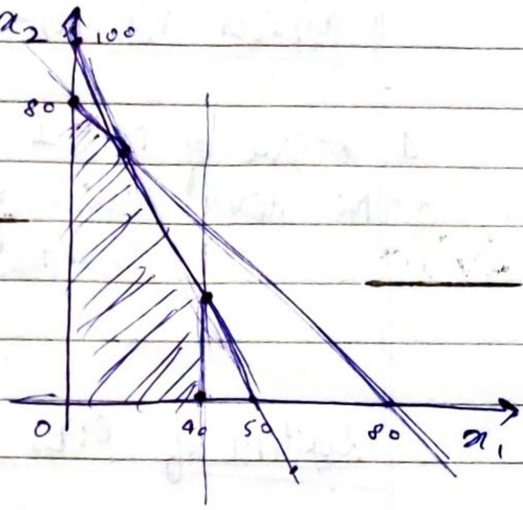
x_1 = no. of soldiers produced / week
x_2 = no. of trains produced / week

Maximise = ~~$270x_1 + 210x_2$~~ $\quad 30x_1 + 20x_2$
$= 30(20) + 20(60)$
$= 600 + 1200$
$= 1800$

Constraints, $x_1 + x_2 \leq 80$
$2x_1 + x_2 \leq 100$
$x_1 \leq 40$
~~x_1+x_2~~ $x_1, x_2 \geq 0$

Profit of $x_1 = 270 - 100 - 140 = 30$
$x_2 = 210 - 90 - 100 = 20$

Thus, we obtain 5 corner points
(0,0) (0,80) (40,0) (40,20)
(20,60)



∴ The objective function is maximised at (20, 60).
Thus, no. of soldiers = 20
no. of trains = 60.

1. Empirical models: Regression model ML model
2. Stochastic Models: Involves randomness or uncertainity.
   - Monte Carlo Simulation
   - Brownian Motion
   - Queing theory.
   - Simulation models
3. Deterministic models: NO randomness / uncertainity
4. Statistical Models: - Sampling Theory
   - Hypothesis Testing.

MM :

Deterministic Modelling

ODE : Only one independent variable
PDE : More than 1 independent variable

Order & Degree of DE : $y'' + ay + by = x^4$

(Max derivative) Order = 2
(Power of highest derivative) Degree = 1

Linear v/s non-linear DE's:

1. Degree of DE = 1
2. No term having product of independent variable and its derivative. $y \frac{dy}{dx}$

System of ODEs

$$\frac{dy_1}{dt} = F_1(y_1, y_2 \ldots y_n)$$
$$\frac{dy_2}{dt} = F_2(y_1, y_2 \ldots y_n)$$
$$\frac{dy_3}{dt} = F_3(y_1, y_2 \ldots y_n)$$
$$\vdots$$
$$\frac{dy_n}{dt} = F_4(y_1, y_2 \ldots y_n)$$

Compartmental Model: CM framework is extremely natural and valuable which formulate models for processes having inputs & /or output over time

Includes the process of incoming or outgoing over time

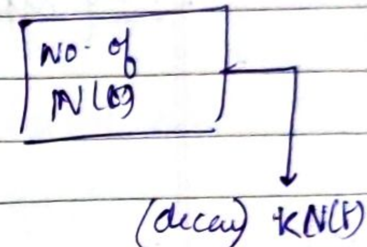
Example: Radioactive decay problem

Balance Law $=$ $\begin{Bmatrix}\text{Net rate change} \\ \text{of a substance}\end{Bmatrix} = \{\text{rate in}\} - \{\text{rate out}\}$

Let $N(t)$ is no. of radioactive nuclei at time t and $kN(t)$

$$\frac{dN}{dt} = -kN(t)$$

$$\boxed{N(t_0) = N_0}$$



Suppose t is current time, Δt is time step.

$N(t) \rightarrow$ current nuclei

$N(t+\Delta t) \rightarrow$ no. of nuclei in future time

$$N(t+\Delta t) = N(t) - kN(t)\Delta t$$

$$N(t+\Delta t) - N(t) = -kN(t)\Delta t$$

$$\lim_{\Delta t \to 0} \frac{N(t+\Delta t) - N(t)}{\Delta t} = -kN(t)$$

$$\boxed{\frac{dN}{dt} = -kN(t)}$$

$$\boxed{N(t_0) = N_0}$$

$$\int \frac{dN}{N(t)} = \int -k\,dt$$

$$\log N = -kt + c$$

$$= \boxed{N = e^{-kt+c}} =$$

$$N = Ae^{-kt}$$

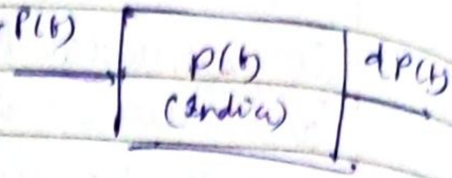
$$N(0) = N_0$$

$$N(0) = Ae^{0} = A = N_0$$

$$N = N_0 e^{-kt}$$

MM with case studies by, barnes

Population growth



$$\frac{dP}{dt} = rP(t) - dP(t)$$

$$\frac{dP}{dt} = (r-d)\,P(t) \qquad \boxed{P(0)=P_0} \text{ (Initial population)}$$

$$\frac{1}{P}\frac{dP}{dt} = (r-d)$$

Per capita growth rate of P in unit time

Half life : Let τ is half life of $N(t)$.

$$N(t+\tau) = \frac{N(t)}{2}$$

$$= \cancel{N_0} e^{-(t+\tau-t_0)k} = \frac{\cancel{N_0} e^{-(t-t_0)k}}{2}$$

$$e^{-\tau k} = \frac{1}{2}$$

$$-\tau k = \ln(1/2)$$

$$\boxed{\tau = \frac{\ln(2)}{k}}$$

Radioactive Elements in Painting: Uranium - 45 8 years
C^{14} 5568 year
Lead 20 22 year
Polonium P^{14} < 1s.

: All painting contain small amount of lead - 210
lead white contain lead (extracted from the rocks) and extremely small amount of Radium 226 (1600 years)

Let $N(t)$ be the amount of lead-210.

Then $\frac{dN}{dt} = -\lambda N + R(t)$. $n_0 = N(t_0)$

where $R(t)$ is the rate of disintegration of radium-226 per minute /g of white lead.

Let us assume that $R(t) = R$ (since half-life of radium is very large)

$$\frac{dN}{dt} = -\lambda N + R$$

$$N(t) = \frac{R}{\lambda}\left(1-e^{-\lambda(t-t_0)}\right) + n_0 e^{-\lambda(t-t_0)}$$

Rearranging the terms, $\lambda n_0 = \lambda N e^{\lambda(t-t_0)} - R\left(e^{\lambda(t-t_0)} - 1\right)$

λn_0 is disintegration rate from the initial time,

half life $\tau = \frac{\ln(2)}{\lambda}$

$$\lambda = \frac{\ln(2)}{\tau} = \frac{\ln(2)}{22 \text{ year}} = \frac{150}{2} = 0.0315$$

$R = 0.8$ amount rate disintegration

lead-210 $= 8.5\ m^{-1}g^{-1}$

$\lambda n_0 = 98\,000_m < 30,000_m$ (new painting)
(given in question)

Modelling Pollution in Lake

mass inflow → [mass of pollutant] → mass outflow

metal

Assumptions: 1. Lake has constant volume, and it is continuously well mixed so that pollutant is uniform throughout ode

2. $\begin{Bmatrix} \text{rate change} \\ \text{in mass of lake} \end{Bmatrix} = \{\text{rate enter}\} - \begin{Bmatrix} \text{rate } \cancel{\text{entering}} \\ \text{leaving} \end{Bmatrix}$

3. Let $C(t)$ be the concentration of polutant in lake at time t
Let F be the rate at which the water flows out of lake in m^3/day. Since volume is constant

$$\therefore \begin{Bmatrix} \text{Flow of mixture} \\ \text{into lake} \end{Bmatrix} = \begin{Bmatrix} \text{Flow of mixture} \\ \text{out of lake} \end{Bmatrix} = F$$

$\therefore$ Now $M'(t) = F c_{in} - \frac{F M(t)}{V}$

$$M(t) = C(t) \cdot V$$

$$M'(t) = C'(t) \cdot V$$

$$M'(t) = F c_{in} - F C(t)$$

$$\boxed{C'(t) = \frac{F c_{in}}{V} - \frac{F\, C(t)}{V}}$$

$$C(0) = C_0$$

$$\frac{dc}{dt} = \frac{F c_{in}}{V} - \frac{FC}{V}$$

$$\int \frac{dc}{c_{in} - c} = \int dt \left(\frac{F}{V}\right)$$

$c_{in} - C_0 < 0$ $(C_0 > c_{in})$

$$-\ln(c_{in} - C) = \frac{F}{V} t + k$$

$$\ln\left(\frac{1}{c_{in} - c}\right) = \frac{Ft}{V} + k$$

$$\therefore \frac{1}{c_{in} - c} = e^{\frac{Ft}{V} + k}$$

$$c_{in} - c = e^{-Ft/V + k}$$

$$C(t) = c_{in} - e^{-Ft/V \mp k}$$

$$C(0) = C_0$$

$$C_0 = c_{in} - e^{-k}$$

$$c_{in} - C_0 = e^{-k}$$

Solution: $\boxed{C(t) = c_{in} - (c_{in} - C_0) e^{-Ft/V}}$

$$C(t) = c_{in}(1 - e^{-Ft/V}) + C_0 e^{-Ft/V}$$

Let

Example $C_{in} = 0$, we have to solve how long will it take for lake's pollution level to reach 5 % of its initial level.

$$C(t) = C_0 e^{-Ft/V}$$

$$\frac{C}{C_0} = e^{-Ft/V}$$

$$\ln\left(\frac{C}{C_0}\right) = \frac{-Ft}{V}$$

$$t = -\frac{V}{F} \ln\left(\frac{C}{C_0}\right)$$

$$\therefore t = -\frac{V}{F} \ln(0.05) \approx \frac{3V}{F}$$

$\therefore$ Lets take Lake Erie, $V = 458 \times 10^9 \ m^3$

$F = 480 \times 10^6 \ m^3/day$

$$\frac{3(458 \times \cancel{10^9})10^3}{480 \times \cancel{10^6}} \approx \begin{matrix} 7.8 \text{ years} \\ 2850 \text{ days.} \end{matrix}$$

Let's take Lake Ontario, $V = 1636 \times 10^9 \ m^3$

$F = 572 \times 10^6 \ m^3/day$

$$t = \frac{3 \times 1636 \times \cancel{10^9}\ 10^3}{572 \times \cancel{10^6}} = \underline{23.5 \text{ year}}$$

<u>Economic growth Model (EGM)</u>

: Classical EG, increases in population, is available labour increase, and production thrives.

Classical, EGM, considered increased stocks of capital goods that are dependent on available labour & investment capital.

Let us assume that output (production) $Y(t)$ stock capital $K(t)$, available labour force $L(t)$, at time t

$$F(K, L) = Y$$

$$\frac{dK}{dt} = sY(t).$$

$$\frac{dK}{dt} = sF(K, L)$$

(Suppose $L(t) = L_0 e^{rt}$)

$$\frac{dK}{dt} = sF(K, L_0 e^{rt}) \quad \text{— (i)}$$

Let us assume $P(t) = \frac{K(t)}{L(t)}$, ratio of capital to labour

$$K(t) = P(t) L(t)$$

$$K(t) = P(t) L_0 e^{rt}$$

$$\frac{dK}{dt} = r P(t) \cdot L_0 e^{rt} + P'(t) L_0 e^{rt})$$

$$\frac{dK}{dt} = L_0 r P(t) e^{rt} + \frac{dP}{dt} L_0 e^{rt} \quad \text{— (ii)}$$

$$sF(K, L_0 e^{rt}) = \left(\frac{dP}{dt} + Pr\right) L_0 e^{rt}$$

$$s \cancel{L_0 e^{rt}} F\left(\frac{K}{L_0 e^{rt}}, 1\right) = \cancel{L_0 e^{rt}} \left(\frac{dP}{dt} + Pr\right)$$

$$sF\left(\frac{K}{L_0 e^{rt}}, 1\right) = \frac{dP}{dt} + Pr$$

$$s f(P) = \frac{dP}{dt} + Pr$$

$$\frac{dP}{dt} = s f(P) - Pr$$

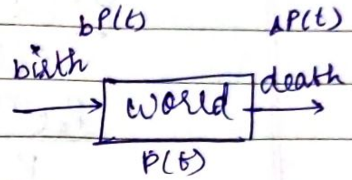
f is a function of capital per worker: Thus rate of change in capital-labour ratio, ρ, difference between the increment of capital and increment of labour.

Cobb-Douglas model, $\frac{d\rho}{dt} = s\rho^{a} - r\rho$

$$f(\rho) = \rho^{a}$$

$$F(L, K) = K^{a} L^{1-a}, \text{ where } a < 1.$$

Population growth Model

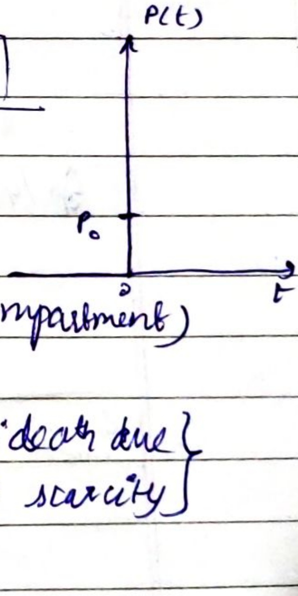


Per capita birth rate b
Per capita death rate d

$$\begin{Bmatrix}\text{rate change} \\ \text{of population}\end{Bmatrix} = \begin{Bmatrix}\text{rate of} \\ \text{birth}\end{Bmatrix} - \begin{Bmatrix}\text{rate of} \\ \text{death}\end{Bmatrix}$$

$$\frac{dP}{dt} = (b-d)P \qquad P(0) = P_0$$

$d < b$, $\quad \frac{dP}{dt} = \underset{\text{growth rate}}{\underset{\uparrow}{r}} P \qquad \boxed{P(t) = P_0 e^{rt}}$



∴ ## Logistic Growth Model

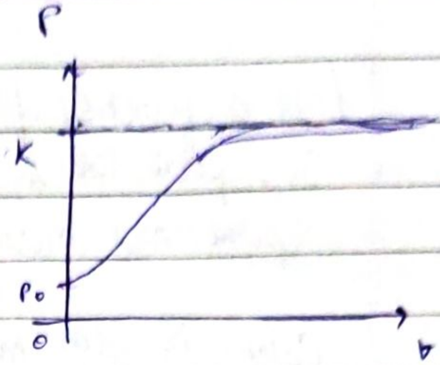
Scarcity of food (Limited resource in the compartment)

$$\begin{Bmatrix}\text{rate change} \\ \text{of population}\end{Bmatrix} = \underset{b}{\begin{Bmatrix}\text{rate of} \\ \text{birth}\end{Bmatrix}} - \underset{d}{\begin{Bmatrix}\text{rate of} \\ \text{death}\end{Bmatrix}} - \begin{Bmatrix}\text{death due} \\ \text{to scarcity}\end{Bmatrix}$$

$$\frac{dP}{dt} = bP - dP - \alpha P^2$$

$$= rP - \alpha P^2 = rP\left(1 - \frac{\alpha P}{r}\right)$$

K: carrying capacity of environment



Let $K = \frac{r}{\alpha}$ $\quad P(0) = P_0$

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

Example: Let $P(t)$ be the population at time 't'.

$$P(t+T) = 2P(t) \qquad P(t) = P_0 e^{r(t-t_0)}$$

$$\frac{P(t+T)}{P(t)} = 2$$

$$\frac{P_0 e^{r(t+T-t_0)}}{P_0 e^{r(t-t_0)}} = 2$$

$$= \cancel{P_0} e^{rT} = 2$$

$$e^{rT} = 2/\cancel{P_0}$$

$$rT = \ln\left(\frac{2}{\cancel{P_0}}\right)$$

$$T = \frac{1}{r} \ln\left(\frac{2}{\cancel{P_0}}\right) = \frac{\ln(2)}{r}$$

$$\boxed{T_n = \frac{\ln(n)}{r}}$$

In 1990, $r = 0.017/yr$ $\qquad T_2 = ?$

$P_0 = 5$[illegible]B $\qquad T_2 = \frac{\ln(2)}{r} =$

$\boxed{\text{Exact Solution}}$: $\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right)$

$$\boxed{\frac{K}{X(K-X)} \frac{dX}{dt} = r}$$

$$\frac{K}{X(K-X)} = \frac{A}{X} + \frac{B}{K-X}$$

$$A(K-X) + B(X) = K$$
$$AK - AX + BX = K$$
$$AK + (B-A)X = K$$
$$B - A = 0$$
$$\boxed{B = A}$$
$$A = 1 = B.$$

$$\int \frac{K}{X(K-X)}\,dX = \int r\,dt$$

$$= \int \frac{1}{X} + \frac{1}{K-X}\,dX = \int r\,dt$$

$$\int \frac{1}{X}\,dX + \int \frac{1}{K-X}\,dX = \int r\,dt$$

$$= \ln X - \ln(K-X) = rt + c$$

$$\ln\left(\frac{X}{K-X}\right) = rt + c$$ Taking antilog

$$\frac{X}{K-X} = e^{rt+c}$$

$$= \frac{X}{K-X} = c_1 e^{rt} \quad (c_1 = e^c)$$

$$X = c_1 (K-X) e^{rt}$$

$$X(1 + c_1 e^{rt}) = c_1 K e^{rt}$$

Let $X(0) = X_0$

$$X_0 (1 + c_1) = c_1 K$$

$$X_0 = \frac{c_1 K}{1 + c_1}$$

$$X_0 + c_1 X_0 = c_1 K.$$

$$\boxed{\left(\frac{X_0}{K - X_0}\right) = c_1}$$

$$X = \frac{c_1 K e^{rt}}{1 + c_1 e^{rt}}$$

$$= \frac{(X_0 / K - X_0) K e^{rt}}{1 + \left(\frac{X_0}{K - X_0}\right) e^{rt}}$$

$$= \frac{X_0 K e^{rt}}{K - X_0 + X_0 e^{rt}} = \frac{X_0 K e^{rt}}{K + (e^{rt} - 1) X_0}$$

$$= \frac{K}{1 + m e^{-rt}} \qquad \boxed{m = \frac{K}{X_0} - 1}$$

$$= \frac{K}{\frac{K + (e^{rt} - 1) X_0}{X_0 e^{rt}}} = \frac{K}{1 + \left(\frac{K}{X_0 e^{rt}} - \frac{1}{e^{rt}}\right)}$$

It $t \to \infty$, $X(t) = K$

Harvesting Model

HW

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right) - H$$ (H: constant rate of harvesting)

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right) - \alpha X$$ (Proportional Harvesting)

$$\frac{dX}{dt} = \frac{rX(K-X) - HK}{K}$$

$$\int \frac{K\, dX}{rX(K-X) - HK} = \int dt$$

Which 20/1

Dynamical systems

- Dynamics is primarily the study of the time revolutionary systems and corresponding system equation is known as dynamical system.

- A system of n-finite ordered DE's in $\mathbb{R}^n$ space is called a dynamical system of dimension-n which determines the time behavior of the system

Deterministic Process - A process is called deterministic, if its entire future and past are completely determined by its state of current/present time.

Types of Dynamical Systems:
- Continuous — differential equation
- Discrete — difference equation

A continuous DS mathematically can be written as:

$$\dot{X} = F(X,t)$$
$$X = (X_1, X_2 \dots X_n) \in R^n$$
$$F = (f_1, f_2 \dots f_n)$$

F : must be continuous smooth function

∴ Here X is known as state variable
F is a smooth function defined on domain of the function $(R^n \times R) = (R^{n+1})$

$$\dot{X} = f(X,t)$$

if independent of time → $\dot{x} = f(X)$ (Autonomus system)

dependent on time → $\dot{x} = f(X,t)$ (Non-autonomus models)

linear

(Fixed Points / Critical points / Equilibrium Point)

$$\dot{X} = f(X) = 0$$
$$x' = 0 = f(X) \rightarrow \text{critical points / state of rest.}$$

It is a constant solution of a dynamical systems.

∴ Stability of equilibrium points

Stability refers to sustaiability. Stability of a equilibrium point determines whetha the solution of dynamical system remains near the equilibrium point after a small perturbation.

Stable: An equilibrium point is stable if initial condition that starts nearby equilibrium point stay close to the equilibrium point

$\dot{x} = f(X)$ - $X(t) \rightarrow$ exact solution of DE
$x_e \rightarrow$ equilibrium point
$x(0) \rightarrow$ initial condition

∴ The equilibrium point x_e is stable if for a $\epsilon > 0$, $\exists$ a $\delta > 0$ such that

$$\underbrace{\|X(0) - X_e\|}_{\text{Norm}} < \delta \Rightarrow \|X(t) - X_e\| < \epsilon \quad \forall t > 0$$

If $\|X(t) - X_e\| = 0$, the asymptotically stable.

∴ <u>How to find stability of a Dynamical system.</u>

<u>1D system</u>: $\dot{X} = f(X)$; $X(0) = X_0$

If $f'(X_e) < 0$, X_e is stable
If $f'(X_e) > 0$, X_e is unstable
If $f'(X_e) = 0$, X_e is degenerate equilibrium point

∴ Logistic growth: $\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) = 0 = rx - \frac{rx^2}{K}$

$x = 0, K$

$\frac{d^2x}{dt^2} = r - \frac{2rx}{K}$

$f'(x_e^1) = r - \frac{2r(0)}{K} = r$ (Unstable)

$f'(x_e^2) = r - \frac{2r(K)}{K} = -r$ (stable)

$\therefore$ <u>Stability</u> 1) <u>Radioactive decay Model:</u>

$$\frac{dN}{dt} = -kN = f(N)$$

$$f'(N) = -k.$$

$\therefore$ Equilibrium point is $N=0$, putting it in $f'(N)$

$$f'(0) = -k < 0$$

$\therefore$ Hence, $N=0$ is a stable equilibrium point

2) <u>Population Growth Model:</u> $\frac{dP}{dt} = (r-d)P = f(P)$

$$f'(P) = (r-d)$$

$\therefore$ Equilibrium point is $P=0$, putting it in $f'(P)$

$$f'(0) = (r-d)$$

$\therefore$ Hence, if $r>d$ then the model is unstable since $f'(0)>0$
if $r<d$ then the model is stable since, $f'(0)<0$

3) <u>Radioactive Elements in Painting:</u> $\frac{dN}{dt} = -\lambda N + R = f(N)$

$$f'(N) = -\lambda$$

$\therefore$ Equilibrium point is $N = R/\lambda$, putting it in $f'(N)$

$$f'(R/\lambda) = -\lambda < 0$$

$\therefore$ Hence, $N = R/\lambda$ is a stable equilibrium point

4) <u>Lake Pollution Model:</u> $\frac{dC}{dt} = \frac{FC_{in}}{V} - \frac{FC}{V} = f(C)$

$$f'(C) = \frac{-F}{V}$$

$\therefore$ Equilibrium point is $C = C_{in}$, putting it in $f'(C)$

$$f'(C_{in}) = \frac{-F}{V} < 0$$

$\therefore$ Hence, $C = C_{in}$ is a stable equilibrium point

5) Logistic Growth Model : $\frac{dX}{dt} = rX\left(1-\frac{X}{K}\right) = f(X)$

$$f'(X) = r - \frac{2rX}{K}$$

$\therefore$ Equilibrium points are $X=0, K$, putting them in $f'(X)$

CASE I : $X = 0$, $f'(0) = r > 0$

$\therefore$ Hence, $X=0$ is an unstable equilibrium point

CASE II : $X = K$, $f'(K) = -r < 0$

Hence, $X = K$ is a stable equilibrium point

6) Constant Harvesting Model : $\frac{dX}{dt} = rX\left(1-\frac{X}{K}\right) - h = f(X)$

$$f'(X) = r - \frac{2rX}{K}$$

$\therefore$ Equilibrium points are $X = \frac{K}{2}\left(1 \pm \sqrt{1-\frac{4h}{rK}}\right)$

CASE I : when $X = \frac{K}{2}\left(1+\sqrt{1-\frac{4h}{rK}}\right) = X_1$

$$f'(X_1) = r - \frac{2r}{\cancel{K}}\left[\frac{\cancel{K}}{\cancel{2}}\left(1+\sqrt{1-\frac{4h}{rK}}\right)\right]$$

$$= r - r - r\sqrt{1-\frac{4h}{rK}} = -r\sqrt{1-\frac{4h}{rK}} < 0$$

$\therefore$ Hence X_1 is a stable equilibrium point

CASE II : when $X = \frac{K}{2}\left(1-\sqrt{1-\frac{4h}{rK}}\right) = X_2$

$$f'(X_2) = r - \frac{\cancel{2}r}{\cancel{K}}\left[\frac{\cancel{K}}{\cancel{2}}\left(1-\sqrt{1-\frac{4h}{rK}}\right)\right] = r\sqrt{1-\frac{4h}{rK}} > 0$$

$\therefore$ Hence, X_2 is an unstable equilibrium point.

2) Proportional Harvesting: $\frac{dX}{dt} = rX\left(1-\frac{X}{K}\right) - \alpha X = f(X)$

$$f'(X) = r - \frac{2rX}{K} - \alpha$$

$\therefore$ equilibrium point is $X = 0,\ (r-\alpha)K/r$.

$\therefore$ CASE I : $X = 0$, $f'(0) = r - \alpha$

Hence, when $r > \alpha$, $X = 0$ is an unstable equilibrium point

when $r < \alpha$, $X = 0$ is a stable equilibrium point

$\therefore$ CASE II, $X = \frac{(r-\alpha)K}{r}$, $f'\left(\frac{(r-\alpha)K}{r}\right) = r - \frac{2r(r-\alpha)K}{Kr} - \alpha$

$$= (r-\alpha) - 2(r-\alpha)$$

$$= -(r-\alpha) = \alpha - r$$

$\therefore$ When, $\alpha > r$, $X = \frac{(r-\alpha)K}{r}$ is an unstable equilibrium point

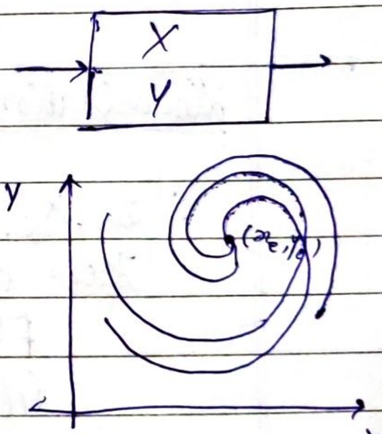
when, $\alpha < r$, $X = \frac{(r-\alpha)K}{r}$ is a stable equilibrium point.

$\therefore$ 2-D Dynamical System

Depend on both X and Y

$\frac{dX}{dt} = \underbrace{f(X,Y)}_{\text{Depends on both } X \,\&\, Y}$ $\qquad \frac{dY}{dt} = G(X,Y)$.

$\therefore$ Coupled Dynamical System



Phase potrait *

Phase - ~~plane~~ Space

Analysis / Diagram

ζ → jai

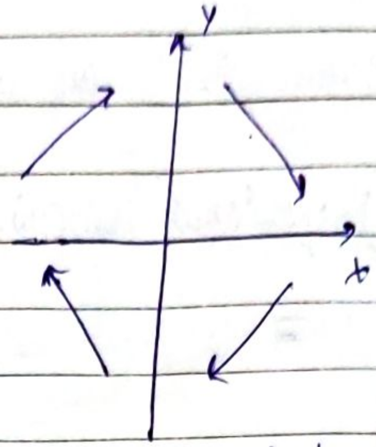
oscalatr
intragate
equibilibrium
Chao

Date / /
Page No.

$$\frac{dX}{dt} = Y \qquad \frac{dY}{dt} = -X$$

Chain Rule: $dY/dX = \dfrac{dY/dt}{dX/dt}$

$$= \frac{-X}{Y}$$



$$\int Y\,dY = \int -X\,dX$$

Clockwise-behaviour

$$\frac{y^2}{2} = \frac{-X^2}{2} + c$$

$$\boxed{y^2 = -x^2 + c}$$

$$\boxed{x^2 + y^2 = c}$$

Linearisation: $\bar{X} = AX$

$$\bar{X} = \begin{bmatrix} dX/dt \\ dY/dt \end{bmatrix} \qquad X = \begin{bmatrix} X \\ Y \end{bmatrix}$$

$AX = B$

$AX = \lambda X$

$|A - \lambda I| = 0$

$(A - \lambda I)X = 0$

Linearisation: $\dfrac{dX}{dt} = F(X,Y) \qquad \dfrac{dY}{dt} = G(X,Y)$ — (i)

∴ Let (X_e, Y_e) is the equilibrium point.
From the dynamics of equilibrium point,

$$\left.\begin{aligned} F(X_e, Y_e) &= 0 \\ G(X_e, Y_e) &= 0 \end{aligned}\right\} \text{— (ii)}$$

Let us suppose $X(t), Y(t)$ is the solution of system (2)

$X(t) = X_e + \xi(t)$, $Y(t) = Y_e + \eta(t)$

∴ Here $\xi(t)$ and $\eta(t)$ be small perturbation that goes to zero when $[X(t), Y(t)] \to [X_e, Y_e]$

$$\frac{dx}{dt} = \frac{d(X_e + \xi(t))}{dt} = F(X_e + \xi(t), Y_e + \eta(t))$$

$$\frac{dY}{dt} = \frac{d(Y_e + \eta(t))}{dt} = G(X_e + \xi(t), Y_e + \eta(t))$$

$$\frac{dx}{dt} = \frac{d\xi}{dt} = F(X_e + \xi(t), Y_e + \eta(t))$$

$$\frac{dY}{dt} = \frac{d\eta}{dt} = G(X_e + \xi(t), Y_e + \eta(t))$$

higher order terms

∴ Functions F and G can be expanded using Taylor series,

$$\frac{d\xi}{dt} = F(X_e, Y_e) + F_x(X_e, Y_e)\xi + F_y(X_e, Y_e)\eta + O(\eta^2)\ldots$$

$$\frac{d\eta}{dt} = G(X_e, Y_e) + G_x(X_e, Y_e)\xi + G_y(X_e, Y_e)\eta + O(\eta^2)\ldots$$

$$\frac{d\xi}{dt} = F_x(X_e, Y_e)\xi + F_y(X_e, Y_e)\eta$$

$$\frac{d\eta}{dt} = G_x(X_e, Y_e)\xi + G_y(X_e, Y_e)\eta$$

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} \xi' \\ \eta' \end{bmatrix} = \underbrace{\begin{bmatrix} F_x(X_e, Y_e) & F_y(X_e, Y_e) \\ G_x(X_e, Y_e) & G_y(X_e, Y_e) \end{bmatrix}}_{\text{Jacobian Matrix}} * \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

$$\begin{cases} a_{11} = F_x(X_e, Y_e) \\ a_{12} = F_y(X_e, Y_e) \\ a_{21} = G_x(X_e, Y_e) \\ a_{22} = G_y(X_e, Y_e) \end{cases}$$

$\cancel{(x+5)(x-4)}$ $\cancel{x^2+x-20=0}$ $f(t)$ $(x-5)(n+4)$

$\boxed{x^2+x-20=0}$

$\frac{3 \pm \sqrt{9-4}}{2}$

$\frac{3 \pm \sqrt{5}}{2}$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Eigen values?

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - (\lambda_1+\lambda_2)\lambda + \lambda_1\lambda_2 = 0$$

$$(a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} = 0$$

$$\lambda^2 - (a_{11}+a_{22})\lambda - a_{12}a_{21} + a_{22}a_{11} = 0$$

Characteristic polynomial of matrix A $\longrightarrow$ $\lambda^2 - (a_{11}+a_{22})\lambda + (a_{22}a_{11} - a_{12}a_{21}) = 0$

(Eigen values can be real or complex conjugate)

∴ CASE I: Real Eigen values:

(i) $\lambda_1 < 0$, $\lambda_2 < 0 \Rightarrow$ System is stable.

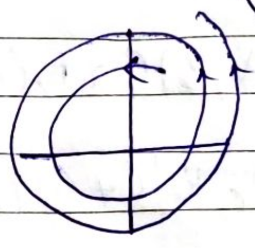
(ii) $\left.\begin{array}{l} \lambda_1 < 0, \lambda_2 > 0 \\ \lambda_1 > 0, \lambda_2 < 0 \end{array}\right\} \Rightarrow$ System is unstable.

(iii) $\lambda_1 > 0$, $\lambda_2 > 0 \Rightarrow$ System is unstable

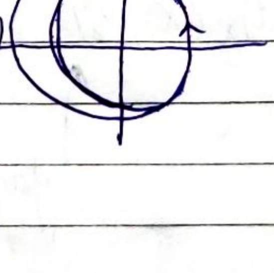
CASE II: Imaginary Eigen Values: $\lambda_{1,2} = \alpha \pm i\beta$

$\alpha < 0$ (stable)
Spiral Inward.

$\alpha > 0$ (Unstable)
Spiral outward



$\alpha = 0$ (Spiral at constant distance)



$\therefore \lambda^2 - \underbrace{(\lambda_1+\lambda_2)}_{p}\lambda + \underbrace{\lambda_1\lambda_2}_{q} = 0$ $\Delta = p^2 - 4q$.

- Both eigen values are real, $p^2 > 4q \Rightarrow \Delta > 0$

Δ	p	q	E_p
$\Delta > 0$	$p < 0$	$q > 0$	stable node $(\lambda_1, \lambda_2 < 0)$
$\Delta > 0$	$p > 0$	$q > 0$	unstable node $(\lambda_1, \lambda_2 > 0)$
$\Delta > 0$	-	$q < 0$	saddle point
$\Delta < 0$	$p < 0$	-	stable spiral
$\Delta < 0$	$p = 0$	-	centre spiral
$\Delta < 0$	$p > 0$	-	unstable spiral

Example: $\frac{dS}{dt} = -\beta SI = F(S,I)$ $\frac{dI}{dt} = \beta SI - \gamma I = G(S,I)$

For equilibrium points, $\frac{dS}{dt} = 0 = -\beta SI$ $\quad \frac{dI}{dt}$ $\beta SI - \gamma I = 0$

$\cancel{S=0}$ or $I=0$ $\quad (\beta S - \gamma)I = 0$

$S=$

$I=0$ and $S=0 = E_1$

$S=0$
$I=0$

$I=0$ and $S = \gamma/\beta = E_2$.

$\cancel{J} =$ $F_S = -\beta I$ $\quad G_S = \beta I$ $\quad \lambda^2 - (-\gamma)\lambda = 0$

$F_I = -\beta S$ $\quad G_I = \beta S - \gamma$ $\quad \lambda(\lambda+\gamma) = 0$ $\quad \boxed{\lambda = 0, -\gamma}$

$$J = \begin{bmatrix} F_S(E_1) & F_I(E_1) \\ G_S(E_1) & G_I(E_1) \end{bmatrix} = \begin{bmatrix} -\beta I & -\beta S \\ \beta I & \beta S - \gamma \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -\gamma \end{bmatrix}$$

$(0,0)$

$\cancel{\lambda^2 - (-\gamma)\lambda + (-\gamma) = 0}$ $\quad \cancel{\frac{-\gamma \pm \sqrt{\gamma^2+4\gamma}}{2}}$

$\cancel{\lambda = 0}$ $\quad \cancel{\lambda^2 + \gamma\lambda - \gamma = 0}$

$\cancel{\Delta = \gamma^2 + 4\gamma > 0}$ $\quad \frac{-\gamma \pm \sqrt{\gamma}\sqrt{\gamma+4}}{2}$

$\frac{\gamma}{2}$

$\Delta = \sqrt{b^2 - 4ac}$

Date / /
Page No.

$$J\left(\frac{\gamma}{\beta}, 0\right) = \begin{bmatrix} 0 & -\gamma \\ 0 & 0 \end{bmatrix}$$

$\cancel{\lambda^2 + \gamma = 0}$ $\boxed{\lambda = 0}$ Hence, its an equilibrium point

$\cancel{\lambda^2 = -\gamma}$

$\cancel{\lambda = \pm i\sqrt{\gamma}}$

Prey & Predator Model

$X(t) \longrightarrow$ population of prey

$Y(t) \longrightarrow$ population of predator

Assumption: Population of predator grows only due to predation.

~~Assumption 1: No predator, $\frac{dX}{dt} = rX$ (exponential growth model)~~

~~Assumption 2: Predator eats no other food, $\frac{dY}{dt}$~~

$$\frac{dX}{dt} = rX - a_1XY \qquad X(0) = X_0 > 0$$

$$\frac{dY}{dt} = a_2XY - dY \qquad Y(0) = Y_0 > 0$$

$\therefore$ Equilibrium point: $\frac{dX}{dt} = 0$ and $\frac{dY}{dt} = 0$

$(r - a_1Y)X = 0 \qquad (a_2X - d)Y = 0$

$Y = \frac{r}{a_1}$ or $X = 0 \qquad Y = 0$ or $X = \frac{d}{a_2}$

$\therefore$ when $Y = 0$, $X = 0$ (E_1)

when $Y = \frac{r}{a_1}$, $X = \frac{d}{a_2}$ (E_2)

$$\begin{bmatrix} r - a_1Y & -a_1X \\ a_2Y & a_2X - d \end{bmatrix} = J$$

$$J_{(0,0)} = \begin{bmatrix} r & 0 \\ 0 & -d \end{bmatrix}$$

$$|J - \lambda I| = 0 = \begin{vmatrix} r-\lambda & 0 \\ 0 & -d-\lambda \end{vmatrix} = 0$$

$$(r-\lambda)(d+\lambda) = 0$$

$$\boxed{\lambda = r, -d}$$ (Saddle Point)

- Unstable equilibrium

$$J_{E_2} = \begin{bmatrix} r - a_1(r/a_1) & -a_1(d/a_2) \\ a_2(r/a_1) & a_2(d/a_2) - d \end{bmatrix} = \begin{bmatrix} 0 & -a_1 d/a_2 \\ a_2 r/a_1 & 0 \end{bmatrix}$$

$$|J - \lambda I| = 0 = \begin{vmatrix} -\lambda & -ad/a_2 \\ a_2 r/a_1 & -\lambda \end{vmatrix} = \lambda^2 - \left(\frac{-a_1 d}{a_2}\right)\left(\frac{a_2 r}{a_1}\right) = 0$$

$$\lambda^2 = -rd$$

$$\lambda = \pm i\sqrt{rd}$$ (Centre Spiral)