0$ $\lambda_2 > 0$
$\Delta > 0$	$p > 0$	$q < 0$	$\Rightarrow$ saddle point $\lambda_1 \lambda_2 < 0$
$\Delta < 0$	$p < 0$	$q > 0$	$\Rightarrow$ stable spiral.
$\Delta < 0$	$p = 0$	$q > 0$	$\Rightarrow$ centre spiral.
$\Delta < 0$	$p > 0$	$q > 0$	$\Rightarrow$ unstable spiral.

$[p = \lambda_1 + \lambda_2]$

$[q = \lambda_1 \lambda_2]$

$p^2 - 4q$

$p^2 + 4q$

$q > 0$

Que) $x^2 - 3x + 1 = 0$
$p \quad q$

$\Delta \Rightarrow 9 - 4 \Rightarrow 5 > 0$

$p < 0$

$x = \left(\frac{3 \pm \sqrt{5}}{2}\right)$

$x^2 + x - 20 = 0$

-20

$-5 \quad 4$

(1)

$$\frac{dS}{dt} = -\beta S I = F$$

$$\frac{dI}{dt} = \beta S I - dI = G$$

$S(0) > 0$, $I(0) > 0$.

$\Rightarrow$ $\frac{dS}{dt} = 0 = -\beta S I = 0 \rightarrow (1)$

$\frac{dI}{dt} = 0 = \beta S I - dI = 0 - (1)$

$(\beta S - d) I = 0$

$I = 0$ & $S = \frac{d}{\beta}$

$E_0 = (0, 0)$

$E_1 = (\frac{d}{\beta}, 0)$

$$J(0,0) = \begin{bmatrix} -\beta I & -\beta S \\ \beta I & \beta S - d \end{bmatrix}_{(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & -d \end{bmatrix}$$

$\lambda^2 - \lambda(0-d) = 0$

$[\lambda^2 + d\lambda = 0]$ $\quad$ $\lambda_1, \lambda_2 = 0, -d$

• $p > 0$ $\quad$ $\Delta = d^2 > 0$

$\rightarrow$ unstable.

$$J\left(\frac{d}{\beta}, 0\right) = \begin{bmatrix} 0 & -d \\ 0 & 0 \end{bmatrix}$$

$\rightarrow$ eigen values are zero

$\Rightarrow E_1$ is also a degenerate equilibrium point.

$\lambda^2 - 0\lambda + 0 - (\cancel{0}) = 0$

$\lambda^2 + \cancel{d} = 0$

$\boxed{\lambda = \cancel{0}}$ $\quad$ $\lambda_1, \lambda_2 = 0, 0$

Predator Prey Model

9/02/24

$X(t)$ = Population of Prey
$Y(t)$ = — " — Predator.

A-1 → No predator.
Prey grows exponential.
A-2 → No other food for predator

$$\frac{dX}{dt} = rX - a_1XY \quad \to F(x,y)$$

$$\frac{dY}{dt} = a_2XY - dy \quad \to G(x,y)$$

$X(0)>0 \quad Y(0)>0$ → proposed by → Lotka-Voteria Model.

Equilibrium points

$$\frac{dX}{dt} = 0 \qquad rX - a_1XY = 0$$
$$X(r - a_1Y) = 0 \longrightarrow \left(X=0 \text{ or } Y=\frac{r}{a_1}\right) \to (1)$$

$$\frac{dY}{dt} = 0 \qquad a_2XY - dy = 0.$$
$$y(a_2X - d) = 0 \longrightarrow \left(X=\frac{d}{a_2} \text{ or } Y=0\right) \to (2)$$

from (1) & (2)

$E_1\ (0,0)$

$E_2\left(\frac{d}{a_2}, \frac{r}{a_1}\right)$

$$J(0,0) = \begin{bmatrix} r - a_1Y & -a_1X \\ a_2Y & a_2X - d. \end{bmatrix}$$

$\lambda^2 - (\text{trace})\lambda + \text{determinant} = 0$

$$\lambda^2 - (r - a_1Y + a_2X - d)\lambda + [(r-a_1Y)(a_2X-d) + a_1a_2XY] = 0$$

$E(0,0) \Rightarrow \lambda^2 - (r-d)\lambda + (-rd) = 0$

$$\lambda_1, \lambda_2 = \frac{(r-d) \pm \sqrt{(r-d)^2 + 4rd}}{2}$$

$$\frac{(r-d) \pm \sqrt{(r+d)^2}}{2}$$

$\lambda_1, \lambda_2 = r, -d \rightarrow$ saddle point

$E\left(\frac{d}{a_2}, \frac{r}{a_1}\right) \Rightarrow \lambda^2 - (r - r + d - d)\lambda + \left[a_1 a_2 \left(\frac{d}{a_2}\right)\left(\frac{r}{a_1}\right)\right] = 0$

$\lambda^2 = -dr$

$\lambda = \pm\sqrt{-dr}$

$= \pm i\sqrt{dr}$

as determinant less than zero.

$\rightarrow$ centre spiral $\quad \Delta < 0 \quad p = 0$

Aka $\rightarrow$ limit cycle behaviour, periodic behaviour, oscillati behaviou

13/02

Simulation :-

Simulation is a representation of reality through the use of model or other device which will react in the same manner as reality under a given set ~~function~~ of condition.

Simulation is also defined as the use of a model that has the designed characteristic of reality in order to produce the essense of an actual operation.

Types of simulation :-

1) Analogue simulation
2) Computer (system) simulation

Examples → 1) Testing a model
2) Model of traffic system.

Random Variable → The random variable is a real valued function defined over a sample space associated with the outcome of a conceptual chance experiment random variables are classified according to their probability density function.

Ex.

$$f(x) = \begin{cases} 1 , & 0 \le x < 1 \\ 0 , & \text{otherwise.} \end{cases}$$

Random number →
It is a number in a sequence of number whose probability of occurence same as that of any other number in that sequence.

Pseudo Random number → Random numbers are called pseudo random number when they are generated by same deterministic process but they have already qualified the pre determined statistical test for randomness.

Monte Carlo simulation I

• Monte carlo simulation yields a solution very close to the optimal solution but not necessarily the exact solution.

The monte carlo simulation procedure can be explained in following steps.

1) Clearly define the problem.
2) Construct an appropriate model.
~~3) Define the relationship~~ b/n variables & parameters
3) Prepare the model for experiment
4) Using steps 1 to 3, experiment with the model.
   (a) Define : a coding system that will correlate the factor. define in ~~step~~ step 1 with the random number to be generated for simulation.
   (b) Select a random number generator and create the random numbers to be used in simulation
5) Summarize the examine the results obtained in ④
6) Evaluate the results of the simulation.

14/02/24

eg) A tourist car operator finds that during the past few month the cars use has varied so much that the cost of maintaining the car varied considerably during the past 20 days the demand for the car fluctuated as below.

Type of week	Frequency
0	16
1	24
2	30
3	60
4	40
5	50

- Simulate the demand for a 10 week period.

use random no's
82, 96, 18, 96, 20, 84, 56, 1
57, 03.

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Trips/week	Frequency	Probability	Cummulative probability	Tag.no
0	16	$\frac{16}{200}$ 0.08	0.08	0–7
1	24	0.12	0.20	08–19
2	30	0.15	0.35	20–34
3	60	0.30	0.65	35–64
4	40	0.20	0.85	65–84
5	30	0.15	1	85–99

Week	Random no	
1	82	4
2	96	5
3	18	~~2~~ 1
4	96	5
5	20	2
6	84	4
7	86	~~5~~
8	11	1
9	52	3
10	03	0
		28

$$\text{Avg demand} = \frac{28}{10}$$

$\Rightarrow$ 2.8 car/week

Q2) An automobile production line turns out about 100 cars a day by deviations occure owing to many causes the production is more accurate describe by the probability distribution given as

Production /day	Probability	Cumulative	Probability
95	0.03	0.03	00-2
96	0.05	0.08	3-7
97	0.07	0.15	8-14
98	0.10	0.25	15-24
99	0.15	0.40	25-39
100	0.10	0.60	40-59
101	0.15	0.75	60-74
102	0.10	0.85	75-84
103	0.07	0.92	85-91
104	0.05	0.97	92-96
105	0.03	1	97-100

Finished cars are transported across the bay at the end of each day by a ferry. If ferry has space only for 101 cars, what will be the avg no. of cars waiting to be shipped and what will be avg. no. of empty space

use random no → 97, 02, 80, 66, 96, 55, 50, 29, 58, 51, 04, 86, 24, 39, 47

Random no.	Production	No of car waiting	No of empty space
97	105	4	0
02	95	0	6
80	102	1	0
66	101	0	0
96	104	3	0
58	100	0	1
50	100	0	1
29	99	0	2
58	100	0	1
51	100	0	1
04	96	0	5
86	103	2	0
24	98	0	3
39	99	0	2
47	100	0	[illegible]

→ Mid Square Method

→ Linear Congruential $x_n = a x_{n-1} + c \pmod m$

$$\left[\frac{x_n - a x_{n-1}}{m} = c\right]$$

$13 \equiv (7+1) \bmod 5$

$\frac{(13-7)}{5} \equiv 1$

$\left(\frac{6}{5} \equiv 1\right)$

⊛ Mid Square Method →
Von Neumann & Metropolis (1940)

- Start with a 4 digit number Z_0
- Square it to obtain 8 digits (if necessary append zero to the left)
- Take the middle 4 digit to obtain the next 4 digit number Z_1
- Square Z_1 & take middle 4 digits again and so on

– We get uniform random number by placing the decimal point at the left of each Z_i

i	Z_i	$Z_i \times Z_i$
0	7182	51581124
1	5811	33767721
2	7677	58936329
3	9363	87665769
4	6657	44315649
5	3156	09960336
6	9603	92217609
7	2176	04734976
8	7349	54007801
9	0078	00006084
10	0060	00003600
11	0036	00001296
12	0012	00000144
13	1	00000001
14		
15		

19/02/24

Drug Assimilation in Blood

$$\frac{dx}{dt} = -k_1 x \quad x(0) = x_0$$

$$\frac{dy}{dt} = k_1 x - k_2 y \quad y(0) = y_0$$

$x \to$ Amount of drug in blood at time t

$y \to$ Amount of blood stream at time t

Phase Plane Analysis.

$$\frac{dx}{dy} = \frac{dx/dt}{dy/dt} \Rightarrow \frac{-k_1 x}{k_1 x - k_2 y}$$

let $y = vx$

$$dy = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} \Rightarrow \frac{k_1 x - k_2(vx)}{-k_1 x}$$

$$v + x\frac{dv}{dx} = \frac{k_1 - k_2 v}{-k_1}$$

$$v + x\frac{dv}{dx} \Rightarrow -1 + \frac{k_2}{k_1}\cdot v$$

$$x\frac{dv}{dx} = -1 + \frac{k_2}{k_1}v - v$$

$$x\,dv = \left[-1 + v\left(\frac{k_2}{k_1} - 1\right)\right] dx$$

$$\frac{dv}{-1 + v(A-1)} = \frac{dx}{x} \qquad A = \frac{k_2}{k_1}$$

$$\left[\frac{dv}{v(B)-1} = \frac{dx}{x}\right]$$

$$\frac{\log(v(B)-1)}{B} = \log x$$

$$\log x = \log(v(B)-1)^{1/B} \cdot c$$

$$x = c\left[\frac{y}{x}\left(\frac{k_2}{k_1} - 1\right) - 1\right]^{\frac{1}{\frac{k_2}{k_1} - 1}}$$

Scanned with CamScanner

1/02

$[X = 0]$

$K_1 x - K_2 y = 0 \quad \left(y = \frac{K_1 X}{K_2}\right)$

$(0,0) \quad \left(0, \frac{K_1 X}{K_2}\right)$

$$\frac{dx}{dt} = I - K_1 x \qquad x(0) = x_0$$

$$\frac{dy}{dt} = K_1 x - K_2 y \qquad y(0) = 0$$

$I - K_1 X = 0 \qquad X = \frac{I}{K_1}$

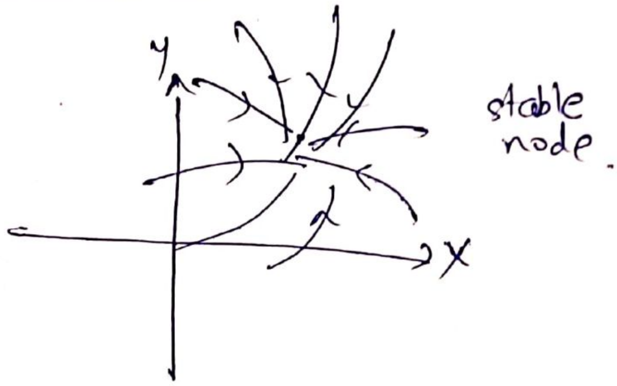
$K_1 X - K_2 y = 0 \rightarrow \left(\cancel{K_1} \frac{I}{\cancel{K_1}} - K_2 y = 0\right) \qquad \left(\frac{I}{K_2}, \frac{I}{K_2}\right)$

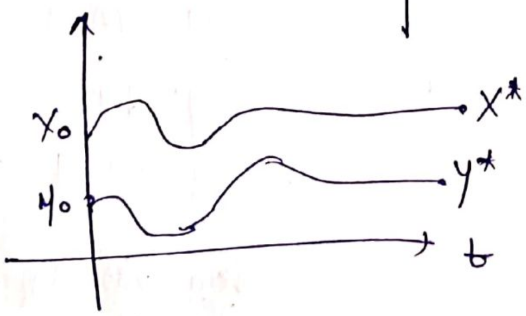
$\left(y = \frac{I}{K_2}\right)$

$(x_*, y_*) = \left(\frac{I}{K_1}, \frac{I}{K_2}\right)$

$$J \Rightarrow \begin{bmatrix} -K_1 & 0 \\ K_1 & -K_2 \end{bmatrix} \qquad \boxed{\text{eigen value} = -K_1, -K_2}$$

$$\lambda^2 + (K_1 + K_2)\lambda + K_1 K_2 = 0$$





Endemic disease model

$$\frac{dS}{dt} = bN - \beta S I - aS \qquad S(0) = S_0$$

$$\frac{dI}{dt} = \beta S I - \gamma I - aI \qquad I(0) = I_0$$

$$\frac{dR}{dt} = \gamma I - aR, \qquad R(0) = R_0$$

natural death.

with $N(t) = S(t) + I(t) + R(t)$