—/—/—

Deterministic Modelling :-

– No randomness, ~~no~~ everything certain

$$\boxed{\frac{dP}{dt} = rP}$$

Order and degree of DEs :-

$$y'' + ay' + by = x^4$$

order $= 2$

Degree $= 1$ (Power of the highest derivative)

$$y'' + (y')^3 + ay^2 = x$$

Ord $= 2$

Degree $= 1$

$$(y')^{3/2} + (y'')^0 + 3 = y$$

Linear vs Non-Linear :-

(I) Degree 1 in Independent variable and its derivative

(II) No terms having product of ~~Ind~~dependent variable and its derivative.

$$y\frac{dy}{dx} \qquad \frac{dy}{dx} + xy = \frac{1}{y^2} \Rightarrow \text{non-linear}$$

System of ODEs :-

Compartmental Models :-

Compartment models framework is extremely natural and valuable which formulates models for processes having inputs and/or output over time.

$\longrightarrow$ Compartment $\longrightarrow$

$\xrightarrow{CO_2}$ atmosphere $\longrightarrow$

(burning) (Plant respiration)

Radioactive decay problem :-

Balance Laws :-

{ Net rate change of substance } = { Rate in } - { rate out }

Let $N(t)$ is the no. of radioactive nuclei at time t and $kN(t)$

$$\boxed{\frac{dN}{dt} = -kN(t)}$$

No. of $N(t)$ $\longrightarrow$ decay $k \cdot N(t)$

Suppose time t is current time. δt is the time step.

$\Rightarrow k \cdot N(t)\,\Delta t$

$$N(t+\Delta t) = N(t) - kN(t)\Delta t$$

$$N(t+\Delta t) - N(t) = -kN(t)\Delta t$$

$$\lim_{\Delta t \to 0} \frac{N(t+\Delta t) - N(t)}{\Delta t} = -kN(t)$$

$$\boxed{\frac{dN}{dt} = -kN(t)} \quad \text{(i)} \qquad \Rightarrow \text{Linear Eq}^n$$

$$\boxed{N(t_0) = N_0} \quad \text{(ii)}$$

_/_/_

Separation of variables :-

$$\int \frac{dN}{N} = \int -k\,dt$$

$$\Rightarrow \log N = -kt + c$$

$$\Rightarrow \boxed{N = Ae^{-kt}}$$

$$N(t) = Ae^{-kt} \quad \cdots\cdots (11)$$

$$N(0) = N_0$$

$$N(0) = Ae^{-0}$$

$$\boxed{A = N(0)}$$

putting A in (11)

$$\boxed{N(t) = N(0)\,e^{-k(t)}}$$

(for initial time $t_0 = 0$)

for initial time t_0

$$\boxed{N(t) = N(0)\,e^{-k(t-t_0)}}$$

Population Growth :-

$$\frac{dP}{dt} = rP(t) - dP(t)$$

$$\frac{dP}{dt} = (r-d)\,P(t)$$

$$\boxed{P(0) = P_0}$$

$\downarrow$

Initial Population

Exact solution :-

$$\boxed{\frac{1}{P}\frac{dP}{dt} = r-d}$$

Per capita growth of P in unit time.

Half life :- Let τ is the half life of $N(t)$

$$\boxed{N(t+\tau) = \frac{N(t)}{2}}$$

$$\frac{N(t+\tau)}{N(t)} = \frac{1}{2}$$

$$\Rightarrow \frac{N(0)\,e^{-k(t+\tau-t_0)}}{N(0)\,e^{-k(t-t_0)}} = \frac{1}{2}$$

$$\Rightarrow \boxed{e^{-k\tau} = 1/2}$$

$\cancel{k\tau}$ $k\tau = \log_e(2)$

$$\Rightarrow \boxed{\tau = \frac{\ln(2)}{k}}$$

Uranium $\Rightarrow$ 45 billion years   Radium - 1600 Years.
C^{14} $\rightarrow$ 5568 Year
Lead 20 $\rightarrow$ 22 year
Polonium p^{14} $\rightarrow$ < 1 sec

* All paintings contains small amount of lead - 210
* Lead white contains lead metal (extracted from the rocks) and extremely small amount of radium - 226 (1600 years)

Let $N(t)$ be the amount of lead 210

then, $\frac{dN}{dt} = -\lambda N + R(t)$, $N(0) = N(t_0)$

where $R(t)$ is the rate of disintegration of radium - 226. Per minute / g of white lead.

We assume that $R(t) = R$ (since half life of Radium is very large)

Apply separation of variables :-

$$N(t) = \frac{R}{\lambda}\left(1 - e^{-\lambda(t-t_0)}\right) + n_0 e^{-\lambda(t-t_0)} \quad \ldots\ldots (11)$$

Rearranging (11)

$$\lambda N_0 = \lambda N e^{+\lambda(t-t_0)} - R\left(e^{\lambda(t-t_0)} - 1\right)$$

* λn_0 is disintegration rate from the initial time
* Half Life $\lambda = \frac{\log(2)}{\tau} = 2^{150/11}$

__/__/__

Q. $R = 0.8$, current rate disintegration

Lead 210 $= 8.5\ m^{-1} g^{-1}$

* Use eq.^{n} (ii) from last page.

$\lambda N_0 \geq 98000$

If $\lambda N_0 \leq 30{,}000$; claim $\Rightarrow$ ~~very old~~ New painting.

Modelling pollution in Lake :-

Mass Incoming $\rightarrow$ $\boxed{\text{Mass of pollutant}}$ $\rightarrow$ Mass outgoing

i) The lake has constant volume and if ~~const~~ continuously well mixed so that the pollutant is uniform throughout.

ii) $\{\text{rate change in mass}\} = \{\text{rate enter}\} - \{\text{rate leaving}\}$

iii) Let $c(t)$ be the concentration of pollutant in lake at time t.
Let F be the rate of the water flows out of lake in m^3/day.
Since value is constant.

$$\left\{\begin{matrix}\text{flow mixtures}\\ \text{into lake}\end{matrix}\right\} = \left\{\begin{matrix}\text{flow of mixtures}\\ \text{out of lake}\end{matrix}\right\} = F$$

We know that,
$M(t) = C(t)V$
$\Rightarrow M'(t) = C'(t)V$

Now we get a problem of change of mass

$$M'(t) = F c_{in} - F\frac{M}{V} \Rightarrow c'(t)\,V = F c_{in} - F\frac{C(t)\,\cancel{V}}{\cancel{V}}$$

$$\Rightarrow \boxed{c'(t) = \frac{F c_{in}}{V} - \frac{F\,c(t)}{V}}$$

$$\boxed{C(0) = C_0}$$

__/__/__

$$\frac{dc}{dt} = \frac{F C_{in}}{V} - \frac{F C(t)}{V}, \quad C(0) = C_0$$

$$V \frac{dc}{dt} = F(C_{in} - c(t))$$

$$\int \frac{1}{C_{in} - c(t)}\, dc = \frac{F}{V}\int dt$$

$$\Rightarrow \log(C_{in} - c(t)) = \cancel{E}\,\frac{F}{V} t + k$$

$$\Rightarrow \boxed{c(t) = C_{in} - e^{-k} e^{-Ft/V}}$$

After applying initial condition

$$t = 0$$

$$\cancel{\frac{dc}{dt}} = C(0) = C_{in} - e^{-k} e^{\cancel{-Ft/V}}$$

$$\frac{C_0 - C_{in}}{e^{\cancel{-Ft/V}}} = -e^{-k}$$

$$e^{-k} = \frac{C_{in} - C_0}{e^{\cancel{-Ft/V}}}$$

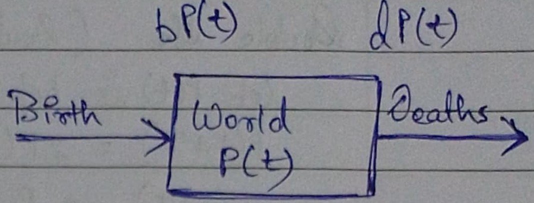
$$\boxed{e^{-k} = C_{in} - C_0}$$

$$\boxed{C(t) = C_{in} - (C_{in} - C_0)\, e^{-Ft/V}}$$

Matlab

ODE45 solver.

Population : Growth Models

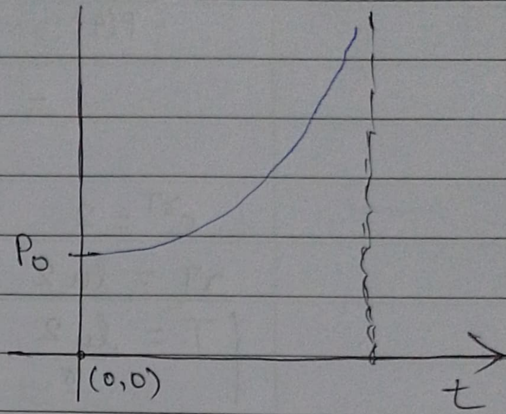


$$\{\text{rate change in population}\} = \left\{\begin{matrix}\text{rate change}\\ \text{birth}\end{matrix}\right\} - \left\{\begin{matrix}\text{rate}\\ \text{death}\end{matrix}\right\}$$

$$\frac{dP}{dt} = bP - dP \quad , \; P(0) = P_0$$

$$\boxed{dP/dt = (b-d)P} \qquad b > d$$

$$\frac{dP}{dt} = rP \longrightarrow \boxed{P(t) = P_0 e^{rt}}$$



* Scarcity of the foods (limited resources in compartment)

$$\boxed{\frac{dP}{dt} = rP}$$

$$\left\{\begin{matrix}\text{rate change}\\ \text{Per capita}\end{matrix}\right\} = \overset{b}{\left\{\begin{matrix}\text{birth}\\ \text{rate}\end{matrix}\right\}} - \overset{d}{\left\{\begin{matrix}\text{death}\\ \text{rate}\end{matrix}\right\}} - \{\alpha P\}$$

(Intra specific competition)

$$\Rightarrow \frac{1}{P}\frac{dP}{dt} = b - d - \alpha P$$

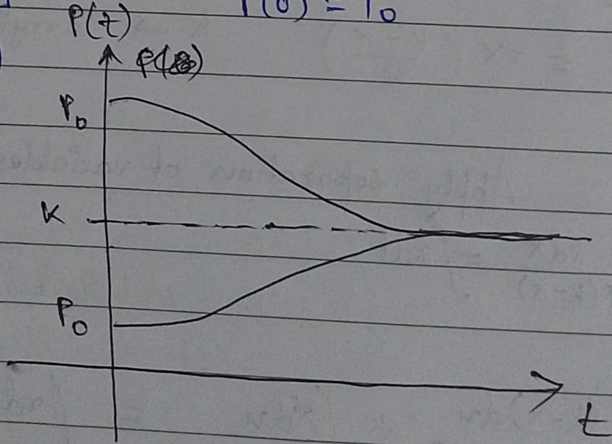
$$\boxed{\frac{dP}{dt} = bP - dP - \alpha P^2}$$

$$= rP\left(1 - \frac{\alpha P}{r}\right)$$

Let $k = \frac{r}{d}$

$$\boxed{\frac{dP}{dt} = rP\left(1 - \frac{P}{k}\right)}$$

$$P(0) = P_0$$



$k \to$ Carrying capacity
$r \to$ Growth rate
$P(t) \to$ Population at time.

$\Longrightarrow$ Logistic Growth Model.

Double life (Double Population size)

Let $P(t)$ is population at time t.

$$P(t+T) = 2P(t) \qquad\qquad P(t) = P_0 e^{r(t-t_0)}$$

$$\frac{P(t+T)}{P(t)} = \frac{2 P_0 e^{r(t+T-t_0)}}{P_0 e^{r(t-t_0)}} \qquad\qquad P(t_0) = P_0$$

$$= 2$$

$$e^{rT} = 2$$

$$rT = \ln 2$$

$$\boxed{T = \frac{\ln 2}{r}}$$

Thus, n times of current Population

$$\boxed{T_n = \frac{\log(n)}{r}}$$

In 1990, Growth rate of world $\boxed{x_0 = 5.3b}$

$r = 0.017/Yr$

Logistic growth Model :-

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right)$$

$$\frac{dx}{dt} = rx\left(\frac{K-x}{K}\right)$$

$x \rightarrow$ is the population of the Individual

$r \rightarrow$ Growth rate of population

$K \rightarrow$ Carrying capacity of the Environment.

Apply separation of variables :-

$$\int \frac{K\,dx}{x(K-x)} = \int r\,dt$$

$$\Rightarrow \int \frac{(K-\cancel{x})\,dx}{x(K-\cancel{x})} + \int \frac{\cancel{x}\,dx}{\cancel{x}(K-x)} = \int r\,dt$$

$$\Rightarrow \ln x + (-\ln(K-x)) = rt + C$$

$$\Rightarrow \ln \frac{x}{K-x} = rt + C$$

$$\Rightarrow \boxed{x = c_1 (K-x) e^{rt}} \qquad (c_1 = e^C)$$

__/__/__

$x(t) = C_1 (K-x) e^{rt}$

$x(0) = C_1 (K - x_0)$.

$\boxed{C_1 = \frac{x_0}{K - x_0}}$ $\boxed{x(t) = \frac{K}{1 + Me^{-rt}}}$ $\boxed{M = \frac{K}{x_0} - 1}$

$\boxed{t \to \infty}$ for t very large

then, $\boxed{x(t) = K}$

for
$$\begin{cases} 0 < x_0 < K & \Rightarrow m = \frac{K - x_0}{x_0} \\ x_0 > K & \Rightarrow -m \Rightarrow x(t) = K \\ x_0 = K & \Rightarrow x(t) = K \end{cases}$$

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right)$$

$r = 1$, $K = 1000$

Harvesting Models :-

Q. find Exact sol.ⁿ of Harvesting Model?

x - No. of individuals

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - H$$

$$\boxed{\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - \alpha x}$$

__/__/__

Dynamical Systems :-

Dynamics is primarily the study of the time-evolutionary systems and corresponding system eq^n is known as dynamical systems.

A system of n-first order DE's in R^n space is called a dynamical system of dimension n, which determines the time behaviour of the system.

Deterministic process :-

A process is called deterministic process if its entire future and past are uniquely determined by its state of current / present time.

Types of Dynamical systems :-

a) Continuous type DS. (Differential Eq^n)
b) Discrete type DS. (Difference Eq^n)

A continuous DS mathematically can be written as

$$\dot{x} = f(\bar{x}, t)$$

$$x = (x_1, x_2 \ldots\ldots x_n) \in \mathbb{R}^n$$

$$f = (f_1, f_2 \ldots\ldots f_n)$$

$$\dot{x}_1 = f_1(x_1, x_2, \ldots\ldots x_n, t)$$

$$\dot{x}_2 = f_2(x_1, x_2, t)$$

$$\dot{x} = f_3(x_1, x_2 \ldots\ldots x_{n-1}, t)$$

Here x is known as state variable. f is a smooth function defined on domain of the function $(\mathbb{R}^n \times \mathbb{R}) = \mathbb{R}^{n+1}$

$$\boxed{\dot{x} = f(x, t)}$$

_/_/_

$$\boxed{\dot{x} = f(x)} \leftarrow \text{Autonomous.}$$

(not dependent upon time)

Kirchoff's voltage Law :-

$\frac{dv}{dt} = \frac{1}{C}$ and $\frac{dI}{dt} = \frac{-RI}{L} - \frac{V}{L}$

$$\boxed{\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)} \qquad \boxed{\frac{dN}{dt} = r(t)N\left(1 - N/K\right)}$$

Fixed Points / Critical Points / Equilibrium points :-

$$\boxed{\dot{x} = f(x)}$$

$\dot{x} = 0 = f(x) \longrightarrow x^{*}$ (critical point)

Critical points are state of the rest.

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

$$\frac{dN}{dt} = 0$$

Either

$rN = 0$

$\boxed{N = 0}$

or,

$1 - \frac{N}{K} = 0$

$\Rightarrow \boxed{N = K}$

Q find Critical points of all models discussed.

1) Radioactive decay :-

$\frac{dN}{dt} = -KN(t) = 0$

$-KN(t) = 0$

$\boxed{N(t) = 0}$

2) Population growth model

$$\frac{dP}{dt} = rP(t) - dP(t) = 0$$

or, $\boxed{P(t) = 0}$

$$\frac{dN}{dt} = -\lambda N(t) + R = 0$$

$N(t) = \frac{R}{\lambda}$ is critical model.

$$\vec{n}(t) = f \times C_{in} - \frac{f N(t)}{V} = 0$$

$$\boxed{N(t) = V C_{in}}$$

3) Lake population model :-

$$\frac{dc}{dt} = \frac{f}{V} C_{in} - \frac{F}{V} C \Rightarrow 0$$

$$\boxed{C = C_{in}}$$

*) $\frac{dP}{dt} = sp^{a} - rp \Rightarrow 0$

$\boxed{p^{a-1} = r/s}$ or $\boxed{P = 0}$

Logistic growth model :-

$$\frac{dp}{dt} = rP(1 - P/K) = 0$$

$\boxed{P = 0}$ or $\boxed{P = K}$

Harvesting model :-

$$\frac{dx}{dt} = rx(1 - x/K) - \alpha x \Rightarrow 0$$

$$x\left[r\left(1 - \frac{x}{K}\right) - \alpha\right] \Rightarrow 0$$

$\boxed{x = 0}$ or $\boxed{x = \left(1 - \frac{\alpha}{r}\right)K}$

_/_/_

Q. Exact solution of Harvesting model :-

$$\frac{dx}{dt} = \frac{rx(K-x)}{K} - K\alpha x$$

$$\frac{K\,dx}{x[r(K-x) - Kx]} = dt$$

$$\frac{K\,dx}{x[K(r-x) - rx]} = dt$$

$$K\left[\frac{A}{x} + \frac{B}{K(r-\alpha) - rx}\right] = dt$$

$$\frac{K(AKr - AK\alpha - Arx + Bx)}{x[K(r-\alpha) - rx]} = dt$$

$$KA(r-x) + x(B - Ar) = 1$$

$$A = \frac{1}{r-\alpha} \qquad B = \frac{r}{r-\alpha}$$

thus, $$\left[\frac{1}{(r-\alpha)x} + \frac{r}{(r-\alpha)[K(r-\alpha) - rx]}\right] dx = dt$$

$$\ln x - \ln\left[1 - \frac{rx}{K(r-\alpha)}\right] = t(r-\alpha) + c(r-\alpha)$$

At $t = 0$, $x = x_0$

$$\ln\left[\frac{x_0 K(r-\alpha)}{K(r-\alpha) - rx_0}\right] = c(r-\alpha)$$

$$\ln\left[\frac{x(Kr - K\alpha)(Kr - K\alpha - rx_0)}{(Kr - K\alpha - rx)(Kr - K\alpha)x_0}\right] = t(r-\alpha)$$

$$\frac{x}{kr - k\alpha - rx} = \frac{x_0}{kr - k\alpha - rx_0} e^{t(r-\alpha)}$$

$$\frac{x}{kr - k\alpha - rx} = c$$

$$x(1 + rc) = (kr - k\alpha)c$$

$$\boxed{x = \frac{kr - k\alpha}{1 + rc} \times c}$$

2/feb/2024

Equilibrium Points :-

x_e is said to be equilibrium point of ① if
$f(x_e) = 0$

It is a constant solution of dynamical system.

* Stability of equilibrium points :-

Stability refers to sustainibility

Stability of a equilibrium point determines whether the solution of dynamical system remains near the equilibrium point after a small perturbation.

An equilibrium point is stable if initial condition that starts nearby equilibrium points stay close to that equilibrium point.

$\dot{x} = f(x)$ $\quad x(t) \longrightarrow$ solution of Dynamic systems.

$x_e \longrightarrow$ Equilibrium point

$x(0) \rightarrow$ Initial value.

The equilibrium point x_e is stable if for a $\epsilon > 0$, $\exists$ a $\delta > 0$ such that,

$||x(0) < x(\epsilon)$

$$||x(0) - x_e|| < \delta \Rightarrow ||x(t) - x_e|| < \epsilon \quad \forall t > 0$$

Norm Distance.

Asymptotically stable

$$||x(t) - x_e|| = 0$$

Q How to find stability of a Dynamical system.

1D system :-

$$\dot{x} = \frac{dx}{dt} = f(x)$$

$$\dot{x} = f(x) , \quad x(0) = x_0$$

* $f'(x_e) < 0$, x_e is stable
$f'(x_e) > 0$, x_e is unstable
$f'(x_e) = 0$, x_e is a degenerate equilibrium point.

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{k}\right) \quad N_1 = 0 , \quad N_2 = K$$

$$f(x) = rN\left(1 - N/K\right)$$

$$f'(N) = r - \frac{2rN}{K}$$

$$f'(N) = f'(0) = r > 0$$

N_1 is unstable

$$f'(k) = r - \frac{2rK}{k}$$

$$= r - 2r = -r$$

$$\boxed{f'(k) = -r < 0}$$

Jacobian Matrix

__/__/__

Equilibrium of all models :-

① Compartment Model :-

(i) Radioactive decay :-

$$\frac{dN}{dt} = -kN$$

$$\boxed{\frac{d^2N}{dt^2} = -k}$$

$N \Rightarrow -kN = 0$

$$\boxed{N = 0}$$

$k > 0 \quad \frac{d^2N}{dt^2} < 0$

$\Rightarrow$ stable at all points.

(ii) Population growth :-

$$\frac{dP}{dt} = (r-d)P$$

$(r-d)P = 0$

$$\boxed{P = 0}$$

② Lake Pollution Model

$$\frac{dc}{dt} = \frac{FC_{in}}{V} - \frac{FC}{V}$$

$\Rightarrow \frac{FC_{in}}{V} - \frac{FC}{V} = 0$

$\Rightarrow \boxed{C_{in} = C}$

$$\frac{d^2C}{dt^2} = -F/V$$

$-F/V > 0$

$\Rightarrow \frac{d^2C}{dt^2} < 0$

$\Rightarrow$ stable at all points.

③ Economic growth :-

$$\frac{dP}{dt} = Sp^a - rp = 0$$

$\Rightarrow Sp^a = rp$

$p^{a-1} = \frac{r}{S}$

$$\boxed{p = \left(\frac{r}{S}\right)^{1/a-1}}$$

$$\frac{d^2p}{dt^2} = aSp^{a-1} - r$$

$p = (r/S)^{1/a-1}$

$aS\left(r/S\right)^{\cancel{1/a-1} \cdot \cancel{a-1}} - r$

$ar - r$

$a(r-1)$

thus, $r < 1 \Rightarrow$ stable

$r > 1 \Rightarrow$ unstable.

Limited Resources Market :-

$$\frac{dP}{dt} = rp\left[1 - \frac{\alpha}{r}p\right] = 0$$

$$\boxed{p = 0}$$

$$1 - \frac{\alpha}{r}p = 0$$

$$\boxed{p = r/\alpha}$$

$$\frac{d^2P}{dt^2} = r - \frac{2\alpha p}{r}$$

$\boxed{p=0} \Rightarrow$ Unstable

$$p = r/\alpha \Rightarrow r - \frac{2\alpha}{r} \cdot \frac{r}{\alpha}$$

$$\Rightarrow (r-2)$$

thus, $r > 2 \Rightarrow$ Unstable

$r < 2 \Rightarrow$ stable.

Logistic Growth Model :-

$$\frac{dx}{dt} = rx(1 - x/k) = 0$$

$$x = 0 \quad , \quad 1 - x/k = 0$$

$$\boxed{x = k}$$

$$rx - \frac{rx^2}{k}$$

$$\frac{d^2P}{dt^2} = r - \frac{2rx}{k}$$

thus, $x = 0 \rightarrow r \rightarrow r > 0$ (unstable)

$x = k \rightarrow r - 2r \Rightarrow -r$ (stable)

Harvesting Model :-

$$\frac{dx}{dt} = rx(1 - x/k) - H = 0$$

$$rx - \frac{rx^2}{k} - H = 0$$

$$krx - rx^2 - kH = 0$$

$$rx - krx + kH = 0$$

$$x^2 - kx + kH/r = 0$$

$$\boxed{x = \frac{k \pm \sqrt{k^2 - 4kH/r}}{2}}$$

$$\frac{dx}{dt} = rx(1 - x/k) - \alpha x = 0$$

$$rx - \frac{rx^2}{k} - \alpha x = 0$$

$$krx - rx^2 - k\alpha x = 0$$

$$rx^2 - krx + k\alpha x = 0$$

$$x^2 - kx + \frac{k\alpha}{r} = 0 \qquad x^2 - \left(k - \frac{k\alpha}{r}\right)x = 0$$

$$x\left(x - \left[k - \frac{k\alpha}{r}\right]\right) = 0$$

$$\boxed{x = 0} \qquad \boxed{x = k + \frac{k\alpha}{r}}$$

$$\frac{d^2x}{dt^2} = rx - \frac{rx^2}{k} - \alpha x$$

$$\Rightarrow r - \frac{2rx}{k} - \alpha$$

for, $x \Rightarrow 0 \longrightarrow r - \alpha$ $\begin{cases} r > \alpha \Rightarrow \text{Unstable} \\ r < \alpha \Rightarrow \text{stable} \end{cases}$

for, $x = k - \frac{kd}{r}$, $r - \frac{2r}{k}\left[k - \frac{k\alpha}{r}\right] - \alpha$

$$= r - 2r + 2\alpha - \alpha$$

$$\Rightarrow \alpha - r$$

$$\begin{cases} r > \alpha \Rightarrow \text{Stable} \\ r < \alpha \Rightarrow \text{Unstable.} \end{cases}$$

_/_/_

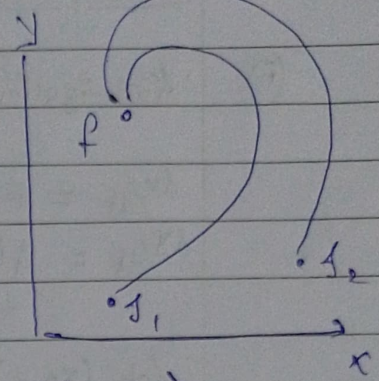
A. dynamical system in 2D

$$\frac{dx}{dt} = F(x, y)$$

$$\frac{dy}{dt} = G(x, y)$$

Coupled dynamical system

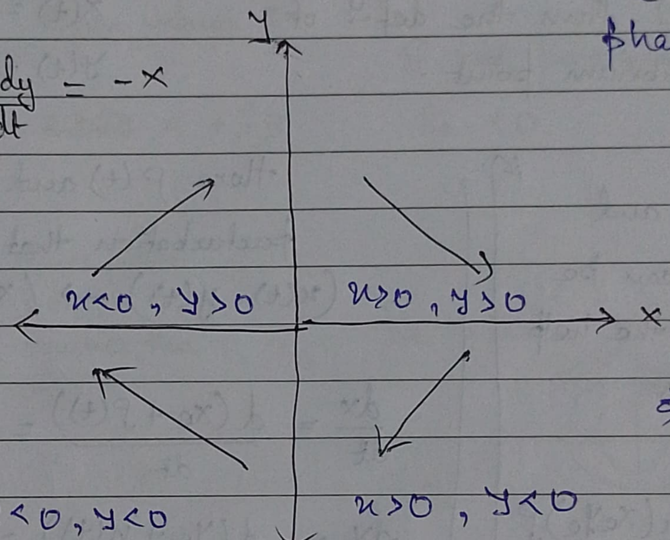
* System of differential equation.



F(x, y)

Phase space Analysis

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x$$



$\Rightarrow$ clockwise behaviour of the system.

<u>Chain rule</u> :-

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\Rightarrow dy/dx = -x/y$$

$$\Rightarrow y\,dy = -x\,dx$$

$$\Rightarrow x^2 + y^2 = 2c$$

$$\Rightarrow \boxed{x^2 + y^2 = k}$$

Linearization

$Ax = b \rightarrow$ Linear

$Ax = \lambda x$

$\Rightarrow |A - \lambda I| = 0$

$\Rightarrow (A - \lambda I)x = 0$

6th feb 2024

① Phase plane Analysis

(ii) Linearization

$$\frac{dx}{dt} = F(x,y)$$
$$\frac{dy}{dt} = G(x,y) \quad \text{—①}$$

Let (x_e, y_e) is equilibrium point of ① from the def.^{n} of equilibrium point.

$$\left.\begin{aligned} F(x_e, y_e) &= 0 \\ G(x_e, y_e) &= 0 \end{aligned}\right\} \cdots \text{(ii)}$$

Let us suppose $(x(t), y(t))$ is solution of system ②

$x(t) = x_e + \rho(t)$,
$y(t) = y_e + n(t)$

Here $\rho(t)$ and $n(t)$ are small perturbation that goes to zero when $(x(t), y(t)) \rightarrow (x_e, y_e)$

$$\frac{dx}{dt} = \frac{d(x_e + \rho(t))}{dt} = F(x_e + \rho, y_e + n)$$

$$\frac{dy}{dt} = \frac{d(y_e + n(t))}{dt} = G(x_e + \rho, y_e + n)$$

$F(x_e + \rho, y_e + n)$ and $G(x_e + \rho, y_e + n)$ can be expanded with the help of Taylor series.

$$\frac{d\rho}{dt} = F(x_e y_e) + F_x(x_e y_e)\rho + F_y(x_e y_e)n + O(h^2)$$

$$\frac{dn}{dt} = G(x_e y_e) + G_x(x_e y_e)\rho + G_y(x_e y_e)n + O(h^2)$$

$O(h^2) \rightarrow$ higher order terms (Ignoring them)

$x_e, y_e \rightarrow$ constant of equilibrium state.

thus, $\frac{dx}{dt} = \frac{d\rho(t)}{dt}$; $\frac{dy}{dt} = \frac{dn}{dt}$

$$\frac{d\rho}{dt} = F_x(x_e, y_e)\rho + F_y(x_e, y_e)n$$

$$\frac{dn}{dt} = G_x(x_e, y_e)\rho + G_y(x_e, y_e)n$$

$$\begin{bmatrix} \rho' \\ n' \end{bmatrix} = \begin{bmatrix} F_x(x_e, y_e) & F_y(x_e, y_e) \\ G_x(x_e, y_e) & G_y(x_e, y_e) \end{bmatrix} \begin{bmatrix} \rho \\ n \end{bmatrix}$$

Jacobian Matrix.

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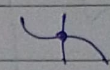
Case :- Real Eigen Values :-

(i) $\lambda_1 < 0$, $\lambda_2 < 0$ $\Rightarrow$ System is stable.

(ii) $\lambda_1 < 0$, $\lambda_2 > 0$
$\lambda_1 > 0$, $\lambda_2 < 0$ $\Rightarrow$ System is unstable.

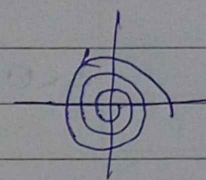
(iii) $\lambda_1 > 0$, $\lambda_2 > 0$ $\Rightarrow$ " " (Saddle Point) $\Rightarrow$ 

Imaginary Eigen values :-

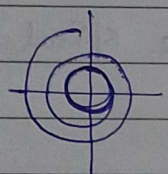
$\lambda_{1,2} = \alpha + i\beta$

If real part, becomes -ve

Re < 0

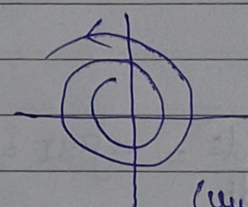


Re = 0



Comes closer but never touches the origin.

Re > 0



(unstable)

$\overset{\circ}{x} = Jx$

$$J = \begin{bmatrix} f_x(x_e, y_e) & F_y(x_e, y_e) \\ G_x(x_e, y_e) & G_y(x_e, y_e) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\lambda^2 - \underbrace{(a_{11} + a_{22})}_{\text{Trace}}\lambda + \underbrace{(a_{11}a_{22} - a_{12}a_{21})}_{\text{Determinant}} = 0$$

$$\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0$$

$\alpha + i\beta$ $\quad \Delta^2 = p^2 - 4q$

Discriminant of quadratic polynomial.

$\Rightarrow$ Both ~~of~~ eigenvalues are real $\Rightarrow$ $p^2 > 4q$ $\Rightarrow$ $\Delta > 0$

(1) Trace

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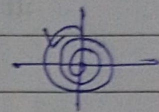
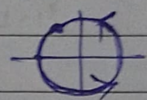
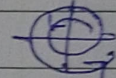
	Δ	P	q	Ep
Real EV	$\Delta > 0$	$P < 0$	$q > 0$	$\Rightarrow$ Stable node $\lambda_1, \lambda_2 < 0$
	$\Delta > 0$	$P > 0$	$q > 0$	$\Rightarrow$ Unstable $\lambda_1, \lambda_2 > 0$
	$\Delta > 0$	$-$	$q < 0$	$\Rightarrow$ Saddle point
	$\Delta < 0$	$P < 0$		$\Rightarrow$ Stable spiral 
	$\Delta < 0$	$P = 0$		$\Rightarrow$ Cetre spiral 
		$P > 0$		$\Rightarrow$ Unstable spiral 

$$\frac{dS}{dt} = -\beta SI = F \qquad \frac{dI}{dt} = \beta SI - dI = G$$

$S(0) > 0$, $I(0) > 0$

In order to find equilibrium points :-

$$\frac{dS}{dt} = 0 \Rightarrow -\beta SI = 0 \quad \ldots\ldots \text{(ii)}$$

$$\frac{dI}{dt} = 0 \Rightarrow \beta SI - dI = 0 \quad \ldots\ldots \text{(iii)}$$

$$(\beta S - d) I = 0$$

$$I = 0, \quad S = d/\beta$$

stability for linear tests

Stability :-

$$J(0,0) = \begin{bmatrix} -\beta I & -\beta S \\ \beta I & \beta S - d \end{bmatrix}_{(0,0)}$$

$$\Rightarrow \boxed{J(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & -d \end{bmatrix}}$$

$(-d-\lambda)\lambda = 0$

$\lambda(d+\lambda) = 0$

$\lambda = 0, -d$

_/_/_

$$J(d/\beta, 0) = \begin{bmatrix} -\beta I & -\beta S \\ \beta I & \beta S - d \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\cancel{\beta}\, d/\cancel{\beta} \\ 0 & \frac{\cancel{\beta}}{\cancel{\beta}} d - d \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -d \\ 0 & 0 \end{bmatrix}$$

Roots = 0

$\lambda_1 = \lambda_2 = 0$

Eigen values are 0

$\Rightarrow E_1$ is also a degenerate equilibrium point.

Prey & Predator Model :-

$x(t) \longrightarrow$ Population of prey.

$y(t) \longrightarrow$ Population of predator.

$A \rightarrow$ No predator. Prey grows exponential.

$$\frac{dx}{dt} = rx - a_1 xy$$

$$\frac{dy}{dt} = a_2 xy - dy$$

$x(0) > 0$

$y(0) > 0$

Equilibrium points :-

$dx/dt = 0 \Rightarrow rx - a_1 xy = 0 \Rightarrow x(r - a_1 y) = 0$ ..... (I)

$dy/dt = 0 \Rightarrow a_2 xy - dy = 0 \Rightarrow y(a_2 x - d) = 0$ ..... (II) from (1) and (11)

(0,0) is the equilibrium points

~~x~~ $r - a_1 y = 0 \Rightarrow y = r/a_1$ (from (1))

$a_2 x^2 - d = 0 \Rightarrow x^* = d/a_2$

$$(x_n, y_n) = \left(d/a_2 ,\ r/a_1\right)$$

$\frac{dx}{dt} = F(x,y)$ $\qquad$ $\frac{dy}{dt} = G(x,y)$

$$\text{thus, } J = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}_{(x_0, y_0)}$$

$$= \begin{bmatrix} r - a_1 y & -a_1 x \\ a_2 y & a_2 x - d \end{bmatrix}_{(x_e, y_e)}$$

$$J(0,0) = \begin{bmatrix} r & 0 \\ 0 & -d \end{bmatrix} \qquad \text{trace} = r - d$$

~~Det~~ $\qquad$ Det $= -rd$

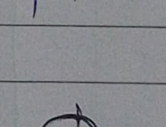
$|J - \lambda| = 0$

$(r-\lambda)(-d-\lambda) = 0$

$\underline{\lambda = -d ,\ \lambda = r} \Rightarrow$ (0,0) is a saddle point.

$$J(x_u, y_u) = \begin{bmatrix} r - a_1 \cdot r/a_1 & -\frac{a_1 d}{a_2} \\ \frac{a_2 r}{a_1} & a_2 \frac{d}{a_2} - d \end{bmatrix}$$

~~$J(x^*, y^*)$~~



$$= \begin{bmatrix} 0 & -a_1/a_2\, d \\ \frac{a_2 r}{a_1} & 0 \end{bmatrix}$$

$\lambda^2 + rd = 0$

$\lambda^2 = -rd$

$\boxed{\lambda = \pm\sqrt{rd}\, i}$

Center Spiral $\Leftarrow$ real part = zero $\Leftarrow$

Q2: An automobile production line turns out about 100 cars a day but deviations occurs owing to many causes. The production is more accurately describe by the probability distribution given as

Production/day	Prob.
95	0.03
96	0.05
97	0.07
98	0.10
99	0.15
100	0.20
101	0.15
102	0.10
103	0.07
104	0.05
105	0.03

Finished cars are transported across the bay at the end of each day by a ferry.
If ferry has space only for 101 cars, what will be the average no. of cars waiting to be shipped and what will be the average no. of empty space.

use random no.s :- 97, 02, 80, 66, 96, 55, 50, 29, 58, 51, 04, 86, 24, 39, 47

		Avg.
total waiting	= 10	0.67
total Empty	= 23	1.53

	CP	Tag
95	0.03	0.0 - 0.02
96	0.08	0.03 - 0.07
97	0.15	0.08 - 0.014
98	0.25	0.15 - 0.024
99	0.40	0.25 - 0.39
100	0.60	0.40 - 0.59
101	0.75	0.60 - 0.74
102	0.85	0.75 - 0.84
103	0.92	0.85 - 0.91
104	0.97	0.92 - 0.96
105	0.10	0.97 - 0.99

~~Production~~

Random no.	Production/day	waiting	Empty
97	105	4	
02	95	–	6
80	102	1	–
66	101	0	0
96	104	3	–
55	100	–	1
50	100	–	1
29	99	–	2
58	100	–	1
51	100	–	1
04	96	–	5
86	103	2	–
24	98	–	3
39	99	—	2
47	100	—	1

Generation of Random Numbers :-

1. Mid square method
2. Linear conguiential : $x_n = ax_{n-1} + c \pmod m$

$$\frac{m}{x_n} - ax_{n-1} = c$$

Mid square method :-

⇒ Von Neumann and Metropolis (1940)

- Start with a 4-digit number z_0 (seed)
- Square it to obtain 8 digits (if necessary, append zero on the left)
- Take the middle 4 digits to obtain the next 4 digit, number z_1
- Then square z_1 and take middle 4-digits again and so on.

we get uniform random number by placing the decimal point at the left of each z_1.

i	z_i	z_i^2
0	7182	51581124
1	5811	33767721
2	7677	58936329
3	9363	87665769
4	6657	44315649
5	3156	09960336
6	9603	92217609
7	2176	04734976
8	7349	54007801
9	0078	00006084
10	0060	00003600
11	0036	00001296
12	0012	00000144
13	0001	0000 0001
14	0000	00000000

Length of cycle = 13

for 2100

$z_0 = 2100$

z_1 – 4100

z_2 – 8100

z_3 – 6100

z_4 – 2100

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Linear congruential method :-

$$x_{i+1} = ax_i + c \pmod m$$
$$x_1 = ax_0 + c \pmod m \qquad i = 0, 1, 2 \ldots\ldots$$

$x_0, x_1, x_2, x_3, x_4 =$

Q// $x_0 = 27$, $a = 17$, $c = 43$

$x_1 = 17 \times 27 + 43 \pmod{100}$

$x_1 = 2 \bmod 100$

$x_2 = 17 \times 2 + 43 \pmod{100}$
$= 77 \bmod 100$

$x_3 = 17 \times 77 + 43 \pmod{100}$
$= 52 \bmod 100$

$x_4 = 17 \times 52 + 43 \pmod{100}$
$= 27$

Q $x_0 = 0$, $a = 5$, $c = 3$ and $m = 7$

$x_1 = 5 \times 0 + 3 \pmod 7$
$= 3 \pmod 7$

$x_2 = 5 \times 3 + 3 \pmod 7$
$= 4 \bmod 7$

$x_3 = 5 \times 4 + 3 \pmod 7$
$= 2$

Sim $x_4 = 6$, $x_5 = 5$, $x_6 = 0$

Algorithm :-

We can generate random numbers from random integers x_i of the LCM by

$$x_{i+1} = ax_i + c \pmod m, \quad i = 0, 1, 2, \ldots\ldots$$

convert integers x_i to random numbers

$$\left[(ax_i + c)/m\right] = x_{i+1}$$

$R_i = \frac{x_i}{m}$, $i = 0, 1, 2$

Note :- $x_i \in [0, m-1]$

$R_i \in [0, 1]$

_/_/_

Theorem: The sequence defined by the congruence relation has full period m, provided that

i) c is relatively prime to m
ii) $a \equiv 1 \pmod{p}$ if p is a prime factor of m
iii) $a \equiv 1 \pmod{4}$ if 4 is a factor of m.

Lewis et al 1969,

$a = 7^5 = \cancel{1608}\ 16807$

$c = 0$

$m = 2^{31} - 1 = 2147483647$

$X_0 \rightarrow$ any integer

period $= m - 1 = 2^{31} - 2$